Monodromy Groups associated to Non-Isotrivial Drinfeld Modules in Generic Characteristic

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Summary. Let φ be a non-isotrivial family of Drinfeld A-modules of rank r in generic characteristic with a suitable level structure over a connected smooth algebraic variety X. Suppose that the endomorphism ring of φ is equal to A. Then we show that the closure of the analytic monodromy group of X in $\mathrm{SL}_r(\mathbb{A}_F^f)$ is open, where $\mathbb{A}_F^f$ denotes the ring of finite adèles of the quotient field F of A.

From this we deduce two further results: (1) If X is defined over a finitely generated field extension of F, the image of the arithmetic étale fundamental group of X on the adèlic Tate module of φ is open in $\mathrm{GL}_r(\mathbb{A}_F^f)$. (2) Let ψ be a Drinfeld A-module of rank r defined over a finitely generated field extension of F, and suppose that ψ cannot be defined over a finite extension of F. Suppose again that the endomorphism ring of ψ is A. Then the image of the Galois representation on the adèlic Tate module of ψ is open in $\mathrm{GL}_r(\mathbb{A}_F^f)$.

Finally, we extend the above results to the case of arbitrary endomorphism rings.

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1 Analytic monodromy groups

Let $\mathbb{F}_p$ be the finite prime field with p elements. Let F be a finitely generated field of transcendence degree 1 over $\mathbb{F}_p$. Let A be the ring of elements of F which are regular outside a fixed place ∞ of F. Let M be the fine moduli space over F of Drinfeld A-modules of rank r with some sufficiently high level structure. This is a smooth affine scheme of dimension $r-1$ over F.

Let F_∞ denote the completion of F at ∞, and $\mathbb{C}$ the completion of an algebraic closure of F_∞. Then the rigid analytic variety $M_{\mathbb{C}}^{\mathrm{an}}$ is a finite disjoint union of spaces of the form $\Delta\backslash\Omega$, where $\Omega \subset (\mathbb{P}^{r-1}_{\mathbb{C}})^{\mathrm{an}}$ is Drinfeld's upper half space and Δ is a congruence subgroup of $\mathrm{SL}_r(F)$ commensurable with $\mathrm{SL}_r(A)$.

Let $X_{\mathbb{C}}$ be a smooth irreducible locally closed algebraic subvariety of $M_{\mathbb{C}}$. Then $X^{\mathrm{an}}_{\mathbb{C}}$ lies in one of the components $\Delta\backslash\Omega$ of $M^{\mathrm{an}}_{\mathbb{C}}$. Fix an irreducible component $\Xi \subset \Omega$ of the pre-image of $X^{\mathrm{an}}_{\mathbb{C}}$. Then $\Xi \to X^{\mathrm{an}}_{\mathbb{C}}$ is an unramified Galois covering whose Galois group $\Delta_\Xi := \mathrm{Stab}_\Delta(\Xi)$ is a quotient of the analytic fundamental group of $X^{\mathrm{an}}_{\mathbb{C}}$.

Let φ denote the family of Drinfeld modules over $X_{\mathbb{C}}$ determined by the embedding $X_{\mathbb{C}} \subset M_{\mathbb{C}}$. We assume that $\dim X_{\mathbb{C}} \geq 1$. Since M is a fine moduli space, this means that φ is non-isotrivial. It also implies that $r \geq 2$. Let $\eta_{\mathbb{C}}$ be the generic point of $X_{\mathbb{C}}$ and $\bar\eta_{\mathbb{C}}$ a geometric point above it. Let $\varphi_{\bar\eta_{\mathbb{C}}}$ denote the pullback of φ to $\bar\eta_{\mathbb{C}}$. Let $\mathbb{A}^f_F$ denote the ring of finite adèles of F. The main result of this article is the following:

Theorem 1. *In the above situation, if* $\mathrm{End}_{\bar\eta_{\mathbb{C}}}(\varphi_{\bar\eta_{\mathbb{C}}}) = A$*, then the closure of* Δ_Ξ *in* $\mathrm{SL}_r(\mathbb{A}^f_F)$ *is an open subgroup of* $\mathrm{SL}_r(\mathbb{A}^f_F)$*.*

The proof uses known results on the $\mathfrak{p}$-adic Galois representations associated to Drinfeld modules [Pi97] and on strong approximation [Pi00].

Theorem 1 leaves open the following natural question:

Question 1. If $\mathrm{End}_{\bar\eta_{\mathbb{C}}}(\varphi_{\bar\eta_{\mathbb{C}}}) = A$, is Δ_Ξ an arithmetic subgroup of $\mathrm{SL}_r(F)$?

Theorem 1 has applications to the analogue of the André-Oort conjecture for Drinfeld moduli spaces: see [Br]. Consequences for étale monodromy groups and for Galois representations are explained in Sections 2 and 3. The proof of Theorem 1 will be given in Sections 4 through 7. Finally, in Section 8 we outline the case of arbitrary endomorphism rings.

For any variety Y over a field k and any extension field L of k we will abbreviate $Y_L := Y \times_k L$.

2 Étale monodromy groups

We retain the notations from Section 1. Let $k \subset \mathbb{C}$ be a subfield that is finitely generated over F, such that $X_{\mathbb{C}} = X \times_k \mathbb{C}$ for a subvariety $X \subset M_k$. Let K denote the function field of X and K^{sep} a separable closure of K. Then $\eta := \mathrm{Spec}\, K$ is the generic point of X and $\bar\eta := \mathrm{Spec}\, K^{\mathrm{sep}}$ a geometric point above η. Let k^{sep} be the separable closure of k in K^{sep}. Then we have a short exact sequence of étale fundamental groups

$$1 \longrightarrow \pi_1(X_{k^{\mathrm{sep}}}, \bar\eta) \longrightarrow \pi_1(X, \bar\eta) \longrightarrow \mathrm{Gal}(k^{\mathrm{sep}}/k) \to 1.$$

Let $\hat{A} \cong \prod_{\mathfrak{p}\neq\infty} A_\mathfrak{p}$ denote the profinite completion of A. Recall that $\mathbb{A}_F^f \cong F \otimes_A \hat{A}$ and contains $\hat{A}$ as an open subring. Let φ_η denote the Drinfeld module over K corresponding to η. Its adèlic Tate module $\hat{T}(\varphi_\eta)$ is a free module of rank r over $\hat{A}$. Choose a basis and let

$$\rho:\ \pi_1(X,\bar{\eta}) \longrightarrow \mathrm{GL}_r(\hat{A}) \subset \mathrm{GL}_r(\mathbb{A}_F^f)$$

denote the associated monodromy representation. Let $\Gamma^{\mathrm{geom}} \subset \Gamma \subset \mathrm{GL}_r(\hat{A})$ denote the images of $\pi_1(X_{k^{\mathrm{sep}}},\bar{\eta}) \subset \pi_1(X,\bar{\eta})$ under ρ.

Lemma 1. *Γ^{geom} is the closure of $g^{-1}\Delta_\Xi\, g$ in $\mathrm{SL}_r(\hat{A})$ for some element $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$.*

Proof. Choose an embedding $K^{\mathrm{sep}} \hookrightarrow \mathbb{C}$ and a point $\xi \in \Xi$ above $\bar{\eta}$. Let $\Lambda \subset F^r$ be the lattice corresponding to the Drinfeld module at ξ. This is a finitely generated projective A-module of rank r. The choice of a basis of $\hat{T}(\varphi_\eta)$ yields a composite embedding

$$\hat{A}^r \ \cong\ \hat{T}(\varphi_\eta)\ \cong\ \Lambda \otimes_A \hat{A}\ \hookrightarrow\ F^r \otimes_A \hat{A}\ \cong\ (\mathbb{A}_F^f)^r,$$

which is given by left multiplication with some element $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$. Since the discrete group $\Delta \subset \mathrm{SL}_r(F)$ preserves Λ, we have $g^{-1}\Delta g \subset \mathrm{SL}_r(\hat{A})$.

For any non-zero ideal $\mathfrak{a} \subset A$ let $M(\mathfrak{a})$ denote the moduli space obtained from M by adjoining a full level $\mathfrak{a}$ structure. Then $\pi_\mathfrak{a}\colon M(\mathfrak{a}) \twoheadrightarrow M$ is an étale Galois covering with group contained in $\mathrm{GL}_r(A/\mathfrak{a})$, and one of the connected components of $M(\mathfrak{a})^{\mathrm{an}}_\mathbb{C}$ above the connected component $\Delta\backslash\Omega$ of $M^{\mathrm{an}}_\mathbb{C}$ has the form $\Delta(\mathfrak{a})\backslash\Omega$ for

$$\Delta(\mathfrak{a}) := \big\{\delta \in \Delta \bigm| g^{-1}\delta g \equiv \mathrm{id} \bmod \mathfrak{a}\hat{A}\big\}.$$

Let $X(\mathfrak{a})_{k^{\mathrm{sep}}}$ be any connected component of the inverse image $\pi_\mathfrak{a}^{-1}(X_{k^{\mathrm{sep}}}) \subset M(\mathfrak{a})_{k^{\mathrm{sep}}}$. Since k^{sep} is separably closed, the variety $X(\mathfrak{a})_\mathbb{C}$ over $\mathbb{C}$ obtained by base change is again connected. The associated rigid analytic variety $X(\mathfrak{a})^{\mathrm{an}}_\mathbb{C}$ is then also connected (cf. [Lü74, Kor. 3.5]) and therefore a connected component of $\pi_\mathfrak{a}^{-1}(X^{\mathrm{an}}_\mathbb{C})$. But one of these connected components is $\big(\Delta_\Xi \cap \Delta(\mathfrak{a})\big)\backslash\Xi$, whose Galois group over $X^{\mathrm{an}}_\mathbb{C} \cong \Delta_\Xi\backslash\Xi$ is $\Delta_\Xi/\big(\Delta_\Xi \cap \Delta(\mathfrak{a})\big)$. This implies that $g^{-1}\Delta_\Xi\, g$ and $\pi_1(X_{k^{\mathrm{sep}}},\bar{\eta})$ have the same images in $\mathrm{GL}_r(A/\mathfrak{a}) = \mathrm{GL}_r(\hat{A}/\mathfrak{a}\hat{A})$. By taking the inverse limit over the ideal $\mathfrak{a}$ we deduce that the closure of $g^{-1}\Delta_\Xi\, g$ in $\mathrm{SL}_r(\hat{A})$ is Γ^{geom}, as desired. □

Lemma 2. $\mathrm{End}_{K^{\mathrm{sep}}}(\varphi_\eta) = \mathrm{End}_{\bar{\eta}_\mathbb{C}}(\varphi_{\bar{\eta}_\mathbb{C}})$.

Proof. By construction $\bar{\eta}_\mathbb{C}$ is a geometric point above η, and $\varphi_{\bar{\eta}_\mathbb{C}}$ is the pullback of φ_η. Any embedding of K^{sep} into the residue field of $\bar{\eta}_\mathbb{C}$ induces a morphism $\bar{\eta}_\mathbb{C} \to \bar{\eta}$. Thus the assertion follows from the fact that for every Drinfeld module over a field, any endomorphism defined over any field extension is already defined over a finite separable extension. □

Theorem 2. *In the above situation, suppose that* $\mathrm{End}_{K^{\mathrm{sep}}}(\varphi_\eta) = A$. *Then*

(a) Γ^{geom} *is an open subgroup of* $\mathrm{SL}_r(\mathbb{A}_F^f)$*, and*
(b) Γ *is an open subgroup of* $\mathrm{GL}_r(\mathbb{A}_F^f)$.

Proof. By Lemma 2 the assumption implies that $\mathrm{End}_{\bar{\eta}_\mathbb{C}}(\varphi_{\bar{\eta}_\mathbb{C}}) = A$. Thus part (a) follows at once from Theorem 1 and Lemma 1. Part (b) follows from (a) and the fact that $\det(\Gamma)$ is open in $\mathrm{GL}_1(\mathbb{A}_F^f)$. This fact is a consequence of work of Drinfeld [Dr74, §8 Thm. 1] and Hayes [Ha79, Thm. 9.2] on the abelian class field theory of F, and of Anderson [An86] on the determinant Drinfeld module. Note that Anderson's paper only treats the case $A = \mathbb{F}_q[T]$; the general case has been worked out by van der Heiden [He03, Chap. 4]. Compare also [Pi97, Thm. 1.8]. □

3 Galois groups

Let F and A be as in Section 1. Let K be a finitely generated extension field of F of arbitrary transcendence degree, and let $\psi : A \to K\{\tau\}$ be a Drinfeld A-module of rank r over K. Let K^{sep} denote a separable closure of K and

$$\sigma : \mathrm{Gal}(K^{\mathrm{sep}}/K) \longrightarrow \mathrm{GL}_r(\mathbb{A}_F^f)$$

the natural representation on the adèlic Tate module of ψ. Let $\Gamma \subset \mathrm{GL}_r(\mathbb{A}_F^f)$ denote its image.

Theorem 3. *In the above situation, suppose that* $\mathrm{End}_{K^{\mathrm{sep}}}(\psi) = A$ *and that* ψ *cannot be defined over a finite extension of* F *inside* K^{sep}. *Then* Γ *is an open subgroup of* $\mathrm{GL}_r(\mathbb{A}_F^f)$.

Proof. The assertion is invariant under replacing K by a finite extension. We may therefore assume that ψ possesses a sufficiently high level structure over K. Then ψ corresponds to a K-valued point on the moduli space M from Section 1. Let η denote the underlying point on the scheme M, and let $L \subset K$ be its residue field. Then ψ is already defined over L, and σ factors through the natural homomorphism $\mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Gal}(L^{\mathrm{sep}}/L)$, where L^{sep} is the separable closure of L in K^{sep}. Since K is finitely generated over L, the intersection $K \cap L^{\mathrm{sep}}$ is finite over L; hence the image of this homomorphism is open. To prove the theorem we may thus replace K by L, after which K is the residue field of η.

The assumption on ψ implies that even after this reduction, K is not a finite extension of F. Therefore its transcendence degree over F is ≥ 1. Let k denote the algebraic closure of F in K. Then η can be viewed as the generic point of a geometrically irreducible and reduced locally closed algebraic subvariety $X \subset M_k$ of dimension ≥ 1. After shrinking X we may assume that

X is smooth. We are then precisely in the situation of the preceding section, with $\psi = \varphi_\eta$. The homomorphism σ above is then the composite

$$\mathrm{Gal}(K^{\mathrm{sep}}/K) \cong \pi_1(\eta, \bar\eta) \twoheadrightarrow \pi_1(X, \bar\eta) \stackrel{\rho}{\to} \mathrm{GL}_r(\mathbb{A}_F^f)$$

with ρ as in Section 2. It follows that the groups called Γ in this section and the last coincide. The desired openness is now equivalent to Theorem 2 (b). □

Note. The adèlic openness for a Drinfeld module ψ as in Theorem 3, but defined over a *finite* extension of F, is conjectured yet still unproved.

4 $\mathfrak{p}$-Adic openness

This section and the next three are devoted to proving Theorem 1. Throughout we retain the notations from Sections 1 and 2 and the assumptions $\dim X \geq 1$ and $\mathrm{End}_{\bar\eta_{\mathbb{C}}}(\varphi_{\bar\eta_{\mathbb{C}}}) = A$. In this section we recall a known result on $\mathfrak{p}$-adic openness. For any place $\mathfrak{p} \neq \infty$ of F let $\Gamma_\mathfrak{p}$ denote the image of Γ under the projection $\mathrm{GL}_r(\mathbb{A}_F^f) \twoheadrightarrow \mathrm{GL}_r(F_\mathfrak{p})$.

Theorem 4. *$\Gamma_\mathfrak{p}$ is open in $\mathrm{GL}_r(F_\mathfrak{p})$.*

Proof. By construction $\Gamma_\mathfrak{p}$ is the image of the monodromy representation

$$\rho_\mathfrak{p} : \pi_1(X, \bar\eta) \longrightarrow \mathrm{GL}_r(F_\mathfrak{p})$$

on the rational $\mathfrak{p}$-adic Tate module of φ_η. This is the same as the image of the composite homomorphism

$$\mathrm{Gal}(K^{\mathrm{sep}}/K) \cong \pi_1(\eta, \bar\eta) \twoheadrightarrow \pi_1(X, \bar\eta) \stackrel{\rho_\mathfrak{p}}{\to} \mathrm{GL}_r(F_\mathfrak{p}).$$

Since K is a finitely generated extension of F, and $\mathrm{End}_{K^{\mathrm{sep}}}(\varphi_\eta) = A$ by the assumption and Lemma 2, the desired openness is a special case of [Pi97, Thm. 0.1]. □

Next let $\Gamma_\mathfrak{p}^{\mathrm{geom}}$ denote the image of Γ^{geom} under the projection $\mathrm{GL}_r(\mathbb{A}_F^f) \twoheadrightarrow \mathrm{GL}_r(F_\mathfrak{p})$. Note that this is a normal subgroup of $\Gamma_\mathfrak{p}$. Lemma 1 immediately implies:

Lemma 3. *$\Gamma_\mathfrak{p}^{\mathrm{geom}}$ is the closure of $g^{-1}\Delta_\Xi\, g$ in $\mathrm{SL}_r(F_\mathfrak{p})$ for some element $g \in \mathrm{GL}_r(F_\mathfrak{p})$.*

5 Zariski density

Lemma 4. *The Zariski closure H of Δ_Ξ in* $\mathrm{GL}_{r,F}$ *is a normal subgroup of* $\mathrm{GL}_{r,F}$.

Proof. Choose a place $\mathfrak{p} \neq \infty$ of F. Then by base extension $H_{F_\mathfrak{p}}$ is the Zariski closure of Δ_Ξ in $\mathrm{GL}_{r,F_\mathfrak{p}}$. Thus Lemma 3 implies that $g^{-1}H_{F_\mathfrak{p}}g$ is the Zariski closure of $\Gamma_\mathfrak{p}^{\mathrm{geom}}$ in $\mathrm{GL}_{r,F_\mathfrak{p}}$. Since $\Gamma_\mathfrak{p}$ normalizes $\Gamma_\mathfrak{p}^{\mathrm{geom}}$, it therefore normalizes $g^{-1}H_{F_\mathfrak{p}}g$. But $\Gamma_\mathfrak{p}$ is open in $\mathrm{GL}_r(F_\mathfrak{p})$ by Theorem 4 and therefore Zariski dense in $\mathrm{GL}_{r,F_\mathfrak{p}}$. Thus $\mathrm{GL}_{r,F_\mathfrak{p}}$ normalizes $g^{-1}H_{F_\mathfrak{p}}g$ and hence $H_{F_\mathfrak{p}}$, and the result follows. □

Lemma 5. *Δ_Ξ is infinite.*

Proof. Let X, K, k and φ_η be as in Section 2. Then, as M_k is affine and $\dim X \geq 1$, there exists a valuation v of K, corresponding to a point on the boundary of X not on M_k, at which φ_η does not have potential good reduction. Denote by $I_v \subset \mathrm{Gal}(K^{\mathrm{sep}}/Kk^{\mathrm{sep}})$ the inertia group at v. By the criterion of Néron-Ogg-Shafarevich [Go96, §4.10], the image of I_v in $\Gamma_\mathfrak{p}^{\mathrm{geom}}$ is infinite for any place $\mathfrak{p} \neq \infty$ of F. In particular, Δ_Ξ is infinite by Lemma 3, as desired.

Alternatively, we may argue as follows. Suppose that Δ_Ξ is finite. Then after increasing the level structure we may assume that $\Delta_\Xi = 1$. Then $\Gamma_\mathfrak{p}^{\mathrm{geom}} = 1$ by Lemma 3, which means that $\rho_\mathfrak{p}$ factors as

$$\pi_1(X, \bar{\eta}) \twoheadrightarrow \mathrm{Gal}(k^{\mathrm{sep}}/k) \longrightarrow \mathrm{GL}_r(F_\mathfrak{p}).$$

After a suitable finite extension of the constant field k we may assume that X possesses a k-rational point x. Let φ_x denote the Drinfeld module over k corresponding to x. Via the embedding $k \subset K$ we may consider it as a Drinfeld module over K and compare it with φ_η. The factorization above implies that the Galois representations on the $\mathfrak{p}$-adic Tate modules of φ_x and φ_η are isomorphic. By the Tate conjecture (see [Tag95] or [Tam95]) this implies that there exists an isogeny $\varphi_x \to \varphi_\eta$ over K. Its kernel is finite and therefore defined over some finite extension k' of k. Thus φ_η, as a quotient of φ_x by this kernel, is isomorphic to a Drinfeld module defined over k'. But the assumption $\dim X \geq 1$ implies that η is not a closed point of M_k; hence φ_η cannot be defined over a finite extension of k. This is a contradiction. □

Proposition 1. *Δ_Ξ is Zariski dense in* $\mathrm{SL}_{r,F}$.

Proof. By construction we have $H \subset \mathrm{SL}_{r,F}$, and Lemma 5 implies that H is not contained in the center of $\mathrm{SL}_{r,F}$. From Lemma 4 it now follows that $H = \mathrm{SL}_{r,F}$, as desired. □

The above results may be viewed as analogues of André's results [An92, Thm. 1, Prop. 2], comparing the monodromy group of a variation of Hodge structures with its generic Mumford-Tate group. Our analogue of the former is Δ_Ξ, and by [Pi97] the latter corresponds to $\mathrm{GL}_{r,F}$. In our situation, however, we do not need the existence of a special point on X.

6 Fields of coefficients

Let $\bar{\Delta}_\Xi$ denote the image of Δ_Ξ in $\mathrm{PGL}_r(F)$. In this section we show that the field of coefficients of $\bar{\Delta}_\Xi$ cannot be reduced.

Definition 1. *Let L_1 be a subfield of a field L. We say that a subgroup $\bar{\Delta} \subset \mathrm{PGL}_r(L)$ lies in a model of $\mathrm{PGL}_{r,L}$ over L_1, if there exist a linear algebraic group G_1 over L_1 and an isomorphism $\lambda_1 : G_{1,L} \xrightarrow{\sim} \mathrm{PGL}_{r,L}$, such that $\bar{\Delta} \subset \lambda_1(G_1(L_1))$.*

Proposition 2. *$\bar{\Delta}_\Xi$ does not lie in a model of $\mathrm{PGL}_{r,F}$ over a proper subfield of F.*

Proof. As before we use an arbitrary auxiliary place $\mathfrak{p} \neq \infty$ of F. Let $\bar{\Gamma}_\mathfrak{p}^{\mathrm{geom}} \triangleleft \bar{\Gamma}_\mathfrak{p}$ denote the images of $\Gamma_\mathfrak{p}^{\mathrm{geom}} \triangleleft \Gamma_\mathfrak{p}$ in $\mathrm{PGL}_r(F_\mathfrak{p})$. Lemma 3 implies that $\bar{\Gamma}_\mathfrak{p}^{\mathrm{geom}}$ is conjugate to the closure of $\bar{\Delta}_\Xi$ in $\mathrm{PGL}_r(F_\mathfrak{p})$. By Proposition 1 it is therefore Zariski dense in $\mathrm{PGL}_{r,F_\mathfrak{p}}$. On the other hand Theorem 4 implies that $\bar{\Gamma}_\mathfrak{p}$ is an open subgroup of $\mathrm{PGL}_r(F_\mathfrak{p})$. It therefore does not lie in a model of $\mathrm{PGL}_{r,F_\mathfrak{p}}$ over a proper subfield of $F_\mathfrak{p}$. Thus $\bar{\Gamma}_\mathfrak{p}^{\mathrm{geom}}$ is Zariski dense and normal in a subgroup that does not lie in a model over a proper subfield of $F_\mathfrak{p}$, which by [Pi98, Cor. 3.8] implies that $\bar{\Gamma}_\mathfrak{p}^{\mathrm{geom}}$, too, does not lie in a model over a proper subfield of $F_\mathfrak{p}$.

Suppose now that $\bar{\Delta}_\Xi \subset \lambda_1(G_1(F_1))$ for a subfield $F_1 \subset F$, a linear algebraic group G_1 over F_1, and an isomorphism $\lambda_1 : G_{1,F} \xrightarrow{\sim} \mathrm{PGL}_{r,F}$. Since $\bar{\Delta}_\Xi$ is Zariski dense in $\mathrm{PGL}_{r,F}$, it is in particular infinite. Therefore F_1 must be infinite. As F is finitely generated of transcendence degree 1 over $\mathbb{F}_p$, it follows that F_1 contains a transcendental element, and so F is a finite extension of F_1. Let $\mathfrak{p}_1$ denote the place of F_1 below $\mathfrak{p}$. Since $\bar{\Gamma}_\mathfrak{p}^{\mathrm{geom}}$ is the closure of $\bar{\Delta}_\Xi$ in $\mathrm{PGL}_r(F_\mathfrak{p})$, it is contained in $\lambda_1(G_1(F_{1,\mathfrak{p}_1}))$. The fact that $\bar{\Gamma}_\mathfrak{p}^{\mathrm{geom}}$ does not lie in a model over a proper subfield of $F_\mathfrak{p}$ thus implies that $F_{1,\mathfrak{p}_1} = F_\mathfrak{p}$.

But for any proper subfield $F_1 \subsetneqq F$, we can choose a place $\mathfrak{p} \neq \infty$ of F above a place $\mathfrak{p}_1$ of F_1, such that the local field extension $F_{1,\mathfrak{p}_1} \subset F_\mathfrak{p}$ is non-trivial. Thus we must have $F_1 = F$, as desired. □

7 Strong approximation

The remaining ingredient is the following general theorem.

Theorem 5. *For $r \geq 2$ let $\Delta \subset \mathrm{SL}_r(F)$ be a subgroup that is contained in a congruence subgroup commensurable with $\mathrm{SL}_r(A)$. Assume that Δ is Zariski dense in $\mathrm{SL}_{r,F}$ and that its image $\bar{\Delta}$ in $\mathrm{PGL}_r(F)$ does not lie in a model of $\mathrm{PGL}_{r,F}$ over a proper subfield of F. Then the closure of Δ in $\mathrm{SL}_r(\mathbb{A}_F^f)$ is open.*

Proof. For finitely generated subgroups this is a special case of [Pi00, Thm. 0.2]. That result concerns arbitrary finitely generated Zariski dense subgroups of $G(F)$ for arbitrary semisimple algebraic groups G, but it uses the finite generation only to guarantee that the subgroup is integral at almost all places of F. For Δ as above the integrality at all places $\neq \infty$ is already known in advance, so the proof in [Pi00] covers this case as well.

As an alternative, we will deduce the general case by showing that every sufficiently large finitely generated subgroup $\Delta_1 \subset \Delta$ satisfies the same assumptions. Then the closure of Δ_1 in $\mathrm{SL}_r(\mathbb{A}_F^f)$ is open by [Pi00], and so the same follows for Δ, as desired.

For the Zariski density of Δ_1 note first that the trace of the adjoint representation defines a dominant morphism to the affine line $\mathrm{SL}_{r,F} \to \mathbb{A}_F^1$, $g \mapsto \mathrm{tr}(\mathrm{Ad}(g))$. Since Δ is Zariski dense, this function takes infinitely many values on Δ. As the field of constants in F is finite, we may therefore choose an element $\gamma \in \Delta$ with $\mathrm{tr}(\mathrm{Ad}(\gamma))$ transcendental. Then γ has infinite order; hence the Zariski closure $H \subset \mathrm{SL}_{r,F}$ of the abstract subgroup generated by γ has positive dimension. Let H° denote its identity component. Since Δ is Zariski dense and $\mathrm{SL}_{r,F}$ is almost simple, the Δ-conjugates of H° generate $\mathrm{SL}_{r,F}$ as an algebraic group. By noetherian induction finitely many conjugates suffice. It follows that finitely many conjugates of γ generate a Zariski dense subgroup of $\mathrm{SL}_{r,F}$. Thus every sufficiently large finitely generated subgroup $\Delta_1 \subset \Delta$ is Zariski dense.

Consider such Δ_1 and let $\bar{\Delta}_1$ denote its image in $\mathrm{PGL}_r(F)$. Consider all triples (F_1, G_1, λ_1) consisting of a subfield $F_1 \subset F$, a linear algebraic group G_1 over F_1, and an isomorphism $\lambda_1 : G_{1,F} \xrightarrow{\sim} \mathrm{PGL}_{r,F}$, such that $\bar{\Delta}_1 \subset \lambda_1(G_1(F_1))$. By [Pi98, Thm. 3.6] there exists such a triple with F_1 minimal, and this F_1 is unique, and G_1 and λ_1 are determined up to unique isomorphism. Consider another finitely generated subgroup $\Delta_1 \subset \Delta_2 \subset \Delta$ and let (F_2, H_2, λ_2) be the minimal triple associated to it. Then the uniqueness of (F_1, G_1, λ_1) implies that $F_1 \subset F_2$, that $G_2 \cong G_{1,F_2}$, and that λ_2 coincides with the isomorphism $G_{2,F} \cong G_{1,F} \to \mathrm{PGL}_{r,F}$ obtained from λ_1. In other words, the minimal model (F_1, G_1, λ_1) is monotone in Δ_1.

For any increasing sequence of Zariski dense finitely generated subgroups of Δ we thus obtain an increasing sequence of subfields of F. This sequence must become constant, say equal to $F_1 \subset F$, and the associated model of $\mathrm{PGL}_{r,F}$ over F_1 is the same up to isomorphism from that point onwards. Thus we have a triple (F_1, G_1, λ_1) with $\bar{\Delta}_1 \subset \lambda_1(G_1(F_1))$ for every sufficiently large finitely generated subgroup $\bar{\Delta}_1 \subset \bar{\Delta}$. But then we also have $\bar{\Delta} \subset \lambda_1(G_1(F_1))$, which by assumption implies that $F_1 = F$. Thus every sufficiently large finitely generated subgroup of Δ satisfies the same assumptions as Δ, as desired. □

Proof of Theorem 1. In the situation of Theorem 1 we automatically have $r \geq 2$, so the assertion follows by combining Propositions 1 and 2 with Theorem 5 for Δ_Ξ. □

8 Arbitrary endomorphism rings

Set $E := \mathrm{End}_{\bar{\eta}_{\mathbb{C}}}(\varphi_{\bar{\eta}_{\mathbb{C}}})$, which is a finite integral ring extension of A. Write $r = r' \cdot [E/A]$; then the centralizer of E in $\mathrm{GL}_r(\mathbb{A}_F^f)$ is isomorphic to $\mathrm{GL}_{r'}(E \otimes_A \mathbb{A}_F^f)$. Lemma 2 implies that all elements of E are defined over some fixed finite extension of K. This means that an open subgroup of $\rho\big(\pi_1(X, \bar{\eta})\big)$ is contained in $\mathrm{GL}_{r'}(E \otimes_A \mathbb{A}_F^f)$. Thus by Lemma 1 the same holds for a subgroup of finite index of Δ_Ξ. The following results can be deduced easily from Theorems 1, 2, and 3, using the same arguments as in [Pi97, end of §2].

Theorem 6. *In the situation of before Theorem 1, for* $E := \mathrm{End}_{\bar{\eta}_{\mathbb{C}}}(\varphi_{\bar{\eta}_{\mathbb{C}}})$ *arbitrary, the closure in* $\mathrm{GL}_r(\mathbb{A}_F^f)$ *of some subgroup of finite index of* Δ_Ξ *is an open subgroup of* $\mathrm{SL}_{r'}(E \otimes_A \mathbb{A}_F^f)$.

Theorem 7. *In the situation of before Theorem 2, for* $E := \mathrm{End}_{K^{\mathrm{sep}}}(\varphi_\eta)$ *arbitrary,*

(a) some open subgroup of $\Gamma^{\mathrm{geom}} := \rho\big(\pi_1(X_{k^{\mathrm{sep}}}, \bar{\eta})\big)$ *is an open subgroup of* $\mathrm{SL}_{r'}(E \otimes_A \mathbb{A}_F^f)$*, and*
(b) some open subgroup of $\Gamma := \rho\big(\pi_1(X, \bar{\eta})\big)$ *is an open subgroup of* $\mathrm{GL}_{r'}(E \otimes_A \mathbb{A}_F^f)$.

Theorem 8. *In the situation of before Theorem 3, for* $E := \mathrm{End}_{K^{\mathrm{sep}}}(\psi)$ *arbitrary, suppose that* ψ *cannot be defined over a finite extension of* F *inside* K^{sep}*. Then some open subgroup of* $\Gamma := \sigma\big(\mathrm{Gal}(K^{\mathrm{sep}}/K)\big)$ *is an open subgroup of* $\mathrm{GL}_{r'}(E \otimes_A \mathbb{A}_F^f)$.

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