Image of the Group Ring of the Galois Representation associated to Drinfeld modules

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Abstract

Let φ be a Drinfeld A-module of arbitrary rank and arbitrary characteristic over a finitely generated field K, and set $\mathrm{G}_K = \mathrm{Gal}(K^{sep}/K)$. Let $E = \mathrm{End}_K(\varphi)$. We show that for almost all primes $\mathfrak{p}$ of A the image of the group ring $A[\mathrm{G}_K]$ in $\mathrm{End}_A\big(T_\mathfrak{p}(\varphi)\big)$ is the commutant of E. In the special case $E = A$ it follows that the representation of G_K on the $\mathfrak{p}$-torsion points $\varphi[\mathfrak{p}](K^{sep})$ of φ is absolutely irreducible for almost all $\mathfrak{p}$.

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0 Introduction

Let $\mathbb{F}_q$ be a finite field with q elements and of characteristic p. Let F be a finitely generated field of transcendence degree 1 over $\mathbb{F}_q$. Let A be the ring of elements of F which are regular outside a fixed place ∞ of F. Let K be another finitely generated field over $\mathbb{F}_q$ of arbitrary transcendence degree. Denote by K^{sep} the separable closure of K inside a fixed algebraic closure $\bar{K}$ and by $\mathrm{G}_K := \mathrm{Gal}(K^{sep}/K)$ the absolute Galois group of K. Let

$$\varphi : A \to K\{\tau\},\ a \mapsto \varphi_a$$

be a Drinfeld A-module over K of rank r and arbitrary characteristic $\mathfrak{p}_0$. (For the general theory of Drinfeld modules see for example Drinfeld [2], Deligne and Husemöller [1], Hayes [5] or Goss [4, Chapter 4].) For any ideal $\mathfrak{a} \not\subset \mathfrak{p}_0$ of A, the $\mathfrak{a}$-torsion

$$\varphi[\mathfrak{a}] := \bigcap_{a\in\mathfrak{a}} \mathrm{Ker}(\varphi_a : \mathbb{G}_{a,K} \longrightarrow \mathbb{G}_{a,K})$$

is a finite étale subgroup scheme of $\mathbb{G}_{a,K}$. By Lang's theorem, its geometric points

$$\varphi[\mathfrak{a}](K^{sep}) = \{x \in K^{sep} \mid \forall a \in \mathfrak{a} : \varphi_a(x) = 0\}$$

form a free $A/\mathfrak{a}$-module of rank r. For any prime $\mathfrak{p} \neq \mathfrak{p}_0$ of A, the $\mathfrak{p}$-adic Tate module

$$T_\mathfrak{p}(\varphi) := \varprojlim \varphi[\mathfrak{p}^n](K^{sep})$$

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of φ is a free $A_{\mathfrak{p}}$-module of rank r, where $A_{\mathfrak{p}}$ denotes the completion of A at $\mathfrak{p}$. It carries a continuous Galois representation

$$\rho_{\mathfrak{p}} : \mathrm{G}_K \longrightarrow \mathrm{Aut}_{A_{\mathfrak{p}}}\big(T_{\mathfrak{p}}(\varphi)\big) \cong \mathrm{GL}_r(A_{\mathfrak{p}}).$$

By construction, its reduction modulo $\mathfrak{p}$ is the continuous Galois representation on the module of $\mathfrak{p}$-torsion

$$\bar{\rho}_{\mathfrak{p}} : \mathrm{G}_K \longrightarrow \mathrm{Aut}_{\kappa_{\mathfrak{p}}}\big(\varphi[\mathfrak{p}](K^{sep})\big) \cong \mathrm{GL}_r(\kappa_{\mathfrak{p}})$$

over the residue field $\kappa_{\mathfrak{p}} := A/\mathfrak{p}$. We call it the residual representation at $\mathfrak{p}$.

If K is of transcendence degree 1 and $\mathrm{End}_K(\varphi) = A$, this representation is known to be irreducible for almost all $\mathfrak{p}$ (see Corollary 1.4). The aim of this paper is to strengthen and generalize this result. Our first main result is the following

Theorem 0.1 (Absolute irreducibility of the residual representation) *Let φ be a Drinfeld A-module of rank r over a finitely generated field K. Assume that $\mathrm{End}_K(\varphi) = A$. Then the residual representation*

$$\bar{\rho}_{\mathfrak{p}} : \mathrm{G}_K \longrightarrow \mathrm{GL}_r(\kappa_{\mathfrak{p}})$$

is absolutely irreducible for almost all primes $\mathfrak{p}$ of A.

By contrast, for Drinfeld modules with $\mathrm{End}_K(\varphi) \neq A$, the residual representation at $\mathfrak{p}$ is never absolutely irreducible. Thus, in the general case, we describe the image of the group ring $A[\mathrm{G}_K]$ under the Galois representation. For this, let Z be the center of $E := \mathrm{End}_K(\varphi)$. By the theory of central simple algebras, there exist integers c, d, e such that $\mathrm{rank}_A(Z) = c$ and $\mathrm{rank}_Z(E) = e^2$ and $r = cde$. If φ has generic characteristic, we have $E = Z$ and $e = 1$.
For any $\mathfrak{p} \neq \mathfrak{p}_0$, let $B_{\mathfrak{p}}$ be the image of the natural homomorphism

$$A_{\mathfrak{p}}[\mathrm{G}_K] \longrightarrow \mathrm{End}_{A_{\mathfrak{p}}}\big(T_{\mathfrak{p}}(\varphi)\big).$$

The natural homomorphism

$$E_{\mathfrak{p}} := E \otimes_A A_{\mathfrak{p}} \longrightarrow \mathrm{End}_{A_{\mathfrak{p}}}\big(T_{\mathfrak{p}}(\varphi)\big)$$

is injective (see [10, Proposition 4.1]) and by the Tate conjecture (Theorem 1.1) its image is the commutant of $B_{\mathfrak{p}}$. Define $Z_{\mathfrak{p}} := Z \otimes_A A_{\mathfrak{p}}$.

Theorem 0.2 (Image of the group ring) *Let φ be a Drinfeld A-module of rank r over a finitely generated field K. Then for almost all primes $\mathfrak{p}$ of A the rings $E_{\mathfrak{p}}$ and $B_{\mathfrak{p}}$ are commutants of each other in $\mathrm{End}_{A_{\mathfrak{p}}}\big(T_{\mathfrak{p}}(\varphi)\big)$. More precisely, for almost all $\mathfrak{p}$ we have $E_{\mathfrak{p}} \cong \mathrm{M}_e(Z_{\mathfrak{p}})$ and $B_{\mathfrak{p}} \cong \mathrm{M}_d(Z_{\mathfrak{p}})$ and an isomorphism of $B_{\mathfrak{p}} \otimes_{Z_{\mathfrak{p}}} E_{\mathfrak{p}}$-modules $T_{\mathfrak{p}}(\varphi) \cong Z_{\mathfrak{p}}^{\oplus d} \otimes_{Z_{\mathfrak{p}}} Z_{\mathfrak{p}}^{\oplus e}$.*

Remark Theorem 0.1 and Theorem 0.2 have already been proven for the case where the Drinfeld module φ is of special characteristic and K has transcendence degree 1. See [10, Theorems A and B].

The article has six parts. In Section 1 we list some known results on Drinfeld modules. Section 2 contains results on the action of inertia groups on torsion points of φ. In Section 3 we use abelian class field theory to prove an interpolation result on characters of a certain algebraic group. The main work is done in Section 4, where we prove Theorem 0.1 in the case that $\mathfrak{p}_0 = 0$ and $\mathrm{End}_{\bar{K}}(\varphi) = A$ and K has transcendence degree 1. In Section 5 we prove Theorem 0.2 in the case that $\mathfrak{p}_0 = 0$ and K has transcendence degree 1. The general case of both theorems is proved in Section 6. The above notations and assumptions will remain in force throughout the article.

The material in this article was part of the doctoral thesis of the second author [11]. There it was applied to prove the adelic openness for Drinfeld modules in generic characteristic. This application will be the subject of our article [9].

1 Known results on Drinfeld modules

The first stated result was proved independently by Taguchi [15], [17] and Tamagawa [18].

Theorem 1.1 (Tate conjecture for Drinfeld modules) *Let φ_1 and φ_2 be two Drinfeld A-modules over K of the same characteristic. Then for all primes $\mathfrak{p}$ of A different from the characteristic of K, the natural map*

$$\mathrm{Hom}_K(\varphi_1,\varphi_2)\otimes_A A_{\mathfrak{p}} \longrightarrow \mathrm{Hom}_{A_{\mathfrak{p}}[\mathrm{G}_K]}\big(T_{\mathfrak{p}}(\varphi_1),T_{\mathfrak{p}}(\varphi_2)\big)$$

is an isomorphism.

The next result was proved by the first author ([6, Proposition 2.6]).

Theorem 1.2 *Assume that $\mathfrak{p}_0 = 0$ and that $\mathrm{End}_{\bar{K}}(\varphi) = A$. Then for all $\mathfrak{p}$ the image of $\rho_{\mathfrak{p}}$ is Zariski dense in $\mathrm{GL}_{r,F_{\mathfrak{p}}}$.*

In the same article, an even stronger result was proved, the openness of the image of Galois. Analogous results in the case $\mathfrak{p}_0 \neq 0$ can be found in [7] and [8]. Theorems 0.1 and 0.2 in the case $\mathfrak{p}_0 \neq 0$ and K of transcendence degree 1 were proved in [10]. The following result ([10, Proposition 2.3]) is essentially a translation of the isogeny conjecture for Drinfeld modules proved by Taguchi in [14], [16].

Proposition 1.3 *Assume that K is of transcendence degree 1. Then for almost all primes $\mathfrak{p}$ of A and all natural numbers $n > 0$, every G_K-invariant $A/\mathfrak{p}^n$-submodule of $\varphi[\mathfrak{p}^n](K^{sep})$ has the form $\alpha\big(\varphi[\mathfrak{p}^n](K^{sep})\big)$ for some $\alpha \in \mathrm{End}_K(\varphi)$.*

In particular, for $n = 1$, we obtain the following

Corollary 1.4 *Assume that K is of trans-cendence degree 1 and that $\mathrm{End}_K(\varphi) = A$. Then the representation $\bar{\rho}_{\mathfrak{p}}$ is irreducible for almost all primes $\mathfrak{p}$ of A.*

2 Action of inertia groups on torsion points

Throughout this section we assume that $\mathfrak{p}_0 = 0$ and that K is a finite extension of F. We begin by recalling the following fundamental fact (see Goss [4, Theorem 4.12.12 (2)]).

Theorem 2.1 *Let $\mathfrak{Q}$ be a place of K where φ has good reduction. Then for every prime $\mathfrak{p}$ not lying below $\mathfrak{Q}$, the representation $\rho_{\mathfrak{p}}$ is unramified at $\mathfrak{Q}$, and the characteristic polynomial of $\rho_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{Q}})$ has coefficients in A and is independent of $\mathfrak{p}$.*

The next result is proved in [10, Proposition 2.7] in the case $\mathfrak{p}_0 \neq 0$, but the proof works in general and is omitted here.

Proposition 2.2 *After replacing K by a suitable finite extension, for all primes $\mathfrak{p}$ of A and all places $\mathfrak{Q}$ of K not lying above $\mathfrak{p}$, the restriction of $\rho_{\mathfrak{p}}$ to the inertia group at $\mathfrak{Q}$ is unipotent.*

We now study the action of the inertia group at a place $\mathfrak{P}$ of K on $\varphi[\mathfrak{p}]$ if $\mathfrak{p}$ lies below $\mathfrak{P}$. For this, fix a place $\mathfrak{P}$ of K, a place $\bar{\mathfrak{P}}$ of $\bar{K}$, and denote by $v_{\mathfrak{P}}$ the associated normalized valuation on the completion $K_{\mathfrak{P}}$ and also its extension to $\bar{K}_{\bar{\mathfrak{P}}}$. Denote the respective residue fields by $k_{\mathfrak{P}}$ and $k_{\bar{\mathfrak{P}}}$. The field $k_{\bar{\mathfrak{P}}}$ is an algebraic closure of $k_{\mathfrak{P}}$. Let $K_{\mathfrak{P}}^{nr} \subset K_{\mathfrak{P}}^{t} \subset K_{\mathfrak{P}}^{sep}$ be the maximal subfields of $\bar{K}_{\bar{\mathfrak{P}}}$ which are unramified, respectively tamely ramified, respectively separable over $K_{\mathfrak{P}}$.

Then $\mathrm{I}_{\mathfrak{P}} := \mathrm{Gal}(K_{\mathfrak{P}}^{sep}/K_{\mathfrak{P}}^{nr})$ is the inertia group at $\mathfrak{P}$, and its quotient $\mathrm{I}_{\mathfrak{P}}^t := \mathrm{Gal}(K_{\mathfrak{P}}^t/K_{\mathfrak{P}}^{nr})$ is the tame inertia group at $\mathfrak{P}$.

Fundamental characters. Let $\pi \in K_{\mathfrak{P}}$ be a uniformizer at $\mathfrak{P}$. Let λ be a finite extension of $k_{\mathfrak{P}}$ inside $k_{\bar{\mathfrak{P}}}$ of cardinality $|\lambda|$, and let π_λ be any nonzero solution in $K_{\mathfrak{P}}^t$ of the equation $X^{|\lambda|} - \pi X = 0$. The *fundamental character associated to* λ is the homomorphism

$$\zeta_\lambda : \mathrm{I}_{\mathfrak{P}} \longrightarrow \lambda^*,\ \sigma \mapsto \sigma(\pi_\lambda)/\pi_\lambda \mod \pi.$$

Proposition 2.3 *The character ζ_λ factors through $\mathrm{I}_{\mathfrak{P}}^t$ and is independent of the choices of π and π_λ.*

Proof. For any other uniformizer π' and any nonzero solution π'_λ of the equation $X^{|\lambda|} - \pi' X = 0$, the elements π_λ and π'_λ have the same valuation and therefore differ by a unit $u \in K_{\mathfrak{P}}^t$. The value $\zeta_\lambda(\sigma)$ then changes by $\sigma(u)/u$, which is congruent to 1 modulo $\mathfrak{P}$ because σ acts trivially on the residue field. Therefore ζ_λ is independent of the choices of π and π_λ. Moreover, it factors through the tame inertia group $\mathrm{I}_{\mathfrak{P}}^t$ because $\pi_\lambda \in K_{\mathfrak{P}}^t$. **q.e.d.**

Note that for all $\tau \in \mathrm{Gal}(K_{\mathfrak{P}}^{sep}/K_{\mathfrak{P}})$ we have

$$(2.4)\qquad \tau\big(\zeta_\lambda(\sigma)\big) = \tau\big(\sigma(\pi_\lambda)/\pi_\lambda\big) = \tau\sigma\tau^{-1}\big(\tau(\pi_\lambda)\big)/\tau(\pi_\lambda) = \zeta_\lambda(\tau\sigma\tau^{-1}).$$

Also, the fundamental characters form a projective system with respect to the norm maps, i.e., for any finite extension λ' of λ inside $k_{\bar{\mathfrak{P}}}$ we have the equality

$$(2.5)\qquad \zeta_\lambda = \mathrm{N}_{\lambda'/\lambda} \circ \zeta_{\lambda'},$$

where $\mathrm{N}_{\lambda'/\lambda} : \lambda' \to \lambda$ is the Norm map. Furthermore, in the case $\lambda = k_{\mathfrak{P}}$ the fundamental character $\zeta_{k_{\mathfrak{P}}}$ extends with the same formula to a homomorphism $\mathrm{Gal}(K_{\mathfrak{P}}^{sep}/K_{\mathfrak{P}}) \to k_{\mathfrak{P}}^*$. Since the target is an abelian group, it follows that $\zeta_{k_{\mathfrak{P}}}$ factors through a homomorphism $\mathrm{Gal}(K_{\mathfrak{P}}^{ab}/K_{\mathfrak{P}}^{nr}) \to k_{\mathfrak{P}}^*$. Its composite with the inverse of the local norm residue symbol $\mathcal{O}_{\mathfrak{P}}^* \xrightarrow{\sim} \mathrm{Gal}(K_{\mathfrak{P}}^{ab}/K_{\mathfrak{P}}^{nr})$ is equal to

$$(2.6)\qquad \mathcal{O}_{\mathfrak{P}}^* \longrightarrow k_{\mathfrak{P}}^*,\ u \mapsto u^{-1} \mod \mathfrak{P}.$$

Action on torsion points. Recall that K is a finite extension of F. Let $\mathfrak{p}$ be the place of F below $\mathfrak{P}$, let $\kappa_{\mathfrak{p}}$ be its residue field, and let $q_{\mathfrak{p}}$ be the cardinality of $\kappa_{\mathfrak{p}}$. For any power m of p denote by k_m be the subfield of $k_{\bar{\mathfrak{P}}}$ with m elements.
Assume that φ has good reduction at $\mathfrak{P}$. This means that φ is isomorphic to a Drinfeld module having coefficients in $\mathcal{O}_{K_{\mathfrak{P}}}$ whose reduction is a Drinfeld module of rank r over $k_{\mathfrak{P}}$. We replace φ by this Drinfeld module. Then $\varphi[\mathfrak{p}]$ extends to a finite flat group scheme over the discrete valuation ring $\mathcal{O}_{K_{\mathfrak{P}}}$, and its connected-étale decomposition gives an exact sequence of finite flat group schemes

$$0 \longrightarrow \varphi[\mathfrak{p}]^0 \longrightarrow \varphi[\mathfrak{p}] \longrightarrow \varphi[\mathfrak{p}]^{et} \longrightarrow 0.$$

Let $h_{\mathfrak{P}}$ denote the height of the Drinfeld module φ modulo $\mathfrak{P}$. Then $\varphi[\mathfrak{p}]^0(K^{sep})$ is a $\kappa_{\mathfrak{p}}$ vector space of dimension $h_{\mathfrak{P}}$ and cardinality $n := {q_{\mathfrak{p}}}^{h_{\mathfrak{P}}}$.
The following result is an analogue of Proposition 9 in Serre's paper [13]. The analogue of Corollary 2.8 for τ-sheaves has been proven by Gardeyn in [3].

Proposition 2.7 *Assume that the extension $K_{\mathfrak{P}}/F_{\mathfrak{p}}$ is unramified and that φ has good reduction at $\mathfrak{P}$. Then the following properties hold.*

(i) The inertia group $\mathrm{I}_{\mathfrak{P}}$ acts trivially on $\varphi[\mathfrak{p}]^{et}(K^{sep})$.

(ii) The $\kappa_{\mathfrak{p}}$ vector space structure of $\varphi[\mathfrak{p}]^0(K^{sep})$ extends uniquely to a one dimensional k_n vector space structure such that the action of $\mathrm{I}_{\mathfrak{P}}$ on $\varphi[\mathfrak{p}]^0(K^{sep})$ is given by the fundamental character ζ_{k_n}.

(iii) The action of the wild inertia group at $\mathfrak{P}$ on $\varphi[\mathfrak{p}]^0(K^{sep})$ is trivial.

Proof. Assertion (i) follows immediately from the definition of an étale group scheme. Assertion (iii) follows from (ii) by Proposition 2.3. To prove (ii), define

$$\begin{aligned} \alpha &:= 1/(n-1), \\ U_\alpha &:= \{x \in K_{\mathfrak{P}}^{sep} \mid v_{\mathfrak{P}}(x) \geq \alpha\}, \\ U'_\alpha &:= \{x \in K_{\mathfrak{P}}^{sep} \mid v_{\mathfrak{P}}(x) > \alpha\}, \text{ and} \\ V_\alpha &:= U_\alpha / U'_\alpha. \end{aligned}$$

Let π_n be a nonzero solution of the equation $X^n - \pi X = 0$. The set V_α is a one dimensional $k_{\bar{\mathfrak{P}}}$ vector space generated by the residue class of π_n. By construction, $\mathrm{I}_{\mathfrak{P}}$ acts on V_α through the fundamental character ζ_{k_n}.

We claim that for every non-zero element $s \in \varphi[\mathfrak{p}]^0(K^{sep})$ we have $v_{\mathfrak{P}}(s) = \alpha$. This can be shown by considering an appropriate Newton polygon. Let $a \in A$ be a function with a zero of order one at $\mathfrak{p}$. Then $(a) = \mathfrak{p}\mathfrak{a}$ for an ideal $\mathfrak{a}$ of A which is prime to $\mathfrak{p}$. This implies that

$$\varphi[a] = \varphi[\mathfrak{p}] \oplus \varphi[\mathfrak{a}],$$

where $\varphi[\mathfrak{a}]$ is étale, and therefore

$$\varphi[a]^0 = \varphi[\mathfrak{p}]^0$$

as group schemes over $\operatorname{Spec} \mathcal{O}_{K_{\mathfrak{P}}}$. Write

$$\varphi_a = \sum \varphi_{a,i} \tau^i.$$

Then

$$v_{\mathfrak{P}}(\varphi_{a,0}) = v_{\mathfrak{P}}\big(\iota(a)\big) = 1,$$

because $\mathrm{ord}_{\mathfrak{p}}(a) = 1$ and $\mathfrak{P}|\mathfrak{p}$ is unramified. Moreover, since φ has good reduction at $\mathfrak{P}$, there exists a unique integer $i_0 > 0$ such that

$$\begin{aligned} v_{\mathfrak{P}}(\varphi_{a,i}) &\geq 1 \text{ for } 0 < i < i_0, \\ v_{\mathfrak{P}}(\varphi_{a,i_0}) &= 0 \text{ and} \\ v_{\mathfrak{P}}(\varphi_{a,i}) &\geq 0 \text{ for } i > i_0. \end{aligned}$$

Thus

$$q^{i_0} = |\varphi[a]^\circ| = |\varphi[\mathfrak{p}]^\circ| = n,$$

and so the points $(1,1)$ and $(n,0)$ are vertices of the Newton polygon of the polynomial

$$\varphi_a(x) = \sum \varphi_{a,i} x^{q^i}.$$

Since every non-zero element $s \in \varphi[\mathfrak{p}]^0(K^{sep})$ has valuation > 0, this valuation must therefore be equal to α, proving the claim.

The claim is equivalent to $\varphi[\mathfrak{p}]^\circ(K^{sep}) \subset U_\alpha$ and $\varphi[\mathfrak{p}]^\circ(K^{sep}) \cap U'_\alpha = 0$. Thus the inclusion induces an injective homomorphism

$$\varphi[\mathfrak{p}]^0(K^{sep}) \hookrightarrow V_\alpha.$$

By construction, this homomorphism is $\mathrm{I}_{\mathfrak{P}}$-equivariant; let W be its image. The fact that $\mathrm{I}_{\mathfrak{P}}$ acts on V_α through the fundamental character ζ_{k_n} implies that W is invariant under multiplication by k_n^*. Since, moreover, $|W| = |k_n|$, it follows that W is a k_n vector subspace of dimension 1. Via the inclusion, we obtain a unique one-dimensional k_n vector space structure on $\varphi[\mathfrak{p}]^0(K^{sep})$ such that the action of $\mathrm{I}_{\mathfrak{P}}$ on it is multiplication by ζ_{k_n}.

It remains to show that this vector space structure is an extension of the previously given $\kappa_{\mathfrak{p}}$ vector space structure on $\varphi[\mathfrak{p}]^0(K^{sep})$. For this, consider any element $\bar{b} \in \kappa_{\mathfrak{p}}$, and let b be an element of A whose residue class in $\kappa_{\mathfrak{p}}$ is equal to $\bar{b}$. Then the action of $\bar{b}$ on any element $s \in \varphi[\mathfrak{p}]^0(K^{sep})$ is given by

$$s \mapsto \varphi_b(s).$$

On the other hand, the inclusion $\kappa_{\mathfrak{p}} \hookrightarrow k_{\mathfrak{P}}$ is given by $\bar{b} \mapsto \iota(b) \mod \mathfrak{P}$. Thus, by the construction of the k_n vector space structure on $\varphi[\mathfrak{p}]^0(K^{sep})$, we must prove that

$$v_{\mathfrak{P}}\big(\varphi_b(s) - \iota(b)s\big) > \alpha.$$

Write

$$\varphi_b = \sum \varphi_{b,i}\tau^i,$$

and note that $\varphi_{b,0} = \iota(b)$ and $v_{\mathfrak{P}}(\varphi_{b,i}) \geq 0$ for all i. Since $v_{\mathfrak{P}}(s) \geq \alpha$, it follows that

$$\varphi_b(s) - \iota(b)s = \sum \varphi_{b,i} s^{q^i}$$

has valuation $\geq q\alpha > \alpha$, as desired. **q.e.d.**

Corollary 2.8 *Assume that the extension $K_{\mathfrak{P}}/F_{\mathfrak{p}}$ is unramified and that φ has good reduction at $\mathfrak{P}$. Then the action of $\mathrm{I}_{\mathfrak{P}}$ on $\varphi[\mathfrak{p}]^0(K^{sep}) \otimes_{\kappa_{\mathfrak{p}}} k_{\bar{\mathfrak{P}}}$ is diagonalizable and given by the $h_{\mathfrak{P}}$ distinct characters $\bar{\sigma} \circ \zeta_{k_n}$ where $\bar{\sigma}$ runs through $\mathrm{Hom}_{k_{\mathfrak{p}}}(k_n, k_{\bar{\mathfrak{P}}})$.*

3 An interpolation result from class field theory

In this section, we assume that K is a finite extension of F. We introduce algebraic groups $\mathbb{T}$ and $\mathbb{S}$ in the same way as Serre did in [12, Chapter II] and [13, §3]. Then we relate algebraic characters of $\mathbb{S}$ with compatible systems of abelian $\mathfrak{p}$-adic Galois representations. Although Serre's construction applies to characters with arbitrary conductor, we restrict ourselves to characters with trivial conductor because that suffices for our purposes.

Let $\mathbb{A}_K$ denote the ring of adeles of K and $\mathbb{A}_K^*$ the group of ideles of K. For any place $\mathfrak{P}$ of K let $\mathcal{O}_{\mathfrak{P}}$ be the discrete valuation ring of $K_{\mathfrak{P}}$. Define

$$U := \prod_{\mathfrak{P}\nmid\infty} \mathcal{O}_{\mathfrak{P}}^* \times \prod_{\infty'|\infty} K_{\infty'}^* \subset \mathbb{A}_K^*,$$

and

$$C := \mathbb{A}_K^*/K^*U.$$

Then C is a finite abelian group and sits in the exact sequence

$$1 \longrightarrow K^*/(K^* \cap U) \longrightarrow \mathbb{A}_K^*/U \longrightarrow C \longrightarrow 1.$$

The Serre groups $\mathbb{T}$ and $\mathbb{S}$. Consider the Weil restriction $H := \mathrm{Res}_F^K(\mathbb{G}_{m,K})$ of the multiplicative group over K to F. By definition, its points over any F-algebra B are given by

$$H(B) := (B \otimes_F K)^*.$$

Let $\overline{K^* \cap U}$ be the Zariski closure of $K^* \cap U$ inside H and consider the quotient

$$\mathbb{T} := H/\overline{K^* \cap U}.$$

Let $\mathbb{S}$ be the push-out of $\mathbb{T}$ and $\mathbb{A}_K^*/U$ over $K^*/(K^* \cap U)$. This is an algebraic group with the universal property that, for any algebraic group H' over F together with homomorphisms $\mathbb{T} \to H'$ and $\mathbb{A}_K^*/U \to H'(F)$ such that the following diagram

$$\begin{tikzcd}
K^*/(K^* \cap U) \arrow[r] \arrow[d] & \mathbb{A}_K^*/U \arrow[d] \\
\mathbb{T}(F) \arrow[r] & H'(F)
\end{tikzcd}$$

commutes, there exists a unique homomorphism $\mathbb{S} \to H'$ through which both $\mathbb{T} \to H'$ and $\mathbb{A}_K^*/U \to H'(F)$ factor. A more explicit construction of the algebraic group $\mathbb{S}$ can be done as in Serre [12, Chapter II]. The definitions of $\mathbb{T}$ and $\mathbb{S}$ give us a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & K^*/(K^* \cap U) \arrow[r] \arrow[d] & \mathbb{A}_K^*/U \arrow[r] \arrow[d, "\gamma'"] & C \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
1 \arrow[r] & \mathbb{T}(F) \arrow[r] & \mathbb{S}(F) \arrow[r] & C \arrow[r] & 1
\end{tikzcd}$$

with exact rows. By composition of γ' with $\mathbb{A}_K^* \to \mathbb{A}_K^*/U$ we get a homomorphism $\gamma : \mathbb{A}_K^* \to \mathbb{S}(F)$.

Homomorphism at a prime $\mathfrak{p}$. Let $\mathfrak{p}$ be any prime of A, and fix a place $\bar{\mathfrak{p}}$ of $\bar{F}$ above $\mathfrak{p}$. Define

$$U^{\mathfrak{p}} := \prod_{\mathfrak{Q} \nmid \{\mathfrak{p},\, \infty\}} \mathcal{O}_{\mathfrak{Q}}^* \times \prod_{\infty' | \infty} K_{\infty'}^* \subset \mathbb{A}_K^*,$$

$$K_{\mathfrak{p}} := \prod_{\mathfrak{P}|\mathfrak{p}} K_{\mathfrak{P}},$$

and

$$\mathcal{O}_{\mathfrak{p}} := \prod_{\mathfrak{P}|\mathfrak{p}} \mathcal{O}_{\mathfrak{P}}.$$

The composite of γ with the inclusion $\mathbb{S}(F) \hookrightarrow \mathbb{S}(F_{\mathfrak{p}})$ is a continuous, even locally constant, homomorphism

$$\gamma_{\mathfrak{p}} : \mathbb{A}_K^* \longrightarrow \mathbb{S}(F_{\mathfrak{p}}).$$

On the other hand, combining the projection from $\mathbb{A}_K^*$ to its direct factor

$$K_{\mathfrak{p}}^* = (F_{\mathfrak{p}} \otimes_F K)^* = H(F_{\mathfrak{p}})$$

with the algebraic homomorphism $H \twoheadrightarrow \mathbb{T} \hookrightarrow \mathbb{S}$ yields a continuous homomorphism

$$\delta_{\mathfrak{p}} : \mathbb{A}_K^* \longrightarrow K_{\mathfrak{p}}^* = H(F_{\mathfrak{p}}) \longrightarrow \mathbb{S}(F_{\mathfrak{p}}).$$

The commutativity of the above diagram implies $\gamma_{\mathfrak{p}}|_{K^*} = \delta_{\mathfrak{p}}|_{K^*}$. Thus the homomorphism

$$\gamma_{\mathfrak{p}} \delta_{\mathfrak{p}}^{-1} : \mathbb{A}_K^* \longrightarrow \mathbb{S}(F_{\mathfrak{p}})$$

is trivial on K^*. Since both $\gamma_{\mathfrak{p}}$ and $\delta_{\mathfrak{p}}$ are trivial on $U^{\mathfrak{p}}$, the continuous homomorphism $\gamma_{\mathfrak{p}} \delta_{\mathfrak{p}}^{-1}$ is trivial on the closure $\overline{K^* U^{\mathfrak{p}}}$ of $K^* U^{\mathfrak{p}}$ in $\mathbb{A}_K^*$ and therefore factors through a continuous homomorphism

$$\varepsilon_{\mathfrak{p}} : \mathbb{A}_K^*/\overline{K^* U^{\mathfrak{p}}} \longrightarrow \mathbb{S}(F_{\mathfrak{p}}).$$

Characters of $\mathbb{T}$ and $\mathbb{S}$. Define $\Sigma := \mathrm{Hom}_F(K, \bar{F})$. Every $\sigma \in \Sigma$ extends to an $\bar{F}$-algebra homomorphism $\bar{F} \otimes_F K \to \bar{F}$ and thus gives rise to a character $[\sigma] : H_{\bar{F}} \to \mathbb{G}_{m,\bar{F}}$ of H. These $[\sigma]$ form a $\mathbb{Z}$-basis of the character group $\mathrm{X}(H)$. Since $\mathbb{T} = H/\overline{K^* \cap U}$, its character group is given by

$$\mathrm{X}(\mathbb{T}) = \left\{ \prod_{\sigma\in\Sigma} [\sigma]^{n_\sigma} \;\middle|\; \prod_{\sigma} \sigma(x)^{n_\sigma} = 1 \text{ for all } x \in K^* \cap U \right\}.$$

The character groups of C, $\mathbb{T}$, and $\mathbb{S}$ lie in an exact sequence

$$1 \longrightarrow \mathrm{X}(C) \longrightarrow \mathrm{X}(\mathbb{S}) \longrightarrow \mathrm{X}(\mathbb{T}) \longrightarrow 1$$

where $\mathrm{X}(C)$ is the finite group $\mathrm{Hom}(C, \bar{F}^*)$. Thus any character μ of $\mathbb{T}$ can be extended to a character θ of $\mathbb{S}$ in precisely $|C|$ ways.

Compatible system associated to a character. Let θ be a character of $\mathbb{S}$. It induces a continuous homomorphism $\mathbb{S}(F_\mathfrak{p}) \to \bar{F}_{\bar{\mathfrak{p}}}^*$, whose composite with $\varepsilon_\mathfrak{p}$ is a continuous homomorphism

$$\theta_\mathfrak{p} : \mathbb{A}_K^*/\overline{K^* U^\mathfrak{p}} \longrightarrow \bar{F}_{\bar{\mathfrak{p}}}^*.$$

Since $\mathbb{A}_K^*/\overline{K^* U^\mathfrak{p}}$ is compact, the image of $\theta_\mathfrak{p}$ is contained in the multiplicative group of the valuation ring of $\bar{F}_{\bar{\mathfrak{p}}}$. Therefore we can reduce it modulo $\bar{\mathfrak{p}}$ and obtain a continuous homomorphism

$$\bar{\theta}_\mathfrak{p} : \mathbb{A}_K^*/\overline{K^* U^\mathfrak{p}} \longrightarrow \kappa_{\bar{\mathfrak{p}}}^*.$$

Let $K^{ab,\mathfrak{p}}$ be the maximal abelian extension of K which splits completely at all places above ∞ and is unramified at all places not lying above $\mathfrak{p}$. Then the Artin reciprocity map of global class field theory induces a surjective homomorphism

$$\mathrm{G}_K \twoheadrightarrow \mathrm{Gal}(K^{ab,\mathfrak{p}}/K) \cong \mathbb{A}_K^*/\overline{K^* U^\mathfrak{p}},$$

whose composite with $\theta_\mathfrak{p}$ is a continuous homomorphism $\mathrm{G}_K \to \bar{F}_{\bar{\mathfrak{p}}}^*$. As $\mathfrak{p}$ varies, these homomorphisms form a system of strictly compatible $\mathfrak{p}$-adic representations in the sense of Serre [12, Chapter II], i.e., the image of $\mathrm{Frob}_\mathfrak{Q}$ lies in $\bar{F}^*$ and is independent of $\mathfrak{p}$ for all $\mathfrak{Q}$ not lying above $\mathfrak{p}$.

Interpolation of characters. We now reverse the above process. Let S be an infinite set of primes of A. For any $\mathfrak{p} \in S$, fix a place $\bar{\mathfrak{p}}$ of $\bar{F}$ above $\mathfrak{p}$ and consider a continuous homomorphism

$$\bar{\eta}_\mathfrak{p} : \mathbb{A}_K^*/\overline{K^* U^\mathfrak{p}} \longrightarrow \kappa_{\bar{\mathfrak{p}}}^*.$$

Every $\sigma \in \Sigma$ determines a place $\sigma^{-1}(\bar{\mathfrak{p}})$ of K above $\mathfrak{p}$ and an embedding $K_{\sigma^{-1}(\bar{\mathfrak{p}})} \hookrightarrow \bar{F}_{\bar{\mathfrak{p}}}$. Let

$$\sigma_\mathfrak{p} : K_\mathfrak{p}^* = \prod_{\mathfrak{P}|\mathfrak{p}} K_\mathfrak{P}^* \longrightarrow \bar{F}_{\bar{\mathfrak{p}}}^*$$

be the homomorphism which is the above embedding on $K_{\sigma^{-1}(\bar{\mathfrak{p}})}$ and identically 1 on all other factors.

Proposition 3.1 *In the above situation, assume that there exist integers $n(\sigma, \mathfrak{p})$ whose absolute values are bounded, such that for all $\mathfrak{p} \in S$ and all $x \in \mathcal{O}_\mathfrak{p}^*$ we have*

$$\bar{\eta}_\mathfrak{p}(x) = \left(\prod_{\sigma\in\Sigma} \sigma_\mathfrak{p}(x^{-1})^{n(\sigma,\mathfrak{p})} \mod \bar{\mathfrak{p}} \right).$$

Then there exist a character $\theta \in X(\mathbb{S})$ and an infinite subset S' of S such that for all $\mathfrak{p} \in S'$ we have

$$\bar{\theta}_\mathfrak{p} = \bar{\eta}_\mathfrak{p}.$$

Proof. Since the numbers $n(\sigma,\mathfrak{p})$ are bounded and Σ is finite, there exists an infinite subset S'' of S such that for all $\mathfrak{p}\in S''$ the value $n(\sigma,\mathfrak{p})$ is independent of $\mathfrak{p}$. Denote this value by n_σ. Consider the character $\alpha := \prod_{\sigma\in\Sigma}[\sigma]^{n_\sigma} \in X(H)$. Then for any $x\in K^*\cap U$, we have $\eta_{\bar{\mathfrak{p}}}(x)=1$ and

$$\prod_\sigma \sigma_{\mathfrak{p}}(x^{-1})^{n(\sigma,\mathfrak{p})} = \prod_\sigma \sigma(x^{-1})^{n(\sigma,\mathfrak{p})} = \alpha(x^{-1}).$$

Thus $\alpha(x^{-1})\equiv 1 \mod \bar{\mathfrak{p}}$ for all $\mathfrak{p}\in S''$. Since S'' is infinite, we find that $\alpha(x)=1$, and hence $\alpha\in \mathrm{X}(\mathbb{T})$.
Extend α to a character $\theta'\in \mathrm{X}(\mathbb{S})$. Then for any $\mathfrak{p}\in S''$, the character

$$\beta_{\mathfrak{p}} := \bar{\eta}_{\mathfrak{p}}\bar{\theta}_{\mathfrak{p}}'^{-1} : \mathbb{A}_K^*/\overline{K^*U^{\mathfrak{p}}} \longrightarrow \kappa_{\bar{\mathfrak{p}}}^*$$

factors through C. Therefore it takes values in the group of m-th roots of unity $\mu_m(\kappa_{\bar{\mathfrak{p}}})$ for $m := |C|$. Since the reduction map $\mu_m(\bar{F}) \longrightarrow \mu_m(\kappa_{\bar{\mathfrak{p}}})$ is an isomorphism, we can lift $\beta_{\mathfrak{p}}$ uniquely to a homomorphism into $\mu_m(\bar{F})$, and thus to an element of $\mathrm{X}(C)$. This is a finite group; hence there exist $\beta\in\mathrm{X}(C)$ and an infinite subset S' of S'' such that for all $\mathfrak{p}\in S'$ we have $\beta_{\mathfrak{p}}=\beta$. Define θ as the product of θ' with the image of β in $X(\mathbb{S})$. Then for all $\mathfrak{p}\in S'$ we have $\bar{\theta}_{\mathfrak{p}} = \bar{\eta}_{\mathfrak{p}}$, as desired **q.e.d.**

4 Absolute irreducibility of the residual representation

Throughout this section, we assume that K is a finite extension of F, so that $\mathfrak{p}_0=0$. We also assume that $\mathrm{End}_{\bar{K}}(\varphi)=A$ (in order to apply Theorem 1.2). We prove the following special case of Theorem 0.1.

Theorem 4.1 *In the above situation, the residual representation*

$$\bar{\rho}_{\mathfrak{p}} : \mathrm{G}_K \longrightarrow \mathrm{GL}_r(\kappa_{\mathfrak{p}})$$

is absolutely irreducible for almost all primes $\mathfrak{p}$ *of* A.

By Corollary 1.4 we know that $\bar{\rho}_{\mathfrak{p}}$ is irreducible for almost all primes $\mathfrak{p}$ of A. By Schur's lemma, for these primes the ring $\mathrm{End}_{\kappa_{\mathfrak{p}}}(\bar{\rho}_{\mathfrak{p}})$ is a finite dimensional division algebra over $\kappa_{\mathfrak{p}}$. As $\kappa_{\mathfrak{p}}$ is finite, this division algebra is commutative and hence a finite field extension of $\kappa_{\mathfrak{p}}$. Call it $\lambda_{\mathfrak{p}}$, and set $s_{\mathfrak{p}} := [\lambda_{\mathfrak{p}}/\kappa_{\mathfrak{p}}]$ and $t_{\mathfrak{p}} = \dim_{\lambda_{\mathfrak{p}}}(\bar{\rho}_{\mathfrak{p}})$. Then $r = s_{\mathfrak{p}}t_{\mathfrak{p}}$, and, in a suitable basis, the representation $\bar{\rho}_{\mathfrak{p}}$ amounts to a homomorphism

$$\bar{\rho}_{\mathfrak{p}} : \mathrm{G}_K \longrightarrow \mathrm{GL}_{t_{\mathfrak{p}}}(\lambda_{\mathfrak{p}}) \subset \mathrm{GL}_r(\kappa_{\mathfrak{p}}).$$

Its composite with the determinant map $\det_{\lambda_{\mathfrak{p}}} : \mathrm{GL}_t(\lambda_{\mathfrak{p}}) \to \lambda_{\mathfrak{p}}^*$ is a character

$$\det\nolimits_{\lambda_{\mathfrak{p}}} \circ \bar{\rho}_{\mathfrak{p}} : \mathrm{G}_K \longrightarrow \lambda_{\mathfrak{p}}^*.$$

For any prime $\mathfrak{p}\in S$ we fix a place $\bar{\mathfrak{p}}$ of $\bar{F}$ above $\mathfrak{p}$. The residue field $\kappa_{\bar{\mathfrak{p}}}$ at $\bar{\mathfrak{p}}$ is an algebraic closure of $\kappa_{\mathfrak{p}}$. We choose an embedding $\beta_{\mathfrak{p}} : \lambda_{\mathfrak{p}} \hookrightarrow \kappa_{\bar{\mathfrak{p}}}$, obtaining a character

$$\bar{\chi}_{\mathfrak{p}} := \beta_{\mathfrak{p}} \circ \det\nolimits_{\lambda_{\mathfrak{p}}} \circ \bar{\rho}_{\mathfrak{p}} : \ \mathrm{G}_K \longrightarrow \kappa_{\bar{\mathfrak{p}}}^*.$$

To prove Theorem 4.1 we must show that $s_{\mathfrak{p}}=1$ for almost all $\mathfrak{p}$. If not, since $s_{\mathfrak{p}}$ is one of finitely many divisors of r, some value of $s_{\mathfrak{p}}>1$ must occur infinitely often. To give an indirect proof, we make the following assumption and derive a contradiction.

Assumption 4.2 There exist integers $s > 1$ and t with $st = r$ and an infinite set S of primes of A, such that for all $\mathfrak{p} \in S$ the residual representation $\bar{\rho}_\mathfrak{p}$ factors through $\mathrm{GL}_t(\lambda_\mathfrak{p})$ where $\lambda_\mathfrak{p}$ is a field extension of $\kappa_\mathfrak{p}$ of degree s.

Reduction steps. We can replace S by any infinite subset, without changing the assumptions. Thus after removing finitely many primes, we may assume that for all $\mathfrak{p} \in S$

(a) φ has good reduction at all places of K lying above $\mathfrak{p}$,

(b) $\mathfrak{p}$ is unramified in K, and

(c) the residue field $\kappa_\mathfrak{p}$ has at least 3 elements.

It is also enough to prove Theorem 4.1 for any open subgroup of G_K. This allows us to replace K by any finite extension. Thus by Proposition 2.2, we may assume that the restriction of $\bar{\rho}_\mathfrak{p}$ to the inertia group at any place not lying above $\mathfrak{p}$ is unipotent. Then

(d) for all $\mathfrak{p} \in S$ and all places $\mathfrak{Q}$ of K not lying above $\mathfrak{p}$ we have $\bar{\chi}_\mathfrak{p}|_{\mathrm{I}_\mathfrak{Q}} = 1$.

Next, recall that at any place ∞' of K above ∞, the Drinfeld module is uniformized by a lattice on which the decomposition group $\mathrm{D}_{\infty'}$ acts through a finite quotient. Thus, after replacing K by a finite extension, we may assume that

(e) for all $\mathfrak{p} \in S$ and all places ∞' of K lying above ∞ we have $\bar{\chi}_\mathfrak{p}|_{\mathrm{D}_{\infty'}} = 1$.

Ramification behavior of $\bar{\chi}_\mathfrak{p}$. Now we describe the ramification behavior of $\bar{\chi}_\mathfrak{p}$ at places above $\mathfrak{p}$. Recall that $\Sigma = \mathrm{Hom}_F(K, \bar{F})$. Then for any place $\mathfrak{P}$ of K above $\mathfrak{p}$, the set $\Sigma_\mathfrak{P} := \{\sigma \in \Sigma \mid \mathfrak{P} = \sigma^{-1}(\bar{\mathfrak{p}})\}$ is non-empty. Any element $\sigma \in \Sigma_\mathfrak{P}$ induces an embedding $k_\mathfrak{P} \hookrightarrow \kappa_{\bar{\mathfrak{p}}}$. As in Section 2, we write $q_\mathfrak{p} = |\kappa_\mathfrak{p}|$ and let $\zeta_{k_{q_\mathfrak{p}^s}} : \mathrm{I}_\mathfrak{P} \to k_{q_\mathfrak{p}^s}^*$ denote the fundamental character associated to $k_{q_\mathfrak{p}^s}$, the subfield of $k_{\bar{\mathfrak{P}}}$ with $q_\mathfrak{p}^s$ elements.

Lemma 4.3 *For any place $\mathfrak{P}$ of K above $\mathfrak{p} \in S$, the following properties hold.*

(i) We have $s \mid [k_\mathfrak{P}/\kappa_\mathfrak{p}]$, and so any $\sigma \in \Sigma_\mathfrak{P}$ induces an embedding $\bar{\sigma} : k_{q_\mathfrak{p}^s} \hookrightarrow \kappa_{\bar{\mathfrak{p}}}$.

(ii) There exists an element $\sigma \in \Sigma_\mathfrak{P}$ such that

$$\bar{\chi}_\mathfrak{p}|_{\mathrm{I}_\mathfrak{P}} = \bar{\sigma} \circ \zeta_{k_{q_\mathfrak{p}^s}}.$$

Proof. By (a) above, the Drinfeld module φ has good reduction at $\mathfrak{P}$, say of height $h_\mathfrak{P}$. We thus have an exact sequence of $\kappa_\mathfrak{p}$ vector spaces

$$0 \longrightarrow \varphi[\mathfrak{p}]^0(K^{sep}) \longrightarrow \varphi[\mathfrak{p}](K^{sep}) \longrightarrow \varphi[\mathfrak{p}]^{et}(K^{sep}) \longrightarrow 0,$$

where $\varphi[\mathfrak{p}]^0(K^{sep})$ has dimension $h_\mathfrak{P}$. By Proposition 2.7, the group $\mathrm{I}_\mathfrak{P}$ acts trivially on $\varphi[\mathfrak{p}]^{et}(K^{sep})$ and, in view of (b) and (c) above, it has no coinvariants on $\varphi[\mathfrak{p}]^0(K^{sep})$. Thus the group of $\mathrm{I}_\mathfrak{P}$-coinvariants of $\varphi[\mathfrak{p}](K^{sep})$ is $\varphi[\mathfrak{p}]^{et}(K^{sep})$. Since the representation factors through $\mathrm{GL}_t(\lambda_\mathfrak{p})$, it follows that the exact sequence is a sequence of $\lambda_\mathfrak{p}$ vector spaces. In particular, the degree $s = [\lambda_\mathfrak{p}/\kappa_\mathfrak{p}]$ must divide $h_\mathfrak{P}$.

Moreover, the determinant over $\lambda_\mathfrak{p}$ of the representation $\bar{\rho}_\mathfrak{p}|_{\mathrm{I}_\mathfrak{P}}$ is equal to the determinant of the subrepresentation on $\varphi[\mathfrak{p}]^0(K^{sep})$. Abbreviate $n := q_\mathfrak{p}^{h_\mathfrak{P}}$. Then by Proposition 2.7 (ii) the $\kappa_\mathfrak{p}$ vector space structure of $\varphi[\mathfrak{p}]^0(K^{sep})$ extends to a one dimensional k_n vector space structure such that $\mathrm{I}_\mathfrak{P}$ acts through the fundamental character $\zeta_{k_n} : \mathrm{I}_\mathfrak{P} \to k_n^*$. The action of $\lambda_\mathfrak{p}$ amounts to an embedding $\lambda_\mathfrak{p} \hookrightarrow k_n$ and thus to an identification $\lambda_\mathfrak{p} \cong k_{q_\mathfrak{p}^s}$ over $\kappa_\mathfrak{p}$. Via this identification, the determinant over $\lambda_\mathfrak{p}$ of an element $x \in k_n^*$ is the norm $\mathrm{N}_{k_n/\lambda_\mathfrak{p}}(x) \in \lambda_\mathfrak{p}^*$. Thus from (2.5) it follows that $\det_{\lambda_\mathfrak{p}} \circ \bar\rho_\mathfrak{p}|_{\mathrm{I}_\mathfrak{P}}$ is the fundamental character $\zeta_{k_{q_\mathfrak{p}^s}} : \mathrm{I}_\mathfrak{P} \to \lambda_\mathfrak{p}^*$.

In particular $\zeta_{k_{q_\mathfrak{p}^s}}$ extends to an abelian character of G_K. Since it is also surjective, equation (2.4) implies that $\mathrm{Gal}(K_\mathfrak{P}^{sep}/K_\mathfrak{P})$ acts trivially on $\lambda_\mathfrak{p}^*$. Therefore $\lambda_\mathfrak{p}$ is contained in the residue field $k_\mathfrak{P}$, and so s divides $[k_\mathfrak{P}/\kappa_\mathfrak{p}]$, proving (i).

Finally, the given embedding $\beta_\mathfrak{p} : \lambda_\mathfrak{p} \hookrightarrow \kappa_{\bar{\mathfrak{p}}}$ extends to some embedding $k_\mathfrak{P} \hookrightarrow \kappa_{\bar{\mathfrak{p}}}$ over $\kappa_\mathfrak{p}$. Any such embedding is induced by some element $\sigma \in \Sigma_\mathfrak{P}$, which then satisfies (ii), as desired. **q.e.d.**

Translation via class field theory. We use the same notations as in Section 3. Since the character $\bar\chi_\mathfrak{p}$ is abelian and unramified at all places not lying above $\mathfrak{p}$ and trivial at all places above ∞ by (d) and (e), it factors through $\mathrm{Gal}(K^{ab,\mathfrak{p}}/K)$. Therefore its composite with the Artin reciprocity map

$$\mathbb{A}_K^*/\overline{K^*U^\mathfrak{p}} \longrightarrow \mathrm{Gal}(K^{ab,\mathfrak{p}}/K)$$

is a character

$$\bar\psi_\mathfrak{p} : \mathbb{A}_K^*/\overline{K^*U^\mathfrak{p}} \longrightarrow \kappa_{\bar{\mathfrak{p}}}^*.$$

Lemma 4.4 *For any $\mathfrak{p} \in S$ there exist $n(\sigma,\mathfrak{p}) \in \{0,1\}$ such that for all $u \in \mathcal{O}_\mathfrak{p}^*$ we have*

$$\bar\psi_\mathfrak{p}(u) = \left(\prod_{\sigma\in\Sigma} \sigma_\mathfrak{p}(u^{-1})^{n(\sigma,\mathfrak{p})} \mod \bar{\mathfrak{p}}\right).$$

Proof. Fix a prime $\mathfrak{p} \in S$ and consider any place $\mathfrak{P}$ of K above $\mathfrak{p}$. Then for any $\sigma \in \Sigma_\mathfrak{P}$ as in Lemma 4.3 (ii), using (2.5) we find that

$$\bar\chi_\mathfrak{p}|_{\mathrm{I}_\mathfrak{P}} = \bar\sigma \circ \zeta_{k_{q_\mathfrak{p}^s}} = \bar\sigma \circ \mathrm{N}_{k_\mathfrak{P}/k_{q_\mathfrak{p}^s}} \circ \zeta_{k_\mathfrak{P}}.$$

Since the norm is the product of all Galois conjugates, and $\mathfrak{P}$ is unramified over $\mathfrak{p}$, the latter is equal to

$$\prod_{\sigma'\in\Sigma'_\mathfrak{P}} \bar\sigma' \circ \zeta_{k_\mathfrak{P}}$$

where $\Sigma'_\mathfrak{P} := \{\sigma' \in \Sigma_\mathfrak{P} : \sigma'|_{k_{q_\mathfrak{p}^s}} = \sigma|_{k_{q_\mathfrak{p}^s}}\}$. Using (2.6) this is equivalent to

$$\bar\psi_\mathfrak{p}(u) \equiv \prod_{\sigma'\in\Sigma'_\mathfrak{P}} \sigma'(u^{-1}) \mod \bar{\mathfrak{p}}$$

for all $u \in \mathcal{O}_\mathfrak{P}^*$. Set $n(\sigma',\mathfrak{p}) := 1$ whenever $\sigma' \in \Sigma'_\mathfrak{P}$ for some $\mathfrak{P}$ above $\mathfrak{p}$, and $:= 0$ otherwise. Then for all $u = (u_\mathfrak{P}) \in \mathcal{O}_\mathfrak{p}^* = \prod_{\mathfrak{P}|\mathfrak{p}} \mathcal{O}_\mathfrak{P}^*$, we have

$$\begin{aligned}\bar\psi_\mathfrak{p}(u) &\equiv \prod_{\mathfrak{P}|\mathfrak{p}} \prod_{\sigma'\in\Sigma'_\mathfrak{P}} \sigma'(u_\mathfrak{P}^{-1}) \mod \bar{\mathfrak{p}} \\ &= \prod_{\sigma\in\Sigma} \sigma_\mathfrak{p}(u^{-1})^{n(\sigma,\mathfrak{p})},\end{aligned}$$

as desired. **q.e.d.**

Interpolation of characters. By Lemma 4.4 the characters $\bar\psi_\mathfrak{p}$ satisfy the assumptions of Proposition 3.1. Thus after replacing S by an infinite subset, there exists $\theta \in X(\mathbb{S})$ such that for all $\mathfrak{p} \in S$ we have $\bar{\theta}_{\mathfrak{p}} = \bar{\psi}_{\mathfrak{p}}$. By abuse of notation, we denote the composite homomorphism

$$\mathrm{G}_K \twoheadrightarrow \mathrm{Gal}(K^{ab,\mathfrak{p}}/K) \cong \mathbb{A}_K^*/\overline{K^* U^{\mathfrak{p}}} \xrightarrow{\theta_{\mathfrak{p}}} \bar{F}_{\bar{\mathfrak{p}}}^*$$

again by $\theta_{\mathfrak{p}}$. Then for all $\mathfrak{p} \in S$ we have

$$(\theta_{\mathfrak{p}} \bmod \bar{\mathfrak{p}}) = \bar{\chi}_{\mathfrak{p}}. \tag{4.5}$$

Construction of an algebraic relation. Let n be an integer, and let $f(T) := \prod_{i=1}^n (T - \alpha_i) = \sum_{i=0}^n \beta_i T^i$ be any monic polynomial of degree n. For any integer $m \leq n$ define

$$f^{(m)}(T) := \prod_I \left(T - \prod_{i \in I} \alpha_i \right),$$

where the outer product ranges over all subsets I of $\{1, \ldots, n\}$ of cardinality m. The coefficients of $f^{(m)}(T)$ are symmetric polynomials in the α_i and are therefore polynomials in $\beta_1, \ldots, \beta_n$ with coefficients in $\mathbb{Z}$. The construction can thus be applied to any monic polynomial with coefficients in any commutative ring. If f is the characteristic polynomial of an endomorphism M of a finite dimensional vector space, then $f^{(m)}$ is the characteristic polynomial of $\bigwedge^m M$. We have $f^{(m)}(\alpha) = 0$ if and only if f has m zeros with product α.

Consider any place $\mathfrak{Q}$ of K where φ has good reduction. Denote by $f_{\mathfrak{Q}}$ the characteristic polynomial of $\rho_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{Q}})$ for any prime $\mathfrak{p}$ of A not lying below $\mathfrak{Q}$. By Theorem 2.1 it has coefficients in A and is independent of $\mathfrak{p}$. On the other hand, recall that the $\theta_{\mathfrak{p}}$ form a system of strictly compatible $\mathfrak{p}$-adic representations, which means that $\theta_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{Q}})$ lies in $\bar{F}^*$ and is independent of $\mathfrak{p}$. It is integral outside ∞.

Lemma 4.6 *For all places $\mathfrak{Q}$ of K where φ has good reduction we have*

$$f_{\mathfrak{Q}}^{(t)}\big(\theta_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{Q}})\big) = 0.$$

Proof. For any prime $\mathfrak{p} \in S$ not lying below $\mathfrak{Q}$, let $\bar{f}_{\mathfrak{Q},\mathfrak{p}} \in \kappa_{\mathfrak{p}}[T]$ denote the characteristic polynomial of $\bar{\rho}_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{Q}}) \in \mathrm{GL}_r(\kappa_{\mathfrak{p}})$. Let $\bar{g}_{\mathfrak{Q},\mathfrak{p}} \in \lambda_{\mathfrak{p}}[T]$ denote the characteristic polynomial of the same element $\bar{\rho}_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{Q}}) \in \mathrm{GL}_t(\lambda_{\mathfrak{p}})$ over $\lambda_{\mathfrak{p}}$. Then we have

$$\begin{aligned} \bar{f}_{\mathfrak{Q},\mathfrak{p}} &= (f_{\mathfrak{Q}} \bmod \mathfrak{p}) \quad \text{and} \\ \bar{f}_{\mathfrak{Q},\mathfrak{p}} &= \mathrm{N}_{\lambda_{\mathfrak{p}}/\kappa_{\mathfrak{p}}} \bar{g}_{\mathfrak{Q},\mathfrak{p}}. \end{aligned}$$

By construction the product of the t zeros of $\bar{g}_{\mathfrak{Q},\mathfrak{p}}$ is equal to $\bar{\chi}_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{Q}})$. Therefore we find that $\bar{f}_{\mathfrak{Q},\mathfrak{p}}^{(t)}\big(\bar{\chi}_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{Q}})\big) = 0$. Since $\bar{f}_{\mathfrak{Q},\mathfrak{p}}^{(t)} = (f_{\mathfrak{Q}}^{(t)} \bmod \mathfrak{p})$ and $\bar{\chi}_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{Q}}) = \big(\theta_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{Q}}) \bmod \bar{\mathfrak{p}}\big)$ by (4.5), it follows that $f_{\mathfrak{Q}}^{(t)}\big(\theta_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{Q}})\big) \equiv 0 \mod \bar{\mathfrak{p}}$. As this happens for the infinitely many $\mathfrak{p} \in S$, the lemma follows. **q.e.d.**

Proof of Theorem 4.1. Now we fix an arbitrary prime $\mathfrak{p}$ of A. Let $\Gamma_{\mathfrak{p}}$ denote the image of the representation

$$\rho_{\mathfrak{p}} \times \theta_{\mathfrak{p}} : \mathrm{G}_K \longrightarrow \mathrm{GL}_r(F_{\mathfrak{p}}) \times \mathrm{GL}_1(\bar{F}_{\bar{\mathfrak{p}}}).$$

Consider the algebraic morphism

$$\nu : \ \mathrm{GL}_r \times \mathrm{GL}_1 \longrightarrow \mathbb{A}^1, \ (g, h) \mapsto \det(\Lambda^t g - h 1_{\binom{r}{t}}).$$

Lemma 4.6 implies $\nu\big(\rho_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{Q}}), \theta_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{Q}})\big) = 0$ for all places $\mathfrak{Q}$ of K with $\mathfrak{Q} \nmid \mathfrak{p}$ and $\mathfrak{Q} \nmid \infty$ and where φ has good reduction. Since these $\mathrm{Frob}_{\mathfrak{Q}}$ form a dense set of conjugacy classes of G_K and the morphism ν is conjugation-invariant, we obtain

$$\nu|_{\Gamma_{\mathfrak{p}}} = 0.$$

Next the commutator morphism of $\mathrm{GL}_r \times \mathrm{GL}_1$ factors through the commutator morphism

$$\mathrm{GL}_r \times \mathrm{GL}_r \to \mathrm{SL}_r,$$

which by [10, Lemma 3.7] is dominant. Moreover Theorem 1.2 asserts that the projection of $\Gamma_{\mathfrak{p}}$ to the first factor is Zariski dense in $\mathrm{GL}_{r,F_{\mathfrak{p}}}$. Therefore the commutator subgroup $\Gamma_{\mathfrak{p}}^{\mathrm{der}}$ of $\Gamma_{\mathfrak{p}}$ is Zariski dense in $\mathrm{SL}_{r,F_{\mathfrak{p}}} \times 1$. Since ν is an algebraic morphism, it follows that ν vanishes on $\mathrm{SL}_{r,F_{\mathfrak{p}}} \times 1$.

But for any matrix of the form

$$g := \begin{pmatrix} \alpha & & & \\ & \ddots & & \\ & & \alpha & \\ & & & \alpha^{1-r} \end{pmatrix}$$

the endomorphism $\Lambda^t g$ has the eigenvalue α^t with multiplicity $\binom{r-1}{t}$ and the eigenvalue α^{t-r} with multiplicity $\binom{r-1}{t-1}$. Therefore

$$\nu(g,1) = (\alpha^t - 1)^{\binom{r-1}{t}} \cdot (\alpha^{t-r} - 1)^{\binom{r-1}{t-1}}.$$

Since $s > 1$ and thus $t < r$, we find that ν is not identically zero on matrices of the above form. In particular ν does not vanish on $\mathrm{SL}_{r,F_{\mathfrak{p}}} \times 1$. This is a contradiction, and so Assumption 4.2 is false, as desired. **q.e.d.**

5 The case of a finite extension of F

In this section, we assume that K is a finite extension of F, so that $\mathfrak{p}_0 = 0$. Then $E = \mathrm{End}_K(\varphi)$ is commutative, but we impose no further condition on it. Recall that $r = dc$, where c is the rank of E as an A-module. Recall also that

$$E_{\mathfrak{p}} = E \otimes_A A_{\mathfrak{p}} \subset \mathrm{End}_{A_{\mathfrak{p}}}\big(T_{\mathfrak{p}}(\varphi)\big),$$

and that $B_{\mathfrak{p}}$ denotes the image of the natural homomorphism

$$A_{\mathfrak{p}}[\mathrm{G}_K] \longrightarrow \mathrm{End}_{A_{\mathfrak{p}}}\big(T_{\mathfrak{p}}(\varphi)\big).$$

Theorem 5.1 *In the above situation, for almost all primes $\mathfrak{p}$ of A we have $B_{\mathfrak{p}} \cong \mathrm{M}_d(E_{\mathfrak{p}})$, and Theorem 0.2 holds in this case.*

Proof. Since E is commutative, we have $e = 1$ and $E_{\mathfrak{p}} = Z_{\mathfrak{p}}$ for all $\mathfrak{p}$. All other arguments from the proof of [10, Theorem B] also work in generic characteristic with the center Z of E replaced by E. The only missing part is the absolute irreducibility of the residual representation in the case that $\mathrm{End}_{\bar{K}}(\varphi) = A$ and that K is a finite extension of F, which is Theorem 4.1. **q.e.d.**

Note that this implies Theorem 0.1 when K is a finite extension of F.

6 The general case

We reduce the general case to the case of transcendence degree 1 in the same way as in [6].

We choose an integral scheme X of finite type over $\mathbb{F}_p$ with function field K such that φ defines a family of Drinfeld A-modules of rank r over X and such that $\mathrm{End}_K(\varphi)$ acts on the whole family of Drinfeld A-modules over X. For any point $x \in X$, we then get a Drinfeld A-module φ_x of rank r over the residue field k_x at x. Its characteristic is the image λ_x of x under the morphism $X \longrightarrow \mathrm{Spec}(A)$.

Let $k_{\bar{x}}$ be a separable closure of k_x and $\bar{x} := \mathrm{Spec}(k_{\bar{x}})$ the associated geometric point of X over x. The morphisms $\mathrm{Spec}(K) \hookrightarrow X \hookleftarrow x$ induce homomorphisms of the étale fundamental groups

$$\mathrm{G}_K \twoheadrightarrow \pi_1^{et}(X, \bar{x}) \leftarrow \pi_1^{et}(x, \bar{x}) = \mathrm{G}_{k_x}\,. \tag{6.1}$$

For any prime $\mathfrak{p} \neq \lambda_x$ of A, the specialization map induces an isomorphism

$$T_\mathfrak{p}(\varphi) \stackrel{\sim}{\longrightarrow} T_\mathfrak{p}(\varphi_x). \tag{6.2}$$

The action of G_K on $T_\mathfrak{p}(\varphi)$ factors through $\pi_1^{et}(X, \bar{x})$, and the isomorphism is equivariant under the above étale fundamental groups. Moreover, since $\mathrm{End}_K(\varphi)$ acts faithfully on $T_\mathfrak{p}(\varphi_x)$, we obtain a natural embedding $\mathrm{End}_K(\varphi) \hookrightarrow \mathrm{End}_{k_x}(\varphi_x)$. Recall that $\mathfrak{p}_0$ denotes the characteristic of φ over K.

Proposition 6.3 *Assume that $K/\mathbb{F}_p$ has transcendence degree at least 1. Then there exists a point $x \in X$ such that the following properties hold.*

(i) k_x *has transcendence degree* 1 *over* $\mathbb{F}_p$.

(ii) x *lies over* $\mathfrak{p}_0$.

(iii) $\mathrm{End}_K(\varphi)$ *has finite index in* $\mathrm{End}_{k_x}(\varphi_x)$.

Proof. Fix any prime $\mathfrak{p}$ different from $\mathfrak{p}_0$, and let $\Gamma_\mathfrak{p}$ be the image of G_K under the representation $\rho_\mathfrak{p} : \mathrm{G}_K \to \mathrm{GL}_r(A_\mathfrak{p})$. By [6, Lemma 1.5], there exists an open normal subgroup $\Gamma_1 \subset \Gamma_\mathfrak{p}$ such that for any subgroup $\Delta \subset \Gamma_\mathfrak{p}$ with $\Delta\Gamma_1 = \Gamma_\mathfrak{p}$ we have $F_\mathfrak{p}\Delta = F_\mathfrak{p}\Gamma_\mathfrak{p}$ as subalgebras of the matrix ring $\mathrm{M}_r(F_\mathfrak{p})$. Let K' be the associated finite Galois extension of K, and let X' be the normalization of X in K'. Denote the morphism $X' \to X$ by π.
By [6, Lemma 1.6], there exists a point $x \in X$ satisfying (i) and (ii) and such that $\pi^{-1}(x) \subset X'$ is irreducible. Denote by $\Delta_\mathfrak{p}$ the image of G_{k_x} in the representation on $T_\mathfrak{p}(\varphi_x)$. Since $\mathfrak{p} \neq \lambda_x$, the specialization isomorphism (6.2) turns $\Delta_\mathfrak{p}$ into a subgroup of $\Gamma_\mathfrak{p}$. The irreducibility of $\pi^{-1}(x)$ means that $\mathrm{Gal}(k_{\pi^{-1}(x)}/k_x) \cong \mathrm{Gal}(K'/K)$, and hence $\Delta_\mathfrak{p}\Gamma_1 = \Gamma_\mathfrak{p}$. The choice of Γ_1 thus implies $F_\mathfrak{p}\Delta_\mathfrak{p} = F_\mathfrak{p}\Gamma_\mathfrak{p}$. Therefore their commutants in $\mathrm{M}_r(F_\mathfrak{p})$ coincide, and by Theorem 1.1 we deduce that $\mathrm{End}_K(\varphi) \otimes_A F_\mathfrak{p} = \mathrm{End}_{k_x}(\varphi_x) \otimes_A F_\mathfrak{p}$. The structure theorem for finitely generated modules over Dedekind rings implies that $\mathrm{End}_K(\varphi)$ has finite index in $\mathrm{End}_{k_x}(\varphi_x)$.
q.e.d.

Proof of Theorem 0.2. Choose a point x be as in Proposition 6.3. Set $E = \mathrm{End}_K(\varphi)$ and $E' := \mathrm{End}_{k_x}(\varphi_x)$, and let Z and Z' be their centers. By the property 6.3 (iii) they possess the same invariants c, d, e, and for almost all $\mathfrak{p}$ we have natural isomorphisms

$$\begin{aligned} E_\mathfrak{p} = E \otimes_A A_\mathfrak{p} \quad &\stackrel{\sim}{\longrightarrow} \quad E'_\mathfrak{p} := E' \otimes_A A_\mathfrak{p}, \\ Z_\mathfrak{p} = Z \otimes_A A_\mathfrak{p} \quad &\stackrel{\sim}{\longrightarrow} \quad Z'_\mathfrak{p} := Z' \otimes_A A_\mathfrak{p}. \end{aligned}$$

Define $B_\mathfrak{p}$ and $B'_\mathfrak{p}$ as the images of the natural homomorphisms

$$\begin{array}{rcl} A_\mathfrak{p}[\mathrm{G}_K] & \twoheadrightarrow & B_\mathfrak{p} \subset \mathrm{End}_{A_\mathfrak{p}}\big(T_\mathfrak{p}(\varphi)\big), \\ A_\mathfrak{p}[\mathrm{G}_{k_x}] & \twoheadrightarrow & B'_\mathfrak{p} \subset \mathrm{End}_{A_\mathfrak{p}}\big(T_\mathfrak{p}(\varphi_x)\big). \end{array}$$

The equivariance of the specialization isomorphism (6.2) under the homomorphisms (6.1) implies that $B'_\mathfrak{p} \subset B_\mathfrak{p}$. By Theorem 5.1 in generic characteristic and [10, Theorem B] in special characteristic, Theorem 0.2 holds for φ_x. Thus $B'_\mathfrak{p}$ is the commutant of $E'_\mathfrak{p}$ for almost all $\mathfrak{p}$. Since $B_\mathfrak{p}$ contains $B'_\mathfrak{p}$ and is contained in the commutant of $E_\mathfrak{p} = E'_\mathfrak{p}$, it follows that $B_\mathfrak{p}$ *is* the commutant of $E_\mathfrak{p}$ and equal to $B'_\mathfrak{p}$ for almost all $\mathfrak{p}$. Thus Theorem 0.2 follows for φ. **q.e.d.**

Proof of Theorem 0.1. In this case, we have $\mathrm{End}_K(\varphi) = E = Z = A$ and $d = r$. Thus Theorem 0.2 asserts that the natural homomorphism

$$A_\mathfrak{p}[\mathrm{G}_K] \longrightarrow \mathrm{End}_{A_\mathfrak{p}}\big(T_\mathfrak{p}(\varphi)\big)$$

is surjective for almost all $\mathfrak{p}$. By reduction modulo $\mathfrak{p}$ the same follows for the natural homomorphism

$$\kappa_\mathfrak{p}[\mathrm{G}_K] \longrightarrow \mathrm{End}_{\kappa_\mathfrak{p}}\big(\varphi[\mathfrak{p}](K^{sep})\big).$$

For these $\mathfrak{p}$ the representation $\bar\rho_\mathfrak{p}$ is absolutely irreducible, as desired. **q.e.d.**

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