The Galois Representations Associated to a Drinfeld Module in Special Characteristic, I: Zariski Density

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Abstract

Let φ be a Drinfeld A-module of rank r over a finitely generated field K. Assume that φ has special characteristic $\mathfrak{p}_0$ and consider any prime $\mathfrak{p} \neq \mathfrak{p}_0$ of A. If $\mathrm{End}_{K^{\mathrm{sep}}}(\varphi) = A$, we prove that the image of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ in its representation on the $\mathfrak{p}$-adic Tate module of φ is Zariski dense in GL_r.

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1 The main result

Let $\mathbb{F}_q$ be a finite field with q elements and of characteristic p. Let F be a finitely generated field of transcendence degree 1 over its constant field $\mathbb{F}_q$. Let A be the ring of elements of F which are regular outside a fixed place ∞ of F. Let K be another finitely generated field over $\mathbb{F}_q$ of arbitrary transcendence degree. Then the ring of $\mathbb{F}_q$-linear endomorphisms of the additive algebraic group over K is the non-commutative polynomial ring in one variable $K\{\tau\}$, where τ represents the endomorphism $u \mapsto u^q$ and satisfies the commutation relation $\tau u = u^q \tau$ for all $u \in K$. Consider a Drinfeld A-module

$$\varphi :\ A \to \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,K}) \cong K\{\tau\},\ a \mapsto \varphi_a$$

of rank $r \geq 1$ over K. (For the general theory of Drinfeld modules see Drinfeld [2], [3] or, e.g., Goss [5, §4]). Throughout this article we assume that φ has special characteristic. This means that the kernel $\mathfrak{p}_0$ of the homomorphism $A \to K$ determined by the lowest coefficient of φ is non-zero and therefore a maximal ideal of A.

Let $\mathfrak{p} \subset A$ be any maximal ideal different from $\mathfrak{p}_0$ and let $A_\mathfrak{p} \subset F_\mathfrak{p}$ denote the completions of $A \subset F$ at $\mathfrak{p}$. Let K^{sep} be a separable closure of K. Then the $\mathfrak{p}$-power torsion points of φ over K^{sep} form an A-module

$$\varphi(K^{\mathrm{sep}})[\mathfrak{p}^\infty] := \big\{x \in K^{\mathrm{sep}} \ \big|\ \exists i \geq 0\ \forall a \in \mathfrak{p}^i :\ \varphi_a(x) = 0\big\}$$

that is isomorphic to a direct sum of r copies of $F_\mathfrak{p}/A_\mathfrak{p}$. Thus the rational $\mathfrak{p}$-adic Tate module

$$V_\mathfrak{p}(\varphi) := \mathrm{Hom}_{A_\mathfrak{p}}\big(F_\mathfrak{p}, \varphi(K^{\mathrm{sep}})[\mathfrak{p}^\infty]\big)$$

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is an $F_\mathfrak{p}$-vector space of dimension r. The natural action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on $\varphi(K^{\mathrm{sep}})[\mathfrak{p}^\infty]$ translates into a continuous representation

$$\rho_\mathfrak{p} : \mathrm{Gal}(K^{\mathrm{sep}}/K) \longrightarrow \mathrm{Aut}_{F_\mathfrak{p}}\big(V_\mathfrak{p}(\varphi)\big) \cong \mathrm{GL}_r(F_\mathfrak{p}).$$

Let $\Gamma_\mathfrak{p} \subset \mathrm{GL}_r(F_\mathfrak{p})$ denote its image. The aim of this article is to prove:

Theorem 1.1 *If* $\mathrm{End}_{K^{\mathrm{sep}}}(\varphi) = A$, *then* $\Gamma_\mathfrak{p}$ *is Zariski dense in* GL_r.

The analogous result for Drinfeld modules in generic characteristic was proved in [7]. Both proofs rely on

- results of Taguchi and Tamagawa on the absolute irreducibility of $\rho_\mathfrak{p}$ (see §2),
- known facts on the valuations of Frobenius eigenvalues (see §3), and
- the classification of certain representations of linear algebraic groups (see §6).

In generic characteristic one first shows that φ has good ordinary reduction at many places of K. The Frobenius element at any such place has precisely one eigenvalue which is not a unit at $\mathfrak{p}_0$; only a little representation theory suffices to deduce from this that the Zariski closure of $\Gamma_\mathfrak{p}$ is GL_r. But in special characteristic one cannot argue like this (unless φ itself is ordinary), which makes things significantly more difficult. The main additional tools needed are

- an adaptation of Serre's theory of Frobenius tori (see §4),
- the formalism and basic properties of Anderson's t-motives (see §5),
- the construction of certain t-submotives of tensor powers of φ that are characterized by representation theoretic data alone (see Proposition 5.6) and an integrality result for them (Proposition 5.3), and
- finer results from representation theory (see §6).

The actual proof of Theorem 1.1 is given in §7.

Notations: The above notations remain in force throughout the paper. Furthermore, for any field L we let $L^{\mathrm{sep}} \subset \bar{L}$ denote a separable, respectively an algebraic closure of L. For any field extension L'/L and any algebraic group H over L we abbreviate $H_{L'} := H \times_L L'$. The character group of H is defined as $X(H) := \mathrm{Hom}(H_{\bar{L}}, \mathbb{G}_{m,\bar{L}})$. The cocharacter group of a torus T over L is defined as $Y(T) := \mathrm{Hom}(\mathbb{G}_{m,\bar{L}}, T_{\bar{L}})$. The corresponding $\mathbb{Q}$-vector spaces are denoted $X(H)_\mathbb{Q} := X(H) \otimes_\mathbb{Z} \mathbb{Q}$, respectively $Y(T)_\mathbb{Q} := Y(T) \otimes_\mathbb{Z} \mathbb{Q}$.

2 Absolute irreducibility

The following facts are known:

Theorem 2.1 *The representation of* $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ *on* $V_\mathfrak{p}(\varphi)$ *is semisimple over* $F_\mathfrak{p}$.

Proof. For K of transcendence degree 1 over $\mathbb{F}_q$ the theorem was proved by Taguchi [11, Th.0.1]. His proof trivially applies to finite K as well, and it can be extended easily to arbitrary transcendence degree. But one can also reduce the case of transcendence degree > 1 to the case of transcendence degree 1, as in [7, Th.1.4].

For this note first that the semisimplicity of the action of a subgroup $\Delta \subset \mathrm{GL}_r(F_\mathfrak{p})$ depends only on the subalgebra $F_\mathfrak{p}\Delta$ of the matrix algebra $\mathrm{Mat}_{r\times r}(F_\mathfrak{p})$. By [7, Lemma 1.5] there exists an open normal subgroup $\Gamma_1 \subset \Gamma_\mathfrak{p}$ such that for any subgroup $\Delta \subset \Gamma_\mathfrak{p}$ with $\Delta\Gamma_1 = \Gamma_\mathfrak{p}$ we have $F_\mathfrak{p}\Delta = F_\mathfrak{p}\Gamma_\mathfrak{p}$. Let K_1 be the finite Galois extension of K corresponding to the open subgroup $\rho_\mathfrak{p}^{-1}(\Gamma_1) \subset \mathrm{Gal}(K^{\mathrm{sep}}/K)$. Let X be a model of K of finite type over $\mathbb{F}_q$ over which φ has good reduction (cf. §3), and let $\pi: X_1 \to X$ be the normalization of X in K_1. By standard Bertini type arguments as in [7, Lemma 1.6] one finds an irreducible closed curve $Y \subset X$ for which $\pi^{-1}(Y) \subset X_1$ is irreducible.

Let y be the generic point of Y with function field L and φ_y the reduction of φ over L. Then the characteristic of φ_y is still $\mathfrak{p}_0 \neq \mathfrak{p}$ and the reduction map induces an isomorphism of Tate modules $V_\mathfrak{p}(\varphi) \xrightarrow{\sim} V_\mathfrak{p}(\varphi_y)$ (cf. §3). The image $\Delta_\mathfrak{p}$ of $\mathrm{Gal}(L^{\mathrm{sep}}/L)$ on $V_\mathfrak{p}(\varphi_y)$ can thus be identified with a closed subgroup of $\Gamma_\mathfrak{p}$. The irreducibility of $\pi^{-1}(Y)$ now means that $\Delta_\mathfrak{p}\Gamma_1 = \Gamma_\mathfrak{p}$. By the construction of Γ_1 this implies that $F_\mathfrak{p}\Delta_\mathfrak{p} = F_\mathfrak{p}\Gamma_\mathfrak{p}$. But by Taguchi [11, Th.0.1] the left hand side acts semisimply on $V_\mathfrak{p}(\varphi)$; hence so does the right hand side, as desired. **q.e.d.**

Next the endomorphism ring $\mathrm{End}_K(\varphi)$ consists of the elements of $K\{\tau\}$ which commute with φ_a for all $a \in A$. The action of endomorphisms on $\varphi(K^{\mathrm{sep}})[\mathfrak{p}^\infty]$ and hence on $V_\mathfrak{p}(\varphi)$ yields a natural homomorphism

$$\mathrm{End}_K(\varphi) \otimes_A F_\mathfrak{p} \longrightarrow \mathrm{End}_{F_\mathfrak{p}}\big(V_\mathfrak{p}(\varphi)\big).$$

This homomorphism commutes with the action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. The following result, the 'Tate conjecture' for Drinfeld modules, was proved independently by Taguchi [12] and Tamagawa [13], [14], [15] as a special case of Theorem 5.4 below:

Theorem 2.2 *The natural homomorphism*

$$\mathrm{End}_K(\varphi) \otimes_A F_\mathfrak{p} \longrightarrow \mathrm{End}_{F_\mathfrak{p},\mathrm{Gal}(K^{\mathrm{sep}}/K)}\big(V_\mathfrak{p}(\varphi)\big)$$

is an isomorphism.

Now let $G_\mathfrak{p}$ denote the Zariski closure of $\Gamma_\mathfrak{p}$, which is an algebraic subgroup of $\mathrm{GL}_{r,F_\mathfrak{p}}$. For Theorem 1.1 we must prove that $G_\mathfrak{p} = \mathrm{GL}_{r,F_\mathfrak{p}}$. The preceding results yield a first approximation to this:

Proposition 2.3 *If $\mathrm{End}_{K^{\mathrm{sep}}}(\varphi) = A$, then the identity component of $G_\mathfrak{p}$ is reductive and acts absolutely irreducibly on $V_\mathfrak{p}(\varphi)$.*

Proof. Let $G_\mathfrak{p}^\circ$ denote the identity component of $G_\mathfrak{p}$. Then $G_\mathfrak{p}^\circ(F_\mathfrak{p}) \cap \Gamma_\mathfrak{p}$ is the image of $\mathrm{Gal}(K^{\mathrm{sep}}/K')$ for some finite subextension $K' \subset K^{\mathrm{sep}}$ of K, and by construction it is Zariski dense in $G_\mathfrak{p}^\circ$. Thus replacing K by K' amounts to replacing $G_\mathfrak{p}$ by $G_\mathfrak{p}^\circ$, after which $G_\mathfrak{p}$ is connected.

Note that by assumption we still have $\mathrm{End}_K(\varphi) = A$. Thus Theorems 2.1 and 2.2 say that $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ acts absolutely irreducibly on $V_\mathfrak{p}(\varphi)$. By construction the same then follows for $\Gamma_\mathfrak{p}$ and hence for $G_\mathfrak{p}$. This implies that $G_\mathfrak{p}$ is reductive (cf. [7, Fact A.1]). **q.e.d.**

3 Good reduction and Frobenius elements

Since A is a finitely generated $\mathbb{F}_q$-algebra, the homomorphism φ factors through $R\{\tau\} \subset K\{\tau\}$ for some finitely generated $\mathbb{F}_q$-algebra $R \subset K$. As K is a finitely generated field extension of $\mathbb{F}_q$, after enlarging R we may assume that $\mathrm{Quot}(R) = K$. After enlarging R further we may also assume that for some non-constant $a \in A$ the highest non-zero coefficient of φ_a is a unit in R. For every point $x \in X := \mathbf{Spec}\, R$ with residue field k_x we consider the induced homomorphism

$$\varphi_x : A \to k_x\{\tau\}.$$

Then by construction the degree of $\varphi_{x,a}$ over k_x is equal to the degree of φ_a over K, which implies that φ_x is a Drinfeld module over k_x of the same rank as φ. Thus φ defines a family of Drinfeld modules of rank r over X, which is a model of K of finite type over $\mathbf{Spec}\, \mathbb{F}_q$.

We fix such R and X for all that follows and say that φ has good reduction at all points $x \in X$. Since φ has characteristic $\mathfrak{p}_0$ and $A/\mathfrak{p}_0$ is a field, the composite homomorphism $A/\mathfrak{p}_0 \hookrightarrow R \to k_x$ induced by the lowest coefficient of φ is still injective; hence φ_x again has characteristic $\mathfrak{p}_0$.

Next for any $i \geq 0$ consider an element $a \in \mathfrak{p}^i \setminus \mathfrak{p}_0$. The lowest coefficient of φ_a is then invertible in k_x for all $x \in X$ and therefore a unit in R. This implies that the kernel of φ_a on $\mathbb{G}_{a,R}$ is a finite étale commutative group scheme over X, and all its sections are defined over finite étale coverings of X. Varying i and a we deduce that all elements of $\varphi(K^{\mathrm{sep}})[\mathfrak{p}^\infty]$ extend to sections over finite étale coverings of X. It follows that the Galois representation $\rho_\mathfrak{p}$ factors through the étale fundamental group $\pi_1^{\text{ét}}(X)$ and the reduction maps induce isomorphisms of Tate modules $V_\mathfrak{p}(\varphi) \xrightarrow{\sim} V_\mathfrak{p}(\varphi_x)$.

Now suppose that x is a closed point of X, so that its residue field k_x is finite. The Galois group $\mathrm{Gal}(k_x^{\mathrm{sep}}/k_x)$ is then pro-cyclic and generated by the Frobenius automorphism $u \mapsto u^{|k_x|}$. Any element $\mathrm{Frob}_x \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ in a decomposition group above x which acts like this on the residue field k_x^{sep} is called a *Frobenius element at* x. Its image $\rho_\mathfrak{p}(\mathrm{Frob}_x) \in \Gamma_\mathfrak{p}$ acts on $V_\mathfrak{p}(\varphi)$ in the same way as the Frobenius automorphism acts on $V_\mathfrak{p}(\varphi_x)$. It possesses the following useful properties:

Theorem 3.1 *(cf. [4, Th.3.2.3 (b)]) The characteristic polynomial f_x of $\rho_\mathfrak{p}(\mathrm{Frob}_x)$ has coefficients in A and is independent of $\mathfrak{p}$.*

Next let $\alpha_1, \ldots, \alpha_r$ be the roots of f_x in an algebraic closure $\bar{F}$ of F, with repetitions if necessary. Consider any valuation v of F, normalized so that a uniformizer has valuation 1, and consider an extension $\bar{v}$ of v to $\bar{F}$. Let k_v denote the residue field at v.

Theorem 3.2 *(cf. Drinfeld [3, Prop.2.1] or [4, Th.3.2.3 c–d])*

(a) If v does not correspond to $\mathfrak{p}_0$ or ∞, then for all $1 \leq i \leq r$ we have

$$\bar{v}(\alpha_i) = 0.$$

(b) If v corresponds to ∞, then for all $1 \leq i \leq r$ we have

$$\bar{v}(\alpha_i) = -\frac{1}{r} \cdot \frac{[k_x/\mathbb{F}_q]}{[k_v/\mathbb{F}_q]}.$$

(c) If v corresponds to $\mathfrak{p}_0$, then there exists an integer $0 < s_x \leq r$, called the height *of φ_x, such that*

$$\bar{v}(\alpha_i) = \begin{cases} \dfrac{1}{s_x} \cdot \dfrac{[k_x/\mathbb{F}_q]}{[k_v/\mathbb{F}_q]} & \textit{for precisely } s_x \textit{ of the } \alpha_i, \textit{ and} \\[2ex] 0 & \textit{for the remaining } r - s_x \textit{ of the } \alpha_i. \end{cases}$$

4 Frobenius tori

A vital tool in Serre's study of Galois representations over $\mathbb{Q}_\ell$ is that of Frobenius tori [10]. We adapt this concept to the present situation as far as necessary. All the ideas in this section are due to Serre.

For every closed point $x \in X$ we fix an element $h_x \in \mathrm{GL}_r(F)$ with characteristic polynomial f_x. We let $H_x \subset \mathrm{GL}_{r,F}$ denote the Zariski closure of the discrete subgroup generated by h_x, and T_x the identity component of H_x.

Proposition 4.1 *(a) T_x is a torus, called the* Frobenius torus at x.

(b) The $\mathrm{GL}_r(F)$-conjugacy class of T_x depends only on f_x.

(c) Some $\mathrm{GL}_r(F_\mathfrak{p})$-conjugate of $T_{x,F_\mathfrak{p}}$ is contained in $G_\mathfrak{p}$.

(d) There exists a positive integer n such that for all $x \in X$ the element $\rho_\mathfrak{p}(\mathrm{Frob}_x)^n$ lies in some $\mathrm{GL}_r(F_\mathfrak{p})$-conjugate of $T_{x,F_\mathfrak{p}}$.

Proof. Let $h_x = su = us$ be the Jordan-Chevalley decomposition into a semisimple element s and a unipotent element u of $\mathrm{GL}_r(\bar{F})$. Recall that $\bar{F}$ has positive characteristic p and fix an integer m so that $p^m \geq r$. The binomial formula then implies that $u^{p^m} = (1+(u-1))^{p^m} = 1+(u-1)^{p^m} = 1$; hence $h_x^{p^m} = s^{p^m}$ is diagonalizable over $\bar{F}$. It follows that the Zariski closure H'_x of the discrete subgroup generated by $h_x^{p^m}$ is diagonalizable over $\bar{F}$ and of finite index dividing p^m in H_x. In particular T_x is the identity component of H'_x, so it is diagonalizable over $\bar{F}$, proving (a).

Next the characteristic polynomial of $h_x^{p^m}$ depends only on f_x. Since any two semisimple elements of $\mathrm{GL}_r(F)$ with the same characteristic polynomial are conjugate under $\mathrm{GL}_r(F)$, the $\mathrm{GL}_r(F)$-conjugacy class of $h_x^{p^m}$ and hence of H'_x and T_x is independent of the choice of h_x, proving (b).

On the other hand $\rho_\mathfrak{p}(\mathrm{Frob}_x)^{p^m}$ is an element of $G_\mathfrak{p}(F_\mathfrak{p}) \subset \mathrm{GL}_r(F_\mathfrak{p})$ with the same characteristic polynomial as $h_x^{p^m}$. The same arguments as in (a) show that $\rho_\mathfrak{p}(\mathrm{Frob}_x)^{p^m}$ is semisimple. Thus both elements are semisimple over $F_\mathfrak{p}$ with the same characteristic polynomial, so they are conjugate under $\mathrm{GL}_r(F_\mathfrak{p})$. The same element conjugates $H'_{x,F_\mathfrak{p}}$ and hence $T_{x,F_\mathfrak{p}}$ into $G_\mathfrak{p}$, proving (c).

Finally, we already know that $\rho_\mathfrak{p}(\mathrm{Frob}_x)^{p^m}$ lies in some $\mathrm{GL}_r(F_\mathfrak{p})$-conjugate of $H'_{x,F_\mathfrak{p}}$. To prove (d) it thus suffices to show that the finite quotient H'_x/T_x has bounded order. By construction this group is cyclic and diagonalizable over $\bar{F}$. Choose any faithful character χ of H'_x/T_x. Then $\zeta := \chi\big(h_x^{p^m}T_x\big) \in \bar{F}^*$ is a root of unity of order $|H'_x/T_x|$. On the other hand ζ is a multiplicative $\mathbb{Z}$-linear combination of the eigenvalues of $\rho_\mathfrak{p}(\mathrm{Frob}_x)$. Let $F_x \subset \bar{F}$ be the field extension of F generated by these eigenvalues. Then ζ is a root of unity in F_x, so it lies in the finite constant field of F_x. Now F_x is the splitting field of the polynomial f_x of degree r, so its degree over F is $\leq r!$. In particular the extension of the constant fields in F_x and F has degree $\leq r!$, so the order of the constant field of F_x is bounded independently of x. Thus ζ and hence H'_x/T_x has bounded order, as desired. **q.e.d.**

Next recall that the character and cocharacter groups of T_x are free $\mathbb{Z}$-modules of finite rank and related to each other by the natural perfect pairing

$$\begin{array}{ccccc} X(T_x) \times Y(T_x) & \longrightarrow & \mathrm{End}(\mathbb{G}_{m,\bar{F}}) & \cong & \mathbb{Z} \\ (\chi,\lambda) & \longmapsto & \chi \circ \lambda & \hat{=} & \langle \chi,\lambda \rangle\,. \end{array}$$

Since the restriction homomorphism $X(H_x)_{\mathbb{Q}} \to X(T_x)_{\mathbb{Q}}$ is an isomorphism, the pairing induces an isomorphism $Y(T_x)_{\mathbb{Q}} \cong \mathrm{Hom}_{\mathbb{Z}}\big(X(H_x), \mathbb{Q}\big)$. Let $\bar{v}$ be an extension to $\bar{F}$ of the normalized valuation of F at $\mathfrak{p}_0$. Then $\chi \mapsto \bar{v}\big(\chi(h_x)\big)$ defines a homomorphism $X(H_x) \to \mathbb{Q}$; hence there exists a unique element $y_x \in Y(T_x)_{\mathbb{Q}}$ such that for all $\chi \in X(H_x)$ we have

$$\bar{v}\big(\chi(h_x)\big) = \langle \chi | T_x, y_x \rangle. \tag{4.2}$$

Since this 'rational cocharacter' determines the Newton polygon of f_x at $\mathfrak{p}_0$, we call it the *Newton cocharacter of T_x*.

Proposition 4.3 *The $\mathrm{Aut}(\bar{F}/F)$-conjugates of y_x generate $Y(T_x)_{\mathbb{Q}}$.*

Proof. If not, there exists a character $\chi \in X(H_x)$ of infinite order such that

$$^{\sigma}\bar{v}\big(\chi(h_x)\big) \;=\; \bar{v}\big({}^{\sigma^{-1}}\chi(h_x)\big) \;=\; \langle {}^{\sigma^{-1}}\chi | T_x, y_x \rangle \;=\; \langle \chi | T_x, {}^{\sigma}y_x \rangle \;=\; 0$$

for all $\sigma \in \mathrm{Aut}(\bar{F}/F)$. Thus the element $\chi(h_x) \in \bar{F}^*$ is a unit at all places above $\mathfrak{p}_0$. Since $\chi(h_x)$ is a product of eigenvalues of $\rho_{\mathfrak{p}}(\mathrm{Frob}_x)$, using Theorem 3.2 we deduce that $\chi(h_x)$ is a unit at all places outside ∞ and that its valuations at all places above ∞ are equal. With the product formula this implies that $\chi(h_x)$ is a unit everywhere and therefore a constant function. It follows that $\chi(h_x)$ is a root of unity in $\bar{F}^*$. Let n be its order, then the relation $\chi^n(h_x) = \chi(h_x)^n = 1$ implies that χ^n vanishes on h_x, and therefore on H_x. This contradicts the assumption that χ has infinite order in $X(H_x)$. **q.e.d.**

Next we note that Theorem 3.2 (c) and the characterization 4.2 of y_x imply:

Proposition 4.4 *The weights of y_x in the tautological representation $T_x \hookrightarrow \mathrm{GL}_{r,F}$ take exactly one non-zero value and, perhaps, the value* 0.

Proposition 4.5 *As $x \in X$ varies, there are only finitely many possibilities for the $\mathrm{GL}_r(\bar{F})$-conjugacy class of $T_{x,\bar{F}}$.*

Proof. Let us conjugate $T_{x,\bar{F}}$ into the diagonal torus $\mathbb{G}^r_{m,\bar{F}}$. The conjugation identifies the cocharacter space $Y(T_x)_{\mathbb{Q}}$ with a subspace $W \subset Y(\mathbb{G}^r_m)_{\mathbb{Q}} \cong \mathbb{Q}^r$. By Proposition 4.3 this subspace is generated by the tuples corresponding to all Galois conjugates ${}^{\sigma}y_x$ of the Newton cocharacter. By Proposition 4.4 all non-zero entries of such a tuple are equal. But up to rational multiples there are only finitely many such tuples in $\mathbb{Q}^r$. Thus W is generated by a subset of a fixed finite subset of $\mathbb{Q}^r$; hence there are only finitely many possibilities for it, as desired. **q.e.d.**

Theorem 4.6 *There exists a Zariski open dense subset U of the identity component $G^{\circ}_{\mathfrak{p}}$ of $G_{\mathfrak{p}}$ such that for all closed points $x \in X$ with $\rho_{\mathfrak{p}}(\mathrm{Frob}_x) \in U(F_{\mathfrak{p}})$ some $\mathrm{GL}_r(F_{\mathfrak{p}})$-conjugate of $T_{x,F_{\mathfrak{p}}}$ is a maximal torus of $G^{\circ}_{\mathfrak{p}}$.*

Proof. Fix a maximal torus $T_{\mathfrak{p}} \subset G^{\circ}_{\mathfrak{p}}$. Since $T_{x,F_{\mathfrak{p}}}$ is conjugate to a subtorus of $G^{\circ}_{\mathfrak{p}}$, it is conjugate over $\bar{F}_{\mathfrak{p}}$ to a subtorus of $T_{\mathfrak{p},\bar{F}_{\mathfrak{p}}}$. In particular we always have $\dim T_x \leq \dim T_{\mathfrak{p}}$, and $T_{x,F_{\mathfrak{p}}}$ is conjugate to a maximal torus of $G^{\circ}_{\mathfrak{p}}$ if and only if $\dim T_x = \dim T_{\mathfrak{p}}$. Thus we must study those x for which $\dim T_x < \dim T_{\mathfrak{p}}$.

By Proposition 4.5 the associated tori $T_{x,\bar{F}}$ lie in only finitely many $\mathrm{GL}_{r,\bar{F}}$-conjugacy classes. For each such conjugacy class there are only finitely many ways to conjugate $T_{x,\bar{F}_{\mathfrak{p}}}$ into a proper subtorus of $T_{\mathfrak{p},\bar{F}_{\mathfrak{p}}}$. Let Z denote the finite union of the resulting proper subtori of $T_{\mathfrak{p},\bar{F}_{\mathfrak{p}}}$; as an algebraic subvariety it is defined over $F_{\mathfrak{p}}$. Let n be as in Proposition 4.1 (d) and set $Z' := \{t \in T_{\mathfrak{p}} \mid t^n \in Z\}$. By construction this is a proper closed subvariety of $T_\mathfrak{p}$. Since $T_\mathfrak{p}$ is a maximal torus of $G_\mathfrak{p}^\circ$, it follows that the set of points in $G_\mathfrak{p}^\circ$ that are not conjugate under $G_\mathfrak{p}^\circ$ to a point of Z' contains an open dense subset U. We claim that U has the desired property.

To see this recall from Proposition 4.1 (d) that $\rho_\mathfrak{p}(\mathrm{Frob}_x)^n$ lies in some $\mathrm{GL}_r(F_\mathfrak{p})$-conjugate of $T_{x,F_\mathfrak{p}}$. Thus if $\dim T_x < \dim T_\mathfrak{p}$, some $G_\mathfrak{p}^\circ$-conjugate of $\rho_\mathfrak{p}(\mathrm{Frob}_x)^n$ lies in Z. It follows that some $G_\mathfrak{p}^\circ$-conjugate of $\rho_\mathfrak{p}(\mathrm{Frob}_x)$ lies in Z'; hence $\rho_\mathfrak{p}(\mathrm{Frob}_x) \notin U$. This proves that for $\rho_\mathfrak{p}(\mathrm{Frob}_x) \in U$ we have $\dim T_x = \dim T_\mathfrak{p}$, and so $T_{x,F_\mathfrak{p}}$ is conjugate to a maximal torus of $G_\mathfrak{p}^\circ$, as desired. **q.e.d.**

Corollary 4.7 *The set of closed points $x \in X$ for which some* $\mathrm{GL}_r(F_\mathfrak{p})$*-conjugate of $T_{x,F_\mathfrak{p}}$ is a maximal torus of $G_\mathfrak{p}$ has Dirichlet density* > 0.

Proof. Since the subset $U \subset G_\mathfrak{p}$ from Theorem 4.6 is Zariski open non-empty and $\Gamma_\mathfrak{p} \subset G_\mathfrak{p}$ is Zariski dense, the intersection $U \cap \Gamma_\mathfrak{p}$ contains a coset of an open normal subgroup $\Gamma_1 \subset \Gamma_\mathfrak{p}$. Thus the corollary follows by applying the Čebotarev density theorem to the finite quotient $\Gamma_\mathfrak{p}/\Gamma_1$. (For the concept of Dirichlet density and the Čebotarev density theorem in the case $\dim X > 1$ see [7, Appendix B].) **q.e.d.**

Remark 4.8 By passing to the $\mathfrak{p}$-adic limit as in Serre [9, §4, Th.10] one can surely prove: If $G_\mathfrak{p}$ is connected, the set of closed points $x \in X$ for which some $\mathrm{GL}_r(F_\mathfrak{p})$-conjugate of $T_{x,F_\mathfrak{p}}$ is a maximal torus of $G_\mathfrak{p}$ has Dirichlet density 1.

5 A-motives

In this section we review the formalism and some basic properties of A-motives. All concepts and results except Proposition 5.6 are due to Anderson [1, §1], who concentrated on the case $A = \mathbb{F}_q[t]$ and used the term t-motives.

Let $A \subset F$ and K be as in the introduction. We are interested in modules over $A_K := A \otimes_{\mathbb{F}_q} K$ with a certain additional structure. Note that since $\mathbb{F}_q$ is the constant field of F, the ring A_K is an integral domain. We fix a homomorphism of $\mathbb{F}_q$-algebras $\iota : A \to K$ and let $I \subset A_K$ be the ideal generated by the elements $a \otimes 1 - 1 \otimes \iota(a)$ for all $a \in A$.

Definition 5.1 *An A*-motive over K of characteristic ι and of rank r *is an A_K-module M together with an additive endomorphism $\tau : M \to M$ satisfying*

$$\tau((a \otimes u)(m)) = (a \otimes u^q)(\tau(m))$$

for all $a \in A$, $u \in K$, and $m \in M$, such that

(a) M is finitely generated and projective over A_K of rank r,

(b) the A_K-module $M/A_K\tau(M)$ is annihilated by a power of I.

Anderson assumed moreover that M is finitely generated over the non-commutative polynomial ring $K\{\tau\}$; but that property is irrelevant for our purposes.

Next let M, M' be two A-motives of characteristic ι. A *homomorphism of A-motives* $M \to M'$ is simply a homomorphism of the underlying A_K-modules that commutes with τ. The set of all homomorphisms $M \to M'$ is a finitely generated projective A-module denoted $\mathrm{Hom}(M, M')$.

Any A_K-submodule $N \subset M$ satisfying $\tau(N) \subset N$ is itself an A-motive and called an *A-submotive of M*. Clearly the image of a homomorphism is an A-submotive.

The *tensor product* $M \otimes M'$ is simply the tensor product of A_K-modules together with the induced semi-linear endomorphism $\tau \otimes \tau$. Similarly the ℓ^{th} *tensor* and *exterior powers* $M^{\otimes \ell}$ and $\Lambda^\ell M$ are obtained by the corresponding construction of A_K-modules together with their semi-linear endomorphisms $\tau^{\otimes \ell}$ and $\tau \wedge \ldots \wedge \tau$.

Next we define weights. Note that F is the function field of a geometrically connected smooth projective algebraic curve C over $\mathbb{F}_q$, and A is the affine coordinate ring of $C \smallsetminus \{\infty\}$. Let C_K be the algebraic curve over K obtained by base change from C. Let $\infty_1, \ldots, \infty_f$ be the points of C_K above ∞, then A_K is the affine coordinate ring of $C_K \smallsetminus \{\infty_1, \ldots, \infty_f\}$. Let $\mathscr{O}_{\infty,K}$ denote the direct sum of the completed local rings of C_K at ∞_i, and $F_{\infty,K}$ the direct sum of their quotient fields. Note that we have a natural embedding $A_K \hookrightarrow F_{\infty,K}$ and that the endomorphism $a \otimes u \mapsto a \otimes u^q$ of A_K extends to a natural endomorphism σ of $F_{\infty,K}$ and $\mathscr{O}_{\infty,K}$. Thus $\tau : M \to M$ extends to a semi-linear endomorphism of $M_\infty := M \otimes_{A_K} F_{\infty,K}$ satisfying $\tau(xm) = {}^\sigma x \cdot \tau(m)$ for all $x \in F_{\infty,K}$ and $m \in M$.

Let v_∞ denote the normalized valuation of F at ∞ for which a uniformizer has valuation 1. For any non-zero element $a \in A$ we set

$$\deg a \;:=\; -[k_\infty/\mathbb{F}_q] \cdot v_\infty(a) \;\in\; \mathbb{Z}^{\geq 0}.$$

Definition 5.2 *An A-motive M is called* pure of weight $\mu \in \mathbb{Q}$ *if and only if there exist integers $r > 0$ and s with $\frac{s}{r} = \mu$ and an $\mathscr{O}_{\infty,K}$-lattice $L_\infty \subset M_\infty$, such that for all non-zero $a \in A$ we have*

$$\mathscr{O}_{\infty,K} \cdot \tau^{r \cdot \deg a}(L_\infty) = a^s L_\infty.$$

One easily shows that if M is pure of weight μ, then so is any A-submotive of M, and so is the image of any homomorphism of A-motives $M \to M'$. Moreover the tensor product of two pure A-motives of weights μ and μ' is pure of weight $\mu + \mu'$, and the ℓ^{th} tensor and exterior powers of a pure A-motive of weight μ are pure of weight $\ell\mu$.

Proposition 5.3 *If M is of rank r and pure of weight μ, then $r\mu \in \mathbb{Z}$.*

Proof. Since $\Lambda^r M$ is of rank 1 and pure of weight $r\mu$, it suffices to show the proposition for all M of rank 1. Take any non-zero $m \in M$ and write $\tau(m) = xm$ for $x \in \mathrm{Quot}(A_K)$. For any non-zero $a \in A$ we then have

$$\tau^{\deg a}(m) = x \cdot {}^\sigma x \cdot \ldots \cdot {}^{\sigma^{\deg a - 1}} x \cdot m.$$

Now the points ∞_i correspond to the simple summands of $k_\infty \otimes_{\mathbb{F}_q} K$, whose number f divides $[k_\infty : \mathbb{F}_q]$ and which are permuted transitively by σ. Moreover let v_{∞_i} denote the normalized valuation at ∞_i extending v_∞. Since σ fixes a uniformizer at ∞ in F, we have $v_{\infty_i}\big({}^{\sigma^j} x\big) = v_{\sigma^j(\infty_i)}(x)$ for all i, j. Thus for all i we have

$$v_{\infty_i}\big(x \cdot {}^\sigma x \cdot \ldots \cdot {}^{\sigma^{\deg a - 1}} x\big) \;=\; \frac{\deg a}{f} \cdot \sum_{j=1}^{f} v_{\infty_j}(x) \;=\; s \cdot v_\infty(a)$$

where

$$s \;:=\; -\frac{[k_\infty/\mathbb{F}_q]}{f} \cdot \sum_{j=1}^{f} v_{\infty_j}(x) \;\in\; \mathbb{Z}.$$

For the lattice $L_\infty := \mathscr{O}_{\infty,K} \cdot m \subset M_\infty$ this implies that

$$\mathscr{O}_{\infty,K} \cdot \tau^{\deg a}(L_\infty) \;=\; x \cdot {}^\sigma x \cdot \ldots \cdot {}^{\sigma^{\deg a - 1}} x \cdot L_\infty \;=\; a^s \cdot L_\infty,$$

so that M is pure of weight $s \in \mathbb{Z}$, as desired. **q.e.d.**

Next fix any prime ideal $\mathfrak{p} \subset A$ with $\iota(\mathfrak{p}) \neq 0$. Let M be an A-motive over K of characteristic ι and of rank r, and let $M^{\mathrm{sep}} := M \otimes_K K^{\mathrm{sep}}$ denote the induced A-motive over K^{sep}. Then for every positive integer i the quotient $M^{\mathrm{sep}}/\mathfrak{p}^i M^{\mathrm{sep}}$ is a free module over $(A/\mathfrak{p}^i) \otimes_{\mathbb{F}_q} K^{\mathrm{sep}}$ of rank r. The endomorphism τ of M induces a semi-linear endomorphism of $M^{\mathrm{sep}}/\mathfrak{p}^i M^{\mathrm{sep}}$, denoted again by τ, which satisfies $\tau(uv) = u^q \tau(v)$ for all $u \in K^{\mathrm{sep}}$ and all vectors v. The assumptions $\iota(\mathfrak{p}) \neq 0$ and 5.1 (b) imply that the image of τ generates $M^{\mathrm{sep}}/\mathfrak{p}^i M^{\mathrm{sep}}$. Using this one easily proves that

$$M[\mathfrak{p}^i] := \bigl\{ v \in M^{\mathrm{sep}}/\mathfrak{p}^i M^{\mathrm{sep}} \bigm| \tau(v) = v \bigr\}$$

is a free $A/\mathfrak{p}^i$-module of rank r. The *rational $\mathfrak{p}$-adic Tate module of M*

$$V_{\mathfrak{p}}(M) := \Bigl(\varprojlim_i M[\mathfrak{p}^i] \Bigr) \otimes_{A_{\mathfrak{p}}} F_{\mathfrak{p}}$$

is then an $F_{\mathfrak{p}}$-vector space of dimension r. By construction it possesses a natural continuous $F_{\mathfrak{p}}$-linear representation of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$.

Let M, M' be two A-motives over K of characteristic ι. Then any homomorphism $h : M \to M'$ induces a $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-equivariant $F_{\mathfrak{p}}$-linear homomorphism $V_{\mathfrak{p}}(h) : V_{\mathfrak{p}}(M) \to V_{\mathfrak{p}}(M')$. Its image is $V_{\mathfrak{p}}(N)$, where $N := h(M) \subset M'$ denotes the image of h. The following result, the 'Tate conjecture' for A-motives, was proved independently by Taguchi [12] and Tamagawa [13], [14], [15]:

Theorem 5.4 *The natural homomorphism*

$$\mathrm{Hom}(M, M') \otimes_A F_{\mathfrak{p}} \longrightarrow \mathrm{Hom}_{F_{\mathfrak{p}}, \mathrm{Gal}(K^{\mathrm{sep}}/K)} \bigl(V_{\mathfrak{p}}(M), V_{\mathfrak{p}}(M') \bigr)$$

is an isomorphism.

Furthermore, there are natural $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-equivariant isomorphisms

$$\begin{aligned} V_{\mathfrak{p}}(M \otimes M') &\cong V_{\mathfrak{p}}(M) \otimes_{F_{\mathfrak{p}}} V_{\mathfrak{p}}(M'), \\ V_{\mathfrak{p}}(M^{\otimes \ell}) &\cong V_{\mathfrak{p}}(M)^{\otimes \ell}, \quad \text{and} \\ V_{\mathfrak{p}}(\Lambda^{\ell} M) &\cong \Lambda^{\ell}(V_{\mathfrak{p}}(M)). \end{aligned} \tag{5.5}$$

The following criterion will play an important role in §7:

Proposition 5.6 *Consider two A-motives M and M' over K of characteristic ι and a positive integer k. Assume that up to scalar multiples there exists exactly one non-zero* $\mathrm{Gal}(K^{\mathrm{sep}}/K)$*-equivariant homomorphism*

$$V_{\mathfrak{p}}(M) \otimes_{F_{\mathfrak{p}}} F_{\mathfrak{p}}^{\mathrm{sep}} \longrightarrow V_{\mathfrak{p}}(M') \otimes_{F_{\mathfrak{p}}} F_{\mathfrak{p}}^{\mathrm{sep}}$$

of rank $\leq k$, and that the same holds with $\bar{F}_{\mathfrak{p}}$ in place of $F_{\mathfrak{p}}^{\mathrm{sep}}$. Then this homomorphism comes from a homomorphism of A-motives $M \to M'$.

Proof. For any homomorphism h of A-motives or of vector spaces we let $\Lambda^{k+1} h := h \wedge \ldots \wedge h$ denote the induced homomorphism of the $(k+1)^{\mathrm{st}}$ exterior power. The proof rests on the fact that a homomorphism of vector spaces h has rank $\leq k$ if and only if $\Lambda^{k+1} h = 0$, together with the Tate conjecture 5.4 and the relation between the functors Λ^{k+1} and $V_{\mathfrak{p}}$. The latter is given by the following commutative diagram resulting by functoriality:

$$\begin{tikzcd}
\mathrm{Hom}\big(M, M'\big) \otimes_A F_{\mathfrak{p}} \arrow[r, "\sim"] \arrow[dd, "\Lambda^{k+1}"'] & \mathrm{Hom}_{\mathrm{Gal}}\big(V_{\mathfrak{p}}(M), V_{\mathfrak{p}}(M')\big) \arrow[d, "\Lambda^{k+1}"'] \\
 & \mathrm{Hom}_{\mathrm{Gal}}\big(\Lambda^{k+1}(V_{\mathfrak{p}}(M)), \Lambda^{k+1}(V_{\mathfrak{p}}(M'))\big) \arrow[d, equal, "\wr"'] \\
\mathrm{Hom}\big(\Lambda^{k+1}M, \Lambda^{k+1}M'\big) \otimes_A F_{\mathfrak{p}} \arrow[r, "\sim"] & \mathrm{Hom}_{\mathrm{Gal}}\big(V_{\mathfrak{p}}(\Lambda^{k+1}M), V_{\mathfrak{p}}(\Lambda^{k+1}M')\big)
\end{tikzcd}$$

where the horizontal isomorphisms are instances of Theorem 5.4. We obtain analogous commutative diagrams after tensoring with $F_{\mathfrak{p}}^{\mathrm{sep}}$ or with $\bar{F}_{\mathfrak{p}}$. Now

$$\Lambda^{k+1} : \; H := \mathrm{Hom}\big(M, M'\big) \otimes_A F \longrightarrow \mathrm{Hom}\big(\Lambda^{k+1}M, \Lambda^{k+1}M'\big) \otimes_A F$$

is a homogeneous map of degree $k+1$ of finite dimensional F-vector spaces. Thus its zero set is the affine cone over a closed subscheme Z of the projective space associated to H. By the above commutative diagram the assumption over $\bar{F}_{\mathfrak{p}}$ is equivalent to saying that Z possesses exactly one $\bar{F}_{\mathfrak{p}}$-valued point. Thus Z is a finite scheme over F possessing a single geometric point; hence its reduced subscheme is $\mathbf{Spec}\, F'$ for a finite totally inseparable field extension F'/F. On the other hand, by the assumption over $F_{\mathfrak{p}}^{\mathrm{sep}}$ it possesses a point over $F_{\mathfrak{p}}^{\mathrm{sep}}$; hence $F' \subset F_{\mathfrak{p}}^{\mathrm{sep}}$. But $F_{\mathfrak{p}}^{\mathrm{sep}}$ does not contain any non-trivial totally inseparable finite extension of F. Therefore $F' = F$, which means that the homomorphism in question comes from an element of H and thus from an element of $\mathrm{Hom}(M, M')$, as desired. **q.e.d.**

Finally, every Drinfeld A-module corresponds to an A-motive, as follows. Let $\varphi : A \to K\{\tau\}$, $a \mapsto \varphi_a$ be a Drinfeld A-module of rank $r \geq 1$ over K. Set $M_\varphi := K\{\tau\}$ and $(a \otimes u)(m) := u \cdot m \cdot \varphi_a$ and $\tau(m) := \tau \cdot m$ for all $a \in A$, $u \in K$, and $m \in M_\varphi$. Let $\iota : A \to K$ be the homomorphism determined by the lowest coefficient of φ.

Proposition 5.7 *M_φ is an A-motive over K of characteristic ι and of rank r. Moreover M_φ is pure of weight $\frac{1}{r}$.*

Proof. Clearly M_φ is a torsion free A_K-module generated by the finitely many elements $1, \tau, \ldots, \tau^n$ for any sufficiently large integer n. Since A_K is a Dedekind domain, this implies that M_φ is projective. Now the fact that the rank of φ is r means that the degree of φ_a with respect to τ is $r \cdot \deg a$. Using this one easily finds that the rank of M_φ over A_K is r. On the other hand we have $M_\varphi / A_K \tau(M_\varphi) \cong K$ on which A acts through ι. Thus all the conditions in Definition 5.1 are satisfied, which implies the first assertion. The second assertion follows directly from Definition 5.2 by letting $L_\infty \subset M_{\varphi,\infty}$ be the $\mathscr{O}_{\infty,K}$-lattice generated by $1, \tau, \ldots, \tau^n$ for any sufficiently large integer n. **q.e.d.**

Let $\mathfrak{p} \subset A$ be a prime ideal not contained in $\mathfrak{p}_0 := \ker(\iota)$. Let Ω_A denote the module of differentials of A over $\mathbb{F}_q$. By [1, Prop.1.8.3] we have:

Proposition 5.8 *There exists a natural* $\mathrm{Gal}(K^{\mathrm{sep}}/K)$*-equivariant isomorphism*

$$V_{\mathfrak{p}}(\varphi) \cong \mathrm{Hom}_{F_{\mathfrak{p}}}\big(V_{\mathfrak{p}}(M_\varphi), \Omega_A \otimes_A F_{\mathfrak{p}}\big).$$

In particular there exists a $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-equivariant isomorphism $V_{\mathfrak{p}}(\varphi)^* \cong V_{\mathfrak{p}}(M_\varphi)$ which is natural up to multiplication by a scalar.

6 Some facts from representation theory

In this section all algebraic groups and all representations are defined over a separably closed field L of arbitrary characteristic. Recall that every torus and hence every reductive linear algebraic group over L is split and that every irreducible representation over $\bar{L}$ of a reductive group over L can be defined over L. We begin with a classification result due to Serre.

Theorem 6.1 *Let G be a connected simple semisimple group and V a faithful absolutely irreducible representation of G. Assume that G possesses a cocharacter y which has precisely two distinct weights on V. Then the pair (G,V) is isomorphic to one from the following table:*

Root system of G	Type of V	$\dim V$	Conditions
A_ℓ	Λ^m(Standard)	$\binom{\ell+1}{m}$	$\frac{\ell+1}{2} \geq m \geq 1$
B_ℓ	Spin	2^ℓ	$\ell \geq 2$
C_ℓ	Standard	2ℓ	$\ell \geq 3$
D_ℓ	Standard	2ℓ	$\ell \geq 4$
D_ℓ	Spin^+	$2^{\ell-1}$	$\ell \geq 5$

Proof. Let $r,\ s \in \mathbb{Z}$ be the two distinct weights of y on V. Let $G' \subset \mathrm{Aut}(V)$ be the product of G with the scalar torus $\mathbb{G}_m$. Then $t \mapsto t^{-r}y(t)$ is a cocharacter of G' whose weights on V are 0 and $s-r$. It is therefore the $(s-r)^{\text{th}}$ multiple of a cocharacter y' of G' whose weights on V are 0 and 1. The possibilities for (G',V) possessing such a cocharacter were determined by Serre [8, §3] when L has characteristic zero, and his proof extends verbatim to arbitrary characteristic. The above table summarizes what we need from [8], with all duplicities due to symmetries of the root system purged. **q.e.d.**

Corollary 6.2 *Let G be a connected reductive group and V a faithful absolutely irreducible representation of G. Let $T \subset G$ be a maximal torus and Δ the group of automorphisms of T that preserve the formal character of V. Assume that T possesses a cocharacter y whose weights on V take exactly one non-zero value and, perhaps, the value 0, and whose Δ-conjugates generate the $\mathbb{Q}$-vector space $Y(T)_{\mathbb{Q}}$. Then we can write G as an almost direct product $G = G_0 \cdot G_1 \cdots G_d$ and V as a tensor product $V \cong V_0 \otimes V_1 \otimes \cdots \otimes V_d$ for some $d \geq 0$, such that*

(a) $G_0 \cong \mathbb{G}_m$ with its tautological 1-dimensional representation V_0, and

(b) for $1 \leq i \leq d$ the pair (G_i, V_i) is isomorphic to one from the table in Theorem 6.1.

Proof. Let G_0 be the identity component of the center of G and $G_1, \ldots, G_d$ the connected simple constituents of G^{der}. Then G is the almost direct product $G_0 \cdot G_1 \cdots G_d$. Every faithful absolutely irreducible representation V of G is a tensor product of faithful absolutely irreducible representations V_i of the G_i. After replacing y by a multiple we may assume that y is a product of cocharacters of $G_i \cap T$ for all i.

Since G_0 is a torus, we must have $\dim V_0 = 1$ and $G_0 \subset \mathrm{Aut}(V_0) = \mathbb{G}_m$. If G_0 is trivial, then G is semisimple, so it acts trivially on the highest exterior power of V. But the assumptions imply that the weight of y on the highest exterior power is non-zero. Thus G_0 is non-trivial, which implies (a).

Next consider any $1 \leq i \leq d$. Since the Δ-conjugates of y generate $Y(T)_{\mathbb{Q}}$, at least one conjugate y' has a non-trivial constituent y'_i in G_i. Then y'_i has at least two distinct weights on V_i. If the same happens for some other constituent of y', one easily shows that y' and hence y has at least three distinct weights on V, contrary to the assumptions. Thus y' lands in G_0G_i, and y'_i has precisely two distinct weights on V_i. It follows that (G_i, V_i) satisfies the assumptions of Theorem 6.1, proving (b). **q.e.d.**

Proposition 6.3 *In Corollary 6.2 we furthermore have for* $1 \leq i \leq d$*:*

(a) If one pair (G_i, V_i) *has type* $(A_1$*,Standard) or* $(B_\ell$*,Spin) for any* $\ell \geq 2$*, then every pair* (G_i, V_i) *has one of these types (where* ℓ *can vary).*

(b) If one pair (G_i, V_i) *has type* $(C_3$*,Standard) or* $(A_3, \Lambda^2($*Standard*$))$*, then every pair* (G_i, V_i) *has one of these types.*

(c) If one pair (G_i, V_i) *has type* $(C_\ell$*,Standard) or* $(D_\ell$*,Standard) for fixed* $\ell \geq 4$*, then every pair* (G_i, V_i) *has one of these types.*

(d) If none of the cases (a–c) occurs, then all pairs (G_i, V_i) *have the same type.*

Proof. For every i let $\Phi_i \subset X(T)$ be the root system of G_i and Φ_i° its subset of short roots. By Larsen-Pink [6, §4] the union $\Phi^\circ := \Phi_1^\circ \cup \ldots \cup \Phi_d^\circ$ is determined uniquely by the formal character of V; hence it is permuted by Δ. We claim that the action of Δ on Φ° is transitive. To see this note first that Δ contains the Weyl group of every Φ_i, which permutes Φ_i° transitively. Thus every Δ-orbit in Φ° is a union of some of the Φ_i°. Suppose that there exists $1 \leq i \leq d$ and a Δ-orbit $\Psi \subset \Phi^\circ$ which does not contain Φ_i°. In the proof of Corollary 6.2 we saw that some Δ-conjugate y' of y lands in G_0G_i. Then y' is orthogonal to all roots in Ψ. Since Ψ is Δ-invariant, this implies that all Δ-conjugates of y' and hence of y are orthogonal to Ψ. But this contradicts the assumption that the Δ-conjugates of y generate $Y(T)_{\mathbb{Q}}$. Therefore Δ acts transitively on Φ°.

Now Φ° itself is a root system. Since the action of Δ is transitive, it follows that Φ° is isotypic. The following table lists the possibilities for Φ_i and Φ_i°:

Φ_i	Φ_i°	Conditions
A_ℓ	A_ℓ	$\ell \geq 1$
B_ℓ	ℓA_1	$\ell \geq 2$
C_3	A_3	
C_ℓ	D_ℓ	$\ell \geq 4$
D_ℓ	D_ℓ	$\ell \geq 4$

Thus if Φ° is isotypic of type A_1, all Φ_i must have type A_1 or B_ℓ, where ℓ can vary. For each of these root systems the table in Theorem 6.1 lists only one representation; this yields the case (a). If Φ° is isotypic of type $\neq A_1$, every Φ_i° is irreducible. Then Δ permutes the Φ_i° and hence the formal characters of the V_i. In particular $\dim V_i$ is independent of i. Using this information, the rest of the proof is achieved simply by comparing the above table with that in Theorem 6.1. **q.e.d.**

Proposition 6.4 *In Corollary 6.2 we have one of the following cases:*

(a) The representation V_i *of* G_i *is self-dual for all* $1 \leq i \leq d$*.*

(b) All pairs (G_i, V_i) *for* $1 \leq i \leq d$ *are of the same type* $(A_\ell, \Lambda^m($*Standard*$))$ *for some* $\frac{\ell+1}{2} \geq m \geq 1$*.*

(c) *All pairs (G_i, V_i) for $1 \leq i \leq d$ are of the same type $(D_\ell$,Spin$^+)$ for some odd $\ell \geq 5$.*

Proof. The pairs in Theorem 6.1 where the representation is not self-dual are precisely $(A_\ell, \Lambda^m(\text{Standard}))$ for $\frac{\ell+1}{2} > m \geq 1$ and (D_ℓ,Spin^+) for odd $\ell \geq 5$. If some (G_i, V_i) has one of these types, Proposition 6.3 implies that every (G_i, V_i) has this type; hence we have case (b) or (c). Otherwise all V_i are self-dual, so we have case (a). **q.e.d.**

In the next section we will use additional information to exclude all pairs in Theorem 6.1 except $(A_\ell$,Standard). In the self-dual case the following easy result will suffice:

Proposition 6.5 *Let V be a self-dual absolutely irreducible representation of a connected semisimple linear algebraic group G. Then up to scalar multiples there exists exactly one G-equivariant endomorphism of $V^{\otimes 2}$ of rank 1.*

Proof. The image of the desired endomorphism is a G-invariant subspace W of dimension 1. As G is connected semisimple, it must act trivially on W. Thus letting G act trivially on L the desired assertion is equivalent to

$$\dim \operatorname{Hom}_G\bigl(V^{\otimes 2}, L\bigr) = \dim \operatorname{Hom}_G\bigl(L, V^{\otimes 2}\bigr) = 1.$$

Since V is self-dual, both dimensions are equal to $\dim \operatorname{Hom}_G(V, V)$, which is 1 by the absolute irreducibility of V. **q.e.d.**

In the A_ℓ-case we will need the following results:

Proposition 6.6 *Let n be a positive integer and V the standard representation of SL_n of dimension n. Then the space of invariants $(V^{\otimes n})^{\mathrm{SL}_n}$ and the space of coinvariants $(V^{\otimes n})_{\mathrm{SL}_n}$ each has dimension 1.*

Proof. Since the dual representation V^* becomes isomorphic to V via an outer automorphism of SL_n, it follows that

$$\dim(V^{\otimes n})^{\mathrm{SL}_n} = \dim\bigl((V^*)^{\otimes n}\bigr)^{\mathrm{SL}_n} = \dim\bigl((V^{\otimes n})_{\mathrm{SL}_n}\bigr)^* = \dim(V^{\otimes n})_{\mathrm{SL}_n}.$$

The natural SL_n-equivariant surjection $V^{\otimes n} \twoheadrightarrow \Lambda^n V \cong L$ shows that this common dimension is ≥ 1. To prove the reverse inequality let $v_1, \ldots, v_n$ be a basis of V and $T \subset \mathrm{SL}_n$ the maximal torus with these eigenvectors. Let N be the normalizer of T in SL_n, then the Weyl group N/T is isomorphic to the symmetric group S_n, which permutes the v_i in the natural way. Now the tensors $v_{i_1} \otimes \cdots \otimes v_{i_n}$ form a basis of $V^{\otimes n}$ of eigenvectors under T, and the associated weight is 0 if and only if every index occurs exactly once in the tuple $(i_1, \ldots, i_n)$. Thus the weight space of weight 0 has the basis $v_{\sigma 1} \otimes \cdots \otimes v_{\sigma n}$ for all $\sigma \in S_n$. It is therefore isomorphic to the regular representation of $N/T \cong S_n$ over L. This implies that $\dim(V^{\otimes n})^{\mathrm{SL}_n} \leq \dim(V^{\otimes n})^N = 1$, as desired. **q.e.d.**

Proposition 6.7 *Let n be a positive integer and V the standard representation of SL_n of dimension n.*

(a) *For all positive integers m, ℓ with $m\ell \leq n$ we have*

$$\dim \operatorname{Hom}_{\mathrm{SL}_n}\bigl((\Lambda^m V)^{\otimes \ell}, \Lambda^{m\ell} V\bigr) = \dim \operatorname{Hom}_{\mathrm{SL}_n}\bigl(\Lambda^{m\ell} V, (\Lambda^m V)^{\otimes \ell}\bigr) = 1.$$

(b) *For all positive integers $m \leq n$ we have*

$$\dim \operatorname{Hom}_{\mathrm{SL}_n}\bigl(\Lambda^m V \otimes \Lambda^{n+1-m} V, V\bigr) = \dim \operatorname{Hom}_{\mathrm{SL}_n}\bigl(V, \Lambda^m V \otimes \Lambda^{n+1-m} V\bigr) = 1.$$

Proof. Since the dual of $\Lambda^{m\ell}V$ is isomorphic to $\Lambda^{n-m\ell}V$, assertion (a) is equivalent to

$$\dim\big((\Lambda^{n-m\ell}V)\otimes(\Lambda^m V)^{\otimes\ell}\big)_{\mathrm{SL}_n} = \dim\big((\Lambda^{n-m\ell}V)\otimes(\Lambda^m V)^{\otimes\ell}\big)^{\mathrm{SL}_n} = 1.$$

To prove these equalities observe that the natural surjections

$$V^{\otimes n} \;\cong\; (V^{\otimes(n-m\ell)})\otimes(V^{\otimes m})^{\otimes\ell} \;\twoheadrightarrow\; (\Lambda^{n-m\ell}V)\otimes(\Lambda^m V)^{\otimes\ell} \;\twoheadrightarrow\; \Lambda^n V \;\cong\; L$$

induce surjections between the associated spaces of coinvariants. Thus the equation for the coinvariants follows from Proposition 6.6. The equation for the invariants follows from that for the coinvariants by dualizing and using the isomorphy $(\Lambda^k V)^* \cong \Lambda^k(V^*)$ for any $0 \le k \le n$. This proves (a).

Since $(\Lambda^m V)^* \cong \Lambda^{n-m}V$ and $(\Lambda^{n+1-m}V)^* \cong \Lambda^{m-1}V$, assertion (b) is equivalent to

$$\dim\big(V\otimes\Lambda^{n-m}V\otimes\Lambda^{m-1}V\big)^{\mathrm{SL}_n} = \dim\big(V\otimes\Lambda^{n-m}V\otimes\Lambda^{m-1}V\big)_{\mathrm{SL}_n} = 1.$$

Since the natural surjections

$$V^{\otimes n} \;\cong\; V\otimes V^{\otimes(n-m)}\otimes V^{\otimes(m-1)} \;\twoheadrightarrow\; V\otimes\Lambda^{n-m}V\otimes\Lambda^{m-1}V \;\twoheadrightarrow\; \Lambda^n V \;\cong\; L$$

induce surjections between the associated spaces of coinvariants, the equation for the coinvariants follows from Proposition 6.6. Again the equation for the invariants follows by dualizing. **q.e.d.**

Proposition 6.8 *Let n be a positive integer and V the standard representation of SL_n of dimension n. Let m, ℓ be positive integers with $n-m < m\ell \le n$. Then up to scalar multiples there exists exactly one non-zero SL_n-equivariant endomorphism of $(\Lambda^m V)^{\otimes\ell}$ of rank $\le \binom{n}{m\ell}$, and its image is isomorphic to the representation $\Lambda^{m\ell}V$.*

Proof. The image of the desired endomorphism is a non-zero SL_n-invariant subspace W of dimension $\le \binom{n}{m\ell}$. We first determine its possible weights. For this recall that in the standard notation the weights of $\Lambda^m V$ are n-tuples of integers $(\lambda_1,\ldots,\lambda_n)$ with m entries 1 and $n-m$ entries 0. Thus for every weight $\mu = (\mu_1,\ldots,\mu_n)$ of $(\Lambda^m V)^{\otimes\ell}$ we deduce that at least m entries are positive and their sum is $m\ell$. We apply this to a weight μ of W and let k be the number of entries 0 in μ. Then the size of the orbit of μ under the Weyl group S_n of SL_n is $\ge \binom{n}{k}$, while on the other hand it must be $\le \dim W \le \binom{n}{m\ell}$; hence $\binom{n}{k} \le \binom{n}{m\ell}$. Since we also have $n-m\ell \le k \le n-m < m\ell$, the only way to satisfy this inequality is with $n-m\ell = k$. Thus precisely $m\ell$ entries of μ are positive. Since their sum is $m\ell$, the value of these entries must be 1. This shows that all weights of W are conjugate to the highest weight of the irreducible representation $\Lambda^{m\ell}V$; hence W is an extension of copies of $\Lambda^{m\ell}V$. As $0 < \dim W \le \binom{n}{m\ell} = \dim\Lambda^{m\ell}V$, we deduce that $W \cong \Lambda^{m\ell}V$. The desired assertion thus follows from Proposition 6.7 (a). **q.e.d.**

Proposition 6.9 *Let n be a positive integer and V the standard representation of SL_n of dimension n. Consider a positive integer $m \le \frac{n}{2}$. Then up to scalar multiples there exists exactly one non-zero SL_n-equivariant endomorphism of $\Lambda^m V \otimes \Lambda^{n+1-m}V$ of rank $\le n$, and its image is isomorphic to V.*

Proof. The image of the desired endomorphism is a non-zero SL_n-invariant subspace W of dimension $\le n$. We first determine its possible weights. The weights of $\Lambda^m V$ are n-tuples of integers with m entries 1 and $n-m$ entries 0. Similarly, the weights of $\Lambda^{n+1-m}V$ are tuples with $n+1-m$ entries 1 and $m-1$ entries 0. Thus every weight μ of $\Lambda^m V \otimes \Lambda^{n+1-m}V$ has entries 2, 1, 0 with respective multiplicities

k, $n+1-2k$, $k-1$ for some k satisfying $1 \leq k \leq m$. We apply this to a weight μ of W. Then the size of the S_n-orbit of μ is $\geq \binom{n}{k}$, while on the other hand it must be $\leq \dim W \leq n$; hence $\binom{n}{k} \leq n$. Since we also have $1 \leq k \leq m \leq \frac{n}{2}$, the only way to satisfy this inequality is with $k=1$. This shows that all weights of W are S_n-conjugate to $(2,1,\ldots,1)$. On the maximal torus of SL_n this weight coincides with $(1,0,\ldots,0)$. Thus all weights of W are conjugate to the highest weight of the irreducible representation V; hence W is an extension of copies of V. As $0 < \dim W \leq n = \dim V$, we deduce that $W \cong V$. The desired assertion thus follows from Proposition 6.7 (b). **q.e.d.**

In the remaining D_ℓ-case we will need:

Proposition 6.10 *Consider an odd integer $\ell \geq 5$. Let V^+ denote the positive Spin representation of dimension $2^{\ell-1}$ of the connected semisimple group G of type D_ℓ, and let V denote the standard representation of $G \twoheadrightarrow \mathrm{SO}(2\ell)$ of dimension 2ℓ. Then*

$$\dim \mathrm{Hom}_G\big((V^+)^{\otimes 2}, V\big) = \dim \mathrm{Hom}_G\big(V, (V^+)^{\otimes 2}\big) = 1.$$

Proof. Since V is self-dual, the assertion is equivalent to

$$\dim\big(V \otimes (V^+)^{\otimes 2}\big)_G = \dim\big(V \otimes (V^+)^{\otimes 2}\big)^G = 1.$$

As ℓ is odd, the dual of V^+ is isomorphic to the negative Spin representation V^-, which corresponds to V^+ again under an outer automorphism of G that fixes the equivalence class of V. Thus by dualizing we find that the two dimensions are equal.

Saying that this common dimension is ≥ 1 amounts to saying that there exists a non-zero G-equivariant homomorphism $(V^+)^{\otimes 2} \to V$. In characteristic zero this follows directly from the construction of V^+ by means of the Clifford algebra of V. Alternatively, it is equivalent to saying that V^- is a constituent of $V \otimes V^+$, which can be proved easily by direct calculation using the Weyl character formula. To show that the assertion extends to characteristic $p>0$ let G_0 be the split simply connected Chevalley group of type D_ℓ over $\mathbb{Q}$, and let V_0^+ and V_0 be its positive Spin and its standard representation. Let $\mathscr{G}$ be the associated Chevalley group scheme over $\mathbf{Spec}\,\mathbb{Z}$ and let $\mathscr{V}^+ \subset V_0^+$ and $\mathscr{V} \subset V_0$ be any $\mathscr{G}$-invariant $\mathbb{Z}$-lattices. Then the weights show that $\mathscr{V}^+/p\mathscr{V}^+$ and $\mathscr{V}/p\mathscr{V}$ are precisely the positive Spin and the standard representation of $\mathscr{G}_{\mathbb{F}_p}$. Take any non-zero G_0-equivariant homomorphism $h : (V_0^+)^{\otimes 2} \to V_0$. After multiplying it by a rational number we may assume that $h\big((\mathscr{V}^+)^{\otimes 2}\big)$ is contained in $\mathscr{V}$ but not in $p\mathscr{V}$. Then the induced $\mathscr{G}_{\mathbb{F}_p}$-equivariant homomorphism $(\mathscr{V}^+/p\mathscr{V}^+)^{\otimes 2} \to \mathscr{V}/p\mathscr{V}$ is non-zero, as desired.

It remains to prove that the common dimension is ≤ 1. Let $T \subset G$ be a maximal torus and $N \subset G$ its normalizer. Then the space of G-invariants is contained in the space of N-invariants, and as in the proof of Proposition 6.6 it suffices to show that the latter has dimension ≤ 1. Recall that in the standard notation the weights of V are the ℓ-tuples $\pm e_i$, where the i^{th} entry of e_i is 1 and all other entries 0, and each such weight occurs with multiplicity 1. Choose a basis of associated eigenvectors $v_{\pm e_i}$. Similarly, the weights of V^+ are the tuples $\underline{\varepsilon}/2 = (\varepsilon_1,\ldots,\varepsilon_\ell)/2$ with $\varepsilon_i \in \{\pm 1\}$ and $\prod \varepsilon_i = 1$, and again each of them occurs with multiplicity 1. Choose a basis of associated eigenvectors $v_{\underline{\varepsilon}/2}$. Then the tensors $v_{\pm e_i} \otimes v_{\underline{\varepsilon}/2} \otimes v_{\underline{\varepsilon}'/2}$ form a basis of eigenvectors of $V \otimes (V^+)^{\otimes 2}$. Recall that the Weyl group of G is

$$N/T \cong S_\ell \ltimes \ker\big(\Pi\colon \{\pm 1\}^\ell \to \{\pm 1\}\big).$$

Here S_ℓ permutes transitively all possible e_i, and $\ker\big(\Pi\colon \{\pm 1\}^\ell \to \{\pm 1\}\big)$ permutes transitively all possible $\underline{\varepsilon}$. Thus each of the above basis vectors is conjugate under

N to one of the form $v_{\pm e_1} \otimes v_{\underline{1}/2} \otimes v_{\underline{\varepsilon}''/2}$ with $\underline{1} = (1, \ldots, 1)$. Now the subspace of T-invariants is generated by all basis vectors of weight zero. Clearly the weight $\pm e_1 + \underline{1}/2 + \underline{\varepsilon}''/2$ is zero if and only if $\pm e_1 = -e_1$ and $\varepsilon''_1 = 1$ and $\varepsilon''_2 = \ldots = \varepsilon''_\ell = -1$. In particular there is precisely one possible choice for the sign of $\pm e_1$ and for $\underline{\varepsilon}''$; hence the subspace of T-invariants is, as a representation of N, induced from a 1-dimensional representation of some subgroup of N. This implies that the space of N-invariants has dimension ≤ 1, as desired. **q.e.d.**

Proposition 6.11 *Consider an odd integer $\ell \geq 5$. Let V^+ denote the positive Spin representation of dimension $2^{\ell-1}$ of the connected semisimple group G of type D_ℓ, and let V denote the standard representation of $G \twoheadrightarrow \mathrm{SO}(2\ell)$ of dimension 2ℓ. Then up to scalar multiples there exists exactly one non-zero G-equivariant endomorphism of $(V^+)^{\otimes 2}$ of rank $\leq 2\ell$, and its image is isomorphic to V.*

Proof. The image of the desired endomorphism is a non-zero G-invariant subspace W of dimension $\leq 2\ell$; we will determine its possible weights. For this recall that the weights of V^+ are the tuples $\underline{\varepsilon}/2 = (\varepsilon_1, \ldots, \varepsilon_\ell)/2$ with $\varepsilon_i \in \{\pm 1\}$ and $\prod \varepsilon_i = 1$. Thus every weight μ of $(V^+)^{\otimes 2}$ is a tuple with all entries in $\{\pm 1, 0\}$ and an even number of entries 0. Since by assumption ℓ is odd, the number k of non-zero entries of μ is > 0. Note also that the Weyl group orbit of μ has size $\binom{\ell}{k} \cdot 2^k$ if $k < \ell$, respectively $2^{\ell-1}$ if $k = \ell$. Now if μ is a weight of W, this size must be $\leq \dim W \leq 2\ell$. If $k = \ell$ this implies that $2^{\ell-1} \leq 2\ell$, which is never true for $\ell \geq 5$. Thus $0 < k < \ell$, and the inequality $\binom{\ell}{k} \cdot 2^k \leq 2\ell$ implies that $k = 1$. This shows that all weights of W are Weyl group conjugate to the highest weight $(1, 0, \ldots, 0)$ of the standard representation V; hence W is an extension of copies of V. Since $0 < \dim W \leq 2\ell = \dim V$, it follows that $W \cong V$. The desired assertion thus follows from Proposition 6.10. **q.e.d.**

7 Proof of the main result

Now we return to the situation of §§1–4. To prove Theorem 1.1 we assume that $\mathrm{End}_{K^{\mathrm{sep}}}(\varphi) = A$ and must show that $G_\mathfrak{p} = \mathrm{GL}_{r,F_\mathfrak{p}}$. As in the proof of Proposition 2.3 we replace K by a finite separable extension to make $G_\mathfrak{p}$ connected. By Proposition 2.3 it is reductive and acts absolutely irreducibly on $V_\mathfrak{p}(\varphi)$. Set $L := F_\mathfrak{p}^{\mathrm{sep}}$ and abbreviate $G := G_\mathfrak{p} \times_{F_\mathfrak{p}} L$ and $V := V_\mathfrak{p}(\varphi) \otimes_{F_\mathfrak{p}} L$.

Fix a maximal torus $T \subset G$. By Corollary 4.7 we can find a closed point $x \in X$ whose associated Frobenius torus T_x becomes conjugate to T over $\mathrm{GL}_r(L)$. Choose an integral multiple my_x of the rational Newton cocharacter of T_x which is a true cocharacter, and let y be its conjugate cocharacter of T. Then Proposition 4.4 implies that the weights of y on V take exactly one non-zero value and, perhaps, the value 0. Furthermore, the tautological representation $T_x \hookrightarrow \mathrm{GL}_{r,F}$ is defined over F; hence its formal character is preserved by the action of $\mathrm{Aut}(\bar{F}/F)$ on $Y(T_x)_\mathbb{Q}$. Thus if Δ_x denotes the group of automorphisms of $T_{x,\bar{F}}$ that preserve this formal character, Proposition 4.3 implies that the Δ_x-conjugates of y_x generate $Y(T_x)_\mathbb{Q}$. Let Δ be the group of automorphisms of T that preserve the formal character of V. Then by conjugation it follows that the Δ-conjugates of y generate $Y(T)_\mathbb{Q}$. Altogether this shows that (G, V) satisfies the assumptions of Corollary 6.2.

From this point onwards we will forget Frobenius tori and concentrate on the representation theory of G. Let $G = G_0 \cdot G_1 \cdots G_d$ and $V \cong V_0 \otimes V_1 \otimes \cdots \otimes V_d$ be the decompositions from Corollary 6.2. By 6.2 (a) we have $G_0 = \mathbb{G}_m$ acting tautologically on V. Thus in the case $r = \dim V = 1$ we have $G = \mathbb{G}_m = \mathrm{GL}_1$, as desired.

So in the following we assume that $r > 1$. Then $d \geq 1$, and to prove Theorem 1.1 we must show that $d = 1$ and that (G_1, V_1) is of type (SL_r,Standard).

Using the theory of A-motives we can prove:

Lemma 7.1 *Consider positive integers ℓ and k. Assume that up to scalar multiples there exists exactly one non-zero G^{der}-equivariant endomorphism of $V^{\otimes \ell}$ of rank $\leq k$ and that its rank is k. Assume moreover that the same statement holds over $\bar{L} = \bar{F}_{\mathfrak{p}}$. Then $r \,|\, k\ell$.*

Proof. Since G is the product of G^{der} with the scalar torus $\mathbb{G}_m$, the same assumptions hold with G in place of G^{der}. Moreover, by dualizing they also hold for endomorphisms of $(V^{\otimes \ell})^* \cong (V^*)^{\otimes \ell}$. Furthermore by the construction of $G_{\mathfrak{p}}$ and G the G-equivariance is equivalent to the equivariance under $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. Let M_φ be the A-motive over K corresponding to φ as in Proposition 5.7. Then Proposition 5.8 and the isomorphy 5.5 imply that

$$(V^*)^{\otimes \ell} \cong \big(V_{\mathfrak{p}}(\varphi)^*\big)^{\otimes \ell} \otimes_{F_{\mathfrak{p}}} L \cong V_{\mathfrak{p}}(M_\varphi)^{\otimes \ell} \otimes_{F_{\mathfrak{p}}} L \cong V_{\mathfrak{p}}\big(M_\varphi^{\otimes \ell}\big) \otimes_{F_{\mathfrak{p}}} L.$$

Applying Proposition 5.6 to $M = M' = M_\varphi^{\otimes \ell}$ we deduce that the endomorphism in question comes from an endomorphism h of the A-motive $M_\varphi^{\otimes \ell}$. Let $N \subset M_\varphi^{\otimes \ell}$ denote its image. Then $V_{\mathfrak{p}}(N)$ is the image of the endomorphism $V_{\mathfrak{p}}(h)$ of $V_{\mathfrak{p}}\big(M_\varphi^{\otimes \ell}\big)$, whose dimension is k; hence N is an A-motive of rank k. On the other hand M_φ is a pure A-motive of weight $\frac{1}{r}$; hence $M_\varphi^{\otimes \ell}$ and N are pure A-motives of weight $\frac{\ell}{r}$. Thus Proposition 5.3 implies that $k \cdot \frac{\ell}{r} \in \mathbb{Z}$, as desired. **q.e.d.**

In the rest of the proof we distinguish cases according to Proposition 6.4.

The self-dual case: In the case 6.4 (a) the representation V_i of G_i is self-dual for every $1 \leq i \leq d$; hence V is self-dual as a representation of G^{der}. Thus Proposition 6.5 implies that up to scalar multiples there exists exactly one G^{der}-equivariant endomorphism of $V^{\otimes 2}$ of rank 1. Moreover, again by Proposition 6.5 the same holds over $\bar{L}$. Thus Lemma 7.1 for $\ell = 2$ and $k = 1$ implies that $r \,|\, 2$. For $r > 1$ the only possibility is $\dim V = r = 2$. Since $V \cong V_0 \otimes V_1 \otimes \cdots \otimes V_d$ with $\dim V_0 = 1$ and $\dim V_i \geq 2$ for all $1 \leq i \leq d$, this shows that $d = 1$ and $\dim V_1 = 2$. Thus the only possibility for G_1 is SL_2; hence $G = \mathbb{G}_m \cdot \mathrm{SL}_2 = \mathrm{GL}_2 = \mathrm{GL}_r$, as desired.

The SL_n-case: In the case 6.4 (b) there exist integers n, m with $\frac{n}{2} \geq m \geq 1$ such that each G_i for $1 \leq i \leq d$ is a quotient of SL_n and V_i comes from the representation $\Lambda^m(\text{Standard})$ of SL_n. Thus $r = \dim V = \binom{n}{m}^d$. Let ℓ be the largest integer such that $m\ell \leq n$. Then $m \leq n - m < m\ell \leq n$ and therefore $\ell \geq 2$. Proposition 6.8 thus states that up to scalar multiples there exists exactly one non-zero G_i-equivariant endomorphism of $V_i^{\otimes \ell}$ of rank $\leq \binom{n}{m\ell}$, and its image is isomorphic to the representation $\Lambda^{m\ell}(\text{Standard})$. In particular its rank is $\binom{n}{m\ell}$. Since this is so in each factor, we deduce that up to scalar multiples there exists exactly one non-zero G^{der}-equivariant endomorphism of $V^{\otimes \ell}$ of rank $\leq k := \binom{n}{m\ell}^d$, and its rank is k. Moreover, again by Proposition 6.8 the same statement holds over $\bar{L}$. Thus Lemma 7.1 implies that $r \,|\, k\ell$.

Lemma 7.2 *We have $n = m(\ell + 1) - 1$ and $d = 1$.*

Proof. In the relation

$$\binom{n}{m}^d \;=\; r \;\Big|\; k\ell \;=\; \binom{n}{m\ell}^d \cdot \ell \;\Big|\; \Big[\binom{n}{m\ell} \cdot \ell\Big]^d$$

we take d^{th} roots and deduce that

$$\binom{n-1}{m-1}\cdot\frac{n}{m} \;=\; \binom{n}{m} \;\Big|\; \binom{n}{m\ell}\cdot\ell \;=\; \binom{n-1}{m\ell-1}\cdot\frac{n}{m\ell}\cdot\ell \;=\; \binom{n-1}{n-m\ell}\cdot\frac{n}{m}$$

and hence

$$\binom{n-1}{m-1} \;\leq\; \binom{n-1}{n-m\ell}.$$

Since $n-m\ell \leq m-1 \leq \frac{n}{2}-1 < \frac{n-1}{2}$, the only way to satisfy this inequality is with $n-m\ell = m-1$. This proves the first equality. It also implies that

$$\binom{n}{m} \;=\; \binom{n}{m-1}\cdot\frac{n+1-m}{m} \;=\; \binom{n}{n-m\ell}\cdot\frac{m\ell}{m} \;=\; \binom{n}{m\ell}\cdot\ell$$

and hence

$$\binom{n}{m\ell}^d\cdot\ell^d \;=\; \binom{n}{m}^d \;=\; r \;\Big|\; k\ell \;=\; \binom{n}{m\ell}^d\cdot\ell.$$

Thus $\ell^d \,|\, \ell$, which by $\ell \geq 2$ implies that $d=1$, as desired. **q.e.d.**

Since $d=1$, we already know that $G^{\text{der}} = G_1$ is simple. To finish this case we repeat the arguments in Lemma 7.1 with a different representation to prove:

Lemma 7.3 *We have $r \,|\, n(\ell+1)$.*

Proof. Let $N \subset M_\varphi^{\otimes\ell}$ be the A-submotive from the proof of Lemma 7.1. Then $W := \big(V_{\mathfrak{p}}(N)\otimes_{F_{\mathfrak{p}}} L\big)^*$ is the image of the unique G^{der}-equivariant endomorphism of $V^{\otimes\ell}$ of rank $k=\binom{n}{m\ell}$, which in the present case is isomorphic to the representation $\Lambda^{m\ell}(\text{Standard})$ by Proposition 6.8. Since $m\ell = n+1-m$, the representation $V\otimes W$ of G^{der} is therefore isomorphic to $\Lambda^m(\text{Standard})\otimes\Lambda^{n+1-m}(\text{Standard})$. Proposition 6.9 thus shows that up to scalar multiples there exists exactly one non-zero G^{der}-equivariant endomorphism of $V\otimes W$ of rank $\leq n$, and its rank is n. Moreover, again by Proposition 6.9 the same statement holds over $\bar{L}$.

The rest of the proof proceeds as in Lemma 7.1. Since $G_0 = \mathbb{G}_m$ acts by scalars on V and W, the same statements follow with G in place of G^{der}. By dualizing the same holds again for $V^*\otimes W^*$ in place of $V\otimes W$. Now Proposition 5.8 and the isomorphy 5.5 imply that

$$V^*\otimes W^* \cong V_{\mathfrak{p}}(M_\varphi)\otimes_{F_{\mathfrak{p}}} V_{\mathfrak{p}}(N)\otimes_{F_{\mathfrak{p}}} L \cong V_{\mathfrak{p}}(M_\varphi\otimes N)\otimes_{F_{\mathfrak{p}}} L.$$

Applying Proposition 5.6 to $M = M' = M_\varphi\otimes N$ we deduce that the endomorphism in question comes from an endomorphism h' of the A-motive $M_\varphi\otimes N$. Let N' denote its image. Then $V_{\mathfrak{p}}(N')$ is the image of the endomorphism $V_{\mathfrak{p}}(h')$, whose dimension is n; hence N' is an A-motive of rank n. On the other hand M_φ and N are pure A-motives of respective weights $\frac{1}{r}$ and $\frac{\ell}{r}$; hence $M_\varphi\otimes N$ and N' are pure A-motives of weight $\frac{\ell+1}{r}$. Thus Proposition 5.3 implies that $n\cdot\frac{\ell+1}{r}\in\mathbb{Z}$, as desired. **q.e.d.**

From Lemmas 7.2 and 7.3 we now deduce

$$\binom{n-1}{m-1}\cdot\frac{n}{m} \;=\; \binom{n}{m} \;=\; r \;\Big|\; n(\ell+1) \;=\; n\cdot\frac{n+1}{m}$$

and hence

$$\binom{n-1}{m-1} \;\Big|\; n+1.$$

One easily shows that the only pairs of integers n, m with $\frac{n}{2}\geq m\geq 1$ and this property are those with $m=1$. Thus $m=1$ and $r=n$ and V_1 is the standard representation of SL_n. Since $d = 1$ we deduce that $G = \mathbb{G}_m \cdot \mathrm{SL}_n = \mathrm{GL}_n = \mathrm{GL}_r$, as desired.

The D_ℓ-case: In the case 6.4 (c) there exists an odd integer $\ell \geq 5$ such that each G_i is a Spin group of type D_ℓ and V_i its Spin^+ representation. Thus $r = \dim V = \left(2^{\ell-1}\right)^d = 2^{(\ell-1)d}$. Moreover Proposition 6.11 states that up to scalar multiples there exists exactly one non-zero G_i-equivariant endomorphism of $V_i^{\otimes 2}$ of rank $\leq 2\ell$, and its rank is 2ℓ. Since this is so in each factor, we deduce that up to scalar multiples there exists exactly one non-zero G^{der}-equivariant endomorphism of $V^{\otimes 2}$ of rank $\leq k := (2\ell)^d$, and its rank is k. Again the same statement holds over $\bar{L}$. Thus Lemma 7.1 with $\ell = 2$ implies that

$$2^{(\ell-1)d} \;=\; r \mid 2k \;=\; 2^{d+1}\ell^d.$$

Since ℓ is odd, this means that $(\ell-1)d \leq d+1$, which is impossible for $\ell \geq 5$ and $d \geq 1$. Thus the case 6.4 (c) does not occur. This finishes the proof of Theorem 1.1.

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