Floer homology, Novikov rings and clean intersections

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Summary

The thesis consists of two chapters. The first one is devoted to the finite dimensional Morse theory. It gives an exposition of Witten's approach to Novikov homology of a closed form. The main result of this chapter is the isomorphism between Novikov homology and singular homology with local coefficients. In the second chapter the local Floer-Conley index near the "clean" intersection of Lagrangian manifolds is computed. The example of a torus is discussed in details. Also, the Floer homology of a symplectic perturbation of the zero section in the cotangent bundle is proved to be isomorphic to the Novikov homology of the flux form of the perturbation.

Declaration

The work in this thesis is, to my best knowledge, original, except where attributed to others.

Chapter 1

Introduction

Let $f\colon M \to \mathbb{R}$ be a Morse function on a compact Riemannian manifold (M, g). The Morse inequalities provide lower estimates for the number of critical points of f using topological information about the manifold, namely its Betti numbers. In fact, it turns out that the critical points and the connecting trajectories of the gradient flow carry the complete information about the topology of the manifold. In particular it is possible to recover the homology of the manifold as the homology of a chain complex generated by the critical points. This idea goes back to Smale [Sma60] and, more recently, has been formulated by Witten [Wit82]. Consider the gradient flow generated by the equation

$$\dot{\gamma} = -\nabla f(\gamma)$$

If x is a nondegenerate critical point of f then its stable and unstable manifolds $W^s(x)$ and $W^u(x)$ intersect transversally and $\dim W^u(x) = \mathrm{ind}(x)$. For each critical point x of f choose an orientation $\langle x \rangle$ of the tangent space $E^u_x := T_x W^u(x)$ and define a graded $\mathbb{Z}$-module $C_*(M, g, f)$ by setting $C_k(M, g, f)$ to be the free $\mathbb{Z}$-module generated by the set of all $\langle x \rangle$ where x is a critical point of f of index k

$$C_k = \bigoplus_{\mathrm{ind}(x)=k} \mathbb{Z}\langle x \rangle.$$

If we assume that the gradient flow is of the Morse-Smale type, then $W^u(x) \cap W^s(y)$ is a manifold of dimension $\mathrm{ind}(x) - \mathrm{ind}(y)$. In particular, if $\mathrm{ind}(x) - \mathrm{ind}(y) = 1$ then $W^u(x) \cap W^s(y)$ consists of isolated orbits. In fact, since the the manifold is compact and the function f decreases along the nonconstant orbits, the number of these orbits is finite. With each of them one can associate a sign determined by the choice of the orientations $\langle x \rangle$ and $\langle y \rangle$. Let $n(y,x)$ denote the number of orbits connecting x and y, counted with signs. We define the homomorphism $\partial_k^f \colon C_k \to C_{k-1}$ by

$$\partial_f^k \langle x \rangle = \sum_{\mathrm{ind}(y)=k-1} n(x,y) \langle y \rangle \tag{1.1}$$

Then $(C_*(M,f,g), \partial_*^f)$ is a chain complex and

$$H_*(C_*(M,f)) \cong H_*(M). \tag{1.2}$$

An immediate consequence is that the number of critical points of a Morse function is greater or equal to the sum of the Betti numbers of the manifold. In the original proof of (1.2) (see [Mil63, Flo89b, Sal90]) one constructs a filtration $\emptyset = N_{-1} \subset N_0 \subset N_1 \subset \cdots \subset N_m = M$ of the manifold M such that all critical points of index k belong to $N_k \setminus N_{k-1}$. The set of critical points of index k can be viewed as an isolated invariant set for the gradient flow and then (N_k, N_{k-1}) form its Conley index pair. The homology groups $H_i(N_k, N_{k-1})$ do not depend on the choice of the filtration and all of them vanish except for $H_k(N_k, N_{k-1})$ which is a free module whose generators can be identified with the critical points. Hence $H_k(N_k, N_{k-1}) \cong C_k(M,f,g)$ and moreover it can be shown that the boundary homomorphism $\Delta_k \colon H_k(N_k, N_{k-1}) \to H_k(N_{k-1}, N_{k-2})$ in the exact sequence of the triple (N_k, N_{k-1}, N_{k-2}) coincides with ∂_k^f. On the other hand $\Delta_k \circ \Delta_{k+1} = 0$ and a standard algebraic argument shows that the homology of the complex $\{H_k(N_k, N_{k+1}), \Delta_k\}$ is isomorphic to that of the manifold M.

The isomorphism (1.2) can be proved without referring to the identification above, via Fredholm theory. Here, $W^u(x) \cap W^s(y)$ is identified with the space of connecting orbits $\mathcal{M}(x, y)$ which arises as the zero set of a Fredholm section of a Banach vector bundle over the manifold of paths connecting x and y. This approach was invented by Floer ([Flo88b, Flo89a, Flo89b]) who applied it to a symplectic action functional a_H on an infinite dimensional manifold of loops whose critical points are the periodic orbits of a Hamiltonian system on an underlying symplectic manifold. If (M, ω) is a symplectic manifold then a critical point of f can be seen as a 1–periodic orbit of a Hamiltonian vector field associated with f. This is one of the reasons behind the Arnold conjecture which states that for any 1–periodic Hamiltonian $H\colon S^1 \times M \to \mathbb{R}$ the number of closed orbits is greater or equal to the sum of the Betti numbers provided that all periodic solutions are nondegenerate. Floer proves the conjecture developing the Morse theory for the functional

$$a_H(\gamma) = \int_{D^2} u^*\omega + \int_0^1 H_t(\gamma(t))\, dt$$

Here $\gamma\colon \mathbb{R} \to M$ and $u\colon D^2 \to M$ extends γ over the disc D^2. Thus a_H is defined locally on the space of loops in M. The classical methods of Morse theory fail for this variational problem. The L^2-gradient of a_H does not define a global flow and moreover the index of any critical point is not finite. However, the Fredholm approach does not make use of the gradient flow and detects only the "relative" Morse index between two critical points. Moreover, even though the ambient manifold is infinite-dimensional, the spaces of connecting orbits $\mathcal{M}(x, y)$ enjoy the compactness property modulo "bubbling off" of pseudoholomorphic spheres (a phenomenon described by Gromov [Gro85]), which can be prevented by additional assumptions on the symplectic manifold. This way, Floer was able to construct a chain complex $CF_*(M, H_t)$ generated by the periodic orbits such that its homology is a symplectic invariant of M. Moreover, there is a direct link with the finite dimensional Morse theory here. If the Hamiltonian is time-independent then its Morse complex is a subcomplex of $CF_*(M, H)$.

There are many generalizations of Floer's original results, for example [SZ94] exploits the Maslov index to prove the existence of infinitely many periodic orbits, [HS92], [Ono93] covers the case of weekly monotone manifolds and [LO93] the nonexact case.

This thesis consists of two separate parts evolving around Floer's theory. The first part describes the construction of Novikov homology, that is the generalization of the Morse complex to the case of a non-exact closed form. Since in general one cannot expect the existence of a suitable filtration $\{N_i\}$ an alternative proof that $\partial^2 = 0$ is needed. Thus there is a good reason to give a presentation of Floer's ideas in the simplest finite dimensional case. This part has mainly expository character (see also [Sal90] and [Sch93] for a very detailed presentation of Morse homology). Then we show that the Novikov homology is isomorphic to the homology of the manifold with coefficients in a local system (Theorem 2.2.2), a result proved independently by V. Lê Hông and K. Ono ([LO93]). The second part uses Floer homology in the context of the Lagrangian intersections. These homology groups have not been computed in general. Theorem 3.4.11 provides the method of computing them when the intersection of the Lagrangian submanifolds is connected and sufficiently regular. As an example we describe the intersection of linear tori in T^{2n}. Also, in this part we show that the Floer homology for a non-exact perturbation of the zero section in the cotangent bundle is isomorphic to the Novikov homology of its flux form.

Chapter 2

Novikov homology

2.1 Introduction

In [Nov81, Nov82] Novikov gives a generalization of Morse theory to the case of a general closed 1–form α. He considers a covering of M, a Morse function f whose derivative is a lift of α and the cell complex built of the "surfaces of steepest descent" of f i.e. the unstable manifolds. The "multiplicity" of the critical points and noncompactness of the covering are dealt with by introducing a special coefficient ring and restricting attention to the "regions of finite energy". Novikov's papers do not include the proof that this construction defines a chain complex. Hofer and Salamon [HS92] employed Novikov's ideas to generalize Floer homology to a wider class of symplectic manifolds. They also remarked that one could define the Witten variant of Novikov homology in the finite dimensional situation. This construction, described below, is also carried out by V. Lê Hông and K. Ono in [LO93]. They show the existence of an isomorphism between Novikov homology and the homology with local coefficients using an analytical method which they developed to prove the Arnold conjecture in the non-exact case. In this thesis we present an independent proof by the methods of differential topology. It seems, however, to be less suitable for generalization to the infinite dimensional situation.

2.2 Novikov complex

Let α be a smooth closed 1–form on a compact Riemannian manifold (M, g). If $\nabla\alpha$ denotes the covariant derivative of α then for any zero point x of α the *Hessian* $\nabla\alpha(x) \colon T_xM \otimes T_xM \to \mathbb{R}$ does not depend on the particular choice of the metric. A zero point x is called *nondegenerate* iff $\nabla\alpha(x)$ is nonsingular. Then the *index* $\mathrm{ind}(x)$ of x is defined as the index of $\nabla\alpha(x)$ i.e. the dimension of its maximal negative subspace. The form α is called *nondegenerate* if all zeros of α are nondegenerate.

We want to construct an analogue of the Morse complex for α. Let X_α be the vector field dual to α:

$$g(X_\alpha(x), \xi) = \alpha(x)\xi \quad \text{for } \xi \in T_xM,\ x \in M$$

and consider the flow $\psi_\alpha \colon \mathbb{R} \times M \to M$ generated by the equation

$$\dot{\gamma} = -X_\alpha(\gamma) \tag{2.1}$$

i.e. $\psi_\alpha(\cdot, x) \colon \mathbb{R} \to M$ is a solution to (2.1) such that $\psi_\alpha(0, x) = x$. For a critical point x of (2.1) (a zero of α) define stable and unstable sets

$$W^s(x) := \{y \in M : \ \psi_\alpha(y, t) \to x \text{ as } t \to +\infty\}$$
$$W^u(x) := \{y \in M : \ \psi_\alpha(y, t) \to x \text{ as } t \to -\infty\}.$$

A flow is said to be of *Morse-Smale type* iff stable and unstable manifolds of any two critical points intersect transversally. This condition is met for a generic metric g if the form α is nondegenerate (Section 2.5). Then the sets $W^u(x^-) \cap W^s(x^+)$ are manifolds of dimension $\mathrm{ind}(x^-) - \mathrm{ind}(x^+)$. Moreover $\mathbb{R}$ acts freely on $W^u(x^-) \cap W^s(x^+)$ via ψ_α and so for $\mathrm{ind}(x^-) - \mathrm{ind}(x^+) = 1$ the set $W^u(x^-) \cap W^s(x^+)/\mathbb{R}$ is discrete. However, it may be infinite if α is not exact. There is a ($\mathbb{R}$ equivariant) bijective correspondence between

$W^u(x^-) \cap W^s(x^-)$ and the set $\mathcal{M}(x^-, x^+)$ of all orbits $\gamma\colon \mathbb{R} \to M$ of the gradient flow satisfying the limit conditions

$$\lim_{s\to-\infty} \gamma(s) = x^- \qquad \lim_{s\to+\infty} \gamma(s) = x^+ \tag{2.2}$$

given by the evaluation map

$$ev\colon \mathcal{M}(x^-, x^+) \ni \gamma \mapsto \gamma(0) \in W^u(x^-) \cap W^s(x^+) \subset M.$$

We denote $\widehat{\mathcal{M}} := \mathcal{M}/\mathbb{R}$. To ensure the finiteness property we impose some restriction on the orbits $\gamma \in \mathcal{M}(x^-, x^+)$. Define the "energy" of a differentiable path $\gamma\colon \mathbb{R} \to M$ by

$$\ell^2(\gamma) := \int_{-\infty}^{+\infty} |\dot\gamma(s)|^2\, ds.$$

In particular if $\gamma \in \mathcal{M}(x^-, x^+)$ then

$$\ell^2(\gamma) = -\int_{-\infty}^{+\infty} \langle X_\alpha(\gamma), \dot\gamma\rangle\, ds = -\int_\gamma \alpha \tag{2.3}$$

and so the energy is locally constant on $\mathcal{M}(x^-, x^+)$. We say that a sequence $\{\gamma_\nu\}_{\nu=1}^\infty \subset C^\infty(\mathbb{R}, M)$ converges to a *split trajectory* $(\gamma^1, \dots, \gamma^m)$ if there are collections of critical points $x^+ = x_0, x_1, \dots x_m = x^-$, orbits $\gamma^k \in \mathcal{M}(x_k, x_{k-1})$ and sequences of time shifts $\{s_\nu^k\}_{\nu=1}^\infty$, $k = 1, \dots, m$ such that $\gamma_\nu(\cdot + s_\nu^k)$ converges with its all derivatives on compact sets to γ^k.

Proposition 2.2.1 *For every sequence $\{\gamma'_\nu\}_{\nu=1}^\infty \subset \mathcal{M}(x^-, x^+)$ satisfying*

$$\sup_{\nu\in\mathbb{N}} \ell^2(\gamma'_\nu) < \infty$$

there is a subsequence γ_ν converging to a split trajectory $(\gamma^1, \dots, \gamma^m)$ with

$$\ell^2(\gamma_\nu) = \sum_{k=1}^m \ell^2(\gamma^k)$$

if ν is sufficiently large. If the flow is of Morse-Smale type then $\operatorname{ind}(x^+) < \operatorname{ind}(x_1) < \cdots < \operatorname{ind}(x^-)$.

Proof.(cf. [Sal90, p.136]) First note that it is enough to find s_ν^k and γ^k such that $\gamma_\nu(s_\nu^k) \to \gamma^k(0)$. Then the required convergence will follow from the differentiable dependence on initial conditions for solutions of o.d.e.'s. Take an $\varepsilon > 0$ such that $d(x, y) > 2\varepsilon$ for any two different critical points x, y of f and set

$$s_\nu^1 := \sup\{s \in \mathbb{R} : \ d(\gamma_\nu(s), x_0) > \varepsilon\}.$$

Passing to a subsequence we may assume that $\gamma_\nu(s_\nu^1)$ converges and let γ^1 be the solution to (2.1) such that $\gamma^1(0) = \lim_{\nu\to\infty} \gamma_\nu(s_\nu^1)$. Then $d(\gamma^1(s), x_0) \leq \varepsilon$ for $s > 0$ and $\ell^2(\gamma^1) < \infty$ which implies that $\gamma^1 \in \mathcal{M}(x_1, x_0)$ for some critical point x_1. For assume the contrary i.e. that there exists an ε_1 and a sequence s_ν converging to $-\infty$ such that $d(\gamma^1(s_\nu), x) > 2\varepsilon_1$ for every critical point x of α. We may assume that $s_\nu - s_{\nu+1} > 1$. If we define $M_\varepsilon := \{y \in M : \ d(y, x) \geq \varepsilon$ for any critical point $x\}$ then $c := \inf_{M_{\varepsilon_1}} |X_\alpha| > 0$. Set $t_\nu = \inf\{t \in [s_\nu, s_\nu + 1] : \ \gamma^1(t) \notin M_\varepsilon\}$ if this set is nonempty and $t_\nu = s_\nu + 1$ otherwise. Then $d(\gamma^1(s_\nu), \gamma^1(t_\nu)) > \varepsilon_1$ in the first case and so

$$\int_{s_\nu}^{t_\nu} |\dot\gamma^1|^2 \geq c \int_{s_\nu}^{t_\nu} |\dot\gamma^1| \geq \min\left\{c^2, c\varepsilon_1\right\}$$

which yields a contradiction. Proceeding by induction assume that we have found x_j, $\{s_\nu^j\}$ and $\gamma^j \in \mathcal{M}(x_j, x_{j-1})$ for $j = 1, \ldots, k$. If $x_k \neq x^-$ then $d(\gamma_\nu(s), x_k) > \varepsilon$ for large $-s$. On the other hand there is an $s^* < 0$ such that $d(\gamma^k(s), x_k) \leq \frac{\varepsilon}{2}$ for $s \leq s^*$ and so $d(\gamma_\nu(s_\nu^k + s^*), x_k) < \varepsilon$ for sufficiently large ν. Hence

$$s_\nu^{k+1} := \inf\{s \in \mathbb{R} : \ s < s_\nu^k + s^* \text{ and } d(\gamma_\nu(\sigma), x_k) < \varepsilon \text{ for } s < \sigma < s_\nu^k + s^*\}$$

is finite and the sequence $s_\nu^k + s^* - s_\nu^{k+1}$ converges to infinity since for any $T > 0$ and $\sigma' \in [s^* - T, s^*]$ we have $d(\gamma_\nu(s_\nu^k + \sigma'), \gamma^k(\sigma')) < \frac{\varepsilon}{2}$ if ν is large enough. Consequently a subsequence of $\gamma_\nu(\cdot + s_\nu^{k+1})$ converges to an orbit $\gamma^{k+1} \in$

$\mathcal{M}(x_{k+1}, x_k)$ for some critical point x_{k+1} which completes the induction step. The energy identity follows immediately from (2.3). □

Now we are ready to construct the *Novikov complex* for the form α. Choose an orientation $\langle x \rangle$ of E^u_x for each critical point of α. Let $\pi\colon \widetilde{M} \to M$ be a covering such that $\pi_1(\widetilde{M}) = ker[\alpha]$ i.e. $\pi^*\alpha$ is exact: $\pi^*\alpha = df_\alpha$ where $f_\alpha\colon \widetilde{M} \to \mathbb{R}$ is a Morse function. Let $Z_k(f_\alpha) \subset \widetilde{M}$ denote the set of all critical points of f_α of index k and define the graded $\mathbb{Z}$–module $C_* = C_*(M, \alpha, g)$ where

$$C_k \subset \prod_{\tilde{x} \in Z_k(f_\alpha)} \mathbb{Z}\langle \tilde{x} \rangle$$

consists of all $\xi = \{\xi_{\tilde{x}}\}_{\tilde{x} \in Z_k(f_\alpha)}$ such that

$$\#\{\tilde{x} \in Z_k(f_\alpha) : f_\alpha(\tilde{x}) > c\,, \xi_{\tilde{x}} \neq 0\} < \infty \quad \text{for all } c \in \mathbb{R}.$$

We also require that $\langle \tilde{x} \rangle = \pi^*\langle \pi(\tilde{x}) \rangle$. The gradient flow of f_α w.r.t. the metric $\tilde{g} = \pi^* g$ satisfies the Morse-Smale condition and clearly

$$\mathcal{M}(x, y) \cong \bigcup_{\tilde{y} \in \pi^{-1}(y)} \mathcal{M}(\tilde{x}, \tilde{y})$$

for any critical points x, y of α and $\tilde{x} \in \pi^{-1}(x)$. Moreover if $\tilde{\gamma} \in \mathcal{M}(\tilde{x}, \tilde{y})$ then

$$\ell^2(\pi \circ \tilde{\gamma}) = -\int_{\pi \circ \tilde{\gamma}} \alpha = -\int_{\tilde{\gamma}} df_\alpha = f_\alpha(\tilde{x}) - f_\alpha(\tilde{y}).$$

It follows from Proposition 2.2.1 that if $\mathrm{ind}(\tilde{x}) - \mathrm{ind}(\tilde{y}) = 1$ the number of orbits connecting $\tilde{x}$ and $\tilde{y}$ is finite. To each of them we assign a number $n_{\tilde{\gamma}} \in \{-1, +1\}$ in the following way. Since the manifold $W^u(x)$ is contractible every vector bundle over $W^u(x)$ is trivial and so $\langle x \rangle$ induces an orientation in the tangent bundle $TW^u(x)$. Similarly $\langle y \rangle$ induces an orientation of $T_y W^u(y) \cong T_y M / T_y W^s(y)$ and hence an orientation of the normal bundle $TM/TW^s(y)$ of $W^s(y)$. For any point $p \in \mathcal{M}(x, y) \cong W^u(x) \cap W^s(y)$ the transversality condition implies the isomorphism

$$T_p W^u(x) / T_p W^u(x) \cap T_p W^s(y) \cong T_p M / T_p W^s(y)$$

and we chose an orientation of $T_p\mathcal{M}(x,y) = T_pW^u(x) \cap T_pW^s(y)$ so that this isomorphism is orientation preserving. Then the number $n_{\tilde\gamma}$ is chosen so that $d\pi(n_{\tilde\gamma}\dot{\tilde\gamma})$ is a positively oriented basis of $T_{\pi\circ\tilde\gamma}\mathcal{M}(x,y)$. Denote by $n(\tilde x, \tilde y)$ the sum of $n_{\tilde\gamma}$ over all $[\tilde\gamma] \in \widehat{\mathcal{M}}(\tilde x, \tilde y)$ and define the operator $\partial\colon C_* \to C_*$

$$(\partial\xi)_{\tilde y} = \sum_{\mathrm{ind}(\tilde x)=\mathrm{ind}(\tilde y)+1} n(\tilde x, \tilde y)\xi_{\tilde x} \quad \text{for } \xi \in C_k.$$

This sum is finite since $n(\tilde x, \tilde y) = 0$ for all $\tilde x$ with $f_\alpha(\tilde y) \ge f_\alpha(\tilde x)$ and

$$\#\{\tilde x \in Z_k : \ f_\alpha(\tilde y) < f_\alpha(\tilde x)\ , \xi_{\tilde y} \ne 0\} < \infty.$$

For a similar reason $\partial\xi \in C_{k+1}$.

The $\mathbb{Z}$–modules C_k are in general infinite dimensional since there may be an infinite number of lifts of a critical point of α. We take this into account by introducing an appropriate coefficient ring. Set $\Gamma = \pi_1(M)/ker[\alpha]$ i.e. Γ is the group of deck automorphisms for $\widetilde M$ and let $\chi_\alpha\colon \Gamma \to \mathbb{R}$ be a homomorphism induced by $[\alpha]\colon \pi_1(M) \to \mathbb{R}$. Then Γ is necessarily isomorphic to $\mathbb{Z}^m$ and there is a $v \in \mathbb{R}^m$ with v_i rationally independent such that $\chi_\alpha(A) = v \cdot A$ for $A = (A_1, \ldots, A_m) \in \Gamma$. Define a subgroup $\Lambda_\alpha = \Lambda_\alpha(\mathbb{Z})$ of the product $\Pi_{A\in\Gamma}\,\mathbb{Z}$ which consists of elements $\lambda = \{\lambda_A\}_{A\in\Gamma}$ satisfying

$$\#\{A \in \Gamma : \ \chi_\alpha(A) < c; \lambda_A \ne 0\} < \infty \quad \text{for all } c \in \mathbb{R}. \tag{2.4}$$

It is a ring with the multiplication given by

$$(\lambda * \mu)_A = \sum_{B\in\Gamma} \lambda_B \mu_{B^{-1}A}$$

and a principal ideal domain ([HS92]). It can be identified with the ring of formal power series $\lambda(t) = \sum_{A\in\Gamma} \lambda_A t_1^{A_1} \ldots t_m^{A_m}$ with λ satisfying (2.4). The groups C_k have a structure of a Λ_α-module defined in the similar manner:

$$(\lambda * \xi)_{\tilde x} = \sum_{A\in\Gamma} \lambda_A \xi_{A^{-1}\tilde x} \quad \text{for } \lambda \in \Lambda_\alpha,\ \xi \in C_*.$$

This is well defined since $f_\alpha(A^{-1}\tilde{x}) = f_\alpha(\tilde{x}) - \chi_\alpha(A)$ and clearly C_k is isomorphic to a free Λ_α-module generated by the critical points of α of index k. An isomorphism is given by picking an element $\tilde{x} \in \pi^{-1}(x)$ for each critical point x. Moreover $\partial\colon C_* \to C_*$ is a Λ_α-homomorphism. Indeed

$$\begin{aligned}(\partial\lambda * \xi)_{\tilde{y}} &= \sum_{\mathrm{ind}(\tilde{x})=\mathrm{ind}(\tilde{y})+1} n(\tilde{x},\tilde{y}) \sum_{A\in\Gamma} \lambda_A \xi_{A^{-1}\tilde{x}} \\ &= \sum_{A\in\Gamma} \lambda_A \sum_{\mathrm{ind}(\tilde{x})=\mathrm{ind}(\tilde{y})+1} n(A^{-1}\tilde{x}, A^{-1}\tilde{y}) \xi_{A^{-1}\tilde{x}} \\ &= \sum_{A\in\Gamma} \lambda_A (\partial\xi)_{A^{-1}\tilde{y}} = (\lambda * \partial\xi)_{\tilde{y}}\end{aligned}$$

where the second equality follows since the gradient flow of f_α is Γ–equivariant and $n_{A\tilde{\gamma}} = n_{\tilde{\gamma}}$.

The ring Λ_α contains the group ring $\mathbb{Z}\Gamma$ of Γ as a subring. Hence we have the representation

$$\pi_1(M) \to \Gamma \to \mathbb{Z}\Gamma \to \mathrm{Hom}(\Lambda_\alpha)$$

which in turn defines a local system $\mathcal{L}_\alpha$ on M (see e.g. [Whi78]). The aim of this chapter is to prove the following

Theorem 2.2.2 (Novikov homology, cf. [Nov81, Nov82, HS92, LO93]) *Let α be a closed nondegenerate form on a compact Riemannian manifold (M, g) such that the flow generated by the dual vector field X_α satisfies the transversality condition. Then $(C_*(M,\alpha,g), \partial^*)$ is a Λ_α–chain complex and*

$$H_*(C_*(M,\alpha_0,g_0);\Lambda_{\alpha_0}) \cong H_*(C_*(M,\alpha_1,g_1);\Lambda_{\alpha_1})$$

for any two forms in the same de Rham homology class. We call it Novikov homology and denote by $HN_*(M,[\alpha];\mathbb{Z})$ where $[\alpha] \in H^1_{DR}(M,\mathbb{R})$. It is isomorphic to the homology of the manifold with coefficients in the local system $\mathcal{L}_\alpha$.

Let μ_k denote the number of the critical points of α of index k and β_k the rank of $H_k(M, \mathcal{L}_\alpha)$. The standard algebraic argument (see e.g. [Hir70]) gives then

Corollary 2.2.3 *There exists a polynomial Q with nonnegative integral coefficients such that*

$$\sum_{i=1}^{n} \mu_i t^i = \sum_{i=1}^{n} \beta_i t^i + (1+t)Q(t).$$

In particular for $k \leq n$

$$\sum_{i=1}^{k} (-1)^i \mu_i \geq \sum_{i=1}^{k} (-1)^i \beta_i.$$

2.3 Connecting orbits and glueing

Theorem 2.3.1 *Let (M, g) be a compact Riemannian manifold and α a nondegenerate closed form on M such that the flow generated by (2.1) satisfies the Morse-Smale condition for critical points. Let $C_*(M, g, \alpha)$ be the graded $\Lambda_\alpha(\mathbb{Z})$ module defined as in the previous section together with the degree -1 homomorphism ∂. Then $\partial^2 = 0 (mod\ 2)$.*

Proof. If we write down the formula for ∂^2:

$$(\partial^2 \xi)_{\tilde{x}} = \sum_{\substack{\text{ind}(\tilde{y})=k+1 \\ \text{ind}(\tilde{z})=k}} n(\tilde{x}, \tilde{y}) n(\tilde{y}, \tilde{z}) \xi_{\tilde{z}} \quad \xi \in C_k$$

we see that the condition $\partial^2 = 0(\text{mod } 2)$ means that the number of pairs of orbits $([\gamma^-], [\gamma^+]) \in \bigcup_{\text{ind}(\tilde{y})=k} \widehat{\mathcal{M}}(\tilde{x}, \tilde{y}) \times \widehat{\mathcal{M}}(\tilde{y}, \tilde{z})$ is even for every critical points $\tilde{x}$, $\tilde{z}$ of f_α with $\text{ind}(\tilde{x}) = \text{ind}(\tilde{z}) + 2 = k + 1$. In fact, we will prove that such pairs of orbits occur in pairs. If $\text{ind}(x) - \text{ind}(z) = 2$ then the quotient manifold $\widehat{\mathcal{M}}(x, z) = \mathcal{M}(x, z)/\mathbb{R}$ is one-dimensional i.e. consists of a number of circles and open intervals. Since the energy $\ell^2(\gamma)$ is constant on a

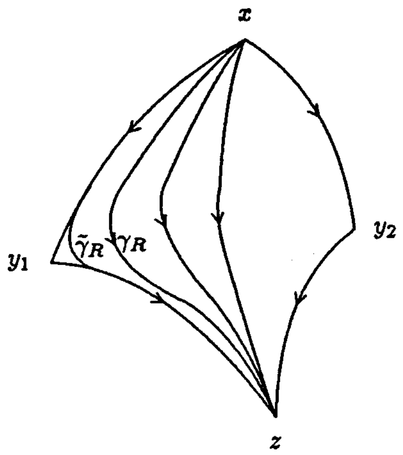

Figure 2.1:

connected component of $\mathcal{M}(x,z)$ each "end" of such an interval converges to a split trajectory $(\gamma^-,\gamma^+) \in \mathcal{M}(x,y)\times\mathcal{M}(y,z)$ (strictly speaking, if $[\gamma_\nu] \in \widehat{\mathcal{M}}$ does not converge in $\widehat{\mathcal{M}}$ then γ_ν converges to (γ^-,γ^+)). Hence in order to prove that $\partial^2 = 0$ it is enough to show that this is a $1-1$ correspondence. For this sake we employ Floer's concept of glueing orbits i.e. a family of orbits $\gamma_R \in \mathcal{M}(x,z)$, with $R \in \mathbb{R}$, $R > R_0$ converging to (γ^-,γ^+). This family is unique i.e. any orbit connecting x and z sufficiently close to γ^- and γ^+ belongs to the family γ_R.

We introduce the setup for the construction of γ_R. For any critical points x^- and x^+ of α define the space $\mathcal{P} = \mathcal{P}(x^-,x^+)$:

$$\begin{aligned}\mathcal{P}(x^-,x^+) = \Big\{\gamma\colon \mathbb{R}\to M : \gamma \in W^{1,2}_{loc}(M),\\ &\exists T>0,\ \exists \xi_- \in W^{1,2}((-\infty,-T];T_{x^-}M),\\ &\exists \xi_+ \in W^{1,2}([T,\infty);T_{x^+}M)\\ &\text{such that } \gamma(s) = exp_{x^\pm}\xi_\pm(s) \text{ for } \pm s \ge T\Big\}.\end{aligned}$$

Clearly any solution to (2.1) with limit conditions (2.2) belongs to $\mathcal{P}$. For $\gamma \in \mathcal{P}$ we will write $\mathcal{F}(\gamma) := \dot{\gamma} + X_\alpha(\gamma)$ i.e. $\mathcal{M}(x^-, x^+) = \mathcal{F}^{-1}(0)$. Define the Banach spaces of vector fields along $\gamma \in \mathcal{P}(x^-, x^+)$:

$$L^2(\gamma) = \left\{\xi\colon \mathbb{R} \to TM \ : \xi(s) \in T_{\gamma(s)}M \text{ and } \int_{-\infty}^{+\infty} |\xi|^2 < \infty\right\},$$
$$W^{1,2}(\gamma) = \left\{\xi \in L^2(\gamma) \ : \nabla\xi \in L^2(\gamma)\right\},$$
$$W^{2,2}(\gamma) = \left\{\xi \in L^2(\gamma) \ : \nabla\xi \in W^{1,2}(\gamma)\right\}$$

where ∇ denotes Levi-Civita connection on M. We define the operator $D_\gamma\colon W^{1,2}(\gamma) \to L^2(\gamma)$ by the formula:

$$D_\gamma\xi = \nabla\xi + \nabla_\xi X_\alpha.$$

Let $D^*\colon W^{1,2}(\gamma) \to L^2(\gamma)$ be the formal adjoint of D. Since α is closed ∇X_α is a symmetric tensor

$$\langle \nabla_\xi X_\alpha, \zeta\rangle = \langle \xi, \nabla_\zeta X_\alpha\rangle \quad \text{for } \xi,\ \zeta \in T_xM,\ x \in M$$

an so $D_\gamma^*\xi = -\nabla\xi + \nabla_\xi X_\alpha$.

Theorem 2.3.2 ([Sal90]) *If x^- and x^+ are nondegenerate critical points of a closed form α and $\gamma\colon \mathbb{R} \to M$ is a smooth path satisfying (1.3) then D_γ is a Fredholm operator and*

$$\operatorname{ind} D_\gamma = \operatorname{ind}(x^-) - \operatorname{ind}(x^+).$$

If $\mathcal{F}(\gamma) = 0$ then D_γ is onto iff $W^u(x^-)$ and $W^s(x^+)$ intersect transversally along γ.

The Fredholm property allows us to formulate an "existence and uniqueness" result, a version of the implicit function theorem which is the main tool for the construction of glueing orbits.

Theorem 2.3.3 *For any $c > 0$ there are positive constants c_1, ε_0 such that the following holds.*

a) if $0 < \varepsilon < \varepsilon_0$ and $\gamma \in \mathcal{P}$ satisfies

$$\|\dot{\gamma}\|_{L^2} \leq c,$$

$$c\|D_\gamma^* \eta\|_{L^2} \geq \|\eta\|_{W^{1,2}} \quad \text{for any } \eta \in W^{1,2}(\gamma)$$

and

$$\|\mathcal{F}(\gamma)\|_{L^2} < \varepsilon$$

then there is a unique vector field $\zeta \in D_\gamma^*(W^{2,2}(\gamma))$ such that $\|\zeta\|_{W^{1,2}} \leq c_1\varepsilon$ the path $\gamma_0 = exp_\gamma \zeta$ satisfies $\mathcal{F}(\gamma_0) = 0$ and

$$4c\|D_{\gamma_0}^* \eta\|_{L^2} \geq \|\eta\|_{W^{1,2}} \quad \text{for any } \eta \in W^{1,2}(\gamma_0)$$

b) For γ as above if $\mathcal{F}(\gamma) = 0$ then there is a C^1-function ϕ defined on the neighbourhood of zero in $\ker D_\gamma$, $\phi\colon \ker D_\gamma \to D_\gamma^*(W^{1,2}(\gamma))$ such that for any $\zeta \in W^{1,2}(\gamma)$ we have $\zeta = \xi + \phi(\xi)$ for some ξ in the domain of ϕ iff $\mathcal{F}(exp_\gamma \zeta) = 0$ and $\|\zeta\|_{W^{1,2}} < \varepsilon_0$.

The proof of this theorem based on Newton method is given in the next section.

Let x, y, z be critical points of α and $\gamma^- \in \mathcal{M}(x, y), \gamma^+ \in \mathcal{M}(y, z)$ and let $\beta\colon \mathbb{R} \to [0, 1]$ be a cut-off function:

$$\beta(s) := \begin{cases} 0, & \text{for } s \leq -1 \\ 1, & \text{for } s \geq 1. \end{cases}$$

For a positive $R \in \mathbb{R}$ define an approximate solution to (2.1):

$$\tilde{\gamma}_R(s) := \begin{cases} \gamma^-(s+R), & \text{for } s \leq -1 \\ exp_y(\beta(s)\xi^+(s-R) + (1-\beta(s))\xi^-(s+R)), & \text{for } s \in [-1, 1] \\ \gamma^+(s-R), & \text{for } s \geq 1 \end{cases}$$

where $\gamma^\pm(s) = exp_y \xi^\pm(s)$ for $|s| > R - 1$ an R sufficiently large.

Proposition 2.3.4 *There are positive constants k, C, R_0 such that for $R > R_0$ the family $\tilde{\gamma}_R$ defined by satisfies the estimates*

$$\|\dot{\tilde{\gamma}}\|_{L^2} \leq C$$
$$C\|D^*_{\tilde{\gamma}_R}\eta\|_{L^2} \geq \|\eta\|_{W^{1,2}} \qquad \text{for } \eta \in W^{1,2}(\gamma)$$
$$\|\mathcal{F}(\tilde{\gamma}_R)\|_{L^2} \leq Ce^{-kR}.$$

Thus $\tilde{\gamma}_R$ satisfies the assumptions of the part a) of Theorem 2.3.3 and leads to a family γ_R of glueing orbits. Then the uniform estimates on the norm of $D^*_{\gamma_R}$ yields the uniqueness result. The properties of γ_R are summarized below

Proposition 2.3.5 *The family γ_R of solutions of* (2.1) *converges (γ^-, γ^+). Moreover, for any sequence $\{\gamma'_j\} \subset \mathcal{M}(x, z)$ converging to the pair (γ^-, γ^+) a curve γ'_j belongs to the same component of $\mathcal{M}$ as the family γ_R for j sufficiently large.*

In other words there exists precisely one end of a component of $\widehat{\mathcal{M}}(x, z)$ which converges to (γ^-, γ^+). This proves Theorem 2.3.1.

2.3.1 Proof of Theorem 2.3.3

In the following we will — abusing notation — denote by K subsequent constants depending only on the Riemannian metric and the function f.

Let ε_0 be less than the injectivity radius for the metric g. For $\gamma \in \mathcal{P}(x^-, x^+)$ define

$$U_\varepsilon(\gamma) := \{\xi \in W^{1,2}(\gamma) \; : \; \|\xi\|_{L^\infty} < \varepsilon\}.$$

Since $\|\xi\|_{L^\infty} \leq \|\xi\|_{W^{1,2}}$ the set $U_\varepsilon(\gamma)$ is open in $W^{1,2}(\gamma)$. We denote also $TM(\varepsilon_0) = \{v \in TM : \; |v| < \varepsilon\}$. The set $\mathcal{P}(x^-, x^+)$ is a Banach manifold modeled on $W^{1,2}(\gamma)$ where $\gamma \in C^\infty(M) \cap \mathcal{P}$ with the charts induced by the exponential map. The proof of this fact is essentially given in [Kli78] and we summarize it below for the sake of completeness. We need the following

Lemma 2.3.6 *There is a positive constant K such that for any interval $J \subset \mathbb{R}$ of length greater than 1, a path $\gamma_0 \in W^{1,2}_{loc}(J; M)$ and a vector field $\xi_0 \in W^{1,2}(\gamma_0)$, $\|\xi_0\|_{W^{1,2}} \leq \frac{\varepsilon_0}{2}$*

$$\|\dot\gamma_1\|_{L^2} \leq \|\dot\gamma_0\|_{L^2} + K(1 + \|\dot\gamma_0\|_{L^2})\|\xi_0\|_{W^{1,2}} \tag{2.5}$$

where $\gamma_1 = exp_{\gamma_0}\xi_0$ and the following holds.

If $\xi_1 \in W^{1,2}(exp_{\gamma_0}\xi_0)$, with $\|\xi_1\|_{W^{1,2}} \leq \frac{\varepsilon_0}{2}$ then there is a vector field $\xi_2 \in W^{1,2}(\gamma_0)$ such that

$$exp_{\gamma_0}\xi_2 = exp_{exp_{\gamma_0}\xi_0}\xi_1 \tag{2.6}$$

and

$$\|\xi_2 - \xi_0\|_{W^{1,2}} \leq K(1 + \|\dot\gamma_0\|_{L^2} + \|\xi_0\|_{W^{1,2}})\|\xi_1\|_{W^{1,2}}.$$

Conversely if $\xi_2 \in W^{1,2}(\gamma_0)$ is such that $\|\xi_2 - \xi_0\|_{W^{1,2}} \leq \frac{\varepsilon_0}{2}$ then there is a $\xi_1 \in W^{1,2}(exp_{\gamma_0}\xi_0)$ satisfying (2.6) and

$$\|\xi_1\|_{W^{1,2}} \leq K(1 + \|\dot\gamma_0\|_{L^2} + \|\xi_0\|_{W^{1,2}})\|\xi_2 - \xi_0\|_{W^{1,2}}.$$

Proof. We denote by $E(x, \xi)$ the representation of $exp_x\xi$ in local coordinates x^i, ξ^i. Then

$$\dot\gamma_1 = \dot\gamma_0 + (E_x(\gamma_0, \xi_0) - id)\dot\gamma_0 + E_\xi(\gamma_0, \xi_0)\dot\xi$$
$$E_x(\gamma_0, \xi_0) - id = \int_0^1 E_{x\xi}(\gamma_0, t\xi_0)\xi_0\, dt$$

and since $\nabla\xi = \dot\xi + \Gamma\xi\dot\gamma$

$$|\dot\gamma_1| \leq |\dot\gamma_0| + K(|\xi|\,|\dot\gamma_0| + |\nabla\xi|)$$

where K bounds the derivatives of the exponential map and the Christoffel symbols Γ on $TM(\varepsilon_0)$. This yields (2.5).

We prove first of the remaining inequalities. The vector field ξ_2 exists if L^∞ norms of ξ_0 and ξ_1 are small and clearly $\xi_2 \in W^{1,2}(\gamma_0)$. In normal coordinates at γ_0 we have $\xi_2 = E(\xi_0, \xi_1)$ and

$$\xi_2 - \xi_0 = E(\xi_0, \xi_1) - E(\xi_0, 0) = \int_0^1 E_\xi(\xi_0, t\xi_1)\xi_1 \, dt$$
$$|\xi_2 - \xi_0| \leq K|\xi_1|$$

This implies the L^2 inequality. Differentiating (2.6) in local coordinates we obtain

$$\begin{aligned} E_\xi(\gamma_0, \xi_2)(\dot{\xi}_2 - \dot{\xi}_0) &= E_\xi(E(\gamma_0, \xi_0), \xi_1)\dot{\xi}_1 \\ &+ (E_x(E(\gamma_0, \xi_0), \xi_1) - id)\,(E_\xi(\gamma_0, \xi_0)\dot{\xi}_0 + E_x(\gamma_0, \xi_0)\dot{\gamma}_0) \\ &+ (E_x(\gamma_0, \xi_2) - (E_x(\gamma_0, \xi_0))\,\dot{\gamma}_0 + (E_\xi(\gamma_0, \xi_0) - E_\xi(\gamma_0, \xi_2))\,\dot{\xi}_0. \end{aligned}$$

As E_ξ is invertible and $\|E_\xi^{-1}\|$ is bounded on $TM(\varepsilon_0)$ it is enough to estimate the RHS in terms of ξ_1. We have

$$E_\xi(\gamma_0, \xi_0) - E_\xi(\gamma_0, \xi_2) = \int_0^1 E_{\xi\xi}(\gamma, \xi_0 + t(\xi_2 - \xi_0))(\xi_2 - \xi_0)\, dt$$
$$\|E_\xi(\gamma_0, \xi_0) - E_\xi(\gamma_0, \xi_2)\| \leq K|\xi_2 - \xi_0|$$

Similarly

$$\|E_x(\gamma_0, \xi_2) - (E_x(\gamma_0, \xi_0)\| \leq K|\xi_2 - \xi_0|$$
$$\|E_x(E(\gamma_0, \xi_0), \xi_1) - id\,\| \leq K|\xi_1|.$$

Putting these estimates together we obtain

$$|\nabla\xi_2 - \nabla\xi_0| \leq |\dot{\xi}_2 - \dot{\xi}_0| + |\Gamma\dot{\gamma}_0(\xi_2 - \xi_0)| \leq K(|\nabla\xi_1| + |\xi_1|(|\dot{\gamma}_0| + |\xi_0| + |\nabla\xi_0|))$$

where Γ are the Christoffel symbols. This yields the result. The second inequality is proved in a similar manner.

From the lemma follows that we can define the exponential map for $\gamma \in \mathcal{C}^\infty \cap \mathcal{P}$

$$exp_\gamma \colon U_{\varepsilon_0}(\gamma) \to \mathcal{P}(x^-, x^+)$$
$$exp_\gamma \xi(s) = exp_{\gamma(s)}\xi(s).$$

Indeed if $\eta \in U_{\varepsilon_0}(\gamma)$ and $\gamma(s) = exp_{x^\pm}\xi_\pm(s)$ then Lemma 2.3.6 guarantees the existence of $\eta_\pm \in W^{1,2}(J, T_{x^\pm}M)$ with $exp_{x^\pm}\eta_\pm = exp_\gamma \eta$. Also, if $\gamma_0, \gamma_1 \in \mathcal{C}^\infty \cap \mathcal{P}$ then

$$exp_{\gamma_0}^{-1}(exp_{\gamma_0}(U_{\varepsilon_0}(\gamma_0) \cap exp_{\gamma_1}(U_{\varepsilon_0}(\gamma_1))$$

is open in $U_{\varepsilon_0}(\gamma_0)$. The transition map can be described as follows. Let U be an open subset of γ^*TM defined by

$$U(s) = U \cap T_{\gamma_0(s)}M = exp_{\gamma_0(s)}^{-1}(exp_{\gamma_1(s)}T_{\gamma_1(s)}M(\varepsilon_0)).$$

Thus we have a bundle map $\Psi \colon U \to \gamma_1^*TM$, $\Psi = exp_{\gamma_1}^{-1} \circ exp_{\gamma_1}$ and the transition map is given by $\xi \mapsto \Psi \circ \xi$.

Proposition 2.3.7 (cf. [Kli78]) *Let $\pi_0 \colon E_0 \to \mathbb{R}$, $\pi_1 \colon E_1 \to \mathbb{R}$ be two Riemannian vector bundles over $\mathbb{R}$ and $\Psi \colon U \to E_1$ a smooth bundle map defined in an open set $U \subset E_0$ with bounded derivatives in vertical direction. Then the assignment $\xi \mapsto \hat{\Psi}(\xi) := \Psi \circ \xi$ defines a C^1 map*

$$\hat{\Psi} \colon \hat{U} \to W^{1,2}(E_1)$$

on any open set $\hat{U} \subset \{\xi \in W^{1,2}(E_0) : \xi(s) \in U\}$

proof. Let $D_v^k\Psi(x)$ denote the k-th derivative of $\Psi|_{\pi_0^{-1}(\pi_0(x))}$ at x. Then for

$\xi, \eta \in \hat{U}$ we have $\nabla\Psi \circ \xi = D_v(\xi).\nabla\xi$ and

$$|\Psi \circ \xi - \Psi \circ \eta| \leq \sup_U |D_v\Psi| \cdot |\xi - \eta|$$

$$\begin{aligned}|\nabla\Psi \circ \xi - \nabla\Psi \circ \eta| &\leq |D_v(\xi) - D_v(\eta)| \cdot |\nabla\xi| + |D_v(\eta)| \cdot |\nabla\xi - \nabla\eta| \\ &\leq \sup_U |D_v^2\Psi| \cdot |\xi - \eta| \cdot |\nabla\xi| + \sup_U |D_v\Psi| \cdot |\nabla\xi - \nabla\eta|\end{aligned}$$

which shows that $\hat{\Psi}$ is continuous. Similarly the bundle map $D_v\Psi\colon U \to L(E_0, E_1)$ induces a continuous map

$$\widehat{D_v\Psi}\colon \hat{U} \to W^{1,2}(L(E_0, E_1)).$$

If $A \in W^{1,2}(L(E_0, E_1))$, $\xi \in W^{1,2}(E_0)$ and $A\xi(s) = A(s).\xi(s)$ then

$$\|A\xi\|_{L^2} \leq \|A\|_{L^\infty} \cdot \|\xi\|_{L^2}$$

$$\|\nabla(A\xi)\|_{L^2} = \|\nabla A\xi + A\nabla\xi\|_{L^2} \leq \|\nabla A\|_{L^2} \cdot \|\xi\|_{L^\infty} + \|A\|_{L^\infty} \cdot \|\nabla\xi\|_{L^2}$$

and so we have a linear continuous map

$$\#\colon W^{1,2}(L(E_0, E_1)) \to L(W^{1,2}(E_0), W^{1,2}(E_1))$$

with $\|\#\| \leq 2$. We define derivative of $\hat{\Psi}$ as $D\hat{\Psi} = \# \circ \widehat{D_v\Psi}$. From Taylor formula

$$\Psi(\xi(s)) - \Psi(\eta(s)) - D_v\Psi(\eta(s)).(\xi(s) - \eta(s)) = r(\xi(s), \eta(s)).(\xi(s) - \eta(s))$$

where $r\colon U \times U \to L(E_0, E_1)$ is a bundle map with bounded derivatives. Hence

$$\hat{\Psi}(\xi) - \hat{\Psi}(\eta) - D\hat{\Psi}(\eta).(\xi - \eta) = \#\hat{r}(\xi, \eta).(.(\xi - \eta)$$

and $\|\#\hat{r}(\xi, \eta)\|_{W^{1,2}} \to 0$ as $\|\xi - \eta\|_{W^{1,2}} \to 0$. □

Thus the maps exp_γ introduce a differentiable structure on $\mathcal{P}$. Note, that this structure is compatible for different metrics if they are C^1 close. Indeed, this reduces to the observation that for such metrics $W^{1,2}$ norms are equivalent.

If $\gamma \in \mathcal{P}(x^-, x^+)$ then for large $|s|$ we have $\gamma(s) = exp_{x^\pm}\xi_\pm(s)$ with $|\xi_\pm(s)| < \varepsilon_0$ and since X_α is locally Lipschitz

$$|\mathcal{F}(\gamma)(s)| \leq |dexp_{x^\pm}\xi_\pm(s)| \left(|\dot{\xi}_\pm| + |X_\alpha(\xi_\pm) - X_\alpha(0)|\right) \leq K(|\dot{\xi}_\pm| + |\xi_\pm|)$$

and hence $\mathcal{F}(\gamma) \in L^2(\gamma)$. For $\xi \in B_{\varepsilon_0}(\gamma)$ denote by $\tau^\xi(t_0, t_1)$ the parallel transport along $exp_\gamma t\xi$ from $exp_\gamma t_0\xi$ to $exp_\gamma t_1\xi$ and define the function $\mathcal{F}_{\gamma_0} : W^{1,2}(\gamma_0) \to L^2(\gamma_0)$

$$\mathcal{F}_{\gamma_0}(\xi) = \tau^\xi(1,0)\mathcal{F}(exp_{\gamma_0}\xi).$$

Proposition 2.3.8 *The function $\mathcal{F}_{\gamma_0}$ is continuously differentiable in $B_{\varepsilon_0}(\gamma_0)$ with differential $d\mathcal{F}_{\gamma_0}(\xi) := D_\xi : W^{1,2}(\gamma_0) \to L^2(\gamma_0)$*

$$D_\xi\eta := \tau^\xi(1,0)D_{\gamma_1} \circ dexp(\xi)\eta + A_\xi(\eta, \mathcal{F}_{\gamma_0}(\xi))$$

where $\gamma_1 = exp_{\gamma_0}\xi$ and A is a bounded bilinear form on $TM(\varepsilon_0)$ and $A_0 = 0$. More specifically there is a constant K independent of γ_0 such that for every $\xi \in B_{\varepsilon_0}(\gamma_0)$ and $\eta \in W^{1,2}(\gamma_0)$

$$\|\mathcal{F}_{\gamma_0}(\xi + \eta) - \mathcal{F}_{\gamma_0}(\xi) - D_\xi\eta\|_{L^2} \leq K \,\|\eta\|_{W^{1,2}} \,\|\eta\|_{L^\infty}(\|\dot{\gamma}_0\|_{L^2} + 1)$$

and moreover

$$\|D_\xi\eta - D_0\eta\|_{L^2} \leq K \left(\|\xi\|_{W^{1,2}} + \|\mathcal{F}_{\gamma_0}(\xi)\|_{L^2}\right) \|\eta\|_{W^{1,2}}.$$

Proof. Let $\xi, \eta \in W^{1,2}(\gamma_0)$ and set

$$u(r,t) = u(r,t,s) = exp_{\gamma_0(s)}r(\xi(s) + t\eta(s)), \quad \gamma_r = u(r,0) \quad \text{and } \xi_t = \xi + t\eta.$$

Denote by τ^η the parallel transport along $u(1,t)$ and consider the difference

$$\begin{aligned}(2.7)\quad \mathcal{F}_{\gamma_0}(\xi + \eta) - \mathcal{F}_{\gamma_0}(\xi) &= \tau^{\xi_1}(1,0)\mathcal{F}(u(1,1)) - \tau^\xi(1,0)\mathcal{F}(u(1,0)) \\ &= \left(\tau^{\xi_1} - \tau^\xi(1,0)\tau^\eta(1,0)\right)\mathcal{F}(u(1,1)) \\ &\quad - \tau^\xi(1,0)\left(\tau^\eta(1,0)\mathcal{F}(u(1,1)) - \mathcal{F}(u(1,0))\right).\end{aligned}$$

The first step is to obtain the estimate

$$\left\| \tau^{\eta}(1,0)\mathcal{F}\left(u(1,1)\right) - \mathcal{F}\left(u(1,0)\right) - D_{\gamma_1}\frac{\partial u}{\partial t}(1,0) \right\|_{L^2} \leq K\, \|\eta\|_{W^{1,2}}\, \|\eta\|_{L^\infty} (\|\dot{\gamma}_0\|_{L^2} + 1). \tag{2.8}$$

Denote

$$\Phi_s(t) = \tau^{\eta}(t,0)\left(\frac{\partial u}{\partial s}(1,t) + X_\alpha(u)\right),$$
$$\Phi_s(0) = \mathcal{F}(\gamma)(s) \qquad \Phi_s(1) = \tau^{\xi}(1,0)\mathcal{F}(u(1,1))(s).$$

Since the derivative of the parallel transport of a vector field along a curve is the covariant derivative of the vector field,

$$\Phi'(t_0) = \frac{d}{dt}\Big|_{t=t_0} \tau^{\eta}(t_0,0)\tau^{\eta}(t,t_0)\left(\frac{\partial u}{\partial s} + X_\alpha(u)\right)$$
$$= \tau^{\eta}(t_0,0)\nabla_t\left(\frac{\partial u}{\partial s} + X_\alpha(u)\right) = \tau^{\eta}(t_0,0)\left(\nabla_s\frac{\partial u}{\partial t} + \nabla_t X_\alpha(u)\right),$$
$$\Phi'(0) = D_{\gamma_1}\frac{\partial u}{\partial t}(1,0).$$

Here we use the fact that $[\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}] = u^*[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = 0$ and so $\nabla_s\frac{\partial u}{\partial t} = \nabla_t\frac{\partial u}{\partial s}$. To prove (2.8) we estimate second derivative of Φ

$$\Phi''(t) = \tau^{\eta}(t,0)\nabla_t\left(\nabla_s\frac{\partial u}{\partial t} + \nabla_t X_\alpha(u)\right).$$

We have

$$\nabla_t\nabla_s\frac{\partial u}{\partial t} - \nabla_s\nabla_t\frac{\partial u}{\partial t} = R\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}\right)\frac{\partial u}{\partial t},$$
$$\nabla_t\left(\nabla_t X_\alpha(u)\right) = \nabla_{\nabla_t\frac{\partial u}{\partial t}} X_\alpha(u) + \nabla_t\left(\nabla X_\alpha(u)\right)\frac{\partial u}{\partial t}.$$

Since both R and $\nabla(X_\alpha)$ are smooth sections of the appropriate tensor bundles and parallel transport is an isometry

$$\left|\tau^{\eta}(t,0)R\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}\right)\frac{\partial u}{\partial t}\right| \leq \|R\|_{L^\infty}\left|\frac{\partial u}{\partial t}\right|^2\left|\frac{\partial u}{\partial s}\right| \tag{2.9}$$

$$\left|\tau^{\eta}(t,0)\nabla_t\left(\nabla X_\alpha(u)\right)\frac{\partial u}{\partial t}\right| \leq \|\nabla(X_\alpha)\|_{L^\infty}\left|\frac{\partial u}{\partial t}\right|^2, \tag{2.10}$$

and moreover $|\frac{\partial u}{\partial t}| \leq \|dexp(\xi + t\eta)\|_{L^\infty}|\eta|$. We estimate $|\frac{\partial u}{\partial s}|$ in local coordinates

$$\left|\frac{\partial u}{\partial s}\right| = \left|\frac{d}{ds}tE(\gamma_1, \eta)\right| \leq |E_x(\gamma_1, \eta)\dot{\gamma}_1| + |E_\xi(\gamma_1, \eta)\dot{\eta}| \leq K\left(|\dot{\gamma}_1| + |\nabla\eta| + |\dot{\gamma}_1|\,|\eta|\right)$$

since $|\dot{\eta}| \leq |\nabla\eta| + |\dot{\gamma}_1|\,|\eta||\Gamma|$ where Γ denote Christoffel symbols. Similarly

$$|\dot{\gamma}_1| \leq K\left(|\dot{\gamma}_0| + |\nabla\xi| + |\dot{\gamma}_0|\,|\xi|\right).$$

It remains to estimate $\nabla_t\frac{\partial u}{\partial t}(1,t)$ and $\nabla_s\nabla_t\frac{\partial u}{\partial t}(1,t)$. We have

$$\nabla_t\frac{\partial u}{\partial t}(1,t) = E_{\xi^i\xi^j}(\gamma, \xi + t\eta)\eta^i\eta^j + E_{\xi^i}(\gamma, \xi + t\eta)E_{\xi^j}(\gamma, \xi + t\eta)\eta^i\eta^j\Gamma_{ij}.$$

It follows that $\left|\nabla_t\frac{\partial u}{\partial t}\right| \leq K|\eta|^2$ and similarly

$$\left|\nabla_s\nabla_t\frac{\partial u}{\partial t}\right| \leq K\left(|\eta|^2 + (|\nabla\eta| + |\dot{\gamma}_0|\,|\eta|)|\eta|\right)$$

where K is a constant bounding Γ and its derivatives and the derivatives of the exponential map on $TM(\varepsilon_0)$. Note that $\nabla_t\frac{\partial u}{\partial t}$ and $\nabla_s\nabla_t\frac{\partial u}{\partial t}$ vanish if $\xi = 0$. Putting all estimates together and assuming that $\varepsilon_0 < 1$

$$\begin{aligned}
&|\Phi(1) - \Phi(0) - \Phi'(0)| \\
&\qquad \leq K\left(|\eta|^2(|\dot{\gamma}_1| + |\nabla\eta| + |\dot{\gamma}_1||\eta|) + (1 + |\gamma_0|)|\eta|^2 + |\nabla\eta|\,|\eta|\right) \\
&\qquad\qquad \leq K\left(|\eta|^2(|\dot{\gamma}_0| + 1) + |\nabla\eta|\,|\eta|\right)
\end{aligned}$$

which implies (2.8).

To deal with the first term of (2.7) let $v(t) = \tau^\eta(1,t)\mathcal{F}(u(1,1))$ and $w(r,t) = \tau^{\xi_t}(1,r)v(t)$. Thus we are to estimate the difference

$$w(0,1) - w(0,0) = \nabla_t w(0,0) + \int_0^1 (1-t)\tau^\eta(t,0)\nabla_t\nabla_t w(0,t)\,dt.$$

The second term is quadratic in η by the same argument as above applied to $u_1(\theta,t)$ such that $w(0,t) = \frac{\partial u_1}{\partial \theta}(0,t)$ and $u_1(0,t) = u(0,t)$. On the other hand since $\nabla_r w = 0$ and $\nabla_t w(1,t) = 0$

$$\begin{aligned}\nabla_t w(0,0) &= \nabla_t w(0,0) - \tau^\xi(1,0)\nabla_t w(1,0)\\ &= -\int_0^1 \tau^\xi(r,0)\nabla_r\nabla_t w(r,0)\,dr = -\int_0^1 \tau^\xi(r,0)R\left(\frac{\partial u}{\partial r},\frac{\partial u}{\partial t}\right)\tau^\xi(1,r)v(0)\,dr\\ &= -\int_0^1 \tau^\xi(r,0)R\left(dexp(r\xi)\xi, dexp(r\xi)\eta\right)\tau^\xi(1,r)v(0)\,dr\\ &= A_\xi(\eta,\tau^\xi(1,0)v(0))\end{aligned}$$

and A is a bounded bilinear form on $TM(\varepsilon_0)$ (more precisely, if $\pi\colon TM \to M$ is the projection then A is a form on $\pi^*TM_{|TM(\varepsilon_0)}$). Summarizing

$$\begin{aligned}(2.11)\quad &\left\|\left(\tau^{\xi_1} - \tau^\xi(1,0)\tau^\eta(1,0)\right)\mathcal{F}(u(1,1)) - A_\xi\left(\eta,\mathcal{F}_{\gamma_0}(\xi)\right)\right\|_{L^2}\\ &\leq \|A\|_{L^\infty}\|\eta\|_{L^\infty}\|\tau^\eta(1,0)\mathcal{F}(u(1,1)) - \mathcal{F}(u(1,0))\|_{L^2}\\ &\qquad + K\|\eta\|_{W^{1,2}}\|\eta\|_{L^\infty}(\|\dot\gamma_0\|_{L^2}+1)\end{aligned}$$

which together with (2.8) implies that the linear operator

$$D_\xi\colon W^{1,2}(\gamma_0) \ni \eta \mapsto \tau^\xi(1,0)D_{\gamma_1}\circ dexp(\xi)\eta + A_\xi(\eta,\mathcal{F}_{\gamma_0}(\xi)) \in L^2(\gamma_0)$$

is the differential of $\mathcal{F}_{\gamma_0}$ at the point ξ. Moreover the quadratic estimates follows since in view of (2.8)

$$\|\tau^\eta(1,0)\mathcal{F}_{\gamma_0}(\xi+\eta) - \mathcal{F}_{\gamma_0}(\xi))\|_{L^2} \leq \|D_{\gamma_1}\frac{\partial u}{\partial t}(1,0)\|_{L^2} + K\|\eta\|^2_{W^{1,2}}(\|\dot\gamma_0\|_{L^2}+1)$$

and the norm of $D_{\gamma_1}\colon W^{1,2}(\gamma_1) \to L^2(\gamma_1)$ is bounded by a universal constant. To estimate $D_\xi - D_0$ define $w_1(r) = D_{\gamma_r}\frac{\partial u}{\partial t}(r,0)$. Thus $w_1(0) = 0$ and $D_\xi\eta = \tau^\xi(1,0)w_1(1) + A_\xi(\eta,\mathcal{F}_{\gamma_0}(\xi))$ and

$$\nabla_r w_1 = \nabla_s\nabla_r\frac{\partial u}{\partial t} + R\left(\frac{\partial u}{\partial r},\frac{\partial u}{\partial s}\right)\frac{\partial u}{\partial t} + \nabla_{\nabla_r\frac{\partial u}{\partial t}}X_\alpha + \nabla_r(\nabla X_\alpha)\frac{\partial u}{\partial t}.$$

On the other hand $\frac{\partial u}{\partial t}\Big|_{r=0} = 0$ and $\nabla_r\frac{\partial u}{\partial t}\Big|_{r=0} = \eta$ so $\nabla_r w_1(0) = D_0\eta$. We can also apply the same kind of argument as previously to obtain the estimate

$$|\nabla_r\nabla_r w_1| \leq K(|\xi|\cdot|\eta| + |\nabla\xi|\cdot|\eta| + |\xi|\cdot|\nabla\eta|).$$

Therefore

$$(2.12)\quad \|D_\xi\eta - D_0\eta\|_{L^2} \leq \|\tau^\xi(1,0)w_1(1) - w_1(0) - \nabla_\tau w_1(0)\|_{L^2} + \|A_\xi(\eta, \mathcal{F}_{\gamma_0}(\xi))\|_{L^2} \leq K\left(\|\xi\|_{W^{1,2}} + \|\mathcal{F}_{\gamma_0}(\xi)\|_{L^2}\right)\|\eta\|_{W^{1,2}}$$

for $\xi, \eta \in B_{\varepsilon_0}(\gamma_0)$. Similar argument proves continuity of the derivative in $B_{\varepsilon_0}(\gamma_0)$. □

If the flow satisfies the Morse-Smale condition then Proposition 2.3.9 together with Theorem 2.3.2 implies that for every $\gamma \in \mathcal{M}(x^-, x^+)$ there is the implicit function

$$\phi_\gamma\colon \ker D_\gamma \to D^*(W^{2,2}(\gamma))$$

such that $\mathcal{F}(exp_\gamma(\xi + \phi_\gamma(\xi))) = 0$ for ξ in a neighbourhood of 0 in $\ker D_\gamma$ which provides locally the manifold structure for $\mathcal{M}(x^-, x^+)$ near γ. The evaluation map

$$ev\colon \mathcal{M}(x^-, x^+) \ni \gamma \mapsto \gamma(0) \in M$$

is an embedding w.r.t. this structure. Clearly, the evaluation $W^{1,2}(\gamma) \ni \xi \mapsto \xi(0) \in T_{\gamma(0)}M$ is a continuous linear map and so $\psi_\gamma\colon \ker D_\gamma \to M$ defined by

$$\psi_\gamma(\xi) = ev \circ exp_\gamma(\xi + \phi_\gamma(\xi)) = exp_{\gamma(0)}\xi(0) + \phi_\gamma(\xi)(0)$$

is differentiable. Moreover $d\psi_\gamma(0)\xi = \xi(0)$ as $d\phi_\gamma(0) = 0$. Since $D_\gamma\xi = 0$ is a first order linear equation $\xi(0) = 0$ implies $\xi = 0$ for $\xi \in \ker D_\gamma$.

Proposition 2.3.9 *There is a constant K such that for any $c > 1$ the following holds. If γ_0 satisfies*

$$c\|D^*_{\gamma_0}\zeta\|_{L^2} \geq \|\zeta\|_{W^{1,2}} \qquad \textit{for all } \zeta \in W^{1,2}(\gamma_0)$$

then

1. *the operator* $D_{\gamma_0} D^*_{\gamma_0} : W^{2,2}(\gamma_0) \to L^2(\gamma_0)$ *is invertible;*

2. $\|D^*_{\gamma_0}\eta_0\|_{W^{1,2}} \leq KC_4(\|\dot\gamma_0\|^2_{L^2} + 1)\|D_{\gamma_0} D^*_{\gamma_0}\eta_0\|_{L^2}$ *for all* $\eta_0 \in W^{2,2}(\gamma_0)$*;*

3. *if* $\xi_0 \in W^{1,2}(\gamma_0)$*,* $\|\xi_0\|_{W^{1,2}} \leq K^{-1}C_{-1}$ *and* $\gamma_1 = exp_{\gamma_0}\xi_0$ *then*

$$4c\|D^*_{\gamma_1}\zeta\|_{L^2} \geq \|\zeta\|_{W^{1,2}} \qquad \text{for all } \zeta \in W^{1,2}(\gamma_1).$$

Proof. We first establish some preliminary properties of D and D^*. Consider the norm of D:

$$\|\nabla\xi + \nabla_\xi X_\alpha\|_{L^2} \leq \|\xi\|_{W^{1,2}} + \|\nabla_\xi X_\alpha\|_{L^2}.$$

The covariant derivative ∇X_α is a smooth section of $T^*M \otimes TM$ and so $\|\nabla X_\alpha\|$ is bounded by some constant K. Hence

$$\|\nabla_\xi X_\alpha\|_{L^2} \leq K\|\xi\|_{L^2} \tag{2.13}$$

and $\|D\|$ is bounded by some universal constant K. The operators D and D^* are also bounded as operators from $W^{2,2}$ to $W^{1,2}$. We have

$$\|\nabla_{\dot\gamma}(D\xi)\|_{L^2} \leq \|\xi\|_{W^{2,2}} + \|\nabla_{\dot\gamma}\nabla_\xi\nabla f\|_{L^2}$$

and $\nabla(\nabla_\xi X_\alpha) = \nabla(\nabla X_\alpha)\xi + \nabla_{\nabla\xi}X_\alpha$ so

$$\|\nabla_{\dot\gamma}(\nabla_\xi X_\alpha)\|_{L^2} \leq \|\nabla(\nabla X_\alpha)\|_{L^\infty}\|\dot\gamma\|_{L^2}\|\xi\|_{L^\infty} + \|\nabla X_\alpha\|_{L^\infty}\|\nabla\xi\|_{L^2}$$

i.e.

$$\|\nabla_\xi X_\alpha\|_{W^{1,2}} \leq K(1 + \|\dot\gamma\|_{L^2})\|\xi\|_{W^{1,2}} \tag{2.14}$$

We can extend D and D^* to bounded operators from L^2 to $W^{-1,2} := (W^{1,2})^*$. Indeed, the mapping $\xi \mapsto -\nabla\xi$ extends to the adjoint of $\nabla : W^{1,2} \to L^2$ and from (2.13) follows that $\nabla X_\alpha\colon L^2 \to L^2$ is bounded. Moreover ∇X_α is selfadjoint and so the operator $D^*\colon L^2 \to W^{-1,2}$ is the adjoint of $D\colon W^{1,2} \to L^2$.

We can also obtain the following estimates for $\|\nabla_\xi X_\alpha\|_{W^{-1,2}}$ with $\xi \in L^2 \subset W^{-1,2}$. For any $\zeta \in W^{1,2}$

$$(2.15) \quad |\nabla_\xi X_\alpha(\zeta)| = |\langle \nabla_\xi X_\alpha, \zeta\rangle_{L^2}| = |\langle \xi, \nabla_\zeta X_\alpha\rangle_{L^2}| \le \|\xi\|_{W^{-1,2}}\|\nabla_\zeta X_\alpha\|_{W^{1,2}}$$
$$\le \|\xi\|_{W^{-1,2}} K(1+\|\dot\gamma\|_{L^2})\|\xi\|_{W^{1,2}}$$

and hence

$$(2.16) \qquad \|\nabla_\xi X_\alpha\|_{W^{-1,2}} \le K(1+\|\dot\gamma\|_{L^2})\|\xi\|_{W^{-1,2}}.$$

Since all operators under consideration clearly have closed range the condition $\|D^*_{\gamma_0}\eta\|_{L^2} \ge C_{-1}\|\eta\|_{W^{1,2}}$ implies that $DD^*\colon W^{1,2} \to W^{-1,2}$ is invertible and

$$\|\eta\|^2_{W^{1,2}} \le c^2\langle D^*\eta, D^*\eta\rangle_{L^2} = c^2\,|DD^*\eta(\eta)| \le c^2\|\eta\|_{W^{1,2}}\|DD^*\eta\|_{W^{-1,2}}$$

i.e.

$$(2.17) \qquad \|(DD^*)^{-1}\| \le c^2.$$

Similarly, $DD^*\colon W^{2,2} \to L^2$ is invertible since $D^*\colon W^{2,2} \to W^{1,2}$ is injective. We shall investigate the norm of $(DD^*)^{-1}\colon L^2 \to W^{2,2}$. For $\eta \in L^2$ we have inequalities

$$(2.18) \qquad \|\nabla\eta\|_{W^{-1,2}} \le \|\eta\|_{L_2} \quad \text{and} \quad \|\eta\|_{W^{-1,2}} \le \|\eta\|_{L_2}$$

where $\nabla\colon L^2 \to W^{-1,2}$ is the adjoint of $\nabla\colon W^{1,2} \to L^2$. Hence, using (2.17)

(2.19)
$$\|\eta\|_{W^{2,2}} \le \|\nabla\eta\|_{W^{1,2}} + \|\eta\|_{W^{1,2}} \le c^2\left(\|DD^*\eta\|_{W^{-1,2}} + \|DD^*\nabla\eta\|_{W^{-1,2}}\right)$$
$$\le c^2\left(2\|DD^*\eta\|_{L_2} + \|\nabla DD^*\eta - DD^*\nabla\eta\|_{W^{-1,2}}\right).$$

In the last expression the 3rd order terms cancel out, namely

$$\begin{aligned}&\nabla DD^*\eta - DD^*\nabla\eta\\ &= -\nabla_{\nabla\nabla_\eta}X_\alpha + \nabla_{\nabla_{\nabla_\eta}X_\alpha}X_\alpha + 2\nabla(\nabla_{\nabla\eta}X_\alpha) - \nabla\nabla\nabla_\eta X_\alpha + \nabla(\nabla_{\nabla_\eta X_\alpha}X_\alpha).\end{aligned}$$

Using several times inequalities (2.18) (2.13) (2.14) and (2.16) we obtain

$$\|\nabla DD^*\eta - DD^*\nabla\eta\|_{W^{-1,2}} \le K(1+\|\dot\gamma\|^2_{L_2})\|\eta\|_{W^{1,2}}.$$

Combining this with (2.17) yields

$$\|\eta\|_{W^{2,2}} \le Kc^4(1+\|\dot\gamma\|^2_{L_2})\|DD^*\eta\|_{L_2}.$$

This proves part (2) of Proposition 2.3.9 as $\|D^*\eta\|_{W^{1,2}} \le K\|\eta\|_{W^{2,2}}$.

Proof of (3). As in the proof of Proposition 2.4.1 let

$$u(s,t) = exp_{\gamma_0(s)}t\xi(s)$$

thus $\gamma_i(s) = u(s,i)$ for $i = 0,1$. Let $\zeta_1 \in W^{1,2}(\gamma_1)$ and let $\zeta_t(s) = \zeta(s,t)$ be the continuous section of u^*TM such that $\zeta(s,\cdot)$ is a parallel transport of ζ_1 along $u(s,\cdot)$ i.e. $\nabla_t\zeta = 0$. Assume that $\|\xi\|_{W^{1,2}} \le \varepsilon$. We will find a constant K such that $\|\dot\gamma_0\|_{L^2} < c$ and $c\|D^*_{\gamma_0}\zeta_0\|_{L^2} \ge \|\zeta_0\|_{W^{1,2}}$ for $\zeta_0 = \zeta(\cdot,0)$ implies $4c\|D^*_{\gamma_1}\zeta_1\|_{L^2} \ge \|\zeta_1\|_{W^{1,2}}$ provided that $Kc\varepsilon < 1$. We first establish relation between $W^{1,2}$ norms of ζ_0 (resp. $D^*_{\gamma_0}\zeta_0$) and ζ_1 (resp. $D^*_{\gamma_1}\zeta_1$). Since $\nabla_t\zeta = 0$ the length $|\zeta|$ does not depend on t, in particular $|\zeta_0| = |\zeta_1|$ and

$$|\nabla_t\nabla_s\zeta_t| = \left|R\left(\frac{\partial u}{\partial t},\frac{\partial u}{\partial s}\right)\zeta_t\right| \le K\left(|\dot\gamma_0| + t\,|\nabla\xi|\right)|\xi|\,|\zeta|$$

if only $\varepsilon \le \varepsilon_0$. Hence

$$\begin{aligned}|\nabla_s\zeta_1| &= \left|\tau^\xi(1,0)\nabla_s\zeta_1\right| \le |\nabla_s\zeta_0| + \sup_{t\in I}|\nabla_t\nabla_s\zeta_t|,\\ \|\nabla_s\zeta_1\|_{L^2} &\le \|\nabla_s\zeta_0\|_{L^2} + K\left(\|\dot\gamma_0\|_{L^2} + \|\nabla\xi\|_{L^2}\right)\|\xi\|_{L^\infty}\|\zeta\|_{L^\infty}\end{aligned}$$

and

$$\|\zeta_1\|_{W^{1,2}} \leq \|\nabla_s \zeta_0\|_{L^2} + 2K(c+\varepsilon)\varepsilon\|\zeta_0\|_{W^{1,2}} + \|\zeta_0\|_{L^2} \leq 2\|\zeta_0\|_{W^{1,2}}$$

if $2K(c+\varepsilon)\varepsilon \leq 1$. Analogously, using (2.7) and (2.8),

$$\nabla_t D^*_{\gamma_r}\zeta_r = R\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}\right)\zeta_r + \nabla_t\left(X_\alpha(u)\right)\zeta_r$$

and we obtain the estimate

$$\|D^*_{\gamma_0}\zeta_0\|_{L^2} \leq \|D^*_{\gamma_1}\zeta_1\|_{L^2} + K(c+\varepsilon+1)\varepsilon\|\zeta_1\|_{W^{1,2}}.$$

Combining (2.3.1), (2.3.1) and the assumed inequality $c\|D^*_{\gamma_0}\zeta_0\|_{L^2} \geq \|\zeta_0\|_{W^{1,2}}$ yields

$$\|\zeta_1\|_{W^{1,2}} \leq 2c\left(\|D^*_{\gamma_1}\zeta_1\|_{L^2} + K(c+\varepsilon+1)\varepsilon\|\zeta_1\|_{W^{1,2}}\right)$$

and so $\|\zeta_1\|_{W^{1,2}} \leq 4c\|D^*_{\gamma_1}\zeta_1\|_{L^2}$ if $Kc(c+\varepsilon+1)\varepsilon < \frac{1}{4}$.

Proof of Theorem 2.3.3. We will use the notation of Proposition 2.3.8. We apply Newton method of successive approximations in order to find a zero of the function $f := \mathcal{F}_\gamma \circ D^*_\gamma \colon W^{2,2}(\gamma) \to L^2(\gamma)$ in a neighbourhood of the origin in $W^{2,2}(\gamma)$. By Proposition 2.3.9 its derivative $Df(\eta) = D_{D^*_\gamma\eta}D^*_\gamma$ is invertible at 0. Thus we have to find $\lambda, \beta > 0$ such that if $\|f(0)\| < \frac{\beta}{\lambda}$ and $\|\eta\| < \beta$ then

(2.20) $$\|Df(\eta) - Df(0)\| < \lambda^{-1}$$

(2.21) $$Df(\eta)^{-1} \text{ exists and } \|Df(\eta)^{-1}\| < 2^{-1}\lambda.$$

Then the condition

$$\eta_{n+1} = \eta_n - Df(0)^{-1}f(\eta_n), \qquad \eta_0 = 0$$

defines a cauchy sequence, $\|\eta_n - \eta_{n-1}\| < 2^{-n}\beta$ and $\|f(\eta_n)\| < 2^{-n}\lambda^{-1}\beta$. This is proved inductively since $\|\eta_{n+1} - \eta_n\| < 2^{-1}\lambda\|f(\eta_n)\|$ and

$$\|f(\eta_{n+1})\| = \|f(\eta_{n+1}) - f(\eta_n) - Df(0)(\eta_{n+1} - \eta_n)\| < \lambda^{-1}\|\eta_{n+1} - \eta_n\|$$

by the mean value theorem. Then the limit $\eta := \lim_{n\to\infty} \eta_n$ is a unique solution to $\mathcal{F}(exp_\gamma D_\gamma^* \eta) = 0$ in the ball $B_\beta(0) \subset W^{2,2}$ and $\|D_\gamma^*\eta\| \le \|D_\gamma^*\| \, \|\eta\| \le K(1+c^2)\beta$.

By the assumptions and Proposition 2.3.9 $\|Df(0)^{-1}\| \le Kc^6 =: \lambda_1$. For any bounded operator $L\colon W^{2,2}(\gamma) \to L^2(\gamma)$ such that $\|L - Df(0)\| < \lambda_1^{-1}$ the inverse L^{-1} exists and $\|L^{-1}\| \le \frac{\lambda_1}{1-\lambda_1\|L-Df(0)\|}$. Hence if we set $\lambda = 3\lambda_1$ then the condition (2.20) implies (2.21). On the other hand by Proposition 2.3.8 we have

$$\|f(\eta) - f(0)\| \le \|D_\gamma D_\gamma^*\| \cdot \|\eta\| + K\|\eta\|^2(1+c^2)$$

and so

$$\begin{aligned}\|Df(\eta) - Df(0)\| &\le K\|D_\gamma^*\|(\|D_\gamma^*\eta\| + \|f(\eta) - f(0)\| + \|f(0)\|) \\ &\le K(1+c^2)\left((1+c^2)\|\eta\| + \|f(0)\|\right) < \frac{1}{\lambda}\end{aligned}$$

the last inequalities following if $\|\eta\| < \beta$, $\|f(0)\| < \frac{\beta}{\lambda}$ and

$$\beta < \left(K(1+c^2)(\lambda(1+c^2)+1)\right)^{-1}.$$

This together with the Proposition 2.3.9(3) completes the proof of the Theorem 2.3.3a).

In order to prove the remaining assertions of the theorem we have to look closer at the size of the domain in which the implicit function is defined. Denote $G = D_\gamma^*(D_\gamma D_\gamma^*)^{-1}$ i.e. $D_\gamma \circ G = id_{L^2(\gamma)}$. If we go through the proof of the implicit function theorem (see e.g. [Die69]) we see that it is enough to find $\varepsilon_1 \ge \varepsilon_0 > 0$ such that

$$\|\mathcal{F}_\gamma(\xi_1) - \mathcal{F}_\gamma(\xi_2) - D_\gamma(\xi_1 - \xi_2)\|_{L^2} \le (2\|G\|)^{-1}\|\xi_1 - \xi_2\|_{W^{1,2}}$$

for $\|\xi_i\|_{W^{1,2}} \le \varepsilon_1$ and $\|\mathcal{F}_\gamma(\xi)\|_{L^2} \le \frac{\varepsilon_1}{2}$ for $\|\xi\|_{W^{1,2}} \le \varepsilon_0$.

1° By Propositions 2.3.9 $\|G\| \le Kc^6$

2° From the Proposition 2.4.1 follows that

$$\begin{aligned}
&\|\mathcal{F}_\gamma(\xi_1) - \mathcal{F}_\gamma(\xi_2) - D_\gamma(\xi_1 - \xi_2)\|_{L^2} \\
&\le \|\mathcal{F}_\gamma(\xi_1) - \mathcal{F}_\gamma(\xi_2) - D_{\xi_2}(\xi_1 - \xi_2)\|_{L^2} + \|D_{\xi_2}(\xi_1 - \xi_2) - D_0(\xi_1 - \xi_2)\|_{L^2} \\
&\qquad \le Kc^2\left(\|\mathcal{F}_\gamma(\xi_2)\|_{L^2} + \|\xi_2\|_{W^{1,2}} + \|\xi_1 - \xi_2\|_{W^{1,2}}\right)\|\xi_1 - \xi_2\|_{W^{1,2}}.v
\end{aligned}$$

3° Again, by Proposition 2.3.8

$$\|\mathcal{F}_\gamma(\xi)\|_{L^2} \le \|D_\gamma \xi\|_{L^2} + Kc^2\|\xi\|^2_{W^{1,2}} \le Kc^2\|\xi\|_{W^{1,2}}.$$

Putting these estimates together yields the result.

Proof of Proposition 2.3.4. Since y is a hyperbolic fixed point of the flow there are positive constants k and c such that

$$|\dot\gamma^-(s+R)| < c|\dot\gamma^-(s)|e^{-kR}, \qquad |\dot\gamma^+(s-R)| < c|\dot\gamma^+(s)|e^{-kR}$$

and

$$|\gamma^-(s+R)| < c|\gamma^-(s)|e^{-kR}, \qquad |\gamma^+(s-R)| < c|\gamma^+(s)|e^{-kR}$$

in normal coordinates at y, for R sufficiently large. It follows that $\tilde\gamma_R$ converges locally uniformly to y with $R \to +\infty$ and since for $s \in [-1,1]$

$$\frac{d}{ds}\tilde\gamma_R(s) = \beta(s)\dot\gamma^-(s+R) + (1-\beta(s))\dot\gamma^+(s-R) + \dot\beta(s)\left(\gamma^-(s+R) - \gamma^+(s-R)\right)$$

and $|\dot\beta|$ is bounded,

$$\left|\dot{\tilde\gamma}_R(s)\right| < c_1 e^{-k_1 R} \qquad \text{for } s \in [-1,1].$$

Hence $\|\dot{\tilde\gamma}\|^2_{L^2} \le 2c_1 e^{-2k_1 R} + \ell^2(\gamma^-) + \ell^2(\gamma^+)$. Also, since X_α is locally Lipschitz

$$|X_\alpha(\tilde\gamma_R(s))| < L\,|\tilde\gamma_R(s)| < c_2 e^{-k_2 R} \qquad \text{for } s \in [-1,1].$$

It follows that

$$\begin{aligned}
(2.22)\qquad \|\mathcal{F}(\tilde\gamma_R)\|^2_{L^2} &= \int_{-\infty}^{+\infty} \left|\dot{\tilde\gamma}_R(s) + X_\alpha(\tilde\gamma_R(s))\right|^2 ds \\
&\le \int_{-1}^{1} \left|\dot{\tilde\gamma}_R(s)\right|^2 + |X_\alpha(\tilde\gamma_R(s))|^2\, ds \le c_3 e^{-k_3 R}.
\end{aligned}$$

In order to prove the second inequality take a $\zeta \in W^{1,2}(\tilde{\gamma}_R)$. We first establish the estimate for $\alpha\zeta$ and $(1-\alpha)\zeta$ where $\alpha \in W^{1,2}(\mathbb{R},\mathbb{R})$ is given by

$$\alpha(s) = \begin{cases} 0, & \text{for } |s| \geq T+1 \\ 2 \pm \frac{s}{T}, & \text{for } 1 \leq \pm s \leq T+1 \\ 1, & \text{for } |s| \leq 1 \end{cases}$$

with some $T > 0$. Since γ^-, γ^+ and y are solutions of (2.1), D_γ is onto and

$$c_0 \|D_\gamma^* \zeta\|_{L^2} \geq \|\zeta\|_{W^{1,2}} \qquad \text{for} \qquad \zeta \in W^{1,2}(\gamma)$$

where $\gamma = \gamma^-, \gamma^+$ or y. Consequently, if we write $(1-\alpha)\zeta = \zeta^- + \zeta^+$ with $\zeta^\mp \in W^{1,2}(\gamma^\mp)$, $\zeta^\mp(s) = 0$ for $\mp s \leq 0$ then

$$\|(1-\alpha)\zeta\|_{W^{1,2}} = \|\zeta^-\|_{W^{1,2}} + \|\zeta^+\|_{W^{1,2}}$$
$$\|D_{\tilde{\gamma}_R}^*(1-\alpha)\zeta\|_{L^2} = \|D_{\gamma^-}^*\zeta^-\|_{L^2} + \|D_{\gamma^+}^*\zeta^+\|_{L^2}$$

and hence

$$c_0 \|D_{\tilde{\gamma}_R}^*(1-\alpha)\zeta\|_{L^2} \geq \|(1-\alpha)\zeta\|_{W^{1,2}}.$$

Next, define $\xi_R \in W^{1,2}(y)$ by

$$\xi_R(s) = \begin{cases} e^{s+T+1}\tilde{\gamma}_R(-T-1), & \text{for } s \leq -T-1 \\ \tilde{\gamma}_R(s), & \text{for } s \in [-T-1, T+1] \\ e^{-s+T+1}\tilde{\gamma}_R(T+1), & \text{for } s \geq T+1. \end{cases}$$

Since ξ_R is a vector field along the constant path y, $\nabla_s \xi_R = \dot{\xi}_R$ and

$$\text{(2.23)} \quad \|\xi_R\|_{W^{1,2}}^2 = \frac{3}{2}\left(|\tilde{\gamma}_R(-T-1)|^2 + |\tilde{\gamma}_R(T+1)|^2\right) + \int_{-T-1}^{T+1} \left|\dot{\tilde{\gamma}}_R(s)\right|^2 + |\tilde{\gamma}_R(s)|^2 \, ds \leq c_4 e^{-k_4 R}.$$

Clearly, $D_{\tilde{\gamma}_R}^* \alpha\zeta = D_{\xi_R}^* \alpha\zeta$ and if R is big enough it follows by Proposition 2.3.9(2) that

$$4c_0 \|D_{\tilde{\gamma}_R}^* \alpha\zeta\|_{L^2} \geq \|\alpha\zeta\|_{W^{1,2}}.$$

Putting (2.3.1) and (2.3.1) together:

(2.24)

$$\|\zeta\|_{W^{1,2}} \leq \|(1-\alpha)\zeta\|_{W^{1,2}} + \|\alpha\zeta\|_{W^{1,2}} \leq c_0\|D^*_{\tilde{\gamma}_R}(1-\alpha)\zeta\|_{L^2} + 4c_0\|D^*_{\tilde{\gamma}_R}\alpha\zeta\|_{L^2}$$
$$\leq 5c_0\left(\|D^*_{\tilde{\gamma}_R}\zeta\|_{L^2} + \|\dot{\alpha}\|_{L^\infty}\|\zeta\|_{L^2}\right) \leq 5c_0\left(\|D^*_{\tilde{\gamma}_R}\zeta\|_{L^2} + \frac{1}{T}\|\zeta\|_{W^{1,2}}\right).$$

Therefore Proposition 2.3.4 follows for $T > 5c_0$ and R sufficiently large.

Proof of Proposition 2.3.5. The family $\tilde{\gamma}_R$ converges to (γ^-, γ^+) and since

$$\sup_{s\in\mathbb{R}} d(\tilde{\gamma}_R(s), \gamma_R(s)) < K_C e^{-kR}$$

γ_R converges to (γ^-, γ^+) with $R \to +\infty$ as well. Note that since

$$\lim_{R\to+\infty} \|\tilde{\gamma}_R\|^2_{L^2} = \ell^2(\gamma^-) + \ell^2(\gamma^+).$$

we have $\ell^2(\gamma_R) = \ell^2(\gamma^-) + \ell^2(\gamma^+)$. Define

$$U_\varepsilon = \left\{x \in M : \exists s \in \mathbb{R} \text{ s.t. } d(x, \gamma^-(s)) < \varepsilon \text{ or } d(x, \gamma^+(s)) < \varepsilon\right\}.$$

It is easy to see that for any $\varepsilon > 0$ and a sequence $\{\gamma'_j\} \subset \mathcal{M}(x,z)$ converging to (γ^-, γ^+) there is a $j_0 \in \mathbb{N}$ such that $\gamma'_j(\mathbb{R}) \subset U_\varepsilon$ for $j \geq j_0$. Consequently, in order to finish the proof of Proposition 2.3.5 it is enough to show the following

Lemma 2.3.10 *For any $\varepsilon_0 > \varepsilon > 0$ there is an $R_0 > 0$ such that the following holds. If $\gamma \in \mathcal{M}(x,z)$ and $\gamma(\mathbb{R}) \subset U_\varepsilon$ then for any $R > R_0$ there is an element $\xi \in W^{1,2}(\tilde{\gamma}_R)$ such that $\gamma = exp_{\tilde{\gamma}_R}\xi$ (modulo time shift) and $\|\xi\|_{W^{1,2}} \leq K\varepsilon$ where K is independent of ε and γ.*

Proof. We may assume that $d(\gamma(0), y) < \varepsilon$. There is an $R_0 > 0$ such that for $R > R_0$ we have $d(\gamma(0), \gamma^-(R)) < 2\varepsilon$ and $d(\gamma(0), \gamma^+(-R)) < 2\varepsilon$. Since the manifolds $W^u(x)$ and $W^s(z)$ are hyperbolic sets there are positive constants C_1, k_1 such that

$$d(\gamma(-s), \gamma^-(-s+R)) < 2\varepsilon C_1 e^{-k_1 s}$$
$$d(\gamma(s), \gamma^+(s-R)) < 2\varepsilon C_1 e^{-k_1 s}$$

for $s \geq 0$. Therefore the required $\xi \in W^{1,2}(\tilde{\gamma}_R)$ satisfies $\|\xi\|_{L^2}^2 \leq (2\varepsilon)^2(2T + \frac{C_1^2}{k_1})$, $\|\xi\|_{L^\infty} < 2C_1\varepsilon$ and from Proposition 2.4.1 follows that

$$\|\nabla\xi\|_{L^2} - \|\mathcal{F}(\tilde{\gamma}_R) + \nabla_\xi X_\alpha\|_{L^2} \leq K\|\xi\|_{L^\infty}\|\xi\|_{W^{1,2}}(\|\dot{\tilde{\gamma}}_R\|_{L^2}^2 + 1)$$
$$\frac{1}{2}\|\xi\|_{W^{1,2}} \leq \|\mathcal{F}(\tilde{\gamma}_R)\|_{L^2} + \|\nabla_\xi X_\alpha\|_{L^2} \leq Ce^{-kR} + K_2\|\xi\|_{L^2} \leq K_3\varepsilon$$

if $2\varepsilon < (K(\|\dot{\tilde{\gamma}}_R\|_{L^2}^2 + 1))^{-1}$ and R is sufficiently large.

2.4 Orientation

To complete the proof of the identity $\partial^2 = 0$ we need the following

Proposition 2.4.1 *Let x and z be two critical points of α, $\mathrm{ind}(x) = \mathrm{ind}(z) + 2 = k$. Assume that there is a connected component of $\widehat{\mathcal{M}}(x,z)$ whose ends converge to (γ_1^-, γ_1^+) and (γ_2^-, γ_2^+) with $\gamma_i^- \in \mathcal{M}(x, y_i)$, $\gamma_i^+ \in \mathcal{M}(y_i, z)$ and $\mathrm{ind}(y_1) = \mathrm{ind}(y_2) = k - 1$. Then*

$$n_{\gamma_1^-}n_{\gamma_1^+} + n_{\gamma_2^-}n_{\gamma_2^+} = 0.$$

Since we deal with a connected component of $\widehat{\mathcal{M}}(x,z)$ we may assume that $\alpha = df$. For $f(x) > a > f(z)$ we have a diffeomorphism $\widehat{\mathcal{M}}(x,z) \cong \mathcal{M}_a(x,z) := W^u(x) \cap W^s(z) \cap f^{-1}(a)$. Indeed, let $\psi \colon \mathbb{R} \times M \to M$ denote the gradient flow of f. Since f decreases along nonconstant trajectories of the flow ψ for any noncritical point $x \in M$ there is at most one real number $\tau_a(x)$ such that $\psi(\tau_a(x), x) \in M_a$. It follows from the implicit function theorem applied to $f \circ \psi$ that τ_a is a smooth function defined on an open subset U_a of M. Then the diffeomorphism is induced by the map

$$\mathcal{M}(x,z) \ni \gamma \mapsto \gamma(\tau_a(\gamma(0))) \in f^{-1}(a).$$

If we take $a = f(\gamma_i^-(0))$ then $\gamma_i^-(0)$ is a boundary point of $\mathcal{M}_a(x,z)$ and let ν_i be the unit vector field tangent to $\mathcal{M}_a(x,z)$ pointing inward at $\gamma_i^-(0)$.

Then $(\frac{\dot\gamma}{\|\dot\gamma\|}, \nu_i)$ form an orthonormal basis for $T\mathcal{M}(x,z)$. Thus the proposition follows immediately from

Lemma 2.4.2

$$n_{\gamma_i^-} n_{\gamma_i^+} = \begin{cases} +1 & \text{if } (\frac{\dot\gamma}{\|\dot\gamma\|}, \nu_i) \text{ matches the orientation of } \mathcal{M}(x,z), \\ -1 & \text{otherwise.} \end{cases}$$

Proof. Recall that the orientation of $\mathcal{M}(x,y)$ is determined by the orientations $\langle x\rangle$, $\langle y\rangle$ of $T_xW^u(x)$ and $T_yW^u(y)$ using the isomorphism

(2.25)
$$T_pW^u(x)/T_p\mathcal{M}(x,y) \cong T_pM/T_pW^s(y) \cong T_yM/T_yW^s(y) \cong T_yW^u(y)$$

and for $\mathrm{ind}(x) = \mathrm{ind}(y)+1$ the numbers n_γ are defined so that $n_\gamma\dot\gamma$ forms a positively oriented basis of $\mathcal{M}(x,y)$. We assume the convention that for oriented vector spaces V, W the natural isomorphism $V \cong (W\times V)/W$ induces the orientation of the quotient. Then we have the orientation preserving isomorphisms ($\langle w\rangle$ denotes the vector space spanned by w)

$$\begin{aligned} T_xW^u(x) &\cong T_{\gamma_i^-}W^u(x)/\langle n_{\gamma_i^-}\dot\gamma_i^-\rangle \times \langle(-1)^k n_{\gamma_i^-}\dot\gamma_i^-\rangle \\ &\cong T_{y_i}W^u(y_i)\times\langle(-1)^k n_{\gamma_i^-}\dot\gamma_i^-\rangle \\ T_{y_i}W^u(y_i) &\cong T_{\gamma_i^+}W^u(y_i)/\langle n_{\gamma_i^+}\dot\gamma_i^+\rangle \times \langle(-1)^{k-1} n_{\gamma_i^+}\dot\gamma_i^+\rangle \\ &\cong T_zW^u(z)\times\langle(-1)^{k-1} n_{\gamma_i^+}\dot\gamma_i^+\rangle \end{aligned}$$

and consequently

$$T_xW^u(x) \cong T_zW^u(z)\times\langle n_{\gamma_i^-}\dot\gamma_i^-\rangle\times\langle n_{\gamma_i^+}\dot\gamma_i^+\rangle$$

(2.26)
$$T\mathcal{M}(x,z) \cong \langle n_{\gamma_i^-}\dot\gamma_i^-\rangle\times\langle n_{\gamma_i^+}\dot\gamma_i^+\rangle.$$

Consider a solution of the equation

$$\nabla_{\dot\gamma_i^-}\xi = -\nabla_\xi X_\alpha(\gamma_i^-).$$

In local coordinates

(2.27)
$$\dot\xi(s) = A(s)\xi(s), \qquad A(s)\to A_0$$

where A_0 is the Hessian $-\nabla X_\alpha(y_i)$. We assume for simplicity that A_0 has the simple eigenvalues $\lambda_1, \ldots, \lambda_n$ (this holds for a generic metric). Let $p_1, \ldots, p_n$ denote the corresponding orthonormal basis of eigenvectors. Then the fundamental solution to (2.27) has form

$$X(t) = \sum_{i=1}^{n} e^{\lambda_i t} (P_i + R_i(t))$$

where P_i denote the projection onto p_n and the matrices R_i converge to zero with $t \to +\infty$ (see e.g. [Hil65]). If $\xi(t)$ is a solution with initial value

$$\xi(0) = \xi_0^{i_1} p_{i_1} + \cdots + \xi_0^{i_m} p_{i_m}, \qquad \xi_0^{i_j} \neq 0$$

then it follows that

$$\lim_{t\to\infty} \frac{\xi(t)}{|\xi(t)|} = p_{i_{max}} \tag{2.28}$$

where $\lambda_{i_{max}} = \max\{\lambda_{i_1}, \ldots, \lambda_{i_m}\}$. In particular, we have the limits

$$\lim_{s\to\infty} \frac{\dot\gamma_i^-}{\|\dot\gamma_i^-\|} = v_i^- \in T_{y_i} W^s(y_i)$$
$$\lim_{s\to-\infty} \frac{\dot\gamma_i^+}{\|\dot\gamma_i^+\|} = v_i^+ \in T_{y_i} W^u(y_i).$$

Let η_i^0 be a limit point of ν_i at γ_i^- and $\eta_i(s)$ a solution to (2.27) with the initial value $\eta_i(0) = \eta_i^0$. Define

$$\eta_i^\infty := \lim_{s\to\infty} \frac{\eta_i}{\|\eta_i\|}.$$

Since γ is a solution to (2.1) η_i is an orbit of the tangent flow

$$\eta_i(s) = d\psi_s(\gamma_i^-(0)).\eta_i^0.$$

Moreover $\eta_i(s)$ is not colinear with $\dot\gamma_i^-$ and $d\psi_s.\nu_i \in TW^u(x)$. Therefore from the transversality of the intersection $TW^u(x) \cap TW^s(y_i)$ follows that $\eta_i^\infty \in T_{y_i} W^u(y_i)$ (note that the limit (2.28) belongs either to $T_{y_i} W^s(y_i)$ or

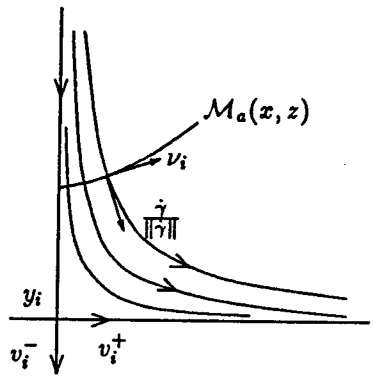

Figure 2.2:

$T_{y_i}W^u(y_i))$. On the other hand $d\psi_s.\nu_i \in TW^s(z)$ and the Morse-Smale condition for y_i and z implies that $\eta_i^\infty = v_i^+$. Consequently, we have an orientation preserving isomorphism

$$\langle \frac{\dot\gamma}{\|\dot\gamma\|}, \nu_i\rangle \cong \langle v_i^-, v_i^+\rangle \cong \langle \dot\gamma_i^-, \dot\gamma_i^+\rangle$$

which together with (2.26) proves the lemma.

2.5 Transversality

In this section we show that the Morse-Smale condition can be achieved by an arbitrarily small perturbation of the metric g. This result is due to Smale ([S1]) and we will follow the argument presented in [F2].

Let $\mathcal{G}$ denote the space of all smooth metrics on M. $\mathcal{G}$ is a subset of the space $C^\infty(S^2TM)$ where $S^2TM \subset T^*M \otimes T^*M$ is the bundle of symmetric tensors. Thus with the identification $T^*M \otimes T^*M = Hom(TM, T^*M)$ we may write $X_\alpha^g = g^{-1}\alpha$.

Let $\varepsilon = \{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers and define the subspace $C_\varepsilon^\infty(S^2TM) \subset C^\infty(S^2TM)$ of all sections $A \in C^\infty(S^2TM)$ satisfying

$$\|A\|_\varepsilon := \sum_{k=0}^{\infty} \varepsilon_k \|A\|_{C^k(M)} < \infty$$

where the C^k norms are taken w.r.t. a fixed metric h_0. This is a separable Banach space which is dense in $L^2(S^2TM)$ for sufficiently rapidly decreasing sequence ε_k ([Flo88b]).

Theorem 2.5.1 *Let α be a closed nondegenerate form on a compact manifold M. There is a dense set $\mathcal{G}_{reg} \subset \mathcal{G}$ such that for every $g \in \mathcal{G}_{reg}$ and $\gamma\colon \mathbb{R} \to M$ satisfying*

$$\mathcal{F}(\gamma, g) = \dot{\gamma} + g^{-1}\alpha(\gamma) = 0 \tag{2.29}$$

the operator $D^g_\gamma\colon W^{1,2}(\gamma) \to L^2(\gamma)$, $D^g_\gamma \xi = \nabla^g_{\dot{\gamma}}\xi + \nabla^g_\xi X^g_\alpha$ is onto.

Let x^-, x^+ be critical points of α. Given a metric $g_0 \in \mathcal{G}$ let N be an open subset of M such that $\gamma^{-1}(N) \neq \emptyset$ for any $\gamma \in \mathcal{M}_{g_0}(x^-, x^+)$. We define the closed subspace

$$G(N) \subset C^\infty_\varepsilon(S^2TM)$$

of those sections which vanish outside N. Let U_{g_0} be a neighbourhood of zero in $G(N)$ such that $g_A = g_0 + A$ is a metric for $A \in U_{g_0}$. Moreover if we take U_{g_0} small enough then g_A are uniformly equivalent to g_0 i.e.

$$\sigma g_0(v,v) \leq g_A(v,v) \leq \sigma^{-1} g_0(v,v) \qquad \text{for all } v \in TM$$

where σ is a constant independent of A. Then Theorem 2.5.1 is a consequence of the following

Theorem 2.5.2 *With the notation as above, the set of all $A \in U_{g_0}$ such that $D^{g_A}_\gamma$ is onto for every $\gamma \in \mathcal{M}_{g_A}(x^-, x^+)$ is residual in U_{g_0}.*

Proof. We claim that the set

$$\mathcal{Z} = \{(\gamma, A) \in \mathcal{P} \times U_{g_0} : \mathcal{F}(\gamma, g_A) = 0\}$$

is a manifold. For $(\gamma, A_0) \in \mathcal{Z}$ consider the function

$$\bar{\mathcal{F}} = \bar{\mathcal{F}}_{\gamma,A_0} : B_{\varepsilon_0}(\gamma) \times U_{g_0} \to L^2(\gamma)$$
$$\bar{\mathcal{F}}(\xi, A) = \tau^{\xi}(1,0)\mathcal{F}(exp_\gamma \xi, A)$$

where the exponential map and the parallel transport are taken w.r.t. g_A and the L^2-product — w.r.t. g_{A_0}. As g_A are uniformly equivalent to g_0, from the proof of Theorem 2.3.3,b) follows that $\bar{\mathcal{F}}$ is continuously differentiable and

$$d\bar{\mathcal{F}}(0, A_0)(\xi, A) = \nabla^{g_{A_0}}\xi + \nabla_\xi^{g_{A_0}} X_\alpha^{g_{A_0}} - g_{A_0}^{-1} \circ A \circ g_{A_0}^{-1}\alpha = D_\gamma^{g_{A_0}}\xi + d_2\bar{\mathcal{F}}(A_0)A.$$

Since $D_\gamma^{g_{A_0}}$ is a Fredholm operator the kernel of $d\bar{\mathcal{F}}(0, A_0)$ splits. Indeed we can write

$$W^{1,2}(\gamma) = \ker D_\gamma^{g_{A_0}} \oplus V_1$$

where $V_1 \subset W^{1,2}(\gamma)$ is a closed subspace. Clearly $\ker d\bar{\mathcal{F}}(0, A_0) \cap V_1 = \{0\}$ and

$$\ker d\bar{\mathcal{F}}(0, A_0) \oplus V_1 = d\bar{\mathcal{F}}(0, A_0)^{-1}(D_\gamma^{g_{A_0}}(W^{1,2}(\gamma))$$

Hence $\operatorname{codim} \ker d\bar{\mathcal{F}}(0, A_0) \oplus V_1 < \infty$ and so $\ker d\bar{\mathcal{F}}(0, A_0)$ has a closed complement in $W^{1,2}(\gamma) \times G(N)$. In order to show that $\mathcal{Z}$ is a submanifold of $\mathcal{P} \times U_{g_0}$ we prove that $d\bar{\mathcal{F}}(0, A_0)$ is onto. The range of $d\bar{\mathcal{F}}(0, A_0)$ is closed since D_γ is Fredholm. Hence it is enough to proof that for any $\eta \in L^2(\gamma)$ the condition

$$\left\langle d\bar{\mathcal{F}}(\xi, A), \eta \right\rangle_{L^2} = 0 \qquad \text{for all } \xi \in W^{1,2}(\gamma) \text{ and } A \in G(N)$$

implies $\eta = 0$. The above condition means that

$$\langle D_\gamma \xi, \eta \rangle = 0$$

and

$$\int_{-\infty}^{+\infty} g_{A_0}(g_{A_0}^{-1} A X_\alpha^{g_{A_0}}, \eta) = 0.$$

First equation yields

$$|\langle \nabla \xi, \eta \rangle| = |\langle \nabla_\xi X_\alpha, \eta \rangle| \leq K \|\eta\|_{L^2} \|\xi\|_{L^2}$$

i.e. $\eta \in W^{1,2}(\gamma)$ and it satisfies first order linear equation $D^*\eta = 0$. Hence it is enough to show that the second equation implies that η vanishes at least at one point. We have

$$g_{A_0}(g_{A_0}^{-1} \circ AX_\alpha, \eta) = (AX_\alpha)\eta = (X_\alpha)^i A^i_j \eta^j$$

where the last equality is understood w.r.t. some basis $\{e_i\}$ along γ. Suppose that $\eta^{j_0}(s_0) \neq 0$ for some index j_0 and $s_0 \in \gamma^{-1}(N)$. We find a symmetric matrix $\hat{A}$ such that

$$(X_\alpha(\gamma(s_0)))^T \hat{A} \eta(s_0) \neq 0.$$

If $(X_\alpha)^{j_0} \neq 0$ we take $\hat{A}^{j_0}_{j_0} = 1$ and $\hat{A}^i_j = 0$ otherwise. Else, there is an index $i_0 \neq j_0$ such that $(X_\alpha)^{i_0} \neq 0$ since X_α does not vanish along γ and we may take $\hat{A}^{i_0}_{j_0} = \hat{A}^{j_0}_{i_0} = 1$ and $\hat{A}^i_j = 0$ otherwise. Choosing an appropriate cutoff function we can extend $\hat{A}$ to an element of $G(N)$ such that

$$\int_{-\infty}^{+\infty} X_\alpha(\gamma(s))^T \hat{A}(\gamma(s))\eta(s)\, ds \neq 0$$

which leads to contradiction.

To conclude the proof of Theorem 2.5.2 consider the projection

$$\pi : \mathcal{Z} \to U_{g_0}.$$

This is a differentiable map between Banach manifolds. The tangent space to $\mathcal{Z}$ at the point (γ, A_0) is the subspace of $W^{1,2}(\gamma) \times G(N)$ given by the equation

$$D_\gamma \xi + d_2\bar{\mathcal{F}}(A_0)A = 0.$$

Hence $\ker d\pi(\gamma, A_0) = \ker D_\gamma$ and $im\, d\pi = d_2\bar{\mathcal{F}}^{-1}(im\, D_\gamma)$. Consequently, we have $\dim \operatorname{coker} d\pi = \dim \operatorname{coker} D_\gamma$ as $d\bar{\mathcal{F}}$ is onto. It follows that $d\pi$ is

Fredholm and A is a regular value of π iff $D_\gamma^{g_A}$ is onto for every γ satisfying $\dot\gamma + g_A^{-1}\alpha(\gamma) = 0$. By Sard-Smale theorem ([Sma73]) the set of regular values of π is residual in U_{g_0} which yields the result.

2.6 Continuation

The homology groups we have constructed depend on the form α as well as on the metric g. The aim of this section is to prove

Theorem 2.6.1 *Let α_0 and α_1 be closed nondegenerate forms on a compact manifold M such that the form $\alpha_1 - \alpha_0$ is exact and g_0, g_1 Riemannian metrics such that the corresponding flows generated by X_{α_i}, $i = 0, 1$ satisfy the transversality condition for critical points. Then $\Lambda_{\alpha_0} \cong \Lambda_{\alpha_1}$ and*

$$H_*(\alpha_0, g_0) \cong H_*(\alpha_1, g_1).$$

Proof. We begin with defining the Novikov complex in slightly more general situation. An open set U in M is called an isolating neighbourhood for a flow ψ if the closure of the sum of all orbits contained in U is contained in U i.e.

$$S(U) := cl\,\{x \in U : \psi^s(x) \in U \text{ for all } s \in \mathbb{R}\} \subset U. \tag{2.30}$$

If U is such a neighbourhood for the flow of X_α we may define the Novikov complex relative to U setting $C_*(U, \alpha, g)$ to be the submodule of elements $\xi \in C_*(M, \alpha, g)$ such that $\xi_{\tilde x} = 0$ whenever $\tilde x \notin \pi^{-1}(U)$ with the boundary operator $\partial^*(U, \alpha, g)$ defined as before except that we count only these connecting orbits which lie in $\pi^{-1}(U)$. That $\partial^2 = 0$ is clear because $\mathcal{M}(x, z) \cap U$ is open and closed in $\mathcal{M}(x, z)$.

In order to construct a chain homomorphism between $C_*(\alpha_0, g_0)$ and $C_*(\alpha_1, g_1)$ pick an $\varepsilon_1 > 0$ and smooth homotopies α_t, g_t which are constant near 0 and 1 i.e. $\alpha_t = \alpha_0$, $g_t = g_0$ if $t < \varepsilon_1$ and $\alpha_t = \alpha_1$, $g_t = g_1$ if

$t > 1 - \varepsilon_1$ and such that $\frac{d}{dt}\alpha_t$ is exact. Thus α_t induce the same homomorphism $\chi_{\alpha_t} = \chi_\alpha\colon \Gamma \to \mathbb{R}$ and the coefficient ring Λ_α. Let $\pi\colon \widetilde{M} \to M$ be a covering with $\pi_1(\widetilde{M}) = ker[\alpha_t]$. Then there is a smooth function $f\colon [0,1] \times \tilde{M} \to \mathbb{R}$ with $df_t = \pi^*\alpha_t$ for $t \in [0,1]$. The following function is well defined and smooth on $\widetilde{M} \times S^1$ where S^1 is parametrized by $t \mapsto e^{\pi i t}$

$$F\colon \widetilde{M} \times S^1 \to \mathbb{R}$$
$$F(x,t) = f_{|t|} + \frac{K}{\pi} cos \pi t \qquad \text{for } t \in [-1,1].$$

Moreover $F(A\tilde{x},t) = F(\tilde{x},t) + \chi_\alpha(A)$ for any $A \in \Gamma$ and the differential dF descends to a closed form β on $M \times S^1$ with $ker[\beta] = ker[\alpha_t] \times \pi_1(S^1)$ and $\chi_\beta = \chi_\alpha\colon \Gamma \to \mathbb{R}$. Since M is compact all critical points of β have form (x,i) where x is a critical point of α_i, $i = 0,1$ if only K is sufficiently large. Moreover $\beta(x,t) = \alpha_i(x) - K \sin \pi t dt$ for $|t-i| < \varepsilon_1$ and therefore these points are nondegenerate with $\operatorname{ind}(x,i) = \operatorname{ind} x + 1 - i$. Next we define the "product-like" metric on $M \times S^1$

$$G_{(x,t)} = g_t \oplus \mathbf{1}$$

where $\mathbf{1}$ denotes the metric on S^1 provided by the parametrization. For $|t-i| < \varepsilon_1$ the dual vector field X_β has form

$$X^G_\beta(x,t) = (X^{g_i}\alpha_i(x), -K \sin \pi t \partial^t) \tag{2.31}$$

and so the submanifolds $M_i = M \times \{i\}$, $i = 0,1$ are invariant w.r.t. the flow and the restricted flow satisfies the Morse-Smale condition. If $t \neq 0,1$ and $\gamma = (\gamma_1,\gamma_2)\colon \mathbb{R} \to M \times S^1$ is a solution to $\dot\gamma = -X_\beta(\gamma)$ through a point (x,t) then from (7.1) follows that $\lim_{s\to-\infty} \gamma_2(s) = 0$ and $\lim_{s\to+\infty} \gamma_2(s) = 1$. In particular there are no orbits going from M_1 to M_0 and those connecting two critical points in M_0 (resp. M_1) are contained in M_0 (resp M_1). Hence by Theorem 2.5.2 we can obtain the Morse-Smale condition by an arbitrarily small perturbation $\tilde{G}$ of G on the set $N_0 = \{(x,t) \in M \times S^1 : t \neq 0,1\}$. A homotopy α_t with such a metric will be called admissible.

The set $U = M \times (-\varepsilon_2, 1+\varepsilon_2) \subset M \times S^1$ with $0 < \varepsilon_2 < \frac{1}{2}$ is clearly an isolating neighbourhood since X_β is transversal to ∂U. Thus we may consider the complex $(C_*^\beta, \Delta) := (C_*(U,\beta,\tilde{G}), \partial(U,\beta,\tilde{G}))$. Then $C_*^\beta = C_*^0 \oplus C_*^1$ with C_*^i generated by $\langle(\tilde{x}, i)\rangle$ where $\tilde{x}$ is a critical point of F and we assume the orientations $\langle(\tilde{x}, i)\rangle$ to be induced by the obvious isomorphisms

$$T_{(p,0)}W^u(x,0) \cong T_pW^u(x) \times T_0S^1$$
$$T_{(p,1)}W^u(x,1) \cong T_pW^u(x)$$

for any $p \in M$. Using this identification and (2.25) we obtain the orientation preserving isomorphisms

$$\begin{aligned} T_pW^u(x) \times T_0S^1/T_{(p,0)}\mathcal{M}((x,0),(y,0)) &\cong T_{(y,0)}W^u((y,o)) \\ &\cong T_pW^u(x)/T_p\mathcal{M}(x,y) \times T_0S^1 \cong T_pW^u(x) \times T_0S^1/T_p\mathcal{M}(x,y) \end{aligned}$$
$$\begin{aligned} T_{(p,1)}W^u(x,1)/T_{(p,1)}\mathcal{M}((x,1),(y,1)) &\cong T_{(y,1)}W^u(y,1) \\ &\cong T_pW^u(x)/T_p\mathcal{M}(x,y) \end{aligned}$$

and hence the orientation preserving inclusions $\mathcal{M}(x,y) \hookrightarrow \mathcal{M}((x,i),(y,i))$. (Recall that we adopt the convention that the natural isomorphism $V \cong (W \times V)/W$ is orientation preserving.) If we denote $\Delta = \sum_{i,j=0,1} \Delta_{ij}$ where $\Delta_{ij} \colon C_*^j \to C_*^i$ it follows that we have the identifications

$$\chi_i \colon (C_k(\alpha_i, g_i), \partial_{\alpha_i}) \to (C_{k+1-i}^i, \Delta_{ii})$$
$$\chi_i(\xi)_{\tilde{x}} := \xi_{(\tilde{x},i)}.$$

Clearly χ_i are well defined Λ_α homomorphisms since the injections $\widetilde{M}_i \hookrightarrow \widetilde{M} \times S^1$ are Γ–equivariant. Since in view of the remarks above $\Delta_{01} = 0$ the equality $\Delta^2 = 0$ reduces to

$$\Delta_{ii}^2 = 0 \qquad \text{for } i = 0,1$$
$$\Delta_{10} \circ \Delta_{00} + \Delta_{11} \circ \Delta_{10} = 0.$$

In particular the second equation means that

$$\psi_{10} = (-1)^k \chi_1^{-1} \circ \Delta_{10} \circ \chi_0 \colon C_*(\alpha_0, g_0) \to C_*(\alpha_1, g_1)$$

is a degree 0 chain homomorphism. If $\alpha_t = \alpha_0$, $g_t = g_0$ for all $t \in [0,1]$ and $\tilde{G} = G = g_0 \oplus 1$ then the set $\mathcal{M}((x,0),(y,1)) \subset U$ is nonempty and consists of a single orbit precisely when $x = y$. For any point (x,t) in $\mathcal{M}((x,0),(x,1))$ we have isomorphisms

$$(2.32) \quad T_{(x,t)}W^u(x,0)/T_{(x,t)}M((x,0),(x,1)) \cong T_{(x,1)}W^u((x,1)) \\ \cong T_xW^u(x) \cong T_xW^u(x) \times T_tS^1/T_tS^1$$

and the last one preserves the orientation iff $\mathrm{ind}(x)$ is even. Hence in the case of the constant homotopy ψ_{10} is an identity.

Note that in terminology of [Spa66] C_*^β is a chain cylinder of the homomorphism ψ_{01}.

We may repeat the whole construction for 2-parameter homotopies $\alpha_{t,r}$, $g_{t,r}$ joining four forms and metrics α_i, g_i for $i = 2t + r$, $t, r = 0, 1$ to obtain the Novikov complex $C_*^\beta = (C_*(\beta, \tilde{G}, U), \Delta)$ associated with the form β on the manifold $M \times S^1 \times S^1$ such that $\pi^*\beta = dF$ and

$$F \colon \widetilde{M} \times S^1 \times S^1 \to \mathbb{R}$$
$$F(x,t,r) = f_{t,r}(x) + \frac{K}{\pi}(\cos \pi t + \cos \pi r).$$

where $df_{t,r} = \pi^*\alpha_{t,r}$. Here $U = M \times (-\varepsilon_2, 1 + \varepsilon_2)^2$. As previously, $C_*^\beta = \bigoplus_{0 \le i \le 3} C_*^i$,

$$\Delta = \begin{pmatrix} \Delta_{00} & 0 & 0 & 0 \\ \Delta_{10} & \Delta_{11} & 0 & 0 \\ \Delta_{20} & 0 & \Delta_{22} & 0 \\ \Delta_{30} & \Delta_{31} & \Delta_{32} & \Delta_{33} \end{pmatrix}$$

where $\Delta = \partial^\beta_*$ denotes the boundary operator and we have natural identifications

$$\chi_0 \colon (C_k^{\alpha_0}, \partial^{\alpha_0}) \to (C_0^{k+2}, \Delta_{00})$$
$$\chi_i \colon (C_k^{\alpha_i}, \partial^{\alpha_i}) \to (C_i^{k+1}, \Delta_{ii}) \quad \text{for } i = 1, 2$$
$$\chi_3 \colon (C_k^{\alpha_3}, \partial^{\alpha_3}) \to (C_3^{k}, \Delta_{33}).$$

We also obtain chain homomorphisms

$$\psi_{10} = (-1)^{k+1}\chi_1^{-1}\,\Delta_{10}\,\chi_0, \qquad \psi_{30} = -\chi_3^{-1}\,\Delta_{30}\,\chi_0$$
$$\psi_{ij} = (-1)^{k}\chi_i^{-1}\,\Delta_{ij}\,\chi_j \quad \text{otherwise,}$$

which fit into the diagram

$$\begin{array}{ccc} C_*^{\alpha_0} & \xrightarrow{\psi_{10}} & C_*^{\alpha_1} \\ {\scriptstyle \psi_{20}}\downarrow & \searrow^{\psi_{30}} & \downarrow{\scriptstyle \psi_{31}} \\ C_*^{\alpha_2} & \xrightarrow[\psi_{32}]{} & C_*^{\alpha_3} \end{array}$$

The condition $\Delta^2 = 0$ gives, among the others, the equation

$$\Delta_{30} \circ \Delta_{00} + \Delta_{31} \circ \Delta_{10} + \Delta_{32} \circ \Delta_{20} + \Delta_{33} \circ \Delta_{30} = 0$$

which means that the degree $+1$ homomorphism ψ_{30} provides the chain homotopy between $\psi_{31} \circ \psi_{10}$ and $\psi_{32} \circ \psi_{20}$.

Note that if all one-parameter homotopies (i.e. for t or $r = 0, 1$) are admissible then we can obtain the transversality condition perturbing the metric in the set $N_0 = \{(x, t) \in M \times S^1 \times S^1 : \ t \neq 0, 1 \text{ and } r \neq 0, 1\}$ thus without affecting the homomorphisms $\psi_{10}, \psi_{20}, \psi_{31}$ and ψ_{32}. If we take $\alpha_0 = \alpha_2$ and $\alpha_1 = \alpha_3$ and a homotopy which is constant w.r.t. t for $r = 0, 1$ then we can conclude that $h_{\alpha_0\alpha_1} = H_*(\psi_{10}) = H_*(\psi_{32})$ is independent of the particular choice of the homotopy joining α_0 and α_1. Similarly, with $\alpha_2 = \alpha_3$ we obtain that

$$h_{\alpha_2\alpha_1} \circ h_{\alpha_1\alpha_0} = h_{\alpha_2\alpha_0}$$

and hence all $h_{\alpha_i\alpha_j}$ are isomorphisms as $h_{\alpha_i\alpha_i} = id_{C_*^{\alpha_i}}$. This proves Theorem 2.6.1.

2.7 Twisted coefficients

The reason for Novikov construction is that for a nonexact form one cannot expect that the number of orbits connecting two critical points is always finite.

Example 2.7.1 *Consider a function on $\mathbb{R}^3$ given by*

$$f(x,y,z) = y + (1-\cos x)\cos y + \cos z.$$

Its differential descends to a 1-form α on the 3-torus T^3 (parametrized by $x \mapsto e^{ix}$ etc.)

$$\alpha = \sin x \cos y \, dx + (1 + (\cos x - 1)\sin y)\; dy - \sin z \, dz$$

which is homologous to the constant form dy. Hence $\Gamma = \mathbb{Z}$ and the ring Λ_α consists of formal series $\sum_{i\in\mathbb{Z}} \lambda_i t^i$ in one variable t with the finite number of nonzero λ_i for negative i. The flow generated by the vector field dual to α w.r.t. the Euclidean metric on T^3 splits in x, y and in z direction. Its behaviour on the invariant 2-torus given by $\{z = 0\}$ or $\{z = \pi\}$ is sketched above (fig.3a). We restrict our attention to an isolating neighbourhood $U_\varepsilon := \{(x,y,z) \in T^3 : -\varepsilon < z < \pi + \varepsilon\}$. There are eight critical points:

$$w = \left(\pi, \frac{\pi}{3}, 0\right) \qquad \textit{of index } 3$$

$$x_0 = \left(\pi, \frac{\pi}{3}, \pi\right);\; x_1 = \left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right);\; x_2 = \left(\frac{3\pi}{2}, \frac{\pi}{2}, 0\right) \qquad \textit{of index } 2$$

$$y_0 = \left(\pi, \frac{2\pi}{3}, 0\right);\; y_1 = \left(\frac{\pi}{2}, \frac{\pi}{2}, \pi\right);\; y_2 = \left(\frac{3\pi}{2}, \frac{\pi}{2}, \pi\right) \qquad \textit{of index } 1$$

$$z = \left(\pi, \frac{2\pi}{3}, \pi\right) \qquad \textit{of index } 0$$

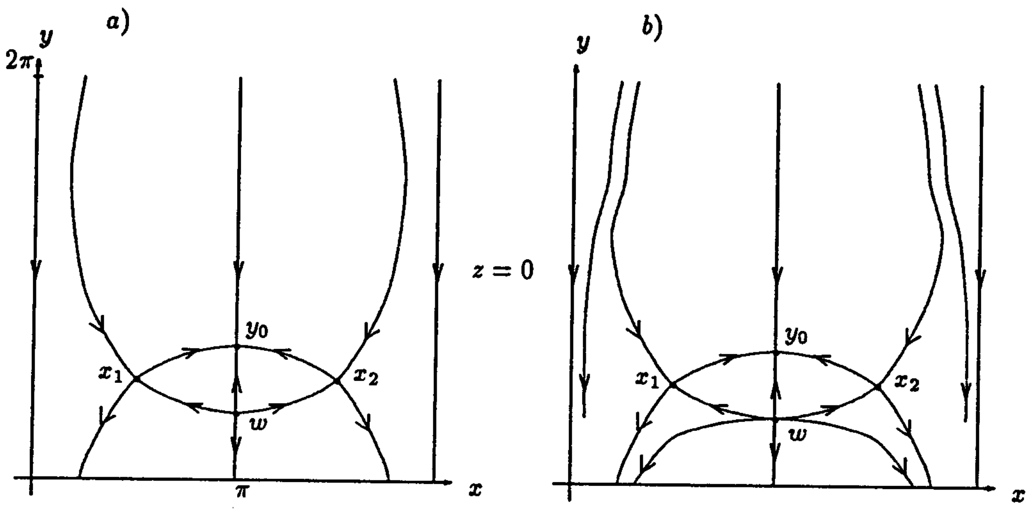

Figure 2.3:

and two periodic orbits ($t \mapsto (0, -t, 0)$ and $t \mapsto (0, -t, \pi)$). From the symmetry of the equation follows that there are homoclinic orbits (i.e. connecting a point with itself) at the critical points x_i, y_i, $i = 1, 2$. For the same reason the unstable manifold of x_1 intersects the stable manifold of y_1 along a 2-dimensional manifold. These are the only cases where the Morse-Smale condition fails. It is easy to see that there is one orbit connecting (in U_ε) each of the following pairs of points: w and x_i, $i = 0, 1, 2$, x_0 and y_i, x_i and y_0, y_i and z for $i = 1, 2$, y_0 and z. We perturb the metric to destroy the homoclinic orbits and create one more orbit connecting w with x_i and y_i with z, $i = 1, 2$ as follows. Pick an $0 < \varepsilon < \frac{\pi}{2}$ and an $0 < \alpha < 1$ and let $\psi \colon \mathbb{R} \to [0, \alpha]$ be a cutoff function satisfying

$$\psi(y) = \begin{cases} 0, & \text{if } |y - \frac{3\pi}{2}| \geq \varepsilon \\ \alpha, & \text{if } |y - \frac{3\pi}{2}| \leq \frac{\varepsilon}{2}. \end{cases}$$

Define new metric $g = dz^2 + dx^2 + dy^2 - \psi(y) \cos z \sin x \, dx dy$. The perturbed system is pictured at the figure 3b. The stable manifold of x_1 is contained in the unstable manifold of w and the part of the unstable manifold of x_1 which *is contained in the 2-torus* $\{z = 0\}$ *consists of one orbit converging to* y_0 *and the another one winding around the periodic orbit. The analogous picture can be drawn for the torus* $\{z = \pi\}$. *Further perturbation of* g *in order to secure the transversality condition for* x_i *and* y_i, $i = 1, 2$ *can be made in the set* $N = \{(x, y, z) \in T^3 : \ x \neq 0, |y - \frac{3\pi}{2}| \leq \varepsilon, 0 < z < \pi\}$ *thus without affecting any of the connecting orbits mentioned above. By considering the global section* $S = \{y = \frac{3\pi}{2}\}$ *of the flow and the behaviour of the Poincaré map of the two periodic orbits defined on an open subset of the section one can give a topological proof of the existence of infinitely many orbits connecting* x_1 *and* y_1. *There is, however, a simple algebraic argument proving this fact. We consider the Morse complex restricted to* U_ε *and write down the boundary operator (with the appropriate choice of the orientations)*

$$\begin{aligned}
\partial\langle w\rangle &= \langle x_0\rangle + (1-t)\langle x_1\rangle - (1-t)\langle x_2\rangle,\\
\partial\langle x_0\rangle &= \langle y_1\rangle - \langle y_2\rangle,\\
\partial\langle x_1\rangle &= \langle y_0\rangle + a_1\langle y_1\rangle,\\
\partial\langle x_2\rangle &= \langle y_0\rangle + a_2\langle y_2\rangle,\\
\partial\langle y_0\rangle &= \langle z\rangle,\\
\partial\langle y_1\rangle &= (1-t)\langle z\rangle,\\
\partial\langle y_2\rangle &= (1-t)\langle z\rangle
\end{aligned}$$

where a_1, a_2 *are some coefficients in* Λ_α. *The condition* $\partial^2 = 0$ *implies*

$$1 + (1-t)a_1 = 1 + (1-t)a_2 = 0$$

and so $a_1 = a_2 = -(1-t)^{-1} = -\sum_{k=0}^{\infty} t^k$. *Thus there must be an infinite number of orbits connecting* x_1 *and* y_1 *(*x_2 *and* y_2*) each in a different homotopy class.*

Example 2.7.2 *If* M *is a torus* T^n *then* $H^1_{DR}(M, \mathbb{R}) \cong Hom(\mathbb{Z}^n, \mathbb{R})$ *every homology class is represented by a constant form and so the Novikov homology*

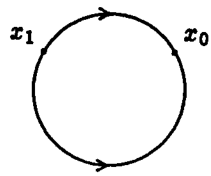

Figure 2.4:

is trivial unless α is exact.

Example 2.7.3 *Consider the constant form $dx \in C^\infty(T^*S^1)$ where the circle is parametrized by $\pi\colon \mathbb{R} \ni x \mapsto e^{ix} \in S^1$, and its exact perturbation $\alpha = (1 - 2\sin x)dx$. The associated flow has two critical points x_0 and x_1 of index 0 and 1 and two connecting orbits. Therefore it is a gradient flow for some Morse function $f\colon S^1 \to \mathbb{R}$ w.r.t. a different metric (see fig.4). The ring Λ_α is as in Example 2.7.1 and if we consider the complex $C_*(S^1, \alpha)$ as a free Λ_α-module generated by x_0 and x_1 then the boundary operator for Novikov homology ∂_*^N has the form (with an appropriate choice of $\tilde{x}_i \in \pi^{-1}(x_i)$, $i = 0, 1$)*

$$\partial_0^N \langle x_1 \rangle = (t-1)\langle x_0 \rangle.$$

This is an isomorphism since $t-1$ is invertible in Λ_α and so $HN_*(S^1, [\alpha]) = 0$. On the other hand we may consider the Morse complex for the function f tensored with Λ_α with the boundary

$$\mathrm{id}_{\Lambda_\alpha} \otimes \partial_0^N\colon \Lambda_\alpha \otimes_{\mathbf{Z}} C_1(S^1, f) \to \Lambda_\alpha \otimes_{\mathbf{Z}} C_0(S^1, f).$$

This operator is trivial and in fact it is an evaluation of ∂_0^N at $t = 1$. Thus $H_i(C_*(S^1, f); \Lambda_\alpha) = \Lambda_\alpha$ for $i = 0, 1$ and we obtain the homology of S^1 with coefficients in Λ_α.

More general if we have a form α and a metric g such that the corresponding flow has no other limit sets than critical points then it is a gradient flow for some Morse function f and a different metric ([Sma61]). In this case the number of the connecting orbits in the definition of the boundary is finite and the operator has form

$$\partial_*^N\langle x\rangle = \sum_{\mathrm{ind}(y)=\mathrm{ind}(x)-1} \left(\sum_A n(y,x;A)t_1^{A_1}\dots t_m^{A_m}\right)\langle y\rangle$$

where $n(y,x;A)$ counts the orbits in the homotopy class corresponding to A (there are several choices involved here) and A ranges in a finite set. Note that taking $t=(1,\dots,1)$ we obtain the Morse complex for f with coefficients in Λ_α. In fact by making a special choice of the gradient flow X_α we will construct an isomorphism between the Novikov complex and the Morse complex with twisted coefficients.

Proposition 2.7.4 *Let M be an n–dimensional closed manifold and $a \in H^1(M,\mathbb{R})$. There exists a 1–form $\alpha \in a$, a Morse function f and Riemannian metrics g_1, g_2 such that $g_1^{-1}\alpha = g_2^{-1}df$ and the flow generated by this vector field is of Morse-Smale type.*

Proof. Step 1. Let $u_1,\dots,u_k$ be a basis for the free part of $H_1(M,\mathbb{Z})$. We choose the corresponding dual manifolds $N_1,\dots.N_k$ so that they intersect transversally.

Each u_i can be represented as a differential of a function $\tilde{u}_i\colon M\to S^1=\mathbb{R}/\mathbb{Z}$ (unique up to homotopy) and an inverse image $N_i=\tilde{u}_i^{-1}(p_i)$ of any regular value p_i is a dual manifold of u_i i.e. the value of u_i on any 1–cycle z is equal to the intersection number $z\cdot N_i$ (see [Mil68]). Set $\tilde{u}\colon M\to \mathbb{T}^k$, $\tilde{u}=(\tilde{u}_1,\dots,\tilde{u}_k)$ and let $pr_{i_1\dots i_p}\colon \mathbb{T}^k\to\mathbb{T}^p$ denote the projection of the torus $\mathbb{T}^k$ onto $\mathbb{T}^p$, $pr_{i_1\dots i_p}(x_1,\dots,x_k)=(x_{i_1},\dots,x_{i_p})$. Choose a point $p=(p_1,\dots,p_k)\in\mathbb{T}^k$ such that $(p_{i_1},\dots,p_{i_p})$ is a regular value of $pr_{i_1\dots i_p}\circ\tilde{u}$ for any collection of subscripts $\{i_1\dots i_p\}\subset\{1,\dots,k\}$ (such p exists by Sards theorem since the projections are open maps). Then

$$\bigcap_{j=1}^{p} N_{i_j} = \bigcap_{j=1}^{p} \tilde{u}_{i_j}^{-1}(p_{i_j}) = (pr_{i_1 \dots i_p} \circ \tilde{u})^{-1}(p_{i_1}, \dots, p_{i_p})$$

are manifolds of dimension $\dim \ker d(pr_{i_1 \dots i_p} \circ \tilde{u}) = n-p$. If $x \in N_l \cap \bigcap_{j=1}^{p} N_{i_j}$ for $l \neq i_j$, $j = 1, \dots, p$ then it follows that $T_x(N_l \cap \bigcap_{j=1}^{p} N_{i_j}) \subsetneq T_x \bigcap_{j=1}^{p} N_{i_j}$. Therefore $N_l \pitchfork_x \bigcap_{j=1}^{p} N_{i_j}$ since $\operatorname{codim} N_l = 1$.

Step 2. We construct vector fields X_i defined in some neighbourhoods U_i of N_i such that

$$X_i \pitchfork N_i, \tag{2.33}$$

$$X_i\Big|_{N_j} \subset TN_j \text{ for } i \neq j \tag{2.34}$$

$$[X_i, X_j] = 0 \tag{2.35}$$

for $1 \leq i, j \leq k$.

First notice that for any $x_0 \in \bigcup_{i=1}^{k} N_i$ there is a neighbourhood V of x_0 and a chart $\psi_V \colon V \to \mathbb{R}^n$, $\psi_V(x_0) = 0$ such that if $N_i \cap V \neq \emptyset$ then $x_0 \in N_i$ and $\psi_V(N_i \cap V)$ is contained in a hyperplane $\{(x_1, \dots, x_n) \in \mathbb{R}^n : \ x_j = 0\}$ for some $1 \leq j \leq n$. In the neighbourhood V we define $X_i^V = \psi_V^* \frac{\partial}{\partial x_j}$ where $\frac{\partial}{\partial x_j} = (0, \dots, \underset{j}{1}, \dots, 0)$. We also choose ψ_V so that $d\tilde{u}_i(X_i^V)$ induces the positive orientation of S^1. Now we can use a partition of unity to obtain vector fields X_i' defined on some neighbourhood U_i' of N_i satisfying (2.33), (2.34), transversality following from the fact that $d\tilde{u}_i(X_i') \neq 0$ as a convex combination of $d\tilde{u}_i(X_i^V)$. Proceeding by induction let us assume an existence of vector fields $X_1, \dots, X_l$ satisfying (2.33), (2.34) for $i \leq l$, $j \leq k$ and (2.35) for $i, j \leq l$. Let ϕ_t^i be the local group of diffeomorphisms generated by X_i and $\varepsilon_0 > 0$ such that N_i is contained in the domain of ϕ_t^i for $|t| < \varepsilon_0$ and $i \leq l$. If we set $U_i^\varepsilon = \phi_{(-\varepsilon,\varepsilon)}^i(N_i)$ for $0 < \varepsilon < \varepsilon_0$ then $[X_i, X_j] = 0$ implies

$\phi_t^i(U_j^\varepsilon) \subset U_j^\varepsilon$. If we denote

$$L_p = \bigcup_{\substack{\{i_1,\dots,i_p\}\\ \subset\{1,\dots,l\}}} \bigcap_{j=1}^{p} N_{i_j} \cap N_{l+1}$$

then $L_{l+1} = \emptyset$, $L_l = \bigcap_{j=1}^{l+1} N_j$ and $L_0 = N_{l+1}$. Starting with $p = l+1$ we shall construct inductively vector fields Y_p in some neighbourhoods W_p of L_p and set $U_{l+1} = W_0$, $X_{l+1} = Y_0$.

Thus assume that Y_p satisfies (2.33), (2.34) and (2.35). W.l.o.g. we may take

$$W_p = W_p^\varepsilon = \bigcup_{\substack{\{i_1,\dots,i_p\}\\ \subset\{1,\dots,l\}}} \bigcap_{j=1}^{p} \phi^{i_j}_{(-\varepsilon,\varepsilon)}(N_{i_j} \cap V)$$

where $V \subset U'_{l+1}$ is a neighbourhood of N_{l+1} and $0 < \varepsilon < \varepsilon_0$. Let $V' \subset U'_{l+1}$ be a neighbourhood of N_{l+1} such that $\phi^i_{(-\varepsilon/2,\varepsilon/2)}(V') \subset V$ for $i \leq l$. For any $\{i_1,\dots,i_{p-1}\} \subset \{1,\dots,l\}$ the set $K = K_{i_1\dots i_{p-1}} := \bigcap_{j=1}^{p-1} N_{i_j} \cap V'$ is a submanifold of V' and $(K \setminus \overline{W}_p^{\varepsilon/2}, K \cap W_p^\varepsilon)$ is an open cover in $K_{i_1\dots i_{p-1}}$. Using cutoff functions we can glue together vector fields $X'_{l+1|K\setminus\overline{W}_p^{\varepsilon/2}}$ and $Y_{p|K\cap W_p^\varepsilon}$ to obtain a vector field $Y_{i_1,\dots,i_{p-1}} : K_{i_1\dots i_{p-1}} \to TM$ satisfying (2.33) and (2.34). Now notice that the map

$$\Phi = \Phi^{i_1\dots i_{p-1}} : (-\varepsilon/2,\varepsilon/2)^{p-1} \times K_{i_1\dots i_{p-1}} \to M,$$
$$(t_1,\dots,t_{p-1},x) \mapsto \phi^{i_1}_{t_1} \circ \dots \circ \phi^{i_{p-1}}_{t_{p-1}}(x)$$

is a diffeomorphism onto some neighbourhood $V_{i_1\dots i_{p-1}}$ of $\bigcap_{j=1}^{p-1} N_{i_j} \cap N_{l+1}$. On $V_{i_1\dots i_{p-1}}$ we set $Y_{p-1} = \Phi_* Y_{i_1\dots i_{p-1}}$. Since $\phi_{i_j}(N_q) \subset N_q$ for all $q \leq k$ the vector field Y_{p-1} satisfies (2.33) and (2.34) and clearly $[Y_{p-1}, X_{i_j}] = 0$ for $j = 1,\dots,p-1$. On the set $\Phi((-\varepsilon/2,\varepsilon/2)^{p-1} \times (K \cap W_p^{\varepsilon/2}))$ we have $Y_{p-1} = Y_p$ since $[Y_p, X_{i_j}] = 0$. On the other hand $\Phi((-\varepsilon/2,\varepsilon/2)^{p-1} \times (K \setminus W_p^{\varepsilon/2})) \cap U_q^{\varepsilon/2} = \emptyset$ if $q \neq i_j$, $j = 1,\dots,p-1$ and $q \leq l$. Hence $[Y_{p-1}, X_i] = 0$ on $V_{i_1\dots i_{p-1}}$ for all $q \leq l$. Moreover our choice of V' guarantees that $V_{i_1\dots i_{p-1}} \cap V_{i_1\dots i_{p-2}i_p} \subset$

$\Phi((-\varepsilon/2,\varepsilon/2)^{p-1}\times(K\cap W_p^{\varepsilon/2}))$ and therefore Y_{p-1} is well defined in

$$W_{p-1}=\bigcup_{\substack{\{i_1,\dots,i_{p-1}\}\\ \subset\{1,\dots,l\}}} V_{i_1\dots i_{p-1}}.$$

This concludes the inductive step.

Step 3. We finish the proof with a simple rescaling argument. Define a function $\tau_i\colon U_i^\varepsilon\to\mathbb{R}$ by the condition $\tau_i(\phi_t^i(x))=t$ where $x\in N_i$. If $\beta\colon\mathbb{R}\to\mathbb{R}$ is a smooth function satisfying $\operatorname{supp}\beta'\subset(-\varepsilon,\varepsilon)$ then $d(\beta\circ\tau_i)=\beta'\circ\tau_i d\tau_i$ extends to a closed form on M. It is easy to see that $d(\beta\circ\tau_i$ is homologous to $(\beta(\varepsilon)-\beta(-\varepsilon))\tilde{u}_i$. Indeed, w.l.o.g. assume that $N_i=\tilde{u}_i^{-1}(0)$ and $\beta(\pm\varepsilon)=\pm\frac{1}{2}$ i.e. $\beta\circ\tau_i\colon M\to S^1$, $\beta\circ\tau_{i|M\setminus U_i^\varepsilon}=\frac{1}{2}$. We have to show that $\tilde{u}_i$ and $\beta\circ\tau_i$ are homotopic. If $\varepsilon_1>0$ is such that $\tilde{u}_i^{-1}([-\varepsilon_1,\varepsilon_1])\subset U_i^\varepsilon$ and $\chi\colon S^1\to S^1$ satisfies $\chi(S^1\setminus(-\varepsilon_1,\varepsilon_1))=\frac{1}{2}$ and $\chi^{-1}(0)=0$ then $\tilde{v}_i=\chi\circ\tilde{u}_i$ is homotopic to $\tilde{u}_i$ and $\tilde{v}_i^{-1}(S^1\setminus\{\frac{1}{2}\})\subset U_i^\varepsilon$. Hence $\tilde{v}_i$ agrees with $\beta\circ\tau_i$ outside U_i^ε and the homotopy $t\beta\circ\tau_i+(1-t)\tilde{v}_i$ is well defined in U_i^ε.

Choose $0<\varepsilon_1<\varepsilon_2<\varepsilon$ and smooth functions $\rho,h\colon\mathbb{R}\to\mathbb{R}$ such that $\operatorname{supp}h'\subset(\varepsilon_1,\varepsilon_2)$, $h(\varepsilon_1)=0$, $h(\varepsilon_2)=2$, $\rho>0$, $\rho(1)=\rho(0)=1$, $\operatorname{supp}\rho'\subset$ $int(\operatorname{supp}h')$ and $\int_0^1\rho h'=1$. Define

$$\beta(t)=\begin{cases}h(t) & \text{if } t\geq 0,\\ \frac{1}{2}h(-t) & \text{if } t\leq 0.\end{cases}$$

Thus $[d(\beta\circ\tau_i)]=u_i$ and $(\rho\beta')\circ\tau_i d\tau_i$ is exact since $\int_0^1\rho\beta'=0$. Therefore we may define $\alpha\in a$ by

$$\alpha=\sum_{i=1}^k\alpha_i d(\beta\circ\tau_i)$$

where $a=\sum\alpha_i u_i\in H^1_{DR}(M,\mathbb{R})$ and $\alpha_i\in\mathbb{R}$. Next we construct metrics g_1 and g_2 such that $g_1(X_i,X_j)=\delta_{ij}$, $g_2(X_i,X_j)=\delta_{ij}\rho\circ\tau_i$ and X_i are perpendicular to $\ker d\tau_i$ w.r.t. both metrics. This can be done locally using coordinates induced by X_i since $[X_i,X_j]=0$ and $X_j\in TN_i$ implies $d\tau_i(X_j)=$

0 and then we may use a partition of unity. Hence we have $g_1(X_i,\cdot) = d\tau_i$ and

$$g_1(\sum \alpha_i \beta' \circ \tau_i X_i, \cdot) = \alpha$$
$$g_2(\sum \alpha_i \beta' \circ \tau_i X_i, \cdot) = \sum \alpha_i (\beta' \rho) \circ \tau_i d\tau_i =: df.$$

i.e. $\sum \alpha_i \beta' \circ \tau_i X_i = g_1^{-1}\alpha = g_2^{-1} df$. Notice that α does not vanish on the closure of the set $W = \{x \in M : \ g_1(x) \neq g_2(x)\}$. Therefore we may add a small Morse function with the support away from W so that α becomes nondegenerate. Since no orbit connecting two critical points is contained entirely in $\overline{W}$ we may obtain the Morse-Smale condition perturbing the metric in $M \setminus \overline{W}$ which does not affect the construction. □

In order to apply the above proposition we recall the concept of equivariant homology (see e.g. [Whi78]). If $\pi\colon \tilde{M} \to M$ is a covering with the group of deck automorphisms Γ then the action of Γ on the singular chain complex $\Delta_*(\tilde{M})$ of $\tilde{M}$ turns it into a $\mathbb{Z}\Gamma$–module where $\mathbb{Z}\Gamma$ is the group ring of Γ. Let Λ be a ring on which Γ acts on right thus inducing on Λ a structure of $\mathbb{Z}\Gamma$–module. The (singular) equivariant homology $H_*^\Gamma(\tilde{M};\Lambda)$ is defined as the homology of the $\mathbb{Z}\Gamma$–complex Δ_* with coefficients in Λ or equivalently $H_*^\Gamma(\tilde{M};\Lambda) = H_*(\Lambda \otimes \Delta_*/Q)$ where the submodule $Q = Q(C_*,\Lambda) \subset \Lambda \otimes_{\mathbb{Z}} \Delta_*$ is generated by the elements of the form $\lambda A \otimes c - \lambda \otimes Ac$ with $A \in \Gamma$, $\lambda \in \Lambda$. It is isomorphic to the singular homology of M with local coefficients system induced by the action of Γ on Λ ([Whi78]). Similarly we define the equivariant Morse complex. Although $\tilde{M}$ is not compact a Morse complex $C_*(M,f)$ lifts to a complex $C_*(\tilde{M},\tilde{f},\tilde{g})$, $\tilde{f} = f \circ \pi$, $\tilde{g} = \pi^* g$ which is equivariant. We obtain the homology groups $H_*^\Gamma(\tilde{M},\tilde{f},\tilde{g};\Lambda)$ defined analogously.

Proposition 2.7.5 *The groups $H_*^\Gamma(\tilde{M},\tilde{f},\tilde{g};\Lambda)$ are independent of the choice of the function f and the metric g. They are isomorphic to the singular equivariant homology $H_*^\Gamma(\tilde{M};\Lambda)$.*

Proof. The proof of independence of the choice of f and g is the same as for the ordinary Morse complex The prove of the remaining part of the proposition is very much the same as in the ordinary case (see [Flo89b, Sal90].). Thus we may choose a Morse function $f\colon M \to \mathbb{R}$ and a filtration $\emptyset = N_{-1} \subset \cdots \subset N_n = M$ so that (N_i, N_{i-1}) is an index pair for an isolated invariant set consisting of critical points of index i. We may also choose an index pair (N_x, L_x) homeomorphic to $(D^i \times D^{n-i}, S^{i-1} \times D^{n-i})$ for every critical point x of f ($\mathrm{ind}(x) = i$) so that $(N_x, L_x) \subset (N_i, N_{i-1})$ and this inclusion is a composition of an excision map and a homotopy equivalence, homotopy being induced by the gradient flow. If we set $\tilde{N}_i = \pi^{-1}N_i$, $\tilde{N}_x = \pi^{-1}N_x$, $\tilde{L}_x = \pi^{-1}L_x$ it follows that the inclusion

$$\bigcup_{\mathrm{ind}(x)=i} (\tilde{N}_x, \tilde{L}_x) \subset (\tilde{N}_i, \tilde{N}_{i-1})$$

being a composition of an equivariant excision map and an equivariant homotopy equivalence induces isomorphism in the equivariant homology. Thus $H_k^\Gamma(\tilde{N}_i, \tilde{N}_{i-1})$ is nonzero only if $k = i$ and the natural transformation from the universal coefficients theorem

$$\chi x\colon \Lambda \otimes_{\mathbb{Z}\Gamma} H_*(\tilde{N}_i, \tilde{N}_{i-1}) \to H_*^\Gamma(\tilde{N}_i, \tilde{N}_{i-1}; \Lambda)$$

is an isomorphism. Thus there is a commutative diagram

$$\begin{array}{ccc}
\bigoplus_{\mathrm{ind}(x)=i} \Lambda \otimes_{\mathbb{Z}\Gamma} H_i(\tilde{N}_x, \tilde{L}_x) & \xrightarrow{\cong} & \bigoplus_{\mathrm{ind}(x)=i} H_i^\Gamma(\tilde{N}_x, \tilde{L}_x; \Lambda) \\
\cong\downarrow & & \downarrow\cong \\
\Lambda \otimes_{\mathbb{Z}\Gamma} H_i(\tilde{N}_i, \tilde{N}_{i-1}) & \xrightarrow{\cong} & H_i^\Gamma(\tilde{N}_i, \tilde{N}_{i-1}) \\
1\otimes\partial_i\downarrow & & \downarrow\partial_i^\Gamma \\
\Lambda \otimes_{\mathbb{Z}\Gamma} H_{i-1}(\tilde{N}_{i-1}, \tilde{N}_{i-2}) & \xrightarrow{\cong} & H_{i-1}^\Gamma(\tilde{N}_{i-1}, \tilde{N}_{i-2}) \\
\cong\uparrow & & \uparrow\cong \\
\bigoplus_{\mathrm{ind}(x)=i-1} \Lambda \otimes_{\mathbb{Z}\Gamma} H_i(\tilde{N}_x, \tilde{L}_x) & \xrightarrow{\cong} & \bigoplus_{\mathrm{ind}(x)=i-1} H_i^\Gamma(\tilde{N}_x, \tilde{L}_x; \Lambda).
\end{array}$$

where $\partial, \partial^\Gamma$ are the boundary homomorphisms in the exact sequence of the triple $(\tilde{N}_i, \tilde{N}_{i-1}, \tilde{N}_{i-2})$. Now notice that $(\tilde{N}_x, \tilde{L}_x)$ is a disjoint union of pairs $(N_{\tilde{x}}, L_{\tilde{x}})$ over all $\tilde{x} \in \pi^{-1}(x)$. Therefore we may consider the composition $\Lambda \otimes_\Gamma \Delta$ of the left vertical arrows separately for each pair $(\tilde{x}, \tilde{y})$ of critical points of $\tilde{f}$, $\mathrm{ind}(\tilde{x}) = \mathrm{ind}(\tilde{y}) + 1$. But these are contained in a compact invariant set and therefore Δ is isomorphic to the boundary operator in the Morse complex ([Flo89b]). On the other hand a standard algebraic argument shows that the homology of the complex $H_*^\Gamma(\tilde{N}_i, \tilde{N}_{i-1}), \partial^\Gamma)$ is isomorphic to the equivariant homology of $\tilde{M}$.

Corollary 2.7.6 *If $a \in H_1(M, \mathbb{R})$ and $\Gamma = \pi_1/\ker a$ then*

$$HN_*(M, a; \mathbb{Z}) \cong_{\Lambda_a} H_*^\Gamma(\tilde{M}; \Lambda_a(\mathbb{Z}))$$

with the action of Γ on Λ_a given by $(\lambda A)(B) = \lambda(BA^{-1})$.

Proof. We describe the chain isomorphism between the Novikov complex of α and the equivariant Morse complex of f with $\alpha \in a$ and f chosen as in the Proposition 2.7.4. Thus the corresponding flows are identical and we define

$$\Phi : \Lambda_a \otimes C_*(\tilde{M}, \tilde{f}) \to C_*(M, \alpha)$$

$$\Phi\left(\sum_x \lambda_x \otimes x\right)(x_0) = \sum_A \lambda_{Ax_0}(A^{-1}).$$

It is easy to see that Φ is an epimorphism and $Q \subset \ker \Phi$. If $\sum_x \lambda_x \otimes x \in \ker \Phi$ then $0 = \Phi(\sum_x \lambda_x \otimes x)(Ax_0) = \sum_B \lambda_{BAx_0}(B^{-1}) = \sum_B \lambda_{Bx_0}(AB^{-1}) = (\sum_B \lambda_{Bx_0} B)(A)$ i.e. $\sum_B \lambda_{Bx_0} B = 0$. Hence $\sum_x \lambda_x \otimes x = \sum_B \sum_{x_0} \lambda_{Bx_0} \otimes Bx_0 - \sum_{x_0} \sum_B \lambda_{Bx_0} B \otimes x_0 \in Q$ where we pick one x_0 in every orbit of Γ. Thus Φ is an isomorphism Furthermore, we have

$$\begin{aligned}\partial\Phi\left(\sum_x \lambda_x \otimes x\right)(y) &= \sum_{x'} n(x', y_0)\Phi\left(\sum_x \lambda_x \otimes x\right)(x') \\ &= \sum_{x'}\sum_A n(x', y)\lambda_{Ax'}(A^{-1}) = \sum_A \sum_x n(A^{-1}x, y)\lambda_x(A^{-1})\end{aligned}$$

$$\begin{aligned}\Phi\partial^\Gamma\left(\sum_x \lambda_x \otimes x\right)(y) &= \Phi\left(\sum_x\sum_{y'} \lambda_x n(x, y') \otimes y'\right)(y) \\ &= \sum_A \sum_x n(x, Ay)\lambda_x(A^{-1})\end{aligned}$$

and so Φ is a chain map since $n(Ax, Ay) = n(x, y)$. This proves the last assertion of Theorem 2.2.2.

Chapter 3

Floer theory

3.1 Introduction

In this chapter we prove a technical result which we hope can be used in computing Floer homology for Lagrangian intersection whenever it has a simple structure. The two extreme cases are on the one hand, when two Lagrangian submanifolds L_0, L_1 intersect transversally at a single point and on the other hand, when they are equal. If we are interested in the "local" behaviour of Floer's complex near the intersection then the first case is trivial: there is one generator of the complex and no connecting orbits. When $L_0 = L_1$ then locally it can be seen as the zero section of the cotangent bundle T^*L_0. This has been investigated in [Flo88a] and it turns out that the Floer complex can be identified with the Morse complex of a function on L. If we want to draw an analogy with the finite dimensional Morse theory then the first case corresponds to a single critical point of a Morse function while the second to a constant function i.e. the whole manifold consists of critical points. An "intermediate" stage is when the critical set of a function consists only of nondegenerate critical manifolds. Then there is a spectral sequence whose entries are the homology of critical manifolds and which converges to the homology of the manifold ([Bot82]). We think that the clean intersection should play the role of the nondegenerate critical manifold in Floer theory. The main theorem of this chapter (Theorem 3.4.11) states that the Floer homology in a neighbourhood of a connected clean intersection is isomorphic to the singular homology of the intersection. The difficulty one encounters trying to obtain a global result is the possibility of the existence of "large" gradient lines i.e. connecting orbits which cannot be deformed into a small neighbourhood of $L_0 \cap L_1$. We can avoid this by imposing quite restrictive topological assumptions or sometimes these additional solutions do not exist for "dimension" reasons.

The set-up is described in the following two sections. Floer presented his theory in a series of papers ([Flo88b],[Flo89b],[Flo88a]) under the assumption that the symplectic form vanishes over $\pi_2(M, L_i)$. Details in the monotone case have been carried out in [Oh93a] and we shall base our set-up on these papers. The analysis necessary to construct Floer homology is a vast subject and we do not make an attempt to reproduce all details here. We state the facts necessary for understanding and proving the results of Section 3.4. Section 3.5 contains some straightforward consequences of Theorem 3.4.11.

3.2 Preliminaries and notation

A 2-form ω on a manifold M is said to be *symplectic* iff it is closed and nondegenerate. This means that M is necessary of even dimension $2n$ and that the exterior power $\omega^n = \omega \wedge \cdots \wedge \omega$ is a volume form on M. An example of a symplectic manifold is the total space of the cotangent bundle $\pi_N \colon T^*N \to N$ of any manifold N, equipped with the symplectic form $\omega_{T^*N} = -d\lambda_{T^*N}$ where λ_{T^*N} is the canonical 1-form on T^*N:

$$\lambda_{T^*N}(\alpha) = \alpha \circ d\pi_N, \qquad \alpha \in T^*N.$$

A diffeomorphism $\phi\colon (M_1,\omega_1) \to (M_2,\omega_2)$ between two symplectic manifolds is called a *symplectomorphism* if $\phi^*\omega_2 = \omega_1$. The Darboux theorem asserts that any symplectic manifold is locally symplectomorphic to an open subset of $\mathbb{R}^{2n}$ with the standard symplectic form

$$\omega_{\mathbb{R}^{2n}} = \sum_{i=1}^{n} dq_i \wedge dp_i.$$

An n-dimensional submanifold L of (M,ω) is called *Lagrangian* if ω vanishes on TL. The ingenious argument by Moser, used in the proof of the Darboux theorem, also yields the Lagrangian Neighbourhood Theorem: *A neighbourhood of a Lagrangian submanifold L in M is symplectomorphic to a neighbourhood of the zero section in T^*L.*

Any symplectic manifold (M,ω) admits an almost complex structure, that is an endomorphism J of the tangent bundle such that $J^2 = I$. This structure can be chosen to be compatible with ω, meaning that

$$\omega(Jv,w) + \omega(v,Jw) = 0 \quad \text{for } v,w \in TM$$

and $g_J := \omega \circ (id \times J)$ is a Riemannian metric on M. This gives the reduction of the structure group of TM to $U(n)$. Moreover, the space of ω-compatible almost complex structures is contractible.

3.3 Floer homology

Let L_0, L_1 be two compact Lagrangian submanifolds of a symplectic manifold (M,ω). Define the space of paths

$$\Omega(L_0,L_1) = \{\gamma \in C^\infty([0,1],M) :\ \gamma(i) \in L_i,\ i = 0,1\} \tag{3.1}$$

and let $\Omega(L_0,L_1,\gamma_0)$ denote the path component of $\gamma_0 \in \Omega(L_0,L_1)$. Since L_i are Lagrangian, the formula

$$\hat{\omega}(\gamma)\xi = \int_0^1 \omega(\xi,\dot{\gamma})\,dt \tag{3.2}$$

where $\gamma \in \Omega(L_0, L_1)$ and ξ is a vector field along γ, defines a closed 1-form on $\Omega(L_0, L_1)$ in the sense that if a map $u\colon S^1 \times I \to M$, $u(s,\cdot) \in \Omega(L_0, L_1)$ represents a contractible loop in $\Omega(L_0, L_1)$ then

$$[\hat{\omega}](u) = \int u^*\omega = 0.$$

This way ω induces a homomorphism

$$I_\omega\colon \pi_1(\Omega(L_0, L_1), \gamma_0) \to \mathbb{R}$$

If I_ω vanishes then $da_\omega = \hat{\omega}$ where a_ω is a real valued function. In general, the symplectic action functional is defined only on the universal cover $\tilde{\Omega}$ of Ω

$$a_\omega(\tilde{\gamma}) = \int u^*\omega \tag{3.3}$$

$$\begin{array}{ccc} \tilde{\Omega}(L_0, L_1, \gamma_0) & \xrightarrow{a_\omega} & \mathbb{R} \\ \downarrow & & \downarrow \\ \Omega(L_0, L_1, \gamma_0) & \longrightarrow & \mathbb{R}/\operatorname{im} I_\omega \end{array}$$

where $u\colon I \times I \to M$ represents an element $\tilde{\gamma} \in \tilde{\Omega}$ that is $u(0,t) = \gamma_0(t)$, $u(1,t) = \gamma(t)$ and $u(s,i) \in L_i$.

3.3.1 Description of $\pi_1(\Omega(L_0, L_1))$ and $\pi_0(\Omega(L_0, L_1))$

The path components, the fundamental group of $\Omega(L_0, L_1)$ and the homomorphism induced on $\pi_1(\Omega(L_0, L_1))$ by the symplectic form ω can be easily visualized. On the other hand it may be convenient to describe them more directly in terms of $\pi_k(M)$ and $\pi_k(L_i)$. In the following we may assume that $\Omega(L_0, L_1, \gamma_0)$ consists of all continuous paths connecting L_0 and L_1, since this space has the same weak homotopy type.

Fix a base point $\gamma_0 \in \Omega(L_0, L_1)$ and set $x_i = \gamma_0(i)$ for $i = 0, 1$. We introduce the notation

$$l_i \colon L_i \hookrightarrow M$$
$$k_i \colon (M, x_i) \hookrightarrow (M, L_i)$$
$$p_i \colon \Omega(L_0, L_1) \ni \gamma \mapsto \gamma(i) \in L_i$$

where k_i, l_i are the inclusions.

Lemma 3.3.1 *If L_i are connected then there is a bijection*

$$\pi_0(\Omega(L_0, L_1)) \cong \operatorname{im}\pi_1(l_0)\backslash\pi_1(M)/\operatorname{im}\pi_1(l_1) \tag{3.4}$$

where on the right side two elements $a, b \in \pi_1(M)$ are in the same equivalence class iff there are $c_i \in \pi_1(L_i)$ such that $a = \pi_1(l_0)(c_0)b\pi_1(l_1)(c_1)$.

Moreover, there are short exact sequences

(3.5)
$$0 \to \pi_2(M)/\operatorname{im}(\pi_2(l_1) - \pi_2(l_0)) \to \pi_1(\Omega(L_0, L_1)) \to \ker \pi_1(l_0)^{-1}\pi_1(l_1) \to 0$$

and

(3.6)
$$0 \to \pi_2(M, L_0) \oplus \pi_2(M, L_1)/(\pi_2(k_0), \pi_2(k_1))\pi_2(M) \to$$
$$\to \pi_1(\Omega(L_0, L_1)) \to \operatorname{im}\pi_1(l_0) \cap \operatorname{im}\pi_1(l_1) \to 0.$$

Proof. The evaluation map

$$p := (p_0, p_1) \colon \Omega(L_0, L_1) \ni \gamma \mapsto (\gamma(0), \gamma(1)) \in L_0 \times L_1$$

is a Serré fibration and the fiber $p^{-1}(\gamma_0) \overset{j}{\hookrightarrow} \Omega(L_0, L_1)$ is homotopy equivalent to the space of loops

$$\Omega(M, x_0) = \{\gamma \colon [0,1] \to M : \ \gamma(0) = x_0 = \gamma(1) \in L_1\}$$
$$p_0^{-1}(\gamma_0) \ni \gamma \mapsto \gamma * \gamma_0^{-1} \in \Omega(M, x_0).$$

Therefore $\pi_k(p_0^{-1}(\gamma_0)) \cong \pi_{k+1}(M, x_0)$ and there is the exact homotopy sequence of homotopy groups

$$(3.7) \quad \longrightarrow \pi_{k+1}(M) \xrightarrow{(\pi_k(j))} \pi_k(\Omega(L_0, L_1)) \xrightarrow{(\pi_k(p_0), \pi_k(p_1))} \longrightarrow \pi_k(L_0) \oplus \pi_k(L_1) \xrightarrow{\pi_k(l_1) - \pi_k(l_0)} \pi_k(M) \longrightarrow$$

from which (3.4) and (3.5) follow. Note that, although the map

$$\pi_1(l_0)^{-1}\pi_1(l_1) \colon \pi_1(L_0) \oplus \pi_1(L_1) \ni (a, b) \mapsto \pi_1(l_0)a^{-1}\pi_1(l_1)b \in \pi_1(M)$$

is not a homomorphism, its kernel is a subgroup of $\pi_1(L_0) \oplus \pi_1(L_1)$. Then the map

$$(\pi_2(k_0), 0) \colon \pi_2(M) \to \pi_2(M, L_0) \oplus \pi_2(M, L_1)$$

induces the isomorphism

$$\pi_2(M)/(\mathrm{im}\, \pi_2(l_0) + \mathrm{im}\, \pi_2(l_1)) \cong \pi_2(M, L_0) \oplus \pi_2(M, L_1)/(\pi_2(k_0), \pi_2(k_1))\pi_2(M).$$

We have also the isomorphism

$$\ker \pi_1(l_0)^{-1}\pi_1(l_1) \ni (a_0, a_1) \mapsto \pi_1(l_0)a_0 = \pi_1(l_1)a_1 \in \mathrm{im}\, \pi_1(l_0) \cap \mathrm{im}\, \pi_1(l_1)$$

and thus (3.6). □

3.3.2 Monotonicity

There is another homomorphism defined on π_1 of the path space

$$I_\mu = I_{\mu, L_0, L_1} \colon \pi_1(\Omega(L_0, L_1), \gamma_0) \to \mathbb{Z}$$

namely, $I_\mu(u)$ is the Maslov class of the bundle $u^*TM \to S^1 \times I$. This may be described as follows. The bundle u^*TM admits a symplectic trivialization $\Psi \colon u^*TM \to S^1 \times I \times \mathbb{C}^n$ which restricted to $S^1 \times \{i\}$ gives a loop

$$\lambda_i \colon S^1 \ni t \mapsto \Psi(u^*TL_i(t, i)) \subset \mathbb{C}^n$$

of Lagrangian planes in $\mathbb{C}^n$. We set

$$I_\mu^\Psi(u) = \mu(\lambda_1) - \mu(\lambda_0)$$

where μ is the Maslov index. If Ψ' is another symplectic trivialization then

$$\Psi' \circ \Psi^{-1}(t, s, z) = (t, s, \psi_s(t)z)$$

and $\psi_s(t) \in Symp(2n)$. We have then the formula for the Maslov index of $\lambda_i' \colon t \mapsto \Psi'(u^*TL_i(t, i))$

$$\mu(\lambda_i') = \mu(\psi_i \circ \lambda_i) = \mu(\lambda_i) + \mu(\psi_i)$$

where in the last term μ denotes the isomorphism between $\pi_1(Symp)$ and $\mathbb{Z}$. But the loops ψ_i are homotopic in $Symp(2n)$ and so $I_\mu^{\Psi'}(u) = I_\mu^\Psi(u)$. We say that a triple (M, L_0, L_1, γ_0) is *monotone* if

$$I_\omega = \lambda I_\mu$$

for some $\lambda \geq 0$.

Remark 3.3.2 *In view of the decomposition (3.6) the monotonicity means that I_ω and I_μ are proportional over $\pi_2(M, L_i)$ and that the induced homomorphism*

$$(I_\omega - \lambda I_\mu)_\# \colon \operatorname{im} \pi_1(l_0) \cap \operatorname{im} \pi_1(l_1) \to \mathbb{R}$$

vanishes. This happens e.g. if one of the groups $\operatorname{im} \pi_1(l_i)$ is torsion or if $L_0 = L_1$ since in the latter case any element in $\operatorname{im} \pi_1(l_i)$ is represented in Ω by a loop u of constant paths and so $I_\omega[u] = I_\mu[u] = 0$. Note that these are the topological assumptions in [Oh93a].

If $A \in \pi_2(M)$ and $\pi_1(j) \colon \pi_2(M) \to \pi_1(\Omega)$ is as in (3.7) then

$$I_\mu \circ \pi_1(j)(A) = 2c_1(A)$$

where c_1 is the first Chern class of TM (see [Flo88a, Oh93a]). In particular, if N_{c_1} is the minimal Chern number and N_μ the minimal Maslov number, i.e. the positive generator of $\operatorname{im} I_\mu$, *then N_μ necessarily divides $2N_{c_1}$. Similarly, if Σ_i denote the generators of* $\operatorname{im} I_{\mu|\pi_2(M,L_i)}$, *then $N_\mu|\Sigma_i|2N_{c_1}$. Also, if a triple (M, L_0, L_1) is monotone then M is itself monotone, i.e.*

$$[\omega]_{\pi_2(M)} = 2\lambda c_{1\,\pi_2(M)}.$$

Conversely, if M is monotone and the groups $\pi_1(L_i)$ are torsion then it follows from (3.5) that (M, L_0, L_1) is monotone as well.

3.3.3 The gradient flow

In order to develop the Morse theory we have to introduce a Hamiltonian perturbation of (3.2) since the form $\tilde{\omega}$ may be degenerate. Let X_t be a Hamiltonian vector field on M with the Hamiltonian $H\colon [0,1] \times M \to \mathbb{R}$, that is

$$i_{X_t}\omega = dH_t \qquad t \in [0,1]$$

and let ϕ_t be the family of symplectomorphisms generated by X_t. Consider the perturbed form

$$\hat{\omega}_H(\gamma)\xi = \hat{\omega}(\gamma)\xi + \int_0^1 dH_t(\xi(t))\,dt = \int_0^1 \omega(\xi, \dot{\gamma} - X_t(\gamma))\,dt$$

It is exact if $\hat{\omega}$ is exact and $\hat{\omega}_H = da_H$ with

$$a_H(\gamma) = \int u^*\omega + \int_0^1 H_t(\gamma(t))\,dt$$

where u is as in (3.3). The critical points of a_H are precisely the paths $x \in \Omega(L_0, L_1)$ satisfying

$$\dot{x}(t) = X_t(x(t)) \tag{3.8}$$

which in turn are in the 1—1 correspondence with the intersection points of L_0 and $\phi_1^{-1}(L_1)$. A solution to (3.8) is a *nondegenerate* critical point of a_H if

$$d\phi^{-1}(T_{x(1)}L_1) \cap T_{x(0)}L_0 = \{0\}$$

that is if L_0 intersects $\phi_1^{-1}(L_1)$ transversally. We shall say that a Hamiltonian H *regular* if all critical points of a_H are nondegenerate. This holds for a generic function in $\mathcal{H} := C^\infty(I \times M, \mathbb{R})$. Let $\mathcal{J}_\omega$ denote the space of all smooth t-dependent families $J\colon [0,1] \times M \to End(TM)$ of ω-compatible a. c. structures on M. For a given $J \in \mathcal{J}_\omega$ the L^2-gradient of a_H is given by

$$grad\, a_H(\gamma) = J(\dot\gamma - X_t(\gamma))$$

and we may write the gradient equation

$$\overline{\partial}_{J,H}(u) = \frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} + \nabla H_t(u) = 0 \tag{3.9}$$

$$u\colon \mathbb{R} \times I \to M, \quad u(s,i) \in L_i \tag{3.10}$$

where $\nabla H_t = -J_t X_t$ is the gradient of H_t w.r.t. the metric g_{J_t}.

This is an elliptic equation and so the Cauchy problem is not well posed. On the other hand, by the elliptic regularity, any $W^{1,p}_{loc}$ solution to (3.9), (3.10) with $p > 2$ is smooth ([Flo88b, Lemma 2.1]). Instead of trying to define a gradient flow of a_H globally, Floer considers the solutions with bounded energy ℓ^2:

$$\ell^2(u) := \int_{-\infty}^{+\infty}\int_0^1 \left\|\frac{\partial u}{\partial s}\right\|^2_{L^2} dt\, ds$$

$$\mathcal{M}_{J,H}(L_0, L_1) := \{u:\ \overline{\partial}_{J,H}(u) = 0 \text{ and } \ell^2(u) < \infty\}$$

It turns out that these are precisely the solutions which satisfy the limit conditions

$$u(s,\cdot) \to x^\pm, \quad s \to \pm\infty$$

for some $x^-, x^+ \in Crit\, a_H$. We have then

$$\ell^2(u) = \int u^*\omega + \int_0^1 H_t(x^+(t)) - H_t(x^-(t))\, dt$$

We denote the set of these solutions by $\mathcal{M}_{J,H}(x^-, x^+)$. We fix a $p > 2$ and similarly as in the previous chapter we introduce the space of paths $\mathcal{P}(x^-, x^+)$ connecting x^- and x^+ in Ω:

$$\begin{aligned}\mathcal{P}(x^-, x^+) = \Big\{u\colon \mathbb{R} \times I \to M :\ & u \in W^{1,p}_{loc}(\mathbb{R} \times I, M),\\ & \exists T > 0,\ \exists \xi_- \in W^{1,p}((-\infty, -T] \times I; (x^-)^*TM),\\ & \exists \xi_+ \in W^{1,p}([T, \infty) \times I; (x^+)^*TM) \text{ such that}\\ & u(s,t) = exp_{x^\pm(t)}\xi_\pm(s,t) \text{ for } \pm s \geq T \text{ and } t \in I\Big\}.\end{aligned}$$

It is clear that the definition does not depend on the choice of a metric on M. If $x^\pm$ are nondegenerate then $\mathcal{P}(x^-, x^+)$ is a Banach manifold ([Flo88b, Theorem 3]), whose tangent space at $u \in \mathcal{P}(x^-, x^+) \cap C^\infty(\mathbb{R} \times I, M)$ consists of $W^{1,p}$-sections of the pullback bundle u^*TM satisfying $\xi(s,i) \in T_{u(s,i)}L_i$, which we denote by $W^{1,p}_{L_0,L_1}(u^*TM)$. Then $\overline{\partial}_{J,H}$ can be seen as a section of the Banach vector bundle $\mathcal{E} \to \mathcal{P}$ with fibres $\mathcal{E}_u = L^p(u^*TM)$. For any $u \in \mathcal{P}$ the linearization of $\overline{\partial}_{J,H}$ along $u \in \mathcal{P}(x^-, x^+)$,

$$D_{J,H}(u)\xi = \nabla_s\xi + J_t(u)\nabla_t\xi + \nabla_\xi J_t(u)\frac{\partial u}{\partial t} + \nabla_\xi \nabla H(u) \tag{3.11}$$

defines a bounded operator $D_{J,H}(u)\colon W^{1,p}_{L_0,L_1}(u^*TM) \to L^p(u^*TM)$.

Theorem 3.3.3 *If x^-, x^+ are nondegenerate critical points of a_H then the operator $D_{J,H}(u)$ is Fredholm. There is a generic set $(\mathcal{J} \times \mathcal{H})_{reg}(L_0, L_1) \subset \mathcal{J} \times \mathcal{H}$ such that $D_{J,H,}(u)$ is onto for any $(J, H) \in (\mathcal{J} \times \mathcal{H})_{reg}$ and $u \in \mathcal{M}_{J,H}(L_0, L_1)$ and then $\mathcal{M}_{J,H}(x^-, x^+)$ is a collection of finite dimensional manifolds with*

$$\dim_u \mathcal{M}_{J,H}(x^-, x^+) = \operatorname{ind} D_{J,H}(u)$$

That $D_{J,H}(u)$ is Fredholm was proved in [Flo88b]. The proof of the transversality result in that paper contains some gaps which were filled in [FHS94]. The main idea is essentially as in Section 2.5. We may assume w.l.o.g. that $H = 0$ (see Remark 3.13 below). We take $\mathcal{J}$ as the parameter space and consider the Fredholm section

$$\mathcal{F}: \mathcal{P} \times \mathcal{J} \ni (u, J) \mapsto \overline{\partial}_J(u) \in \mathcal{E}$$

We have to prove that $\mathcal{F}$ is transversal to the zero section. Then the set

$$\mathcal{Z} = \{(u, J) \in \mathcal{P} \times \mathcal{J} : \ \overline{\partial}_J(u) = 0\}$$

is an infinite dimensional Banach manifold and the projection

$$j: \mathcal{Z} \to \mathcal{J}$$

is a Fredholm map. The set $\mathcal{J}_{reg}$ consists precisely of the regular values of j and thus is residual in $\mathcal{J}$ by the Sard-Smale theorem. It remains to prove that the vertical part of $d\mathcal{F}$:

$$D\mathcal{F}(u, J)(\xi, Y) = D_J(u)\xi + Y(u)\frac{\partial u}{\partial t}$$

is onto for any $(u, J) \in \mathcal{P} \times \mathcal{J}$. As in Section 2.5 it is enough to show that $D\mathcal{F}(u, J)$ has a dense range. Suppose then by contradiction that a non-zero vector field $\eta \in L^q(u)$, where $\frac{1}{q} + \frac{1}{p} = 1$, satisfies

$$D_J^*(u)\eta = 0$$

$$\int_{-\infty}^{+\infty} \int_0^1 \langle \eta, Y_t(u) \frac{\partial u}{\partial t} \rangle \, dt \, ds = 0 \tag{3.12}$$

Since $D_J^*(u)$ is an elliptic operator with smooth coefficients, vanishing of η in a non-empty open set yields, by the Aronszajn unique continuation theorem, $\eta \equiv 0$. Now, define

$$R(u) : = \{(s, t) \in \mathbb{R} \times (0, 1) : \ \frac{\partial u}{\partial s} \neq 0, \\ u(s, t) \neq x^{\pm} \text{ and } u(s, t) \notin u(\mathbb{R} - \{s\}, t)\}$$

A calculation in [SZ94] shows that if $\eta(s,t) \neq 0$ for some $(s,t) \in R(u)$ then there exists $Y \in T_J\mathcal{J}$ such that the integral (3.12) is positive. The subtlety of the argument lies in proving that $R(u)$ is non-empty. The proof based on the Carleman similarity principle is formulated in [FHS94] although in slightly different set-up, for pseudoholomorphic maps defined on the whole $\mathbb{R}^2$. Theorem 4.3 in that paper states that $R(u)$ is open and dense in $\mathbb{R}^2$. However, the argument has local character and so we may simply disregard the boundary points in our case.

Remark 3.3.4 *We assumed above that $H = 0$. It is easy to see that this does not affect generality of the proof: Let ϕ be an exact symplectomorphism and ϕ_t a Hamiltonian isotopy joining id and ϕ. There is a bijection*

$$\Phi^*\colon \Omega(L_0, L_1, \gamma_0) \to \Omega(L_0, \phi^{-1}(L_1), \phi_\cdot^{-1} \circ \gamma_0) \tag{3.13}$$

given by $\Phi(\gamma)(t) = \phi_t^{-1}(\gamma(t))$ and this gives the 1–1 correspondence between the trajectory spaces

$$\mathcal{M}_{J,H}(L_0, L_1) \cong \mathcal{M}_{\tilde{J},0}(L_0, \phi^{-1}(L_1))$$

where $\tilde{J}_t = \phi_t^* J_t$. Indeed, if $u\colon \mathbb{R} \times I \to M$ satisfies (3.9) and (3.10) then $\tilde{u}(s,t) := \phi_t^{-1}(u(s,t))$ defines a solution to

$$\frac{\partial \tilde{u}}{\partial s} + \tilde{J}_t(u)\frac{\partial \tilde{u}}{\partial t} = 0$$

with $\tilde{u}(s,0) \in L_0$ and $\tilde{u}(s,1) \in \phi^{-1}(L_1)$. Similarly, the correspondence $\xi \mapsto \tilde{\xi}$, $\tilde{\xi}(s,t) := d\phi_t^{-1}\xi(s,t)$ defines the isomorphisms $\Phi_L^*\colon L^p(u^*TM) \to L^p(\tilde{u}^*TM)$, $(\Phi_W^*\colon W^{1,p}_{L_0,L_1}(u^*TM) \to W^{1,p}_{L_0,\psi^{-1}L_1}(\tilde{u}^*TM))$ and

$$D_{\tilde{J},0}(\tilde{u}) \circ \Phi_W^* = \Phi_L^* \circ D_{J,H}(u).$$

This is because the Levi-Civita connection $\nabla^{\tilde{J}_t}$ associated with $g_{\tilde{J}_t}$ is equal to *the pullback $\phi_t^* \nabla^{J_t}$ and also because*

$$\bar{\partial}_{\tilde{J},0}(\tilde{u}) = \frac{\partial}{\partial s}\left(\phi_t^{-1} \circ u\right) + d\phi_t^{-1} \circ J \circ d\phi_t \left(\frac{\partial}{\partial t}\left(\phi_t^{-1} \circ u\right)\right) = d\phi_t^{-1}\left(\frac{\partial u}{\partial s} + J_t(u)\left(\frac{\partial u}{\partial t} - X_t(u)\right)\right) = \widetilde{\bar{\partial}_{J,H}(u)}$$

Hence the operators $D_{\tilde{J},0}(\tilde{u})$ and $D_{J,H}(u)$ have the same index and $(\tilde{J},0) \in (\mathcal{J} \times \mathcal{H})_{reg}(L_0, \phi^{-1}(L_1))$ iff $(J,H) \in (\mathcal{J} \times \mathcal{H})_{reg}(L_0, L_1)$ (more generally, a pair (J, H_1) corresponds to $(\tilde{J}, \tilde{H}_1)$ with $\tilde{H}_1 = (H - H_1) \circ \phi$).

Clearly, the map (3.13) preserves also the homomorphisms I_ω and I_μ

$$I_{\omega,L_0,\phi^{-1}(L_1)} \circ \Phi^* = I_{\omega,L_0,L_1} \quad \text{and} \quad I_{\mu,L_0,\phi^{-1}(L_1)} \circ \Phi^* = I_{\mu,L_0,L_1}$$

The index $\mu_u := \operatorname{ind} D_{J,H}(u)$ depends on the homotopy class of u and is equal to the Viterbo index. This is defined as follows ([Vit87, Flo88a]). In view of the remark above we may assume w.l.o.g. that $H = 0$ and let $u\colon I^2 \to M$ be a paths in $\Omega(L_0, L_1)$ joining two nondegenerate intersection points, $u(s,i) \in L_i$, $u(0,t) = x^-$, $u(1,t) = x^+$. We may choose a symplectic trivialisation $\Psi\colon u^*TM \to I^2 \times \mathbb{C}^n$ which is constant at the end points $u(i,t)$ and such that $\Psi(T_{x^\pm} L_1) = i\Psi(T_{x^\pm} L_0)$. Then the index m_u is defined as the Maslov class of the loop

$$\lambda(\tau) = \begin{cases} \Psi(T_{u(4\tau,0)}L_0) & \text{for } 0 \le \tau \le \frac{1}{4}, \\ -ie^{2\pi i\tau}\Psi(T_{x^+}L_0) & \text{for } \frac{1}{4} \le \tau \le \frac{1}{2}, \\ \Psi(T_{u(3-4\tau,1)}L_1) & \text{for } \frac{1}{2} \le \tau \le \frac{3}{4}, \\ ie^{2\pi i\tau}\Psi(T_{x^-}L_1) & \text{for } \frac{3}{4} \le \tau \le 1. \end{cases}$$

In [Flo88a] Floer proves that these indices coincide. More precisely, let $\beta\colon \mathbb{R} \to I$ be a cutoff function such that $\beta((-\infty,0]) = 0$ and $\beta([1,+\infty)) = 1$ and let $u \in \mathcal{M}_{J,0}(L_0, L_1)$. Then

$$m_{u'} = \mu_u$$

where $u'\colon I^2 \to M$ is chosen so that u and $u'(\beta(s),t)$ are homotopic in $\Omega(L_0, L_1)$ with the end points fixed. There is a straightforward relation between the Viterbo index and I_μ:

Lemma 3.3.5 *If $u_1, u_2\colon I^2 \to M$ connect two intersection points x^-, x^+ then*

$$I_\mu(u_1 * (-u_2)) = m_{u_1} - m_{u_2}$$

Here we use the notation

$$(u_1 * (-u_2))(s,t) = \begin{cases} u_1(2s,t) & s \leq \frac{1}{2} \\ u_2(1-2s,t) & s \geq \frac{1}{2} \end{cases}$$

This results together with Remark 3.3.4 implies

Corollary 3.3.6 *If the triple (M, L_0, L_1) is monotone and H is regular then for any $u_1, u_2 \in \mathcal{M}_{J,H}(L_0, L_1)$*

$$\ell^2(u_1) = \ell^2(u_2) \Leftrightarrow \mu_{u_1} = \mu_{u_2}.$$

In order to construct the boundary operator we need a compactness property for 1- and 2-dimensional parts of $\mathcal{M}_{J,H}(L_0, L_1)$ similar as in Proposition 2.2.1. We say that a sequence $\{u_\nu\}_{\nu=1}^\infty \subset C^\infty(\mathbb{R} \times I, M)$ converges to a split trajectory $(u^1, \ldots, u^m)$ if there are collections of critical points $x^+ = x_0, x_1, \ldots x_m = x^-$ of $\hat{\omega}_H$, trajectories $u^k \in \mathcal{M}_{J,H}(x_k, x_{k-1})$ and sequences of time shifts $\{s_\nu^k\}_{\nu=1}^\infty$, $k = 1, \ldots, m$ such that $u_\nu(\cdot + s_\nu^k, \cdot)$ converges with its all derivatives on compact sets to u^k.

The following propositions summarize the compactness results. For the proof we refer to [Flo88b, Flo89a, Oh93a]. Let $u_\nu \in \mathcal{M}_{J_\nu, H_\nu}(L_0, L_1)$ be a sequence of trajectories with (J_ν, H_ν) converging to some (J, H) in C^1.

Proposition 3.3.7 *Assume that $\omega_{|\pi_2(M,L_\nu)} = 0$ and the energy of u_ν is bounded*

$$\ell^2(u_\nu) < C$$

Then a subsequence of u_ν converges to a split trajectory $\underline{u} \subset \mathcal{M}_{J,H}(L_0, L_1)$ and

$$\sum \ell^2(u^i) \leq \limsup \ell^2(u_\nu)$$

Proposition 3.3.8 *Assume that the triple (M, L_0, L_1) is monotone, $L_0 \pitchfork L_1$ and $\Sigma_i \geq 3$, for $i = 0, 1$. Let $u_\nu \in \mathcal{M}_{J_\nu, H_\nu}(L_0, L_1)$ be a sequence of trajectories of index $\mu_{u_\nu} = 1$ or 2. Then a subsequence of u_ν converges to a split trajectory $\underline{u} \in \mathcal{M}_{J,H}(L_0, L_1)$.*

Lemma 3.3.9 *If $\ell^2(u_\nu) \to 0$ then u_ν converges uniformly to a constant map.*

In the statements above we assume the energy bound for u_ν that is a bound in $W^{1,2}$ topology. If the derivatives of u_ν are bounded in L^∞ then they are bounded in L^p for $p > 2$ and the elliptic bootstrapping argument yields a subsequence convergent in C^∞_{loc}. Otherwise, a non-constant pseudoholomorphic disc or sphere would bubble off. Such disc must have positive energy, which is impossible under the assumptions of Proposition 3.3.7 or Lemma 3.3.9. If the intersection is transverse, then the index of the disc is well defined and cannot exceed the index of u_ν. This guarantees convergence in Proposition 3.3.8 (recall that Σ_i denote the generators of $\operatorname{im} I_{\mu|\pi_2(M,L_i)}$).

The Floer's complex $C_*(L_0, L_1, \gamma_0, J, H)$ is defined for a regular pair (J, H). It is generated as a $\mathbb{Z}_2$ module by the critical points of a_H and the boundary operator $\partial \colon C_* \to C_*$ is given by

$$\partial x = \sum \langle y, \partial x \rangle y \tag{3.14}$$

where $\langle y, \partial x \rangle$ is the number modulo 2 of one-dimensional components of $\mathcal{M}_{J,H}(x, y)$. Propositions 3.3.7 and 3.3.8 imply that the number of these components is finite and thus (3.14) is well defined. Then compactness of the 2-dimensional parts together with the gluing argument yields $\partial \circ \partial = 0$ as in the finite dimensional case. The following theorem is the central result of Floer's papers [Flo88a, Flo88b, Flo89b]. Details in the monotone case have been carried out in [Oh93a].

Theorem 3.3.10 *Assume that* $I_\omega = 0$ *or that the triple* (M, L_0, L_1, γ_0) *is monotone with* $\Sigma_i \geq 3$. *Then the boundary operator is well defined and* $\partial \circ \partial = 0$. *For any two regular pairs* (J^0, H^0) *and* (J^1, H^1) *there is a natural isomorphism between the resulting homology groups*

$$HF_*(L_0, L_1, \gamma_0, J^0, H^0) \cong HF_*(L_0, L_1, \gamma_0, J^1, H^1). \tag{3.15}$$

We denote these groups by $HF_*(L_0, L_1, \gamma_0)$.

Remark 3.3.11 *a) Obviously the Floer complex is symplectically invariant i.e. for any symplectomorphism* ϕ

$$C_*(L_0, L_1, \gamma_0, J, H) \cong C_*(\phi(L_0), \phi(L_1), \phi \circ \gamma_0, \phi_* J, H \circ \phi^{-1}).$$

b) The map (3.13) yields the isomorphism between the Floer complexes

$$C_*(L_0, L_1, \gamma_0, J, H) \cong C_*(L_0, \phi^{-1}(L_1), \phi_\cdot^{-1} \circ \gamma_0, \phi_t^* J_t, 0)$$

for any regular pair (J, H). *Consequently, we obtain an isomorphism*

$$HF_*(L_0, L_1) \cong HF_*(L_0, \phi^{-1}(L_1)).$$

However, unlike the isomorphism in (3.15), this is not naturally defined, because it depends a priori *on the homotopy class of the Hamiltonian isotopy* ϕ_t.

3.3.4 Floer homology of an isolated invariant set

We can define Floer homology in slightly more general setting. Let $\mathcal{U} \subset \Omega(L_0, L_1)$ be a closed subset and assume that $\mathcal{U}$ is *bounded* i.e. the evaluation map $\mathcal{U} \times I \ni (\gamma, t) \mapsto \gamma(t) \in M$ has precompact image. Let $\mathcal{M}_{J,H}(L_0, L_1, \mathcal{U})$ denote the set of such $u \in \mathcal{M}_{J,H}(L_0, L_1)$ that $u(s, \cdot) \in \mathcal{U}$ for all $s \in \mathbb{R}$. Then the maximal invariant subset $\mathcal{S}_{J,H}(\mathcal{U})$ of $\mathcal{U}$ is defined to be the image of $\mathcal{M}_{J,H}(L_0, L_1, \mathcal{U})$ under the evaluation map

$$ev\colon \mathbb{R} \times \mathcal{M}_{J,H}(L_0, L_1) \to \Omega(L_0, L_1), \quad ev(s,u)(t) = u(s,t).$$

We say that $\mathcal{S}_{J,H}(\mathcal{U})$ is *isolated* if its closure is contained in the interior of $\mathcal{U}$ (cf (2.30)).

If $\mathcal{S}_{J,H}(\mathcal{U})$ is isolated and $u_\nu \in \mathcal{M}_{J,H}(L_0, L_1, \mathcal{U})$ converges to some $u \in \mathcal{M}_{J,H}(L_0, L_1)$ then $u \in \mathcal{M}_{J,H}(L_0, L_1, \mathcal{U})$. Therefore we can build the local Floer homology $H_*(C_*(L_0, L_1, \mathcal{U}, J, H))$ out of all critical points of $\hat{\omega}_H$ in $\mathcal{U}$ and the connecting orbits in $\mathcal{M}_{J,H}(L_0, L_1, \mathcal{U})$. The following proposition is the analogue of [Flo89a, Proposition 1.2] in the Lagrangian intersection setting.

Proposition 3.3.12 *Assume that $\mathcal{U}$ is bounded, $\mathcal{S}_{J,H}(\mathcal{U})$ is isolated and the symplectic action a_H is defined on $\mathcal{U}$. There is an $\varepsilon > 0$ such that if $\|J' - J\|_{C^1} < \varepsilon$ and $\|H' - H\|_{C^1} < \varepsilon$ then $\mathcal{S}_{J',H'}(\mathcal{U})$ is isolated and if moreover the pairs (J, H), (J', H') are regular then*

$$H_*(C_*(L_0, L_1, \mathcal{U}, J', H')) \cong H_*(C_*(L_0, L_1, \mathcal{U}, J, H)).$$

In this context we say that $\mathcal{S}_{J',H'}(\mathcal{U})$ is a *continuation* of $\mathcal{S}_{J,H}(\mathcal{U})$. This is the general abstract situation. Lemma 3.4.12 suffices for our purpose.

3.4 Clean intersection

The aim of this section is to prove that the local Floer homology in a neighbourhood of a Lagrangian intersection is isomorphic to its singular homology provided that the intersection is appropriately regular. We say that two submanifolds L_0, L_1 of M have a *clean intersection along a manifold* N iff $N \subset L_0 \cap L_1$ and $T_xN = T_xL_0 \cap T_xL_1$ for $x \in N$. A special case where $L_0 = L_1 = N$ has been considered by Floer who proved the existence of the isomorphism in [Flo88a]. The results of this section should be seen as a generalization of Floer's theorem.

We sketch the main points of the proof. If two Lagrangian submanifolds L_0, L_1 of M intersect cleanly along a compact manifold N then, seen as a subset of the pathspace, $N \subset \Omega(L_0, L_1)$ is an isolated invariant set for the unperturbed symplectic action. This is because N is a strong deformation retract of its neighbourhood $\mathcal{U}$ in $\Omega(L_0, L_1)$ and therefore $\hat{\omega}$ is exact in $\mathcal{U}$. Moreover the critical points of the action functional a in $\mathcal{U}$ are the constant paths at the points of N and if N is connected they lie in the same level set of a. Since a decreases along its gradient lines there are no connecting orbits in $\mathcal{U}$ except for the constant ones and $\mathcal{U}$ isolates N. The critical points in N are degenerated, in fact the kernel of the Hessian of a at x is exactly T_xN. We must then, by adding a Hamiltonian term, perturb the functional in the direction of N only. In other words the Hamiltonian H may be constant in the direction orthogonal to N. But then the gradient of H is tangent to N and the critical points and the gradient lines of $H_{|N}$ coincide with those of a_H which are t–independent. The crucial point of the proof is that they are the only ones in $\mathcal{U}$ and therefore the Floer complex coincides with the Morse complex of H_N.

The section begins with the description of a normal form neighbourhood of a clean Lagrangian intersection. In this form we then construct the appropriate Hamiltonian H which, however, is not compactly supported. Finally, putting these results together we formulate and prove the main theorem of this section — Theorem 3.4.11.

3.4.1 The normal form of a clean Lagrangian intersection

Let L be a manifold and N a submanifold of L. Let $TN^{annih} \subset T^*L$ be a subbundle of $T^*L_{|N}$ annihilating TN:

$$TN^{annih} := \{\alpha \in T^*L_{|N} : \alpha_{|TN} = 0\}$$

This is a Lagrangian submanifold of (T^*L, ω_{T^*L}) since $\dim TN^{annih} = \dim L$ and the canonical 1-form λ_{T^*L} vanishes on TN^{annih}. Clearly $N = L \cap TN^{annih}$ is a clean intersection, since $T_x(TN^{annih}) = T_xN \oplus T_xN^{annih}$ for $x \in N$. We prove that every clean Lagrangian intersection can be put in this form. Thus the following proposition may be seen as a generalization of the Lagrangian neighbourhood theorem.

Proposition 3.4.1 *Let (M, ω) be a symplectic manifold and L_0, L_1 two Lagrangian submanifolds of M which intersect cleanly along a compact manifold N. There exist a vector bundle $\tau\colon L \to N$, a neighbourhood V_0 of N in T^*L, a neighbourhood U_0 of N in M and a symplectomorphism $\phi\colon U_0 \to V_0$ such that*

$$\phi(L_0 \cap U_0) = L \cap V_0 \quad \textit{and} \quad \phi(L_1 \cap U_0) = TN^{annih} \cap V_0. \tag{3.16}$$

Proof. By the Lagrangian neighbourhood theorem there is a symplectomorphism $\chi_0\colon (W, \omega_{T^*L_0}) \to (M, \omega)$ defined on a neighbourhood W of the zero section in T^*L_0 such that $\chi_{0|L_0} = id$. Fix any metric on L_0 and take $L = TN^\perp \subset TL_0$ to be the orthogonal complement of TN in TL_0. Then the exponential map is a diffeomorphism between neighbourhoods of N in L_0 and in L, which induces a natural symplectomorphism of the cotangent bundles (restricted to these neighbourhoods). Therefore we may assume w.l.o.g. that $L_0 = L$ and $M = T^*L$.

Denote $L_2 = TN^{annih} \subset T^*L$. The proof of the proposition is based on the observation that both L_1 and L_2 are transversal to the "complement" of N in T^*L and therefore L_1 can be seen as a graph over L_2. More precisely, we shall prove

Lemma 3.4.2 *There exists a symplectomorphism $\chi_1 : U_1 \to V_1$ where $U_1 \subset T^*L$, $V_1 \subset T^*L_2$ are neighbourhoods of N, such that*

$$\chi_{1|L_2 \cap U_1} = id, \qquad \chi_1(L \cap U_1) \subset T^*L_{2|N} \quad \mathit{and} \quad \chi_1(L_1 \cap U_1) = graph\, \alpha$$

where α is a 1-form on L_2.

Then α is closed as L_1 is Lagrangian and moreover $N \subset graph\, \alpha$ or in other words $\alpha_{|N} = 0$. Hence the map

$$\psi_\alpha : T_x^*L_2 \ni \beta \mapsto \beta - \alpha(x) \in T_x^*L_2, \qquad x \in L_2 \cap U_1$$

is a symplectomorphism. We have $\psi_\alpha(graph\, \alpha) \subset L_2$ and ψ_α restricted to $T^*L_{2|N}$ is the identity. In particular $\psi_\alpha(V_2) \subset \chi_1(U_1)$ for a sufficiently small neighbourhood V_2 of N in T^*L_2. Now, a symplectomorphism

$$\phi(x) = \chi_1^{-1} \psi_\alpha \chi_1(x)$$

is well defined and clearly satisfies (3.16) if we take $U_0 = \chi_1^{-1}(V_2)$. □

Proof of Lemma 3.4.2. A submanifold of T^*L_2 is (locally) a graph of a 1-form if it has the dimension of L_2 and is transversal to the fibres $T_x^*L_2$, $x \in L_2$. Since transversality is an open condition it is enough to show that $\chi_1(L_1) \pitchfork_x T_x^*L_2$ for $x \in N$.

Let $E = \ker d\tau$ be the vertical subbundle of TL and E^{annih} the subbundle of T^*L annihilating E. For $x \in N$ consider the following decomposition

$$T_x(T^*L) = T_xL \oplus T_x^*L = E_x \oplus T_xN \oplus TN_x^{annih} \oplus E_x^{annih}.$$

The clean intersection $L_1 \cap L$ implies that $T_xL_1 \cap E_x = \{0\}$. The first three summands of the above decomposition form the symplectic complement $T_xN^\perp$ of T_xN. On the other hand $T_xL_1 \subset T_xN^\perp$ and so $T_xL_1 \cap E_x^{annih} = \{0\}$ as well. Now, $E_x \oplus E_x^{annih}$ is the tangent space at x to the restricted bundle $E_{|L_x}^{annih}$ (seen as a submanifold of T^*L). Thus L_1 is transversal to $E_{|L_x}^{annih}$ at x. Hence it is enough to construct a symplectomorphism χ_1 in such way that

$$\chi_1(E_{|L_x}^{annih} \cap U_1) \subset T_x^*L_2.$$

Note that the last condition implies also that $\chi_1(L) \subset T^*L_{2|N}$. Now the existence of the required symplectomorphism χ_1 follows immediately from the two lemmas below.

Lemma 3.4.3 *There is a vector bundle $\sigma \colon T^*L \to TN^{annih}$, whose fibers are Lagrangian submanifolds of T^*L. In particular, $\sigma^{-1}(x) = E_{|L_x}^{annih}$ for $x \in N$.*

Proof. Firstly, we have the restriction map $T^*L \to E^*$. Secondly, E is isomorphic to the pullback bundle τ^*L, which gives the isomorphism of the dual bundles, $E^* \cong \tau^*L^*$. Thirdly, $TL_{|N} = L \oplus TN$ and the restriction

$$TN^{annih} \ni \alpha \to \alpha_{|L} \in L^*$$

is the bundle isomorphism. Putting all these facts together, we obtain a vector bundle map σ

$$\sigma \colon T^*L \to E^* \cong \tau^*L^* \to L^* \cong TN^{annih}.$$

Clearly, $\sigma^{-1}(x) = E_{|L_x}^{annih}$ if $x \in N$. To see that σ carries a vector bundle structure consider the local coordinates

$$\psi \colon W \times \mathbb{R}^k \ni (q, q') = (q_1, \ldots, q_n, q'_1, \ldots, q'_k) \mapsto \psi(q, q') \in L, \quad W \subset \mathbb{R}^n$$

induced by a local trivialization of $\tau \colon L \to N$, that is $\psi(q, 0) \in N$, $L_{\psi(q,0)} = \psi(\{q\} \times \mathbb{R}^k)$ and if $\bar{\psi} \colon \overline{W} \times \mathbb{R}^k \to L$ is another chart of this type then the change of the coordinates takes form

$$\bar{\psi}^{-1}\psi(q,q') = (A(q), B(q)q') = (A_1(q), \dots, A_n(q), B_i^1(q)q'_i, \dots, B_i^k(q)q'_i)$$

where $A\colon W \cap \psi^{-1}\bar{\psi}(\overline{W}) \to \mathbb{R}^n$ is a diffeomorphism and $B\colon W \cap \psi^{-1}\bar{\psi}(\overline{W}) \to GL(\mathbb{R}^k)$. Let

$$\Psi\colon W \times \mathbb{R}^k \times \mathbb{R}^{n+k} \ni (q,q',p,p') \mapsto \Psi(q,q',p,p') \in T^*L \tag{3.17}$$

denote the natural coordinates on T^*L induced by ψ. In these coordinates the map σ becomes the projection

$$\sigma_\Psi = \Psi^{-1}\sigma\Psi\colon (q,q',p,p') \mapsto (q,0,0,p')$$

and this induces the vector structure in the fibre

$$\sigma^{-1}(\Psi(q,0,0,p')) \cong \{q\} \times \mathbb{R}^k \times \mathbb{R}^n \times \{p'\}.$$

This is because the change of the coordinates $\Theta = \bar{\Psi}^{-1} \circ \Psi$ is given by

$$\begin{aligned}\Theta(q,q',p,p') &= \left(\bar{\psi}^{-1}\psi(q,q'), (p,p')d_{(q,q')}(\bar{\psi}^{-1}\psi)^{-1}\right) = \\ &= \left(A(q), B(q)q', pC(q)^{-1} - D(q,p')q', p'B(q)^{-1}\right)\end{aligned}$$

where $C\colon W \to GL(\mathbb{R}^n)$, $D\colon W \times \mathbb{R}^k \to L(\mathbb{R}^k, \mathbb{R}^n)$,

$$C(q)^i_j = \frac{\partial A_i}{\partial q_j} \quad \text{and} \quad D(q,p')^i_j = p'_k \left(B(q)^{-1}\right)^k_l \frac{\partial B(q)^i_l}{\partial q_m} \left(C(q)^{-1}\right)^m_j.$$

Thus Θ restricted to $\{q\}\times\mathbb{R}^k\times\mathbb{R}^n\times\{p'\}$ is linear and so the atlas consisting of the charts as above yields the structure of a vector bundle on $\sigma\colon T^*L \to L_2$. Moreover, from the local description of σ follows clearly that a fibre $\sigma^{-1}(x)$ is a Lagrangian submanifold of T^*L. □

Lemma 3.4.4 *Let $\sigma\colon V \to L$ be a vector bundle such that (V, ω_0) is a symplectic manifold and the zero section $L \subset V$ and the fibres V_x are Lagrangian.*

Then for every compact subset $K \in L$ there is a fibre preserving symplectomorphism χ defined in a neighbourhood U of K in V

$$\chi\colon (U,\omega_0) \to (T^*L, \omega_{T^*L}), \quad \pi_L \circ \chi = \sigma_{|U} \quad \textit{and} \quad \chi_{|L} = id.$$

Proof. This is a special case of [Wei71, Theorem 7.1]. Define a vector bundle isomorphism

$$\eta\colon V \to V \oplus TL = TV_{|L} \xrightarrow{-i\omega} TV^*_{|L} = (V \oplus TL)^* \to T^*L$$

that is $\eta(v) = -i_v\omega_{|TL}$. Now, $\omega_{0|L} = \eta^*\omega_{T^*L|L}$ and so ω_0 and $\omega_1 = \eta^*\omega_{T^*L}$ are diffeomorphic in some neighbourhood of K. Moreover, a close look at the Moser's proof reveals that this diffeomorphism is fibre preserving. For the sake of the completeness we recall the argument. Let $\theta = \omega_1 - \omega_0$. Then $d\theta = 0$ and θ vanishes on $TV_{|L}$. Moreover $\theta_{|V_x} = 0$ as the fibres V_x are Lagrangian w.r.t. both ω_0 and ω_1. Then θ is exact in a neighbourhood of K, $\theta = d\rho$ and one defines the vector field X_t by

$$i_{X_t}\omega_t = -\rho, \text{ where } \omega_t = \omega_0 + t\theta.$$

If ϕ_t is the family of diffeomorphisms generated by X_t then

$$\frac{d}{dt}\phi_t^*\omega_t = \phi_t^*(L_{X_t}\omega_t + \theta) = \phi_t^*(di_{X_t}\omega_t + d\rho) = 0$$

and so $\phi_1^*\omega_1 = \omega_0$. The diffeomorphism ϕ_1 is fibre preserving if only X_t is tangent to the fibres. As fibres are Lagrangian w.r.t. ω_t this is equivalent to $\rho_{|V_x} = 0$. Define a smooth map $\psi\colon \mathbb{R} \times V \to V$, $\psi(t,w) = tw$ and the family of 1-forms $\rho_t\colon TV \to \mathbb{R}$

$$\rho_t(y)w = \theta(\psi(t,w))\left(\frac{\partial\psi}{\partial t}(t,y), d\psi_t(y)w\right)$$

Then $\rho = \int_0^1 \rho_t$. But $\theta_{|V_x} = 0$ and $\frac{\partial\psi}{\partial t}$ is tangent to the fibres and so $\rho_{t|V_x} = 0$ which completes the proof of the lemma and of the proposition. □

3.4.2 The t–independence of the gradient lines of a_H in the normal form

Let N be a compact, connected manifold and $\tau\colon L \to N$ a vector bundle over N. Recall that $\pi\colon T^*L \to L$ denotes the projection.

For any function $f\colon N \to \mathbb{R}$ we define its extension to T^*L by composing it with the projections

$$f_L := f \circ \tau\colon L \to \mathbb{R} \qquad H_f := f_L \circ \pi\colon T^*L \to \mathbb{R}$$

Let g_N be a metric on N. Our next step is to extend g_N to T^*L. Consider a metric g on the manifold L such that

- $g_{|N} = g_N$,
- the fibres $L_x = \tau^{-1}(x)$ are orthogonal to N,
- N is a totally geodesic submanifold of L.

Note that the second condition implies that

$$\nabla^g f_L(x) = \nabla^{g_N} f(x)$$

for $x \in N$. The Levi-Civita connection associated with g provides the splitting of $T_\xi(T^*L)$ into horizontal and vertical subspaces $T_\xi(T^*L) = H_\xi \oplus V_\xi$ and $V_\xi = T_\xi(T^*_{\pi(\xi)}L)$ is canonically isomorphic to $T^*_{\pi(\xi)}L$ while $d\pi$ gives the isomorphism $d_\xi\pi_{|H_\xi}\colon H_\xi \to T_{\pi(\xi)}L$. These give the "diagonal" lift of g to a metric g^D on T^*L (the Kaluza–Klein metric) such that $H_\xi \perp V_\xi$ and the above isomorphisms become isometries. In the canonical coordinates (q,p) we have

$$g^D(\partial_{q_i}, \partial_{q_j}) = g_{ij} + g^{kl}\Gamma^k_{ir}\Gamma^l_{js}p_r p_s$$

$$g^D(\partial_{p_i}, \partial_{q_j}) = g^{ik}\Gamma^k_{jl}p_l$$

$$g^D(\partial_{p_i}, \partial_{p_j}) = g^{ij}.$$

Here Γ are the Christoffel symbols in the cotangent bundle. In particular, if $x \in L$ then $T(T^*L) = TL \oplus T^*L$ and $g^D(x) = g(x) \oplus g^*(x)$. The metric g^D is compatible with the canonical symplectic structure and the induced almost complex structure $J = J_g$ maps a horizontal vector $w \in H_\xi$ to the vertical lift of $i_{d_\xi \pi w} g \in T^*_{\pi(\xi)}L$. Thus in the canonical coordinates

$$J = \begin{pmatrix} -A & -I \\ I + A^2 & A \end{pmatrix}$$

where $A_i^k = \Gamma_{ij}^k p_j$ is the connection matrix.

Lemma 3.4.5 *The action functional is well defined in $\Omega(T^*L, L, TN^{annih})$. Moreover*

$$|a_\omega(\gamma) - a_\omega(x)| \leq \|\dot{\gamma}\|^2_{L^2}$$

for any $x \in N$ and $\gamma \in \Omega$.

Proof. The symplectic form ω is exact, $\omega = d\lambda$ and $\lambda_{|L} = \lambda_{|TN^{annih}} = 0$. Therefore, if $u\colon I \times S^1 \to T^*L$, $u_0(s) = u(0,s) \in L$, $u_1(s) \in TN^{annih}$ represents a loop in Ω then

$$[\hat{\omega}][u] = \int u^*\omega = \int u_1^*\lambda - \int u_0^*\lambda = 0$$

and $\hat{\omega}$ is exact. We can also assume that $a_\omega(\gamma) = 0$ if $\gamma \subset L$. Let $\gamma \in \Omega$. In the canonical coordinates $\gamma = (q,p)$ with $p(0) = 0$ and $p(1) \in T_{q(1)}N^{annih}$. Define $u\colon I \times I \to T^*L$, $u(s,t) = (q(t), sp(t))$. Then

$$a_\omega(\gamma) = \int_{I^2} u^*\omega = \int_{\partial I^2} u^*\lambda = \int_0^1 p\dot{q}\,dt = \langle p, \dot{q}\rangle_{L^2}$$

Now $|\dot{\gamma}^H| = |\dot{q}|$ and $|\dot{\gamma}^V| = |\nabla p|$ and $|p|_{L^2} \leq 2|\nabla p|_{L^2}$ since $p(0) = 0$ and so

$$|a_\omega(\gamma)| \leq |p|_{L^2}|\dot{q}|_{L^2} \leq 2|\nabla p|_{L^2}|\dot{q}|_{L^2} \leq |\dot{\gamma}^V|^2_{L^2} + |\dot{\gamma}^H|^2_{L^2} = |\dot{\gamma}|^2_{L^2}. \square$$

The main result of this section is the following proposition which is a generalization of Floer's [Flo88a, Theorem 2]

Proposition 3.4.6 *Let (N, g_N) be a compact, Riemannian manifold, $\tau: L \to N$ a vector bundle over N and $f: N \to \mathbb{R}$ a C^2 function on N. Let $J = J_g$ and $H = H_f$ be constructed as above and suppose that there is a neighbourhood U of N in L such that*

$$\|\nabla^g df_L(x)\| \leq 1 \tag{3.18}$$

for $x \in U$. Then the following holds

(a) all critical points and gradient lines (w.r.t. J) of the action functional a_H in $\Omega(\pi^{-1}(U), U, TN^{annih})$ are t–independent and so they are in 1—1 correspondence with the critical points and the gradient lines of f (w.r.t. g_N),

(b) the critical points of a_H are nondegenerate if f is a Morse function. In this case if $x^-, x^+ \in Crit\, f$ and $u: \mathbb{R} \to N$ is a t-independent element of $\mathcal{P}(x^-, x^+)$ then the linearized operator $D_{J,H}(u)$ is onto iff the operator $D_f(u): W^{1,p}(u^*TN) \to L^p(u^*TN)$

$$D_f(u)\xi = \nabla_s \xi + \nabla_\xi \nabla f(u)$$

is onto and the assignment $\xi \mapsto \xi'(s,t) = \xi(s)$ gives the isomorphism $\ker D_f(u) \cong \ker D_{J,H}(u)$.

Proof. Consider the local coordinates (q, q', p, p') as in (3.17). Then

$$\frac{\partial H}{\partial p} = \frac{\partial H}{\partial p'} = \frac{\partial H}{\partial q'} = 0 \text{ and } X_H = (0, 0, df(q), 0)$$

and the Hamiltonian flow associated with H takes form $\phi_t(q, q', p, p') = (q, q', p + tdf(q), p')$. If $x = (q, q', p, p') \in L$ then $p = p' = 0$. If $\phi_1(x) \in TN^{annih}$ then $q' = 0$ and $p + df(q) = 0$. Thus the only solutions to (3.8) with the boundary conditions $x(0) \in L$, $x(1) \in TN^{annih}$ are the constant paths

$x(t) = (q, 0, 0, 0)$ where q is a critical point of f. Furthermore, we have in these coordinates

$$D\phi_1(x) = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ D^2 f(q) & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

where the Hessian $D^2 f(q)$ is nondegenerate. Let $v = (Q, Q', 0, 0)$ be a vector tangent to L. The vector $D\phi_1(x)v = (Q, Q', D^2 f(q)Q, 0)$ is tangent to TN^{annih} iff $Q' = D^2 f(q)Q = 0$. But then also $Q = 0$ which shows that $D\phi_1(x)(T_x L) \cap T_x TN^{annih} = \{0\}$ i.e. x is nondegenerate.

Remark 3.4.7 *Obviously, the established correspondence implies that the $\mathbb{Z}_2$ intersection number $\#_2(L_0, L_1)$ is equal to the Euler characteristic $\chi(N, \mathbb{Z}_2)$ of N. It is not difficult to see that this is true generally, in the differential topology category and also, up to an appropriate sign, over $\mathbb{Z}$.*

Since $dH = df_L \circ d\pi$ vanishes on the vertical subbundle $V \subset T(T^*L)$, the gradient of H w.r.t. the metric g^D belongs to the horizontal subspace, $\nabla H(\xi) \in H_\xi$. Furthermore $d_\xi \pi_{|H_\xi}$ is an isometry and hence $d_\xi \pi(\nabla H(\xi)) = \nabla H(\pi(\xi)) = \nabla^g f_L(\pi(\xi))$, that is ∇H is the horizontal lift of $\nabla^g f_L$. In particular, if $x \in N$ then

$$\nabla H(x) = \nabla^{g_N} f(x) \in TN.$$

Therefore if $u\colon \mathbb{R} \to N$ is a gradient line of f then $u(s,t) := u(s)$ is a solution to (3.9) satisfying the appropriate boundary conditions.

Conversely, let $u\colon \mathbb{R} \times I \to U$ be a solution to (3.9),(3.10). We want to show that $\frac{\partial u}{\partial t} \equiv 0$. Let $x(s,t)$ and $y(s,t)$ be the horizontal and the vertical components of $u(s,t)$. Strictly speaking $x = \pi \circ u$ and y is a section of T^*L along x i.e. $y \in \Gamma(x^* T^* L)$. Then $\frac{\partial u}{\partial s}$ (and similarly $\frac{\partial u}{\partial t}$) decomposes into the horizontal and vertical parts, where $(\frac{\partial u}{\partial s})^H$ is the horizontal lift of $\frac{\partial x}{\partial s}$ and

$(\frac{\partial u}{\partial s})^V$ is the vertical lift of $\nabla_s y \in T_x^* L$. Thus we may write the equation (3.9) in the form

$$\frac{\partial x^*}{\partial s} - \nabla_t y + dH_1(x) = 0$$
$$\nabla_s y + \frac{\partial x^*}{\partial t} = 0$$

where $v^* = g(v, \cdot) = J(v)$ for $v \in TL$. The boundary conditions are

$$y(s,0) = 0, \quad x(s,1) \in N, \quad y(s,1) \in T_{x(s,1)} N^{annih},$$
$$\lim_{s \to \pm\infty} y(s,t) = 0 \quad \text{and} \quad \lim_{s \to \pm\infty} x(s,t) = x^{\pm}$$

for some critical points x^+, x^- of f. We have to show that $y \equiv 0$. Then $\frac{\partial x}{\partial t} = 0$ and $x \colon \mathbb{R} \to N$ is a gradient line of f. This is in principle an application of Green's identity since y is a solution of the elliptic equation

$$\Delta y - \langle \nabla df_L, \nabla_s y \rangle = 0.$$

Define

$$\gamma(s) = \frac{1}{2} \int_0^1 |y(s,t)|^2 \, dt = \frac{1}{2} \|y(s)\|_{L^2}^2$$

Since $lim_{s \to \pm\infty} \gamma(s) = 0$ and γ is nonnegative it attains a maximum in $\mathbb{R}$. On the other hand the following lemma shows that γ is convex and so $\gamma \equiv 0$.

Lemma 3.4.8 *If* $\|\nabla^g f_L\|_{L^\infty(U)} < 1$ *then* $\gamma''(s) \geq 0$ *for all* $s \in \mathbb{R}$.

Proof. (Cf. [Flo88a, Lemma 5.2]) As the Levi-Civita connection is torsion free $\nabla_s \frac{\partial x}{\partial t} = \nabla_t \frac{\partial x}{\partial s}$ and we may compute

$$\frac{\partial^2}{\partial s^2} \frac{1}{2} |y(s,t)|^2 = |\nabla_s y|^2 + \langle \nabla_s^2 y, y \rangle = |\nabla_s y|^2 - \left\langle \nabla_t \frac{\partial x}{\partial s}, y \right\rangle$$
$$= |\nabla_s y|^2 - \langle \nabla_t^2 y, y \rangle + \langle \nabla_{\frac{\partial x}{\partial t}} df_L, y \rangle$$

Integrating by parts the second term yields

$$\int_0^1 \langle \nabla_t^2 y, y \rangle \, dt = \langle \nabla_t y(s,1), y(s,1) \rangle - \langle \nabla_t y(s,0), y(s,0) \rangle - \int_0^1 |\nabla_t y|^2 \, dt.$$

Now, $y(s,0) = 0$ and $y(s,1) \in T_{x(s,1)}N^{annih}$. On the other hand $\frac{\partial x}{\partial s}(s,1) \in T_xN$ and $\nabla f_L(x(s,1)) \in T_xN$. We have then

$$\langle \nabla_t y(s,1), y(s,1)\rangle = \left\langle y(s,1), \frac{\partial x}{\partial s}^*(s,1) + df_L(x)\right\rangle$$
$$= y(s,1)\left(\frac{\partial x}{\partial s}(s,1) + \nabla f_L(x)\right) = 0$$

for all $s \in \mathbb{R}$. Consequently

$$\gamma''(s) = \|\nabla_s y(s)\|^2_{L^2} + \|\nabla_t y(s)\|^2_{L^2} - \int_0^1 \langle \nabla_{\nabla_s y} df_L(x), y\rangle \, dt$$
$$\geq \|\nabla_s y(s)\|^2_{L^2} + \|\nabla_t y(s)\|^2_{L^2} - \|\nabla df_L\|_{L^\infty}\|\nabla_s y\|_{L^2}\|y\|_{L^2}.$$

Since $y(s,0) = 0$ the Poincaré inequality gives $\|y(s)\|_{L^2} \leq 2\|\nabla_t y(s)\|_{L^2}$ and therefore $\gamma'' \geq 0$ if only $\|\nabla df_L\|_{L^\infty} = \sup_{x\in U}|\nabla df_L(x)| < 1$. □

The isomorphism $\ker D_f(u) \cong \ker D_{J,H}(u)$ can be proved in the similar manner. A vector field $\xi \in \Gamma(u^*T(T^*L))$ decomposes into

$$\xi = (\zeta, \eta) \in T_uL \oplus T_u^*L.$$

Since $\xi(s,t) \in T_{u(s)}(T^*L)$ for all $t \in [0,1]$ we have $\nabla_t \xi = \frac{\partial \xi}{\partial t}$. Furthermore, direct computation of the Christoffel symbols of the Levi-Civita connection associated with g^D yields

$$\nabla_{\partial_{q_i}}\partial_{q_j}(q,0) = \nabla^g_{\partial_{q_i}}\partial_{q_j}(q) \in T_qL \quad \text{and} \quad \nabla_{\partial_{q_i}}\partial_{p_j}(q,0) \in T_L^*.$$

Therefore

$$\nabla_s\zeta = \nabla^g_s\zeta \in T_uL \quad \text{and} \quad \nabla_s\eta \in T_u^*L.$$

Since $dH(q,p) = \partial_{q_i} f_L dq_i$ we have also

$$\nabla_\zeta dH(u) = \nabla^g_\zeta df_L(u) \quad \text{and} \quad \nabla_\eta dH(u) = 0.$$

Thus the linearized equation (3.14) takes form

$$\nabla^g_s\zeta^* - \frac{\partial \eta}{\partial t} + \nabla^g_\zeta df_L(u) = 0 \tag{3.19}$$
$$\nabla_s\eta + \frac{\partial \zeta^*}{\partial t} = 0$$

with the boundary conditions

(3.20)
$$\eta(s,0)=0,\quad \zeta(s,1)\in T_{u(s)}N,\quad \eta(s,1)\in T_{u(s)}N^{annih},$$
$$\lim_{s\to\pm\infty}\eta(s,t)=0 \quad\text{and}\quad \lim_{s\to\pm\infty}\zeta(s,t)=0.$$

If $\zeta\in T_uN$ then, since N is totally geodesic in L, we have $\nabla^g_s\zeta=\nabla^{g_N}_s\zeta$ and $\nabla^g_\zeta df_L=\nabla^{g_N}_\zeta f$ and so any vector field $\zeta\in\ker D_f(u)$ is a solution to (3.19), (3.20). If we set

$$\gamma_1(s)=\frac{1}{2}\int_0^1\|\eta(s,t)\|^2\,dt$$

then the converse follow from the lemma similar to Lemma 3.4.8.

Lemma 3.4.9 *If $\|\nabla^g df_L\|_{L^\infty(U)}<1$ then $\gamma_1''(s)\geq 0$ for all $s\in\mathbb{R}$.* □

Finally, the same argument provides the isomorphism between the kernels of the adjoint operators

$$\ker D_f^*(u)\cong\ker D_{J,H}^*(u)$$

and so $D_f(u)$ is onto iff $D_{J,H}(u)$ is onto which completes the proof of the proposition. □

Remark 3.4.10 *Consider the case $L=N=TN^{annih}$. It is easy to see that the exactness was not used in the proof, that is the result holds if we replace $df=df_L$ with a closed nondegenerate form α on L such that*

$$\|\nabla^g\alpha\|_g\leq 1$$

w.r.t. some regular metric g on L. But this condition can be always satisfied if we rescale the metric

$$g_\varepsilon:=\varepsilon^2 g$$

This is because the Levi-Civita connection is invariant under rescaling, $\nabla^{g_\varepsilon}=\nabla^g$ and $\|\cdot\|_{g_\varepsilon}=\varepsilon\|\cdot\|_g$. Clearly, g_ε is regular if g is regular. We shall draw the conclusions of this fact in section 3.5.3.

3.4.3 The main result

Theorem 3.4.11 *Let (M,ω) be a symplectic manifold and L_0, L_1 two Lagrangian submanifolds of M. Let N be a compact connected component of $L_0 \cap L_1$ such that L_0 and L_1 intersect cleanly along N. Fix a base point $x_0 \in N$. If U is any relatively compact neighbourhood of N such that*

(a) there are no critical points of a other than those in N in the connected component $\Omega(U, L_0, L_1, x_0)$ of x_0 in the path space $\Omega(U, L_0, L_1)$,

(b) the action functional of ω is well defined in $\Omega(U, L_0, L_1, x_0)$.

Then $\mathcal{U} = \Omega(U, L_0, L_1, x_0)$ is an isolating neighbourhood and $\mathcal{S}_{J,0}(\mathcal{U}) = N$ for any a.c. structure J. There exists an a.c. structure J_0 and a Hamiltonian $H_0 \colon M \to \mathbb{R}$ such that

1. *$\mathcal{S}_{J_0,H_0}(\mathcal{U})$ is a continuation of N,*

2. *(J_0, H_0) is a regular pair and if $g_N = g_{J|N}$, $f = H_{0|N}$ then (g_N, f) is a regular pair in the sense of (2.5.1),*

3. *The Floer complex $CF_*(\mathcal{U}, J_0, H_0)$ coincides with the Morse complex $C_*^{Morse}(N, g_N, f)$ of f and thus*

$$HF_*(\mathcal{U}, \mathbb{Z}_2) \cong H_*^{sing}(N, \mathbb{Z}_2).$$

Proof. Let $\phi \colon U_0 \to V_0$ be as in the Proposition 3.4.1. Then clearly U_0 satisfies (a) and (b). Let U_1 be another neighbourhood of N such that $U_1 \subset \bar{U}_1 \subset U_0$. Fix a Morse function $f \colon N \to \mathbb{R}$ and a regular metric g_N on N such that

$$\|\nabla df\|_{L^\infty} < 1$$

and take an almost complex structure J_0 and a Hamiltonian H_0 on M such that

$$J_0\big|_{U_1} = \phi^* J_g\big|_{U_1} \quad \text{and} \quad H_0\big|_{U_1} = H_f \circ \phi\big|_{U_1}$$

where J_g and H_f are constructed as in Proposition 3.4.6. Thus if u is a critical point or a gradient line of a_{H_0} contained in U_1 then $\phi \circ u$ is a critical point or a gradient line of a_{H_f}. If we choose U_1 small enough then the assumption (3.18) of Proposition 3.4.6 is satisfied and so u is t–independent. Now the following lemma finishes the proof of the theorem.

Lemma 3.4.12 *Let U be a neighbourhood of N satisfying (3.4.11)(a)(b) and let $U_1 \subset U$. Then there exists an $\varepsilon > 0$ such that if $\|H\|_{C^1} < \varepsilon$ then any solution to (3.8) and (3.9) contained in U lies in fact in U_1.*

Proof. Let $x\colon [0,1] \to U$ satisfy $\dot{x} = X_H(x)$ and $x(0) \in L_0$. We then have

$$dist(x(0), x(t)) \leq \int_0^t |\dot{x}(s)|\, ds \leq \|H\|_{C^1}$$

Choose a neighbourhood U_2 of N such that $\overline{U}_2 \subset U_1$. Then

$$d = \min(dist(U_2, M \setminus U_1), dist(L_0 \cap (U \setminus U_2), L_1))$$

is positive. Assume $\|H\|_{C^1} < d$. It follows that $x(0) \in U_2$ implies $x(t) \in U_1$ for all $t \in [0,1]$ and if $x(0) \notin U_2$ then $x(1) \notin L_1$.

Suppose now that there is a sequence $\{\varepsilon_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ converging to 0 and for every $n \in \mathbb{N}$ and a solution $u'_n\colon I \times \mathbb{R} \to U$ to

$$\frac{\partial u'_n}{\partial s} + J_n(u_n)\frac{\partial u'_n}{\partial t} + \nabla H_n(u) = 0$$

where $\|H_n\|_{C^1} < \varepsilon_n$ and J_n is a regular a.c. structure, with the limit conditions $u'_n(s,i) \in L_i$, $i = 0, 1$,

$$\lim_{s \to \pm\infty} u'_n(s) = x_n^{\pm} \subset U_1$$

and such that $u_n'(I\times\mathbb{R})\not\subset U_1$. As $\overline{U}$ is compact there is a subsequence $\{u_n\}\subset\{u_n'\}$ and a sequence (s_n,t_n) such that $\lim_{n\to\infty}u_n(s_n,t_n)=y_0\in U\setminus U_1$. We may assume that $\lim_{n\to\infty}t_n=t_0$ and using time shift, that $s_n=0$ for all $n\in\mathbb{N}$. Now, since $\hat\omega$ is exact in $\mathcal{U}$ we have

$$\ell^2(u_n)=a_\omega(x^-)-a_\omega(x^+)+\int_0^1(H_n\circ x_n^+-H_n\circ x_n^-)\,dt\to 0$$

by Lemma 3.4.5 and it follows from Lemma 3.3.9 that a subsequence of $\{u_n\}$ converges locally uniformly to a constant map u_0. Because of the limit conditions $u_0\in N$. On the other hand $u_0(0,t_0)=\lim u_n(0,t_n)=y_0$, a contradiction. □

Corollary 3.4.13 *Assume that* (M,L_0,L_1,γ_0) *is a monotone triple such that* $\Sigma_i\geq 3$, $i=0,1$. *Assume that* L_0 *intersects* L_1 *cleanly along a connected manifold* $N=L_0\cap L_1$ *such that* $\dim N+1<N_\mu$. *Then*

$$HF_*(M,L_0,L_1,\mathbb{Z}_2)\cong H_*^{sing}(N,\mathbb{Z}_2).$$

Proof. Clearly it is enough to show that all trajectories of index 1 must lie in a small neighbourhood U_1 of N in M if only the Hamiltonian perturbation is sufficiently small. We proceed as in the proof of Lemma 3.4.12. Thus suppose that there are sequences $J_n\to J$, $H_n\to 0$ and $u_n\in\mathcal{M}_{J_n,H_n}(x_n^-,x_n^+)$ such that $\mu_{u_n}=1$ and $u_n(\mathbb{R}\times I)\not\subset U$. We may assume that H is constructed as in Proposition3.4.6 so that $x^\pm$ are t-independent. For each n choose a path $v_n\colon\mathbb{R}\to N$ connecting x_n^- and x_n^+ that is, $v_n\in\mathcal{P}(x_n^-,x_n^+)$. Then Proposition 3.4.6 together with Theorem 2.3.2 implies

$$\mu_{v_n}=\operatorname{ind}D_{J_n,H_n}(v_n)=\operatorname{ind}D_f(v_n)=\operatorname{ind}x_n^--\operatorname{ind}x_n^+$$

Consequently, $|I_\mu(u_n*(-v))|=|\mu_{u_n}-\mu_v|\leq\dim N+1<N_\mu$. But this means that $I_\mu(u_n*(-v))=0$. Since (M,L_0,L_1) is monotone and obviously

$\int v_n^*\omega = 0$ this yields

$$\int u_n^*\omega = \int u_n^*\omega - \int v_n^*\omega = I_\omega(u_n * (-v)) = 0.$$

The conclusion follows as before. □

3.5 Applications

3.5.1 Example: Intersection of the linear Lagrangian tori in T^{2k}

Consider the torus T^{2k} with the standard symplectic structure $\omega_0 = \sum dx_i \wedge dy_i$. For $i = 0, 1$ let $\tilde{A}_i\colon T^k \to T^{2k}$ be an inclusion which is induced by a linear map of the universal covers $A_i\colon \mathbb{R}^k \to \mathbb{R}^{2k}$. We have $rkA_i = k$ and $\tilde{A}_i$ is an injection iff $A_i x \in \mathbb{Z}^{2k}$ implies that $x \in \mathbb{Z}^k$ that is iff

$$A_i\mathbb{Q}^k \cap \mathbb{Z}^{2k}/A_i\mathbb{Z}^k = Tor(\mathbb{Z}^{2k}/A_i\mathbb{Z}^)) = 0$$

where Tor denotes the torsion part of the group. The latter condition is equivalent to the greatest common divisor of the k-minors of A_i being equal 1. Assume that $L_i := \tilde{A}_i(T^k)$ is a Lagrangian submanifold of T^{2k} (that is $(A_i'^T A_i'')^T = A_i'^T A_i''$ where $A_i = (A_i', A_i'')$). In order to describe $\pi_0(\Omega(L_0, L_1))$ and $\pi_1(\Omega(L_0, L_1))$ we use (3.4) and (3.6). We have $\pi_1(L_i) = \mathbb{Z}^k$, $\pi_1(M) = \mathbb{Z}^{2k}$, $\pi_2(M) = 0$ and

$$\pi_1(i_0)^{-1}\pi_1(i_1) = \overline{A} = -A_0 \oplus A_1\colon \mathbb{Z}^{2k} \to \mathbb{Z}^{2k}.$$

Consequently

$$\pi_0(\Omega(L_0, L_1)) \cong \mathbb{Z}^{2k}/\overline{A}\mathbb{Z}^{2k} = \operatorname{coker}\overline{A} \tag{3.21}$$

$$\pi_1(\Omega(L_0, L_1)) \cong \ker\overline{A}$$

The intersection $L_0 \cap L_1$ is a closed subgroup of T^{2k} so $L_0 \cap L_1 = T^r \oplus G$ where G is discrete and $r = \dim \operatorname{im} A_0 \cap \operatorname{im} A_1 = \dim\ker\overline{A}$. The injection

$$\sigma\colon L_0 \cap L_1 \to \pi_0(\Omega(L_0, L_1))$$

can be described as follows. Let $p \in L_0 \cap L_1$ and $[x_p] = \tilde{A}_0^{-1}(p)$ $[y_p] = \tilde{A}_1^{-1}(p)$. Then there is an $n_p \in \mathbb{Z}^{2k}$ such that $A_1 y_p - A_0 x_p + n_p = 0$. Obviously $n_p \in \overline{A}\mathbb{Q}^{2k}$. Moreover if $x' \in [x_p]$ and $y' \in [y_p]$ then $x_p = x' + m_0$, $y_p = y' + m_1$ and

$$A_1 y' - A_0 x' + n_p + A_1 m_1 - A_0 m_0 = 0$$

and so n_p is well defined modulo $\overline{A}\mathbb{Z}^{2k}$. It is easy to see that $\sigma(p) = [n_p] \in Tor(\mathbb{Z}^{2k}/\overline{A}\mathbb{Z}^{2k})$. Indeed, the homotopy $\gamma_s(t) = [(1-s)A_0 x_p + stn_p] \in T^{2k}$ gives a path in $\Omega(L_0, L_1)$ connecting the constant path at p and the loop $[tn_p]$ representing the class $[n_p]$ under the isomorphism (3.21). The fibres of σ are connected: if $A_1 y - A_0 x + n = 0$, $A_1 y' - A_0 x' + n' = 0$ and $n - n' = A_1 m_1 - A_0 m_0$ then

$$\gamma(t) = [A_0(t(x + m_0) + (1-t)x')] = [A_1(t(y + m_0) + (1-t)y')]$$

connects $p = [A_0 x]$ and $p' = [A_0 x']$. Summarizing, $L_0 \cap L_1$ consists of the r-dimensional tori T_α, $\alpha \in Tor(\mathbb{Z}^{2k}/\overline{A}\mathbb{Z}^{2k})$ each in the different component of $\Omega(L_0, L_1)$. There are no intersection points in the components of $\Omega(L_0, L_1)$ represented by the nontorsion elements of $\mathbb{Z}^{2k}/\overline{A}\mathbb{Z}^{2k}$.

Let $[n] \in \mathbb{Z}^{2k}/\overline{A}\mathbb{Z}^{2k}$. Then the isomorphism $\pi_1(\Omega(L_0, L_1, [tn])) \cong \ker \overline{A}$ can be written explicitly

$$\ker \overline{A} \ni m = (m_0, m_1) \mapsto [F_m] \in \pi_1(\Omega(L_0, L_1, [tn]))$$

$$F_m(s,t) = [sA_0 m_0 + tn] = [sA_1 m_1 + tn] \in T^{2k}$$

Thus

$$[\hat{\omega}]: \ker \overline{A} \ni m \mapsto \int_{I \times I} F_m^* \omega = n^T J_0 A_0 m_0 \in \mathbb{R}$$

and $[\hat{\omega}]$ vanishes iff n annihilates $A_0\mathbb{R}^k \cap A_1\mathbb{R}^k$ that is $n \in \overline{A}\mathbb{R}^{2k}$ or, equivalently, $[n] \in Tor(\mathbb{Z}^{2k}/\overline{A}\mathbb{Z}^{2k})$. We may then apply Theorem 3.4.11 to each component of $L_0 \cap L_1$ with $U = T^{2k}$. We summarize the above remarks as

Corollary 3.5.1 *For $i = 0, 1$ let $A_i = (A_i', A_i'') \colon \mathbb{Z}^k \to \mathbb{Z}^{2k}$ satisfy*

— *$rk\, A_i = k$,*

— *the* gcd *of the k-minors of A_i is equal 1,*

— *$(A_i'^T A_i'')^T = A_i''^T A_i'$.*

Then $\tilde{A}_0(T^k)$, $\tilde{A}_1(T^k)$ are Lagrangian submanifolds and

$$HF_*(\tilde{A}_0(T^k), \tilde{A}_1(T^k)) \cong H_*^{sing}(T^r) \times Tor(\mathbb{Z}^{2k}/\overline{A}\mathbb{Z}^{2k})$$

where $\overline{A} = -A_0 \oplus A_1$ and $2k - r = rk\,\overline{A}$. Consequently, if $L_i = \psi_i(\tilde{A}_i(T^k))$ where ψ_i is an exact symplectomorphism and L_0 intersects L_1 transversally then

$$\# L_0 \cap L_1 \geq 2^r \cdot \mathrm{gcd} \text{ of the k-minors of } \overline{A}.$$

3.5.2 Floer homology of a symplectomorphism

Let (M, ω) be a compact symplectic manifold and $\phi \colon M \to M$ a symplectomorphism. There are two ways of constructing Floer homology detecting its fixed points, $Fix\phi$.

Firstly, the graph of ϕ is a Lagrangian submanifold of $(M \times M, (-\omega) \times \omega)$ and its fixed points correspond to the intersection points of $graph\, \phi$ with the diagonal $\Delta = \{(x, x) \in M \times M\}$. Thus we have the Floer homology of the Lagrangian intersection $HF_*(M \times M, \Delta, graph\, \phi)$. This intersection is transversal if the fixed points of ϕ are nondegenerate, i.e. if 1 is not an eigenvalue of $d\phi(x)$, for $x \in$ Fix. The second approach was mentioned in [Flo88a] and presented with details in [DS93b, DS93a, FHS94]. There the action

$$\hat{\omega}_H(\gamma)\xi = \int_0^1 \omega(\xi, \dot{\gamma} - X_t(\gamma))\, dt \tag{3.22}$$

is defined on the space of paths

$$\Omega_\phi := \{\gamma\colon \mathbb{R} \to M:\ \gamma(t+1) = \phi(\gamma(t))\}$$

The tangent vector field $\xi \in \Gamma(\gamma^*TM)$ in (3.22) satisfies $\xi(t+1) = d\phi\xi(t)$ and X_t is a Hamiltonian vector field with the Hamiltonian $H\colon \mathbb{R} \times M \to \mathbb{R}$ satisfying the periodicity conditions $H_{t+1} \circ \phi = H_t$. The critical points of (3.22) correspond to the fixed points of $\psi_H^{-1} \circ \phi$ where ψ_H is the time one map of the Hamiltonian system $\dot{x} = X_t(x)$. Similarly, we choose an ω-compatible almost complex structure $J\colon \mathbb{R} \times M \to End(TM)$ such that $\phi^* J_{t+1} = J_t$ and write down the Cauchy-Riemann equations

$$\text{(3.23)} \qquad \begin{aligned} &\bar{\partial}_{\phi,J,H}(u) = \frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} + \nabla H_t(u) = 0 \\ &u\colon \mathbb{R} \times \mathbb{R} \to M \\ &u(s,t+1) = \phi(u(s,t)) \end{aligned}$$

Finally, we set

$$\mathcal{M}_{J,H}(M,\phi) := \{u:\ \bar{\partial}_{\phi,J,H}(u) = 0 \text{ and } \ell^2(u) < \infty\}$$

As before, for a generic J and H, this is a collection of finite dimensional manifolds and one can construct a $\mathbb{Z}_2$-complex $CF_*(M,\phi)$ generated by the fixed points of ϕ. It is worth mentioning that this construction provides a link between symplectic and instanton Floer homology (see [DS93a, DS93b]).

Lemma 3.5.2 *For a symplectomorphism $\phi\colon M \to M$ and J_t, H_t as above there exist an almost complex structure $\tilde{J}_t$ and a Hamiltonian $\tilde{H}_t$ on $(M \times M, (-\omega) \times \omega)$ and a bijection*

$$\mathcal{M}_{J,H}(M,\phi) \ni u \mapsto \tilde{u} \in \mathcal{M}_{\tilde{J},\tilde{H}}(M \times M, \Delta, graph\,\phi).$$

Moreover, $\tilde{u}$ is regular if only u is regular.

Proof. The lemma is a simple consequence of the observation that there is a bijection

$$\Omega_\phi \ni \gamma \mapsto \tilde{\gamma} \in \Omega(\Delta, graph\,\phi)$$
$$\tilde{\gamma}(t) = (\tilde{\gamma}_1(t), \tilde{\gamma}_2(t)) = (\gamma(\tfrac{1-t}{2}), \gamma(\tfrac{1+t}{2}))$$

between the path spaces. It is easy to see that this correspondence translates all necessary concepts of Floer's construction from one setup to the other. Indeed, let $\gamma \in \Omega_\phi$ satisfy (3.8). Then

$$\dot{\tilde{\gamma}}_1(t) = -\tfrac{1}{2} X_{\frac{1-t}{2}}(\tilde{\gamma}_1(t))$$
$$\dot{\tilde{\gamma}}_2(t) = \tfrac{1}{2} X_{\frac{1+t}{2}}(\tilde{\gamma}_2(t)).$$

The vector field $\tilde{X}_t := \frac{1}{2}(X_{\frac{1-t}{2}}, X_{\frac{1+t}{2}})$ has the Hamiltonian

$$\tilde{H}_t := \tfrac{1}{2}(H_{\frac{1-t}{2}} \circ pr_1 + H_{\frac{1+t}{2}} \circ pr_2)$$

where $pr_i \colon M \times M \to M$ denote the projections. If we set

$$\tilde{J}_t := (-J_{\frac{1-t}{2}}) \times J_{\frac{1+t}{2}}$$

then the map $\tilde{u} \colon \mathbb{R} \times I \to M \times M$

$$\tilde{u}(s,t) = (\tilde{u}_1(s,t), \tilde{u}_2(s,t)) = (u(\tfrac{s}{2}, \tfrac{1-t}{2}), u(\tfrac{s}{2}, \tfrac{1+t}{2}))$$

satisfies

$$\frac{\partial \tilde{u}}{\partial s} + \tilde{J}_t(\tilde{u})(\frac{\partial \tilde{u}}{\partial t} - \tilde{X}_t(\tilde{u})) = 0 \tag{3.24}$$

for any $u \in \mathcal{M}_{J,H}(M,\phi)$. Conversely, any solution to (3.24) defines a $W^{1,p}_{loc}$, and thus smooth, solution $\hat{u}$ to (3.23)

$$\hat{u}(s,t) = \begin{cases} \phi^{[t]} \circ \tilde{u}_1(2s, 1-2t') & \text{for } t' \leq \frac{1}{2} \\ \phi^{[t]} \circ \tilde{u}_2(2s, 2t'-1) & \text{for } t' \geq \frac{1}{2} \end{cases}$$

where $t = [t] + t'$ and $[t]$ is the integer part of t. Similarly, there is an isomorphism between the kernels of the linearized operators

$$\ker D^{\phi}(u)_{J,H} \cong \ker D_{\tilde{J},\tilde{H}}(\tilde{u})$$

and their adjoints which shows that $\operatorname{ind} D^{\phi}(u)_{J,H} = \operatorname{ind} D_{\tilde{J},\tilde{H}}(\tilde{u})$ and $\tilde{u}$ is regular iff u is regular. □

Corollary 3.5.3 *If J, H is a regular pair then there is an isomorphism*

$$HF_*(M, \phi, J, H) \cong HF_*(M \times M, \Delta, graph\, \phi, \tilde{J}, \tilde{H})$$

induced by an isomorphism of the chain complexes.

The concept of clean intersection translates to a nondegenerate manifold of fixed points i.e. a submanifold $N \subset M$ such that $N \subset \operatorname{Fix} \phi$ and $\operatorname{Fix} d\phi(x) = T_xN$ for $x \in N$.

Corollary 3.5.4 *Let (M, ω) be a symplectic manifold and $\phi \colon M \to M$ a symplectomorphism with a connected nondegenerate manifold of fixed points,* $\operatorname{Fix} \phi = N$. *Assume that $I_{\omega,\phi} = 0$. Then*

$$HF_*(M, \phi) \cong H_*^{sing}(N, \mathbb{Z}_2).$$

3.5.3 The Novikov homology in the cotangent bundle

We end this chapter with some remarks on a "non-exact" intersection, based rather on Floer's original result ([Flo88a]) than on the generalization of the previous section. Let L_0, L_1 be two Lagrangian submanifolds of (M, ω). Assume that

$$\omega_{|\pi_2(M, L_i)} = 0 \qquad \text{for } i = 0, 1.$$

Then, according to the decomposition (3.6) of $\pi_1(\Omega(L_0, L_1))$ the homomorphism I_ω descends to

$$\omega_{\#} : im\pi_1(l_0) \cap im\pi_1(l_1) \to \mathbb{R}.$$

This can be described directly as the integral of ω over a cylinder which represents a homotopy in M between two loops $\alpha_i \subset L_i$ such that $\pi_1(l_0)[\alpha_0] = \pi_1(l_1)[\alpha_1]$.

Proposition 3.3.7 provides the compactness property for the subsets of $\mathcal{M}(L_0, L_1)$ with bounded energy. Therefore we may construct the Floer complex over an appropriate Novikov ring similarly as in Section 2.2. Let

$$\Gamma = im\pi_1(l_0) \cap im\pi_1(l_1)/\ker\omega_{\#}$$

and $\phi_\omega \colon \Gamma \to \mathbb{R}$ be the induced homomorphism. Fix a regular pair (J, H) and for any solution x to (3.8) choose a path v_x connecting x with the base point in $\Omega(L_0, L_1)$, i.e.

$$v_x \colon I \times I \to M, \quad v_x(s,i) \in L_i, \quad v_x(0,\cdot) = \gamma_0 \quad \text{and} \quad v_x(1,\cdot) = x.$$

For any class $A \in \Gamma$ and $x^\pm \in Crit\, a_H$ denote

$$\mathcal{M}_1(x^-, x^+, A) := \{u \in \mathcal{M}_1(x^-, x^+) \colon \mu_u = 1,\ v_{x^-} * u * (-v_{x^+}) \in A\}.$$

If $u \in \mathcal{M}_1(x^-, x^+, A)$ then its energy is bounded:

$$\ell^2(u) = \int u^*\omega + \int_0^1 H_t(x^-(t)) - H_t(x^+(t))\, dt \quad \text{and}$$
$$\int u^*\omega = \phi_\omega(A) - \int_0^1 v_{x^-}^*\omega + \int_0^1 v_{x^+}^*\omega$$

We define the Floer complex as a free module generated by the zeros of ω_H over the Novikov ring

$$\Lambda_\omega = \Lambda(\Gamma, \phi_\omega, \mathbb{Z}_2)$$

The boundary operator $\partial = \partial_{\Lambda_\omega}$ is given by (3.14) where $\langle y, \partial x\rangle \in \Lambda_\omega$ and

$$\langle y, \partial x\rangle_A = n(x, y, A) = \#\mathcal{M}_1(x^-, x^+, A)/\mathbb{R}$$

Proposition 3.3.7 assures that ∂ is well defined.

Proposition 3.5.5 *$\partial^2_{\Lambda_\omega} = 0$ and the resulting homology is naturally isomorphic for different choices of a regular pair (J, H)*

Remark 3.5.6 *Let X_t, $t \in [0,1]$ be a symplectic vector field, α the flux form of X_t*

$$\alpha = Flux[X_t] = \int_0^1 i_{X_t}\omega \, dt$$

and ϕ_t the family of symplectomorphisms generated by X_t. Then, as in Remark 3.13, there is a bijection between $\Omega(L_0, L_1)$ and $\Omega(L_0, \phi_1^{-1}(L_1))$ and it is easy to see that

$$\omega^\alpha_\# := \omega_\#(L_0, \phi_1^{-1}(L_1)) = \omega_\#(L_0, L_1) + [\alpha]_{|im\pi_1(l_0)\cap im\pi_1(l_1)}.$$

Moreover, the Floer complex $CF_*(L_0, \phi_1^{-1}(L_1))$ can be identified with the complex $CF_*(L_0, L_1, X_t, \Lambda_{\omega^\alpha})$ which is built from the solutions to

$$\dot{x}(t) = X_t(x(t)) \tag{3.25}$$

$$\frac{\partial u}{\partial s} + J_t(u)\left(\frac{\partial u}{\partial t} - X_t(u)\right) = 0 \tag{3.26}$$

If Y_t is another symplectic vector field with the flux homologous to α, generating the family of symplectomorphisms ψ_t then ϕ_1 and ψ_1 can be connected by a Hamiltonian isotopy (Banyaga, [Ban78]) and thus

$$CF_*(L_0, L_1, X_t, \Lambda_{\omega^\alpha}) \cong CF_*(L_0, L_1, Y_t, \Lambda_{\omega^\alpha})$$

We denote the homology by $HF_*(L_0, L_1, [\alpha])$, where $[\alpha] \in H^1_{DR}(M, \mathbb{R})$, although this is an abuse, since the isomorphism above is not natural (Remark 3.3.11)

Now let L be a compact manifold and α a closed nondegenerate form on L. If we define the family of symplectomorphisms of T^*L by

$$\phi_t \colon T^*L \ni \beta \mapsto \beta + t\alpha(\pi(\beta)) \in T^*L$$

then the flux of the associated vector field is just $\alpha \circ \pi$ and

$$\omega^\alpha = [\alpha] \colon \pi_1(L) \to \mathbb{R}$$

Therefore Proposition 3.4.6 together with Remark 3.4.10 imply that all solutions to (3.25), (3.26) are t–independent. This yields

Corollary 3.5.7 *The Floer homology $HF_*(T^*L, L, L, [\alpha \circ \pi])$ is isomorphic to the Novikov homology $HN_*(L, [\alpha], \mathbb{Z}_2)$.*

Note that this result is not so strong because, unlikely in the exact case, it cannot be immediately generalized to an arbitrary Lagrangian submanifold of M, even under assumption that $\omega_{|\pi_2(M,L)} = 0$. The argument of Lemma 3.4.12 fails simply because we do not control the energy of u_n.

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