ETH Zürich

From Hilbert to Bézout

Bachelor's Thesis

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Abstract

In the projective plane, the number of intersection points of two curves equals the product of their degrees, as postulated by Bézout. This thesis gives a proof of a generalization of this theorem to the intersection of a variety and a hypersurface in higher dimensional projective space applying the notion of the Hilbert polynomial of a graded module to the coordinate ring of the said varieties. As an application, Pascal's Theorem is proved, which predicates the collinearity of the intersection points of opposite sides of a hexagon embedded in a conic.

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1 Introduction

A question that naturally arises while analysing plane curves is whether there exists a rule as to how often certain curves intersect. In the Euclidean plane, the answer generally seems to be negative, as it is easy to find many curves which do not intersect at all. However, if the underlying field is algebraically closed, and the points at infinity are included, the remarkable Theorem of Bézout comes into play.

Presumably first stated by Newton in 1665, and later famously known as Bézout's Theorem (1779), two projective plane curves of degrees n and m intersect at exactly nm points counted with multiplicities. This result was presumed and applied for many years, even though various geometers in the company of Maclaurin, Euler, Cramer and Bézout himself, failed at providing a rigorous proof [4].

The introduction of the *characteristic (Hilbert) function* of a module by Hilbert in 1889 [5] enabled the development of new tools for the study of projective geometry. By cleverly defining the *intersection multiplicity* of algebraic varieties as did Severi [6] and Weil [7], scholars of Hilbert's and Emmy Noether's found a rigorous proof of a generalization of Bézout's Theorem to higher dimensions as for example published by Van der Waerden [8] in 1928.
These results are nicely summarized in the first chapter of the well-known book on algebraic geometry by Hartshorne [1], along the lines of which this thesis is set up. Inspired by various noted books on algebraic geometry, namely Eisenbud's [9] and Matsumura's [10], this thesis aims at providing a generalization of Bézout's Theorem to the intersection of a variety and a hypersurface in n-dimensional projective space.

Bézout's Theorem turned out to be of great importance for the mathematical fields of enumerative geometry and intersection theory. With its help, it is possible to find quite short and direct proofs of various results stated long before a proof to Bézout's Theorem was found. As one of many applications, in Section 7 of this thesis Pascal's Theorem is introduced, which was published in 1640 in an essay [11]. The discussion of Pascal's Theorem, as an application of Bézout's, was enlightened by Tao's essays [12] and [13], as well as Kunz' book on plane algebraic curves [14].

Prerequisites are the elementary principles of commutative algebra, which can for instance be found in Atiyah and Macdonald's Introduction to Commutative Algebra [3], or obtained by attending a lecture course as it was held in the fall semester of 2016 at ETH

Zurich by Richard Pink [2]. The most important results in this field, such as Hilbert's Basis Theorem, Krull's Principal Ideal Theorem and Noether's Normalization Theorem, are employed hereafter. All notions and notations used are introduced in Section 2.

2 Basic Notions and Notations

The framework of the following pages is mainly the projective n-space over an algebraically closed field k, which is denoted by $\mathbb{P}_k^n$. It results from removing the origin from the affine $(n+1)$-space $\mathbb{A}_k^{n+1}$, and taking it modulo the equivalence relation $x \sim \lambda x$ for any unit $\lambda \in k^\times$. The points in $\mathbb{P}_k^n$ are denoted by $(x_0 : \ldots : x_n)$, and $S := k[X_0, \ldots, X_n]$ describes its coordinate ring throughout.

Endowing S with its natural $\mathbb{Z}$-grading by total degree, define S_+ as the maximal ideal $\bigoplus_{d>0} S_d$ of S such that $S = k \oplus S_+$. Furthermore, define the *zero locus* of any set T of homogeneous elements in S_+ as

$$\bar{V}(T) := \{p \in \mathbb{P}_k^n \mid \forall f \in T : f(p) = 0\}.$$

For any homogeneous ideal $\mathfrak{a}$ in S_+, define $\bar{V}(\mathfrak{a}) := \bar{V}(T)$ with T the set of homogeneous elements in $\mathfrak{a}$. Furthermore, set $\bar{V}(S) := \bar{V}(S_+) = \varnothing$. It is worth noting that since S is a noetherian ring, any set of homogeneous elements T in S contains a finite subset $\{f_1, \ldots, f_r\}$ such that $\bar{V}(T) = \bar{V}(f_1, \ldots, f_r)$.

Any subset of $\mathbb{P}_k^n$ of the form $\bar{V}(T)$ will subsequently be referred to as a *projective variety*. In particular, projective varieties need not be irreducible. The projective varieties form the closed sets of the *Zariski topology* on $\mathbb{P}_k^n$. Analogously, the *affine varieties* are the closed sets of the Zariski topology on the affine n-space $\mathbb{A}_k^n$. These are the zero loci $V(Q)$ of all possible subsets Q of $A := k[X_1, \ldots, X_n]$.

The *dimension* of a variety always describes its dimension as a topological space, that is to say the length of the longest chain of irreducible subspaces where indexing starts at 0. The empty variety is set to have dimension $-\infty$. An affine or projective variety of dimension one is called an *affine* or *projective curve*. An *affine* or *projective hypersurface* is a variety of dimension $n-1$ in $\mathbb{A}_k^n$ and $\mathbb{P}_k^n$ respectively. Besides, a variety is said to be *irreducible*, if it is nonempty and not the union of two proper closed subvarieties.

For any subset Y of $\mathbb{P}_k^n$, its *homogeneous ideal* in S is defined as

$$\bar{I}(Y) := (\{f \in S_+ \mid f \text{ homogeneous and } \forall\, p \in Y : f(p) = 0\}) \,,$$

and for any projective variety Y, its *homogeneous coordinate ring* is denoted by

$$S(Y) := S/\bar{I}(Y).$$

The ideal in A induced by a subset X of $\mathbb{A}_k^n$ is denoted by $I(X)$, and $A(X) = A/I(X)$ then describes the affine coordinate ring of an affine variety X.

As they are going to be of use later on, some basic properties of the zero locus and the homogeneous ideal are included hereafter without proofs. For any homogeneous ideals $\mathfrak{a}$, $\mathfrak{b} \subset S_+$ and any subsets X, $Y \subset \mathbb{P}_k^n$, the following equalities and equivalences hold:

(a) $\bar{V}(\mathfrak{a} + \mathfrak{b}) = \bar{V}(\mathfrak{a}) \cap \bar{V}(\mathfrak{b})$

(b) $\bar{V}(\mathfrak{a} \cap \mathfrak{b}) = \bar{V}(\mathfrak{a} \cdot \mathfrak{b}) = \bar{V}(\mathfrak{a}) \cup \bar{V}(\mathfrak{b})$

(c) $\bar{V}(\mathfrak{a}) \subset \bar{V}(\mathfrak{b}) \Leftrightarrow \mathfrak{a} \supset \mathfrak{b}$

(d) $\bar{I}(X \cup Y) = \bar{I}(X) \cap \bar{I}(Y)$.

Note that the dimension of any affine variety is equal to that of its coordinate ring. However, the dimension of any nonempty projective variety Y satisfies

$$\dim Y = \dim S(Y) - 1.$$

This is due to the fact that as in the affine case, any longest chain of irreducible subvarieties $Y_0 \subsetneq \ldots \subsetneq Y_r \subset Y$ of Y corresponds to a chain of prime ideals $0 = \bar{I}(Y) \subset \bar{I}(Y_r) \subsetneq \ldots \subsetneq \bar{I}(Y_0)$ in $S(Y)$, but since $0 \notin \mathbb{P}_k^n$, the homogeneous ideal of Y_0 is not a maximal ideal in $S(Y)$. Therefore, the chain of prime ideals has to be extended by $S_+/\bar{I}(Y)$ in order to be maximal in $S(Y)$.

3 The Dimension Theorems

Even though the intersection of any family of varieties is again a variety, the intersection of two irreducible projective varieties need not be irreducible. This can be shown by the following example:

Example 3.1. Consider the projective varieties $Y = \bar{V}(x^2 - yw)$ and $Z = \bar{V}(xy - zw)$ in $\mathbb{P}^3_k$. Then,

$$\begin{aligned} Y \cap Z &= \bar{V}(x^2 - yw, xy - zw) \\ &= \bar{V}(x(y^2 - zx)) \\ &= \bar{V}(x) \cup \bar{V}(y^2 - zx). \end{aligned}$$

Therefore, the intersection of Y and Z is the union of two closed subsets in $\mathbb{P}^3_k$, and as such not irreducible.

However, it is possible to describe the irreducible components of the intersection of two varieties in more detail.

Lemma 3.2. *Let X, Y be two irreducible affine varieties of dimensions r, s in $\mathbb{A}^n_k$. Then the product $X \times Y$ is an irreducible affine variety of dimension $r + s$ in $\mathbb{A}^{2n}_k$.*

Proof. Note first that $X \times Y$ is closed in $\mathbb{A}^{2n}_k$ as it is the zero locus of the ideal generated by $I(X)$ and $I(Y)$.
Suppose that $X \times Y$ is reducible and equal to $A \cup B$ with non-empty proper closed subsets A and B of $X \times Y$. Fix $y \in Y$ and let $X_y := X \times \{y\}$. Then, X is homeomorphic to X_y via the map $x \mapsto (x, y)$ and thus, X_y is also irreducible. Since $X_y = (A \cap X_y) \cup (B \cap X_y)$, it is clear that X_y has to lie in one of A or B. Define $V_A := \{y \in Y \mid X_y \subset A\}$ and V_B analogously. A is closed in $X \times Y$ and therefore $A = V(f_1, \dots, f_r)$ for some polynomials $f_1, \dots, f_r \in k[X, Y]$. Then, V_A is exactly the zero locus in Y of the set of polynomials $\{f_i(x_0, y) \mid x_0 \in X,\ 1 \leq i \leq r\}$. Hence, V_A is closed in Y. Analogously, it follows that V_B is closed in Y.
Write $Y = V_A \cup V_B$ and then Y equals either V_A or V_B, because Y is irreducible. If $Y = V_A$, the product $X \times Y$ is equal to $\bigcup_{y \in Y} X_y \subset A$ and $B = \varnothing$. Symmetry shows that $X \times Y$ is irreducible.
To show that $\dim(X \times Y) = \dim X + \dim Y$, note that it is equivalent to show that $\dim(A(X \times Y)) = \dim A(X) + \dim A(Y)$. In addition, the tensor product $A(X) \otimes_k A(Y)$ is isomorphic to $A(X \times Y)$ by $(x \otimes y) \mapsto xy$.
By Noether Normalization, $A(X)$ is integral over $k[x_1, \dots, x_r]$ for $x_1, \dots, x_r$ algebraically independent over k and similarly, $A(Y)$ is integral over $k[y_1, \dots, y_s]$. So for any $x \in A(X)$, there exists an integer n and coefficients $a_i \in k[x_1, \dots, x_r]$ such that

$x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$. For any $y \in A(Y)$, it follows that

$$(x \otimes y)^n + a_{n-1}(x \otimes y)^{n-1} + \ldots + a_0 = 0$$

and $x \otimes y$ is thus integral over $k[x_1, \ldots, x_r] \otimes A(Y)$. Similarly, it follows that all $x \otimes y$ in $A(X) \otimes A(Y)$ are integral over $k[x_1, \ldots, x_r] \otimes k[y_1, \ldots, y_s]$, which is isomorphic to a polynomial ring in $r + s$ variables. As the Krull dimension is invariant under integral ring extension, the dimension of $A(X \times Y)$ is thereby equal to $r + s$. □

Proposition 3.3 (Affine Dimension Theorem). *Let Y and Z be irreducible affine varieties of respective dimensions r and s in $\mathbb{A}_k^n$. Then every irreducible component of $Y \cap Z$ is of dimension greater or equal to $r + s - n$.*

Proof. First, suppose that $Z = V(f)$ is an irreducible hypersurface and thus $s = \dim(Z) = n - 1$. If $Y \subset Z$, there is nothing to prove.
So assume $Y \not\subset Z$ and write $Y = V(\mathfrak{a})$ for a prime ideal $\mathfrak{a}$ in $k[X_1, \ldots, X_n]$. As the intersection of two varieties is also a variety, $Y \cap Z$ possesses a decomposition into unique irreducible components. In a noetherian ring, any ideal is decomposable, and so it follows that the radical ideal $I(Y \cap Z) = \mathrm{Rad}(\mathfrak{a} + (f))$ can be uniquely decomposed into prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$, which are the minimal associated primes of $\mathfrak{a} + (f)$. Those minimal primes precisely correspond to the irreducible components $V_1, \ldots, V_r$ of $Y \cap Z$. Let $A(Y)$ be the affine coordinate ring of Y. As prime ideals are preserved under projection, these prime ideals are exactly the minimal prime ideals containing the principal ideal (f) in $A(Y)$.
By Krull's Principal Ideal Theorem, it follows that each such minimal prime ideal $\mathfrak{p}_i$ has height one in $A(Y)$. As $A(Y)$ is finitely generated over a field and noetherian, the sum of the height and the coheight of any prime ideal equals the dimension of $A(Y)$. Therefore, $A(Y)/\mathfrak{p}_i$ has dimension $r - 1$. As $A(Y)/\mathfrak{p}_i$ is the coordinate ring of the corresponding irreducible component V_i of $Y \cap Z$, it follows that $\dim(V_i) = r - 1$.

For the general case, consider the product $Y \times Z \subset \mathbb{A}_k^{2n}$, which is an irreducible variety of dimension $r + s$ by Lemma 3.2.
Let Δ be the diagonal $\{p \times p \mid p \in \mathbb{A}_k^n\} \subset \mathbb{A}_k^{2n}$. Then $\mathbb{A}_k^n$ is isomorphic to Δ via the map $p \mapsto p \times p$, and under this isomorphism, $Y \cap Z$ corresponds to $(Y \times Z) \cap \Delta$. Since Δ has dimension n, and since $r + s - n = (r + s) + n - 2n$, it is possible to reduce to the case of the two varieties $Y \times Z$ and Δ in $\mathbb{A}_k^{2n}$. With $k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ as the coordinate

ring of $\mathbb{A}_k^{2n}$, the diagonal Δ can be written as $V(X_1 - Y_1) \cap \ldots \cap V(X_n - Y_n)$, and Δ is therefore the intersection of n irreducible hypersurfaces. An n-time application of the special case concludes the proof. $\square$

Definition 3.4. For any projective variety $Y \subset \mathbb{P}_k^n$ and the natural projection map $\pi : \mathbb{A}_k^{n+1} \smallsetminus \{0\} \twoheadrightarrow \mathbb{P}_k^n$, the *affine cone over* Y is defined as $C(Y) := \pi^{-1}(Y) \cup \{0\}$.

Lemma 3.5. *Let Y be a projective variety and $C(Y)$ the affine cone over Y. Then,*

(a) $C(Y)$ is an affine variety in $\mathbb{A}_k^{n+1}$, with $\bar{I}(Y) = I(C(Y))$ in $k[X_0, \ldots, X_n]$,

(b) for Y nonempty, $C(Y)$ is irreducible if and only if Y is,

(c) for Y nonempty, $\dim(C(Y)) = \dim(Y) + 1$.

Proof. (a) Let $Y \subset \mathbb{P}_k^n$ be a projective variety. For $Y = \varnothing$, the affine cone of Y is equal to $C(Y) = \{0\}$, which is an affine variety, and $\bar{I}(Y) = \bar{I}(\varnothing) = S_+ = I(\{0\}) = I(C(Y))$. Now assume Y to be nonempty, and denote by T the subset of homogeneous elements of S such that $\bar{V}(T) = Y$. Letting S stand for the coordinate ring of $\mathbb{A}_k^{n+1}$ as well, it needs to be shown that $C(Y) = V(T)$.
Consider an element $y' \in C(Y) = \pi^{-1}(Y) \cup \{0\}$. If $y' = 0$, it follows that $f(y') = 0$ for any $f \in T$, as T does not contain any constant elements apart from zero. Now assume $y' \neq 0$ and choose a representative of $\pi(y') = (y_0 : \ldots : y_n)$. Then $y' = \lambda \cdot (y_0, \ldots, y_n)$ for some $\lambda \in k^\times$ and consequently, for all $f \in T$,

$$f(y') = f(\lambda(y_0, \ldots, y_n)) = \lambda^{\deg f} f((y_0, \ldots, y_n)) = \lambda^{\deg f} f((y_0 : \ldots : y_n)) = 0.$$

Hence $C(Y) \subset V(T)$. Conversely, consider an element $x \in \mathbb{A}_k^{n+1} \smallsetminus \{0\}$ such that $f(x) = 0$ for all $f \in T$. As all $f \in T$ are homogeneous, $f(\lambda x) = 0$ for any $\lambda \in k^\times$ and thus also $f(\pi(x)) = 0$, so $x \in C(Y)$.
To show that $\bar{I}(Y) = I(C(Y))$, note first that $I(C(Y))$ is a homogeneous ideal. Besides, as $V(T) = C(Y)$, it follows that

$$\begin{aligned} \bar{I}(Y) &= (\{f \in S \mid f \text{ homog. and } \forall\, y \in Y :\ f(y) = 0\}) \\ &= (\{f \in S \mid f \text{ homog. and } \forall\, y' \in C(Y) :\ f(\pi(y')) = 0\}) \\ &= (\{f \in S \mid f \text{ homog. and } \forall\, y' \in C(Y) :\ f(y') = 0\}) = I(C(Y)). \end{aligned}$$

(b) The projective variety Y is irreducible if and only if $\bar{I}(Y)$ is a prime ideal properly contained in S_+, and the affine cone $C(Y)$ is irreducible if and only if $I(C(Y))$ is a prime ideal. By (a), the equality $\bar{I}(Y) = I(C(Y))$ holds, and the statement follows.
(c) As discussed in Section 2, the dimension of the affine variety $C(Y)$ satisfies the equality $\dim C(Y) = \dim S/I(C(Y))$. By (a), it follows that $\dim C(Y) = \dim S/\bar{I}(Y)$, which equals $\dim Y + 1$ again by Section 2. □

Theorem 3.6 (Projective Dimension Theorem)**.** *Let Y and Z be irreducible projective varieties of respective dimensions r and s in $\mathbb{P}^n_k$. Then, every irreducible component of $Y \cap Z$ is of dimension greater or equal to $r + s - n$. Furthermore, if $r + s - n \geq 0$, the intersection is nonempty.*

Proof. As $\mathbb{P}^n_k$ can be covered by open subsets $U_i := \mathbb{P}^n_k \setminus V(X_i)$ for $0 \leq i \leq n$ that are isomorphic to affine n-spaces, the first statement follows from the Affine Dimension Theorem 3.3. As to the second claim, let $C(Y)$ and $C(Z)$ be the cones over Y and Z in $\mathbb{A}^{n+1}_k$. From Lemma 3.5, it follows that $C(Y)$ and $C(Z)$ have dimensions $r+1$ and $s+1$ respectively. Furthermore, $C(Y) \cap C(Z) \neq \varnothing$, as both contain the origin of $\mathbb{A}^{n+1}_k$. By the Affine Dimension Theorem 3.3, the affine variety $C(Y) \cap C(Z)$ has dimension greater or equal to $(r+1) + (s+1) - (n+1) = r + s - n + 1 > 0$. Hence, the intersection of the two cones contains some point apart from the origin, and thus $Y \cap Z \neq \varnothing$. □

4 The Hilbert Polynomial

4.1 Numerical Polynomials

Definition 4.1. A *numerical polynomial* is a polynomial $P \in \mathbb{Q}[X]$ such that $P(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$ with $n \gg 0$.

Using the following basis of the vector space of rational polynomials significantly simplifies the upcoming calculations with polynomials. Consider the binomial coefficient function

$$\binom{X}{r} = \frac{1}{r!} X(X-1)\cdots(X-r+1)$$

for an arbitrary variable X and any natural number r. Note that $\binom{X}{r}$ is a polynomial of degree r. Therefore, $\mathcal{B} := \left(\binom{X}{r}, \dots, \binom{X}{0}\right)$ forms a basis of the vector space of rational polynomials of degree less or equal to r.

Furthermore, it is useful to define the *difference polynomial* ΔQ of any polynomial $Q \in \mathbb{Q}[X]$ as $\Delta Q(X) = Q(X+1) - Q(X)$. The difference polynomial of any numerical polynomial is clearly numerical itself. The difference polynomial of a basis element in $\mathcal{B}$ is

$$\Delta\binom{X}{r} = \binom{X+1}{r} - \binom{X}{r} = \binom{X}{r-1}.$$

These arguments already prove the following remark:

Remark 4.2. For any polynomial $Q \in \mathbb{Q}[X]$ with $\deg(Q) > 0$, the difference polynomial ΔQ is of degree $\deg(Q) - 1$.

Proposition 4.3. *(a) For any numerical polynomial $P \in \mathbb{Q}[X]$ of degree $r \geq 0$ there exist unique integers $c_0, \dots, c_r$, such that*

$$P(X) = c_0\binom{X}{r} + c_1\binom{X}{r-1} + \ldots + c_r.$$

In particular $P(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.

(b) Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be any function. If there exists a numerical polynomial $Q(X)$ such that the difference function $\Delta f = f(n+1) - f(n)$ is equal to $Q(n)$ for all $n \gg 0$, then there exists a numerical polynomial $P(X)$ such that $f(n) = P(n)$ for all $n \gg 0$.

Proof. (a) The result follows by induction on the degree r of P, where the case $r = 0$ follows directly. Suppose thus that the claim holds for $r - 1$ and let $c_0, \dots, c_r \in \mathbb{Q}$ be the rational coefficients of $P \in \mathbb{Q}[X]$ with respect to the basis $\mathcal{B}$ as defined above.
According to the calculations at the beginning of this section, the difference polynomial of P is of the form

$$\Delta P(X) = c_0\binom{X}{r-1} + c_1\binom{X}{r-2} + \ldots + c_{r-1}.$$

By induction, the coefficients $c_0, \dots, c_{r-1}$ of this numerical polynomial of degree $r-1$ lie in $\mathbb{Z}$. Since $\binom{n}{r}$ lies in $\mathbb{Z}$ for any integer n, it follows that $P(n) - c_r$ has to lie in $\mathbb{Z}$ as well. As P is a numerical polynomial, c_r is therefore an integer. Furthermore $P(n) \in \mathbb{Z}$ for any integer n.

(b) If $\Delta f = 0$, the zero polynomial serves as a suitable P. Else, let $q_0, \ldots, q_s$ be the integer coefficients of Q with respect to $\mathcal{B}$ with $s := \deg(Q)$. Define another polynomial $P \in \mathbb{Q}[X]$ as follows:

$$P(X) = q_0 \binom{X}{s+1} + \ldots + q_s \binom{X}{1}.$$

Then $\Delta P = Q$, and therefore $\Delta(f - P)(n) = 0$ for all $n \gg 0$ and P is a numerical polynomial as well. Moreover, $f - P$ is constant with $q_{s+1} := (f - P)(n) = (f - P)(n+1)$ for $n \gg 0$. Then $\tilde{P} := P + q_{s+1}$ is a suitable numerical polynomial satisfying $f(n) = \tilde{P}(n)$ for all $n \gg 0$. □

4.2 The Multiplicity of a Module over a Minimal Prime Ideal

Let R be an arbitrary $\mathbb{Z}$-graded ring and M a graded R-module with a decomposition $M = \bigoplus_{d \in \mathbb{Z}} M_d$. Note that the annihilator $\mathrm{Ann}(M) = \{r \in R \mid r \cdot M = 0\}$ of M is a homogeneous ideal in R. This is due to the fact that any r in $\mathrm{Ann}(M)$ also lies in $\mathrm{Ann}(M_d)$ for any $d \in \mathbb{Z}$, and thus $\left(\sum_{i=1}^{n} r_i\right) M_d = 0$, where $r_1, \ldots, r_n$ are the homogeneous components of r. It follows that $r_i M_d = 0$ for all $1 \leq i \leq n$ and hence all the homogeneous components of r lie in $\mathrm{Ann}(M)$.

Lemma 4.4. *For any short exact sequence of graded R-modules*

$$0 \longrightarrow M' \overset{i}{\hookrightarrow} M \overset{p}{\twoheadrightarrow} M'' \longrightarrow 0,$$

the following statements hold:

(i) $\mathrm{Ann}(M) \subset \mathrm{Ann}(M') \cap \mathrm{Ann}(M'')$

(ii) $\mathrm{Ann}(M') \cdot \mathrm{Ann}(M'') \subset \mathrm{Ann}(M)$

(iii) $\bar{V}(\mathrm{Ann}(M)) = \bar{V}(\mathrm{Ann}(M')) \cup \bar{V}(\mathrm{Ann}(M''))$.

Proof. (i) Let s lie in the annihilator of M and let m' lie in M'. Then, $i(sm') = s \cdot i(m') = 0$ and thus $sm' = 0$, as i is injective. Consider now an element $m'' \in M''$. Since p is surjective, there exists an element $m \in M$, such that $p(m) = m''$. Hence $sm'' = s \cdot p(m) = p(sm) = p(0) = 0$. Thus, s lies in the intersection of the annihilators of M' and M''.

(ii) First consider homogeneous elements $a \in \mathrm{Ann}(M')$ and $b \in \mathrm{Ann}(M'')$. For any $m \in M$, it follows that $0 = b \cdot p(m) = p(bm)$. Thus, bm lies in the kernel of p and hence also in the image of i. Choosing an m' in M' with $i(m') = bm$, it follows that $0 = i(am') = a \cdot i(m') = abm$, and thus ab lies in the annihilator of M. As all elements in the product $\mathrm{Ann}(M') \cdot \mathrm{Ann}(M'')$ are R-linear combinations of products of the elements in $\mathrm{Ann}(M')$ and $\mathrm{Ann}(M'')$ and i and p are R-module-homomorphisms, the statement follows by varying a and b.

(iii) Taking the projective varieties of the ideals on both sides of (a) and (b), and using the fact that $\bar{V}(\mathfrak{p}) \subset \bar{V}(\mathfrak{q}) \Leftrightarrow \mathfrak{p} \supset \mathfrak{q}$ for any ideals $\mathfrak{p}, \mathfrak{q} \subset R$, the following inclusions hold:

$$\begin{aligned} \bar{V}(\mathrm{Ann}(M')) \cup \bar{V}(\mathrm{Ann}(M'')) &= \bar{V}(\mathrm{Ann}(M') \cap \mathrm{Ann}(M'')) \\ &\subset \bar{V}(\mathrm{Ann}(M)) \\ &\subset \bar{V}(\mathrm{Ann}(M') \cdot \mathrm{Ann}(M'')) \\ &= \bar{V}(\mathrm{Ann}(M')) \cup \bar{V}(\mathrm{Ann}(M'')). \end{aligned}$$

Therefore $\bar{V}(\mathrm{Ann}(M)) = \bar{V}(\mathrm{Ann}(M')) \cup \bar{V}(\mathrm{Ann}(M''))$. □

Remark 4.5. It follows from Lemma 4.4 that for any graded R-module M and any filtration of graded submodules $0 = M^0 \subset \ldots \subset M^r = M$,

$$\mathrm{Ann}(M) \subset \bigcap_{i=1}^{r} \mathrm{Ann}(M^i/M^{i-1}) \quad \text{and} \quad \prod_{i=1}^{r} \mathrm{Ann}(M^i/M^{i-1}) \subset \mathrm{Ann}(M).$$

Proof. If $M = 0$, the annihilator of M is equal to R, and thus the two statements hold. Then, assume $M \neq 0$. Applied to the short exact sequences

$$0 \longrightarrow M^{i-1} \hookrightarrow M^i \twoheadrightarrow M^i/M^{i-1} \longrightarrow 0$$

for $1 \leq i \leq r$, Lemma 4.4 states that $\mathrm{Ann}(M^i) \subset \mathrm{Ann}(M^{i-1}) \cap \mathrm{Ann}(M^i/M^{i-1})$, and similarly $\mathrm{Ann}(M^{i-1}) \cdot \mathrm{Ann}(M^i/M^{i-1}) \subset \mathrm{Ann}(M^i)$. Using these relations $r-1$ times and the fact that $M^1 = M^1/M^0$, the statements follow. □

Definition 4.6. For any graded module M, the *twisted module* $M(l)$ with $l \in \mathbb{Z}$ is formed by shifting the grading of M by exactly l places to the left, so $M(l)_d = M_{d+l}$ for all integers d.

Remark 4.7. Note that shifting a graded module leads to the following equality of dimensions:

$$\dim(M(l)_d) = \dim(M_{d+l}).$$

Definition 4.8. For any R-module M, the *associated prime ideals of* M are defined as the associated prime ideals of the annhilator $\operatorname{Ann}(M)$ of M.

Proposition 4.9. *Let M be a finitely generated graded module over a noetherian graded ring R. Then, there exists a filtration $0 = M^0 \subset M^1 \subset \ldots \subset M^r = M$ of graded submodules such that for each $1 \leq i \leq r$, the quotient is of the form $M^i/M^{i-1} \cong (R/\mathfrak{p}_i)(l_i)$, where $\mathfrak{p}_i$ is a homogeneous prime ideal in R and $l_i \in \mathbb{Z}$. For any such filtration, the ensuing statements hold.*

(a) Let $\mathfrak{q}$ be a homogeneous prime ideal in R. Then $\mathfrak{q} \supset \operatorname{Ann}(M)$ if and only if $\mathfrak{q} \supset \mathfrak{p}_i$ for some i. In particular, the minimal elements of the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ are exactly the minimal associated prime ideals of M.

(b) For each minimal associated prime ideal $\mathfrak{p}$ of M, the number of times $\mathfrak{p}$ occurs in the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ is equal to the length of $M_\mathfrak{p}$ over the local ring $R_\mathfrak{p}$. Hence, it is independent of the chosen filtration.

Proof. To prove the existence of a filtration with the desired properties, consider the set $\mathcal{S}$ of graded submodules of M which admit such a filtration. The zero module does so trivially, and therefore $\mathcal{S}$ is nonempty. As M is finitely generated, it is noetherian, so there exists a maximal element $M' \in \mathcal{S}$. Consider $M'' := M/M'$. If $M'' = 0$, there is nothing to prove. If not, consider the set of ideals

$$\mathfrak{I} := \{I_m = \operatorname{Ann}(m) \mid m \in M'' \text{ homogenous and } m \neq 0\}.$$

As mentioned at the beginning of this section, each I_m is a homogeneous ideal. Furthermore, $I_m \neq R$, as m is nonzero. Since R is noetherian, every nonempty collection of ideals possesses a maximal element, and therefore so does $\mathfrak{I}$.

Claim. Any maximal element I_m of $\mathfrak{I}$ is a prime ideal.

Proof of the claim. Let $a, b \in R$ with $ab \in I_m$ and $b \notin I_m$. Thus $abm = 0$, but $bm \neq 0$, and it is to be shown that a lies in I_m. As $sbm = bsm = b \cdot 0 = 0$ for any $s \in I_m$, it follows that $I_m \subset I_{bm}$. Since I_m is maximal, $I_m = I_{bm}$. By assumption ab lies in I_m, so $abm = 0$ and therefore $a \in I_{bm} = I_m$. □

Therefore, $\mathfrak{p} := I_m$ is a homogeneous prime ideal of R. Let m have degree l. Then the module $N \subset M''$ generated by m is isomorphic to $(R/\mathfrak{p})(-l)$. This can be easily seen by

identifying $m + M'$ in N with $1 + \mathfrak{p}$ in $(R/\mathfrak{p})(-l)$.
Let $N' \subset M$ be the inverse image of N under the projection map $M \twoheadrightarrow M/M'$. Then $M' \subset N'$, and $N'/M' \cong (R/\mathfrak{p})(-l)$. So N' also admits a filtration of the required type. This contradicts the maximality of M' and therefore $M/M' = 0$, and so $M = M'$, which proves the existence of the filtration.

(a) By assumption $\mathrm{Ann}(M^i/M^{i-1}) = \mathrm{Ann}((R/\mathfrak{p}_i)(l_i))$, which is equal to $\mathfrak{p}_i$. Therefore, it is enough to show the equivalence $\mathfrak{q} \supset \mathrm{Ann}(M) \Leftrightarrow \mathfrak{q} \supset \mathrm{Ann}(M^i/M^{i-1})$ for some i.
To do so, suppose given a filtration of M as above, and that $\mathfrak{q}$ is a homogenous prime ideal of R. If the annihilator of M lies in $\mathfrak{q}$, then Remark 4.5 states that also the product of the annhilators of the factor modules M^i/M^{i-1} lies in $\mathfrak{q}$. Since $\mathfrak{q}$ is a prime ideal, at least one of the factors of the said product has to lie in $\mathfrak{q}$. Conversely, if the annihilator of M^i/M^{i-1} lies in $\mathfrak{q}$ for some $1 \leq i \leq r$, then it follows directly that $\mathrm{Ann}(M)$ has to lie in $\mathfrak{q}$ as well, since $\mathrm{Ann}(M) \subset \bigcap_{i=1}^{r} \mathrm{Ann}(M^i/M^{i-1})$ according to Remark 4.5, and hence the statement follows.

(b) Let $\mathfrak{p}$ be a minimal associated prime ideal of M, and consider the localization $M_\mathfrak{p}$ of M at $\mathfrak{p}$. By (a), the prime ideal $\mathfrak{p}$ is minimal in the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$, and thus $M^i_\mathfrak{p}/M^{i-1}_\mathfrak{p} \cong (R/\mathfrak{p}_i)_\mathfrak{p} = 0$ for any i, if $\mathfrak{p}_i \neq \mathfrak{p}$. This is due to the fact that $\mathfrak{p}_i \cap (R \setminus \mathfrak{p}) \neq \varnothing$, and then some elements of $\mathfrak{p}_i$ are inverted under the localization. When $\mathfrak{p}_i = \mathfrak{p}$ however, $M^i_\mathfrak{p}/M^{i-1}_\mathfrak{p} \cong (R/\mathfrak{p})_\mathfrak{p} = \mathrm{Quot}(R/\mathfrak{p})$ ([2] §2). For any submodule $M^{i-1} \subsetneq N \subset M^i$, the quotient N/M^{i-1} then maps isomorphically onto a prime ideal of $\mathrm{Quot}(R/\mathfrak{p})$, and thus N has to be equal to M^i. This shows that $M_\mathfrak{p}$ is an $R_\mathfrak{p}$-module of length equal to the number of times $\mathfrak{p}$ occurs in the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. □

Definition 4.10. For $\mathfrak{p}$ a minimal prime of a graded R-module M, the *multiplicity* $\mu_\mathfrak{p}(M)$ *of* M *at* $\mathfrak{p}$ is defined to be the length of $M_\mathfrak{p}$ over $R_\mathfrak{p}$.

4.3 The Hilbert–Serre Theorem

Consider again the graded ring $S = k[X_0, \ldots, X_n]$.

Definition 4.11. The *Hilbert function* φ_M of a graded module M over S is defined as

$$\varphi_M(l) = \mathrm{length}(M_l), \quad l \in \mathbb{Z}.$$

Theorem 4.12 (Hilbert–Serre)**.** *Let M be a finitely generated graded S-module. There exists a unique polynomial $P_M \in \mathbb{Q}[Z]$ such that $\varphi_M(l) = P_M(l)$ for all $l \gg 0$. Furthermore,* $\deg(P_M) = \dim(\bar{V}(\operatorname{Ann} M))$*. In particular, $\varphi_M(l) < \infty$ for all integers l.*

Definition 4.13. The polynomial P_M is called the *Hilbert polynomial of M*.

Remark 4.14. For any short exact sequence of graded S-modules,

$$0 \longrightarrow M' \hookrightarrow M \twoheadrightarrow M'' \longrightarrow 0,$$

the respective Hilbert functions satisfy the equality $\varphi_M = \varphi_{M'} + \varphi_{M''}$.

Proof. Consider the short exact sequence from above. For the Hilbert functions of the graded S-modules, it follows that $\varphi_{M'} + \varphi_{M''} = \varphi_M$, due to the additivity of the length of modules in short exact sequences ([2] §7). □

Remark 4.15. The leading coefficient of the Hilbert polynomial of any graded S-module is always non-negative, if it exists.

Proof. Suppose the Hilbert polynomial P_N of a graded S-module N exists. As $P_N(l) = \varphi_N(l) = \operatorname{length}(N_l) \geq 0$ for any integer $l \gg 0$, the leading coefficent of the Hilbert polynomial P_N must be greater or equal to 0. □

Proof of the theorem. If $M = 0$, the corresponding polynomial is $P_M = 0$, and it follows that $\deg(P_M) = \dim(\bar{V}(S)) = \dim(\operatorname{Ann} M)$, where by convention the zero-polynomial is of degree $-\infty$ and the empty set is of dimension $-\infty$. Thus, assume $M \neq 0$, and consider anew the short exact sequence of graded S-modules

$$0 \longrightarrow M' \hookrightarrow M \twoheadrightarrow M'' \longrightarrow 0.$$

Lemma 4.4 states that $\bar{V}(\operatorname{Ann} M) = \bar{V}(\operatorname{Ann} M') \cup \bar{V}(\operatorname{Ann} M'')$, and therefore,

$$\dim(\bar{V}(\operatorname{Ann} M)) = \max\{\dim(\bar{V}(\operatorname{Ann} M')), \dim(\bar{V}(\operatorname{Ann} M''))\}.$$

Consequently and by Remarks 4.14 and 4.15, if the theorem holds for M' and M'', it also holds for M with $P_M = P_{M'} + P_{M''}$.
By Proposition 4.9, M has a filtration $0 = M^0 \subset M^1 \subset \ldots \subset M^r = M$, with quotients of

the form $(S/\mathfrak{p}_i)(l_i)$ where $\mathfrak{p}_i$ are homogeneous prime ideals and $l_i \in \mathbb{Z}$. Consider for any $1 \leq i \leq r$ the short exact sequence

$$0 \longrightarrow M^{i-1} \hookrightarrow M^i \twoheadrightarrow M^i/M^{i-1} \longrightarrow 0,$$

and note that $M^1 = M^1/M^0 \cong (S/\mathfrak{p}_1)(l_1)$. The previous reasoning implies that the proof can be reduced to the case $M \cong (S/\mathfrak{p})(l)$ for $\mathfrak{p}$ a homogeneous prime ideal of S and $l \in \mathbb{Z}$. As the shift l corresponds to a change of variables $z \mapsto z + l$, it is sufficient to only consider $M = S/\mathfrak{p}$.

To prove the theorem, use induction on the coheight of $\mathfrak{p}$. If $\mathfrak{p} = (X_0, \dots, X_n)$, then $M \cong k$ and thus all elements of M are of degree zero. Hence, $\varphi_M(l) = 0$ for $l > 0$, so $P_M = 0$ is again the corresponding polynomial, and additionally $\deg(P_M) = \dim(\bar{V}(\mathfrak{p}))$.

If $\mathfrak{p} \neq (X_0, \dots, X_n)$, consider $X_i \notin \mathfrak{p}$ and the short exact sequence

$$0 \longrightarrow M \xhookrightarrow{\cdot X_i} M \twoheadrightarrow M/X_i \cdot M \longrightarrow 0.$$

Then, $\varphi_{M/X_i \cdot M}(l) = \varphi_M(l) - \varphi_M(l-1) = (\Delta\varphi_M)(l-1)$. On the other hand,

$$\begin{aligned}\bar{V}(\mathrm{Ann}(M/X_iM)) &= \bar{V}(\mathrm{Ann}(\,(S/\mathfrak{p})/(X_i)\,)) \\ &= \bar{V}(\mathrm{Ann}(\,S/(\mathfrak{p} + (X_i))\,)) \\ &= \bar{V}(\mathfrak{p} + (X_i)) = \bar{V}(\mathfrak{p}) \cap \bar{V}(X_i).\end{aligned}$$

As $\bar{V}(\mathfrak{p}) \not\subset \bar{V}(X_i)$ by the choice of X_i and by the Projective Dimension Theorem 3.6, it follows that $\dim(\bar{V}(\mathrm{Ann}(\,M/X_iM\,))) = \dim(\bar{V}(\mathfrak{p})) - 1 = \dim(\bar{V}(\mathrm{Ann}\ M)) - 1$.
By the induction hypothesis, $\varphi_{M/X_iM}(l) = P_{M/X_iM}(l)$ for $l \gg 0$, where P_{M/X_iM} is a polynomial of degree equal to the dimension of $\bar{V}(\mathrm{Ann}(M/X_iM))$. It follows by Proposition 4.3 that φ_M is also equal to a polynomial P_M for all integers large enough. Remark 4.2 implies furthermore that taking the difference polynomial decreases the degree by one, and therefore $\deg(P_M) = \dim(\bar{V}(\mathfrak{p})) = \dim(\bar{V}(\mathrm{Ann}\, M))$. □

Remark 4.16. For any short exact sequence of graded S-modules,

$$0 \longrightarrow M' \hookrightarrow M \twoheadrightarrow M'' \longrightarrow 0,$$

the equality $P_M = P_{M'} + P_{M''}$ is satisfied by the corresponding Hilbert polynomials, as established in the proof above. In particular, $\deg(P_M) = \max\{\deg(P_{M'}), \deg(P_{M''})\}$, which follows by Remark 4.15.

5 The Degree of a Projective Variety

Definition 5.1. For any projective variety $Y \subset \mathbb{P}_k^n$ of dimension $r \geq 0$, the *Hilbert polynomial P_Y of Y* is defined to be the Hilbert polynomial of its homogeneous coordinate ring $S(Y)$. The *degree of Y* is defined to be $r!$ times the leading coefficient of P_Y. Moreover, the degree of the empty projective variety is set to $-\infty$.

Proposition 5.2. *The Hilbert polynomial of Y is a polynomial of degree r.*

Proof. Since $S(Y) = S/\bar{I}(Y)$, the annihilator of $S(Y)$ is equal to $\bar{I}(Y)$. Therefore, $\bar{V}(\mathrm{Ann}(S(Y))) = \bar{V}(\bar{I}(Y)) = \overline{Y} = Y$. It follows by the Theorem of Hilbert–Serre 4.12 that $\deg(P_Y) = \dim(\bar{V}(\mathrm{Ann}\, M)) = \dim(Y) = r$. □

Proposition 5.3. *(a) For any nonempty projective variety $Y \subset \mathbb{P}_k^n$, the degree of Y is a positive integer.*

(b) Let Y be a nonempty projective variety with a decomposition into projective varieties Y_1 and Y_2, i.e. $Y = Y_1 \cup Y_2$. Suppose that Y_1 and Y_2 have the same dimension r, and that $\dim(Y_1 \cap Y_2) < r$. *Then* $\deg Y = \deg Y_1 + \deg Y_2$.

(c) $\deg \mathbb{P}_k^n = 1$.

(d) For any projective hypersurface $H \subset \mathbb{P}_k^n$ whose ideal is generated by a homogeneous polynomial f, the degree of H is equal to the degree of f.

Proof. (a) Since Y is nonempty, P_Y is a nonzero polynomial of degree $r := \dim Y$. By Proposition 4.3, $\deg Y = c_0$ where $\frac{c_0}{r!}$ is the leading coefficient of P_Y and c_0 an integer. By Remark 4.16, it follows that c_0 is positive.

(b) Consider the sequence

$$0 \longrightarrow S/\bar{I}(Y) \xrightarrow{i} S/\bar{I}(Y_1) \oplus S/\bar{I}(Y_2) \xrightarrow{p} S/(\bar{I}(Y_1) + \bar{I}(Y_2)) \longrightarrow 0$$

where i sends any element $s + \bar{I}(Y)$ to $(s + \bar{I}(Y_1),\, s + \bar{I}(Y_2))$, and p sends $(x + \bar{I}(Y_1),\, y + \bar{I}(Y_2))$ onto their difference $x - y + (\bar{I}(Y_1) + \bar{I}(Y_2))$. The kernel of i is equal to the intersection $\bar{I}(Y_1) \cap \bar{I}(Y_2)$, which is equal to $\bar{I}(Y_1 \cup Y_2) = \bar{I}(Y)$ by the basic properties of the homogeneous ideal. Hence, i is injective. Furthermore, all the elements in $S/\bar{I}(Y_1) \oplus S/\bar{I}(Y_2)$ which lie in the image of i are pairs of residue classes of elements of S that share a residue class in $S/\bar{I}(Y)$. Therefore, the image of i is exactly the kernel of

p, and as p is clearly surjective, the sequence is exact.
Remark 4.16 implies that

$$P_{S/\bar{I}(Y)} = P_{S/\bar{I}(Y_1)\oplus S/\bar{I}(Y_2)} - P_{S/(\bar{I}(Y_1)\cap\bar{I}(Y_2))}.$$

The same fact is applicable to the natural split sequence

$$0 \longrightarrow S/\bar{I}(Y_1) \hookrightarrow S\bar{I}(Y_1) \oplus S/\bar{I}(Y_2) \twoheadrightarrow S/\bar{I}(Y_2) \longrightarrow 0,$$

and thus $P_{S/\bar{I}(Y_1)\oplus S/\bar{I}(Y_2)} = P_{S/\bar{I}(Y_1)} + P_{S/\bar{I}(Y_2)}$. On the other hand, the projective variety $\bar{V}(\bar{I}(Y_1) + \bar{I}(Y_2)) = Y_1 \cap Y_2$ is of smaller dimension than Y by assumption. Hence, $P_{Y_1\cap Y_2}$ has degree $< r$. Therefore, the leading coefficient of P_Y is the sum of the leading coefficients of P_{Y_1} and P_{Y_2}.
(c) The Hilbert polynomial of $\mathbb{P}^n_k$ is P_S. For $l > 0$, the dimension of the components S_l is equal to $\binom{l+n}{n}$, which follows by counting the possible ways to form a monomial of degree d in $n+1$ variables. Therefore, $\varphi_S(l) = \binom{l+n}{n}$ for $l > 0$, and so $P_S(z) = \binom{z+n}{n}$, which is of degree n. The leading coefficient of P_S is thus equal to $\frac{1}{n!}$ and so $\deg P_S = 1$.
(d) Set $d := \deg(f)$, and consider the following short exact sequence of graded S-modules

$$0 \longrightarrow S(-d) \xhookrightarrow{\cdot f} S \twoheadrightarrow S/(f) \longrightarrow 0$$

where the multiplication by f is an injective graded homomorphism, for S is an integral domain. Hence, for $l \in \mathbb{Z}$, it follows that $\varphi_{S/(f)}(l) = \varphi_S(l) - \varphi_S(l-d)$. Therefore, the Hilbert polynomial of H can be written as

$$P_H(z) = \binom{z+n}{n} - \binom{z-d+n}{n} = \frac{d}{(n-1)!}z^{n-1} + \dots.$$

Hence, $\deg H = d$. □

6 The Intersection Theorem and Bézout

The previous sections provide the complete set-up needed to prove the main result of this thesis: a generalization of Bézout's Theorem to the intersection of an irreducible projective variety with an irreducible hypersurface in n-dimensional projective space.

Let $Y \subset \mathbb{P}^n_k$ be an irreducible projective variety of dimension r, and let H be an irreducible hypersurface not containing Y. Then, by the Projective Dimension Theorem, $Y \cap H = Z_1 \cup \ldots \cup Z_s$, where the Z_i are irreducible varieties of dimension $r-1$. Let $\mathfrak{p}_i$ be the homogeneous prime ideal of Z_i.

Definition 6.1. The *intersection multiplicity of Y and H along Z_i* is defined as

$$\iota(Y,H;Z_i) := \mu_{\mathfrak{p}_i}\left(S/(\bar{I}(Y)+\bar{I}(H))\right)$$

where $\mu_{\mathfrak{p}_i}$ is the multiplicity defined in 4.10.

Remark 6.2. The annihilator of the module $M = S/(\bar{I}(Y)+\bar{I}(H))$ is $\bar{I}(Y)+\bar{I}(H)$, and the corresponding projective variety $\bar{V}(\bar{I}(Y)+\bar{I}(H))$ is equal to $Y \cap H$. Therefore, $\bar{I}(Y)+\bar{I}(H) \subset \bar{I}(Y \cap H) \subset \mathfrak{p}_i$ and so, $\mathfrak{p}_i$ is a minimal associated prime of M, and thus the intersection multiplicity is well-defined.

Theorem 6.3. *Let Y be an irreducible projective variety of dimension greater or equal to 1 in $\mathbb{P}^n_k$, and let H be an irreducible hypersurface not containing Y. Let $Z_1,\dots,Z_s$ be the irreducible components of $Y \cap H$. Then*

$$\sum_{i=1}^{s} \iota(Y,H;Z_i)\cdot \deg Z_i = (\deg Y)(\deg H).$$

Proof. Let f be the irreducible homogeneous polynomial in S with $\bar{V}(f) = H$, and denote the degree of f by d. To simplify the notation, set $M := S/(\bar{I}(Y)+\bar{I}(H))$ and consider the following sequence of graded S-modules

$$0 \longrightarrow (S/\bar{I}(Y))(-d) \xrightarrow{\cdot f} S/\bar{I}(Y) \xrightarrow{p} M \longrightarrow 0.$$

The multiplication by f is a graded homomorphism, as $f\cdot (S/\bar{I}(Y))(-d)_l \subset (S/\bar{I}(Y))_l$ for any non-negative integer l. It is also injective, since f is not a zero divisor in S and f does not lie in $\bar{I}(Y)$ by assumption. Its image is the residue class of $\bar{I}(H)$ in $S/\bar{I}(Y)$. For the sequence to be exact, p projects the homogeneous coordinate ring of Y onto the module $(S/\bar{I}(Y))/\bar{I}(H)$. This module is equal to M, because for any ring R and any two ideals $\mathfrak{a},\mathfrak{b} \subset R$, the following equality holds:

$$(R/\mathfrak{a})\big/(\mathfrak{b}(R/\mathfrak{a})) = R/(\mathfrak{a}+\mathfrak{b}).$$

Regarding the Hilbert polynomials of the modules in the sequence above, Remarks 4.16 and 4.7 imply that

$$P_M(z) = P_Y(z) - P_Y(z-d). \tag{$\star$}$$

In particular, the leading coefficient of the right hand side equals the one of the left hand side.

Let Y have dimension r and degree e. Then, the leading coefficient of P_Y is $\frac{e}{r!}$ and hence the right hand side of ($\star$) is equal to

$$\frac{e}{r!}\,z^r + \ldots - \left(\frac{e}{r!}\,(z-d)^r + \ldots\right) = \frac{de}{(r-1)!}\,z^{r-1} + \ldots\ .$$

Note that by Proposition 5.3, $\deg(H) = d$, and therefore

$$\frac{de}{(r-1)!} = \frac{\deg(Y)\cdot\deg(H)}{(r-1)!}. \tag{$\star\star$}$$

To determine the leading coefficient of P_M, examine the module M. By Proposition 4.9, the module M has a filtration of submodules $0 = M^0 \subset M^1 \subset \ldots \subset M^t = M$, the quotients M^i/M^{i-1} of which are of the form $(S/\mathfrak{q}_i)(l_i)$ for $\mathfrak{q}_i$ a homogeneous ideal in S and an integer l_i. Applying Remark 4.16 to the short exact sequence

$$0 \longrightarrow M^{i-1} \hookrightarrow M^i \twoheadrightarrow M^i/M^{i-1} \longrightarrow 0$$

for all $1 \leq i \leq t$, it follows that $P_M = \sum_{i=1}^{t} P_i$, where P_i denotes the Hilbert polynomial of $(S/\mathfrak{q}_i)(l_i)$ and of the corresponding projective variety $\bar{V}(\mathfrak{q}_i)$. Note that the shift l_i does not affect the leading coefficient of P_i, as it only shifts the variable.
Denoting the dimension of $\bar{V}(\mathfrak{q}_i)$ by r_i and its degree by f_i , it follows that

$$P_i(z) = \frac{f_i}{r_i!}\,z^{r_i} + \ldots .$$

Since only the leading coefficient of P_M is of interest, the polynomials P_i of degree smaller than $r-1$ can be ignored. For any $\mathfrak{q}_j$ which is not minimal in $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$, there exists a $\mathfrak{q}_i$ such that $\mathfrak{q}_i \subsetneq \mathfrak{q}_j$, and thus $\bar{V}(\mathfrak{q}_i) \supsetneq \bar{V}(\mathfrak{q}_j)$, and by the Hilbert–Serre Theorem 4.12 also $\deg(P_i) > \deg(P_j)$. Thus, the remaining 'P_i's are exactly the ones for which the associated prime ideal $\mathfrak{q}_i$ is a minimal associated prime of M. Denote them by $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$, and note that by definition, these are exactly the minimal prime ideals over the annihilator of M, which is equal to $\bar{I}(Y)+\bar{I}(H)$. The corresponding projective variety $\bar{V}(\bar{I}(Y)+\bar{I}(H))$ is equal to the intersection of $\bar{V}(\bar{I}(Y)) = Y$, and $\bar{V}(\bar{I}(H)) = H$.
Hence, $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ are exactly the prime ideals corresponding to the irreducible components $Z_1, \ldots, Z_s$ of $Y \cap H$. According to Proposition 4.9, each of the irreducible components occurs exactly $\mu_{\mathfrak{p}_j}(M)$ times in the set of minimal associated primes of M. Furthermore, by the Projective Dimension Theorem 3.6, all of these have dimension $\geq r-1$. As H does not contain Y, the dimension of the intersection satisfies $\dim(Y \cap H) < \dim(Y) = r$,

and so $\dim(Z_i) = r - 1$ for all $1 \leq i \leq s$. Thus, the degree of $Y \cap H$ can be computed using Proposition 5.3:

$$\deg(Y \cap H) = \sum_{j=1}^{s} \iota(Y, H; Z_j) \cdot \deg Z_j.$$

The leading coefficient of P_M is then

$$\left(\sum_{j=1}^{s} \iota(Y, H; Z_j) \cdot \deg Z_j \right) \Big/ (r-1)!$$

Comparing with the equation $(\star\star)$, the result follows. $\square$

Corollary 6.4 (Bézout's Theorem)**.** *Let Y and Z be distinct irreducible curves in $\mathbb{P}_k^2$ of respective degrees d and e. Let $Y \cap Z$ be the set of points $\{P_1, \ldots, P_S\}$. Then*

$$\sum_{j=1}^{s} \iota(Y, Z; P_j) = de.$$

Proof. Note that a point P is a projective variety of dimension zero and thus has a constant Hilbert polynomial. Furthermore, the homogeneous coordinate ring of P is isomorphic to k with the trivial grading, and hence $\dim(k_0) = 1$ and $\dim(k_i) = 0$ for all $i > 0$. Thereby, P has Hilbert polynomial 1 and hence degree 1. The claim follows by applying Theorem 6.3. $\square$

Remark 6.5. The proof above extends to the case in which Y and Z are reducible curves, provided they have no common irreducible component. The points of intersection are then exactly the intersection points of all the irreducible components.

7 An Application of Bézout: Pascal's Theorem

The Theorem of Bézout is of great importance in the mathematical field of enumerative geometry and thus has many geometric applications. One of many, Pascal's Theorem concerned with the intersection points of opposite sides of a hexagon embedded in a conic section is introduced and proved, using mainly Bézout.

Hereafter, any curve of degree one in $\mathbb{P}_k^2$ is called a *line*, and one of degree two or three a *quadric* or *cubic*. In particular, lines, quadrics, and curves are allowed to be reducible.

A *conic* is the intersection of the surface of a cone with a plane. In the Euclidean plane, conics can be classified as ellipses, hyperbolas and parabolas. Embedded into the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ however, these three types become equivalent under basis change, and it is then unnecessary to treat them separately.

Lemma 7.1. *Let Q be a quadric in $\mathbb{P}^2_{\mathbb{C}}$. Choose six arbitrary distinct points on Q such that no four of them are collinear.*

(a) If three of the points $P_1, \ldots, P_6$ are collinear, the remaining three are collinear as well, and Q is the union of two lines; otherwise Q is irreducible.

(b) Q is uniquely determined by any five of the points $P_1, \ldots, P_6$.

Proof. (a) Let $P_1, \ldots, P_6$ be six distinct points on Q such that no four of them are collinear. If P_1, P_2, P_3 lie on a common line L, the intersection of L and Q contains at least three points. By Bézout's Theorem 6.4, it follows that Q and L therefore share an irreducible component, and thus $L \subset Q$. Consequently, Q is the union of two lines $L \cup L'$. As no four points of $P_1, \ldots, P_6$ are collinear by assumption, the points P_4, P_5, P_6 cannot lie on L. Therefore, the points P_4, P_5, P_6 lie on L' and are thereby collinear.
If no three points of $P_1, \ldots, P_6$ are collinear and Q is the union of two lines $L \cup L'$, at most two points lie on the lines L and L' each. Then, there exist integers $1 \leq i < j \leq 6$ such that $P_i, P_j \notin L \cup L' = Q$. This is a contradiction.
(b) Let Q' be another quadric containing any five of the points $P_1, \ldots, P_6$ and assume Q to be irreducible. Then the intersection $Q \cap Q'$ contains at least five points and thus they share an irreducible component by Bézout. It immediately follows that $Q = Q'$.
If Q is reducible, Q is decomposable into two distinct lines $L \cup L'$. The assertion (a) implies that exactly three points of $P_1, \ldots, P_6$ lie on each line L and L', and therefore Q' intersects at least one line in more than two points, say L. It follows by Bézout that $L \subset Q'$ and therefore, Q' is decomposable into two distinct lines L and L''. By (a), the remaining two intersection points of Q' and Q thus lie on L''. Therefore, L'' and L' intersect at at least two points. They coincide by Bézout, and hence $Q = Q'$. □

Proposition 7.2. *Consider two cubic curves in the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ intersecting at exactly nine distinct points. If six of them lie on a quadric, the remaining three intersection points are collinear.*

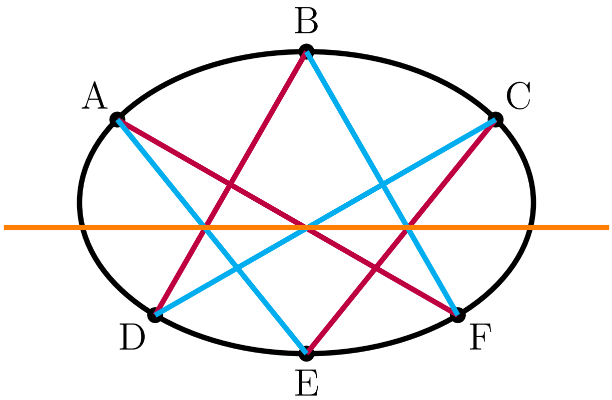
Proof. Let $X = \bar{V}(f)$ and $Y = \bar{V}(g)$ be the two cubics, and denote their points of intersection by $X \cap Y = \{P_1, \ldots, P_9\}$. As $X \cap Y$ contains exactly nine points, it follows by Bézout's Theorem 6.4 that X and Y share no irreducible component. Furthermore, no four of the nine points in $X \cap Y$ are collinear, otherwise both X and Y would have more than three intersection points with some line and by Bézout, this line then would lie in the intersection of X and Y, which contradicts the finiteness of $X \cap Y$. Similarly, no seven of the intersection points lie on a common quadric. Otherwise, X and Y would have more than six intersection points with some quadric, which then had to be contained in both X and Y by Bézout. This is again a contradiction to the finiteness of $X \cap Y$.

Let Q be a quadric containing $P_4, \ldots, P_9$.
In case Q has no common irreducible component with either X or Y, it is possible to choose an arbitrary point P on Q which does not lie in $X \cap Y$. As then neither f nor g is zero at P, set $\lambda := \frac{f(P)}{g(P)}$, which is a unit in $\mathbb{C}$. Consider now the linear combination $h := f - \lambda g$, and $Z := \bar{V}(h)$ the cubic curve it describes in $\mathbb{P}^2_{\mathbb{C}}$. Then Z intersects Q in at least seven points $\{P, P_4, \ldots, P_9\}$, and thus, by Bézout, Z shares an irreducible component with Q.
Assume this component to be a line. By Lemma 7.1, it follows that Q is the union of two lines, say $L_1 \cup L_2$ with $L_1 \subset Z$ such that three of the points $P_4 \ldots, P_9$ lay on the one line and the remaining three on the other. Therefore, Z is the union of a quadric Q' and L_1, with $Q' \cap L_2$ containing three distinct points. By Bézout, $L_2 \subset Q'$ and the whole quadric Q lies in Z. Hence, $Q \subset Z$ regardless of the irreducibility of Q.
It follows that $Z = Q \cup L$ for some line L. All points in the intersection of X and Y also clearly lie in Z. As argued above, no seven of the intersection points $X \cap Y$ lie on a quadric, and therefore none of the three remaining intersection points P_1, P_2, P_3 lies on Q. Hence, $\{P_1, P_2, P_3\} \subset L$.
If Q shares an irreducible component with X or Y, it follows by the argumentation above that Q completely lies in the said cubic. As previously, it follows that P_1, P_2, P_3 are collinear. □

In the special case in which the two cubic curves in Proposition 7.2 are both the union of three lines and the quadratic curve is a conic, the Theorem of Pascal follows.

Corollary 7.3 (Pascal's Theorem)**.** *Consider six arbitrary distinct points on a conic which are joined by line segments in any order to form a hexagon. Then the lines containing opposite sides of this hexagon intersect at three collinear points.*

Definition 7.4. The line in Pascal's Theorem 7.3 is called the *Pascal line*.



An example of the situation described in Pascal's Theorem 7.3 depicted in the Euclidean plane. The Pascal line is pictured in orange.

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