Determining Representations from Invariant Dimensions

This paper is motivated by the following "Tannakian" question: to what extent is a (complex) Lie group, G, and a finite dimensional representation, (ρ, V) of G, determined by the dimensions of the various invariant spaces W^G, where the W are obtained from V by linear linear algebra? That is, given $\dim(\mathrm{Sym}^2(V)^G)$, $\dim((\Lambda^3 V)^G)$, *etc.*, can one determine (G, V)? This problem arises, for instance, in the cohomological study of exponential sums. Given a finite extension E_λ of $\mathbf{Q}_\ell$, the field of ℓ-adic numbers, a base scheme X in characteristic $p \neq \ell$, and a representation $\sigma : \pi_1^{alg}(X, x_0) \to GL(n, E_\lambda)$, we take G to be the group of complex points of the algebraic group which gives the Zariski closure of $\sigma(\pi_1(X))$, and $(\rho, V = \mathbf{C}^n)$ the complexification of (σ, E_λ^n). By [Weil II], the invariant dimensions are determined. If σ is pure of weight zero, then G is actually semi-simple.

We make some simplifying assumptions. If $G \to GL(V)$ has kernel H, the most we can hope for is information about G/H. Hence, we assume V faithful. We also assume that G is connected and semi-simple except where the contrary is explicitly stated. Our main results are the following:

Theorem 1. *For any faithful finite dimensional represntation V of a connected semi-simple Lie group G, Lie(G) is uniquely determined by dimension data.*

Theorem 2. *If V is irreducible, (G, V) is uniquely determined up to abstract isomorphism by dimension data.*

Theorem 3. *In the general connected semi-simple case, (G, V) is not determined up to isomorphism by dimension data.*

There are several possible notions of dimension data. For instance, we might mean the data associating the dimension of W^G to each represntation $GL(V) \to GL(W)$. Or we might mean the data associating

$$\dim\left(\mathrm{Hom}_{S_n}\left(U, V^{\otimes n}\right)^G\right)$$

to every $n \in \mathbf{N}$ and every representation U of the symmetric group S_n (which acts on $V^{\otimes n}$ by permuting factors.) The latter formulation has the advantage that it makes sense even if we don't know $\dim(V)$. We will see that $\dim(V)$ can actually be deduced from $\left\{\dim\left(\left(V^{\otimes n}\right)^G\right)\right\}_n$, so by a standard Young tableau argument, the data of the first kind can be deduced from the data of the second kind. When we speak of dimension data, we intend either of these two equivalent notions.

§1. Sato-Tate Measure

let G be a connected complex reductive Lie group and $\rho : G \to GL(V)$ a faithful representation. Let K be a maximal compact subgroup of G and T a maximal torus of K. We choose a basis of V so that

$$\begin{array}{ccccccc}
\rho(T) & \subseteq & \rho(K) & \subset & \rho(G) & & \\
\cap\!\shortmid & & \cap\!\shortmid & & \cap\!\shortmid & & \\
(1)^n & \subset & U(n) & \subset & GL(\mathbf{C}) & = & GL(V).
\end{array}$$

Let $X^\natural$ denote the space of conjugacy classes in X. Then we have the commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\rho} & U(n) \\ p_K \downarrow & & \downarrow p_U \\ K^\natural & \xrightarrow{\rho^\natural} & U(n)^\natural \\ \pi_T \uparrow & & \uparrow \pi_U \\ T & \xrightarrow{\rho_T} & U(1)^n \end{array},$$

where $K^\natural \cong T/W$, $U(n)^\natural \cong U(1)^n/S_n$, and the maps π_t, π_U are the quotient maps. Let dk denote Haar measure on K. Given a representation $\sigma : GL(V) \to GL(W)$,

$$\dim(W^G) = \dim(W^K) = \int_K tr(\sigma\rho(x))dk = \int_{U(n)} tr(\sigma)\rho_* dk.$$

By the Peter-Weyl theorem, the values of these integrals determine $\rho_* dk$ and hence

$$p_{U_*}\rho_* dk = \rho_*^\natural p_{K_*} dk.$$

(If we restrict to tensor power representations $W = V^{\otimes k}$, by the Weierstrass approximation theorem $tr_*\rho_* dk$ is determined, so $d = \sup(\mathrm{supp}(tr_*\rho_* dk)$ is determined as well; see the introductory remarks on dimension data.) As $\mathrm{supp}(p_{K_*}dk) = K^\natural = T/W$,

$$Y = \mathrm{supp}(\pi_U^*\rho_*^\natural p_{K_*} dk) = \pi_U^{-1}(\rho^\natural(T/W)) = \bigcup_{\sigma\in S_n} \rho_T(T)^\sigma.$$

Now $\rho_T(T)$ is irreducible, so it is one of the irreducible components of Y. These components differ only by renumbering the coordinates of $U(1)^n$. We choose one such component and assume it is $\rho_T(T)$. As ρ is faithful, we can identify T with $\rho_T(T)$.

What we would like to know is the *Sato-Tate measure* $p_{K_*}dk$. From this we could immediately deduce

$$\pi_T^* p_{K_*} dk = \left(\prod_{\alpha\in\Phi}(1-\alpha(t))\right) dt = F_\Phi(t)dt,$$

by the Weyl integration formula. Here Φ denotes the roots of G, the non-zero characters in the restriction to T of the adjoint representation of K. By unique factorization of polynomials, this would give us Φ. Unfortunately, $\rho^\natural$ is not always injective, even on the complement of a set of measure zero. The set of weights of V may have symmetries outside the Weyl group W, in which case $\rho^\natural$ is generically many-one. This reflects the fact that symmetries are preserved by the operations of linear algebra, and that consequently, our dimension data bears only on a subcategory of $\mathrm{Rep}(K)$. This is the central difficulty in proving theorems 1 and 2, and it makes possible the counter-examples of theorem 3.

What we do know is the measure

$$\frac{1}{|W|}\pi_U^*\pi_{U_*}\rho_{T_*}F_\Phi(t)dt = \frac{1}{|W|}\pi_U^*\rho_*^\natural\pi_{T_*}F_\Phi(t)dt = \pi_U^*\rho_*^\natural p_{K_*}dk.$$

Restricting to $T = \rho_T(T)$, we obtain

$$\sum_{\sigma \in Stab_{S_n} T} \sigma^* F_P hi(t) dt. \tag{1}$$

We would like to understand which of the elements of S_n, acting on $U(1)^n$, fix T. A subtorus is determined by the characters which vanish on it, so we want to know which elements of S_n fix $\ker(X(U(1)^n) \to X(T)) \subset \mathbf{Z}^n$. If $\chi_1, \ldots, \chi_n$ are the characters on T obtained by projection onto the $U(1)$ factors, then σ lies in $\mathrm{Stab}_{S_n} T$ if and only if

$$\sum_i a_i \chi_i = 0 \iff \sum_i a_i \chi_{\sigma(i)} = 0.$$

In other words, the necessary and sufficient condition on σ is that, viewing χ_i as elements of $X(T) \otimes \mathbf{Q}$, there exists $g \in GL(X(T) \otimes \mathbf{Q}$ such that $g(\chi_i) = \chi_{\sigma(i)}$. The condition that σ act trivially on T is that $g = 1$. Therefore, if N is the order of the subgroup of $\S_n$ which acts trivially on T, equation (1) becomes

$$N \sum_{\gamma \in \Gamma)} \gamma^* F_P hi(t) dt, \tag{2}$$

where ρ_T is viewed as an element of the group ring $\mathbf{Z}[X(T) \otimes \mathbf{Q}]$, and Γ is the set of automorphisms of $X(T) \otimes \mathbf{Q}$ which preserves it. Thus dimension data determines ρ_T, *i.e.* the set of weights of ρ with multiplicity, and the averaged Weyl product

$$m_G = \frac{1}{N|\Gamma|\,|W|} \sum_{\gamma \in \Gamma} \sigma(F_P hi) \in \mathbf{Q}[X(T)].$$

(Note that this expression is normalized so that the [0] coefficient is 1.) On the other hand, if (ρ', V') is any representation of K such that $\Gamma' \supset \Gamma$, then $\dim(V'^K)$ equals the [0] coefficient in

$$\frac{1}{|W|} F_P hi \rho'_T$$

, or equivalently, by symmetry, the [0] coefficient in $m_G \rho'_T$. Therefore, (m_G, ρ_T) determines the dimension data.

Proposition.. *If G is a connected torus and (ρ, V) a faithful representation, dimension data determines (ρ, V) uniquely.*

Proof. As $T = G$, $\rho_T = \rho$ We have seen that dimension data determines ρ_T.

§2. A Root Argument

Let $\mathbf{g}$ be a complex semi-simple Lie algebra with Cartan subalgebra $\mathbf{t}$. Let X be the $\mathbf{Q}$-vector space spanned by the roots of $(\mathbf{g}, \mathbf{t})$. We can view a finite dimensional representation V of $\mathbf{g}$ as a finite set S of elements $s \in X$ taken with multiplicity $m(s)$. We define an inner product on X^* by setting

$$(x_1^*, x_2^*) = \sum_{s \in S} m(s) x_1^*(s) x_2^*(s).$$

We denote the dual inner product on X, $\langle\, , \rangle$. As long as V is faithful, S spans X and $(\, ,)$, and hence $\langle\, , \rangle$is positive definite. As S is Weyl-invariant, so is $\langle\, , \rangle$. Thus $\langle\, , \rangle$restricts to a non-zero multiple of the Killing form on every simple factor of X.

The main theorem in this section is the following:

Theorem 1'.. *Let X be a $\mathbf{Q}$-vector space with a positive definite inner product $\langle\, , \rangle$. Let Φ be a reduced root system in X with Weyl group W. Let $\mathbf{Z}\Phi \subset X$ denote the root lattice, and*

$$\Lambda_\Phi = \left\{ \lambda \in X \;\middle|\; \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbf{Z} \quad \forall \alpha \in \Phi \right\}$$

the weight lattice. If Λ is a lattice such that $\mathbf{Z}\Phi \subseteq \Lambda \subseteq \Lambda_\Phi$, and Γ is a group of isometries of X such that $W \subseteq \Gamma \subseteq Aut(\Lambda)$, then Φ is determined as an abstract root system by the 1*-dimensional subspace $\mathbf{Q}F \subset \mathbf{Q}[X]$, where*

$$F = \sum_{\gamma \in \Gamma} \gamma(F_\Phi), \quad F_\Phi = \prod_{\alpha \in \Phi} (1 - [\alpha]).$$

We observe that letting Φ be the root system of $\mathbf{g}$ and $X = \mathbf{Q}\Phi$ endowed with the inner product described above, Theorem 1' implies Theorem 1. The rest of this section is devoted to a proof of Theorem 1'. Root systems below will not necessarily be reduced.

Definition.. *A* short root *in a root system Φ is any root which is short in its irreducible component of Φ. We denote the set of short roots Φ°.*

Lemma 1. *The set Φ° is a root system.*

Proof. Every reflection $\sigma \in W(\Phi)$ fixes all but one component, Ψ of Φ. If $\alpha \notin \Psi$, $\sigma(\alpha) = \alpha$. If $\alpha \in \Psi$, $\|\sigma(\alpha)\|^2 = \|\alpha\|^2$, so $\sigma(\alpha)$ is short in Ψ and hence in Φ.

Lemma 2. *if Φ is root system, Λ a lattice such that $\mathbf{Z}\Phi \subseteq \Lambda \subseteq \Lambda_\Phi$, and Γ is a group of isometries such that $W(\Phi) \subseteq \Gamma \subseteq Aut(\Lambda)$, then $\Gamma\Phi$ is a (not necessarily reduced) root system.*

Proof. As Γ is contained in a compact (orthogonal) group and fixes a lattice, it is finite. Hence $\Gamma\Phi$ is finite. As Φ is a root system, it does not contin zero and it is closed under multiplication by -1. These properties are obviously inherited by $\Gamma\Phi$. If $\gamma\alpha \in \Gamma\Phi$, the reflection in $\gamma\alpha^\perp$ is

$$S_{\gamma\alpha} = \gamma S_\alpha \gamma^{-1}.$$

As $\Gamma \supseteq W(\Phi)$, γ and S_α belong to Γ, so $S_{\gamma\alpha} \in \Gamma$ fixes $\Gamma\Phi$. Finally, if $\gamma_1\alpha_1, \gamma_2\alpha_2 \in \Gamma\Phi$, then

$$\frac{2\langle \gamma_1\alpha_1, \gamma_2\alpha_2 \rangle}{\|\gamma_1\alpha_1\|^2} = \frac{2\langle \alpha_1, \gamma_1^{-1}\gamma_2\alpha_2 \rangle}{\|\alpha_1\|^2} \in \mathbf{Z},$$

since $\gamma_1^{-1}\gamma_2\alpha_2 \in \Gamma\Phi \subset \Lambda \subseteq \Lambda_\Phi$.

Lemma 3. *To prove Theorem 1', it suffices to prove it in the case that X is irreducible as Γ-module.*

Proof. Let X_i be a Γ-submodule of X, Λ_i the projection of Λ on X_i, and $\Gamma_i = \mathrm{im}(\Gamma \to GL(X_i))$. As $\Gamma \supseteq W(\Phi)$, irreducible Γ-modules are (orthogonal) direct sums of irreducible $W(\Phi)$-modules. The irreducible $W(\Phi)$-modules are spans of the irrudible components of the root system Φ, so each $X_i = \mathbf{Q}\Phi_i$ for some subroot systems Φ_i, with $\Phi = \sum_i \Phi_i$. Obviously $\Gamma_i\Lambda_i = \Lambda_i$, and $\Lambda_\Phi = \bigoplus \lambda_{\Phi_i}$ implies $\Phi_i \subseteq \Lambda_i \subseteq \Lambda_{\Phi_i}$. We project $\mathbf{Q}[X]$ onto $\mathbf{Q}[X_i]$ by mapping $[x] \mapsto 0 \quad \forall x \in X \backslash X_i$. the image of $\gamma(F_\Phi)$ is

$$\frac{|W(\Phi)|}{|W(\Phi_i)| \gamma(F_{\Phi_i})|} \gamma(F_{\Phi_i}),$$

so the image of F is

$$F_i = \frac{W(\Phi)}{W(\Phi_i)} \sum_{\gamma \in \Gamma} \gamma(F_{\Phi_i}).$$

All of these are determined by Γ and F. The Lie algebra of Φ is just the direct sum of the Lie algebras of the Φ_i, so to determine the former, it suffices to determine the latter.

Henceforth, X will always be assumed Γ-irreducible. This does not, of course, imply that Φ is irreducible as a root system.

Proposition.. *Under the hypotheses of Theorem 1', F determines $(\Gamma\Phi)^\circ$.*

Proof. As

$$\begin{aligned} F_\Phi &= \left[\prod_{\alpha \in \Phi^+} \left(\left[-\frac{\alpha}{2}\right] - \left[\frac{\alpha}{2}\right]\right)\right] \left[\prod_{\alpha \in \Phi^+} \left(\left[\frac{\alpha}{2}\right] - \left[-\frac{\alpha}{2}\right]\right)\right] \\ &= \left[\sum_{w \in W} \mathrm{sgn}(w)[-w\delta]\right] \left[\sum_{w \in W} \mathrm{sgn}(w)[w\delta]\right] \\ &= \sum_{w' \in W} w' \left(\sum_{w \in W} \mathrm{sgn}(w)[\delta - w\delta]\right), \end{aligned}$$

we have

$$F = \sum_{\gamma \in \Gamma} \gamma(F_\Phi) = |W| \sum_{\gamma \in \Gamma} \gamma \left(\sum_{w \in W} \mathrm{sgn}(w)[\delta - w\delta]\right).$$

Now,

$$\|\delta - w\delta\|^2 = 2\|\delta\|^2 - 2\langle \delta, w\delta \rangle = \langle 2\delta, \delta - w\delta \rangle,$$

and

$$\delta - w\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha - \frac{1}{2}\sum_{\alpha\in\Phi^+} w\alpha = \sum_{\alpha\in\Phi^+\cap w\Phi^-},$$

so

$$\|\delta - w\delta\|^2 = \sum_{\Phi^+\cap w\Phi^-} 2\langle\delta,\alpha\rangle.$$

If $\alpha = \sum r_i\alpha_i$, where the α_i are simple roots,

$$2\langle\delta,\alpha\rangle = \sum_i r_i\|\alpha_i\|^2.$$

Thus, if $w \neq 1$,

$$\|\delta - w\delta\|^2 \geq \min_{\alpha\in\Phi}\|\alpha\|^2,$$

with equality if and only if $W = S_\alpha$, and α is a short simple root. As all occurrences of terms of length $\min_{\alpha\in\Phi}\|\alpha\|^2$ in F_Φ (and therefore in F) have sign $\mathrm{sgn}(S_\alpha) = -1$, there can be no cancellation. Therefore the terms of minimal non-zero length in F are precisely the $[\Gamma\alpha]$, where $\gamma\in\Gamma$, and α is a root of minimal length in Φ. All short roots in $\Gamma\Phi$ are of this form since $\Gamma\alpha$ spans X.

Lemma 4. *Given a root system $\Psi = \Psi^\circ$ and a group Γ such that $W(\Psi)\subseteq\Gamma$, there exists a unique root system $\Psi^*\supseteq\Psi$ such that every root system Ω with $\Omega^\circ = \Psi$ and $W(\Omega)\subseteq\Gamma$ is contained in Ψ^*.*

Proof. Consider the set S of root systems Ω in X with $\Omega^\circ = \Psi$ and $W(\Psi)\subseteq\Gamma$. Since Ψ is an element, S is non-empty. We know (for instance by classification) that the short roots always generate the whole root lattice, and this, together with the fact that no root can be more than twice as long as the short roots, means that Ω is contained in a fixed finite set. Thus S is finite, and it suffices to prove that for any $\Omega_1,\Omega_2\in S$ there exists a root system $\Omega_3\in S$ containing both. We let W_{12} denote the subgroup of Γ generated by $W(\Omega_1)$ and $W(\Omega_2)$. We set $\Omega_3 = W_{12}(\Omega_1\cup\Omega_2)$. Given $\gamma\in W_{12}$, $\alpha\in\Omega_1\cup\Omega_2$,

$$S_{\gamma\alpha} = \gamma\S_\alpha\gamma^{-1}\in W_{12}.$$

The root lattices generated by Ψ, Ω_1, and Ω_2 all coincide, and W_{12} preserves this lattice. Therefore, Ω_3 satisfies the integrality condition for root systems. The Weyl group of Ω_3 is generated by the reflection $S_{\gamma\alpha}$, so it is just $W_{12}\subseteq\Gamma$.

Let $\Psi = ((\Gamma\Phi)^\circ)^*$. We have just seen that under the hypotheses of Theorem 1', we can construct Ψ. Moreover, $\Psi\supseteq\Phi$, and $W(\Psi)\subseteq\Gamma$. As Γ is the unique maximal element in the set S of root systems with $\Omega^\circ = \Gamma\Phi$ and $W(\Omega)\subseteq\Gamma$, it must be stable by Γ. Therefore, $\Gamma\subseteq\mathrm{Aut}(\Psi)$, and Ψ is isotypic. We will write it $m\bar\Psi$.

If $\bar\Psi$ is of type B_r, C_r, or D_r, it embeds canonically in $\bar\Psi' = BC_r$, and we replace Ψ in these cases by mBC_r. Note that in the case that $\bar\Psi = D_r$, $W(\bar\Psi')$ is larger than $W(\bar\Psi)$, so we can no longer assume that $\Gamma\supseteq W(\Psi)$. In the case $r = 2$, we cannot distinguish the root systems B_2 and C_2, and we choose arbitrarily, one of the two canonical embeddings of Ψ into mBC_2. In any case, we can construct from the data of Theorem 1' an isotypic root system $\Psi = m\bar{\Psi}$ which contains the root system Φ, and $\bar{\Phi}$ can be taken to be of type A, BC, E, F, or G. Moreover, the ranks of Φ and Ψ coincide. Each factor $\bar{\Psi}$ contains a (not necessarily irreducible) factor of Φ, which must have rank equal to the rank of $\bar{\Psi}$.

Proposition.. *The following list gives for each simple Ψ a complete list of reduced root subsystems Φ of equal rank:*

$$\begin{array}{rl}
A_r: & A_r. \\
BC_r: & \sum B_{b_i} + \sum C_{c_i} + \sum D_{d_i} \quad (\sum b_i + \sum c_i + \sum d_i = r.) \\
E_6: & E_6, A_5 + A_1, 3A_2. \\
E_7: & E_7, D_6 + A_1, A_5 + A_2, 2A_3 + A_1, A_7, D_4 + 3A_1, 7A_1. \\
E_8: & E_8, A_8, D_8, A_7 + A_1, A_5 + A_2 + A_1, 2A_4, 4A_2, A_2 + E_6, A_1 + E_7,\ D_6 + 2A_1, D_5 + \\
 & A_3, 2D_4, D_4 + 4A_1, 2A_3 + 2A_1, 8A_1. \\
F_4: & F_4, B_4, D_4^\ell, B_3 + A_1^s, A_3^\ell + A_1^s, C_4, D_4^s,\ C_3 + A_1^\ell, A_3^s + A_1^\ell, B_2, B_2 + 2A_1^\ell, B_2 + 2A_1^s, 4A_1^\ell, \\
 & 2A_1^\ell + 2A_1^s, 4A_1^s. \\
G_2: & G_2, A_2^\ell, A_2^s, A_1^\ell + a_1^s.
\end{array}$$

Here the superscripts ℓ and s denote long and short embeddings respectively. Moreover, we adopt the convention that $D_3 = A_3$, $B_1 = A_1^s$, $C_1 = A_1^\ell$, and $D_2 = A_1 + A_1$ with its canonical embedding in BC_2.

Proof. If $\Psi = A_r$, and $e_i - e_j, e_j - e_k \in \Phi$, then by reflection $e_k - e_i \in \Phi$. Therefore, the relation $i \sim j$ if $i = j$ or $e_i - e_j \in \Phi$ is an equivalence relation. If the equivalence classes have order $r_1, \ldots, r_k$, then $\Phi = \sum_i A_{r_i}$, and

$$\operatorname{rank}(\Phi) = \sum (r_i - 1) = r + 1 - k < r = \operatorname{rank}(\Psi)$$

unless $k = 1$. In this case, $\Phi = \Psi$.

If $\Psi = BC_r$, we define $i \sim j$ if and only if $i = j$, $e_i - e_j \in \Phi$, or $e_i + e_j \in \Phi$. This is an equivalence relation. Let S be an equivalence class for this relation. Then the roots

$$\{e_i, 2e_i, \pm e_i \pm e_j | i, j \in S\}$$

form a root system Φ_S which is a factor of Φ. Since $\operatorname{rank}(\Phi) = r$, $\operatorname{rank}(\Phi_S) = |S|$. Changing the signs and indices of e_i if necessary, we may assume that $e_1 - e_2, \ldots, e_{|S|-1} - e_{|S|} \in \Phi_S$. Therefore Φ_S contains $A_{|S|-1}$. As $\operatorname{rank}(\Phi_S) = |S|$, some e_i, $2e_i$ or $e_i + e_j$ must belong to Φ_S. In the first case $\Phi_S = B_{|S|}$, in the second, $\Phi_S = C_{|S|}$, and in the third, $\Phi_S = D_{|S|}$. If $\Psi = E_r$, all the roots of Ψ have equal length. Thus, $\alpha, \beta, \alpha + \beta \in \Psi$ implies

$$\|\alpha\|^2 = \|\beta\|^2 = \|\alpha + \beta\|^2; \quad \langle \alpha, \beta \rangle = \frac{-\|\alpha\|^2}{2}.$$

Therefore if $\alpha, \beta \in \Phi$,

$$-S_\alpha(\beta) = \beta - \frac{2\langle \alpha, \beta \rangle}{\|\alpha\|^2} = \alpha + \beta \in \Phi.$$

Hence, root subsystems of Ψ correspond to Lie subalgebras. The entries listed above are taken directly from [D], Table 10.

The root system F_4 has no angles of $\frac{\pi}{3}$, so the only simple root subsystems it has are $B_2 = C_2$, B_3, C_3, B_4, C_4, F_4, which have unique embeddings (up to conjugation), and A_1, D_3, D_4, which have two embedding each, one long and one short. Suppose Φ cannot be realized as a subsystem of $B_4 \subset F_4$. This means that Φ contains two short roots α and β such that $\alpha \neq \pm\beta$ and α and β are non-orthogonal. In other words Φ has a factor of C_3, C_4, D_3, or D_4. We see that $C_3^{\perp}$, $D_3^{s^{\perp}}$, and $D_3^{\ell^{\perp}}$ contain long, long, and short vectors in Φ, respectively. Thus,

$$\Phi \in \{F_4, C_4, D_4^s, D_4^\ell, C_3 + A_1^\ell, D_3^s + A_1^\ell, D_3^\ell + a_1^s\}.$$

Suppose, on the other hand that $\Phi \subseteq B_4$. We know that $B_4 \subset BC_4$, so Φ is a subset of BC_4 without any roots of longest length. (Note that BC_4 has roots of three different lengths, so Φ may still contain "long" roots. We have already enumerated the subsystems of BC_4. The ones that lie in B_4 are

$$\{B_4, D_4^\ell, B_3 + A_1^s, D_3^e ll + a_1^s, 2B_2, B_2 + D_2, 2D_2, B_2 + 2A_1^s, D_2 + 2A_1^s, 4A_1^s\}.$$

In this context, $D_2 = 2A_1^\ell$. This confirms the list given above.

The case $\Psi = G_2$ is trivial.

Given a pair of root systems Ψ_1, Ψ_2, we write

$$F_{\Psi_1,\Psi_2} = \sum_{\gamma \in \mathrm{Aut}(\Psi_2)} \gamma(F_{\Psi_1}).$$

Writing our original root system $\Phi = \sum_{i=1}^m \Phi_i$, where $\Phi_i \subseteq \bar{\Psi}$, the vector space

$$\mathbf{Q} \sum_{\gamma \in \mathrm{Aut}(\Psi)} \gamma(F) = \mathbf{Q}F_{\Phi,\Psi} = \mathbf{Q} \sum_{\sigma \in S_m} \sigma \left(\bigotimes_{i=1}^m (F_{\Phi_i,\bar{\Psi}}) \right)$$

can be computed from the data of Theorem 1'. We write $X = \bigoplus_{i=1}^m \bar{X}$, where $\bar{X} = \mathbf{Q}\bar{\Psi}$, and view $F_{\Phi_i,\bar{\Psi}}$ as an element of the i^th copy of $\mathbf{Q}[\bar{X}]$ in $\mathbf{Q}[X] = \bigotimes_{i=1}^m \mathbf{Q}[\bar{X}]$; the group S_m acts by permuting the factors of $\mathbf{Q}[X]$. If all the possible equal-rank root subsystems $\Omega \subseteq \bar{\Psi}$ have linearly independent $F_{\Omega,\bar{\Psi}}$, then $\mathbf{Q}F_{\Phi,\Psi}$ determines the multiplicity with which each factor occurs as Φ_i. Indeed, setting $Z = \mathrm{Span}(F_{\Omega,\bar{\Psi}})$, $F_{\Phi,\Psi}$ is a monomial in the polynomial algebra

$$\mathrm{Sym}^m(Z) \subset \mathrm{Sym}^m(\mathbf{Q}[\bar{X}]) \subset \mathbf{Q}[\bar{X}]^{\otimes m} = \mathbf{Q}[X],$$

and monomials with different m-tuples of exponents are linearly independent.

Lemma 5. *For $\bar{\Psi} \in \{A_r, E_r, F_4, G_2\}$, the different F_{Ω,Ψ_0} are linearly independent.*

Proof. The case $\bar{\Psi} = A_n$ is trivial. For E, F, and G, we enumerate the values of $\|\lambda\|^2$, where λ is the longest vector appearing in $F_{\Omega,\bar{\Psi}}$.

$\bar{\Psi}$	Ω	$\|\lambda\|^2$
E_6	E_6	312
E_6	$A_5 + A_1$	72
E_6	$3A_2$	24
E_7	E_7	789
E_7	$D_6 + A_1$	222
E_7	A_7	168
E_7	$A_5 + A_2$	78
E_7	$D_4 + 3A_1$	62
E_7	$2A_3 + A_1$	42
E_7	$7A_1$	14
E_8	E_8	2480
E_8	$E_7 + A_1$	800
E_8	D_8	560
E_8	$E_6 + A_2$	320
E_8	A_8	240
E_8	$D_6 + 2A_1$	224
E_8	$A_7 + A_1$	170
E_8	$D_5 + A_3$	140
E_8	$2D_4$	112
E_8	$A_5 + A_2 + A_1$	80
E_8	$2A_4$	80
E_8	$D_4 + 4A_1$	64
E_8	$2A_3 + 2A_1$	44
E_8	$4A_2$	32
E_8	$8A_1$	16
F_4	F_4	156
F_4	C_4	120
F_4	B_4	84
F_4	$C_3 + A_1^\ell$	58
F_4	D_4^ℓ	56
F_4	$B_3 + A_1^s$	36
F_4	D_4^s	28
F_4	$D_3^\ell + A_1^s$	21
F_4	$2B_2$	20
F_4	$B_2 + 2A_1^\ell$	14
F_4	$D_3^s + A_1^\ell$	12
F_4	$B_2 + 2A_1^s$	12
F_4	$4A_1^\ell$	8
F_4	$2A_1^\ell + 2A_1^s$	6

F_4	$4A_1^s$	4
G_2	G_2	28
G_2	A_2^ℓ	12
G_2	A_2^s	4
G_2	$A_1^\ell + A_1^s$	4

By considering the non-zero term with longest maximal vector, we deduce the theorem for E_6 and E_7 immediately from this table. Let $m_{\bar\Psi}(\Omega, \ell)$ be the sum of the coefficents of all the terms with $\| \|^2 = \ell$ in $F_{\Omega,\bar\Phi}$. Let

$$M_{\bar\Psi}(\Omega_1, \Omega_2, \ell_1, \ell_2) = \frac{m_{\bar\Psi}(\Omega_1, \ell_1) m_{\bar\Psi}(\Omega_2, \ell_1)}{m_{\bar\Psi}(\Omega_1, \ell_2) m_{\bar\Psi}(\Omega_2, \ell_2)}.$$

One checks that

$$M_{E_8}(A_5 + A_2 + A_1, 2A_4, 80, 76) \neq 1,$$

$$M_{F_4}(D_3^s + A_1^\ell, B_2 + 2A_2^s, 12, 10) \neq 1,$$

and

$$M_{G_2}(A_2^s, A_1^s + A_1^\ell, 4, 3) \neq 1,$$

which takes care of the remaining cases.

As we shall see in §4, the sets $\{F_{\Phi_i, BC_n}\}$ are not generally linearly independent for large n. We have, however, the following:

Lemma 6. *If $\Psi = BC_n$, $\Phi = \sum b_i B_i + \sum c_i C_i + \sum d_i D_i$, $\lambda = a_1 e_1 + \ldots + a_n e_n \in \mathbf{Z}\Psi$, and $f_{\Phi,\Psi}(\lambda)$ denotes the ratio of the $[\lambda]$ coefficent in $F_{\Phi,\Psi}$ to the $[0]$ coefficient, then*

$$f_{\Phi,\Psi}(ke_1) = \begin{cases} \frac{1}{2n} \sum_{i \geq \frac{k}{2}+1} d_i - \frac{1}{n} \sum_{i \geq \frac{k}{2}} c_i & \text{if } k > 0 \text{ even,} \\ \frac{-1}{2n} \sum_{i > \frac{k}{2}} b_i & \text{if } k > 0 \text{ odd.} \end{cases}$$

Proof. We know that

$$F_{\Phi,\Psi} \in \mathbf{Q} \sum_{\gamma \in \mathrm{Aut}(\Psi)} \gamma \left(\sum_{w \in W} \mathrm{sgn}(w)[\delta - w\delta] \right).$$

The orbit of ke_1 under $\mathrm{Aut}(\Psi)$ is $\pm ke_i$. The coefficent of $[0]$ in $\sum_{w \in W} \mathrm{sgn}(w)[w - \delta w]$ is 1, so $f_{\Phi,\Psi}(ke_1)$ is the average over the $\{\pm ke_i\}$ coefficients in $\sum_{w \in W} \mathrm{sgn}(w)[\delta - w\delta]$. The only way that $\delta - w\delta$ can lie in $\mathbf{Z}e_i$ is if $\Phi^+ \cap w\Phi^-$ is $\{e_i\}$, $\{2e_i\}$, or $\{e_i + e_j, e_i - e_j\}$ (cases B, C, and D respectively). In case B_k, w is a simple reflection and

$$w - w\delta \in \{e_i, 3e_i, \ldots, (2k-1)e_i\},$$

each value occurring once. In case C_k, w is again a simple reflection, and

$$w - \delta w \in \{2e_i, 4e_i, \ldots, 2ke_i\}.$$

In case D_k, $\operatorname{sgn}(w) = 1$, and

$$\delta - w\delta \in \{2e_i, 4e_i, \ldots, (2k-2)e_i\}.$$

The lemma follows immediately.

Lemma 7. *If $\Psi = BC_n$, the element $F_{\Phi,\Psi}$ determines Φ.*

Proof. There is a unique element $w \in W = W(\Phi)$ such that $w\delta = -\delta$. Therefore, $[2\delta]$ appears with sign $\operatorname{sgn}(w)$ in $\sum_{w\in W} \operatorname{sgn}(w)[\delta - w\delta]$; it is, moreover, the unique element of maximal length occurring with non-zero coefficient in $f_{\Phi,\Psi}$. Therefore, the elements of maximal length in $F_{\Phi,\Psi}$ constitute the orbit $\operatorname{Aut}(\Psi)(2\delta)$. Choose a representative from this orbit with non-negative coordinates $(a_1, a_2, \ldots, a_n)$. The number n_i of i with $a_i = k$ is

$$n_i = \begin{cases} \sum_{i\geq\frac{k}{2}} c_i + \sum_{i>\frac{k}{2}} d_i & k > 0 \text{ even}, \\ \sum_{i>\frac{k}{2}} b_i & k > 0 \text{ odd}, \\ \sum_i d_i & k = 0. \end{cases}$$

From this formula and the Lemma 6, this lemma follows immediately.

We now assemble the proof of Theorem 1'. We have already seen that we can construct a $\Psi = \bar{\Psi}^m$ which contains Φ. If $\bar{\Psi} \neq BC_r$, Φ is determined by dimension data. If $\bar{\Psi} = BC_k$, Ψ is canonically embedded in BC_{mk}. We can deduce $F_{\Phi,BC_{mk}}$ from $F_{\Phi,\Psi}$ by averaging over $\operatorname{Aut}(BC_{mk})$. The theorem now follows from Lemma 7.

§3. A Weight Argument

In this section we assume that V is irreducible and use information about the weights of V to find the abstract isomorphism class of (G, V). The idea is that for irreducible representations, (G, V) is almost determined by the $GL(X^*(T) \otimes \mathbf{Q})$-orbit of ρ_T, without any root data at all.

Let G be a connected semi-simple Lie group with Lie algebra $\mathbf{g}$. We write $\mathbf{g} = \bigoplus_{i=1}^k \mathbf{g}_i$, where the $\mathbf{g}_i$ are simple. The character group $X(T)$ satisfies

$$\mathbf{Z}\Phi \subseteq X(T) \subseteq \Lambda_\Phi;$$

indeed the complex connected Lie groups with algebra $\mathbf{g}$ are indexed by the sublattices of Λ_Φ which contain the root lattice. A finite dimensional representation, V, of $\mathbf{g}$ comes from a representation of G if and only if the weights of V lie in $X(T)$; V is faithful as a G-representation if and only if its weights generate $X(T)$. If (ρ, V) is a finite dimensional irreducible G-module, the corresponding $\mathbf{g}$-module is also irreducible; indeed, the weights of any $\mathbf{g}$-submodule lie in $X(T)$. The irreducible representations of $\mathbf{g}$ are indexed by dominant weights, so they are of the form $\bigotimes_{i=1}^k V_i$, where V_i are irreducible $\mathbf{g}_i$-modules. If G admits any faithful, irreducible representation V, then, it must be a product of simple Lie groups G_i, and V must be the exterior tensor product of faithful irreducible representations V_i of G_i.

As in §2, we endow $X = X(T) \otimes \mathbf{Q}$ with a positive definite inner product $\langle\, , \rangle$under which the automorphism group $\Gamma = \operatorname{Aut}(\rho_T)$ acts by isometries. As ρ is faithful and irreducible, the root lattice $\mathbf{Z}\Phi$ is generated by the differences $\alpha - \beta$, where α and β range over the set of weights of V. We say that a lattice Λ in an inner product space *factors* as $\Lambda_1 \times \Lambda_2$ if $\Lambda = \Lambda_1 + \Lambda_2$ and $\Lambda_1 \perp \Lambda_2$. A lattice is *irreducible* if it does not have a non-trivial factorization. It is well-known that with respect to a positive definite inner product, factorization into irreducible lattice is unique.

Proposition. *Every simple root lattice Φ except B_r, $r \geq 2$, is irreducible.*

Proof. By §2, Lemma 1, the set of short roots, Φ°, forms a root system. Except when $\Phi = B_r$, this root system is irreducible. It is known that the shortest non-zero vectors in a simple root lattice is the set of short roots. If $\Lambda = \mathbf{Z}\Phi = \mathbf{Z}\Phi^\circ$ splits as $\Lambda_1 \times \Lambda_2$, every short root α must lie in $\Lambda_1 \cup \Lambda_2$, because the projections of α onto the two factors must have length $\leq \|\alpha\|$. But non-orthogonal pairs of roots must lie in the same orthogonal factor, so if Φ° is irreducible, so is $\mathbf{Z}\Phi$.

As in §2, we may assume, without loss of generality, that $X(T) \otimes \mathbf{Q}$ is irreducible as Γ-module. By the unique factorization property for lattices, we may therefore assume that $\mathbf{Z}\Phi$ is isotypical, *i.e.,* that $\mathbf{Z}\Phi = \Lambda^{\times k}$, where Λ is irreducible. Moreover, Γ must act transitively on the factors Λ, so $\rho_T = \bigotimes_{i=1}^k \rho_{T_i}$, where the ρ_{T_i} are equal. (This, does not, unfortunately, mean that the ρ_i are equal. For instance, the standard ($2n$-dimensional) representations of C_n and D_n are isomorphic as representations of macimal tori $U(1)^n$.)

Lemma. *The lattices $\mathbf{Z}\Phi(A_i)$, $\mathbf{Z}\Phi(C_i)$, $\mathbf{Z}\Phi(D_i)$, $\mathbf{Z}\Phi(E_i)$, $\mathbf{Z}\Phi(F_4)$, and $\mathbf{Z}\Phi(G_2)$ satisfy only the following similarity relations: $\mathbf{Z}\Phi(A_2) \sim \mathbf{Z}\Phi(G_2)$, $\mathbf{Z}\Phi(A_3) = \mathbf{Z}\Phi(D_3) \sim \mathbf{Z}\Phi(C_3)$, $\mathbf{Z}\Phi(C_4) \sim \mathbf{Z}\Phi(D_4) \sim \mathbf{Z}\Phi(F_4)$, and $\mathbf{Z}\Phi(C_n) \sim \mathbf{Z}\Phi(D_n)$, $n \geq 5$.*

Proof. We list the number of vectors of shortest non-zero length in each lattice:

Type	Dimension	Number
A	n	$n(n+1)$
$C \sim D$	n	$2n(n-1)$
E	6	72
E	7	126
E	8	240
F	4	24
G	2	6

We see immediately that the only possible similarities are the ones enumerated above. That they do, in fact, occur, is obvious except for the fact that $\mathbf{Z}\Phi(F_4)$ is similar to $\mathbf{Z}\Phi(C_4) \sim \mathbf{Z}\Phi(D_4)$. To see this, we can view $\mathbf{Z}\Phi(F_4)$ as the ring of Hurwitz quaternions $\mathbf{Z}\left[i, j, k, \frac{1+i+j+k}{2}\right]$, and multiply on the left by $1+i$.

We can therefore, break down the problem into the following seven cases for Φ:

1) $\sum_{i\geq 1} b_i B_i \quad (B_1 = A_1)$
2) $aA_2 + gG_2$
3) $aA_3 + cC_3$
4) $aA_n \quad (n \geq 4)$

5) $cC_4 + dD_4 + fF_4$
6) $cC_n + dD_n \quad (n \geq 5)$
7) eE_n

By Theorem 1, we know the Lie algebra of G, so we know exactly what the constants a, b, c, *etc.*, are. In fact, having come this far, it is easy to give a self-contained proof that Lie algebra is well-determined. For example, in cases 4 and 7 it is obviously so, and cases 2, 3, and 6 follow immediately from an examination of the length of δ (which we know from m_G.) Cases 1 and 5 are slightly more involved, but an examination of 2δ as an element of the root lattice is sufficient.

In any event, we know the Lie algebra of each factor G_i , and for each factor we know the restriction of the representation ρ_i to T_i. More precisely, we know ρ_{T_i} as an element of the group algebra on the vector space spanned by the root lattice of G_i. This determines ρ_{T_i} as a representation of $Lie(G)$, and hence as a representation $\rho_{\tilde{G}}$ of the simply connected form of $Lie(G)$. Then G is determined uniquely by the criterion that ρ be faithful on G; it is the quotient of $\tilde{G}$ by the (finite) kernel of $\rho_{\tilde{G}}$.

§4. Counter-examples

Let $Z_n = \mathbf{Q}[\mathbf{Z}^n]$, $W_n = (\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n$. For $m \leq n$, the injection

$$\mathbf{Z}^m \hookrightarrow \mathbf{Z}^n : (a_1, \ldots, a_m) \mapsto (a_1, \ldots, a_m, 0, \ldots, 0)$$

extends to an injection $i_{m,n} : Z_m \to Z_n$. We define $\phi_{m,n} : Z_m \to Z_n$:

$$\phi_{m,n}(z) = \frac{|W_m|}{|W_n|} \sum_{w \in W_n} w(i_{m,n}(z)).$$

Evidently $\phi_{m,n}\phi_{k,m} = \phi_{k,n}$ for $k \leq m \leq n$. The image of $\phi_{m,n}$ lies in $Y_n = Z_n^{W_n}$, so we can form the direct limit under $\phi_{m,n}$:

$$Y = \varinjlim_n Y_n.$$

We define maps $j_n : Z_n \to Y$ by composing $i_{n,p}$ with the injection $Y_p \subset Y$ for any $p \geq n$. The maps $\phi_{m,n}$ are not ring homomorphisms, so *a priori* Y is only a vector space. It is endowed with an algebra structure as follows: The canonical isomorphism $\mathbf{Z}^m \oplus \mathbf{Z}^n \tilde{\to} \mathbf{Z}^{m+n}$ gives a canonical isomorphism $M : Z_m \otimes Z_n \tilde{\to} \mathbf{Z}^{m+n}$. Given two elements of Y represented by $y \in Y_m$ and $y' \in Y_n$, we define

$$yy' = j_{m+n}(M(y \otimes y')).$$

This product is independent of the choice of m and n and is commutative and associative.

Lemma. *With respect to this product, $Y \cong \mathbf{Q}[x_1, x_2, \ldots]$.*

Proof. Each Z_n is generated by monomials $[e_1]^{a_1} \cdots [e_n]^{a_n}$. Therefore, Y has basis

$$e(a_1, a_2, \ldots a_n) = j\left([e_1]^{a_1} \cdots [e_n]^{a_n}\right),$$

ndexed by n and integers $a_1 \geq a_2 \geq \cdots \geq a_n > 0$. We identify this element with

$$\prod_{i=1}^{n} x_{a_i}$$

. We verify that

$$\begin{aligned} & e(a_1, \ldots, a_m) e(b_1, \ldots, b_n) \\ &= \frac{1}{|W_p||W_q||W_r|} \sum_{w_p \in W_p} w_p \left(\left(\sum_{w_q \in W_q} w_q \left([e_1]^{a_1} \cdots [e_m]^{a_m} \right) \right) \left(\sum_{w_r \in W_r} w_r \left([e_{m+1}]^{b_1} \cdots [e_{m+n}]^{b_n} \right) \right) \right) \\ &= \frac{1}{|W_p|} \sum_{w_p \in W_p} w_p \left([e_1]^{a_1} \cdots [e_m]^{a_m} [e_{m+1}]^{b_1} \cdots [e_{m+n}]^{b_n} \right) \\ &= e(a_1, \ldots, a_m, b_1, \ldots, b_n), \end{aligned}$$

where $p \geq q, r$, $q \geq m$, $r \geq n$. The lemma follows immediately.

For $\Phi = \sum b_i B_i + \sum c_i C_i + \sum d_i D_i$, Φ admits a canonical embedding in $\mathbf{Z}^n$, $n = \operatorname{rank}(\Phi)$. Therefore, we can define $F(\Phi) = j_n(F_\Phi) \in Y$, where F_Φ, as usual, denotes the Weyl product $\prod_{\alpha \in \Phi}(1 - [\alpha])$. By construction, $F(\Phi_1 + \Phi_2) = F(\Phi_1)F(\Phi_2)$.

Lemma. *There exist integers r and k and distinct root systems $\Phi_1, \ldots, \Phi_k$ of rank r such that $F(\Phi_1), \cdots, F(\Phi_k)$ are linearly dependent in Y.*

Proof. We recall that

$$F_\Phi = \sum_{w_1 \in W(\Phi)} w_1 \left(\sum_{w_2 \in W(\Phi)} \operatorname{sgn}(w_2)[\delta - w\delta] \right).$$

For root systems B_n, C_n, and D_n, δ is $\left(\frac{1}{2}, \frac{3}{2}, \ldots, \frac{2n-1}{2}\right)$, $(1, 2, \ldots, n)$, and $(0, 1, \ldots, n-1)$ respectively, so every coordinate in every n-tuple $w_1\delta - w_1 w_2 \delta$ is an integer between $-2n$ and $2n$. Therefore,

$$F(\Phi(B_n)),\ F(\Phi(C_n)),\ F(\Phi(D_n)) \in \mathbf{Q}[x_1, \ldots, x_{2n}].$$

Hence,

$$\frac{F(\Phi(B_k))}{F(\Phi(A_1))^k}, \frac{F(\Phi(C_k))}{F(\Phi(A_1))^k}, \frac{F(\Phi(D_k))}{F(\Phi(A_1))^k} \in \mathbf{Q}\left[x_1, \ldots, x_{2n}, \frac{1}{F(\Phi(A_1))}\right], \tag{1}$$

for $3 \leq k \leq n$. Setting $n = 8$, we obtain 18 elements in a ring with 17 generators. Therefore, the expressions in equation (1) satisfy the a polynomial equation, which can be taken to have rational coefficients. Equivalently, the expressions

$$\prod_i \left(\frac{F(\Phi(B_i))}{F(\Phi(A_1))^i} \right)^{b_i} \prod_j \left(\frac{F(\Phi(C_j))}{F(\Phi(A_1))^j} \right)^{c_j} \prod_k \left(\frac{F(\Phi(D_k))}{F(\Phi(A_1))^k} \right)^{d_k}$$

are linearly dependent. Multiplying through by a sufficiently high power, $F(\Phi(A_1))^r$, to clear denominators, we see that expressions

$$F(\Phi(aA_1 + \sum_i b_i B_i + \sum_j c_j C_j + \sum_k d_k D_k))$$

are linearly dependent. Moreover, all are of the form $F(\Phi)$, for Φ a root system of rank r.

Given such a linear dependence between $F(\Phi(\mathbf{g}_1)), \ldots, F(\Phi(\mathbf{g}_k))$, for rank r Lie algebras $\mathbf{g}_i$, we construct two representations, V and V', of $\mathbf{g} = \bigoplus_{i=1}^k \mathbf{g}_i$ such that the pairs $(\mathbf{g}, V)$ and $(\mathbf{g}, V')$ have the same dimension data but are not abstractly isomorphic. More precisely, we construct elements $v_1, \ldots, v_k$ in Z_r, such that for each i, j there exists a representation $V_{i,j}$ of $\mathbf{g}_i$ with $\rho_T = v_j$. Then

$$V = \bigoplus_{\sigma \in A_k} V_{1\sigma(1)} \otimes \cdots \otimes V_{k\sigma(k)},$$

and

$$V' = \bigoplus_{\sigma \in S_k \setminus A_k} V_{1\sigma(1)} \otimes \cdots \otimes V_{k\sigma(k)}.$$

Lemma. *We can choose v_i and V_{ij} as above so that the subgroup of $GL(\mathbf{Q}^{rk})$ which preserves*

$$\bigoplus_{\sigma \in A_k} V_{1\sigma(1)} \otimes \cdots \otimes V_{k\sigma(k)} \in \mathbf{Z}\left[\mathbf{Z}^{rk}\right]$$

is $W_r^k \rtimes A_k$.

Proof. Given any semi-simple Lie algebra $\mathbf{g}$ with weight lattice $X = \Lambda_{\Phi(\mathbf{g})}$ and Weyl group W, every element $\rho \in \mathbf{Z}[X]^W$ corresponds to a virtual representation of $\mathbf{g}$. The condition that ρ correspond to an effective representation can be expressed by saying that the coefficient of every vector $x \in X$ must be larger than some linear combination of the coefficients of vectors of greater length than x. In particular, if we start with some value of x and declare that its coefficient is 1 and that no longer vector has non-zero coefficient, we can then proceed inward, making each coefficient sufficiently large as we go. Of course, in choosing the v_i we have to satisfy effectivity conditions for many Lie algebras simultaneously, but we can always satisfy a finite number of conditions of the form $x > C_i$. If we choose v_1 with longest vector $(1, 2, \ldots, r)$, v_2 with longest vector $(r+1, r+2, \ldots, 2r)$, and so on, we see that the orbit of $v_1 \otimes \ldots \otimes v_k$ under W_{kr}/W_r^k consists of linearly independent elements of $\mathbf{Z}[\mathbf{Z}^{kr}]$. Indeed, each $\sigma(v_1 \otimes \ldots \otimes v_k)$ possesses a $\sigma([1, 2, \ldots, kr])$ term, which no other $\tau(v_1 \otimes \ldots \otimes v_k)$ can have. We conclude that the trace of $v_1 \otimes \ldots \otimes v_k$ under A_k is invariant by $W_k^r \rtimes A_k$ and no more.

We can now prove Theorem 3. We have constructed representations V and V' of $\mathbf{g}$. The pairs $(\mathbf{g}, V)$ and $(\mathbf{g}, V')$ cannot be isomorphic because the set of automorphisms of the weight lattice, X, of $\mathbf{g}$ which take V to V' is

$$S = W_k^r \rtimes S_k \setminus W_k^r \rtimes A_k;$$

no automorphism of $\mathbf{g}$ can act on X by any element of S, because the $\mathbf{g}_i$ are pairwise non-isomorphic. On the other hand, if $\sigma \in S$,

$$\sigma(\rho_T) = \rho'_T,$$

where ρ_T and ρ'_T are the elements of $\mathbf{Z}[X]$ corresponding to V and V' respectively. On the other hand,

$$\sigma(m_G) = \sigma\left(\frac{1}{|W_r^k \rtimes A_k|}\sum_{\gamma\in W_r^k\rtimes A_k}\gamma(F_\Phi)\right) =$$
$$\sigma\left(\frac{1}{|W_r^k \rtimes A_k|}\sum_{\gamma\in W_r^k\rtimes A_k}\gamma\left(F_{\Phi_1}\otimes\cdots\otimes F_{\Phi_k}\right)\right) =$$
$$\frac{1}{|W_r^k \rtimes A_k|}\sum_{\gamma\in W_r^k\rtimes A_k}\gamma\left(F_{\Phi_1}\otimes\cdots\otimes F_{\Phi_k}\right) = m_G = m'_G,$$

since

$$F_{\Phi_1}\wedge\cdots\wedge F_{\Phi_k} = 0.$$

Therefore,

$$\sigma(\rho_T, m_G) = (\rho'_T, m'_G),$$

so by §1, dimension data is the same for V and V'.

§5. Sharper Results

Proposition. *Our main theorems are effective; that is, a finite amount of dimension data suffices to determine the Lie(G) (resp. (G,V)) under the hypotheses of Theorem 1 (resp. 2).*

Proof. First we determine the order w of the group of scalar matrices in $\rho(G)$; it is the smallest positive integer k for which $\dim\left(\left(V^{\otimes k}\right)^G\right) > 0$. Replacing V by $W = V^{\otimes w}$, we may assume $G \xrightarrow{\rho} GL(W)$. We have the exterior square map $SL(W) \to Sp(W\wedge W)$. It is a classical result [W] that

$$\dim\left(\left(\left(C^{2n}\right)^{\otimes 2m}\right)^{Sp(2n)}\right) = \frac{(2m)!}{2^m m!}, \quad \forall m \le n.$$

Therefore,

$$\dim\left(\left((W\wedge W)^{\otimes 2m}\right)^G\right) \ge \dim\left(\left((W\wedge W)^{\otimes 2m}\right)^{Sp(W\wedge W)}\right) > m^{m/2},$$

if

$$m \le \frac{\dim(W\wedge W)}{2} \le \frac{\dim(V)^2 - \dim(V)}{4}. \tag{1}$$

On the other hand,

$$\dim\left(\left((W\wedge W)^{\otimes 2m}\right)^G\right)\leq \dim\left((W\wedge W)^{\otimes 2m}\right) < \dim(V)^{2mw}.$$

For $m > \dim(V)^{4w}$, then,

$$\dim\left(\left((W\wedge W)^{\otimes 2m}\right)^G\right) < m^{m/2}. \tag{2}$$

As soon as we reach an m for which equation (2) holds, we know that equation (1) cannot hold, which gives us an upper bound on $\dim(V)$. This reduces the possibilites to a finite set, which by our previous results we know we can distinguish.

Proposition. *Theorem 1 holds under the weaker hypothesis that G is semi-simple and has a finite set of connected components.*

Proof. As in §1, we let K denote the compact real form of G and consider $\rho_* dk$. As ρ is faithful, some neighborhood of the identity in $GL(V)$ is disjoint from $\rho(K\setminus K^\circ)$, where K° denotes the identity component of K. Therefore, the components of $\mathrm{supp}(\rho_*^\natural p_{K_*} dk)$ which pass through the identity are precisely the components of the support of

$$\mu = \rho_*^\natural p_{K_*^\circ}\left(dk|_{K^\circ}\right)$$

Moreover, since μ is analytic (in fact polynomial), it is determined by its germ at the identity matrix. This reduces the problem to that of $(K^\circ, \rho|_{K^\circ})$, which is treated in Theorem 1.