ETH zürich

Catenary rings

Anna Bot

Bachelor thesis

supervised by
Prof. Dr. Richard Pink

2.10.2017

Contents

1 Introduction

It is assumed that the reader is at ease with the terminology and concepts from Commutative Algebra, for example prime ideals, noetherian rings, quotient rings, local rings, finitely generated algebras over a ring and localisation of rings, to name a few. For an orientation, one might find the lecture notes of Commutative Algebra [1] useful.

For the entire exposition, all rings are commutative and unitary. The aim of this bachelor thesis is to prove the following two theorems:

Theorem. *Any ring that is finitely generated over a field K, or over $\mathbb{Z}$, or over any Dedekind ring, respectively, is catenary.*

Theorem. *Any integral domain R that is finitely generated over a field K, or over $\mathbb{Z}$, satisfies the following:*

(i) For all prime ideals $\mathfrak{p} \subset R$: $\mathrm{ht}(\mathfrak{p}) + \mathrm{coht}(\mathfrak{p}) = \dim(R)$.

(ii) For all maximal ideals $\mathfrak{m} \subset R$: $\mathrm{ht}(\mathfrak{m}) = \dim(R)$.

There are two approaches to proving the first statement; one can work with complexes, namely the Koszul complex, or — as will be pursued here — one uses regular sequences, depth, Cohen Macaulay rings and catenary rings.

The path we will take is as follows: After introducing and examining regular sequences, we are able to define the depth of an ideal. Then, we anchor the depth to other concepts such as localisation, the height of an ideal, and the Jacobson radical. Subsequently, we define Cohen-Macaulay rings and (universally) catenary rings, and show that Cohen-Macaulay rings are universally catenary. This cumulates in the proof of our first theorem. For the second theorem, one does not necessarily need to understand the previous chapters in detail, and can thus skip to Chapter 8. At the end, we look back on our findings and try to find some meaningful geometric interpretation of them.

Nearly all proofs in this thesis imitate the proofs found in the book by I. Kaplansky [4] in chapter 3, pages 84-100. Said book is primarily concerned with modules, to which the notions introduced in this bachelor thesis can be extended. Here, the proofs given by Kaplansky were adapted to rings in order to fit into the scope of this bachelor thesis.

The proof about polynomial rings over Cohen-Macaulay rings found in Section 6 is taken from the book by D. Eisenbud [3], as are the proofs of Section 7 and the example of a ring that is not Cohen-Macaulay. They are taken from chapter 18, pages 448, 449, 451-453 and 466.

I would like to thank my supervisor Professor Pink for his guidance, the careful corrections and his useful comments on structure, content and the mathematical language. I learnt from him that to the reader, legibility and a good order is as important as correct proofs. Furthermore, I am grateful to Sebastian Schlegel Mejia for his thorough reading of one of my drafts and making me aware of some inconsistencies. Also, I am thankful to Oliver Edtmair for the helpful discussion of the last proof.

2 Prerequisites

We state a few important theorems here that will later be referenced in this thesis. Their proofs can be found in standard textbooks on Commutative Algebra, such as in the book *Introduction to Commutative Algebra* by M.F. Atiyah and I.G. Macdonald [2], or in the book by Eisenbud [3]. Other uses of smaller, but useful facts will be mentioned as we go along.

Definition 2.1. *A* zero-divisor *of a ring R is an element $x \in R$ such that there exists an element $y \in R \smallsetminus \{0\}$ with $xy = 0$. We shall treat 0 as a zero-divisor.*

Definition 2.2. *The* height of a prime ideal $\mathfrak{p} \subset R$ *is defined as*

$$\mathrm{ht}(\mathfrak{p}) := \sup\{\, r \geqslant 0 \,|\, \textit{There exist prime ideals } \mathfrak{p}_0 \subsetneqq \ldots \subsetneqq \mathfrak{p}_r = \mathfrak{p} \,\}.$$

The height of an ideal $\mathfrak{a} \subset R$ *is defined as*

$$\mathrm{ht}(\mathfrak{a}) := \inf\{\, \mathrm{ht}(\mathfrak{p}) \mid \mathfrak{a} \subset \mathfrak{p},\ \textit{where}\, \mathfrak{p} \textit{ is a prime ideal}\,\}.$$

Definition 2.3. *The* coheight of an ideal $\mathfrak{a} \subset R$ *is defined as*

$$\mathrm{coht}(\mathfrak{a}) := \dim(R/\mathfrak{a}).$$

Theorem 2.4 (Artin-Tate)**.** *Suppose R is noetherian and S a finitely generated R-algebra. If $T \subset S$ is an R-algebra such that S is a finitely generated T-module, then T is a finitely generated R-algebra.*

Theorem 2.5. *Let L/K be a field extension which is finitely generated as a ring over K. Then L/K is finite.*

Theorem 2.6 (Hilbert's Basis Theorem)**.** *If R is noetherian, then so is $R[X_1, \ldots, X_n]$ for any $n \in \mathbb{Z}^{\geqslant 0}$.*

Theorem 2.7 (Krull's Principal Ideal Theorem)**.** *For any noetherian ring R and any $a \in R$ which is not a zero-divisor, any minimal prime ideal $\mathfrak{p}$ above (a) satisfies* $\mathrm{ht}(\mathfrak{p}) = 1$.

Theorem 2.8 (Krull's Dimension Theorem)**.** *For any noetherian ring R, any ideal $\mathfrak{a} = (a_1, \ldots, a_r)$ generated by $r \geqslant 0$ elements and any minimal prime ideal $\mathfrak{p}$ above $\mathfrak{a}$ we have* $\mathrm{ht}(\mathfrak{p}) \leqslant r$. *If $\mathfrak{a} \neq (1)$ then* $\mathrm{ht}(\mathfrak{a}) \leqslant r$.

Theorem 2.9. *For any noetherian ring R and any $n \in \mathbb{Z}^{\geqslant 0}$, we have* $\dim(R[X_1, \ldots, X_n]) = \dim(R) + n$.

3 Regular sequences

Definition 3.1. *A* regular sequence *is a sequence $a_1, \dots, a_r \in R$ for which the following two conditions hold:*

(i) For every $i = 1, \dots, r$, the image of a_i in $R/(a_1, \dots, a_{i-1})$ is not a zero-divisor.

(ii) $R/(a_1, \dots, a_r) \neq 0$.

If, in addition, $a_1, \dots, a_r$ all lie in an ideal $\mathfrak{a}$, we call it a regular sequence in $\mathfrak{a}$.

We see that $R = 0$ cannot contain a regular sequence. In fact, the empty sequence is regular if and only if $R \neq 0$, so we exclude the special case of $R = 0$ from our discussion. The reader can convince her- or himself that the statements of this chapter still hold, albeit vacuously so. As soon as we prove Lemma 4.3 however, the assumption that $R \neq 0$ must hold becomes necessary.

Remark 3.2. *We assume $R \neq 0$.*

Let us warm up by proving a few direct observations about regular sequences.

Lemma 3.3. *For every regular sequence $a_1, \dots, a_r \in R$, we get the chain of ideals*

$$(a_1) \subsetneqq (a_1, a_2) \subsetneqq \dots \subsetneqq (a_1, \dots, a_r).$$

Proof. For $i = 2, \dots, r$, $(a_1, \dots, a_{i-1}) \subset (a_1, \dots, a_i)$ is an equality if and only if $a_i \in (a_1, \dots, a_{i-1})$, or equivalently, if $\overline{a_i}$ is zero in $R/(a_1, \dots, a_{i-1})$. This is excluded by Definition 3.1 and Remark 2.1. $\square$

As the conditions for a regular sequence are stated in terms of factor rings, we prove the following claim to be able to better deal with them.

Lemma 3.4. *Let $\mathfrak{a}, \mathfrak{b}$ be ideals of R. Then*

$$\frac{R/\mathfrak{a}}{\mathfrak{b}(R/\mathfrak{a})} \cong \frac{R}{\mathfrak{a} + \mathfrak{b}},$$

where $\mathfrak{b}(R/\mathfrak{a}) = \left\{ \sum' b_i \overline{r_i} \mid \text{ all } b_i \in \mathfrak{b}, \text{ all } \overline{r_i} \in R/\mathfrak{a} \right\}$.

Proof. Consider the composite ring homomorphism

$$\varphi : R \twoheadrightarrow R/\mathfrak{a} \twoheadrightarrow \frac{R/\mathfrak{a}}{\mathfrak{b}(R/\mathfrak{a})}.$$

This homomorphism is surjective, hence $R/\ker(\varphi) \cong \frac{R/\mathfrak{a}}{\mathfrak{b}(R/\mathfrak{a})}$ by the homomorphism theorem. We determine the kernel of φ.

$$r \in \ker(\varphi) \Leftrightarrow r + \mathfrak{a} \in \mathfrak{b}(R/\mathfrak{a}) \Leftrightarrow (\exists\, b_i \in \mathfrak{b}\, \exists\, r_i \in R\, \exists\, a \in \mathfrak{a} : r = a + \sum b_i r_i) \Leftrightarrow r \in \mathfrak{a} + \mathfrak{b}.$$

The conclusion follows. $\square$

Proposition 3.5. *Suppose $a_1, \dots, a_r \in R$ is a sequence and let $i \in \{1, \dots, r\}$. The sequence $a_1, \dots, a_r$ is regular if and only if $a_1, \dots a_{i-1}$ is a regular sequence and $\overline{a}_i, \dots, \overline{a}_r$ is a regular sequence in $R/(a_1, \dots, a_{i-1})$.*

Proof. If $a_1, \dots, a_r$ is a regular sequence, then for any $i \in \{1, \dots, a_r\}$, we see that $a_1, \dots, a_{i-1}$ is a regular sequence, too. Now assume $a_1, \dots, a_{i-1}$ to be a regular sequence. Set $\overline{R} := R/(a_1, \dots, a_{i-1})$. By Lemma 3.4, we get the isomorphism $R/(a_1, \dots, a_r) \cong \overline{R}/(\overline{a}_i, \dots, \overline{a}_r)$. Hence, for any $i \leqslant j \leqslant r$, the image of a_j in $R/(a_1, \dots, a_{j-1})$ is a non-zero-divisor if and only if the image of $\overline{a}_j$ in $\overline{R}/(\overline{a}_i, \dots, \overline{a}_{j-1})$ is a non-zero-divisor. Furthermore, we have $R/(a_1, \dots, a_r) \neq 0$ if and only if $\overline{R}/(\overline{a}_i, \dots, \overline{a}_r) \neq 0$, which concludes the proof. $\square$

Example 3.6. Consider the polynomial ring $\mathbb{Z}[X_1, X_2, X_3]$ and the regular sequence

$$X_2, X_1(X_2 + 1), X_3(X_2 + 1).$$

If we change the order to

$$X_1(X_2 + 1), X_3(X_2 + 1), X_2,$$

we see that $X_3(X_2+1)$ is a zero-divisor of $\mathbb{Z}[X_1, X_2, X_3]/(X_1(X_2+1))$, which implies that the second sequence is not regular.

We see from the above example that we are not allowed to arbitrarily permute the elements of a regular sequence. This is the motivation for why we would like to know under what circumstances the sequence that results from a regular sequence when changing the order is again a regular sequence.

Lemma 3.7. *If $a, b \in R$ is a regular sequence then the image of a in $R/(b)$ is not a zero-divisor.*

Proof. Suppose $\overline{a}$ in $R/(b)$ is a zero-divisor. Then there exists a non-zero $\overline{z} \in R/(b)$ such that

$$\overline{a}\,\overline{z} = \overline{0} \text{ in } R/(b).$$

So, there exists $z \in R \smallsetminus (b)$ such that $az \in (b)$, i.e. $\exists\, t \in R : az = tb$.

As a, b is a regular sequence, the image of b in $R/(a)$ is not a zero-divisor. This implies that $t \in (a)$, so $t = au$ for some $u \in R$. Then,

$$\begin{aligned} & az = tb = aub \\ \Rightarrow\ & a(z - ub) = 0. \end{aligned}$$

Since a is not a zero-divisor of R we can deduce that $z = ub$, which implies $\overline{z} = \overline{0}$ in $R/(b)$, a contradiction. $\square$

Proposition 3.8. *If $a_1, \dots, a_r$ is a regular sequence, then*

$$a_1, \dots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \dots, a_r$$

is one if and only if the image of a_{i+1} is not a zero-divisor in $R/(a_1, \dots, a_{i-1})$.

Proof. "$\Rightarrow$": This follows directly from Defintion 3.1.

"$\Leftarrow$": By Proposition 3.5, we can replace R by $R/(a_1, \dots, a_{i-1})$ and $a_i, \dots, a_r$ by $\overline{a}_i, \dots, \overline{a}_r$. Hence, we would like to show that for a regular sequence $a_1, \dots, a_r$ where a_2 is not a zero-divisor of R, the sequence $a_2, a_1, a_3, \dots, a_r$ is again a regular sequence. This is guaranteed by the assumption that a_2 is not a zero-divisor of R, by Lemma 3.7 and by $a_1, \dots, a_r$ being a regular sequence. $\square$

To finish this section, we look at how regular sequences behave under localisation. The condition is reminiscent of the condition under which the pushforward of a prime ideal is again a prime ideal of the localisation.

Proposition 3.9. *Let R be a ring and $a_1, \dots, a_r \in R$ a regular sequence. Let $S \subset R$ be a multiplicative subset and suppose that $(a_1, \dots, a_r) \cap S = \varnothing$. Then $\frac{a_1}{1}, \dots, \frac{a_r}{1}$ is a regular sequence in $S^{-1}R$.*

Proof. We show that for every $i = 1, \dots, r$, the image of $\frac{a_i}{1}$ in $S^{-1}R/(\frac{a_1}{1}, \dots, \frac{a_{i-1}}{1})$ is not a zero-divisor.

For simplicity of notation, we consider $\frac{a_1}{1} \in S^{-1}R$ — for $i = 2, \dots, r$ it follows analogously. Suppose it is a zero-divisor of $S^{-1}R$, meaning there exists $\frac{r}{s} \in S^{-1}R \setminus \{0_{S^{-1}R}\}$ such that $\frac{ra_1}{s} = 0_{S^{-1}R}$. This holds if and only if:

$$\exists\, t \in S : tra_1 = ts0 = 0.$$

This, in turn, implies

$$\exists\, t \in S : tr = 0$$

as a_1 is not a zero-divisor of R by assumption. But then

$$\frac{r}{s} = \frac{rt}{st} = \frac{0}{st} = 0_{S^{-1}R}$$

contradicting our choice of $\frac{r}{s}$.

Denote by $\iota_*\mathfrak{a}$ the pushforward of an ideal $\mathfrak{a}$ via the map $\iota : R \longrightarrow S^{-1}R$, $a \mapsto \frac{a}{1}$. We know that for any ideal $\mathfrak{a} \subset R$ we have $\iota_*\mathfrak{a} = (1)$ if and only if $\mathfrak{a} \cap S = \varnothing$. By assumption, $(a_1, \dots, a_r) \cap S = \varnothing$, hence $(\frac{a_1}{1}, \dots, \frac{a_r}{1}) \neq S^{-1}R$. This implies the second property of a regular sequence, namely $S^{-1}R/(\frac{a_1}{1}, \dots, \frac{a_r}{1}) \neq 0$.

We see that indeed, $\frac{a_1}{1}, \dots, \frac{a_r}{1}$ is a regular sequence in $S^{-1}R$. $\square$

4 Depth

For the remainder of the thesis, we assume all rings to be noetherian. In this section, our strategy is to define the depth of an ideal and work towards proving that any two maximal regular sequences in a proper ideal have the same length, which means that the depth of a proper ideal is determined by the common length of its maximal regular sequences, and vice versa.

To understand right away why we are interested in this, let us look at the definition of the depth.

Definition 4.1. *For an ideal* $\mathfrak{a} \subset R$ *define the* depth of $\mathfrak{a}$ *as*

$$\operatorname{depth}(\mathfrak{a}) := \sup\{r \geqslant 0 \,|\, a_1, \dots, a_r \in \mathfrak{a} \text{ is a regular sequence}\} \in \mathbb{Z}^{\geqslant 0} \cup \{\pm\infty\}.$$

For the sake of brevity, we define:

Definition 4.2. *Let* $\mathcal{Z}(R)$ *denote the set of zero-divisors of* R.

First, we will have to prove a more general lemma. It will be used as a stepping stone for the main finding of this section, Theorem 4.7, and a few more times in Section 5.

Lemma 4.3. *Consider an ideal* $\mathfrak{a}$ *contained in* $\mathcal{Z}(R)$*. Then there exists an element* $x \in R \smallsetminus \{0\}$ *such that* $x\mathfrak{a} = 0$.

Proof. As mentioned previously, we need the assumption $R \neq 0$ for this proof, as otherwise there could not exist such an element $x \in R \smallsetminus \{0\}$. As R is noetherian, (0) is decomposable, and hence by [1, 23, Prop.9, Lemma 2, Lemma 3, Thm.3] we obtain a finite union

$$\mathcal{Z}(R) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}((0))} \mathfrak{p}$$

where $\mathfrak{p} = ((0) : (x)) = \operatorname{Ann}(x)$ for some $x \in R \smallsetminus \{0\}$. Here, we use the notation $(\mathfrak{a} : \mathfrak{b})$ for the quotient ideal of two ideals $\mathfrak{a}, \mathfrak{b}$. As this is a finite union of prime ideals, we know from [1, 31, Lemma 5] that any ideal lying in this union is contained in one of the associated prime ideals $\mathfrak{p}$ of (0).

So as $\mathfrak{a}$ lies in $\mathcal{Z}(R)$, there exists an $x \in R \smallsetminus \{0\}$ such that $\mathfrak{a} \subset \mathfrak{p} = \operatorname{Ann}(x)$. Using this x we get $x\mathfrak{a} = 0$. □

Definition 4.4. *A regular sequence* $a_1, \dots, a_r$ *in* $\mathfrak{a}$ *is called a* maximal regular sequence *if it cannot be extended to a regular sequence* $a_1, \dots, a_{r+1}$ *in* $\mathfrak{a}$.

The following lemma assures that maximal regular sequences actually exist.

Lemma 4.5. *Every regular sequence* $a_1, \dots, a_r$ *in* $\mathfrak{a}$ *can be extended to a maximal regular sequence in* $\mathfrak{a}$*. This holds especially when* $\mathfrak{a} = R$.

Proof. Suppose there exists an ideal $\mathfrak{a} \subset R$ which contains a regular sequence $a_1, \dots, a_r$ that cannot be extended to a maximal regular sequence. Then we can extend the sequence $a_1, \dots, a_r$ repeatedly with elements of $\mathfrak{a}$. By Lemma 3.3, it follows that we get an ascending chain of properly included ideals

$$(a_1) \subsetneqq (a_1, a_2) \subsetneqq \dots$$

which contradicts R being noetherian. □

The main goal of this section is to prove Theorem 4.7 below, for which we will prove a lemma first.

Lemma 4.6. *Let $\mathfrak{a} \subsetneqq R$ be an ideal and let $a_1, \dots, a_r$ be a regular sequence in $\mathfrak{a}$. Then, $a_1, \dots, a_r$ is a maximal regular sequence if and only if $\mathfrak{a}/(a_1, \dots, a_r) \subset \mathcal{Z}(R/(a_1, \dots, a_r))$.*

Proof. The regular sequence $a_1, \dots, a_r \in \mathfrak{a}$ is a maximal sequence if and only if it can not be extended to a regular sequence $a_1, \dots, a_r, a_{r+1}$ for any $a_{r+1} \in \mathfrak{a}$. This holds if and only if the image of any $a_{r+1} \in \mathfrak{a}$ in $R/(a_1, \dots, a_r)$ is a zero-divisor or if $R/(a_1, \dots, a_r, a_{r+1}) = 0$.

But $R/(a_1, \dots, a_r, a_{r+1}) \neq 0$, since $\mathfrak{a} \subsetneqq R$, which shows the lemma. □

Theorem 4.7. *Any two maximal regular sequences in an ideal $\mathfrak{a} \subsetneqq R$ have the same length.*

Proof. Let $a_1, \dots, a_r \in \mathfrak{a}$ be a maximal regular sequence and let $b_1, \dots, b_r \in \mathfrak{a}$ be another regular sequence of the same length. We want to show that $b_1, \dots, b_r$ is also a maximal regular sequence. We proceed by induction.

The case $r = 0$ is clear. Consider $r = 1$.

Let $a \in \mathfrak{a}$ and $b \in \mathfrak{a}$ be two regular sequences, and suppose a is a maximal regular sequence. By Lemma 4.6, $\mathfrak{a}/(a)$ only contains zero-divisors, and therefore, we can apply Lemma 4.3 to find $s + (a) \in (R/(a)) \smallsetminus \{0\}$ such that

$$\forall\, x + (a) \in \mathfrak{a}/(a) : sx + (a) = 0 \text{ in } R/(a).$$

We want to show that $\mathfrak{a}/(b) \subset R/(b)$ only contains zero-divisors, which again by Lemma 4.6 will imply that b is a maximal regular sequence in $\mathfrak{a}$. Let y be an arbitrary element whose image in $R/(b)$ is non-zero and consider the image of y in $R/(a)$. We get

$$\begin{aligned} sy + (a) &= 0 \\ sb + (a) &= 0 \end{aligned}$$

which means that there exist $u, v \in R$ such that

$$sy = ua,$$

$$sb = va, \tag{4.7}$$

where $v \neq 0$, because $s \neq 0$ and b is by assumption not a zero-divisor of R. This implies

$$bua = bsy = vya.$$

As a is not a zero-divisor, we obtain

$$vy = ub.$$

Suppose that $v \in (b)$. Then, by inserting it in (4.7) we obtain

$$sb = s'ba$$

for some $s' \in R$. By cancelling b on both sides — as b is not a zero-divisor of R — we obtain

$$s = s'a$$

in contradiction to $s + (a) \neq 0$.

We have shown that there exists $v + (b) \neq 0$ such that $(v + (b))(y + (b)) = 0$ in $R/(b)$. Thus, we conclude that the image of y in $R/(b)$ is a zero-divisor. Since y was chosen arbitrarily, b is also a maximal regular sequence.

Suppose that for $r \geqslant 1$, if we have a maximal regular sequence of length r in any proper ideal $\mathfrak{a}'$, then any other regular sequence of the same length in $\mathfrak{a}'$ is also maximal.

Consider $a_1, \ldots, a_{r+1} \in \mathfrak{a}$ a maximal regular sequence and $b_1, \ldots, b_{r+1} \in \mathfrak{a}$ some other regular sequence. Set for every $i = 0, \ldots, r$

$$\begin{aligned} A_i &:= R/(a_1, \ldots, a_i), \\ B_i &:= R/(b_1, \ldots, b_i), \\ \mathcal{D}(A_i) &:= \{\, x \in R \,|\, \text{the image of } x \text{ in } A_i \text{ is a zero-divisor} \,\}, \\ \mathcal{D}(B_i) &:= \{\, x \in R \,|\, \text{the image of } x \text{ in } B_i \text{ is a zero-divisor} \,\}. \end{aligned}$$

We know that for any A_i or B_i, the set of zero-divisors $\mathcal{Z}(A_i)$, $\mathcal{Z}(B_i)$, respectively, is a finite union of prime ideals by [1, 23, Prop.9]. Also, $\mathcal{D}(A_i)/(a_1, \ldots, a_i) = \mathcal{Z}(A_i)$. Therefore, $\mathcal{D}(A_i)$ can be written as the finite union of the corresponding primes in R that contain $(a_1, \ldots, a_i)$, and similarly for $\mathcal{D}(B_i)$. We obtain

$$\bigcup_{i=0}^{r} (\mathcal{D}(A_i) \cup \mathcal{D}(B_i)) = \bigcup_{\substack{\mathfrak{p}_j \text{ prime}, \\ j \in J,\, |J| < \infty}} \mathfrak{p}_j.$$

As both sequences $a_1, \ldots, a_{r+1}$ and $b_1, \ldots, b_{r+1}$ are regular, we see that $\mathfrak{a} \not\subset \mathcal{D}(A_i), \mathcal{D}(B_i)$ for all $i = 0, \ldots, r$. By [1, 31, Lemma 5] we obtain $\mathfrak{a} \not\subset \mathfrak{p}_j$ for all $j \in J$ and therefore $\mathfrak{a} \not\subset \bigcup_{i=0}^{r} (\mathcal{D}(A_i) \cup \mathcal{D}(B_i))$. This means that we can choose $z \in \mathfrak{a}$ whose image in all A_i and B_i is not a zero-divisor.

Now, consider the ring $R/(a_1, \ldots, a_r)$, in which, by assumption, $\overline{a}_{r+1}$ is a maximal regular sequence. As $\overline{z}$ is not a zero-divisor of $R/(a_1, \ldots, a_r)$, we conclude by the case $r = 1$ that $\overline{z}$ is also a maximal regular sequence, and hence $a_1, \ldots, a_r, z$ is a maximal sequence of $\mathfrak{a}$.

Using Proposition 3.8 repeatedly, and the fact that for all $i = 1, \ldots, r$, the image of z in $R/(a_1, \ldots, a_i)$ is not a zero-divisor, we can push z to the very front of the sequence $a_1, \ldots, a_r$ and obtain a regular sequence $z, a_1, \ldots, a_r$, which stays a maximal sequence of $\mathfrak{a}$. Similarly, push z to the very front of $b_1, \ldots, b_r$ to obtain a regular sequence $z, b_1, \ldots, b_r$.

We are ready to use the inductive hypothesis, as we can consider the ring $R/(z)$, in which $\overline{a}_1, \ldots, \overline{a}_r$ and $\overline{b}_1, \ldots, \overline{b}_r$ are regular sequences in $\mathfrak{a}/(z)$, the first one even a maximal one. By the inductive hypothesis, $\overline{b}_1, \ldots, \overline{b}_r$ is also a maximal regular sequence.

Lemma 3.4 implies that $R/(b_1,\ldots,b_r,z) \cong \frac{R/(z)}{(\bar{b}_1,\ldots,\bar{b}_r)}$. Hence $\mathfrak{a}/(b_1,\ldots,b_r,z)$ only contains zero-divisors, and thus $b_1,\ldots,b_r,z$ is a maximal sequence of $\mathfrak{a}$. Applying the base case once more to z and b_{r+1} within $R/(b_1,\ldots,b_r)$ implies that the sequence $b_1,\ldots,b_{r+1}$ is in fact maximal too, concluding our proof. □

Corollary 4.8. *Any maximal regular sequence in* $\mathfrak{a} \subsetneq R$ *has length equal to* $\operatorname{depth}(\mathfrak{a})$.

To conclude this section, we apply Theorem 4.7 to the next proposition, which provides us with the useful possibility to reduce to a suitable prime ideals instead of a general ideal.

Proposition 4.9. *For every ideal* $\mathfrak{a} \subsetneq R$ *there exists a prime ideal* $\mathfrak{p}$ *containing* $\mathfrak{a}$ *such that* $\operatorname{depth}(\mathfrak{a}) = \operatorname{depth}(\mathfrak{p})$.

Proof. Call $a_1,\ldots,a_r$ a maximal regular sequence in $\mathfrak{a}$. By the maximality of the sequence and because $\mathfrak{a} \neq R$, we see using Lemma 4.6 that

$$\mathfrak{a}/(a_1,\ldots,a_r) \subset \mathcal{Z}(R/(a_1,\ldots,a_r)).$$

We use [1, 23, Prop.9, Lemma 2, Lemma 3, Thm.3] to find

$$\mathcal{Z}(R/(a_1,\ldots,a_r)) = \bigcup_{j=1}^{m} \operatorname{Ann}(\overline{x_j})$$

for some $m \geqslant 1$ and suitable $\overline{x_j}$, where the $\operatorname{Ann}(\overline{x_j})$ are prime. So there exists some $k \in \{1,\ldots,m\}$ such that $\overline{\mathfrak{a}} \subset \operatorname{Ann}(\overline{x_k})$. By taking the prime ideal $\mathfrak{p}$ in R corresponding to $\operatorname{Ann}(\overline{x_k})$, we obtain $\mathfrak{a} \subset \mathfrak{p}$. As $\mathfrak{p}/(a_1,\ldots,a_r) = \operatorname{Ann}(\overline{x_k})$ only consists of zero-divisors, we conclude that $a_1,\ldots,a_r$ is a maximal regular sequence in $\mathfrak{p}$. By Theorem 4.7 we have $\operatorname{depth}(\mathfrak{a}) = r = \operatorname{depth}(\mathfrak{p})$. □

5 Depth in light of height, localisation and the Jacobson radical

We branch out more into different directions in this section, and try to understand the depth in connection with the height, localisation and the Jacobson radical. Since in Section 6, we will connect the depth and the height and use localisation extensively, the knowledge we will accumulate in this section will enable us to quickly navigate through the proofs of that section.

We start off by comparing the depth of an ideal to its height.

Proposition 5.1. *For any ideal* $\mathfrak{a}$ *we have* $\operatorname{depth}(\mathfrak{a}) \leqslant \operatorname{ht}(\mathfrak{a})$.

Proof. We show by induction that for a maximal regular sequence in $\mathfrak{a}$ of length r, we have $r \leqslant \mathrm{ht}(\mathfrak{a})$. For $r = 0$, the inequality holds, since we have $0 \leqslant \mathrm{ht}(\mathfrak{a})$ for any ideal $\mathfrak{a}$.

Assume that $\mathfrak{a}$ contains a maximal regular sequence $a_1, \ldots, a_{r+1}$. Consider $\overline{R} := R/(a_1)$ and $\overline{\mathfrak{a}} := \mathfrak{a}/(a_1)$. The sequence $\overline{a}_2, \ldots, \overline{a}_{r+1}$ in $\overline{\mathfrak{a}}$ is a maximal regular sequence by Proposition 3.5. So by the induction hypothesis it follows that

$$r \leqslant \mathrm{ht}(\overline{\mathfrak{a}}).$$

This means that for any prime ideal $\overline{\mathfrak{a}} \subset \overline{\mathfrak{p}}$, we have $r \leqslant \mathrm{ht}(\overline{\mathfrak{p}})$.

Take any prime ideal $\mathfrak{a} \subset \mathfrak{p}$. It holds that $\overline{\mathfrak{a}} \subset \mathfrak{p}/(a_1)$, and hence, by the above,

$$r \leqslant \mathrm{ht}(\mathfrak{p}/(a_1)).$$

Take a chain of prime ideals $\overline{\mathfrak{p}}_0 \subsetneqq \ldots \subsetneqq \overline{\mathfrak{p}}_n = \mathfrak{p}/(a_1)$ in $\mathfrak{p}/(a_1)$. The corresponding prime ideals $\mathfrak{p}_i$ in $\mathfrak{p}$ also form a chain of prime ideals, and satisfy $(a_1) \subset \mathfrak{p}_0$. But a_1 is not a zero-divisor, so by Krull's Principal Ideal Theorem 2.7, it follows that

$$\mathrm{ht}(\mathfrak{p}_0) \geqslant 1.$$

So there exists a prime ideal $\mathfrak{p}' \subsetneqq \mathfrak{p}_0$ and we can extend the induced chain of prime ideals $\mathfrak{p}_0 \subsetneqq \ldots \subsetneqq \mathfrak{p}_n$ by the prime ideal $\mathfrak{p}'$. Hence

$$r + 1 \leqslant \mathrm{ht}(\mathfrak{p}).$$

As $\mathfrak{p}$ above $\mathfrak{a}$ was arbitrary, we obtain

$$r + 1 \leqslant \mathrm{ht}(\mathfrak{a})$$

by taking the infimum over all such prime ideals. Finally, by taking the supremum over all maximal regular sequences if $\mathfrak{a} = R$ or by applying Corollary 4.8 if $\mathfrak{a} \subsetneqq R$, we can deduce the claim of the proposition. $\square$

Corollary 5.2. *For any proper ideal* $\mathfrak{a}$ *of* R *we have* $\mathrm{depth}(\mathfrak{a}) < \infty$.

Proof. In a noetherian ring, any ideal is generated by finitely many elements. Using Krull's Dimension Theorem 2.8, for any ideal $\mathfrak{a}$ we have $\mathrm{ht}(\mathfrak{a}) < \infty$. By Proposition 5.1, we can deduce $\mathrm{depth}(\mathfrak{a}) \leqslant \mathrm{ht}(\mathfrak{a}) < \infty$. $\square$

As we are usually interested in localising at a prime, we apply Proposition 3.9 accordingly.

Lemma 5.3. *Let* S *be a multiplicative set of* R *with* $0 \notin S$, *and consider an ideal* $\mathfrak{a} \subset R \smallsetminus S$. *Then* $\mathrm{depth}_R(\mathfrak{a}) \leqslant \mathrm{depth}_{S^{-1}R}(\mathfrak{a}_{S^{-1}R})$. *As a special case we get* $\mathrm{depth}_R(\mathfrak{a}) \leqslant \mathrm{depth}_{R_\mathfrak{p}}(\mathfrak{a}_\mathfrak{p})$ *for a prime ideal* $\mathfrak{p}$ *with* $\mathfrak{a} \subset \mathfrak{p}$.

Proof. Let $a_1, \dots, a_r \in \mathfrak{a}$ be a regular sequence. The assumptions of Proposition 3.9 are fulfilled, and hence $\frac{a_1}{1}, \dots, \frac{a_r}{1} \in \mathfrak{a}_{S^{-1}R}$ is a regular sequence. This means, that for any regular sequence of $\mathfrak{a}$ we can find a regular sequence in $\mathfrak{a}_{S^{-1}R}$ which is at least as long as the one in $\mathfrak{a}$. So by taking supremums, the inequality follows.

For the second claim notice that $R \setminus \mathfrak{p}$ is a multiplicative set which does not contain 0. $\square$

One might think that the following proposition is just a corollary to Lemma 5.3, however not every maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$ does the trick.

Proposition 5.4. *For any ideal $\mathfrak{a} \subsetneq R$, there exists a maximal ideal $\mathfrak{m}$ such that* $\operatorname{depth}_R(\mathfrak{a}) = \operatorname{depth}_{R_\mathfrak{m}}(\mathfrak{a}_\mathfrak{m})$.

Proof. Consider a maximal regular sequence $a_1, \dots, a_r \in \mathfrak{a}$. By Lemma 4.6, we get

$$\mathfrak{a}/(a_1, \dots, a_r) \subset \mathcal{Z}(R/(a_1, \dots, a_r)).$$

So this implies by Lemma 4.3 that there exists an element $\overline{x} \in (R/(a_1, \dots, a_r)) \setminus \{0\}$ such that

$$\overline{x}\mathfrak{a}/(a_1, \dots, a_r) = 0.$$

This, in turn, gives an element $x \in R \setminus (a_1, \dots, a_r)$ such that

$$x\mathfrak{a} \subset (a_1, \dots, a_r).$$

Now set $\mathfrak{b} := ((a_1, \dots, a_r) : (x))$. By the above, we know that $\mathfrak{a}$ is contained in $\mathfrak{b}$. Choose a maximal ideal $\mathfrak{m} \subset R$ such that

$$\mathfrak{a} \subset \mathfrak{b} \subset \mathfrak{m}.$$

Notice that by Proposition 3.9 the chosen maximal regular sequence gives us a regular sequence $\frac{a_1}{1}, \dots, \frac{a_r}{1} \in \mathfrak{a}_\mathfrak{m}$. We want to show that $\mathfrak{a}_\mathfrak{m}/(\frac{a_1}{1}, \dots, \frac{a_r}{1}) \subset \mathcal{Z}(R_\mathfrak{m}/(\frac{a_1}{1}, \dots, \frac{a_r}{1}))$. Since for all elements $a \in \mathfrak{a}$ we have $xa \in (a_1, \dots, a_r)$ we see that

$$\forall\, a' \in \mathfrak{a}_\mathfrak{m} : \frac{x}{1}a' \in (\frac{a_1}{1}, \dots, \frac{a_r}{1}).$$

Suppose by contradiction that $\frac{x}{1} \in (\frac{a_1}{1}, \dots, \frac{a_r}{1})$, meaning

$$\frac{x}{1} = \sum_{i=1}^{r} \frac{r_i}{s_i}\frac{a_i}{1}$$

for some suitable $r_i \in R$ and $s_i \in R \setminus \mathfrak{m}$. This means that there exists some $t \in R \setminus \mathfrak{m}$ such that

$$xt\left(\prod_{i=1}^{r} s_i\right) = t\left(\sum_{i=1}^{r} r_i \prod_{i \neq j} s_j a_i\right).$$

From this, we can see that $xt\left(\prod_{i=1}^{r} s_i\right) \in (a_1, \ldots, a_r)$, meaning $t\left(\prod_{i=1}^{r} s_i\right) \in \mathfrak{b}$. But this contradicts $t\left(\prod_{i=1}^{r} s_i\right) \notin \mathfrak{m}$.

So we obtain that $\frac{x}{1}$ is non-zero in $R_{\mathfrak{m}}/(\frac{a_1}{1}, \ldots, \frac{a_r}{1})$ and that $\frac{x}{1}\mathfrak{a}_{\mathfrak{m}} = 0$. Hence

$$\mathfrak{a}_{\mathfrak{m}} \subset \mathcal{Z}(R_{\mathfrak{m}}/(\frac{a_1}{1}, \ldots, \frac{a_r}{1}))$$

which, using Lemma 4.6, implies that for any maximal sequence in $\mathfrak{a}$ we get an equally long maximal sequence in $\mathfrak{a}_{\mathfrak{m}}$. In conclusion we obtain

$$\operatorname{depth}_R(\mathfrak{a}) = \operatorname{depth}_{R_{\mathfrak{m}}}(\mathfrak{a}_{\mathfrak{m}}).$$

□

We turn our attention to the Jacobson radical.

Definition 5.5. *The intersection of all maximal ideals of a ring R is called the* Jacobson radical *and is denoted by $j(R)$.*

The following two results are rather technical, but Proposition 5.7 will be crucial in two statements later on. The reason is that if the ring is local, the Jacobson radical is equal to the only maximal ideal of the ring. We will use this when localising at a prime ideal.

Lemma 5.6. *Let $x \in j(R) \setminus \mathcal{Z}(R)$. If $\mathfrak{a}$ is an ideal contained in $\mathcal{Z}(R)$, then*

$$(x, \mathfrak{a})/(x) \subset \mathcal{Z}(R/(x)).$$

Proof. Set

$$\mathfrak{b} := ((0) : \mathfrak{a}) = \{\, b \in R \,|\, b\mathfrak{a} = 0 \,\}.$$

By Lemma 4.3, $(0) \subsetneqq \mathfrak{b}$. If $\mathfrak{b} \not\subset (x)$ we find non-zero elements of $R/(x)$ which annihilate the elements of $(x, \mathfrak{a})/(x)$. Suppose now by contradiction that $\mathfrak{b} \subset (x)$, which means that

$$\forall b \in \mathfrak{b} \, \exists \, r \in R : b = rx. \tag{5.6}$$

This implies

$$rx\mathfrak{a} = b\mathfrak{a} = 0.$$

As x is not a zero-divisor, we obtain

$$r\mathfrak{a} = 0$$

which in turn implies that this r in (5.6) was already in $\mathfrak{b}$. Thus, we obtain

$$\mathfrak{b} = x\mathfrak{b}$$

and by Nakayama's lemma we conclude $\mathfrak{b} = 0$, a contradiction. □

Proposition 5.7. *Let $\mathfrak{a} \subset R$ be an ideal, $x \in R$ and $\mathfrak{b} := (x, \mathfrak{a})$. If $\mathfrak{b} \subset j(R)$, then*

$$\operatorname{depth}(\mathfrak{b}) \leqslant \operatorname{depth}(\mathfrak{a}) + 1.$$

Proof. Consider any maximal regular sequence $a_1, \ldots, a_r$ in $\mathfrak{a}$. By Corollary 4.8, we have $\operatorname{depth}(\mathfrak{a}) = r$. We know that $\mathfrak{a} \subset \mathfrak{b} \subset j(R) \subsetneq R$. Since $a_1, \ldots, a_r$ is a maximal regular sequence in $\mathfrak{a}$, the image of any element of $\mathfrak{a}$ will be a zero-divisor in $\overline{R} := R/(a_1, \ldots, a_r)$ by Lemma 4.6. Therefore, the only contenders for the non-zero-divisors in $\overline{\mathfrak{b}} := \mathfrak{b}/(a_1, \ldots, a_r)$ are images of elements of the form $yx + a$ for $y \in R \setminus \{0\}$ and $a \in \mathfrak{a}$.

We distinguish between the cases when $\overline{\mathfrak{b}} \subset \mathcal{Z}(\overline{R})$ and when $\overline{\mathfrak{b}} \not\subset \mathcal{Z}(\overline{R})$

(i) If $\overline{\mathfrak{b}} \subset \mathcal{Z}(\overline{R})$, it follows directly that $a_1, \ldots, a_r$ is also a maximal regular sequence in $\mathfrak{b}$, hence by Corollary 4.8,
$$\operatorname{depth}(\mathfrak{b}) = \operatorname{depth}(\mathfrak{a})$$
and the proposition is shown.

(ii) If $\overline{\mathfrak{b}} \not\subset \mathcal{Z}(\overline{R})$, then there exist $y \in R \setminus \{0\}$ and $a \in \mathfrak{a}$ such that $\overline{y}\,\overline{x} + \overline{a} \notin \mathcal{Z}(\overline{R})$. This means that $\overline{y}\,\overline{x} + \overline{a} \in j(\overline{R}) \setminus \mathcal{Z}(\overline{R})$. Therefore we can apply Lemma 5.6 to obtain $(\overline{y}\,\overline{x} + \overline{a}, \overline{\mathfrak{b}})/(\overline{y}\,\overline{x} + \overline{a}) \subset \mathcal{Z}(\overline{R}/(\overline{x}\,\overline{y} + \overline{a}))$. From this we can deduce that $\overline{y}\,\overline{x} + \overline{a}$ is a maximal regular sequence in $\overline{\mathfrak{b}}$. Hence, $a_1, \ldots, a_r, yx + a \in \mathfrak{b}$ is a maximal regular sequence, and therefore
$$\operatorname{depth}(\mathfrak{b}) = r + 1 = \operatorname{depth}(\mathfrak{a}) + 1.$$

So in conclusion
$$\operatorname{depth}(\mathfrak{b}) \leqslant \operatorname{depth}(\mathfrak{a}) + 1.$$

□

Remark 5.8. *The above theorem shows that if R is a local noetherian ring, then for any ideal $\mathfrak{a}$ and for any $a \in R$ such that $\mathfrak{a} + (a) \neq (1)$ we have*

$$\operatorname{depth}(\mathfrak{a} + (a)) \leqslant \operatorname{depth}(\mathfrak{a}) + 1.$$

This conclusion can be reached by recalling as mentioned before that in a local ring, $j(R) = \mathfrak{m}$ for the only maximal ideal $\mathfrak{m}$. If $\mathfrak{a} + (a) \neq (1)$, then $\mathfrak{a} + (a) \subset \mathfrak{m} = j(R)$, and therefore the conclusion holds.

6 Cohen-Macaulay rings

In this section we bring the concept of depth and height together, namely in the following definition:

Definition 6.1. *A ring R is called* Cohen-Macaulay *if it is noetherian and if for all maximal ideals $\mathfrak{m} \subset R$ the equality* $\operatorname{depth}(\mathfrak{m}) = \operatorname{ht}(\mathfrak{m})$ *holds.*

Which of our frequently-used rings are Cohen-Macaulay?

Example 6.2. Let K be a field. The ideal (0) is the unique prime — and hence maximal — ideal of K, which implies $\dim(K) = \operatorname{ht}((0)) = 0$. Any element of K is either zero or a unit, and can thus not belong to a regular sequence. Therefore, as any regular sequence in K must have length 0, it follows that K is a Cohen-Macaulay ring.

Example 6.3. Consider the ring $\mathbb{Z}$. All maximal ideals of $\mathbb{Z}$ are of the form (p) for a prime number $p \in \mathbb{Z}$. They do not contain any further prime ideals apart from (0), and hence we have

$$\operatorname{ht}((p)) = 1.$$

To calculate the depth of a prime ideal (p) we observe that any regular sequence in (p) can have length at most one by Proposition 5.1. Also, this length is attained by any non-zero element of (p). This implies that $\mathbb{Z}$ is Cohen-Macaulay.

Example 6.4. A Dedekind ring is a noetherian integral domain of Krull dimension 1. Consider any maximal ideal $\mathfrak{m}$. Since the Krull dimension is 1, we find that $\operatorname{ht}(\mathfrak{m}) = 1$. By Proposition 5.1, we always have $\operatorname{depth}(\mathfrak{m}) \leqslant \operatorname{ht}(\mathfrak{m})$, so we want to show that

$$\operatorname{depth}(\mathfrak{m}) = 1$$

is attained. For this, it suffices that any maximal ideal $\mathfrak{m}$ contains a non-zero-divisor. But this is surely satisfied, as R is an integral domain and not a field. Hence, any Dedekind ring is a Cohen-Macaulay ring.

Example 6.5. Consider a regular local ring R and its unique maximal ideal $\mathfrak{m}$. Since the ring is local, we have $\operatorname{ht}(\mathfrak{m}) = \dim(R)$. Any minimal set of generators of $\mathfrak{m}$ form a regular sequence in $\mathfrak{m}$ because of the minimality. As the ring is regular, we obtain $\dim(R) \leqslant \operatorname{depth}(\mathfrak{m})$. By Proposition 5.1, we also have the inequality $\operatorname{depth}(\mathfrak{m}) \leqslant \operatorname{ht}(\mathfrak{m})$, which implies equality, and therefore that R is Cohen-Macaulay.

In this section, we would like to simplify our definition of a Cohen-Macaulay ring to extend the equality to any proper ideal instead of only maximal ideals. Afterwards, we are interested in what transformations of a Cohen-Macaulay ring results again in a Cohen-Macaulay ring, namely passing to a localisation or a factor ring, or considering a polynomial ring over it.

Before we can show the equality of depth and height for any proper ideal, we need to prove the following lemma.

Lemma 6.6. *Let R be a Cohen-Macaulay ring and consider a maximal ideal $\mathfrak{m} \subset R$. Then $R_{\mathfrak{m}}$ is also Cohen-Macaulay.*

Proof. The localisation $R_\mathfrak{m}$ is noetherian. The only maximal ideal of $R_\mathfrak{m}$ is $\mathfrak{m}_\mathfrak{m}$ and therefore we want to show

$$\mathrm{ht}_{R_\mathfrak{m}}(\mathfrak{m}_\mathfrak{m}) = \mathrm{depth}_{R_\mathfrak{m}}(\mathfrak{m}_\mathfrak{m}).$$

We always have

$$\mathrm{ht}_{R_\mathfrak{m}}(\mathfrak{m}_\mathfrak{m}) = \mathrm{ht}_R(\mathfrak{m})$$

and by Proposition 5.4 we find that

$$\mathrm{depth}_{R_\mathfrak{m}}(\mathfrak{m}_\mathfrak{m}) = \mathrm{depth}_R(\mathfrak{m}).$$

As R is a Cohen-Macaulay ring, we have $\mathrm{ht}_R(\mathfrak{m}) = \mathrm{depth}_R(\mathfrak{m})$ and the conclusion follows. $\square$

Counterexample 6.7. Consider $R := K[X^4, X^3Y, XY^3, Y^4]$ for some field K. It has the maximal ideal $\mathfrak{m} = (X^4, X^3Y, XY^3, Y^4)$, and we want to show that $R_\mathfrak{m}$ is not Cohen-Macaulay. For this, observe that X^4, Y^4 is a system of parameters, and hence $\mathrm{ht}(\mathfrak{m}_\mathfrak{m}) = \mathrm{ht}(\mathfrak{m}) \geqslant \mathrm{ht}((X^4, Y^4)) \geqslant 2$. We claim that X^4 is a maximal regular sequence in $\mathfrak{m}$.

The images of the generators X^3Y, XY^3, Y^4 of $\mathfrak{m}$ are each zero-divisors in $R/(X^4)$. To see this, multiply each by $(X^3Y)^2$, which is not zero in $R/(X^4)$, since $(X^3Y)^2 = X^4 \cdot X^2Y^2$, but $X^2Y^2 \notin R$. Hence, by Lemma 4.6, X^4 is a maximal regular sequence in $\mathfrak{m}$. Therefore, $1 = \mathrm{depth}(\mathfrak{m}) = \mathrm{depth}(\mathfrak{m}_\mathfrak{m}) < \mathrm{ht}(\mathfrak{m}_\mathfrak{m})$. So, $R_\mathfrak{m}$, and hence R by the contraposition of Lemma 6.6, are not Cohen-Macaulay.

Proposition 6.8. *In a Cohen-Macaulay ring R we have for all $\mathfrak{a} \subsetneq R$:*

$$\mathrm{depth}(\mathfrak{a}) = \mathrm{ht}(\mathfrak{a}).$$

Proof. In any noetherian ring, we have by Proposition 5.1

$$\mathrm{depth}(\mathfrak{a}) \leqslant \mathrm{ht}(\mathfrak{a})$$

for any ideal $\mathfrak{a} \subset R$. So we aim to show the reverse inequality.

By Proposition 4.9, we can choose a prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$ such that

$$\mathrm{depth}(\mathfrak{a}) = \mathrm{depth}(\mathfrak{p}).$$

If we can show $\mathrm{depth}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p})$, then the inequality follows for $\mathfrak{a}$, since $\mathrm{ht}(\mathfrak{p}) \geqslant \mathrm{ht}(\mathfrak{a})$. So without loss of generality, replace $\mathfrak{a}$ by a prime ideal $\mathfrak{p}$.

Furthermore, we can restrict ourselves to local Cohen-Macaulay rings. To see why this suffices, consider a maximal ideal $\mathfrak{m}$ as in Proposition 5.4. By Lemma 6.6, the ring $R_\mathfrak{m}$ is Cohen-Macaulay. If we know that the equality holds for local Cohen-Macaulay rings, then

$$\mathrm{ht}_R(\mathfrak{p}) = \mathrm{ht}_{R_\mathfrak{m}}(\mathfrak{p}_\mathfrak{m}) = \mathrm{depth}_{R_\mathfrak{m}}(\mathfrak{p}_\mathfrak{m}) = \mathrm{depth}_R(\mathfrak{p}).$$

This shows that we can assume without loss of generality that R is a local Cohen-Macaulay ring with maximal ideal $\mathfrak{m}$.

Suppose that there exists a prime ideal $\mathfrak{p}$ with

$$\operatorname{depth}(\mathfrak{p}) < \operatorname{ht}(\mathfrak{p}).$$

Choose this $\mathfrak{p}$ maximal among all such prime ideals that satisfy the strict inequality above. Since R is Cohen-Macaulay, $\operatorname{depth}(\mathfrak{m}) = \operatorname{ht}(\mathfrak{m})$, and hence $\mathfrak{p} \subsetneqq \mathfrak{m}$.

As $\mathfrak{p} \subsetneqq \mathfrak{m}$, there exists $x \in \mathfrak{m} \setminus \mathfrak{p}$ such that $(x, \mathfrak{p}) \subset \mathfrak{m} = j(R)$. By Proposition 4.9, there exists a prime ideal $\mathfrak{p}'$ containing $(x, \mathfrak{p})$ such that $\operatorname{depth}(\mathfrak{p}') = \operatorname{depth}((x, \mathfrak{p}))$. After applying Proposition 5.7 we obtain

$$\operatorname{depth}(\mathfrak{p}') = \operatorname{depth}((x, \mathfrak{p})) \leqslant \operatorname{depth}(\mathfrak{p}) + 1.$$

As $\mathfrak{p} \subsetneqq \mathfrak{p}'$, we have $\operatorname{ht}(\mathfrak{p}) + 1 \leqslant \operatorname{ht}(\mathfrak{p}')$. This implies:

$$\operatorname{depth}(\mathfrak{p}') \leqslant \operatorname{depth}(\mathfrak{p}) + 1 < \operatorname{ht}(\mathfrak{p}) + 1 \leqslant \operatorname{ht}(\mathfrak{p}')$$

which contradicts the maximality of $\mathfrak{p}$. $\square$

In Lemma 6.6, we saw how a Cohen-Macaulay ring behaves when localising with respect to a maximal ideal. This can now be extended to any localisation.

Proposition 6.9. *Cohen-Macaulayness is preserved by localisation.*

Proof. Let R be Cohen-Macaulay and let $S \subset R$ be a multiplicative subset. The localisation $S^{-1}R$ is noetherian. Consider a maximal ideal $\mathfrak{m} \subset S^{-1}R$. By Proposition 5.1, we always have

$$\operatorname{depth}_{S^{-1}R}(\mathfrak{m}) \leqslant \operatorname{ht}_{S^{-1}R}(\mathfrak{m}).$$

Any prime ideal of a localisation corresponds to a prime ideal within $R \setminus S$, so call $\widehat{\mathfrak{m}} \subset R$ the prime ideal corresponding to $\mathfrak{m}$. By assumption, R is Cohen-Macaulay, so by Proposition 6.8,

$$\operatorname{depth}_R(\widehat{\mathfrak{m}}) = \operatorname{ht}_R(\widehat{\mathfrak{m}}).$$

We also know from [1, 31, Prop.1] that the height is preserved by localisation, which means

$$\operatorname{depth}_R(\widehat{\mathfrak{m}}) = \operatorname{ht}_R(\widehat{\mathfrak{m}}) = \operatorname{ht}_{S^{-1}R}(\mathfrak{m}).$$

By Lemma 5.3, we obtain

$$\operatorname{depth}_{S^{-1}R}(\mathfrak{m}) \geqslant \operatorname{depth}_R(\widehat{\mathfrak{m}})$$

which shows the desired inequality

$$\operatorname{depth}_{S^{-1}R}(\mathfrak{m}) \geqslant \operatorname{ht}_{S^{-1}R}(\mathfrak{m}).$$

Hence $S^{-1}R$ is also Cohen-Macaulay. $\square$

After having examined how Cohen-Macaulay rings behave under localisation, it is natural to wonder if some sort of reverse also holds. As is possible with other properties such as the ring being normal, Cohen-Macaulayness is a local property.

Proposition 6.10. *Suppose that for all $\mathfrak{m}$ maximal $R_\mathfrak{m}$ is Cohen-Macaulay. Then R is also Cohen-Macaulay.*

Proof. Let $\mathfrak{m} \subset R$ be a maximal ideal. By localising at $\mathfrak{m}$ and using Proposition 5.4 we see that
$$\operatorname{depth}_R(\mathfrak{m}) = \operatorname{depth}_{R_\mathfrak{m}}(\mathfrak{m}_\mathfrak{m}).$$
Since $R_\mathfrak{m}$ is a Cohen-Macaulay ring by assumption, we get
$$\operatorname{depth}_{R_\mathfrak{m}}(\mathfrak{m}_\mathfrak{m}) = \operatorname{ht}_{R_\mathfrak{m}}(\mathfrak{m}_\mathfrak{m}).$$
The height of $\mathfrak{m}$ is preserved by localisation with respect to $\mathfrak{m}$. By combining these facts, we infer
$$\operatorname{depth}_R(\mathfrak{m}) = \operatorname{ht}_R(\mathfrak{m})$$
which shows that R is Cohen-Macaulay, as $\mathfrak{m}$ was arbitrary. $\square$

An arbitrary factor ring of a Cohen-Macaulay ring is not necessarily Cohen-Macaulay again. The following lemma gives us conditions under which we can factor to obtain again a Cohen-Macaulay ring. Though its scope might seem limited, its use will be apparent once we prove Theorem 6.12.

Lemma 6.11. *If x is a non-zero-divisor of a Cohen-Macaulay ring R, then $R/(x)$ is also Cohen-Macaulay.*

Proof. A factor ring of a noetherian ring is again noetherian.

Any maximal ideal in $R/(x)$ is of the form $\mathfrak{m}/(x)$ for a maximal ideal $\mathfrak{m}$ in R which contains x. As R is Cohen-Macaulay, we get $\operatorname{depth}(\mathfrak{m}) = \operatorname{ht}(\mathfrak{m})$. Referring back to the proof of Proposition 5.1 and since x was assumed to be a non-zero-divisor, we can argue similarly to obtain
$$\operatorname{ht}(\mathfrak{m}/(x)) + 1 \leqslant \operatorname{ht}(\mathfrak{m}).$$

Let $a_1 + (x), \ldots, a_r + (x) \in \mathfrak{m}/(x)$ be a maximal regular sequence in $\mathfrak{m}/(x)$, which implies that $\mathfrak{m}/(x, a_1, \ldots, a_r) \subset \mathcal{Z}(R/(x, a_1, \ldots, a_r))$. Hence, $x, a_1, \ldots, a_r$ is a maximal regular sequence in $\mathfrak{m}$. By this argument and by Corollary 4.8, we obtain
$$\operatorname{depth}(\mathfrak{m}) = \operatorname{depth}(\mathfrak{m}/(x)) + 1.$$
Now we concatenate all findings to obtain:
$$\operatorname{ht}(\mathfrak{m}/(x)) + 1 \leqslant \operatorname{ht}(\mathfrak{m}) = \operatorname{depth}(\mathfrak{m}) = \operatorname{depth}(\mathfrak{m}/(x)) + 1$$
which implies
$$\operatorname{ht}(\mathfrak{m}/(x)) \leqslant \operatorname{depth}(\mathfrak{m}/(x)).$$
By Proposition 5.1 we get equality, and hence $R/(x)$ is Cohen-Macaulay. $\square$

Theorem 6.12. *A ring R is Cohen-Macaulay if and only if the polynomial ring $R[X]$ is Cohen-Macaulay.*

Proof. By Hilbert's Basis Theorem 2.6, R is noetherian if and only if $R[X]$ is noetherian. Let $R[X]$ be Cohen-Macaulay. The element X is a non-zero-divisor of $R[X]$, and hence by Lemma 6.11, we see that $R[X]/(X) \cong R$ is also Cohen-Macaulay.

Now suppose that R is Cohen-Macaulay. Consider a maximal ideal $\mathfrak{m} \subset R[X]$. We would like to show that $R[X]_{\mathfrak{m}}$ is Cohen-Macaulay and then apply Proposition 6.10. Since we know that $R[X]_{\mathfrak{m}}$ is noetherian, we aim to show the equality of depth and height on maximal ideal of $R[X]_{\mathfrak{m}}$.

Set $\widehat{\mathfrak{m}} := \mathfrak{m} \cap R$. Then $\widehat{\mathfrak{m}}$ is a prime ideal of R. We would first like to show that we can reduce R to be a local ring with maximal ideal $\widehat{\mathfrak{m}}$.

Since $R \smallsetminus \widehat{\mathfrak{m}} \subset R[X] \smallsetminus \mathfrak{m}$ we obtain by using the universal property of $R[X]_{\mathfrak{m}}$

$$R[X]_{\mathfrak{m}} \cong (R_{\widehat{\mathfrak{m}}}[X])_{\mathfrak{m}}. \tag{6.12}$$

Using Lemma 6.6, without loss of generality, we can suppose that R is a local Cohen-Macaulay ring with maximal ideal $\widehat{\mathfrak{m}}$.

Now, we have

$$R[X]/\widehat{\mathfrak{m}}R[X] \cong (R/\widehat{\mathfrak{m}})[X].$$

As $R/\widehat{\mathfrak{m}}$ is a field, $(R/\widehat{\mathfrak{m}})[X]$ is a principal ideal domain. By the isomorphism above, $R[X]/\widehat{\mathfrak{m}}R[X]$ is also a principal ideal domain. Consider $\mathfrak{m}/(\widehat{\mathfrak{m}}R[X])$, which must be a principal ideal, meaning

$$\mathfrak{m}/(\widehat{\mathfrak{m}}R[X]) = (f + \widehat{\mathfrak{m}}R[X])$$

for a monic polynomial f in $R[X]$. From this, we deduce that $\mathfrak{m} = (\widehat{\mathfrak{m}}, f)$.

Take a maximal regular sequence $a_1, \dots, a_r$ in $\widehat{\mathfrak{m}}$. The elements $a_1, \dots, a_r$ also lie in $\mathfrak{m}$, and since $R[X]/(a_1, \dots, a_r)R[X] \cong (R/(a_1, \dots, a_r))[X]$, they form a regular sequence in $\mathfrak{m}$. Since f is monic, it is not a zero-divisor of $R[X]/(a_1, \dots, a_r)R[X]$. Therefore, we can extend the regular sequence $a_1, \dots, a_r$ in $\mathfrak{m}$ to the regular sequence

$$a_1, \dots, a_r, f \in \mathfrak{m}.$$

By taking suprema, we obtain

$$\operatorname{depth}(\widehat{\mathfrak{m}}) + 1 \leqslant \operatorname{depth}(\mathfrak{m}).$$

We see using Krull's Principal Ideal Theorem 2.7, since f is neither a zero-divisor nor a unit,

$$\operatorname{ht}(\mathfrak{m}/(\widehat{\mathfrak{m}}R[X])) = \operatorname{ht}((f + \widehat{\mathfrak{m}}R[X])) = 1.$$

If we have a chain of prime ideals $\mathfrak{p}_0 \subsetneqq \dots \subsetneqq \mathfrak{p}_n \subset \widehat{\mathfrak{m}}R[X]$, we obtain the chain of prime ideals $\mathfrak{p}_0 \cap R \subsetneqq \dots \subsetneqq \mathfrak{p}_n \cap R \subset \widehat{\mathfrak{m}}$, which implies

$$\operatorname{ht}(\widehat{\mathfrak{m}}R[X]) \leqslant \operatorname{ht}(\widehat{\mathfrak{m}}).$$

For arbitrary ideals $\mathfrak{b} \subset \mathfrak{a}$ we have

$$\operatorname{ht}(\mathfrak{a}) \leqslant \operatorname{ht}(\mathfrak{b}) + \operatorname{ht}(\mathfrak{a}/\mathfrak{b}).$$

Hence,

$$\operatorname{ht}(\mathfrak{m}) \leqslant \operatorname{ht}(\widehat{\mathfrak{m}}R[X]) + \operatorname{ht}(\mathfrak{m}/(\widehat{\mathfrak{m}}R[X])) = \operatorname{ht}(\widehat{\mathfrak{m}}R[X]) + 1 \leqslant \operatorname{ht}(\widehat{\mathfrak{m}}) + 1.$$

Since R is Cohen-Macaulay by Lemma 6.6, we get

$$\operatorname{ht}(\mathfrak{m}) \leqslant \operatorname{ht}(\widehat{\mathfrak{m}}) + 1 = \operatorname{depth}(\widehat{\mathfrak{m}}) + 1 \leqslant \operatorname{depth}(\mathfrak{m})$$

Because of Proposition 5.1 we always have the reverse inequality. So $R[X]$ is Cohen-Macaulay, where R is the local ring with maximal ideal $\widehat{\mathfrak{m}}$. If we pass now to the localisation with respect to $\mathfrak{m}$, the height and depth will stay the same by Proposition 5.4. This implies that $R[X]_{\mathfrak{m}}$ is Cohen-Macaulay as well.

Then, since the maximal ideal $\mathfrak{m}$ was arbitrary, we use the isomorphism in (6.12) and apply Proposition 6.10 to find that our original polynomial ring $R[X]$ is Cohen-Macaulay. □

Remark 6.13. *By induction, we can extend the above to the following: A ring R is Cohen-Macaulay if and only if for any $n \geqslant 0$, the polynomial ring $R[X_1, \ldots, X_n]$ is Cohen-Macaulay.*

For a less obvious Cohen-Macaulay ring, one can have a look at the paper by R. Pink and S. Schieder [5], where one of the algebras they construct is Cohen-Macaulay.

7 (Universally) Catenary rings

Finally we are able to introduce the concepts of a catenary ring and universally catenary, which are what we need in order to understand the first main theorem stated at the beginning of the thesis. It is then our aim to connect Cohen-Macaulay rings from the previous chapter with (universally) catenary rings, namely by showing that every Cohen-Macaulay ring is already catenary.

Definition 7.1. *A ring R is called* catenary *if for any two prime ideals $\mathfrak{p} \subset \mathfrak{p}' \subset R$ any two maximal chains of prime ideals between $\mathfrak{p}$ and $\mathfrak{p}'$ have the same length.*

As the definition is stated in terms of prime ideals, catenary rings are bound to behave well under localisation or passing to the factor ring.

Proposition 7.2. *Every quotient ring $R/\mathfrak{a}$ of a catenary ring R is catenary.*

Proof. The prime ideals of $R/\mathfrak{a}$ correspond to prime ideals of R which contain $\mathfrak{a}$, and the inclusions are preserved. Hence $R/\mathfrak{a}$ is catenary. □

Proposition 7.3. *Every localisation of a catenary ring R is catenary.*

Proof. The prime ideals of $S^{-1}R$ for some multiplicative set $S \subset R$ are in a one-to-one inclusion-preserving correspondence to the prime ideals $\mathfrak{p}$ of R which satisfy $\mathfrak{p} \cap S = \varnothing$. Hence the localisation of a catenary ring is catenary. □

The definition of a catenary ring can be extended further:

Definition 7.4. *A ring for which all finitely generated algebras over it are catenary rings is called* universally catenary*.*

In order to prove that Cohen-Macaulay rings are also catenary — thus linking the two concepts together — we first prove the claim for the particular case of local Cohen-Macaulay rings in the following proposition. As often in Commutative Algebra, one can show a claim for a local ring and then are able to use localisation to arrive at the main result. We follow the same trail of thoughts here.

Proposition 7.5. *In any local Cohen-Macaulay ring, any two maximal chains of prime ideals have the same length.*

Proof. We claim that the maximal chains are precisely those of length $\dim(R)$. To this end consider a maximal chain of prime ideals $\mathfrak{p}_0 \subsetneqq \ldots \subsetneqq \mathfrak{p}_n$, where $\mathfrak{p}_n$ is the only maximal ideal of R. Since $\mathfrak{p}_n$ is the only maximal ideal, we see that $\mathrm{ht}(\mathfrak{p}_n) = \dim(R)$ and therefore we want to show that $n = \mathrm{ht}(\mathfrak{p}_n)$. The height satisfies $n \leqslant \mathrm{ht}(\mathfrak{p}_n)$, which means we aim to show the reverse inequality.

For this, we turn to the depth of $\mathfrak{p}_n$. We do induction on $0 \leqslant i \leqslant n$ to find $\mathrm{depth}(\mathfrak{p}_i) \leqslant i$. For $i = 0$, we have

$$\mathrm{depth}(\mathfrak{p}_0) = \mathrm{ht}(\mathfrak{p}_0) = 0.$$

The first equality is due to the ring being Cohen-Macaulay and Proposition 6.8, and the second equality is because the chain is maximal. Our inductive hypothesis is that $\mathrm{depth}(\mathfrak{p}_i) \leqslant i$.

In the inductive step, one localises at $\mathfrak{p}_i$, which does no harm since the prime ideals below $\mathfrak{p}_i$ will be in a one-to-one order-preserving correspondence with the prime ideals in the localisation. Let $a \in \mathfrak{p}_{i,\mathfrak{p}_i} \smallsetminus \mathfrak{p}_{i-1,\mathfrak{p}_i}$ and consider $\mathfrak{p}_{i-1,\mathfrak{p}_i} + (a) \subset \mathfrak{p}_{i,\mathfrak{p}_i}$. The ideal $\mathfrak{p}_{i,\mathfrak{p}_i}$ is the only maximal ideal of the localisation and there is no prime ideal properly between $\mathfrak{p}_{i,\mathfrak{p}_i}$ and $\mathfrak{p}_{i-1,\mathfrak{p}_i}$. Hence, $\mathfrak{p}_{i,\mathfrak{p}_i}$ is also the only prime ideal above $\mathfrak{p}_{i-1,\mathfrak{p}_i} + (a)$. Hence, we see by Proposition 4.9 that

$$\mathrm{depth}_{R_{\mathfrak{p}_i}}(\mathfrak{p}_{i,\mathfrak{p}_i}) = \mathrm{depth}_{R_{\mathfrak{p}_i}}(\mathfrak{p}_{i-1,\mathfrak{p}_i} + (a)).$$

Since $\mathfrak{p}_{i,\mathfrak{p}_i} = j(R_{\mathfrak{p}_i})$, applying Proposition 5.7 yields

$$\mathrm{depth}_{R_{\mathfrak{p}_i}}(\mathfrak{p}_{i-1,\mathfrak{p}_i} + (a)) \leqslant \mathrm{depth}_{R_{\mathfrak{p}_i}}(\mathfrak{p}_{i-1,\mathfrak{p}_i}) + 1.$$

Because $\mathfrak{p}_i$ is the only maximal ideal above $\mathfrak{p}_{i-1}$, we find thanks to Proposition 5.4 that

$$\mathrm{depth}_{R_{\mathfrak{p}_i}}(\mathfrak{p}_{i-1,\mathfrak{p}_i}) = \mathrm{depth}_R(\mathfrak{p}_{i-1})$$

We see that

$$\operatorname{depth}_R(\mathfrak{p}_i) \leqslant \operatorname{depth}_{R_{\mathfrak{p}_i}}(\mathfrak{p}_{i,\mathfrak{p}_i}) = \operatorname{depth}_{R_{\mathfrak{p}_i}}(\mathfrak{p}_{i-1,\mathfrak{p}_i} + (a)) \leqslant \operatorname{depth}_{R_{\mathfrak{p}_i}}(\mathfrak{p}_{i-1,\mathfrak{p}_i}) + 1$$
$$= \operatorname{depth}_R(\mathfrak{p}_{i-1}) + 1 \leqslant i - 1 + 1 = i,$$

where the first inequality is due to Lemma 5.3, the last inequality is due to the induction hypothesis, and all steps in-between due to the arguments developed above.

This implies the following for $\mathfrak{p}_n$:

$$\operatorname{depth}(\mathfrak{p}_n) \leqslant n.$$

which, since the ring is Cohen-Macaulay, shows

$$\operatorname{ht}(\mathfrak{p}_n) = \operatorname{depth}(\mathfrak{p}_n) \leqslant n.$$

This in turn implies that

$$\operatorname{ht}(\mathfrak{p}_n) = n.$$

□

Proposition 7.6. *Every Cohen-Macaulay ring is catenary.*

Proof. Let $\mathfrak{q} \subset \mathfrak{p}$ be two prime ideals in R. Localising at $\mathfrak{q}$ results in a local Cohen-Macaulay ring thanks to Proposition 6.9. Then, by Proposition 7.5, we can infer that two maximal chains of prime ideals below $\mathfrak{q}$ in the original ring have the same length. We now localise R at $\mathfrak{p}$. Again by Proposition 6.9 and Proposition 7.5 we can now assume without loss of generality that R is a local Cohen-Macaulay ring, where any two maximal chains of prime ideals below $\mathfrak{q}$, or below $\mathfrak{p}$, have the same length.

Consider two maximal chains of prime ideals between $\mathfrak{q}$ and $\mathfrak{p}$,

$$\mathfrak{q} = \mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_{n-1} \subsetneqq \mathfrak{p}_n = \mathfrak{p}$$

and

$$\mathfrak{q} = \mathfrak{p}'_0 \subsetneqq \mathfrak{p}'_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}'_{m-1} \subsetneqq \mathfrak{p}'_m = \mathfrak{p}.$$

Elongate both chains $\{\mathfrak{p}_i\}_{0\leqslant i\leqslant n}, \{\mathfrak{p}'_j\}_{0\leqslant j\leqslant m}$ by a maximal chain of prime ideals below $\mathfrak{q} = \mathfrak{p}_0 = \mathfrak{p}'_0$. This results in two maximal chains of prime ideals below $\mathfrak{p}$, which by Proposition 7.5 must have the same length. This implies $n = m$, which means that any two maximal chains of prime ideals between $\mathfrak{q}$ and $\mathfrak{p}$ have the same length.

Since $\mathfrak{q}$ and $\mathfrak{p}$ were arbitrary, the claim follows. □

One can already see the path leading up to the first statement of our main theorem. We know that $\mathbb{Z}$, any field K, and any Dedekind ring are Cohen-Macaulay, and hence, by what we have just proved, are catenary. What remains to be shown is that if we have a finitely generated algebra over them, the resulting ring is again catenary. More generally:

Theorem 7.7. *Any Cohen-Macaulay ring is universally catenary.*

Proof. Let R' be a finitely generated R-algebra. Since R' is finitely generated over R, we have a surjective ring homomorphism $\varphi : R[X_1, \dots, X_n] \twoheadrightarrow R'$. We find using the homomorphism theorem:

$$R' \cong R[X_1, \dots, X_n]/\ker(\varphi).$$

We recall that by Remark 6.13, any polynomial ring in finitely many variables over a Cohen-Macaulay ring is Cohen-Macaulay. By Proposition 7.6, the polynomial ring $R[X_1, \dots, X_n]$ is also catenary. Proposition 7.2 ensures that $R[X_1, \dots, X_n]/\ker(\varphi)$ is catenary, and hence, R' is catenary. □

8 Proofs of the main Theorems

All results in the previous sections now lead to the proof of the first theorem:

Theorem 8.1. *Any ring that is finitely generated over a field K, or over $\mathbb{Z}$, or over any Dedekind ring, respectively, is catenary.*

Proof. We apply Theorem 7.7 to Examples 6.2, 6.3, 6.4 and 6.5. □

For the second theorem, we will need a few preliminary results.

Lemma 8.2. *For every local catenary integral domain R, and for any of its prime ideals $\mathfrak{p}$, we have*

$$\mathrm{ht}(\mathfrak{p}) + \mathrm{coht}(\mathfrak{p}) = \dim(R).$$

Proof. Call the maximal ideal $\mathfrak{m}$. As R is an integral domain and catenary, any maximal chains of prime ideals between (0) and $\mathfrak{m}$ have the same length.

Let $\mathfrak{p}$ be an arbitrary prime ideal of R. Given any maximal chain between (0) and $\mathfrak{p}$ and any maximal chain between $\mathfrak{p}$ and $\mathfrak{m}$, we can glue them together to form a maximal chain between (0) and $\mathfrak{m}$, since R is catenary. Hence

$$\mathrm{ht}(\mathfrak{p}) + \mathrm{coht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{m}) = \dim(R),$$

because R is local. □

Proposition 8.3. *For any catenary integral domain R, for which all its maximal ideals $\mathfrak{m}$ satisfy*

$$\mathrm{ht}(\mathfrak{m}) = \dim(R),$$

the equality

$$\mathrm{ht}(\mathfrak{p}) + \mathrm{coht}(\mathfrak{p}) = \dim(R)$$

holds for any of its prime ideals $\mathfrak{p}$.

Proof. The assumption on the maximal ideals assures that if we pass to the localisation with respect to some maximal ideal that

$$\dim(R_{\mathfrak{m}}) = \mathrm{ht}(\mathfrak{m}) = \dim(R).$$

If we take some prime ideal $\mathfrak{p}$, we find a maximal ideal $\mathfrak{m}$ containing $\mathfrak{p}$. Upon localising with respect to $\mathfrak{m}$ we find

$$\dim(R) = \dim(R_{\mathfrak{m}}) = \mathrm{ht}(\mathfrak{p}_{\mathfrak{m}}) + \mathrm{coht}(\mathfrak{p}_{\mathfrak{m}}) \leqslant \mathrm{ht}(\mathfrak{p}) + \mathrm{coht}(\mathfrak{p}).$$

The first equality stems from the argument above and the second equality is due to Lemma 8.2. The inequality is because, while the height is preserved under this localisation, the coheight might be reduced. We always have $\dim(R) \geqslant \mathrm{ht}(\mathfrak{p}) + \mathrm{coht}(\mathfrak{p})$, so we obtain the desired equality. □

Proposition 8.4. *Any integral domain R' that is finitely generated over $R \in \{\mathbb{Z}, K\}$ satisfies*

$$\text{(8.4)} \qquad \textit{For all } \mathfrak{m} \subset R' \textit{ maximal}: \mathrm{ht}(\mathfrak{m}) = \dim(R').$$

Proof. We reduce to the case $R' = R[X_1, \ldots, X_n]$: Assume that the property (8.4) holds for $R[X_1, \ldots, X_n]$. As before, for some $n \in \mathbb{Z}^{\geqslant 0}$ and for the associated homomorphism $\varphi : R[X_1, \ldots, X_n] \twoheadrightarrow R'$, we have

$$R' \cong R[X_1, \ldots, X_n]/\ker(\varphi)$$

by the homomorphism theorem. The ideals of $R[X_1, \ldots, X_n]/\ker(\varphi)$ are in a one-to-one correspondence with the ideals of $R[X_1, \ldots, X_n]$ containing $\ker(\varphi)$. Furthermore $\ker(\varphi)$ is prime, because R' is an integral domain.

Let $\overline{\mathfrak{m}} \subset R[X_1, \ldots, X_n]/\ker(\varphi)$ be a maximal ideal and let $\mathfrak{m}$ be the corresponding maximal ideal of $R[X_1, \ldots, X_n]$. The ideal $\ker(\varphi)$ lies in $\mathfrak{m}$, and we can consider a maximal chain of prime ideals

$$(0) \subsetneqq \ldots \subsetneqq \ker(\varphi) \subsetneqq \ldots \subsetneqq \mathfrak{m}.$$

We obtain

$$\mathrm{ht}(\overline{\mathfrak{m}}) = \mathrm{ht}(\mathfrak{m}) - \mathrm{ht}(\ker(\varphi)) = \dim(R[X_1, \ldots, X_n]) - \mathrm{ht}(\ker(\varphi))$$

$$\geqslant \mathrm{coht}(\ker(\varphi)) = \dim(R[X_1, \ldots, X_n]/\ker(\varphi)).$$

The first equality is given because by Theorem 8.1, R' is catenary. The second equality is due to our assumption that the property (8.4) holds for the polynomial ring. The remaining inequality and equality hold by the definition of height, coheight and the Krull dimension. We deduce that $\mathrm{ht}(\overline{\mathfrak{m}}) = \dim(R[X_1, \ldots, X_n]/\ker(\varphi)$. So if we can show the property (8.4) for $R[X_1, \ldots, X_n]$ then (8.4) will also follow for $R[X_1, \ldots, X_n]/\ker(\varphi) \cong R'$. Hence we can just consider the polynomial ring.

We proceed by induction on n. The case $n = 0$ works, since both $\{\mathbb{Z}, K\}$ satisfy the property (8.4) by Examples 6.2 and 6.3. Our inductive hypothesis is that for some $n \geqslant 0$ the ring $R[X_1, \ldots, X_n]$ satisfies the property (8.4).

Consider $R_{n+1} := R[X_1, \ldots, X_{n+1}]$ and some maximal ideal $\mathfrak{m} \subset R_{n+1}$. It always holds that $\mathrm{ht}(\mathfrak{m}) \leqslant \dim(R[X_1, \ldots, X_{n+1}])$. Call $R_n := R[X_1, \ldots, X_n]$ and set $\widetilde{\mathfrak{m}} := \mathfrak{m} \cap R_n$. The ideal $\widetilde{\mathfrak{m}}$ is prime. If we take a chain of prime ideals

$$\mathfrak{p}_0 \subsetneqq \ldots \subsetneqq \mathfrak{p}_n = \widetilde{\mathfrak{m}}$$

in R_n, we obtain a chain of prime ideals in R_{n+1} by taking their pushforward in R_{n+1} via the inclusion

$$\mathfrak{p}_0[X_{n+1}] \subsetneqq \ldots \subsetneqq \mathfrak{p}_n[X_{n+1}] = \widetilde{\mathfrak{m}}[X_{n+1}] \subset \mathfrak{m}.$$

Since $R_{n+1}/(\widetilde{\mathfrak{m}}[X_{n+1}]) \cong (R_n/\widetilde{\mathfrak{m}})[X_{n+1}]$, the ideal $\widetilde{\mathfrak{m}}[X_{n+1}]$ cannot be maximal and hence we see that the inclusion $\widetilde{\mathfrak{m}}[X_{n+1}] \subsetneqq \mathfrak{m}$ is strict. Hence,

$$\mathrm{ht}(\widetilde{\mathfrak{m}}) + 1 \leqslant \mathrm{ht}(\mathfrak{m}).$$

We would like to show that $\widetilde{\mathfrak{m}}$ is a maximal ideal of R_n. Consider the diagram

$$\begin{tikzcd} R \arrow[d, hook] \arrow[dr, hook] & \\ R_n \arrow[r, hook] & R_{n+1}. \end{tikzcd}$$

Define $\mathfrak{n} := \mathfrak{m} \cap R = \widetilde{\mathfrak{m}} \cap R$. Then $\mathfrak{n}$ is prime in R, and we want to show that $\mathfrak{n}$ is maximal too. If $R = K$, it holds.

If $R = \mathbb{Z}$, then $\mathfrak{n} = (p)$ for a prime number $p \in \mathbb{Z}$, which is a maximal ideal. The case $p = 0$ is not possible: If $\mathfrak{m} \cap R = (0)$, consider

$$\mathbb{Z} \hookrightarrow R_{n+1} \twoheadrightarrow R_{n+1}/\mathfrak{m}.$$

The kernel of the composition is $\mathfrak{n} = (0)$. Thus, we obtain

$$\mathbb{Z} \hookrightarrow \mathbb{Q} \hookrightarrow R_{n+1}/\mathfrak{m}.$$

Since $R_{n+1}/\mathfrak{m}$ is a finite ring extension over $\mathbb{Z}$, it is also a finite ring extension over $\mathbb{Q}$. By Theorem 2.5, $R_{n+1}/\mathfrak{m}$ is a finite field extension over $\mathbb{Q}$, and hence it is a finitely generated $\mathbb{Q}$-module. Now, the theorem by Artin-Tate 2.4 implies that $\mathbb{Q}$ is a finitely generated $\mathbb{Z}$-algebra, which is a contradiction.

Consider the induced diagram of injective homomorphisms:

$$\begin{tikzcd} R/\mathfrak{n} \arrow[d, hook] \arrow[dr, hook] & \\ R_n/\widetilde{\mathfrak{m}} \arrow[r, hook] & R_{n+1}/\mathfrak{m}. \end{tikzcd}$$

Both $R/\mathfrak{n}$ and $R_{n+1}/\mathfrak{m}$ are fields, since $\mathfrak{n}$ and $\mathfrak{m}$ are maximal. We obtain inclusions

$$R/\mathfrak{n} \hookrightarrow R_n/\widetilde{\mathfrak{m}} \hookrightarrow R_{n+1}/\mathfrak{m}.$$

By Theorem 2.5, $R_{n+1}/\mathfrak{m}$ is a finite field extension of $R/\mathfrak{n}$, which implies

$$\dim_{R/\mathfrak{n}}(R_n/\widetilde{\mathfrak{m}}) < \infty.$$

We use a result from Algebra which says that if you have a field extension M/L and a subring $L \subset S \subset M$ with $\dim_L(S) < \infty$, then S is a field. Therefore, we find that $R_n/\widetilde{\mathfrak{m}}$ is a field too. Thus, $\widetilde{\mathfrak{m}}$ is a maximal ideal of R_n.

We find

$$\dim(R_{n+1}) = \dim(R) + n + 1 = \dim(R_n) + 1 = \operatorname{ht}(\widetilde{\mathfrak{m}}) + 1,$$

where the first two equalities follow from Theorem 2.9 and the last one follows from the induction hypothesis. Therefore,

$$\dim(R_{n+1}) = \operatorname{ht}(\widetilde{\mathfrak{m}}) + 1 \leqslant \operatorname{ht}(\mathfrak{m}),$$

which implies equality and concludes the proof. □

Finally, we can prove the second theorem:

Theorem 8.5. *Any integral domain R that is finitely generated over a field K, or over $\mathbb{Z}$, satisfies the following:*

(i) For all prime ideals $\mathfrak{p} \subset R$: $\operatorname{ht}(\mathfrak{p}) + \operatorname{coht}(\mathfrak{p}) = \dim(R)$.

(ii) For all maximal ideals $\mathfrak{m} \subset R$: $\operatorname{ht}(\mathfrak{m}) = \dim(R)$.

Proof. By Proposition 8.4, all maximal ideals have height equal to the dimension of R, which shows (ii). Applying Proposition 8.3 we obtain (i). □

9 Geometric interpretation

First, consider an algebraically closed field K, set $R := K[X_1, \dots, X_n]$ and recall the following definitions:

Definition 9.1. *The subset*

$$V(S) := \{x \in K^n \mid \forall f \in S : f(x) = 0\} \subset K^n$$

defined by a subset $S \subset R$ is called an affine algebraic variety*.*

Definition 9.2. *For any subset $X \subset K^n$ we define the subset*

$$I(X) := \{f \in R \mid \forall x \in X : f(x) = 0\} \subset R.$$

Recall that the Zariski-topology on K^n has as closed sets precisely the $V(S)$ for all $S \subset R$. Furthermore, $I(X)$ is an ideal, and the Zariski-closed subsets of K^n are in a mutually inverse bijection to the radical ideals of R via the maps I and V. We also know that the prime ideals and the irreducible Zariski-closed subset are in a one-to-one order-reversing correspondence under this bijection.

Call X a Zariski-closed subset of K^n and denote by $R_X := R/I(X)$ its coordinate ring. Suppose we have a regular sequence $a_1 + I(X), \ldots, a_r + I(X)$ in R_X for $a_1, \ldots, a_r \in R$. We are interested in what happens in K^n. Let us look at $a_1 + I(X)$. By the definition of a regular sequence, we know that $a_1 + I(X)$ is not a zero-divisor of R_X.

We examine the fact that a_1 is not a zero-divisor of R_X further by recalling that for any ring R' we have

$$\{\text{ zero-divisors of } R' \} = \bigcup_{\mathfrak{p} \in \operatorname{Ass}((0))} \mathfrak{p}.$$

The associated prime ideals of (0) are comprised of the minimal prime ideals above (0) — which are also the minimal elements of Ass((0)) — and the embedded prime ideals. Hence, we know that a_1 cannot lie in any of the associated prime ideals of R_X. The minimal prime ideals of R_X correspond to the irreducible components of X, and the remaining prime ideals of Ass((0)) correspond to the embedded components. We are now able to translate the above into a condition on our variety X: If $V(a_1)$ denotes the hypersurface associated to a_1, then for all irreducible components and embedded components $V(\mathfrak{p})$ of X, we have $V(\mathfrak{p}) \not\subset V(a_1)$.

When we look at the next element of the sequence, we require its image in $R_X/(a_1 + I(X)) \cong R/(I(X), a_1)$ to be a non-zero-divisor. The ring $R/(I(X), a_1)$ is the coordinate ring of the new variety $V(I(X), a_1) = X \cap V(a_1)$, which is the intersection of the varieties X and $V(a_1)$. Hence in the next step, we cut the new variety $X \cap V(a_1)$ with another hypersurface $V(a_2)$ such that once more, none of the irreducible or embedded components of $X \cap V(a_1)$ are contained in $X \cap V(a_1) \cap V(a_2)$. This means that at every step, each irreducible or embedded component of the new intersection must be strictly included in an irreducible component of the previous intersection.

We continue in the same manner, until we reach the end of the regular sequence, where the condition that $R/(I(X), a_1, \ldots, a_r) \neq 0$ ensures that the final intersection

$$X \cap V(a_1) \cap \ldots \cap V(a_r)$$

is nonempty. The depth of the ring R_X is therefore the supremum over all such sequences of intersecting the variety X.

Analogously, we can analyse the depth of an ideal $I(X)$ associated to a variety X. Consider a regular sequence $a_1, \ldots, a_r$ in $I(X)$. This time, the fact that $a_i \in I(X)$ implies that $X \subset V(a_i)$, meaning that the variety X lies within each hypersurface. By a similar argument, we find that at each step, every irreducible or embedded component of

$$K^n \cap V(a_1) \cap \ldots \cap V(a_i)$$

must be strictly included in an irreducible component of

$$K^n \cap V(a_1) \cap \ldots \cap V(a_{i-1}),$$

and that

$$K^n \cap V(a_1) \cap \ldots \cap V(a_r) \neq \varnothing.$$

Hence we can think of cutting through K^n one step at a time and approaching the "shape" of the variety X. The depth of the variety is therefore the supremum over all such cuts.

We see that depth is just another way of trying to find a meaningful way of measuring what we intuitively would call dimension of a variety. The depth of the coordinate ring R_X would correspond to the dimension, whereas the depth of the ideal $I(X)$ can be considered as a codimension of X. This means that we could establish a dimension theory using depth. We already have a notion of dimension and codimension though, given by the Krull dimension and the height. Further even, we know thanks to Proposition 5.1 in what relation they stand. Hence, if a ring is Cohen-Macaulay, the two approaches to define a concept of dimension coincide.

We also examined catenary rings, which we can try to visualise. As mentioned before, prime ideals correspond to irreducible subsets, and hence chains of prime ideals

$$\mathfrak{p}_0 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n$$

correspond to chains of irreducible subsets, with the order reversed:

$$V(\mathfrak{p}_0) \supsetneqq \cdots \supsetneqq V(\mathfrak{p}_n).$$

So when we have a catenary ring, for example $K[X_1, \ldots, X_n]$ — as was shown in Theorem 7.7 —, the number of steps it takes to move from one irreducible subset to another that contains it is always the same, no matter along which chain of irreducible subsets we decide to go.

Let us look back at how we interpreted regular sequences and the depth of an ideal. Given prime ideals $\mathfrak{q} \subset \mathfrak{p}$, let us consider a maximal chain of prime ideals

$$\mathfrak{q} = \mathfrak{p}_0 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n = \mathfrak{p}$$

which corresponds to a chain of irreducible subsets

$$V(\mathfrak{q}) = V(\mathfrak{p}_0) \supsetneqq \cdots \supsetneqq V(\mathfrak{p}_n) = V(\mathfrak{p}).$$

We can go back to the proof of Proposition 7.5 to see that for any subsequent irreducible subsets $V(\mathfrak{p}_{i+1}) \subsetneqq V(\mathfrak{p}_i)$ we can find some hypersurface $V(a)$ which contains $V(\mathfrak{p}_{i+1})$ but does not contain $V(\mathfrak{p}_i)$. This hypersurface corresponds to some element in the localisation with respect to $\mathfrak{p}_{i+1}$. This helps us relay the depth of $\mathfrak{p}_{i+1}$ to the depth of $\mathfrak{p}_i$, i.e. we make a connection between the codimension of $V(\mathfrak{p}_{i+1})$ and the codimension of $V(\mathfrak{p}_i)$. Namely, the codimension in each step — going from $V(\mathfrak{p}_0)$ to $V(\mathfrak{p}_n)$ — can increase by at most one, in a sense taking into account the extra hypersurface in-between the two subsets. But then, since the ring is Cohen-Macaulay, we deduced in the proof that any maximal chain of prime ideals must have length equal to the height of $\mathfrak{p}/\mathfrak{q}$. So, in a sense, the depth forces the property of catenary onto the ring.

Lastly, we return to look at the Krull dimension. We found in Theorem 8.5 that for an integral domain that is finitely generated over a field K, or over $\mathbb{Z}$, the dimension and codimension of an irreducible Zariski-closed subset behaves as we expect them to, meaning that they add up to the dimension of the whole space.

To make it easier to imagine the varieties, we discussed the classical algebraic geometry case of the affine coordinate space K^n, but the ideas can be generalised to schemes too.

References

[1] Lecture "Commutative Algebra" held in Autumn 2016 by Professor Dr. R. Pink at ETH Zurich: `https://people.math.ethz.ch/~pink/`

[2] Atiyah, M.F., Macdonald, I.G.: Introduction to Commutative Algebra. (Addison-Wesley Publishing Company, Inc., 1969)

[3] Eisenbud, D.: Commutative Algebra with a View Toward Algebraic Geometry. (Springer-Verlag, New York, 1995)

[4] Kaplansky, I.: Commutative Rings. (Allyn and Bacon, Inc., 470 Atlantic Avenue, Boston, Mass. 02210, 1970)

[5] Pink, R., Schieder, S.: Compactification of a Drinfeld period domain over a finite field. *J. Algebraic Geom.* **23** (2014), no. 2, 201-243.