Eidgenössische Technische Hochschule Zürich

GOOD REDUCTION OF POSTCRITICALLY FINITE QUADRATIC MORPHISMS

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Abstract

We consider quadratic morphisms of smooth curves of genus zero over the field of fractions of a discrete valuation ring. We focus on the case of good reduction, where we study the postcritical orbit over the residue field.

Introduction

Consider a rational map $f \in K(x)$ over a field K as a morphism of the projective line $\mathbb{P}^1_K$. The forward orbit of a point $P \in \mathbb{P}^1_K$ is the set of iterates $f^n(P)$ of P under f for $n \geqslant 0$. In one-dimensional complex dynamics, the orbits of the critical points of a rational map $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ play a fundamental role as they determine the dynamics of the map to a large extent. One source of examples are postcritically finite (pcf) morphisms, where the forward orbit of each critical point is a finite set. In the context of arithmetic dynamics, these maps display certain analogies to elliptic curves with complex multiplication, which is one of the motivations to study pcf morphisms.

Over a field with a valuation, one can further obtain information on the dynamics of the rational map f when it has good reduction. In this case, the dynamics of the reduction over the residue field carry considerable information on the dynamics of f.

In this light, there is an active interest both in criteria for good reduction and in the behaviour of the postcritical orbit after reduction. We hope to provide a contribution to this in the case of quadratic pcf maps, that is, pcf maps of degree two. An example of the overlap of these angles is the following: If a critical point of a quadratic pcf map f is a fixed point, or maps to the other critical point, then the map has good reduction (see Claim 2.1), and if f has good reduction, then a critical point which is not fixed by f cannot reduce to a fixed point of the reduction (see Proposition 10.8).

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In this master's thesis, using the terminology of algebraic geometry, we study the postcritical orbits of quadratic pcf morphisms from a smooth curve of genus zero to itself. Focusing on those with a postcritical orbit of cardinality at least three, which we refer to as *stable*, our main result, Theorem 7.7, states that these morphisms reduce to stable quadratic pcf morphisms whenever they have good reduction. We prove this making use of a combinatorial description in terms of stable marked curves and their associated dual trees. This description was used by Pink in [7] to prove that over any algebraically closed field of characteristic $\neq 2$, there are at most finitely many isomorphism classes of quadratic rational maps of the projective line with a postcritical orbit of size n for any integer n. We will utilise several properties established in that paper.

In Section 1, we endow quadratic morphisms with marked critical points. We define good reduction of a quadratic morphism over the field of fractions of a discrete valuation ring in terms of a smooth model over the ring. The critical marking ensures uniqueness of the smooth model up to unique isomorphism and thus allows us to identify a quadratic morphism of an arbitrary smooth curve of genus zero to itself with a quadratic morphism of $\mathbb{P}^1$.

Following these basic definitions and facts are some examples of pcf morphisms with good reduction in Section 2. In Sections 3-5, we introduce postcritical markings and review the necessary material on stable marked curves and their dual trees, which we use to study the combinatorial properties of the reduction. This is all incorporated in a worked example in Section 6.

We then focus on good reduction of stable quadratic pcf morphisms. Section 7 comprises the proof of Theorem 7.7. As a consequence of this statement, good reduction of a quadratic pcf morphism is equivalent to the existence of a certain unique fixed point of a map describing the combinatorial effect of the morphism on the respective dual tree. This is shown in Section 8. In Section 9 we study good reduction of strictly preperiodic postcritical points in search of a criterion for preperiodicity after reduction. In Section 10 we analyse and give an overview of the dual trees for good reduction, making use of the fixed point from Section 8. The types of trees which arise are in a certain sense well-behaved and reflect the dynamics of the associated morphism, which is not necessarily the case for morphisms with bad reduction, as we demonstrate in several examples in Section 11.

Acknowledgments. Most importantly, I want to express my appreciation to Professor Pink for his outstanding support, inspiration and patience throughout the makings of this thesis. I am lucky to have had an advisor who dedicated so much time and care to my work and who provides such critical insight into the art of mathematics and mathematical writing. I would also like to thank Andreas Wieser for helpful discussions and coffee breaks.

Contents

1 Basic Notions

Let S be a scheme over $\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$.

Definition 1.1. A *critically marked quadratic morphism* over a scheme S is a quadruple $(C, f, \omega_1, \omega_2)$ consisting of a smooth curve C of genus zero over S, an S-morphism $f : C \to C$ which is fibrewise of degree 2 and sections $\omega_1, \omega_2 \in C(S)$ whose images are the ordered critical points of f.

In the special case $S = \operatorname{Spec}(\mathbb{C})$, Milnor [6] refers to these as 'critically marked quadratic rational maps'.

To ease notation, we will often denote a critically marked quadratic morphism by f if the data C, ω_1, ω_2 is clear or not explicitly used, and speak simply of a quadratic morphism. We denote the nontrivial covering automorphism of f by σ, and for a section $s \in C(S)$ we write $f(s) := f \circ s$.

Definition 1.2. An *isomorphism* $\alpha : (C, f, \omega_1, \omega_2) \xrightarrow{\sim} (C', g, \omega_1', \omega_2')$ of quadratic morphisms over S is an isomorphism $\alpha : C \xrightarrow{\sim} C'$ over S such that $\alpha \circ f = g \circ \alpha$ and $\alpha(\omega_1) = \omega_1'$ and $\alpha(\omega_2) = \omega_2'$.

Let R be a discrete valuation ring with field of fractions K, uniformiser π and residue field $k := R/R\pi$ of characteristic $\neq 2$. Further, let $S := \operatorname{Spec} R$.

Definition 1.3. A *smooth model* for a quadratic morphism $(C, f, \omega_1, \omega_2)$ over K is a quadratic morphism $(\mathcal{C}, \varphi, \omega_1, \omega_2)$ over R where the generic fibre of $\mathcal{C}$ is C and φ is an R-morphism extending the K-morphism f to $\mathcal{C}$ and $\omega_1, \omega_2 : S \to \mathcal{C}$ are sections extending the K-valued points ω_1, ω_2.

We will make use of the following assertions in order to prove uniqueness of a smooth model up to unique isomorphism.

Fact 1.4. *Every smooth curve C of genus zero over S together with two disjoint sections $P, Q \in C(S)$ is isomorphic to $(\mathbb{P}^1_S, 0, \infty)$ and the isomorphism is unique up to units in $\mathcal{O}_S$.*

Claim 1.5. *Let $(\mathbb{P}^1_K, f : x \mapsto \frac{ax^2+b}{cx^2+d}, 0, \infty)$ be a quadratic morphism over K with $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(K)$. Then A may be represented by a matrix with coefficients in R with at least one in $R^\times$ and this representation is unique up to multiplication by a unit in R.*

Proof. Choose a matrix representing A and for simplicity denote it again by A. Define $\mu(A) := \min\{\operatorname{ord}_\pi(t) \mid t \text{ is a coefficient of } A\}$ and set $s := \pi^{-\mu(A)}$. Then

sA is of the desired form. Further, we have $\mu(rA) = \mathrm{ord}_\pi(r) + \mu(A)$ for every nonzero $r \in R$. Thus, any other choice $s' \in K^\times$ yields such a form if and only if $s' = rs$ for some $r \in R^\times$. $\square$

This and the next lemma can be found in slightly different terms in Silverman [8, Section 2]. With a representation as in Claim 1.5, we say f (or the matrix A) is in *normalised form.*

Lemma 1.6. *A quadratic morphism $(\mathbb{P}^1_K, f : x \mapsto \frac{ax^2+b}{cx^2+d}, 0, \infty)$ in normalised form extends to a quadratic morphism over R if and only if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R)$.*

Proof. In normalised form, f extends to a rational map $f_R : \mathbb{P}^1_R \dashrightarrow \mathbb{P}^1_R$ again given by $x \mapsto \frac{ax^2+b}{cx^2+d}$. We need to show that the induced map f_k on the closed fibre $\mathbb{P}^1_k$ is a quadratic morphism over k. Set $p(x) := ax^2 + b$ and $q(x) := cx^2 + d$ as well as $\bar{p} := p \bmod \pi$ and $\bar{q} := q \bmod \pi$. Then f_k is a quadratic k-morphism precisely when $\bar{p}$ and $\bar{q}$ have no common zeros in $\bar{k}$. This is true if and only if their resultant $\mathrm{Res}(\bar{p}, \bar{q}) = \overline{\mathrm{Res}(p, q)}$ is nonzero in k and equivalently, if $\mathrm{Res}(p, q)$ is a unit in R. Since $\mathrm{Res}(p, q) = (ad - bc)^2$, we have $\mathrm{Res}(p, q) \in R^\times$ if and only if $\det(A) \in R^\times$. $\square$

Proposition 1.7. *If there exists a smooth model $(\mathcal{C}, \varphi, \omega_1, \omega_2)$ for $(C, f, \omega_1, \omega_2)$ over S, then this model is unique up to unique isomorphism.*

Proof. By Fact 1.4, we can choose a coordinate x such that $\psi : C \xrightarrow{\sim} \mathbb{P}^1_K$ sends (ω_1, ω_2) to $(0, \infty)$. In this coordinate, the covering involution of f is given by $\sigma(x) = -x$ and f is of the form $f(x) = \frac{ax^2+b}{cx^2+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(K)$, where the coefficients are determined by the choice of ψ. Conjugation by an automorphism $x \mapsto ux$ for $u \in K^\times$ changes this into $f(x) = \frac{aux^2+bu^3}{cx^2+du^2}$ with $A_u := \begin{pmatrix} au & bu^3 \\ c & du^2 \end{pmatrix} \in \mathrm{PGL}_2(K)$.
By Claim 1.5, we may represent A_u by a matrix in normalised form, again denoted by A_u and unique up a scalar in $R^\times$. By Lemma 1.6, the rational map f_R induced by f is a quadratic R-morphism - and hence a smooth model for f - if and only if $\det(A_u)$ is a unit in R. This condition determines u and thus ψ up to units in R: If both $\det(A_u)$ and $\det(A_1)$ are units in R, then $\det(A_1)^{-1}\det(A_u) = u^3$ is a unit in R and thus, so is u.
Suppose $(\mathcal{C}, \varphi, \omega_1, \omega_2)$ is another smooth model for f. The choice of ψ from above for the generic fibre $(C, f, \omega_1, \omega_2) \xrightarrow{\sim} (\mathbb{P}^1_K, x \mapsto \frac{ax^2+b}{cx^2+d}, 0, \infty)$ is an isomorphism on a dense subset and thus extends to a unique isomorphism of quadratic R-morphisms $\alpha : (\mathcal{C}, f, \omega_1, \omega_2) \xrightarrow{\sim} (\mathbb{P}^1_R, x \mapsto \frac{ax^2+b}{cx^2+d}, 0, \infty)$. $\square$

Definition 1.8. We say f has *good reduction* if a smooth model for f exists. Since the smooth model is then unique up to unique isomorphism, combined with Fact 1.4, we may identify this model with $(\mathbb{P}^1_R, f_R : x \mapsto \frac{ax^2+b}{cx^2+d}, 0, \infty)$. The restriction of f_R to the closed fibre $\mathbb{P}^1_k$ is denoted by $\bar{f}$ and is given by the reduction of the coefficients of f modulo π. We call $\bar{f}$ *the reduction of* f.

Definition 1.9. A quadratic morphism f over a field is *postcritically finite* if the (strictly) *postcritical orbit* $\{f^n(\omega_1), f^n(\omega_2) \mid n \geq 1\}$ is finite. We refer to such morphisms as *pcf morphisms*.
A quadratic morphism f over S is *stable* if in every fibre the postcritical orbit has cardinality at least three.

Remark 1.10. Let $(\mathcal{C}, f, \omega_1, \omega_2)$ be a smooth model for f. An isomorphism of quadratic morphisms $\alpha : (C, f, \omega_1, \omega_2) \xrightarrow{\sim} (C', g, \omega_1', \omega_2')$ maps the postcritical orbit of f to the postcritical orbit of g: For $i = 1, 2$ and $n \geqslant 0$, we have $\alpha(f^n(\omega_i)) = g^n(\alpha(\omega_i)) = g^n(\omega_i')$. Thus, the identification in Definition 1.7 does not affect the combinatorial type of the postcritical orbit of the morphism.

Two more facts we will need are the following:

Fact 1.11. *A quadratic morphism over a field K is stable if and only if it is not isomorphic to $(\mathbb{P}^1_K, x \mapsto ax^{\pm 2}, 0, \infty)$ for any sign and any $a \in K^\times$, see for example Pink* [7, Prop. 1.4].

Fact 1.12. *For any point $P \in \mathcal{C}(K)$ let $\bar{P} \in \mathcal{C}(k)$ denote the corresponding point in the closed fibre. Then $\bar{f}(\bar{P}) = \overline{f(P)}$ and $\bar{f}^n = \overline{f^n}$ for all $n \geqslant 0$. In particular, the reduction of the postcritical orbit of f coincides with the postcritical orbit of the reduction of f when f has good reduction.*

For a proof of Fact 1.12, see Silverman [8, Thm. 2.18], or Hutz [3, Thm. 8] for a version in the language of schemes.

2 Three Examples of Good Reduction

For quadratic morphisms with postcritical orbit of certain types, one can use pedestrian methods to show that these morphisms have good reduction, as the following proof shows.

Claim 2.1. *For any pcf morphism f given by $x \mapsto (ax^2+1)^{\pm 1}$ or $x \mapsto \frac{x^2+a}{x^2-a}$ for some sign and some $a \in K^\times$, both a and a^{-1} are integral over $\mathbb{Z}[\frac{1}{2}]$ and f thus has good reduction.*

Proof. In all three cases, f has good reduction if a does not reduce to 0 or ∞ in any residue field. So for the reduction assertion it is indeed sufficient to show that a and a^{-1} are integral over $\mathbb{Z}[\frac{1}{2}]$. The postcritical orbit of f is determined by two equations. If one of these is $f^n(0) = -f^m(0)$ for some $n > m \geq 0$, then we can recursively define certain polynomials $p_k, q_k \in \mathbb{Z}[\frac{1}{2}, \alpha]$, where α is an indeterminate, so that $f^k(0) = \frac{p_k}{q_k}$ for all $k \geq 0$. Then $f^n(0) = -f^m(0)$ is equivalent to $p_n q_m + p_m q_n = 0$ and the coefficient a of f is a root of an irreducible factor $P = P_f \in \mathbb{Z}[\frac{1}{2}, \alpha]$ of the polynomial $p_n q_m + p_m q_n$. We claim that both the leading coefficient $\ell c(P)$ and the constant coefficient $cc(P)$ of P are units in $\mathbb{Z}[\frac{1}{2}]$.

Case 1. $f(x) = ax^2 + 1$: In this case ∞ is a fixed point of f (we will see in Lemma 8.3 that this implies that f has good reduction). The second equation is $f^n(0) = -f^m(0)$ for some $n > m \geq 0$. Set $p_0 := 0$ and $p_{k+1} := \alpha p_k^2 + 1$ for $k \geq 0$. Then $p_n = f^n(0) = -f^m(0) = -p_m$ and P is a factor of the polynomial $p_n + p_m$. By induction arguments, the following holds:

(i) $\mathrm{ord}_\pi(p_k) = 0$ for all $k \geq 1$,

(ii) $\deg_\alpha(p_k) = 2^{k-1} - 1$ for all $k \geq 1$,

(iii) $\ell c(p_k) = 1$ and $cc(p_k) = 1$ for all $k \geq 1$.

By (ii), we have $\deg_\alpha(p_k) > \deg_\alpha(p_{k'})$ for all $k > k'$. This, together with (iii) implies that $\ell c(p_m + p_n) = \ell c(p_n) = 1$. Furthermore, the constant coefficient of $p_m + p_n$ is given by $cc(p_m + p_n) = cc(p_m) + cc(p_n) = 2$ by (iii).

Case 2. $f(x) = (ax^2 + 1)^{-1}$: Here we have $f(\infty) = 0$ and the second equation is either $f^n(0) = \infty$ for $n > 1$, or $f^n(0) = -f^m(0)$ for some $n > m \geq 0$. Set $p_0 := 0$, $q_0 := 1$ and $p_{k+1} := q_k^2$, $q_{k+1} := \alpha p_k^2 + q_k^2$ for $k \geq 0$. Then P is a factor of either q_n or $p_n q_m + p_m q_n$. By induction we find that

(i) $\mathrm{ord}_\pi(p_k) = 0 = \mathrm{ord}_\pi(q_k)$ for all $k \geq 1$,

(ii) $\deg_\alpha(q_{2k-1}) = \deg_\alpha(p_{2k-1})$ and $\deg_\alpha(q_{2k}) = \deg_\alpha(p_{2k}) + 1$ for all $k \geq 1$,

(iii) $\ell c(p_k) = 1 = \ell c(q_{k-1})$ and $cc(p_k) = 1 = cc(q_{k-1})$ for all $k \geq 1$.

From this we can calculate that $\ell c(p_m q_n + p_n q_m)$ is 2 if $m \equiv n \bmod (2)$ and $n > 2$ and is 1 otherwise, and $cc(p_m q_n + p_n q_m)$ is 2 for all $m > n > 0$ and is 1 if $n = 0$.

Case 3. $f(x) = \frac{x^2+a}{x^2-a}$: In this case $f(\infty) = -f(0)$ and the second equation is $f^n(0) = -f^m(0)$ for some $n > m \geq 0$. Define $p_0 := 0$, $q_0 := 1$ and for $k \geq 0$ set $p_{k+1} := p_k^2 + \alpha q_k^2$, $q_{k+1} := p_k^2 - \alpha q_k^2$. Then P is a factor of $p_n q_m + p_m q_n$. Again, by induction

(i) $\mathrm{ord}_\pi(p_k) = 2^{k-1} = \mathrm{ord}_\pi(Q_k)$ for all $k \geq 1$,

(ii) $\deg_\alpha(q_k) = 2^k - 1 = \deg_\alpha(p_k)$ for all $k \geq 1$,

(iii) $\ell c(p_k) = (-1)^{2^k-2} = 1$, $\ell c(q_{k-1}) = (-1)^{2^k-1} = -1$ and $cc(\alpha^{-2^{k-1}} p_k) = 1 = \ell c(\alpha^{-2^k} q_{k+1})$ for all $k \geq 1$.

From this we can derive that $\ell c(p_m q_n + p_n q_m) = -2$ for all $m > n \geq 1$, and for the constant coefficient we find $cc((p_m q_n + p_n q_m)\alpha^{-(2^{m-1}+2^{n-1})}) = 2$ for all $m > n \geq 2$. For $n = 1$, the polynomial $(p_m q_1 + q_m p_1)\alpha^{-(2^{m-1}+1)}$ is divisible by α. However, this polynomial corresponds to the equation $f^m(0) = -f(0)$ and since $f(\infty) = -f(0)$, this is equivalent to $f^{m-1}(0) = \infty$, which corresponds to q_m with $cc(q_m \alpha^{-2^{m-1}}) = 1$.
In all of the above cases, P is a factor of a polynomial with leading and constant coefficients 1 or 2, which are units in $\mathbb{Z}[\frac{1}{2}]$. Therefore, in all three cases $\ell c(P)$ and $cc(P)$ are also units, so P is (associated to) a monic polynomial and any root a of P thus integral over $\mathbb{Z}[\frac{1}{2}]$. Moreover, since $cc(P)$ is a unit, the inverse a^{-1} is also integral over $\mathbb{Z}[\frac{1}{2}]$. □

Integrality of the coefficient a and its inverse a^{-1} is, however, not sufficient for good reduction of a large collection of quadratic morphisms, e.g. for morphisms given by $x \mapsto \frac{x^2+a}{x^2+h(a)}$ for any polynomial $h(a) \neq -a$ with a nonzero constant term. For this reason, we will be using additional machinery to analyse good reduction of stable quadratic pcf morphisms on a more general level.

3 Postcritical Marking

To start with, we will add a kind of level structure by marking the postcritical orbit of a stable quadratic pcf morphism following Pink [7, Sections 2 and 7].

Definition 3.1. A *finite mapping scheme* is a quadruple (Γ, τ, i_1, j_1) consisting of a finite set Γ, a map $\tau : \Gamma \to \Gamma$ and two distinct elements $i_1, j_1 \in \Gamma$ such that with $i_n := \tau^{n-1}(i_1)$ and $j_n := \tau^{n-1}(j_1)$ for all integers $n \geq 2$, the following is satisfied:

(i) $\Gamma = \{i_n, j_n \mid n \geq 1\}$.

(ii) Any element γ of Γ has at most two preimages under τ.

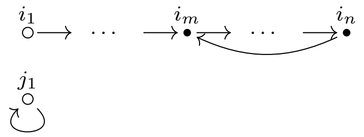
(iii) The distinguished elements i_1 and j_1 have at most one preimage under τ.

For brevity we will denote a finite mapping scheme by Γ if the data τ, i_1, j_1 is understood.

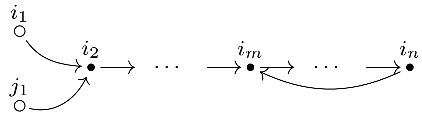
Remark 3.2. For any quadratic morphism f over a field and ω one of the critical points of f, we have $f^{-1}(f(\omega)) = \{\omega\}$, which implies $f(\omega_1) \neq f(\omega_2)$. For any noncritical point $P \in C(K)$ the preimage $f^{-1}(f(\omega))$ is the set $\{P, \sigma(P)\}$. Thus, the postcritical orbit of f with the map induced by f and the distinguished elements $i_1 := f(\omega_1)$ and $j_1 := f(\omega_2)$ is a finite mapping scheme in the sense of the above definition whenever f is postcritically finite.

Example 3.3. The postcritical orbits of the morphisms discussed in Claim 2.1 are the following mapping schemes:

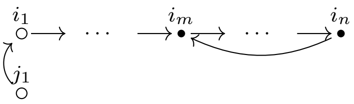
$x \mapsto ax^2 + 1$ and $n > m \geq 1$



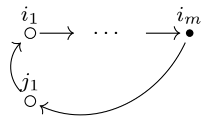
$x \mapsto \frac{x^2+a}{x^2-a}$ and $n > m \geq 1, m \neq 2$



$x \mapsto (ax^2 + 1)^{-1}$ and $n > m \geq 2$



or



Definition 3.4. Let f be a quadratic pcf morphism over K, and let Γ denote the postcritical orbit of f, with a natural map $s : \Gamma \hookrightarrow C(K)$ which sends i_n to $s(i_n) := f^n(\omega_1)$ and j_n to $s(j_n) := f^n(\omega_2)$. We call the quintuple $(C, f, \omega_1, \omega_2, s)$ a *postcritically marked quadratic morphism* and refer to the map s as the *postcritical marking* for f.

For the next section, we extend the postcritical marking $s : \Gamma \to C(K)$ to all points in the non-strictly postcritical orbit and their σ-conjugates.
By definition of the abstract mapping scheme, the elements $i_1, j_1 \in \Gamma$ each have at most one preimage in Γ under τ and every other element has at most two preimages in Γ under τ. If i_1 has a preimage, denote it by i_0. If such an element does not exist in Γ, choose a new symbol $i_0 \notin \Gamma$. Repeat this for j_1. For each $\gamma \in \Gamma \setminus \{i_0, j_0\}$ such that γ is the only preimage of $\tau(\gamma)$, choose a new symbol $\sigma(\gamma) \notin \Gamma \cup \{i_0, j_0\}$. Set $\tilde{\Gamma} := \Gamma \cup \{i_0, j_0, \sigma(\gamma) \mid \gamma \in \Gamma \setminus \{i_0, j_0\}, |\tau^{-1}(\tau(\gamma))| = 1\}$ and define an automorphism $\sigma : \tilde{\Gamma} \to \tilde{\Gamma}$ of order two as follows:

$$\begin{cases} \gamma \mapsto \gamma, & \gamma \in \{i_0, j_0\}, \\ \gamma \mapsto \gamma' \mapsto \gamma, & \gamma \neq \gamma' \text{ and } \{\gamma, \gamma'\} = \tau^{-1}(\tau(\gamma)), \\ \gamma \mapsto \sigma(\gamma) \mapsto \gamma & \text{otherwise.} \end{cases}$$

We then extend τ to a surjective map $\tilde{\tau} : \tilde{\Gamma} \to \Gamma$ satisfying

$$\begin{cases} i_0 \mapsto i_1 \\ j_0 \mapsto j_1 \\ \sigma(\gamma) \mapsto \tau(\gamma), & \sigma(\gamma) \in \tilde{\Gamma} \setminus \Gamma \\ \gamma \mapsto \tau(\gamma) & \text{otherwise.} \end{cases}$$

Since the fixed points of $\sigma : \tilde{\Gamma} \to \tilde{\Gamma}$ are precisely i_0 and j_0, and for each $\gamma \in \Gamma$ the preimage $\tilde{\tau}^{-1}(\gamma)$ is the set $\{\gamma, \sigma(\gamma)\} \subset \tilde{\Gamma}$, this map induces an isomorphism $\tilde{\tau} : \tilde{\Gamma}/\langle\sigma\rangle \xrightarrow{\sim} \Gamma$. Moreover, if $i_0 \in \Gamma$, we have $f(\omega_1) = s(i_1) = s(\tau(i_0)) = f(s(i_0))$ and thus $s(i_0) = \omega_1$. Similarly, if $j_0 \in \Gamma$, then $s(j_0) = \omega_2$. So the following extension of s to an injective map $\tilde{s} : \tilde{\Gamma} \to C(K)$ is welldefined:

$$\begin{cases} i_0 \mapsto \omega_1 \\ j_0 \mapsto \omega_2 \\ \sigma(\gamma) \mapsto \sigma(s(\gamma)), & \sigma(\gamma) \in \tilde{\Gamma} \setminus \Gamma \\ \gamma \mapsto s(\gamma) & \text{otherwise.} \end{cases}$$

The image of $\tilde{s}$ is precisely the set of points in the non-strictly postcritical orbit and their σ-conjugates. The set $\tilde{s}(\Gamma \cup \{i_0, j_0\})$ is the non-strictly postcritical orbit $\{f^n(\omega_1), f^n(\omega_2) \mid n \geq 0\}$ of f. We call $\tilde{s} : \tilde{\Gamma} \to C(K)$ the *extended postcritical marking* for f.

4 Stable marked models

We will use the additional level structure to obtain certain stable marked curves, which we introduce only as far as necessary in this context. The content of this section is derived from Knudsen [5], Keel [4] and Pink [7]. See also Deligne-Mumford [1] for more on stable curves and their moduli, or Gerritzen et al. [2] on stable marked trees of projective lines.

Definition 4.1. A *stable marked curve* $(\mathcal{C}, s)$ *of genus zero* over a scheme S is a flat proper morphism $\mathcal{C} \to S$ together with an injective map $s : I \hookrightarrow \mathcal{C}(S)$, $i \mapsto s(i)$ from a finite set I, and such that

(i) each geometric fibre $\mathcal{C}_x$ is a reduced connected curve with at worst ordinary double points, each irreducible component of which is isomorphic to $\mathbb{P}^1$,

(ii) for all $i \in I$, the sections $s(i)$ are fibrewise distinct and land in the smooth locus of $\mathcal{C}$,

(iii) (stability condition) each irreducible component of $\mathcal{C}_x$ contains at least three points which are either singular or the image of a section $s(i)$, and

(iv) $\dim H^1(\mathcal{C}_x, \mathcal{O}_{\mathcal{C}_x}) = 0$.

Conditions (i) and (iv) imply that each geometric fibre is a tree of copies of $\mathbb{P}^1$. In the following, we abbreviate the expression 'stable marked curve of genus zero' by stable marked curve.

One can obtain one stable marked curve from another by removing a marking:

Definition 4.2 (Contraction, Part I)**.** Let $(\mathcal{C}, s)$ be a stable marked curve over a scheme S and I' a subset of I with $|I'| = |I| - 1 \geqslant 3$. Let $(\mathcal{C}', s')$ be a stable marked curve with marking $s' : I' \hookrightarrow \mathcal{C}'$. Then $(\mathcal{C}', s')$ over S is a *contraction* of $(\mathcal{C}, s)$ if there exists an S-morphism $\kappa : \mathcal{C} \twoheadrightarrow \mathcal{C}'$ such that $\kappa \circ s|_{I'} = s'$ and on every geometric fibre $\mathcal{C}_x$ the following happens:
If the irreducible component Y of $\mathcal{C}_x$ containing the image $P := s(i)(x), i \in I \setminus I'$ has at least three points other than P which are either singular or the image of a section, then the induced morphism κ_x is an isomorphism. Otherwise, κ_x contracts Y to a point and the restriction of κ_x to $\mathcal{C}_x \setminus Y$ is an isomorphism.

Proposition 4.3 ([5, Prop. 2.1])**.** *For any stable marked curve with $n+1$ markings, with $n \geqslant 3$, there exists up to unique isomorphism precisely one contraction to a stable marked curve with n markings.*

This process can be extended to the removal of several markings:

Definition 4.4 (Contraction, Part II)**.** Let $(\mathcal{C}, s)$ be a stable marked curve over S with marking $s : I \hookrightarrow \mathcal{C}(S)$, and $(\mathcal{C}', s')$ a stable marked curve over S with marking $s' : I' \hookrightarrow \mathcal{C}'(S)$ such that $I' \subset I$. We call $(\mathcal{C}', s')$ a *contraction* of $(\mathcal{C}, s)$ if $(\mathcal{C}', s')$ can be obtained from $(\mathcal{C}, s)$ as follows:
Consider a sequence of subsets $I := I_n \supset I_{n-1} \supset \cdots \supset I_{n-k} := I'$, where $|I_\ell| = \ell$ for each $n \geqslant \ell \geqslant n-k \geqslant 3$. Set $(\mathcal{C}_n, s_n) := (\mathcal{C}, s)$ and for each subset, let $(\mathcal{C}_{\ell-1}, s_{\ell-1})$ denote the contraction of $(\mathcal{C}_\ell, s_\ell)$ together with the S-morphism $\kappa_\ell : \mathcal{C}_\ell \twoheadrightarrow \mathcal{C}_{\ell-1}$, in the sense of Definition 4.2. Then $(\mathcal{C}', s')$ is $(\mathcal{C}_{n-k}, s_{n-k})$ given by k successive contraction morphisms $\kappa_{n-k} \circ \cdots \circ \kappa_n : \mathcal{C} \twoheadrightarrow \mathcal{C}'$.

As a consequence of Proposition 4.3, given a stable marked curve, a contraction in the sense of Definition 4.4 is unique up to unique isomorphism.

Example 4.5. Let $(C, f, \omega_1, \omega_2, s)$ be a postcritically marked quadratic morphism over K with extended postcritical marking $\tilde{s}$. If f is stable, then the postcritical orbit Γ contains at least three elements and (C, s) is the contraction of $(C, \tilde{s})$ as stable marked curves over K. In this case C comprises one irreducible component and the morphism $\kappa : C \twoheadrightarrow C$ induced by 'forgetting' the markings for $\tilde{\Gamma} \setminus \Gamma$ is an isomorphism.

There is also an inverse to contraction, namely *stabilisation*: Given a stable marked curve $(\mathcal{C}, s)$ with $n-1$ markings and an arbitrary additional section $\xi \in \mathcal{C}(S)$, there exists up to unique isomorphism precisely one stable marked curve $(\mathcal{C}', s')$ with n markings such that $(\mathcal{C}, s)$ is the contraction of $(\mathcal{C}', s')$ and such that the n^{th} section is mapped to ξ. This can be extended inductively to any number of additional markings, and the stabilisation is obtained (uniquely up to unique isomorphism) by a series of blowups described explicitly in Knudsen [5, Def. 2.3 & Thm. 2.4].

The stable marked curves (C, s) and $(C, \tilde{s})$ associated to a postcritically marked stable quadratic morphism f over K possess extensions to stable marked curves $(\mathcal{C}, s)$ and $(\tilde{\mathcal{C}}, \tilde{s})$ over R, which are unique up to unique isomorphism and which we now construct according to Pink [7, Section 6].

Construction of stable models

As in Section 1, choose a coordinate such that $C \xrightarrow{\sim} \mathbb{P}^1_K$ sends (ω_1, ω_2) to $(0, \infty)$, and f is in normalised form $f(x) = \frac{ax^2+b}{cx^2+d}$. In this coordinate, the non-trivial covering automorphism of f is $\sigma : x \mapsto -x$ with fixed points 0 and ∞.

Starting with the extended marking $(C, \tilde{s})$, we have $\tilde{s}(i_0) = 0$ and $\tilde{s}(j_0) = \infty$. For all $\gamma \in \tilde{\Gamma} \setminus \{i_0, j_0\}$, let $\infty > n_1 > \cdots > n_r > -\infty$ denote the possible orders $\mathrm{ord}_\pi(\tilde{s}(\gamma))$. Define $U_0 := \mathrm{Spec} R[x/\pi^{n_1}]$, $U_r := \mathrm{Spec} R[\pi^{n_r}/x]$ and for $0 < \ell < r$ set $U_\ell := \mathrm{Spec} R[x/\pi^{n_{\ell+1}}, \pi^{n_\ell}/x]$. The points $\tilde{s}(i_0)$ and $\tilde{s}(j_0)$ extend to sections of U_1 and U_r respectively, again denoted by $\tilde{s}(i_0)$ and $\tilde{s}(j_0)$.

For each $0 < \ell \leqslant r$, the schemes $U_{\ell-1}$ and U_ℓ have a common open subscheme $U_{\ell-1} \cap U_\ell = \mathrm{Spec} R[x/\pi^{n_\ell}, \pi^{n_\ell}/x]$ along which we glue $U_{\ell-1}$ and U_ℓ, thus obtaining a projective flat curve Z over S with generic fibre C. The closed fibre $(U_{\ell-1} \cap U_\ell)_0$ of these subschemes is $\mathrm{Spec} R[x/\pi^{n_\ell}, \pi^{n_\ell}/x]/(\pi)$, which is isomorphic to $\mathbb{P}^1_k \setminus \{0, \infty\}$. For each $0 < \ell \leqslant r$ let Y_ℓ denote the closure in Z of $(U_{\ell-1} \cap U_\ell)_0$. Then Y_ℓ is isomorphic to $\mathbb{P}^1_k$ and these are precisely the irreducible components of the closed fibre Z_0 of Z. These components are arranged in sequence such that any two consecutive components meet precisely in an ordinary double point. The automorphism σ induces an automorphism $y \mapsto -y$ on each $Y_\ell \cong \mathbb{P}^1_k$ and thus has precisely two fixed points on each Y_ℓ. These comprise the singular points of Z_0 together with the reductions of the points $\tilde{s}(i_0) = 0$ and $\tilde{s}(j_0) = \infty$ on Y_1 and Y_r respectively. Furthermore, for each K-valued point $\tilde{s}(\gamma), \gamma \in \tilde{\Gamma} \setminus \{i_0, j_0\}$ there is a unique n_ℓ such that $\mathrm{ord}_\pi(\tilde{s}(\gamma)) = n_\ell$ and thus $\tilde{s}(\gamma)/\pi^{-n_\ell}$ can be extended to a section $\tilde{s}(\gamma) : S \to U_{\ell-1} \cap U_\ell$ which meets Y_ℓ

in the closed fibre. Since $\tilde{s}(\gamma)/\pi^{-n_\ell} \bmod (\pi) \notin \{0,\infty\}$, the section lands in the smooth locus of Z, is distinct from $\tilde{s}(i_0), \tilde{s}(j_0)$ and is thus not fixed by σ.
The marked curve (Z, s) is 'almost stable': the only condition on a stable marked curve which is not ensured is that some of the sections may collide in the closed fibre. The *stable extension* $(\tilde{\mathcal{C}}, \tilde{s})$ of $(C, \tilde{s})$ is now obtained from $(Z, \tilde{s})$ by stabilisation as in Knudsen [5], i.e. by blowing up an ideal centered at the (finite) set of points in Z_0 where sections $\tilde{s}(\gamma), \gamma \in \tilde{\Gamma}$ meet in the closed fibre, and $(\tilde{\mathcal{C}}, \tilde{s})$ is thus a stable marked curve, unique up to unique isomorphism. The blowup moves these colliding sections to new irreducible components in the exceptional fibres which are each disjoint from their σ-conjugate. Each irreducible component Y in the closed fibre $\tilde{\mathcal{C}}_0$ of $\tilde{\mathcal{C}}$ is a smooth curve of genus zero over k. Furthermore, the automorphism σ on C extends to an automorphism σ on $\tilde{\mathcal{C}}$ which remains compatible with the marking, ie. $\sigma \circ \tilde{s} = \tilde{s} \circ \sigma$. The sections $\tilde{s}(\gamma) : S \to \tilde{\mathcal{C}}$ are now pairwise disjoint and thus induce an injection $\tilde{s}_0 : \tilde{\Gamma} \hookrightarrow \tilde{\mathcal{C}}_0(k)$. The data $(\tilde{\mathcal{C}}, \tilde{s})$ is stable in the sense that each irreducible component Y contains at least three points which are either singular or marked points. This construction satisfies the following:

Proposition 4.6 ([7, Prop. 6.1])**.**

(i) *The fixed points of σ in the closed fibre $\tilde{\mathcal{C}}_0$ of $\tilde{\mathcal{C}}$ are precisely the reductions of the sections $\tilde{s}(i_0)$ and $\tilde{s}(j_0)$ and the double points of $\tilde{\mathcal{C}}_0$ which separate them.*

(ii) *Any irreducible component Y of $\tilde{\mathcal{C}}_0$ is either equal to $\sigma(Y)$ or disjoint from $\sigma(Y)$.*

(iii) *An irreducible component Y of $\tilde{\mathcal{C}}_0$ is equal to $\sigma(Y)$ if and only if it contains a fixed point of σ. The automorphism induced by σ on it is then non-trivial.*

We call the irreducible components satisfying (iii) *components on the spine* of $\tilde{\mathcal{C}}$. In the notation of the construction, these are precisely the components $Y_1, \ldots, Y_r$. From the construction we also see that each of these corresponds to an integer n_ℓ.

Since $s = \tilde{s}|_\Gamma$, and $|\Gamma| \geqslant 3$, the stable extension $(\mathcal{C}, s)$ of (C, s) can be obtained from $(\tilde{\mathcal{C}}, \tilde{s})$ as the contraction in the sense of Definition 4.4.
An irreducible component of $\mathcal{C}_0$ whose proper transform in $\tilde{\mathcal{C}}_0$ is a component Y_ℓ on the spine of $\tilde{\mathcal{C}}$ is again denoted by Y_ℓ and is called a component on the spine of $\mathcal{C}$. We refer to $(\mathcal{C}, s)$ as the *stable model* for f and to $(\tilde{\mathcal{C}}, \tilde{s})$ as the *extended stable model* for f.

Remark 4.7. Take the postcritical marking $s : \Gamma \hookrightarrow C(K)$ and let Γ' denote a maximal subset of Γ such that for the marking $s' := s|_{\Gamma'} : \Gamma' \hookrightarrow C(K)$, the reductions $\overline{s'(\gamma)}$ of the K-valued points $s'(\gamma)$ are pairwise distinct. If $|\Gamma'| \geq 3$, then the stable extension $(\mathcal{C}', s')$ is a *smooth* stable marked curve with a single irreducible component in the closed fibre $\mathcal{C}'_0$. Furthermore, by uniqueness of stabilisation and the above construction, we see that any stable marked curve $(\mathcal{C}, s)$ over R with the same generic fibre C is a stabilisation of $(\mathcal{C}', s')$, unique up to unique isomorphism.

As the last ingredient in this section, consider the quotient $\bar{\mathcal{C}} := \tilde{\mathcal{C}}/\langle\sigma\rangle$. The projection morphism $p : \tilde{\mathcal{C}} \twoheadrightarrow \bar{\mathcal{C}}$ induces a marking $\bar{s} : \tilde{\Gamma}/\langle\sigma\rangle \hookrightarrow \bar{\mathcal{C}}(R)$ sending $\bar{\gamma} := \{\gamma, \sigma(\gamma)\}$ to $\bar{s}(\bar{\gamma}) := p \circ s(\gamma)$.

Proposition 4.8 ([7, Prop. 7.7, 6.2])**.**

(i) *The morphism f extends to a unique morphism $f : \tilde{\mathcal{C}} \to \mathcal{C}$ and induces an isomorphism $\bar{\mathcal{C}} = \tilde{\mathcal{C}}/\langle\sigma\rangle \xrightarrow{\sim} \mathcal{C}$.*

(ii) *The pair $(\bar{\mathcal{C}}, \bar{s})$ is a stable marked curve over S.*

(iii) *For any double point x_0 of $\tilde{\mathcal{C}}_0$ which is fixed by σ and where $\tilde{\mathcal{C}}$ is étale locally isomorphic to* $\mathrm{Spec}R[y,z]/(yz - \pi^r)$ *for some $r > 0$, the quotient $\bar{\mathcal{C}}$ is étale locally isomorphic to* $\mathrm{Spec}R[u,v]/(uv - \pi^{2r})$ *at $p(x_0)$.*

5 Dual Trees

Let $(\mathcal{C}, s)$ be the stable extension of a smooth marked curve (C, s) over K as in the previous section. We continue to follow [7, Sections 5-7].

Definition 5.1. The *dual tree* of the closed fibre $\mathcal{C}_0$ of $\mathcal{C}$ is a finite graph $T = (V_T, E_T)$ where each vertex $t \in V_T$ corresponds to a unique irreducible component of $\mathcal{C}_0$ and each edge $(t, t') \in E_T$ corresponds to the unique singular point where the two components represented by t and t' intersect. The dual tree is in fact a tree because C has genus zero.

The marking $s_0 : \Gamma \hookrightarrow \mathcal{C}_0(k)$ induces a map $s : \Gamma \to V_T$ where γ is sent to the vertex corresponding to the unique irreducible component Y in $\mathcal{C}_0$ with $s_0(\gamma) \in Y(k)$. This map is not injective because the corresponding irreducible components can (and some must) each contain more than one marked point.

Remark 5.2. The stability condition 4.1.(iii) on $(\mathcal{C}, s)$ translates to a stability condition on T: at each vertex $t \in V_T$ there are at least three objects which are either markings $s(\gamma)$ for $\gamma \in \Gamma$ or edges $(t, t') \in E_T$.

Let f be a stable quadratic pcf morphism over K. Let $(\mathcal{C}, s)$ and $(\tilde{\mathcal{C}}, \tilde{s})$ be the stable model and the extended stable model for f, and let T and $\tilde{T}$ denote their respective dual trees. Mapping each irreducible component of $\mathcal{C}_0$ to its proper transform in $\tilde{\mathcal{C}}_0$ defines an injection $V_T \hookrightarrow V_{\tilde{T}}$ of the corresponding vertices in the dual trees. Identify V_T with its image in $V_{\tilde{T}}$ and call it the set of vertices which *survive in* T. This induces a map $s : \Gamma \to V_T \subset V_{\tilde{T}}$ described below. Furthermore, the automorphism σ on $\tilde{\mathcal{C}}$ induces an involution σ on $\tilde{T}$ which is again compatible with the markings in the sense that $\sigma \circ \tilde{s} = \tilde{s} \circ \sigma$. Let $t_1, \ldots, t_r \in V_{\tilde{T}}$ denote the vertices representing the irreducible components $Y_1, \ldots, Y_r$ on the spine of $\tilde{\mathcal{C}}$.

Proposition 5.3 ([7, Prop. 5.5 (a),(c) and Prop. 6.3 (a)-(d)])**.**

(i) *A vertex $t \in V_{\tilde{T}}$ survives in T if and only if there are at least three objects which are either markings $\tilde{s}(\gamma)$ with $\gamma \in \Gamma$ at t or connected components of $\tilde{T} \setminus \{t\}$ containing such markings.*

(ii) *The map $s : \Gamma \to V_T \subset V_{\tilde{T}}$ is given as follows: For any $\gamma \in \Gamma$ the $s(\gamma)$ is the unique vertex in V_T with minimal distance to $\tilde{s}(\gamma)$ in $\tilde{T}$. In particular, if $\tilde{s}(\gamma)$ survives in T, then $s(\gamma)$ coincides with $\tilde{s}(\gamma)$.*

(iii) *The fixed points of σ on $V_{\tilde{T}}$ are precisely the vertices $t_1, \ldots, t_r$.*

(iv) *The vertices $t_1, \ldots, t_r$ are distinct, connected in the given order by a string of edges, and satisfy $\tilde{s}(i_0) = t_1$ and $\tilde{s}(j_0) = t_r$.*

(v) *All other vertices and edges come in pairs of two σ-conjugates.*

(vi) *Let $\tilde{T}/\langle\sigma\rangle$ denote the graph whose set of vertices is $V_{\tilde{T}}/\langle\sigma\rangle$, and where two vertices $\{t, \sigma(t)\}$ and $\{t', \sigma(t')\}$ are joined by an edge if and only if t is joined by an edge to t' or to $\sigma(t')$. Then the dual tree of $\tilde{\mathcal{C}}/\langle\sigma\rangle$ is naturally isomorphic to $\tilde{T}/\langle\sigma\rangle$.*

In analogy to the irreducible components, we call $t_1, \ldots t_r$ the *vertices on the spine of* $\tilde{T}$ and those vertices among $t_1, \ldots, t_r$ which survive in T are called *vertices on the spine of* T.

Recall from the previous section the quotient $\bar{\mathcal{C}} := \tilde{\mathcal{C}}/\langle\sigma\rangle$ and the map $\tilde{\tau} : \tilde{\Gamma} \to \Gamma$ which induces an isomorphism $\tilde{\Gamma}/\langle\sigma\rangle \xrightarrow{\sim} \Gamma$. Combining this with Propositions 4.8 (i) and 5.3 (vi) and the map $s : \Gamma \to V_T \subset V_{\tilde{T}}$ described in Proposition 5.3 (ii), we obtain a surjective map $\tilde{\tau} : V_{\tilde{T}} \twoheadrightarrow V_{\tilde{T}}/\langle\sigma\rangle \xrightarrow{\sim} V_T$ which sends $\tilde{s}(\gamma)$ to $s(\tilde{\tau}(\gamma))$, and a composite map $\nu : V_T \hookrightarrow V_{\tilde{T}} \twoheadrightarrow V_T$. All in all, we have the following diagram which commutes everywhere except for the leftmost square, where the rule is given by Proposition 5.3 (ii)

$$\begin{tikzcd}
V_T \arrow[r, hook] \arrow[rrrr, dotted, bend left, "\nu"] & V_{\tilde{T}} \arrow[r, "\sigma"] & V_{\tilde{T}} \arrow[r, two heads] & V_{\tilde{T}}/\langle\sigma\rangle \arrow[r, "\sim"] & V_T \\
\Gamma \arrow[r, hook] \arrow[u, hook, "s"] & \tilde{\Gamma} \arrow[r, "\sigma"] \arrow[u, hook, "\tilde{s}"] & \tilde{\Gamma} \arrow[r, two heads] \arrow[u, hook, "\tilde{s}"] \arrow[rr, dotted, bend right, "\tau"'] & \tilde{\Gamma}/\langle\sigma\rangle \arrow[r, "\sim"] \arrow[u, hook, "\bar{s}"] & \Gamma \arrow[u, hook, "s"]
\end{tikzcd}$$

Abbreviate the marked vertices on T as $P_n := s(i_n)$ and $Q_n := s(j_n)$ for $n \geq 1$ and on $\tilde{T}$ as $\tilde{P}_n := \tilde{s}(i_n)$ and $\tilde{Q}_n := \tilde{s}(j_n)$ for $n \geq 0$. The vertices P_n and Q_n can be constructed from $\tilde{P}_n$ and $\tilde{Q}_n$ by the rule for $s : \Gamma \to V_T \subset V_{\tilde{T}}$ from Proposition 5.3, and the map $\tilde{\tau} : V_{\tilde{T}} \twoheadrightarrow V_T$ sends $\tilde{P}_n$ to P_{n+1} and $\tilde{Q}_n$ to Q_{n+1}. The marked vertices $\tilde{P}_0$ and $\tilde{Q}_0$ are precisely the first and last vertices t_1 and t_r respectively, on the spine of $\tilde{T}$.

Lemma 5.4 ([7, Lemma 7.10])**.**

(a) *Any vertex strictly between $\tilde{P}_0$ and $\tilde{Q}_0$ survives in T.*

(b) *If $\tilde{P}_0 \neq \tilde{Q}_0$, then $\tilde{P}_0$ survives in T unless one of the following happens in $\tilde{T}$:*

  (i) *there is only one edge at t_1 and the only other markings at t_1 are $s(\gamma)$ and $s(\sigma(\gamma))$ with $\gamma \in \Gamma$ and $\sigma(\gamma) \notin \Gamma$ or*

  (ii) *there are no other markings at t_1 and the connected components of $\tilde{T} \setminus \{t_1\}$ are precisely that containing $\tilde{Q}_0$ and two others S and $\sigma(S)$ where $s^{-1}(\sigma(S)) \cap \Gamma = \emptyset$.*

6 Worked Example

Consider the mapping scheme Γ

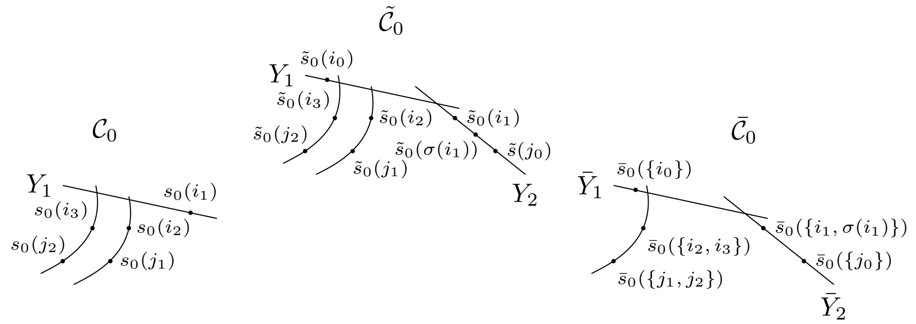
$$\begin{tikzcd}
\overset{i_1}{\circ} \arrow[r] & \overset{i_2}{\bullet} \arrow[r] & \overset{i_3}{\bullet} \arrow[loop below] & \overset{j_1}{\circ} \arrow[r] & \overset{j_2}{\bullet} \arrow[loop below]
\end{tikzcd}$$

For the extended mapping scheme, the construction from Section 3 produces $\tilde{\Gamma} = \{i_0, i_1, i_2, i_3, \sigma(i_1), j_0, j_1, j_2\}$ with the additional elements i_0, j_0 and $\sigma(i_1)$.

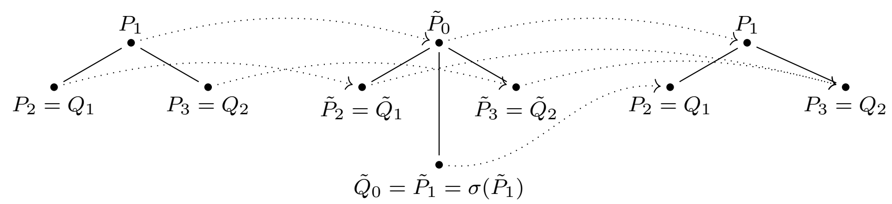
Consider a quadratic morphism $(\mathbb{P}^1_K, x \mapsto \frac{x^2+a}{x^2-(a+2)}, 0, \infty)$ over $K := \mathbb{Q}(a)$, where a is a root of the polynomial $P(\alpha) = \alpha^4 + 9\alpha^3 + 40\alpha^2 + 96\alpha + 128$. This morphism has postcritical orbit Γ and the extended postcritical marking $\tilde{s} : \tilde{\Gamma} \hookrightarrow \mathbb{P}^1_K$ is given by

$$\begin{aligned}
\tilde{s}(i_0) &= 0 & \tilde{s}(j_0) &= \infty \\
\tilde{s}(i_1) &= -\tfrac{1}{20}(a^3+7a^2+26a+64) & \tilde{s}(j_1) &= 1 \\
\tilde{s}(i_2) &= \tfrac{1}{8}a(a^2+5a+12) & \tilde{s}(j_2) &= -1 \\
\tilde{s}(i_3) &= -\tilde{s}(i_2) \\
\tilde{s}(\sigma(i_1)) &= -\tilde{s}(i_1)
\end{aligned}$$

The polynomial P reduces to $\bar{P}(\alpha) = (\alpha+2)(\alpha^3+2\alpha^2+\alpha-1)$ in characteristic 5. Thus, in the ring of integers $\mathcal{O}_K$ of K, the ideal (5) factors into two prime ideals, one of which yields a discrete valuation ring R with uniformiser $\pi = a+2$. Over the residue field, the morphism has good reduction to $x \mapsto 1 - \frac{2}{x^2}$. The distinct orders of the points $\tilde{s}(\gamma) \in \mathbb{P}^1(K)$ are $n_1 := \mathrm{ord}_\pi(\tilde{s}(\gamma)) = 0$ for $\gamma \neq i_1$ and $n_2 := \mathrm{ord}_\pi(\tilde{s}(i_1)) = -1$. Thus, there are two irreducible components Y_1 and Y_2 on the spine of the extended stable model $\tilde{\mathcal{C}}$ over R. Further $\tilde{s}(j_1)$ and $\tilde{s}(i_2)$ are both congruent to 1 modulo π, and $\tilde{s}(j_2)$ and $\tilde{s}(i_3)$ are congruent to -1 modulo π. Replacing $\tilde{s}(i_1)$ by $\tilde{s}(i_1)/\pi^{n_2}$, the image of i_1 in the closed fibre meets Y_2. The stable extension $\tilde{\mathcal{C}}$ is obtained by blowing up Y_1 in the two points ± 1, and then the closed fibre $\tilde{\mathcal{C}}_0$ comprises four irreducible components arranged as below. The stable model $\mathcal{C}$ is obtained from $\tilde{\mathcal{C}}$ by removing the sections $\tilde{s}(i_0), \tilde{s}(\sigma(i_1))$ and $\tilde{s}(j_0)$ and contracting Y_2, which is the only irreducible component that becomes unstable. Thus $\mathcal{C}_0$ is of the form below, and $\mathcal{C}$ is indeed isomorphic to the quotient $\bar{\mathcal{C}} = \tilde{\mathcal{C}}/\langle\sigma\rangle$.



The associated dual trees, their markings and the maps between them are given as follows:



Note that the vertex $\tilde{Q}_0$ on the spine of $\tilde{T}$ does not survive in T as it satisfies Case (i) of Lemma 5.4 (b). Also, the composite map $\nu : V_T \hookrightarrow V_{\tilde{T}} \twoheadrightarrow V_T$ maps the vertex P_1 on the spine to itself.
A maximal marking for f as in Remark 4.7 would, for example, be given by the subset $\Gamma' = \{i_1, j_1, j_2\}$. Obtaining $(\mathcal{C}', s')$ from $(\mathcal{C}, s)$ by removing the sections $s(i_2)$ and $s(i_3)$ and contracting thus unstable irreducible components, we see that $(\mathcal{C}', s')$ is indeed a smooth stable marked curve with a single irreducible component in the closed fibre.

7 Good Reduction and Stable Quadratic pcf Morphisms

In this section we will show that a stable quadratic pcf morphism f over K with good reduction reduces to a stable quadratic pcf morphism. By Fact 1.11, this is equivalent to saying that if there exists a smooth model for f, then there is no choice of coordinate x such that the reduction of f is of the form $\bar{f}(x) = ax^{\pm 2}$ for any $a \in k^\times$ and any sign.

Let $(C, f, \omega_1, \omega_2, s)$ be a postcritically marked stable quadratic pcf morphism over K with good reduction. Let $(\mathbb{P}^1_R, f_R : x \mapsto \frac{ax^2+b}{cx^2+d}, 0, \infty)$ be the smooth model. Further, let $(\mathcal{C}, s)$ and $(\tilde{\mathcal{C}}, \tilde{s})$ denote the stable model and the extended stable model for f.
Suppose the reduction $\bar{f}$ of f is of the form $\bar{f}(x) = ax^{\pm 2}$ for some $a \in k^\times$ and some sign. Since f is stable, there are at least three elements in the postcritical orbit Γ. Denote by $\overline{s(\gamma)}$ the reduction modulo π of the K-valued points $s(\gamma)$, for $\gamma \in \Gamma$, in the smooth model. Recall from Fact 1.12 that, since f has good reduction, the reduction of the postcritical orbit of f coincides with the postcritical orbit of $\bar{f}$. Moreover, the postcritical orbit of $\bar{f}$ consists precisely of the two critical points 0 and ∞. Therefore, the possible orders $\mathrm{ord}_\pi(s(\gamma))$ for all $\gamma \in \Gamma \setminus \{i_0, j_0\}$ of points in the postcritical orbit of f are all nonzero. Recall from the construction in Section 4 that the distinct components in the spine of $\tilde{\mathcal{C}}$ arise from the distinct orders of these points, and thus, so do the distinct vertices on the spine of the dual tree $\tilde{T}$ of $\tilde{\mathcal{C}}$. For these vertices, let the *order of the vertex* be the corresponding order and the *sign of the vertex* be the sign of its order.

Claim 7.1. *If $\bar{f}(x) = ax^2$, then there exist at least two vertices on the spine of $\tilde{T}$ with different signs.*

Proof. Suppose contrapositively that all vertices on the spine of $\tilde{T}$ have the same sign. Then either all noncritical points in the postcritical orbit reduce to 0 in the smooth model, or they all reduce to ∞. It suffices to consider only one of these two cases, otherwise interchange the roles of the critical points. Since $\bar{f}(x) = ax^2$, we have $\overline{s(i_n)} = \bar{f}^n(0) = 0$ and $\overline{s(j_n)} = \bar{f}^n(\infty) = \infty$ for all $n \geq 1$. Therefore, if all noncritical points reduce to 0, then ∞ must be a fixed point of f. In this case, we can assume that f is of the form $a'x^2 + 1$ for some $a' \in K^\times$. By Claim 2.1, both the coefficient a' and its inverse are integral over $\mathbb{Z}[\frac{1}{2}]$ and thus units in R. In particular, the reduction of f, which in this coordinate is given by the reduction of the coefficients of f, is not of the form $\bar{f}(x) = ax^2$. □

Claim 7.2. *If $\bar{f}(x) = a/x^2$, then there exist at least two vertices on the spine of $\tilde{T}$ with different signs.*

Proof. If all vertices on the spine of $\tilde{T}$ have the same sign (in particular, if there is only one vertex on the spine), then again, either all noncritical points in the postcritical orbit reduce to 0 in the smooth model, or they all reduce to ∞. As before, by symmetry we need only consider one of these two cases. Since $\bar{f}(x) = a/x^2$, we have $\overline{s(i_{2n})} = \bar{f}^{2n}(0) = 0 = \bar{f}^{2n+1}(\infty) = \overline{s(j_{2n+1})}$ and $\overline{s(i_{2n+1})} = \bar{f}^{2n+1}(0) = \infty = \bar{f}^{2n}(\infty) = \overline{s(j_{2n})}$ for all $n \geq 0$. Therefore, if all noncritical points reduce to 0, then the postcritical orbit of f must be the set $\{0, \infty, f(\infty)\}$ with 0 a fixed point of f. But this is impossible because then $0 = \overline{f(0)} = \bar{f}(0) = \infty$. Thus, at least one noncritical point in the postcritical orbit reduces to ∞ and $\tilde{T}$ has at least two vertices on the spine with different signs. □

In both cases $\bar{f}(x) = ax^2$ and $\bar{f}(x) = a/x^2$, let t_1 and t_2 denote the neighbouring vertices on the spine of $\tilde{T}$ with $\operatorname{sgn}(t_1) = 1$ and $\operatorname{sgn}(t_2) = -1$.

Claim 7.3. *If $\bar{f}(x) = ax^2$, then both t_1 and t_2 survive in T.*

Proof. It suffices to show that t_1 survives in T because the argument for t_2 is analogous interchanging the roles of $\tilde{P}_0$ and $\tilde{Q}_0$. All markings on the connected component of $\tilde{T} \setminus \{t_1\}$ containing $\tilde{Q}_0$ correspond to points in the postcritical orbit of strictly negative order and all markings on the connected component of $\tilde{T} \setminus \{t_2\}$ containing $\tilde{P}_0$ to those of strictly positive order. Suppose that t_1 does not survive in T. By Lemma 5.4, this implies that t_1 is the first vertex $\tilde{P}_0$ on the spine, that $i_0 \notin \Gamma$ and t_1 satisfies one of the cases in Lemma 5.4 (b).
Suppose Case (i) occurs. Since all markings on $\tilde{T} \setminus \{t_1\}$ represent points of strictly negative order, the unique marking $\tilde{s}(\gamma)$ at t_1 with $\gamma \in \Gamma$ and $\sigma(\gamma) \notin \Gamma$ must represent the only point in the forward orbit of 0. But then $\tilde{s}(\gamma)$ marks a fixed point $s(i_1)$ in the postcritical orbit and thus $s(i_1) = s(i_0)$, contradicting the assumption that $i_0 \notin \Gamma$.
Suppose Case (ii) occurs. Then the two connected components S and $\sigma(S)$

contain all markings for the forward orbit of 0, again because all markings on the connected component of $\tilde{T} \setminus \{t_1\}$ containing $\tilde{Q}_0$ represent points of strictly negative order. Since $\bar{f}(x) = ax^2$, the postcritical orbit of $\bar{f}$ comprises disjoint forward orbits for 0 and ∞. Therefore f must be determined by equations $f^m(0) = -f^n(0)$ and $f^k(\infty) = -f^\ell(\infty)$ for some $m > n \geq 0,\ k > \ell \geq 0$ and in particular, both i_n and $\sigma(i_n)$ lie in Γ (as do j_ℓ and $\sigma(j_\ell)$). The corresponding markings P_n and $\sigma(P_n)$ lie on $S \cup \{t_1\} \cup \sigma(S)$. Since there are no markings at t_1, and P_n and $\sigma(P_n)$ are σ-conjugate, they must lie on the σ-conjugate components S and $\sigma(S)$, ie. $P_n \in S$ and $\sigma(P_n) \in \sigma(S)$ or vice versa, contradicting the assumption that $\sigma(S)$ contains no markings $s(\gamma)$ for $\gamma \in \Gamma$. Hence t_1 must survive in T. $\square$

In order to prove the analogous statement for the case $\bar{f}(x) = a/x^2$, we make use of an additional model:

Construction of fixed point models

Let $\bar{f}(x) = ax^{\pm 2}$ for some sign. First, we construct a fixed point ξ of f_R: The scheme of fixed points of a quadratic morphism is finite and flat over the base. After possibly enlarging the base field k, there exists a fixed point in the closed fibre which is not a critical point and after possibly extending the base ring R, this fixed point can be lifted to a fixed point ξ of the whole scheme by flatness. The image ξ_K of ξ in the generic fibre has order zero with respect to the uniformiser π because in the closed fibre ξ does not meet a critical point. In particular ξ_K cannot lie in the postcritical orbit of f, since all such points have nonzero order. Furthermore ξ and $\sigma(\xi)$ are fibrewise distinct because the only fixed points of σ are the critical points.

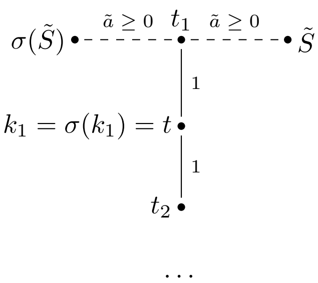
In the smooth model, the reduction of at least two of the points $s(i_1), s(i_2), s(j_1), s(j_2) \in \mathcal{C}'(K)$ must be distinct, namely $s(i_1)$ and $s(j_1)$ for $\bar{f}(x) = ax^2$ and $s(i_1)$ and $s(j_2)$ for $\bar{f}(x) = a/x^2$. Let i and j denote the corresponding indices in Γ and choose new symbols k_1 and $\sigma(k_1)$ corresponding to ξ and $\sigma(\xi)$ respectively. Set $\Gamma' := \{i, j, k_1\}$ and $\tilde{\Gamma}' := \Gamma' \cup \{\sigma(k_1)\}$ and let $(\mathcal{C}', s')$ and $(\tilde{\mathcal{C}}', \tilde{s}')$ denote the smooth stable Γ'-marked curves extending (C, s') and $(C, \tilde{s}')$. Further, define $\Gamma'' := \Gamma \cup \{k_1\}$ and $\tilde{\Gamma}'' := \tilde{\Gamma} \cup \{k_1, \sigma(k_1)\}$ and let $(\mathcal{C}'', s'')$ and $(\tilde{\mathcal{C}}'', \tilde{s}'')$ be the stable extensions of (C, s'') and $(C, \tilde{s}'')$ respectively. Then the fixed point models $(\mathcal{C}', s')$ and $(\tilde{\mathcal{C}}', \tilde{s}')$ can be obtained as the contractions of $(\mathcal{C}'', s''|_{\Gamma'})$ and $(\tilde{\mathcal{C}}'', \tilde{s}'|_{\tilde{\Gamma}'})$, and the closed fibres of both $\mathcal{C}'$ and $\tilde{\mathcal{C}}'$ each comprise one irreducible component $\tilde{\mathcal{C}}_0' \cong \mathcal{C}_0'$. This yields the following commutative diagram, where all unmarked arrows are contractions:

$$\begin{tikzcd}
\mathcal{C}' & \tilde{\mathcal{C}}' \arrow[l, two heads] \arrow[r, two heads, "f"] & \mathcal{C}' \\
\mathcal{C}'' \arrow[u, two heads] \arrow[d, two heads] & \tilde{\mathcal{C}}'' \arrow[l, two heads] \arrow[r, two heads, "f"] \arrow[u, two heads] \arrow[d, two heads] & \mathcal{C}'' \arrow[u, two heads] \arrow[d, two heads] \\
\mathcal{C} & \tilde{\mathcal{C}} \arrow[l, two heads] \arrow[r, two heads, "f"] & \mathcal{C}
\end{tikzcd}$$

Remark 7.4. Since ξ does not collide with the two critical points after reduction, the sections $s(\sigma(k_1))$ and $s(k_1)$ have $\mathrm{ord}_\pi(s(k_1)) = 0 = \mathrm{ord}_\pi(s(\sigma(k_1)))$ unequal to $\mathrm{ord}_\pi(s(\gamma))$ for all $\gamma \in \tilde{\Gamma}$. Thus, the (extended) fixed point model $\tilde{\mathcal{C}}''$ contains precisely one more irreducible component $Y \cong \tilde{\mathcal{C}}'_0$ on the spine and Y lies strictly between two irreducible components Y_1 and Y_2 represented by the neighbouring vertices t_1 and t_2 on the spine of $\tilde{T}$ with different signs. Since the only markings at the vertex representing Y in $\tilde{T}''$ are $\tilde{s}''(k_1)$ and $\tilde{s}''(\sigma(k_1))$, and k_1 and $\sigma(k_1)$ are not in $\tilde{\Gamma}$, this vertex does not survive in $\tilde{T}$ or in T, and $\tilde{\mathcal{C}}$ is obtained from $\tilde{\mathcal{C}}''$ by contracting precisely Y.

Claim 7.5. *If $\bar{f}(x) = a/x^2$, then both t_1 and t_2 survive in T.*

Proof. Let t denote the vertex in $\tilde{T}''$ representing the additional irreducible component Y in the fixed point model. By the above remark t lies precisely between t_1 and t_2 in $\tilde{T}''$ and thus, in particular, survives in T'' by Lemma 5.4 (a). Since the fixed point and its σ-conjugate do not collide with each other or any point in the postcritical orbit after reduction, they are not moved away from Y and thus at t there are precisely the two markings $k_1, \sigma(k_1)$ and two edges (t, t_2) and (t_1, t). Suppose first that precisely one of t_1, t_2 survives in T'', say t_2 (otherwise interchange the roles of $\tilde{P}_0$ and $\tilde{Q}_0$). Lemma 5.4 (b) then implies that t_1 is the first vertex on the spine, that $i_0 \notin \Gamma$ and $\tilde{T}''$ is of the form



where either $\tilde{a} = 0$ and $\tilde{S} = \sigma(\tilde{S})$ comprises the vertex t_1 with markings $s(\gamma), s(\sigma(\gamma))$ according to Case (i), or $\tilde{a} > 0$ and $\tilde{S}$, $\sigma(\tilde{S})$ are connected componenents as in Case (ii). The rest of the tree at t_2 is not depicted.

Since $f(s(k_1)) = s(k_1)$, the vertex t is mapped to itself under the composite $\tilde{T}'' \twoheadrightarrow \tilde{T}''/\langle\sigma\rangle \xrightarrow{\sim} T''$. All markings on the northern hemitree (with respect to t) arise from points with strictly positive order and all markings on the southern hemitree from points with strictly negative order. Further, the orders of the points in the postcritical orbit have alternating sign since $\bar{f}(x) = a/x^2$. Thus, the northern and southern hemitrees are switched under $\tilde{T}'' \twoheadrightarrow \tilde{T}''/\langle\sigma\rangle \xrightarrow{\sim} T''$. It follows that T'' is isomorphic to

$$\begin{tikzcd}
\bar{t}_2 \arrow[d, dash, "1"] \\
t \arrow[d, dash, "1"] \\
\bar{t}_1 \arrow[r, dash, dashed, "\tilde{a}"] & \bar{S}
\end{tikzcd}$$

On the other hand, T'' can be obtained from $\tilde{T}''$ by stabilisation, where t and t_2 survive, the components $\sigma(S)$ and t_1 are contracted, and $\tilde{S}$ is moved to t (possibly contracted). Thus T'' is of the form

$$\begin{tikzcd}
t \arrow[r, dash, dashed, "a \geq 0"] \arrow[d, dash, "1"] & S \\
t_2
\end{tikzcd}$$

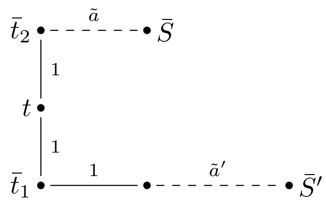
Comparing these two, we deduce that T'' is of the form

$$\begin{tikzcd}
t \arrow[r, "1"] \arrow[d, dash, "1"] & \bullet \arrow[r, dash, dashed, "a' \geq 0"] & S' \\
t_2 \arrow[r, dash, dashed, "\tilde{a}"] & S
\end{tikzcd}$$

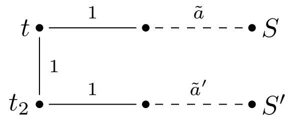
and $\tilde{T}''$ is then

$$\begin{tikzcd}
\sigma(\tilde{S}') \arrow[r, dash, dashed, "\tilde{a}'"] & \bullet \arrow[r, dash, "1"] & t_1 \arrow[r, dash, "1"] \arrow[d, dash, "1"] & \bullet \arrow[r, dash, dashed, "\tilde{a}'"] & \tilde{S}' \\
& & t \arrow[d, dash, "1"] & & \\
& \sigma(\tilde{S}) \arrow[r, dash, dashed, "\tilde{a}"] & t_2 \arrow[r, dash, dashed, "\tilde{a}"] & \tilde{S} &
\end{tikzcd}$$

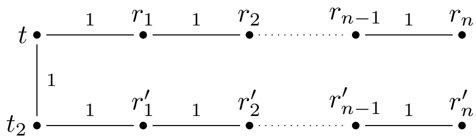
This implies that T'' isomorphic to



and thus T'' equal to

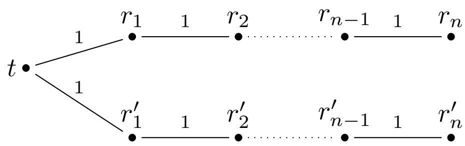


Repeating these arguments, by induction T'' contains a subtree of the form



for every $n \geq 1$, which is impossible for large enough n because f is postcritically finite. Therefore t_1 must survive in T'' and thus in T.

Suppose neither of t_1, t_2 survives in T''. Then an analogous argument shows that T'' contains a subtree of the form



for every $n \geq 1$, which is also impossible for large enough n, finishing the proof. □

Claim 7.6. *In either case $\bar{f}(x) = ax^2$ and $\bar{f}(x) = a/x^2$, the vertices t_1 and t_2 cannot both survive in T.*

Proof. Let x_0 denote the singular point in $\mathcal{C}$ represented by the edge (t_1, t_2) in T and where the irreducible components Y_1 and Y_2 intersect, and let $\tilde{x}_0$ denote the corresponding point in $\tilde{\mathcal{C}}$ represented by (t_1, t_2) in $\tilde{T}$. By construction of the fixed point models $\tilde{\mathcal{C}}''$ and $\mathcal{C}''$, the additional component Y in $\mathcal{C}''$ is contracted to x_0 via $\mathcal{C}'' \twoheadrightarrow \mathcal{C}$ and in $\tilde{\mathcal{C}}''$ to $\tilde{x}_0$ via $\tilde{\mathcal{C}}'' \twoheadrightarrow \tilde{\mathcal{C}}$, so $\tilde{x}_0$ is mapped to x_0 via $\tilde{\mathcal{C}} \twoheadrightarrow \mathcal{C}$

and this contraction is a local isomorphism at x_0 and $\tilde{x}_0$. On the other hand, since Y is mapped to itself via $\mathcal{C}'' \hookrightarrow \tilde{\mathcal{C}}'' \twoheadrightarrow \mathcal{C}''$, the point x_0 is mapped to itself via $\mathcal{C} \hookrightarrow \tilde{\mathcal{C}} \twoheadrightarrow \mathcal{C}$ and thus the projection $\tilde{\mathcal{C}} \twoheadrightarrow \tilde{\mathcal{C}}/\langle\sigma\rangle \xrightarrow{\sim} \mathcal{C}$ also maps $\tilde{x}_0$ to x_0. By Proposition 4.8 (iii), this implies that $\tilde{\mathcal{C}}$ is given locally at $\tilde{x}_0$ by the equation $xy = \pi^r$ for some $r > 0$ and $\mathcal{C}$ by the equation $uv = \pi^{2r}$ at x_0. But this is impossible because locally at these two points $\tilde{\mathcal{C}}$ and $\mathcal{C}$ are isomorphic and these equations are not equivalent for $r > 0$. Therefore, the vertices t_1 and t_2 cannot both survive in T. □

Theorem 7.7. *A quadratic pcf morphism f over K with good reduction is stable if and only if the reduction of f is stable.*

Proof. If the reduction of a quadratic pcf morphism f is stable, then clearly so is f. Conversely, if f reduces to $x \mapsto ax^{\pm 2}$ for some sign, then the vertices t_1 and t_2 both survive in T by Claims 7.3 and 7.5, which is impossible for these models by Claim 7.6. □

8 Good Reduction and the Composite Map ν

Let f be a stable quadratic pcf morphism over K with stable model $\mathcal{C}$, extended stable model $\tilde{\mathcal{C}}$ and dual trees T and $\tilde{T}$. Let $\nu : V_T \hookrightarrow V_{\tilde{T}} \twoheadrightarrow V_T$ be the composite map introduced in Section 5.

Proposition 8.1. *The composite map ν has a fixed point on the spine of T if and only if f has good reduction.*

Proof. Suppose t is a vertex on the spine which is mapped to itself via ν. Then t corresponds to an irreducible component Y on the spine of the closed fibre $\mathcal{C}_0$, on which σ induces a non-trivial automorphism. Since $f : \tilde{\mathcal{C}} \to \mathcal{C}$ is quasifinite, the induced rational map $f : \mathcal{C} \dashrightarrow \mathcal{C}$ has degree one or two on each irreducible component of the closed fibre. Since t is mapped to itself under ν, so is Y via the corresponding map $\mathcal{C} \hookrightarrow \tilde{\mathcal{C}} \twoheadrightarrow \mathcal{C}$ and thus f restricts to a morphism on Y. Since σ is non-trivial on Y, this morphism has degree two, and $(\mathcal{C}', f_R)$ with generic fibre C and closed fibre Y is a smooth model for f.
Conversely, if f has good reduction, then by Theorem 7.7, the reduction is stable and thus, in the smooth model $(\mathcal{C}', f_R)$ for f at least three points in the postcritical orbit remain distinct after reduction. Thus, we may choose a maximal marking $s' : \Gamma' \hookrightarrow \mathcal{C}'$ as in Remark 4.7 such that $(\mathcal{C}', s')$ is a smooth stable marked curve. Since $\mathcal{C}'$ and $\mathcal{C}$ have the same generic fibre C, the stable model $(\mathcal{C}, s)$ is the stabilisation of $(\mathcal{C}', s)$ by Remark 4.7 and so the following diagram commutes:

$$\begin{tikzcd} \mathcal{C} \arrow[d, two heads] & \tilde{\mathcal{C}} \arrow[l, two heads, "\kappa"'] \arrow[r, two heads, "f"] \arrow[d, two heads] & \mathcal{C} \arrow[d, two heads] \\ \mathcal{C}' & \tilde{\mathcal{C}}' \arrow[l, two heads, "\kappa"'] \arrow[r, two heads, "f"] & \mathcal{C}' \end{tikzcd}$$

Since f_R extends f to $\mathcal{C}'$, the restriction of f_R to the closed fibre $\mathcal{C}'_0 \cong Y$ coincides with the restriction of the rational map $f : \mathcal{C} \dashrightarrow \mathcal{C}$ to Y in the sense of the diagram above. Therefore, Y is mapped to itself via the composite $\mathcal{C} \hookrightarrow \tilde{\mathcal{C}} \twoheadrightarrow \mathcal{C}$ and equivalently, the vertex t representing Y in the dual tree T is mapped to itself via ν. Moreover, f_R has degree two on Y and thus, the vertex t must lie on the spine of T. $\square$

We refer to a fixed point of ν on the spine as a *vertex of good reduction.*

Lemma 8.2. *If there exists a vertex of good reduction, then it is unique.*

Proof. As shown in the proof of Proposition 8.1, a vertex of good reduction represents the smooth model, which is unique up to units in R. A change of vertices however, corresponds to a change of coordinates by some power of the uniformiser π. Thus, if a vertex of good reduction exists, then it is unique. $\square$

Lemma 8.3. *If f has a fixed point in the postcritical orbit which is marked at a vertex t on the spine of $\tilde{T}$, then t is a vertex of good reduction.*

Proof. Let ω be a critical point of f and let $n \geq 1$ be minimal such that $f^n(\omega)$ is a fixed point of f with marking at t. Without loss of generality suppose $\omega = \omega_1$. If $n = 1$, then $f(f(\omega_1)) = f(\omega_1)$ implies that ω_1 is a fixed point, thus $t = \tilde{P}_0$ and $i_0 \in \Gamma$. If $n > 1$, then $f^{n-1}(\omega_1) = \sigma(f^n(\omega_1)) \neq f^n(\omega_1)$ is also in the postcritical orbit of f and in $\tilde{T}$, there are at least two markings at t indexed by $i_n, i_{n-1} \in \Gamma$. In both cases, the vertex t must survive in T by Lemma 5.4. Since t lies on the spine of $\tilde{T}$ by assumption, it also lies on the spine in T. On the other hand, $f(f^n(\omega)) = f^n(\omega)$ implies that t is mapped to itself via $V_{\tilde{T}} \twoheadrightarrow V_{\tilde{T}}/\langle\sigma\rangle \xrightarrow{\sim} V_T$ and thus, also by the composite map ν. Therefore f has good reduction by Proposition 8.1. $\square$

Remark 8.4. As a special case of Lemma 8.3, if one of the critical points is a fixed point, then f has good reduction.

Lemma 8.5. *If for some $n > 0$, the n^{th} iterate ν^n has a fixed point on the spine, then f^n reduces to a morphism of degree 2^k for some $0 < k \leqslant n$.*

Proof. If ν^n has a fixed point on the spine, then the corresponding irreducible component Y is mapped to itself by the n^{th} composite of $\mathcal{C} \hookrightarrow \tilde{\mathcal{C}} \twoheadrightarrow \mathcal{C}$, which is a composite of maps of degree 1 or 2, since $\tilde{\mathcal{C}} \twoheadrightarrow \tilde{\mathcal{C}}/\langle\sigma\rangle \xrightarrow{\sim} \mathcal{C}$ has degree 1 or 2 on each irreducible component. Thus, the n^{th} composite is of degree 2^k for some $0 \leqslant k \leqslant n$. Since σ is non-trivial on Y, this map must have even degree and thus $k > 0$. □

9 Good Reduction and Strictly Preperiodic Points

Has anyone ever noticed that strictly preperiodic points are points of no return?

Let Γ be the postcritical orbit of a stable quadratic pcf morphism over K with good reduction to $\bar{f}$. Since f is stable, the reduction $\bar{f}$ is also stable by Theorem 7.7. Let $\bar{\Gamma}$ denote the postcritical orbit of $\bar{f}$, and consider both Γ and $\bar{\Gamma}$ as mapping schemes with the maps τ and $\bar{\tau}$ induced by f and $\bar{f}$ respectively, and write $\Gamma = \{i_k, j_k \mid k \geqslant 1\}$ and $\bar{\Gamma} = \{\bar{\imath}_k, \bar{\jmath}_k \mid k \geqslant 1\}$. The reduction map induces a map $p : \Gamma \to \bar{\Gamma}, i \mapsto \bar{\imath}$ and since f has good reduction, we have $p \circ \tau = \bar{\tau} \circ p$ as well as $p \circ \sigma = \sigma \circ p$.

The isomorphism classes of finite mapping schemes can be classified as follows:

Classification 9.1 ([7, Classification 2.3])**.**

(a) All relations result from two relations $i_n = \sigma(i_{n+k})$ and $j_m = \sigma(j_{m+\ell})$ with $k, \ell \geqslant 1$ and $n, m \geqslant 0$.

(b) All relations result from two relations $i_n = \sigma(j_{m+\ell})$ and $j_m = \sigma(i_{n+k})$ with $k, \ell \geqslant 1$ and $n, m \geqslant 0$.

(c) All relations result from two relations $i_{n'} = \sigma(j_{m'})$ and $i_n = \sigma(i_{n+k})$ with $0 \leqslant n' < n$ and $m' \geqslant 0$ and $k \geqslant 1$ and $(m', n') \neq (0, 0)$.

Definition 9.2. A *loop* in a postcritical orbit is the set of all periodic points in the forward orbit of a critical point. A *spoke* is the set of all strictly preperiodic points in the forward orbit of a critical point. The *length* of a spoke is its cardinality, and we call a spoke *trivial* if it has length 0.

Type (a) mapping schemes consist of two disjoint loops of respective periods k and ℓ with disjoint spokes of respective lengths n and m. Type (b) mapping schemes consist of one loop of period $k+\ell$ with two disjoint spokes of respective lengths n and m. Type (c) mapping schemes consist of one loop of period k and two disjoint spoke segments of respective lengths n' and m' which merge into a common spoke segment of length $n - n' > 0$. In particular, the critical points are always strictly preperiodic in Type (c) mapping schemes.

The reduction map sends periodic points to periodic points. Therefore, if both critical points are in Γ and thus periodic, then so are their images in $\bar{\Gamma}$, and the period of the reduced points divides the period of the original ones. So in this case $\bar{\Gamma}$ consists of either one loop containing both critical points, or of two disjoint loops for each critical point. If, however, one of i_0, j_0 is strictly preperiodic, it is not immediately clear what the structure of the reduced orbit $\bar{\Gamma}$ is.

Recall that for any point $P \in C(K)$ and ω a critical point, we have $f(P) = f(\omega)$ if and only if $P = \omega$. Thus, a critical point is strictly preperiodic if and only if its image under f is, and equivalently, if its image is a spoke point. When necessary, we refer to the spoke and loop comprising the forward orbit of i_1 as the i_1-spoke and the i_1-loop. Similarly, we speak of the j_1-spoke and j_1-loop.

Definition 9.3. We call the points i_n and j_m for Types (a), (b) and $i_n, i_{n'}$ and $j_{m'}$ for Type (c) in the defining relations the *defining points of* Γ. We refer to all other points as *nondefining points.*

Remark 9.4. For any mapping scheme Γ, the points whose σ-conjugates also lie in Γ, ie. the points in the set $\Gamma \cap \sigma(\Gamma)$, are precisely the defining points and their σ-conjugates. By definition, each defining point is one of the following types: (i) the last (strictly preperiodic) point on a spoke, (ii) critical, or (iii) the last point before a common spoke segment.
In Case (i), the σ-conjugate of the point is periodic and nondefining. In Case (ii), the point is periodic or Γ is of Type (c) and one spoke is a subset of the other. In Case (iii), Γ is of Type (c) and the σ-conjugate of the point is also a defining point of Type (iii).
Moreover, a critical point lies in Γ if and only if it is a defining point, and defining points lie on loops only if they are critical.

Suppose the i_1-spoke has length $n \geqslant 1$, so the defining point i_n is the last strictly preperiodic point in the forward orbit of i_0.

Claim 9.5. *If i_n does not reduce to either of the critical points, then the length of the i_1-spoke is preserved after reduction.*

Proof. Depending on the orbit type, the point i_n satisfies either $\sigma(i_n) = i_{n+k}$, or $\sigma(i_n) = j_m$ for some $k, m \geq 1$. In either case, i_n is a Type (i) defining point and thus, i_n and $\sigma(i_n)$ are distinct points, which both lie in the postcritical orbit, and $\sigma(i_n)$ is periodic. If i_n does not reduce to either critical point, then $\bar{\imath}_n$ is not fixed by σ. Thus $\bar{\imath}_n$ and $\sigma(\bar{\imath}_n)$ are also distinct points which both lie in $\bar{\Gamma}$, and $\sigma(\bar{\imath}_n)$ is periodic because $\sigma(i_n)$ is. Since $\bar{\tau}$ is injective on the loops, this implies that $\bar{\imath}_n$ is strictly preperiodic, and thus, so are $\bar{\imath}_1, \ldots, \bar{\imath}_{n-1}$. $\square$

Let $m \geqslant 0$ denote the length of the j_1-spoke in Γ, so j_{m+1} is periodic.

Claim 9.6. *If i_n reduces to a noncritical point, then j_m cannot reduce to $\bar{\imath}_0$.*

Proof. If $\bar{\jmath}_m = \bar{\imath}_0$, then $\bar{\jmath}_{m+1} = \bar{\imath}_1$ is periodic and thus, so is $\bar{\imath}_n$. But this is impossible because $\bar{\imath}_n$ is strictly preperiodic by Claim 9.5. □

Claim 9.7. *If i_n and j_m do not both reduce to the same critical point, then each spoke reduces either to a spoke of the same length, or to a loop.*

Proof. If neither of i_n, j_m reduce to a critical point, then both spoke lengths are preserved by Claim 9.5. If i_n does not reduce to a critical point but j_m does, then the length of the i_1-spoke is preserved by Claim 9.5, the point j_m reduces to $\bar{\jmath}_0$ by Claim 9.6, and $\bar{\jmath}_0$ is periodic, so the j_1-spoke reduces to a loop. The case where $\bar{\jmath}_m$ is noncritical and $\bar{\imath}_n$ is critical is analogous interchanging i_1 and j_1. If i_n reduces to $\bar{\imath}_0$ and j_m reduces to $\bar{\jmath}_0$, then both critical points are periodic in $\bar{\Gamma}$ and thus both spokes reduce to loops. Similarly, if i_n reduces to $\bar{\jmath}_0$ and j_m to $\bar{\imath}_0$, then the postcritical orbit reduces to a single loop without spokes. □

Remark 9.8. If i_n does not reduce to a critical point but j_m does, then $\bar{\jmath}_0$ is periodic of period $\bar{\ell}$ dividing both the period ℓ of the j_1-loop and the length m of the j_1-spoke. Furthermore, if previously disjoint loops are identified, then the period of $\bar{\jmath}_0$ also divides the period k of the i_1-loop.
If i_n reduces to $\bar{\imath}_0$ and j_m reduces to $\bar{\jmath}_0$, and the loops in $\bar{\Gamma}$ are disjoint, then $\bar{\imath}_0$ has period $\bar{k}$ dividing both k and the length n of the i_1-spoke, and $\bar{\jmath}_0$ has period $\bar{\ell}$ dividing both ℓ and m, and $\max\{\bar{k}, \bar{\ell}\} > 1$, since $|\bar{\Gamma}| \geq 3$. If there is only one loop in $\bar{\Gamma}$, then for the same reasons, the gcd of k and ℓ and thus $\gcd(k, \ell, n, m)$ must be at least 3. Furthermore, in this case both n and m are strictly larger than 1: if $\bar{\imath}_1 = \bar{\imath}_0$ is a fixed point, it cannot coincide with a loop containing $\bar{\jmath}_0$.
If i_n reduces to $\bar{\jmath}_0$ and j_m to $\bar{\imath}_0$, then the loop in $\bar{\Gamma}$ has period dividing both k and ℓ as well as m and n, and this case can also only occur if $\gcd(k, \ell, m, n) \geq 3$.

Claim 9.9. *If i_n and j_m both reduce to the same critical point, then both spoke lengths are shorter after reduction.*

Proof. It suffices to consider the case where both reduce to $\bar{\jmath}_0$, otherwise interchange the roles of i_0 and j_0. In this case, the j_1-spoke reduces to a loop. Since this loop contains the point $\bar{\jmath}_0 = \bar{\imath}_n$ and thus also the images of all points on the i_1-loop, the postcritical orbit in the reduction is Type (b) comprising a single loop with at most one nontrivial spoke. Further, since $\bar{\imath}_n$ is periodic, the $\bar{\imath}_1$-spoke has length $\bar{n} < n$. □

Remark 9.10. If i_n and j_m both reduce to the same critical point, and the length $\bar{n}$ of the $\bar{\imath}_1$-spoke is zero, then $\bar{\imath}_0$ lies on the unique loop in $\bar{\Gamma}$ and thus $\bar{\imath}_0 = \bar{\jmath}_{m'}$ for some $m' > 0$. If $\bar{n} > 0$, then the defining point $\bar{\imath}_{\bar{n}}$ in the reduction satisfies $\bar{\imath}_{\bar{n}} = \sigma(\bar{\jmath}_{m'})$ for some $m' > 0$.
The case where two defining points reduce to the same critical point is the only case where we obtain a defining point $\bar{\imath}_{\bar{n}}$ in the reduction which is *not* the image of a defining point of the original orbit. We shall refer to this point as the *new* defining point and to the case when the new defining point is noncritical, ie. when $n > \bar{n} > 0$, as *the special case.*

Combining Claims 9.5 to 9.9, we have proved the following:

Proposition 9.11. *The length of a spoke is preserved after reduction if and only if the last point on the spoke does not reduce to a critical point. If two defining points reduce to distinct critical points, then both spokes reduce to loops.* □

Example 9.12. As an application of Proposition 9.11, for each of the pcf morphisms from Claim 2.1, the lengths of the spokes are preserved:
These morphisms have good reduction and in each of the three cases, in the notation of Claim 2.1, the relevant points are given by $\frac{p_n}{q_n}$ for some $n > 0$, with $\mathrm{ord}_\pi(\frac{p_n}{q_n}) = \mathrm{ord}_\pi(p_n) - \mathrm{ord}_\pi(q_n)$. However, we showed in each case that $\mathrm{ord}_\pi(p_n) = \mathrm{ord}_\pi(q_n)$ for every $k \geqslant 1$. Thus $\mathrm{ord}_\pi(\frac{p_n}{q_n}) = 0$ and the defining points do not collide with critical points after reduction.

10 Good Reduction and Dual Trees

As usual, let f be a stable quadratic pcf morphism over K with good reduction. In the following, we analyse the dual trees for $\mathcal{C}$ and $\tilde{\mathcal{C}}$ according to the isomorphism class of $\bar{\Gamma}$. To determine the dual trees T and $\tilde{T}$, we will analyse the preimages in Γ and in $\tilde{\Gamma} \setminus \Gamma$ of points in $\bar{\Gamma}$ under the reduction map $p : \tilde{\Gamma} \twoheadrightarrow \bar{\tilde{\Gamma}}$.

Consider the dual trees T and $\tilde{T}$ of the stable models $\mathcal{C}$ and $\tilde{\mathcal{C}}$ for f. Applying Proposition 8.1 and Lemma 8.2, we know that there is a vertex of good reduction, namely the vertex t of order zero on the spine of both T and $\tilde{T}$. We refer to the connected components of $T \setminus \{t\}$ or $\tilde{T} \setminus \{t\}$ as *branches at t in T* or $\tilde{T}$, or simply as *branches* if the additional data is understood. We say a branch in $\tilde{T}$ *survives* in T if at least one vertex on the branch survives in T.

To start with, we make some observations based on what we know about the dual trees, the inclusion map $V_T \hookrightarrow V_{\tilde{T}}$ and the surjection $\tilde{\tau} : V_{\tilde{T}} \twoheadrightarrow V_{\tilde{T}}/\langle\sigma\rangle \xrightarrow{\sim} V_T$:

Observations.

1. Each branch at t in $\tilde{T}$ contains precisely the markings of all points in $\tilde{\Gamma}$ which have the same image in $\bar{\tilde{\Gamma}}$ after reduction. Thus, each branch in $\tilde{T}$ corresponds to a unique point in $\bar{\tilde{\Gamma}}$. This is due to the stabilisation in the construction of the extended stable model in Section 4.

2. The stability condition on $\tilde{T}$ from Remark 5.2 implies that each branch at t contains at least two markings in $\tilde{\Gamma}$. Therefore, the point $\bar{\imath}$ in $\bar{\tilde{\Gamma}}$ corresponding to a given branch in $\tilde{T}$ must have at least two preimages in $\tilde{\Gamma}$.

3. Similarly, each branch in T corresponds to a unique point in $\bar{\Gamma}$ and each point in $\bar{\Gamma}$ corresponding to a branch in T is the image of at least two points in Γ. We call a branch in T or $\tilde{T}$ associated to a critical point in $\bar{\Gamma}$ a *critical branch*, all other branches are *noncritical*.

4. A branch in $\tilde{T}$ survives in T if and only if the corresponding point in $\bar{\tilde{\Gamma}}$ is the image of at least two points in Γ, by definition of the inclusion $V_T \hookrightarrow V_{\tilde{T}}$ together with the preceding observations.

5. Given a branch at t in $\tilde{T}$ corresponding to $\bar{\imath} \in \bar{\tilde{\Gamma}}$, the images of its vertices under the surjection $\tilde{\tau}$ are vertices on a branch in T corresponding to $\bar{\tau}(\bar{\imath}) \in \bar{\Gamma}$. This is due to the definition of $\tilde{\tau}$, Observations 1 and 3, the fact that t is fixed under $\tilde{\tau}$, and because the set of points in $\tilde{\Gamma}$ which reduce to the same point $\bar{\imath}$ in $\bar{\tilde{\Gamma}}$ is mapped under f to the set of points in Γ which reduce to $\bar{\tau}(\bar{\imath})$ in $\bar{\Gamma}$.

6. If a branch in $\tilde{T}$ contains no markings for points in $\tilde{\Gamma} \setminus \Gamma$, then the branch is isomorphic to the corresponding branch in T. This is due to the fact that the restriction of the contraction morphism $\kappa : \tilde{\mathcal{C}} \to \mathcal{C}$ to the intersecting components represented by the branch is an isomorphism. Moreover, if the branch is noncritical and associated to $\bar{\imath} \in \bar{\tilde{\Gamma}}$, then it is disjoint from its σ-conjugate by Proposition 5.3 (v) and isomorphic to the branch in T containing the images of its vertices under $\tilde{\tau}$ and thus associated to $\bar{\tau}(\bar{\imath})$. Equivalently, if a point $\bar{\imath}$ in $\bar{\tilde{\Gamma}}$ corresponding to a branch in $\tilde{T}$ lies in $\bar{\Gamma}$ and has no preimages in $\tilde{\Gamma} \setminus \Gamma$, then there is a branch corresponding to $\bar{\imath}$ in T which is isomorphic to the branch in T corresponding to the point $\bar{\tau}(\bar{\imath})$ in $\bar{\Gamma}$.

Proposition 10.1.

(i) *At least two of the defining points in* Γ *reduce to defining points in* $\bar{\Gamma}$.

(ii) *At most two points in* Γ *reduce to the same point on a spoke in* $\bar{\Gamma}$, *and precisely two only if the points lie on distinct spokes before reduction and* $\bar{\Gamma}$ *is of Type (c).*

(iii) *The images of points in* $\tilde{\Gamma} \setminus \Gamma$ *which reduce to points in* $\bar{\Gamma}$ *are defining points plus, in the special case 9.10, the* σ*-conjugate of the new defining point. This* σ*-conjugate has precisely one preimage in* $\tilde{\Gamma} \setminus \Gamma$.

(iv) *The defining point of a spoke has the same number of preimages in* $\tilde{\Gamma} \setminus \Gamma$ *as its* σ*-conjugate does in* $\Gamma \setminus \sigma(\Gamma)$.

Proof. (i) By Classification 9.1, in any postcritical orbit, there are either two defining points and they are not σ-conjugate, or there are three defining points, two of which are σ-conjugate. Thus, it suffices to consider one defining point i and its σ-conjugate and show that at least i reduces to a defining point in $\bar{\Gamma}$. Recall from Remark 9.4 that a critical point lies in the postcritical orbit if and only if it is a defining point. Thus, if i reduces to $\bar{\imath}_0$ or $\bar{\jmath}_0$, then $\bar{\imath}$ is a defining point. Suppose both i and $\bar{\imath}$ are noncritical. Since both i and $\sigma(i)$ are in Γ, both $\bar{\imath}$ and $\sigma(\bar{\imath})$ are in $\bar{\Gamma}$ and thus, at least one of them is a defining point in $\bar{\Gamma}$. If $\sigma(\bar{\imath})$ is periodic, and thus not a defining point because it is noncritical, then $\bar{\imath}$ is a defining point. If both $\bar{\imath}$ and $\sigma(\bar{\imath})$ are strictly preperiodic, then $\bar{\Gamma}$ is of Type (c) and both $\bar{\imath}$ and $\sigma(\bar{\imath})$ are defining points.
(ii) Suppose i and j are two distinct points in Γ with the same strictly preperiodic image in $\bar{\Gamma}$. Then both i and j are also strictly preperiodic and they must lie on distinct spokes because otherwise, their image would be periodic. Indeed, if i and j lie on the same spoke, then $i = i_k$ and $j = i_{k'}$ for some $k, k' \geqslant 0$. Then $\bar{\imath}_k = \bar{\imath}_{k'}$ implies that $\bar{\imath}_k$ and $\bar{\imath}_{k'}$ are periodic, contradicting the assumption. Furthermore, $\bar{\imath} = \bar{\jmath}$ implies that $\bar{\tau}(\bar{\imath}) = \bar{\tau}(\bar{\jmath})$ and therefore, the spoke segments starting at i and j coincide in $\bar{\Gamma}$, which is thus of Type (c).
(iii) Let $\sigma(i) \in \tilde{\Gamma} \setminus \Gamma$. If $\sigma(i)$ is critical, then so is its reduction and if $\sigma(\bar{\imath})$ is in $\bar{\Gamma}$, it is a defining point. If $\sigma(i)$ is noncritical, then the σ-conjugate $i \in \Gamma$ is a nondefining point and reduces to $\bar{\imath} \in \bar{\Gamma}$. Thus, the point $\sigma(\bar{\imath})$ is in $\bar{\Gamma}$ if and only if $\bar{\imath}$ or $\sigma(\bar{\imath})$ is a defining point in $\bar{\Gamma}$. If both $\bar{\imath}$ and $\sigma(\bar{\imath})$ are defining points, then we are done. Otherwise one of $\bar{\imath}, \sigma(\bar{\imath})$ is periodic and the other is a strictly preperiodic point and thus the image of a strictly preperiodic point $i_{n'}$. If $\bar{\imath}$ is periodic, then $\sigma(\bar{\imath})$ is a defining point and we are done. If $\bar{\imath}$ is strictly preperiodic, then so is i and since i is nondefining, the spoke must be strictly shorter, but nontrivial after reduction, which is the special case 9.10. Moreover, $\bar{\imath}$ has precisely one (nondefining) preimage in Γ by (ii) because the spoke not

containing i reduces to a loop in the special case 9.10 and hence $\bar{\Gamma}$ is not of Type (c). Thus, $\sigma(\bar{\imath})$ has precisely one preimage in $\tilde{\Gamma} \setminus \Gamma$.
(iv) The σ-conjugate $\sigma(\bar{\imath})$ of this type of defining point is periodic and noncritical by Remark 9.4. For every point $\sigma(i') \in \tilde{\Gamma} \setminus \Gamma$ which reduces to $\bar{\imath}$, the point i' is noncritical, lies in Γ, is nondefining and reduces to $\sigma(\bar{\imath})$. Conversely, every point i in $\Gamma \setminus \sigma(\Gamma)$ which reduces to $\sigma(\bar{\imath})$ is nondefining, and noncritical because $\sigma(\bar{\imath})$ is. Thus $\sigma(i)$ lies in $\tilde{\Gamma} \setminus \Gamma$ and reduces to $\bar{\imath}$. □

Consequence 10.2. Combining Proposition 10.1 with Observation 6, the only type of branch in $\tilde{T}$ which is not necessarily isomorphic to the corresponding branch in T containing the images of its vertices under $\tilde{\tau}$ are the branches corresponding to defining points and to the σ-conjugate of the new defining point in the special case 9.10.

Lemma 10.3. *If $\bar{\Gamma}$ is of Type (a), then*

(i) *A spoke in Γ is either isomorphic to its image in $\bar{\Gamma}$ or reduces to a loop.*

(ii) *If there is a critical branch in $\tilde{T}$, then it survives in T.*

(iii) *The marking for a defining spoke point is moved away from t in T to a branch in $\tilde{T}$ if and only if every point on the subsequent loop is marked away from t on a branch in T and thus in $\tilde{T}$.*

Proof. If $\bar{\Gamma}$ is of Type (a), then so is Γ.
(i) Since the forward orbits of the critical points are disjoint in $\bar{\Gamma}$, the defining points cannot both reduce to the same critical point. Thus, by Proposition 9.11, a spoke either reduces to a loop or to a spoke of the same length. In the latter case, by Proposition 10.1 (ii), each point in the image of the spoke has precisely one preimage in Γ and thus, the spoke is isomorphic to its image.
(ii) Since the forward orbits of $\bar{\imath}_0$ and $\bar{\jmath}_0$ are disjoint, it suffices to analyse just one of them, say $\bar{\imath}_0$. By Observations 2 and 4, we need to show that if $\bar{\imath}_0$ has more than one preimage in $\tilde{\Gamma}$, then at least two points in Γ reduce to $\bar{\imath}_0$. Let $\bar{n} \geqslant 0$ denote the length of the $\bar{\imath}_1$-spoke. By (i), we have $\bar{n} \in \{0, n\}$. If $\bar{n} = n > 0$, then $\bar{\imath}_0$ is strictly preperiodic and thus lies in $\tilde{\bar{\Gamma}} \setminus \bar{\Gamma}$ and has precisely one preimage in $\tilde{\Gamma}$. Therefore, there is no critical branch corresponding to i_0 in $\tilde{T}$ in this case. If $\bar{n} = n = 0$, then i_0 is periodic in Γ and $\bar{\imath}_0$ has preimages in $\tilde{\Gamma} \setminus \Gamma$ if and only if the loop period is reduced if and only if $\bar{\imath}_0$ has at least two preimages in Γ. If $\bar{n} = 0 \leqslant n$, then i_n reduces to $\bar{\imath}_0$ and the $\bar{\imath}_1$-orbit is a loop, which is the image of both the i_1-spoke and the i_1-loop. Hence, every point on the $\bar{\imath}_1$-loop, in particular $\bar{\imath}_0$, has at least two preimages in Γ, namely a spoke point and a loop point.

(iii) From the arguments in (ii) we only need to consider the case $\bar{n} = n > 0$. By Proposition 10.1 (ii), the defining point $\bar{\imath}_n$ on the spoke has precisely one preimage in Γ and is thus marked at t in T. Proposition 10.1 (iv) implies that $\bar{\imath}_n$ has preimages in $\tilde{\Gamma} \setminus \bar{\Gamma}$ if and only if $\sigma(\bar{\imath}_n)$ has at least two preimages in Γ. In terms of the dual trees, this is equivalent to the assertion by Observations 3, 4 and 5. □

Lemma 10.4. *If $\bar{\Gamma}$ is of Type (c), then*

(i) *The spoke lengths in Γ are always preserved in $\bar{\Gamma}$. If Γ is of Type (a) or (b), then each point on the disjoint segments of the spoke in $\bar{\Gamma}$ is the image of a unique point in Γ, and each noncritical point on the common segment of the spoke is the image of precisely two points in Γ. If Γ is of Type (c), then the spoke in Γ is isomorphic to the spoke in $\bar{\Gamma}$.*

(ii) *If Γ is not isomorphic to $\bar{\Gamma}$, then each point on the loop in $\bar{\Gamma}$ has at least two preimages in $\bar{\Gamma}$.*

(iii) *If there is a critical branch at t in $\tilde{T}$, then it does not survive in T.*

Proof. Let $\bar{\Gamma}$ be of Type (c). Then both $\bar{\imath}_0$ and $\bar{\jmath}_0$ are strictly preperiodic and thus at least one lies in $\tilde{\bar{\Gamma}} \setminus \bar{\Gamma}$, say $\bar{\imath}_0$. Further Γ can be of any type, with i_1, j_1 strictly preperiodic.

(i) Since $\bar{\imath}_0$ and $\bar{\jmath}_0$ are both strictly preperiodic, the defining points in Γ cannot reduce to critical points. Thus, by Proposition 9.11, both spoke lengths are preserved. The assertion for Γ of Type (a) or (b) is immediate from Proposition 10.1 (ii). Let Γ be of Type (c) with defining points $i_n, i_{n'}$ and $j_{m'}$ for $n > n' \geqslant 0$ and $m' \geqslant 0$ and $(m', n') \neq (0, 0)$. Assume $n' \geqslant m'$, otherwise interchange the roles of i_0 and j_0. Since $i_{n'} = \sigma(j_{m'})$, the disjoint spoke segments cannot be further identified: this would only be possible if $\sigma(\bar{\jmath}_{m'}) = \bar{\imath}_{n'} = \bar{\jmath}_{m'}$, which implies that $\bar{\imath}_{n'} = \bar{\imath}_0$ or $\bar{\jmath}_{m'} = \bar{\jmath}_0$. But then we would have a periodic critical point in $\bar{\Gamma}$, which is impossible for $\bar{\Gamma}$ of Type (c). Therefore, the defining points for $\bar{\Gamma}$ are $\bar{\imath}_n, \bar{\imath}_{n'}$ and $\bar{\jmath}_m$, and each spoke point thus has precisely one preimage in Γ, again by Proposition 10.1 (ii).

(ii) If Γ is Type (a), then the two loops coincide after reduction, if Γ is Type (b), then the first periodic points $i_{n+1} \neq j_{m+1}$ on the loop collide after reduction and thus, the period of the loop is strictly reduced. If Γ is of Type (c), then by (i), the spoke is isomorphic to its image in $\bar{\Gamma}$, and thus Γ is isomorphic to $\bar{\Gamma}$ if and only if the period of the loop is preserved after reduction. If the period of the loop is not preserved, then each point on the loop has at least two preimages in Γ.

(iii) A critical point, say $\bar{\jmath}_0$ lies in $\bar{\Gamma}$ if and only if the $\bar{\jmath}_1$-spoke lies on the $\bar{\imath}_1$-spoke. If Γ is Type (c), then by (i), the spoke in Γ is isomorphic to the spoke in $\bar{\Gamma}$. Hence $\bar{\jmath}_0$ has precisely one preimage in Γ and thus is marked at t in both $\tilde{T}$ and T. If Γ is of Type (a) or (b) with defining spoke points i_n and j_m, then $n > m$ and $\bar{\jmath}_0$ has precisely one preimage i_{n-m} in Γ and two in $\tilde{\Gamma} \setminus \Gamma$, namely j_0 and $\sigma(i_{n-m})$. □

Lemma 10.5. *If $\bar{\Gamma}$ is of Type (a) or (c), then the branches in T corresponding to points on a loop in $\bar{\Gamma}$ are all isomorphic.*

Proof. By Lemma 10.3 (i) and Lemma 10.4 (i), the special case 9.10 cannot occur. Thus, combining Observation 6 with Proposition 10.1 (iii) shows that a branch in T corresponding to any nondefining point on a loop in $\bar{\Gamma}$ maps isomorphically to its image, which is a branch corresponding to a point on the same loop. Recall from Remark 9.4 that a point on a loop is defining if and only if it is critical. Since $\bar{\Gamma}$ is not of Type (b), there is at most one critical point on a given loop, say $\bar{\imath}_0$. If $\bar{\imath}_1 = \bar{\imath}_0$, then we are done because there is only one branch to consider. Otherwise, starting at $\bar{\imath}_1$ and iterating along the loop, we find by the above argument that each branch corresponding to a point on the loop is isomorphic to the next, and in particular, so is the critical branch, which is the image of the branch corresponding to $\bar{\imath}_k$ for some $k > 0$ with $\bar{\tau}(\bar{\imath}_k) = \bar{\imath}_0$. □

Lemma 10.6. *If $\bar{\Gamma}$ is of Type (b), then*

(i) *Each spoke point in $\bar{\Gamma}$ has precisely one preimage in Γ.*

(ii) *There is a critical branch at t in $\tilde{T}$ if and only if it survives in T.*

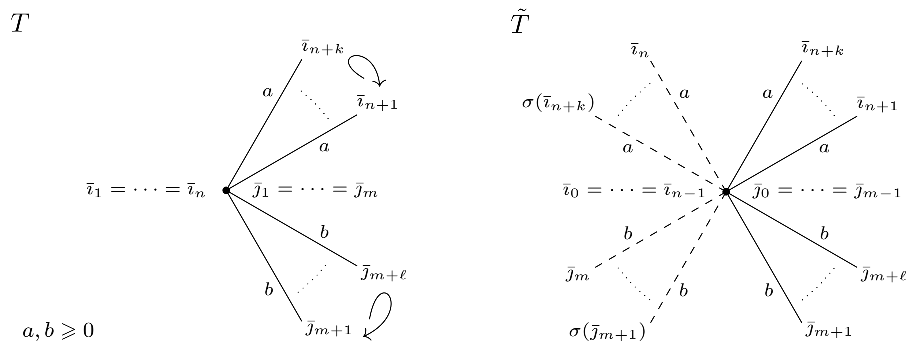
Proof. If $\bar{\Gamma}$ is of Type (b), then Γ can be of any type, as long as neither critical point a fixed point before or after reduction. Further, this is the only type of reduction for which the special case 9.10 can occur.
(i) Two disjoint spokes cannot reduce to a single spoke because $\bar{\Gamma}$ has precisely one nontrivial spoke if and only if the other critical point is periodic. If Γ is of Type (c), then the defining point of the common spoke segment must reduce to a periodic critical point and $\bar{\Gamma}$ has at most one nontrivial spoke which is the image of (part of) one of the disjoint spoke segments in Γ. Thus any spoke point in $\bar{\Gamma}$ has precisely one preimage in Γ.
(ii) Suppose $\bar{\imath}_0$ lies in $\bar{\Gamma}$. Then $\bar{\imath}_0$ is periodic. If i_0 is strictly preperiodic in Γ, and periodic in $\bar{\Gamma}$, then $\bar{\imath}_0$ has at least two preimages in Γ. If i_0 is periodic, then Γ is Type (a) or (b). If Γ is Type (a), then $\bar{\imath}_0$ has at least two preimages because the two disjoint loops must reduce to a single loop. If Γ is Type (b) and $\bar{\imath}_0$ has precisely one preimage, then the defining points in Γ are $i_0 = j_{m+\ell}$ and j_m, the loop length is not reduced, and j_m does not reduce to $\bar{\imath}_0$. If j_m reduces to $\bar{\jmath}_0$, then $\bar{\jmath}_\ell = \bar{\jmath}_{m+\ell} = \bar{\imath}_0$ and $\bar{\imath}_0$ has more than one preimage in Γ. Thus j_m

cannot reduce to a critical point and thus the spoke length in Γ is not reduced. Therefore, $\bar{\imath}_0$ has precisely one preimage in Γ if and only if Γ is Type (b) and isomorphic to $\bar{\Gamma}$, in which case T is trivial. In every other case, whenever $\bar{\imath}_0$ lies in $\bar{\Gamma}$, it has at least two preimages in Γ. □

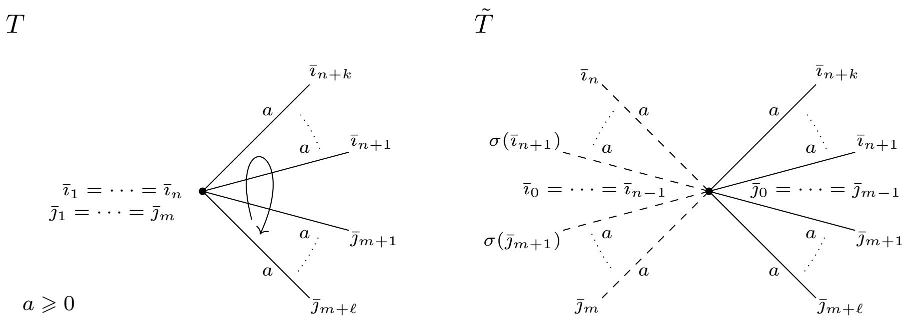
Remark 10.7. If $\bar{\Gamma}$ is of Type (b) and a spoke only partially coincides with a loop after reduction (in particular, when the special case 9.10 occurs), say the $\bar{\imath}_1$-spoke, then $\bar{\jmath}_0$ is periodic and $\bar{\jmath}_0 = \bar{\jmath}_m = \bar{\imath}_n$. Let $\bar{\jmath}_{m'}$ be the σ-conjugate of the new defining point in the special case 9.10, or $\bar{\jmath}_{m'} = \bar{\imath}_0$. Then by the same arguments as in the proof of Lemma 10.5, the branches corresponding to the loop segment given by $\bar{\jmath}_1, \ldots, \bar{\jmath}_{m'}$ are isomorphic. Similarly, the branches corresponding to the loop segment given by $\bar{\jmath}_{m'+1}, \ldots, \bar{\jmath}_m = \bar{\jmath}_0$ are isomorphic and they each have precisely one more marking in Γ than those corresponding to $\bar{\jmath}_1, \ldots, \bar{\jmath}_{m'}$. In particular, this also holds for $m' + 1 = m$, in which case $\bar{\imath}_1 = \bar{\jmath}_0$.

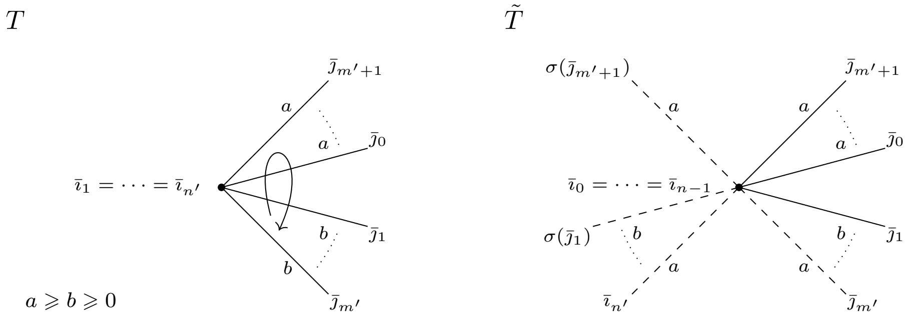
In the figures below, we indicate branches in T or $\tilde{T}$ as follows: an edge of strictly positive length indicates a branch with at least two markings. An edge of length one indicates a branch with precisely two markings, and which thus comprises precisely one edge of length one and one vertex with the two markings. An edge of length zero means that there is only one marking and the indicated branch thus is actually just the vertex t in disguise. Further, we keep the notation from above, including indices, for the defining points, with the convention that a marking of the form $i_k = \cdots = i_n$ is an 'empty' marking if $n < k$.

In summary, the above case analysis yields the following trees for each type of good reduction:

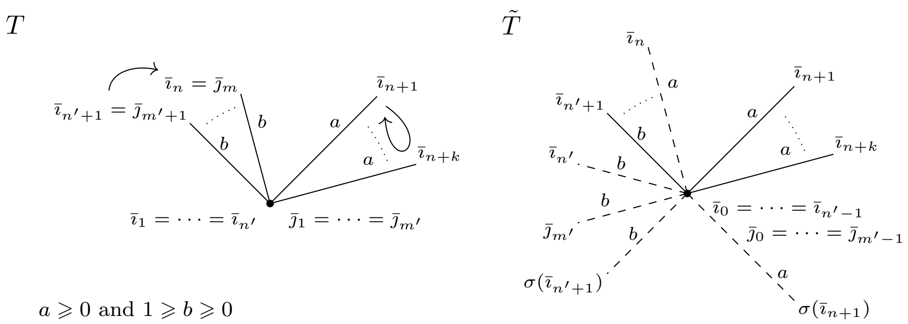


Type (a): These trees have branches corresponding to the two distinct loops in $\bar{\Gamma}$ within each of which all associated branches are isomorphic. Any remaining spoke points are marked at the vertex t. The only possibilities for critical branches are those corresponding to $\bar{\imath}_{n+k}$ or $\bar{\jmath}_{m+\ell}$.





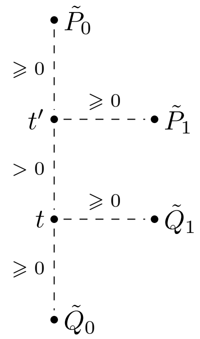
Type (b): These trees have branches corresponding to a single loop comprising two segments: one coincidence segment, which exists if a spoke is only partially identified with the loop after reduction, and the rest of the loop. Within the segments, all corresponding branches are isomorphic. Each branch corresponding to a point on the coincidence segment has precisely one more marking in Γ than each branch associated to a point on the other segment. Any remaining spoke points are marked at the vertex t. The only possibilities for a single critical branch are those corresponding to $\bar{\imath}_{n+k}$ or $\bar{\jmath}_{m+\ell}$. A second critical branch can then be any other branch associated to a point on the loop.



Type (c): These trees have branches corresponding to a single loop and a coincidence segment of the spoke, within each of which all branches are isomorphic. Each branch associated to points on the spoke segment has at most two markings in Γ. The disjoint spoke segments are marked at the vertex t. There can be at most one critical branch in $\tilde{T}$ and then it is that marked either by $\bar{\imath}_{n'} = \bar{\imath}_0$ or by $\bar{\jmath}_{m'} = \bar{\jmath}_0$.

Proposition 10.8. *If f is a stable quadratic pcf morphism with good reduction, then a critical point reduces to a fixed point if and only if it is a fixed point before reduction.*

Proof. Without loss of generality consider the critical point i_0, otherwise interchange the roles of i_0 and j_0. Clearly, if i_0 is a fixed point of f, then $\bar{\imath}_0$ is a fixed point of $\bar{f}$. For the converse, suppose $\bar{\imath}_0$ is fixed and i_0 is not. Since critical values have only one preimage under τ and $\bar{\tau}$, if $\bar{\imath}_0 = \bar{\imath}_1$ is fixed, the forward orbits of the two critical points must be disjoint, ie. $\bar{\Gamma}$ is of Type (a), and thus Γ is also of Type (a). Further, since f is stable, $\bar{\Gamma}$ comprises at least three points by Theorem 7.7 and thus, there are at least two points in the forward orbit of $\bar{\jmath}_1$. Each point in the forward orbit of i_0 reduces to $\bar{\imath}_0$ and since i_0 is not a fixed point, there are at least two such points in Γ. Hence, there is a critical branch in the associated dual tree $\tilde{T}$ which survives in T and which in both trees contains all markings for the points in the forward orbit of i_1. On the other hand, all points in the forward orbit of j_1 are marked away from the critical branch for $\bar{\imath}_0$ because T is Type (a). In particular, the points which reduce to $\bar{\jmath}_1$ are points of order zero, and thus are marked at, or on a branch at, the vertex t of good reduction and marked away from j_0. Therefore, $\tilde{T}$ contains a subtree of the form:



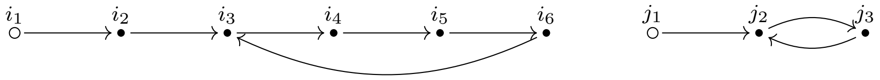
This, however, is impossible by Theorem 8.19 in [7]. □

Corollary 10.9. *Any morphism given by $x \mapsto \frac{ax^2+b}{cx^2+d}$ in normalised form with $0 \neq bc \in R\pi$ has bad reduction or is not postcritically finite. In particular, for $ad \in R^\times$, the morphism is not postcritically finite.*

11 Selected Examples

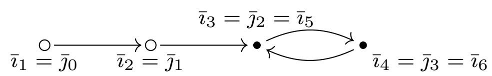
For the calculations omitted from the following examples, consult the appendix.

Example 11.1. Consider the Type (a) mapping scheme Γ with defining points $i_2 = \sigma(i_6)$ and $j_1 = \sigma(j_3)$



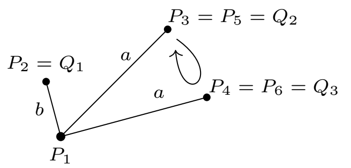
For the extended mapping scheme, the construction from Section 3 produces $\tilde{\Gamma} = \{i_0, i_1, \sigma(i_1), i_2, \sigma(i_2), i_3, i_4, \sigma(i_4), i_5, \sigma(i_5), i_6, j_0, j_1, j_2, \sigma(j_2), j_3\}$.

The quadratic morphism $\left(\mathbb{P}^1_K, f_a : x \mapsto \frac{x^2 - \frac{1}{2}(a^2+1)}{x^2+a}, 0, \infty\right)$ over $K := \mathbb{Q}(a)$ has postcritical orbit Γ when a is a root of the factor of degree 36 of the polynomial derived from the equation $f_\alpha^6(0) = -f_\alpha^2(0)$. This morphism has good reduction to $x \mapsto 1 - \frac{1}{2x^2}$ over $\mathbb{F}_7$, with postcritical orbit $\bar{\Gamma}$ of Type (c) with defining points $\bar{\imath}_2 = \sigma(\bar{\imath}_4)$ and $\bar{\imath}_1 = \bar{\jmath}_0$



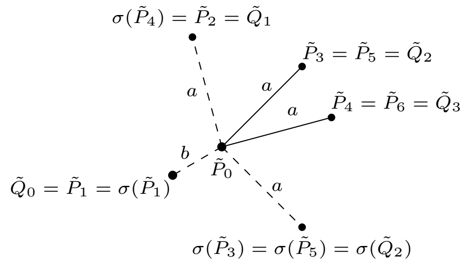
All assertions in Proposition 10.1 and Lemmata 10.4 and 10.5 are satisfied, for example, the spoke lengths are preserved and each point on the loop in $\bar{\Gamma}$ has several preimages in Γ. The critical branch in the dual tree $\tilde{T}$ does not survive in T, the two branches in T corresponding to the points $\bar{\imath}_2$ and $\bar{\imath}_3$ on the loop are isomorphic, and the points on the coinciding spoke segment are marked on a different branch. Further, the vertex of good reduction on the dual tree is the one marked by P_1 and is indeed mapped to itself under the composite map as in Proposition 8.1.

T

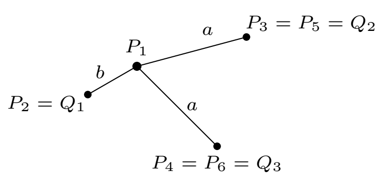


$a = b = 1$

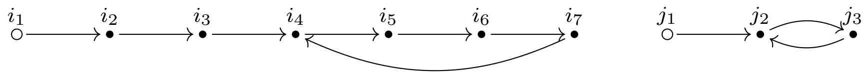
$\tilde{T}$



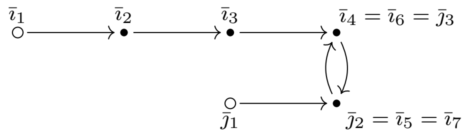
$\tilde{T}/\langle\sigma\rangle \cong T$



Example 11.2. The Type (a) mapping scheme Γ with defining points $i_3 = \sigma(i_7)$ and $j_1 = \sigma(j_3)$ given below is the postcritical orbit of the quadratic morphism $\left(\mathbb{P}^1_K, f_a : x \mapsto \frac{x^2 - \frac{1}{2}(a^2+1)}{x^2+a}, 0, \infty\right)$ over $K := \mathbb{Q}(a)$ when a is a root of the factor of degree 73 of the polynomial derived from the equation $f^7_\alpha(0) = -f^3_\alpha(0)$.

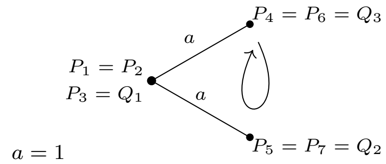


This morphism has good reduction to $x \mapsto \frac{x^2+1}{x^2+\bar{a}}$ over $\mathbb{F}_{25}$ generated by $\bar{a}$, with postcritical orbit $\bar{\Gamma}$ of Type (b) with defining points $\bar{\imath}_3 = \sigma(\bar{\jmath}_2)$ and $\bar{\jmath}_1 = \bar{\imath}_4$

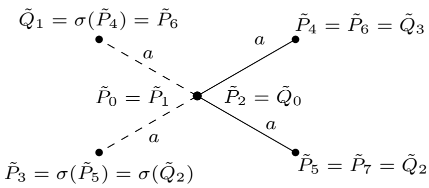


Since neither defining point reduces to a critical point, both spoke lengths are preserved as stated in Proposition 9.11. As in Lemma 10.6, each point on the spoke in Γ is marked at the vertex of good reduction, which is mapped to itself under the composite map and is the vertex of order zero as it should be according to Lemma 8.2. Since the spokes in Γ are isomorphic to the spokes in $\bar{\Gamma}$, there is only one loop segment and thus, the branches at t are isomorphic.

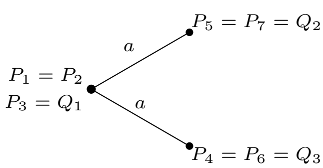
T



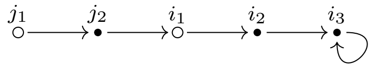
$\tilde{T}$



$\tilde{T}/\langle\sigma\rangle \cong T$

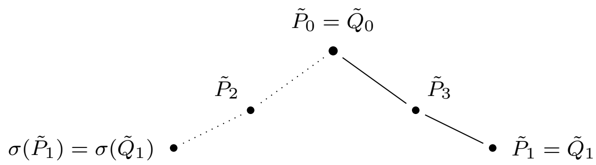


Example 11.3. In contrast, consider the Type (c) mapping scheme Γ with defining points $i_2 = \sigma(i_3)$ and $j_2 = i_0$

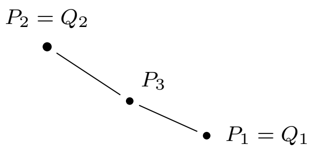


For a root a of the polynomial $P(\alpha) = 2\alpha^6 - 6\alpha^5 + 10\alpha^4 - 8\alpha^3 + 2\alpha^2 + 2\alpha - 1$, the quadratic morphism $\left(\mathbb{P}^1_K, f_a : x \mapsto \frac{x^2-1}{x^2+a}, 0, \infty\right)$ over $K := \mathbb{Q}(a)$ has postcritical orbit Γ. This morphism has bad reduction over $\mathbb{F}_5$ with $\bar{\imath}_1 = \bar{\imath}_3 = \bar{\jmath}_1 = \sigma(\bar{\imath}_2)$. The associated dual trees are as follows (all edges have length one, dotted edges are those which do not survive in T):

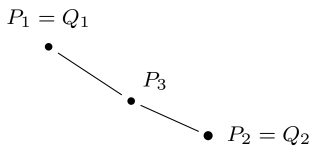
$\tilde{T}$



T

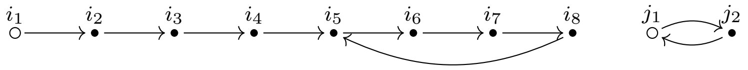


$\tilde{T}/\langle\sigma\rangle \cong T$

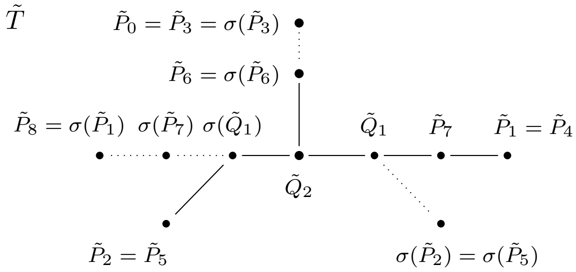


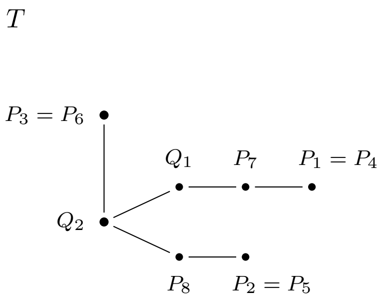
P_2 and Q_2 are marked at the vertex t of order zero. The vertex marked by P_3 is a fixed point of the composite map, but does not lie on the spine of T. The second iterate of the composite maps sends t to itself, and as predicted in Lemma 8.5, the degree 4 morphism f^2 reduces to the quadratic morphism $x \mapsto -2(x^2+2)^{-1}$ over $\mathbb{F}_5$, with postcritical orbit determined by $\bar{\imath}_1 = \sigma(\bar{\imath}_2)$ and $\bar{\jmath}_1 = \bar{\imath}_0$. Further, this provides a counterexample to the properties found for trees of good reduction, as the markings on T are not distributed in a fashion that reflects the postcritical orbit: the branch at t contains markings both for the loop point and for two spoke points, and t is marked by non-consecutive spoke points.

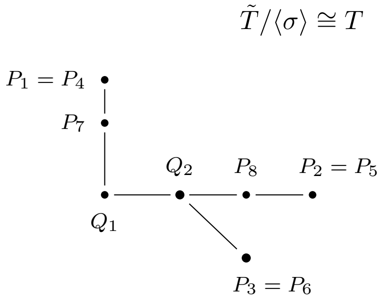
Example 11.4. Another example for bad reduction is the Type (a) mapping scheme Γ with defining points $i_4 = \sigma(i_8)$ and $j_0 = j_2$



For a root a of the factor of degree 60 of the polynomial derived from the equation $f_\alpha^8(0) = -f_\alpha^4(0)$, the quadratic morphism $\left(\mathbb{P}^1_K, f_a : x \mapsto \frac{x^2+a}{x^2-1}, 0, \infty\right)$ over $K := \mathbb{Q}(a)$ has postcritical orbit Γ. This morphism has bad reduction over $\mathbb{F}_3$. Further, we have $\mathrm{ord}_\pi(i_3) = 3$, $\mathrm{ord}_\pi(i_6) = 2$ and $\mathrm{ord}_\pi(i_k) = 0 = \mathrm{ord}_\pi(j_1)$ for $k \neq 3, 6$, and the following trees:







Here, Q_2 is marked at the vertex t of order zero. The composite map has no fixed vertex at all. However, the second iterate maps t to itself, and again as in Lemma 8.5, the degree 4 morphism f^2 reduces to the quadratic morphism $x \mapsto 2x^2 + 1$ over $\mathbb{F}_3$, with postcritical orbit determined by $\bar{\imath}_1 = \sigma(\bar{\imath}_2)$ and $\bar{\jmath}_0 = \bar{\jmath}_1$. On the other hand, this is another counterexample to the properties found for trees of good reduction: each branch at t in T contains markings of both loop and spoke points, and no two branches are isomorphic. In fact, the trees do not seem to reflect the dynamics at all.

Appendix - SageMath Calculations

Example 11.1 $x \mapsto \frac{x^2-\frac{1}{2}(a^2+1)}{x^2+a}$ with $i_2 = \sigma(i_6)$ and $j_1 = \sigma(j_3)$

```
sage: Qx.<x> = PolynomialRing(QQ,1);
sage: P1Qx.<u,v> = ProjectiveSpace(Qx,1);
sage: EndP1Qx = End(P1Qx);
sage: fx=EndP1Qx([u^2-1/2*(x^2+1)*v^2,u^2+x*v^2]);
sage: (fx.nth_iterate(P1Qx(1,0),1)[1])*(fx.nth_iterate(P1Qx(1,0),3)[0])
 == -(fx.nth_iterate(P1Qx(1,0),3)[1])*(fx.nth_iterate(P1Qx(1,0),1)[0])
True
sage: p2 = fx.nth_iterate(P1Qx(0,1),2)[0];
sage: q2 = fx.nth_iterate(P1Qx(0,1),2)[1];
sage: p6 = fx.nth_iterate(P1Qx(0,1),6)[0];
sage: q6 = fx.nth_iterate(P1Qx(0,1),6)[1];
sage: (p2*q6+q2*p6).factor();
(-1/8796093022208) * (x - 1) * x^2 * (x + 1)^37 * (x^2 + 1) * (x^3 +
 5*x^2 - x + 3) * (x^7 + 9*x^6 + 15*x^5 - 33*x^4 + 43*x^3 - 29*x^2 +
 13*x - 3) * (x^36 + 36*x^35 + 598*x^34 + 5804*x^33 + 34433*x^32 +
 116480*x^31 + 150928*x^30 - 202912*x^29 - 140028*x^28 + 2084752*x^27 -
 1771960*x^26 - 6163184*x^25 + 23623188*x^24 - 33435968*x^23 -
 864336*x^22 + 120029728*x^21 - 336166274*x^20 + 612801016*x^19 -
 872157916*x^18 + 1032386632*x^17 - 1047543538*x^16 + 926691456*x^15 -
 721763472*x^14 + 497732768*x^13 - 304687820*x^12 + 165609168*x^11 -
 79774328*x^10 + 33923472*x^9 - 12657052*x^8 + 4108224*x^7 -
 1146800*x^6 + 271328*x^5 - 53431*x^4 + 8580*x^3 -1098*x^2 + 108*x - 7)
sage: K.<a> = NumberField(x^36 + 36*x^35 + 598*x^34 + 5804*x^33 +
 34433*x^32 + 116480*x^31 + 150928*x^30 - 202912*x^29 - 140028*x^28 +
 2084752*x^27 - 1771960*x^26 - 6163184*x^25 + 23623188*x^24 -
 33435968*x^23 - 864336*x^22 + 120029728*x^21 - 336166274*x^20 +
 612801016*x^19 - 872157916*x^18 + 1032386632*x^17 - 1047543538*x^16 +
 926691456*x^15 - 721763472*x^14 + 497732768*x^13 - 304687820*x^12 +
 165609168*x^11 - 79774328*x^10 + 33923472*x^9 - 12657052*x^8 +
 4108224*x^7 - 1146800*x^6 + 271328*x^5 - 53431*x^4 + 8580*x^3 -
 1098*x^2 + 108*x - 7)
sage: P1K.<x,y> = ProjectiveSpace(K,1); EndP1K = End(P1K);
sage: f=EndP1K([x^2-1/2*(a^2+1)*y^2,x^2+a*y^2]); f.is_morphism()
True
sage: i0 = P1K(0,1); nmax = 6;
sage: Orb0 = [0,0,0,0,0,0];
sage: for k in range(1,nmax+1):
            Orb0[k-1] = f.nth_iterate(i0,k)[0]/f.nth_iterate(i0,k)[1];
sage: Orb0[1] == -Orb0[5]
True
```

```
sage: P = K.prime_factors(7);
sage: n=4; K(7).valuation(P[n]);
1
sage: val = [0,0,0,0,0,0];
sage: for k in range(1,nmax+1):
            val[k-1] = Orb0[k-1].valuation(P[n]);
sage: val;
[-1, 0, 0, 0, 0, 0]
sage: pi = K.uniformizer(P[n]); pi;
a
sage: R = (P[n]).residue_field(); R.order();
7
sage: imageOrb0 = [0,0,0,0,0,0];
sage: for k in range(1,nmax+1):
            imageOrb0[k-1] = R(pi^(-val[k-1])*Orb0[k-1]);
sage: imageOrb0;
[3, 1, 4, 6, 4, 6]
```

Example 11.2 $x \mapsto \frac{x^2-\frac{1}{2}(a^2+1)}{x^2+a}$ with $i_3 = \sigma(i_7)$ and $j_1 = \sigma(j_3)$

```
sage: Qx.<x> = PolynomialRing(QQ,1);
sage: P1Qx.<u,v> = ProjectiveSpace(Qx,1);
sage: EndP1Qx = End(P1Qx);
sage: fx=EndP1Qx([u^2-1/2*(x^2+1)*v^2,u^2+x*v^2]);
sage: (fx.nth_iterate(P1Qx(1,0),1)[1])*(fx.nth_iterate(P1Qx(1,0),3)[0])
 == -(fx.nth_iterate(P1Qx(1,0),3)[1])*(fx.nth_iterate(P1Qx(1,0),1)[0])
True
sage: p3 = fx.nth_iterate(P1Qx(0,1),3)[0];
sage: q3 = fx.nth_iterate(P1Qx(0,1),3)[1];
sage: p7 = fx.nth_iterate(P1Qx(0,1),7)[0];
sage: q7 = fx.nth_iterate(P1Qx(0,1),7)[1];
sage: (p3*q7+q3*p7).factor();
(-1/309485009821345068724781056) * (x + 1)^76 * (x^2 + 1) * (x^3 +
 3*x^2 - x + 1)^2 * (x^6 + 4*x^5 - 7*x^4 + 10*x^3 - 7*x^2 + 4*x - 1) *
 (x^16 + 18*x^15 + 128*x^14 + 342*x^13 - 228*x^12 - 838*x^11 +
 3568*x^10 - 6290*x^9 + 7622*x^8 - 6730*x^7 + 4576*x^6 - 2366*x^5 +
 924*x^4 - 258*x^3 + 48*x^2 - 6*x + 1) * (x^73 + 73*x^72 + 2532*x^71 +
 55316*x^70 + 849630*x^69 + 9683038*x^68 + 84061956*x^67 +
 560196596*x^66 + 2838609793*x^65 + 10589280857*x^64 + 26962741152*x^63 +
 38318897184*x^62 + 9420828368*x^61 + 4827571920*x^60 +
 290535565600*x^59 + 422879484832*x^58 - 1226625816140*x^57 -
 557015129644*x^56 + 8715198241584*x^55 - 7786173271952*x^54 -
 34171450315448*x^53 + 107484566773576*x^52 - 418574541477792*x^51 -
 455491395907216*x^50 + 1324988146394340*x^49 - 1258202502594364*x^48 -
```

```
 2694719517334304*x^47 + 13339021685860448*x^46 - 28139711725219664*x^45 +
 31810402998837424*x^44 + 8849985463597664*x^43 -
 141853581723096096*x^42 + 415287253665441070*x^41 -
 854139490242326370*x^40 + 1438749202708104344*x^39 -
 2097103176313052744*x^38 + 2718137082921459764*x^37 -
 3183032632420318924*x^36 + 3402088834357557720*x^35 -
 3341685996050000008*x^34 + 3030824424921250142*x^33 -
 2546519769833105618*x^32 + 1986340820876757088*x^31 -
 1440167357493769248*x^30 + 970970156611241904*x^29 -
 608524070452035664*x^28 + 354098608281860320*x^27 -
 190917648548574624*x^26 + 95073990908804292*x^25 -
 43526468966776220*x^24 + 18196957317762480*x^23 - 6878302058829072*x^22 +
 2314879145910856*x^21 - 676065407105976*x^20 + 163168313062064*x^19 -
 28914441960208*x^18 + 2201996777044*x^17 + 587245021620*x^16 -
 133486421728*x^15 - 111832397408*x^14 +105694642640*x^13 -
 52775812656*x^12 + 19495109024*x^11 - 5802748896*x^10 + 1436598361*x^9 -
 299127071*x^8 + 52345508*x^7 - 7605036*x^6 + 888926*x^5 - 77346*x^4 +
 3908*x^3 + 52*x^2 - 23*x + 1)
sage: K.<a> = NumberField(x^73 + 73*x^72 + 2532*x^71 + 55316*x^70 +
 849630*x^69 + 9683038*x^68 + 84061956*x^67 + 560196596*x^66 +
 2838609793*x^65 + 10589280857*x^64 + 26962741152*x^63 +
 38318897184*x^62 + 9420828368*x^61 + 4827571920*x^60 +
 290535565600*x^59 + 422879484832*x^58 - 1226625816140*x^57 -
 557015129644*x^56 + 8715198241584*x^55 - 7786173271952*x^54 -
 34171450315448*x^53 + 107484566773576*x^52 - 41857454147792*x^51 -
 455491395907216*x^50 + 1324988146394340*x^49 - 1258202502594364*x^48 -
 2694719517334304*x^47 + 13339021685860448*x^46 - 28139711725219664*x^45 +
 31810402998837424*x^44 + 8849985463597664*x^43 -
 141853581723096096*x^42 + 415287253665441070*x^41 -
 854139490242326370*x^40 + 1438749202708104344*x^39 -
 2097103176313052744*x^38 + 2718137082921459764*x^37 -
 3183032632420318924*x^36 + 3402088834357557720*x^35 -
 3341685996050000008*x^34 + 3030824424921250142*x^33 -
 2546519769833105618*x^32 + 1986340820876757088*x^31 -
 1440167357493769248*x^30 + 970970156611241904*x^29 -
 608524070452035664*x^28 + 354098608281860320*x^27 -
 190917648548574624*x^26 + 95073990908804292*x^25 -
 43526468966776220*x^24 + 18196957317762480*x^23 -
 6878302058829072*x^22 + 2314879145910856*x^21 - 676065407105976*x^20 +
 163168313062064*x^19 - 28914441960208*x^18 + 2201996777044*x^17 +
 587245021620*x^16 - 133486421728*x^15 - 111832397408*x^14 +
 105694642640*x^13 - 52775812656*x^12 + 19495109024*x^11 -
 5802748896*x^10 + 1436598361*x^9 - 299127071*x^8 + 52345508*x^7 -
 7605036*x^6 + 888926*x^5 - 77346*x^4 + 3908*x^3 + 52*x^2 - 23*x + 1)
sage: P1K.<x,y> = ProjectiveSpace(K,1); EndP1K = End(P1K);
```

```
sage:  f=EndP1K([x^2-1/2*(a^2+1)*y^2,x^2+a*y^2]); f.is_morphism()
True
sage: i0 = P1K(0,1); nmax = 7;
sage: Orb0 = [0,0,0,0,0,0,0];
sage: for k in range(1,nmax+1):
            Orb0[k-1] = f.nth_iterate(i0,k)[0]/f.nth_iterate(i0,k)[1];
sage: Orb0[2] == -Orb0[6]
True
sage: P = K.prime_factors(5);
sage: n=5; K(5).valuation(P[n]);
1
sage: val = [0,0,0,0,0,0,0];
sage: for k in range(1,nmax+1):
            val[k-1] = Orb0[k-1].valuation(P[n]);
sage: val;
[0, 0, 0, 0, 0, 0, 0]
sage: pi = K.uniformizer(P[n]); pi;
a^2 - 2
sage: R = (P[n]).residue_field(); R.order();
25
sage: imageOrb0 = [0,0,0,0,0,0,0];
sage: for k in range(1,nmax+1):
            imageOrb0[k-1] = R(pi^(-val[k-1])*Orb0[k-1]);
sage: imageOrb0;
[3*abar, 3*abar + 1, 3*abar + 2, 4, 2*abar + 3, 4, 2*abar + 3]
sage: Rz.<z> = PolynomialRing(R,1); (z^2+R(-1/2*(a^2+1)))/(z^2+R(a))
(z^2 + 1)/(z^2 + (abar))
```

Example 11.3 $x \mapsto \frac{x^2-1}{x^2+a}$ with $i_2 = \sigma(i_3)$ and $j_2 = i_0$

```
sage: Qx.<x> = PolynomialRing(QQ,1);
sage: P1Qx.<u,v> = ProjectiveSpace(Qx,1);
sage: EndP1Qx = End(P1Qx);
sage: fx=EndP1Qx([u^2-v^2,u^2+x*v^2]);
sage: (fx.nth_iterate(P1Qx(0,1),0)[1])*(fx.nth_iterate(P1Qx(1,0),2)[0])
 == -(fx.nth_iterate(P1Qx(1,0),2)[1])*(fx.nth_iterate(P1Qx(0,1),0)[0])
True
sage: p2 = fx.nth_iterate(P1Qx(0,1),2)[0];
sage: q2 = fx.nth_iterate(P1Qx(0,1),2)[1];
sage: p3 = fx.nth_iterate(P1Qx(0,1),3)[0];
sage: q3 = fx.nth_iterate(P1Qx(0,1),3)[1];
sage: (p2*q3+q2*p3).factor();
(-1) * (x + 1)^3 * (2*x^6 - 6*x^5 + 10*x^4 - 8*x^3 + 2*x^2 + 2*x - 1)
sage: K.<a> = NumberField(2*x^6-6*x^5+10*x^4-8*x^3+2*x^2+2*x-1);
```

```
sage: P1K.<x,y> = ProjectiveSpace(K,1); EndP1K = End(P1K);
sage: f=EndP1K([x^2-y^2,x^2+a*y^2]); f.is_morphism()
True
sage: i0 = P1K(0,1); nmax = 3;
sage: Orb0 = [0,0,0];
sage: for k in range(1,nmax+1):
            Orb0[k-1] = f.nth_iterate(i0,k)[0]/f.nth_iterate(i0,k)[1];
sage: Orb0[1] == -Orb0[2]
True
sage: P = K.prime_factors(5);
sage: n=1; K(5).valuation(P[n]);
1
sage: val = [0,0,0];
sage: for k in range(1,nmax+1):
            val[k-1] = Orb0[k-1].valuation(P[n]);
sage: val;
[0, 0, 0]
sage: pi = K.uniformizer(P[n]); pi;
-2*a^5 + 4*a^4 - 6*a^3 + 4*a^2 - 2*a + 2
sage: R = (P[n]).residue_field(); R.order();
5
sage: imageOrb0 = [0,0,0];
sage: for k in range(1,nmax+1):
            imageOrb0[k-1] = R(pi^(-val[k-1])*Orb0[k-1]);
sage: imageOrb0;
[1, 4, 1]
```

Example 11.4 $x \mapsto \frac{x^2+a}{x^2-1}$ with $i_4 = \sigma(i_8)$ and $j_2 = j_0$

```
sage: Qx.<x> = PolynomialRing(QQ,1);
sage: P1Qx.<u,v> = ProjectiveSpace(Qx,1);
sage: EndP1Qx = End(P1Qx);
sage: fx=EndP1Qx([u^2+x*v^2,u^2-v^2]);
sage: (fx.nth_iterate(P1Qx(1,0),0)[1])*(fx.nth_iterate(P1Qx(1,0),2)[0])
 == -(fx.nth_iterate(P1Qx(1,0),2)[1])*(fx.nth_iterate(P1Qx(1,0),0)[0])
True
sage: p4 = fx.nth_iterate(P1Qx(0,1),4)[0];
sage: q4 = fx.nth_iterate(P1Qx(0,1),4)[1];
sage: p8 = fx.nth_iterate(P1Qx(0,1),8)[0];
sage: q8 = fx.nth_iterate(P1Qx(0,1),8)[1];
sage: (p4*q8+q4*p8).factor();
x * (x - 1)^2 * (x + 1)^90 * (x^3 - x^2 + 3*x - 1) * (x^4 - 3*x^3 +
 6*x^2 - 4*x + 1) * (2*x^5 - 6*x^4 + 12*x^3 - 12*x^2 + 6*x - 1) * (x^15 -
 8*x^14 + 36*x^13 - 114*x^12 +286*x^11 - 604*x^10 + 1120*x^9 - 1806*x^8 +
```

```
 2460*x^7 - 2712*x^6 + 2300*x^5 - 1426*x^4+ 614*x^3 - 172*x^2 + 28*x - 2) *
 (x^60 - 29*x^59 + 435*x^58 - 4477*x^57 + 35434*x^56- 229387*x^55 +
 1262336*x^54 - 6062841*x^53 + 25904259*x^52 - 99896366*x^51 +
 351673024*x^50 - 1140514867*x^49 + 3433115464*x^48 - 9652048616*x^47 +
 25479950988*x^46 - 63445828673*x^45 + 149605065128*x^44 -
 335214616106*x^43 + 715880391462*x^42 - 1460959303891*x^41 +
 2855580959465*x^40 - 5355800245541*x^39 + 9653111457481*x^38 -
 16736747469622*x^37 + 27930251060647*x^36 - 44863923302670*x^35 +
 69331772604135*x^34 - 102980562941135*x^33 + 146803936549600*x^32 -
 200479385272800*x^31 + 261689608219001*x^30 - 325683896569160*x^29 +
 385396674344672*x^28 - 432369975068060*x^27 + 458478473108819*x^26 -
 458078701028407*x^25 + 429866479821881*x^24 - 377652656155683*x^23 +
 309592145311550*x^22 - 236034665010641*x^21 + 166790438376056*x^20 -
 108857643256119*x^19 + 65384044206513*x^18 - 36006141631104*x^17 +
 18107343419576*x^16 - 8280889987849*x^15 + 3428240790852*x^14 -
 1278441792796*x^13 +427080982024*x^12 - 127014277743*x^11 +
 33388270415*x^10 - 7692731893*x^9 + 1537913771*x^8 -
 263491822*x^7 + 38087869*x^6 - 4550908*x^5 + 437067*x^4 -
 32397*x^3 + 1738*x^2 - 60*x + 1)
sage: K.<a> = NumberField(x^60 - 29*x^59 + 435*x^58 - 4477*x^57 +
 35434*x^56 - 229387*x^55 + 1262336*x^54 - 6062841*x^53 + 25904259*x^52 -
 99896366*x^51 + 351673024*x^50 - 1140514867*x^49 + 3433115464*x^48 -
 9652048616*x^47 + 25479950988*x^46 - 63445828673*x^45 +
 149605065128*x^44 - 335214616106*x^43 + 715880391462*x^42 -
 1460959303891*x^41 + 2855580959465*x^40 - 5355800245541*x^39 +
 9653111457481*x^38 - 16736747469622*x^37 + 27930251060647*x^36 -
 44863923302670*x^35 + 69331772604135*x^34 - 102980562941135*x^33 +
 146803936549600*x^32 - 200479385272800*x^31 + 261689608219001*x^30 -
 325683896569160*x^29 + 385396674344672*x^28 - 432369975068060*x^27 +
 458478473108819*x^26 - 458078701028407*x^25 + 429866479821881*x^24 -
 377652656155683*x^23 + 309592145311550*x^22 - 236034665010641*x^21 +
 166790438376056*x^20 - 108857643256119*x^19 + 65384044206513*x^18 -
 36006141631104*x^17 + 18107343419576*x^16 - 8280889987849*x^15 +
 3428240790852*x^14 - 1278441792796*x^13 + 427080982024*x^12 -
 127014277743*x^11 + 33388270415*x^10 - 7692731893*x^9 + 1537913771*x^8 -
 263491822*x^7 + 38087869*x^6 - 4550908*x^5 + 437067*x^4 - 32397*x^3 +
 1738*x^2 - 60*x + 1)
sage: P1K.<x,y> = ProjectiveSpace(K,1); EndP1K = End(P1K);
sage: f=EndP1K([x^2+a*y^2,x^2-y^2]); f.is_morphism()
True
sage: i0 = P1K(0,1); nmax = 8;
sage: Orb0 = [0,0,0,0,0,0,0,0];
sage: for k in range(1,nmax+1):
          Orb0[k-1] = f.nth_iterate(i0,k)[0]/f.nth_iterate(i0,k)[1];
sage: Orb0[3] == -Orb0[7]
```

```
True
sage: P = K.prime_factors(3);
sage: n=4; K(3).valuation(P[n]);
7
sage: val = [0,0,0,0,0,0,0,0];
sage: for k in range(1,nmax+1):
            val[k-1] = Orb0[k-1].valuation(P[n]);
sage: val;
[0, 0, 3, 0, 0, 2, 0, 0]
sage: pi = K.uniformizer(P[n]); pi;
10478603729234184232898606305 9/314592329665919799*a^59 -
 1099435965696802170952948907 59/11651567765404437*a^58 +
 1452664254315549970861821511 8844/104864109888639933*a^57 -
 4396006780207711973427836517 84307/314592329665919799*a^56 +
 3413865476013958638657410431 481777/314592329665919799*a^55 -
 4280895248595129630910465248 2068/620497691648757*a^54 +
 1173887849785225708465041828 72146198/314592329665919799*a^53 -
 5544857382511598698843102264 03278818/314592329665919799*a^52 +
 2331298134232767336071580868 520798150/314592329665919799*a^51 -
 8851384680827995476543715938 569110783/314592329665919799*a^50 +
 3069274180268407762082990565 9450561068/314592329665919799*a^49 -
 3269590079626859661205835547 3477147731/104864109888639933*a^48 +
 2910620170321487251906704327 40705640044/314592329665919799*a^47 -
 2689877081558061772177515959 59080171646/104864109888639933*a^46 +
 7004723200078624728347206034 35608399497/104864109888639933*a^45 -
 5163316161649693805857058399 115298055984/314592329665919799*a^44 +
 1201731295649825593107030244 1112375838702/314592329665919799*a^43 -
 6816620487292803871565801010 78089941522/8066469991433841*a^42 +
 1868889373457438059229551076 4765728466328/104864109888639933*a^41 -
 1130177560265539647523668785 75884402202542/314592329665919799*a^40 +
 7274411500164078812938131269 2959875338831/104864109888639933*a^39 -
 4043991132930245680631635504 59739655652941/314592329665919799*a^38 +
 8001645374362329627550132173 8375298688363/34954703296213311*a^37 -
 1233534601759443671652429449 812508980732486/314592329665919799*a^36 +
 2033172918884601233633409609 227134662739754/314592329665919799*a^35 -
 3224283739527865919125661373 271530758734139/314592329665919799*a^34 +
 4916271438453089045085001626 371255407719245/314592329665919799*a^33 -
 7198859142667497839089166673 631847204377450/314592329665919799*a^32 +
 1010624763852639767056337537 1057128406059069/314592329665919799*a^31 -
 1357387490146205368948874093 8328903958796520/314592329665919799*a^30 +
 1739956774479095138183237886 9684788360987062/314592329665919799*a^29 -
 7075840973778171424463541285 490538270534663/104864109888639933*a^28 +
 2457488692949237274524440995 2522113208811872/314592329665919799*a^27 -
 2691185449204384866429122614 6530443165882989/314592329665919799*a^26 +
 9262018859093357840252115499 679364536574967/104864109888639933*a^25 -
```

```
 26956874496437371399967028012113062488842710/314592329665919799*a^24 +
 2720962245304221651275474778450025177824872/34954703296213311*a^23 -
 6919175469176681084409775464343386555049993/104864109888639933*a^22 +
 16358019306825674306203766427636391033683206/314592329665919799*a^21 -
 11940474026109806170284359142678523043059633/314592329665919799*a^20 +
 2680839339417516312492289652277579099941164/104864109888639933*a^19 -
 1659606461605193145355116138691762886192513/104864109888639933*a^18 +
 940398562699098110939554527866534056468678/104864109888639933*a^17 -
 485628441589878038181066953063953489463155/104864109888639933*a^16 +
 682486161010260557425545807885779940761681/314592329665919799*a^15 -
 32065041183550063359124192102424993516185/34954703296213311*a^14 +
 4057322743339567326889772257430128497455/11651567765404437*a^13 -
 37107425611833391046359309444193830524370/314592329665919799*a^12 +
 11140219832361444745673155513659536065217/314592329665919799*a^11 -
 2941122324767570592117068737410104449937/314592329665919799*a^10 +
 676631643970479317521119551063527793056/314592329665919799*a^9 -
 134173817315622895292382901193387070707/314592329665919799*a^8 +
 22628057836241539220022740203475844142/314592329665919799*a^7 -
 3191289312551819416440002046234679484/314592329665919799*a^6 +
 122730724308080029229592511290977416/104864109888639933*a^5 -
 2594345158015277956941471570778228/24199409974301523*a^4 +
 2349047373176773266305403039036397/314592329665919799*a^3 -
 116236784315528093007296437355182/314592329665919799*a^2 +
 3614058343806609352807367913890/314592329665919799*a -
 52536393720153604779545747981/314592329665919799
sage: R = (P[n]).residue_field(); R.order();
3
sage: imageOrb0 = [0,0,0,0,0,0,0,0];
sage: for k in range(1,nmax+1):
          imageOrb0[k-1] = R(pi^(-val[k-1])*Orb0[k-1]);
sage: imageOrb0;
[1, 2, 2, 1, 2, 1, 1, 2]
```

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