Lecture 10

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§23 The Dieudonné functor in the local-local case

Recall that k is a perfect field, $W = W_k$ is the Witt group scheme over k, W_n is the cokernel of V^n on W, and W_n^m is the kernel of F^m on W_n. The collection of all W_n^m becomes a direct system via the homomorphisms v and i:

$$\begin{tikzcd} W_n^m \arrow[r, hook, "i"] \arrow[d, hook, "v"] & W_n^{m+1} \arrow[d, hook, "v"] \\ W_{n+1}^m \arrow[r, hook, "i"] & W_{n+1}^{m+1} \end{tikzcd}$$

Let $\sigma : W(k) \longrightarrow W(k)$ denote the ring endomorphism induced by F. (We use a different letter to avoid confusion with F as an endomorphism of the group scheme W!)

Definition. Let E be the ring of "noncommutative polynomials" over $W(k)$ in two variables F and V, subject to the following relations:

- $F \cdot \xi = \sigma(\xi) \cdot F \qquad \forall \xi \in W(k)$
- $V \cdot \sigma(\xi) = \xi \cdot V \qquad \forall \xi \in W(k)$
- $FV = VF = p$

Note that E is a free left, or right, module over $W(k)$ with basis

$$\{\ldots, V^2, V, 1, F, F^2, \ldots\}.$$

Example. If $k = \mathbb{F}_p$, then $E = \mathbb{Z}_p[F, V]/(FV - p)$ is a regular commutative ring of Krull dimension 2. In all other cases, E is non-commutative.

Proposition 23.1. There exist unique ring homomorphisms $E \to \mathrm{Aut}(W_n^m)$ for all m, n such that F and V act as such and $\xi \in W(k)$ acts through multiplication by $\sigma^{-n}(\xi)$. Moreover, these actions of E are compatible with the transition homomorphisms i and v of the direct system.

Proof. For any $\xi \in W(k)$ and $x \in W$, the formulas in Proposition 21.1 imply that $F(\xi x) = \sigma(\xi) \cdot F(x)$ and $\xi \cdot V(x) = V(\sigma(\xi)x)$. On the other hand recall that $V \circ F = F \circ V = p \cdot \mathrm{id}$ by Theorem 14.4. Thus there is a unique action of E

on W, where F and V act as such and $\xi \in W(k)$ acts through multiplication by itself. The above relations also imply that this action induces a unique action of E on W_n and on W_n^m for all n and m. Moreover, the functoriality of F and V shows that the homomorphisms i and r are equivariant.

However, since $V = vr$, the relation $\xi \cdot V(x) = V(\sigma(\xi)x)$ implies that $\xi \cdot v(x) = v(\sigma(\xi)x)$. Thus in order to turn v into an E-linear homomorphism, we must modify the action of $W(k)$ by an appropriate power of σ. This is precisely what we accomplish by letting ξ act on W_n^m through multiplication by $\sigma^{-n}(\xi)$. Then E acts compatibly on the whole direct system. □

Definition. For any finite commutative group scheme G over k of local-local type we define

$$M(G) := \varinjlim_{m,n} \mathrm{Hom}(G, W_n^m),$$

with its induced left E-module structure via the actions of E on the W_n^m. Clearly this defines a left exact additive contravariant functor to the category of left E-modules.

Theorem 23.2. The functor M induces an anti-equivalence of categories

$$\left\{\left\{\begin{array}{l}\text{finite commutative}\\ \text{group schemes over}\\ k \text{ of local-local type}\end{array}\right\}\right\} \xrightarrow{\sim} \left\{\left\{\begin{array}{l}\text{left } E\text{-modules of}\\ \text{finite length with}\\ F \text{ and } V \text{ nilpotent}\end{array}\right\}\right\}.$$

This "main theorem of contravariant Dieudonné theory in the local-local case" is essentially a formal consequence of the results obtained so far. As a preparation note that the action of E on W_n^m via Proposition 23.1 and the embedding of W_n^m into the whole direct system induce homomorphisms of left E-modules

$$E_n^m := E/(EF^m + EV^n) \longrightarrow \mathrm{End}(W_n^m) \longrightarrow M(W_n^m).$$

Proposition 23.3. (a) These homomorphisms are isomorphisms.

(b) $\mathrm{length}_{W(k)} M(G) = \log_p |G|$.

Proof. As $W_n^m \hookrightarrow W_{n'}^{m'}$ is a monomorphism for all $n \leq n'$ and $m \leq m'$, the map $\mathrm{End}(W_n^m) \to M(W_n^m)$ is injective. By Lemma 22.1 it is also surjective, and hence bijective. Next Proposition 16.1 implies that

$$k \xrightarrow{\sim} E/(EF + EV) \xrightarrow{\sim} \mathrm{End}(\boldsymbol{\alpha}_p) \xrightarrow{\sim} M(\boldsymbol{\alpha}_p)$$

and hence (a) for $m = n = 1$. More generally, one easily checks that every non-trivial E-submodule of E_n^m contains the residue class of $F^{m-1}V^{n-1}$ (compare Proposition 23.9 below). Since the image of $F^{m-1}V^{n-1}$ in $\mathrm{End}(W_n^m)$ is non-zero, we deduce that the map $E_n^m \to \mathrm{End}(W_n^m)$ is injective.

Before finishing the proof of (a), we prove (b), using induction on $|G|$. The assertion is trivial when $|G| = 1$, and holds for $G = \boldsymbol{\alpha}_p$ by the above. Whenever $|G| \neq 1$ there exists a short exact sequence

$$0 \longrightarrow G' \longrightarrow G \longrightarrow \boldsymbol{\alpha}_p \longrightarrow 0,$$

and we may assume that (b) holds for G'. The induced sequence

$$0 \longleftarrow M(G') \longleftarrow M(G) \longleftarrow M(\boldsymbol{\alpha}_p) \longleftarrow 0 \tag{23.4}$$

is exact except possibly at $M(G')$. To prove the exactness there consider any element of $M(G')$, say represented by a homomorphism $\varphi : G' \to W_n^m$ for some m, n. Consider the morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & G' \arrow[r] \arrow[d, "\varphi"'] & G \arrow[r] \arrow[d] & \boldsymbol{\alpha}_p \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W_n^m \arrow[r] & H \arrow[r] & \boldsymbol{\alpha}_p \arrow[r] & 0
\end{tikzcd}$$

where H is the pushout of the left hand square. Applying Proposition 22.2 to the lower exact sequence yields a homomorphism $H \to W_{n+1}^{m+1}$ extending the homomorphism $iv \colon W_n^m \to W_{n+1}^{m+1}$. The composite homomorphism $G \to H \to W_{n+1}^{m+1}$ then defines an element of $M(G)$ which maps to the given element of $M(G')$. This proves that the sequence (23.4) is exact, and hence

$$\begin{aligned}
\operatorname{length}_{W(k)} M(G) &= \operatorname{length}_{W(k)} M(G') + \operatorname{length}_{W(k)} M(\boldsymbol{\alpha}_p) \\
&= \log_p |G'| + \log_p |\boldsymbol{\alpha}_p| \\
&= \log_p |G|,
\end{aligned}$$

proving (b).

Returning to (a) one directly calculates that $\operatorname{length}_{W(k)} E_n^m = nm$. By (b) and the beginning of §22, we also have $\operatorname{length}_{W(k)} M(W_n^m) = nm$. Thus $E_n^m \to \operatorname{End}(W_n^m)$ is an injective homomorphism of E-modules of equal finite length; hence it is an isomorphism, finishing the proof of (a). $\square$

Lemma 23.5. The functor M is exact.

Proof. By construction it is left exact. For any exact sequence $0 \to G' \to G \to G'' \to 0$, Proposition 23.3 (b) and the multiplicativity of group orders imply that the image of the induced map $M(G) \to M(G')$ has the same finite length over $W(k)$ as $M(G')$ itself. Thus the map is surjective, and M is exact. $\square$

Lemma 23.6. If $F_G^m = 0$ and $V_G^n = 0$, then F^m and V^n annihilate $M(G)$. In particular, the functor M lands in the indicated subcategory.

Proof. The first assertion follows from the definition of $M(G)$ and the functoriality of F and V, the second from the first and Proposition 23.3 (b). $\square$

Lemma 23.7. The functor M is fully faithful.

Proof. For given G, H choose m, n such that F^m and V^n annihilate G, H, and abbreviate $U := W_n^m$. By Proposition 22.6, we may choose a copresentation

$$0 \longrightarrow H \longrightarrow U^r \longrightarrow U^s$$

for some r, s. By the exactness of M, we obtain a presentation of E-modules

$$0 \longleftarrow M(H) \longleftarrow M(U)^r \longleftarrow M(U)^s.$$

Applying the left exact functors $\mathrm{Hom}(G, -)$ and $\mathrm{Hom}_E(-, M(G))$, we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(G,H) \arrow[r] \arrow[d, "M"'] & \mathrm{Hom}(G,U^r) \arrow[r] \arrow[d, "M"'] & \mathrm{Hom}(G,U^s) \arrow[d, "M"'] \\
0 \arrow[r] & \mathrm{Hom}_E(M(H),M(G)) \arrow[r] & \mathrm{Hom}_E(M(U^r),M(G)) \arrow[r] & \mathrm{Hom}_E(M(U^s),M(G))
\end{tikzcd}$$

where the vertical arrows are induced by the functor M. We must prove that the left vertical arrow is bijective. By the 5-Lemma it suffices to show that the other vertical arrows are bijective. Since M is an additive functor, this in turn reduces to direct summands of U^r and U^s. All in all, it suffices to prove the bijectivity in the case that $H = U = W_n^m$. For this consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(G,W_n^m) \arrow[r, "M"] \arrow[d] & \mathrm{Hom}_E(M(W_n^m),M(G)) \arrow[d, "\wr"', "23.3\,(a)"] \\
M(G) & \mathrm{Hom}_E(E_n^m,M(G)) \arrow[l, "\varphi([1]) \leftarrow\!\shortmid \varphi"']
\end{tikzcd}$$

Here the left vertical arrow is simply that induced by the embedding of W_n^m into the whole direct system; hence it is an isomorphism by Lemma 22.1. The lower horizontal arrow is an isomorphism by Lemma 23.6. Thus the upper horizontal arrow is an isomorphism, as desired. $\square$

Lemma 23.8. The functor M is essentially surjective.

Proof. Let N be a left E-module of finite length with F and V nilpotent. Suppose that F^m and V^n annihilate N. Then there exists an epimorphism of E-modules $(E_n^m)^{\oplus r} \twoheadrightarrow N$ for some r. Its kernel is again annihilated by F^m and V^n; hence there exists a presentation

$$(E_n^m)^{\oplus s} \xrightarrow{\varphi} (E_n^m)^{\oplus r} \longrightarrow N \longrightarrow 0.$$

Since $E_n^m = M(W_n^m)$ and M is fully faithful, we see that $\varphi = M(\psi)$ for a unique homomorphism $(W_n^m)^{\oplus r} \xrightarrow{\psi} (W_n^m)^{\oplus s}$. Setting $G(N) := \ker(\psi)$, the 5-Lemma shows that $N \cong M(G(N))$. $\square$

Piecing together the above results, we see that Theorem 23.2 is proven.

Proposition 23.9. "$\varinjlim_{m,n} W_n^m$" is the injective hull of $\boldsymbol{\alpha}_p$ in the associated category of ind-objects.

Proof. It is injective, because $\mathrm{Hom}(-, \text{“}\varinjlim_{m,n} W_n^m\text{”}) = M(-)$ is an exact functor. To show that is a hull, we must prove that any non-trivial subgroup scheme $G \subset W_n^m$ contains $i^{m-1}v^{n-1}(W_1^1) \cong \boldsymbol{\alpha}_p$. For this note first that W_n^m, and hence G, is an extension of copies of $\boldsymbol{\alpha}_p$. In particular there exists a monomorphism $\boldsymbol{\alpha}_p \hookrightarrow G$. On the other hand, Lemma 22.1 implies that $i^{m-1}v^{n-1}$ induces an isomorphism $\mathrm{Hom}(\boldsymbol{\alpha}_p, W_1^1) \xrightarrow{\sim} \mathrm{Hom}(\boldsymbol{\alpha}_p, W_n^m)$. Thus $i^{m-1}v^{n-1}(W_1^1)$ is the only copy of $\boldsymbol{\alpha}_p$ inside W_n^m, and so this copy must be contained in G, as desired. $\square$

Remark. For any abelian category $\mathfrak{C}$ with an injective cogenerator I one has a faithful exact contravariant functor $X \mapsto \mathrm{Hom}_{\mathfrak{C}}(X, I)$ to the category of left modules over $\mathrm{End}_{\mathfrak{C}}(I)$. If $\mathfrak{C}$ is artinian, i.e., if every object has finite length, one can show that this defines an anti-equivalence of categories from $\mathfrak{C}$ to the category of left modules of finite length over $\mathrm{End}_{\mathfrak{C}}(I)$. Above we have essentially done this for the category of finite commutative group schemes annihilated by F^m and V^n, with $I = W_n^m$ and $\mathrm{End}_{\mathfrak{C}}(I) = E_n^m$, and then taken the limit over all m, n.

Remark. Instead of the contravariant functor M above, one can define a covariant functor $G \mapsto \varinjlim_{m,n} \mathrm{Hom}(W_n^m, G)$ landing in right E-modules, where the W_n^m are viewed as an inverse system with transition epimorphisms r and f, and on which the action of $W(k)$ must be defined differently. The "main theorem of *covariant* Dieudonné theory in the local-local case" is then the direct analogue of Theorem 23.2 and can be proved similarly. It can also be deduced from Theorem 23.2 itself by showing that $N \mapsto \varinjlim_{m,n} \mathrm{Hom}_E(N, E_n^m)$ defines an antiequivalence between left and right E-modules of finite length with F and V nilpotent.