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FINITE DIMENSIONAL COMPENSATORS FOR INFINITE DIMENSIONAL SYSTEMS WITH UNBOUNDED INPUT OPERATORS*

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Abstract. This paper contains a design procedure for constructing finite dimensional compensators for a class of infinite dimensional systems with unbounded input operators. Applications to retarded functional differential systems with delays in the input or the output variable and to partial differential equations with boundary input operators are discussed.

Key words. infinite dimensional systems, unbounded input operators, compensator, retarded functional differential equations, partial differential equations, boundary control

AMS(MOS) subject classifications. 34K20, 93C15, 93C25, 93D15

1. Introduction. In [15], [16], [17] Schumacher presented a design procedure for constructing stabilizing compensators for a class of infinite dimensional systems. The novel feature was that the compensators were finite dimensional and that they could be readily numerically calculated from finitely many system parameters. The class included those systems described by retarded functional differential and partial differential equations provided that the eigenvectors of the system operators were complete and provided that the input and output operators were bounded. In [3] Curtain presented an alternative compensator design which applied to essentially the same class of systems, except that for the special case of parabolic systems unbounded input and output operators were allowed. By means of enlarging the state space of the given distributed boundary control system, Curtain in [2] essentially transformed the original problem with unbounded control into one with bounded control action so that the techniques of either [16] or [3] could be applied. The resulting control, however, was of integral type. Neither of these two compensator designs are applicable to retarded systems with delays in the control or the observation.

In the present paper we make use of the abstract approach developed by Salamon [14] to extend the results of Schumacher [16] to allow for unbounded control action. This is done in a direct way without reformulating the original problem into one with a bounded input operator. In § 2 we outline the abstract formulation and prove a theorem on the existence of a finite dimensional compensator paralleling the development in [16]. In § 3 the general approach is then applied to retarded functional differential systems with delays in either the input or the output variables. The conditions are easy to check and they are quite reasonable, except for the assumption of completeness of the eigenfunctions which seems to be too strong. In a special case we are able to weaken this assumption.

Finally, in § 4, we show how boundary control systems fit into the abstract framework of § 2 and give an example. The results are compared with the approach in [3].

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2. A general result. We consider the abstract Cauchy problem

(2.1.1) $$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in X,$$

(2.1.2) $$y(t) = Cx(t),$$

on the real, reflexive Banach space X where $A: \mathscr{D}(A) \to X$ is the infinitesimal generator of a strongly continuous semigroup $S(t)$ on X and $C \in \mathscr{L}(X, \mathbb{R}^m)$.

In order to give a precise definition of what we mean by an unbounded input operator we need an extended state space $Z \supset X$. For this purpose let us first introduce the subspace

$$Z^* = \mathscr{D}_{X^*}(A^*) \subset X^*$$

endowed with the graph norm of A^*. Then Z^* becomes a real, reflexive Banach space and the injection of Z^* into X^* is continuous and dense. Defining Z to be the dual space of Z^*, we obtain by duality that

$$X \subset Z$$

with a continuous dense injection.

Remarks 2.1. (i) A^* can be regarded as a bounded operator from Z^* into X^* and $S^*(t)$ restricts to a strongly continuous semigroup on Z^*. By duality, A extends to a bounded operator from X into Z. This extension, regarded as an unbounded operator on Z, is the infinitesimal generator of the extended semigroup $S(t) \in \mathscr{L}(Z)$ (see [14, Lemma 1.3.2]).

(ii) If $\mu \notin \sigma(A) = \sigma(A^*)$, then the operator $\mu I - A: X \to Z$ is bijective. Furthermore, this operator commutes with the semigroup $S(t)$ so that it provides a similarity action between the semigroups $S(t) \in \mathscr{L}(X)$ and $S(t) \in \mathscr{L}(Z)$.

(iii) It follows from (ii) that the exponential growth rate of the semigroup $S(t)$ is the same on the state spaces X and Z, i.e.

$$\omega_0 = \lim_{t\to\infty} t^{-1} \log \|S(t)\|_{\mathscr{L}(X)} = \lim_{t\to\infty} t^{-1} \log \|S(t)\|_{\mathscr{L}(Z)}.$$

(iv) It also follows from (ii) that the spectrum of A on the state space X coincides with the spectrum of A on Z (see [14, Lemma 1.3.2]). Furthermore, the generalized eigenvectors for both operators are the same, since the eigenvectors of A and Z are contained in $\mathscr{D}_Z(A) = X$. Finally, the (generalized) eigenvectors of A are complete in X if and only if they are complete in Z.

We will always assume that B is a bounded, linear operator from R^l into Z. However, we want the solutions of (2.1.1) to be in the smaller space X on which the output operator is defined. Therefore we need the following hypothesis.

(H1) For every $T > 0$ there exists a constant $b_T > 0$ such that $\int_0^T S(T-s)Bu(s)\,ds \in X$ and

$$\left\| \int_0^T S(T-s)Bu(s)\,ds \right\|_X \leqq b_T \|u(\cdot)\|_{L^p[0,T;R^l]}$$

for every $u(\cdot) \in L^p[0, T; R^l]$ where $1 \leqq p < \infty$. In the following we collect some important consequences of (H1) which have been established in [14, § 1.3].

Remarks 2.2. (i) (H1) is satisfied if and only if the inequality

$$\|B^*S^*(\cdot)x^*\|_{L^q[0,T;R^l]} \leqq b_T \|x^*\|_{X^*}$$

holds for every $x^* \in Z^*$ and every $T > 0$ where $1/p + 1/q = 1$.

(ii) If (H1) is satisfied, then

$$x(t)=S(t)x_0+\int_0^t S(t-s)Bu(s)\,ds\in X \tag{2.2}$$

is the unique strong solution of (2.1.1) for every $x_0\in X$ and every $u(\cdot)\in L^p[0,T;R^l]$. More precisely $x(t)$ is continuous in X on the interval $[0,T]$ and satisfies

$$x(t)=x_0+\int_0^t [Ax(s)+Bu(s)]\,ds,\qquad 0\leqq t\leqq T,$$

where the integral has to be understood is the state space Z. Thus (2.1.1) is satisfied in the space Z for almost every $t\in[0,T]$.

(iii) If (H1) is satisfied, then for every $w(\cdot)\in\mathscr{C}[0,T;X]$ there exists a unique $x(\cdot)\in\mathscr{C}[0,T;X]$ satisfying the equation

$$x(t)=w(t)+\int_0^t S(t-s)BFx(s)\,ds,\qquad t\geqq 0.$$

This solution $x(\cdot)$ depends continuously on $w(\cdot)$.

Moreover hypothesis (H1) implies the following important perturbation result.

THEOREM 2.3. *Let $F\in\mathscr{L}(X;\mathbb{R}^l)$ be given. Then the following statements hold.*

(i) *There exists a unique strongly continuous semigroup $S_F(t)$ on X satisfying*

$$S_F(t)x=S(t)x+\int_0^t S(t-s)BFS_F(s)\,x\,ds \tag{2.3}$$

for every $x\in X$ and every $t\geqq 0$. Its infinitesimal generator is given by

$$\mathscr{D}(A_F)=\{x\in X\,|\,Ax+BFx\in X\},$$
$$A_Fx=Ax+BFx.$$

(ii) $A_F^*=A^*+F^*B^*:\mathscr{D}(A_F^*)=Z^*\to X^*$.

(iii) *$S_F(t)$ extends to a strongly continuous semigroup on Z and the infinitesimal generator of the extended semigroup is given by $A+BF:X\to Z$.*

(iv) *Let $x_0\in X$ and $v(\cdot)\in L^p[0,T;\mathbb{R}^l]$ be given and let $x(\cdot)\in\mathscr{C}[0,T;X]$ be the unique solution of*

$$x(t)=S(t)x_0+\int_0^t S(t-s)B[Fx(s)+v(s)]\,ds,\qquad 0\leqq t\leqq T. \tag{2.4}$$

Then

$$x(t)=S_F(t)x_0+\int_0^t S_F(t-s)Bv(s)\,ds,\qquad 0\leqq t\leqq T. \tag{2.5}$$

(v) *Hypothesis* (H1) *is satisfied with $S(t)$ replaced by $S_F(t)$ and $S_F(t)$ satisfies*

$$S_F(t)x=S(t)x+\int_0^t S_F(t-s)BFS(s)x\,ds \tag{2.6}$$

for every $x\in X$ and every $t\geqq 0$.

(vi) *Let $x_0\in X$ and $f(\cdot)\in L^p[0,T;X]$ be given and define*

$$x(t)=S_F(t)x_0+\int_0^t S_F(t-s)f(s)\,ds,\qquad 0\leqq t\leqq T. \tag{2.7}$$

Then

$$x(t)=S(t)x_0+\int_0^t S(t-s)[BFx(s)+f(s)]\,ds,\qquad 0\leqq t\leqq T. \tag{2.8}$$

Proof. Statement (i) has been shown in [14, Thm. 1.3.7] and (ii) follows from [14, Thm. 1.3.9] since the input space is finite-dimensional. By (ii), $S_F^*(t)$ restricts to a strongly continuous semigroup on $Z^* = \mathcal{D}(A_F^*)$ and hence $S_F(t)$ extends to a semigroup on Z. By Remark 2.1(i), the extended semigroup is generated by the adjoint operator of A_F^*, where A_F^* is regarded as a bounded operator from Z^* into X^*. This proves statement (iii).

In order to establish statement (iv), let us first assume that $v(\cdot) \in \mathscr{C}^1[0, T; \mathbb{R}^l]$ and let $x(\cdot) \in \mathscr{C}[0, T; X]$ be the unique solution of (2.4). Then it follows from Remark 2.2 (ii) that $x(\cdot) \in \mathscr{C}^1[0, T; Z]$ and

$$\frac{d}{dt} x(t) = (A + BF)x(t) + Bv(t), \qquad 0 \leqq t \leqq T.$$

Hence it follows from (iii) and a classical result in semigroup theory that $x(\cdot)$ is given by (2.5). In general, statement (iv) follows from the fact that the unique solutions of both (2.4) and (2.5), regarded as continuous functions with values in Z, depend continuously on $v(\cdot) \in L^p[0, T; \mathbb{R}^l]$.

It follows immediately from (iv) that (H1) is satisfied with $S(t)$ replaced by $S_F(t)$. Now let $x(t)$, $t \geqq 0$, be defined by the RHS of (2.6). Then it follows from (iv) that

$$\begin{aligned} x(t) &= S(t)x + \int_0^t S(t-s)B\left[F\int_0^s S_F(s-\tau)BFS(\tau)x\,d\tau + FS(s)x\right] ds \\ &= S(t)x + \int_0^t S(t-s)BFx(s)\,ds \end{aligned}$$

for $t \geqq 0$, and hence $x(t) = S_F(t)x$, by the definition of $S_F(t)$. This proves statement (v).

Statement (vi) can be established straightforwardly by inserting (2.3) into (2.7) and interchanging integrals. □

The aim of this section is to give sufficient conditions under which system (2.1) can be stabilized by a finite-dimensional compensator of the form

$$\text{(2.9.1)} \qquad \dot{w}(t) = Mw(t) - Hy(t), \qquad w(0) = w_0,$$

$$\text{(2.9.2)} \qquad u(t) = Kw(t) + v(t),$$

where $M \in \mathbb{R}^{N\times N}$, $H \in \mathbb{R}^{N\times m}$, $K \in \mathbb{R}^{l\times N}$ are suitably chosen matrices. To this end we need the following well-posedness result for the connected system (2.1), (2.9).

PROPOSITION 2.4. *Let* (H1) *be satisfied. Then for all* $x_0 \in X$, $w_0 \in \mathbb{R}^N$, $v(\cdot) \in L^p_{\mathrm{loc}}[0, \infty; \mathbb{R}^l]$ *there exists a unique solution pair* $x(t)$, $w(t)$ *of* (2.1) *and* (2.9). *This means that* $x(t)$ *is continuous in* X *and absolutely continuous in* Z, *that* (2.1.1) *is satisfied for almost every* $t \geqq 0$ *where* $u(t)$ *is given by* (2.9.2) *and that* $w(t) \in \mathbb{R}^N$ *is continuously differentiable and satisfies* (2.9.1) *where* $y(t)$ *is given by* (2.1.2).

Proof. Let us introduce the spaces $X_e = X \times \mathbb{R}^N$, $Z_e = Z \times \mathbb{R}^N$, $U_e = \mathbb{R}^l \times \mathbb{R}^m$ and the operators $S_e(t) \in \mathscr{L}(X_e)$, $B_e \in \mathscr{L}(U_e, Z_e)$, $F_e \in \mathscr{L}(X_e, U_e)$ by

$$S_e(t) = \begin{bmatrix} S(t) & 0 \\ 0 & e^{Mt} \end{bmatrix}, \quad B_e = \begin{bmatrix} B & 0 \\ 0 & -H \end{bmatrix}, \quad F_e = \begin{bmatrix} 0 & K \\ C & 0 \end{bmatrix}.$$

Then hypothesis (H1) is satisfied with X, Z, $S(t)$, B replaced by X_e, Z_e, $S_e(t)$, B_e, respectively. Moreover $x(t) \in X$ and $w(t) \in \mathbb{R}^N$ satisfy (2.1) and (2.9) in the above sense if and only if the following equation holds for every $t \geqq 0$

$$\begin{pmatrix} x(t) \\ w(t) \end{pmatrix} = S_e(t)\begin{pmatrix} x_0 \\ w_0 \end{pmatrix} + \int_0^t S_e(t-s)B_e\left[F_e\begin{pmatrix} x(s) \\ w(s) \end{pmatrix} + \begin{pmatrix} v(s) \\ 0 \end{pmatrix}\right] ds.$$

This proves the statement of the proposition. □

The following hypothesis together with (H1) will turn out to be sufficient for the existence of a stabilizing, finite dimensional compensator for system (2.1). It generalizes the approach of Schumacher [15], [16], [17] to systems with unbounded input operators.

(H2) Suppose that there exist operators $F \in \mathscr{L}(X, \mathbb{R}^l)$, $G \in \mathscr{L}(\mathbb{R}^m, X)$ and a finite dimensional subspace $W \subset X$ such that the following conditions are satisfied.

1. The feedback semigroup $S_F(t) \in \mathscr{L}(X)$, defined by (2.3), is exponentially stable.
2. The observer semigroup $S^G(t) \in \mathscr{L}(X)$, generated by $A + GC$ is exponentially stable.
3. $S_F(t) W \subset W$ for all $t \geqq 0$.
4. Range $G \subset W$.

If (H2) is satisfied and $N = \dim W$, then there exist linear maps $\iota : \mathbb{R}^N \to X$, $\pi : X \to \mathbb{R}^N$ satisfying

$$\pi\iota = \mathrm{id}, \quad \iota\pi x = x, \quad x \in W. \tag{2.10}$$

Moreover, $W \subset \mathscr{D}(A_F)$ and hence $\pi A_F \iota$ is a well defined linear map on $\mathbb{R}^N$. We will show that the system

$$\begin{aligned} \dot{w}(t) &= \pi(A_F + GC)\iota w(t) - \pi G y(t), \quad w(0) = w_0, \\ u(t) &= F\iota w(t), \end{aligned} \tag{2.11}$$

defines a stabilizing compensator for the Cauchy problem (2.1).

THEOREM 2.5. *If* (H1), (H2) *and* (2.10) *are satisfied, then the closed loop system* (2.1), (2.11) *is exponentially stable.*

Proof. By Proposition 2.4, the system (2.1), (2.11) is a well-posed Cauchy problem. Now let $x(t) \in X$, $w(t) \in \mathbb{R}^N$ be any solution pair of (2.1), (2.11) and define

$$z(t) = \iota w(t) - x(t) \in X, \qquad t \geqq 0. \tag{2.12}$$

Then

$$\dot{w}(t) = \pi A_F \iota w(t) + \pi G C z(t), \tag{2.13}$$

and hence, using $\pi S_F(t)\iota = e^{\pi A_F \iota t}$ and Theorem 2.3(vi) with $f(t) = GCz(t)$, we get

$$\begin{aligned} z(t) &= \iota\pi S_F(t)\iota w_0 + \int_0^t \iota\pi S_F(t-s)\iota\pi G C z(s)\, ds - x(t) \\ &= S_F(t)\iota w_0 + \int_0^t S_F(t-s) G C z(s)\, ds - x(t) \\ &= S(t)\iota w_0 + \int_0^t S(t-s)[BF\iota w(s) + GCz(s)]\, ds \\ &\quad - S(t)x_0 - \int_0^t S(t-s) B u(s)\, ds \\ &= S(t) z(0) + \int_0^t S(t-s) G C z(s)\, ds, \qquad t \geqq 0. \end{aligned}$$

This implies $z(t) = S^G(t) z(0)$ and hence, by (2.13), stability of the pair $z(t)$, $w(t)$. Now the stability of the pair $x(t)$, $w(t)$ follows from (2.12). □

Clearly, the hypothesis (H2) is not very useful in the present form since it is rather difficult to check in concrete examples. Following the ideas of Schumacher [16], we transform (H2) into an easily verifiable criterion. The basic idea is to approximate G by generalized eigenvectors of A_F and to show that, if A has a complete set of generalized eigenvectors and is stabilizable through B, then there exists a stabilizing feedback operator F which does not destroy the completeness property of A.

More precisely, we need the following assumptions on A.

(H3) The resolvent operator of A is compact and the set $\Lambda = \{\lambda \in P\sigma(A) | \operatorname{Re} \lambda \geqq -\omega\}$ is finite for some $\omega > 0$.

If (H3) is satisfied, then we may introduce the projection operator

$$P_\Lambda = \frac{1}{2\pi i} \int_T (\mu I - A)^{-1} \, d\mu$$

where Γ is a simple rectifiable curve surrounding Λ but no other eigenvalue of A. Clearly, P_Λ is a projection operator on both X and Z. Correspondingly we obtain the decomposition

$$X = X_\Lambda \oplus X^\Lambda, \qquad Z = X_\Lambda \oplus Z^\Lambda,$$

where $X_\Lambda = \operatorname{range} P_\Lambda$, $Z^\Lambda = \ker P_\Lambda$, $X^\Lambda = Z^\Lambda \cap X$ are invariant subsapces under $S(t)$. If $N_\Lambda = \dim X_\Lambda$, we may identify X_Λ with $\mathbb{R}^{N_\Lambda}$ and obtain two maps

$$\iota_\Lambda : \mathbb{R}^{N_\Lambda} \to X_\Lambda, \qquad \pi_\Lambda : Z \to \mathbb{R}^{N_\Lambda}$$

with the properties

$$\pi_\Lambda \iota_\Lambda = \mathrm{id}, \qquad \iota_\Lambda \pi_\Lambda = P_\Lambda. \tag{2.14}$$

Then the projection $x_\Lambda(t) = \pi_\Lambda x(t)$ of a solution to (2.1) satisfies the finite dimensional ODE

$$\begin{aligned} \dot{x}_\Lambda(t) &= A_\Lambda x_\Lambda(t) + B_\Lambda u(t), \qquad x_\Lambda(0) = \pi_\Lambda x_0, \\ y_\Lambda(t) &= C_\Lambda x_\Lambda(t) \end{aligned} \tag{2.15}$$

where

$$A_\Lambda = \pi_\Lambda A \iota_\Lambda, \quad B_\Lambda = \pi_\Lambda B, \quad C_\Lambda = C \iota_\Lambda. \tag{2.16}$$

Now we can replace (H2) by the following stronger conditions which can in many cases be easily verified. The result has been proved by Schumacher [16] for the case that range $B \subset X$ (bounded input operator).

PROPOSITION 2.6. *Let the operator A satisfy* (H3), *assume that the exponential estimate*

$$\| S(t)|_{X^\Lambda} \|_{\mathscr{L}(X^\Lambda)} \leqq M e^{-\omega t}, \qquad t \geqq 0, \tag{2.17}$$

holds for some $M \geqq 1$ and that the reduced finite dimensional system (2.15) *is controllable and observable. Furthermore, assume that the generalized eigenvectors of A are complete in X. Then* (H2) *is satisfied.*

Proof. Since (2.15) is controllable, there exists a matrix $F_\Lambda \in \mathbb{R}^{l \times N_\Lambda}$ such that the matrix $A_\Lambda + B_\Lambda F_\Lambda$ is stable. Furthermore, the estimate (2.17) implies that

$$\| S(t)|_{Z^\Lambda} \|_{\mathscr{L}(Z^\Lambda)} \leqq M e^{-\omega t}, \qquad t \geqq 0,$$

(Remark 2.1(iii)). It is a well-known result in infinite dimensional systems theory (see e.g. [5] or [16]) that under these assumptions the closed loop semigroup $S_F(t) \in \mathscr{L}(Z)$, generated by $A + BF : X \to Z$ with $F = F_\Lambda \pi_\Lambda : Z \to \mathbb{R}^l$ is exponentially stable. By Theorem 2.3, $S_F(t)$ restricts to a strongly continuous semigroup on X and the operator $\mu I - A - BF : X \to Z$ provides a similarity action between both semigroups (Remark 2.1(ii)). Hence the restricted semigroup $S_F(t) \in \mathscr{L}(X)$ is still exponentially stable (Remark 2.1(iii)).

By assumption, the generalized eigenvectors of A are complete in X and hence they are complete in Z (Remark 2.1(iv)). Therefore it is possible to choose the feedback matrix F_Λ such that $S_F(t)$ is exponentially stable and the generalized eigenvectors of $A+BF: X\to Z$ are complete in Z (Schumacher [16]). It follows again from the above similarity argument, that this completeness property carries over to the restricted operator $A_F: \mathcal{D}(A_F)\to X$ introduced in Theorem 2.3(i).

Now choose $G_\Lambda \in \mathbb{R}^{N_\Lambda\times m}$ such that $A_\Lambda + G_\Lambda C_\Lambda$ is stable and define $G=\iota_\Lambda G_\Lambda: \mathbb{R}^m\to X$. Then it is again a well-known fact from infinite-dimensional linear systems theory that $A+GC: \mathcal{D}(A)\to X$ generates an exponentially stable semigroup on X (see [5] or [16]). It is also well known that $A+\hat{G}C$ still generates an exponentially stable semigroup on X whenever $\|\hat{G}-G\|_{\mathcal{L}(\mathbb{R}^m,X)}$ is sufficiently small. Now we make use of the fact that the generalized eigenvectors of A_F are complete in X. This implies that $G: R^m\to X$ can be approximated arbitrarily close by an operator $\hat{G}: \mathbb{R}^m\to X$ whose range is spanned by finitely many generalized eigenvectors of A_F. We choose $\hat{G}$ in such a way that $A+\hat{G}C$ generates a stable semigroup and denote by W the finite dimensional subspace of X which is invariant under A_F and generated by those generalized eigenvectors which span the range of $\hat{G}$. Since W is a finite dimensional subspace contained in $\mathcal{D}(A_F)$ and invariant under A_F, the restriction of A_F to W is a bounded, linear operator generating a semigroup $S_W(t)$ on W. Since $d/dt\, S_W(t)x = A_F S_W(t)x$, the semigroup $S_W(t)$ coincides with $S_F(t)$ on W. Hence W is also invariant under the semigroup $S_F(t)$. We conclude that the operators $F: Z\to\mathbb{R}^l$ and $\hat{G}: \mathbb{R}^m\to X$ satisfy hypothesis (H2). □

Combining Proposition 2.6 with Theorem 2.5, we obtain a constructive procedure for designing a finite dimensional compensator for the Cauchy problem (2.1). The construction is based on the knowledge of the finite dimensional reduced system (2.15) and on the knowledge of sufficiently many eigenvalues and eigenvectors of the operator A_F. For the case of bounded input operators (range $B\subset X$) the procedure has been described in detail by Schumacher [16]. Precisely the same algorithm applies to the case where range $B\not\subset X$.

3. Retarded systems. In this section we apply the abstract result of the previous section to retarded functional differential equations (RFDE) with delays either in the input or in the output variable. If delays occur in the input and output variables at the same time, the RFDE can still be reformulated as an abstract Cauchy problem (see e.g. Pritchard–Salamon [12]) however, the completeness assumption will no longer be satisfied.

3.1. Retarded systems with output delays. We consider the linear RFDE

$$\begin{aligned} \dot{x}(t) &= Lx_t + B_0 u(t), \\ y(t) &= Cx_t, \end{aligned} \tag{3.1}$$

where $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^l$, $y(t)\in\mathbb{R}^m$ and x_t is defined by $x_t(\tau)=x(t+\tau)$ for $-h\leqq\tau\leqq 0$, $h>0$. Correspondingly B_0 is a real $n\times l$-matrix and L and C are bounded linear functionals on $\mathscr{C}=\mathscr{C}[-h,0;\mathbb{R}^n]$ with values in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. These can be written in the form

$$L\phi=\int_{-h}^{0} d\eta(\tau)\phi(\tau), \quad C\phi=\int_{-h}^{0} d\gamma(\tau)\phi(\tau), \quad \phi\in\mathscr{C},$$

where $\eta(\tau)$ and $\gamma(\tau)$ are normalized functions of bounded variation, i.e. they vanish for $\tau\geqq 0$, are constant for $\tau\leqq -h$ and left continuous for $-h<\tau<0$.

It is well known that equation (3.1) admits a unique solution $x(\cdot)\in L^p_{\rm loc}[-h,\infty;\mathbb{R}^n]\cap W^{1,p}_{\rm loc}[0,\infty;\mathbb{R}^n]$ for every input $u(\cdot)\in L^p_{\rm loc}[0,\infty;\mathbb{R}^l]$ and every initial condition of the form

$$x(0)=\phi^0,\quad x(\tau)=\phi^1(\tau),\quad -h\leqq\tau\leqq 0, \tag{3.2}$$

where $\phi=(\phi^0,\phi^1)\in\mathbb{R}^n\times L^p[-h,0;\mathbb{R}^n]=M^p$. Moreover, in these spaces the solution $x(\cdot)$ of (3.1) and (3.2) depends continuously on ϕ and $u(\cdot)$. This has motivated the definition of the state of system (3.1) at time $t\geqq 0$ to be the pair

$$\hat{x}(t)=(x(t),x_t)\in M^p. \tag{3.3}$$

The evolution of $\hat{x}(t)$ can be described by the variation-of-constants formula

$$\hat{x}(t)=S(t)\phi+\int_0^t S(t-s)Bu(s)\,ds,\qquad t\geqq 0, \tag{3.4}$$

where $B\in\mathscr{L}(\mathbb{R}^l,M^p)$ maps $u\in\mathbb{R}^l$ into the pair $Bu=(B_0u,0)$ and $S(t)\in\mathscr{L}(M^p)$ is the strongly continuous semigroup generated by

$$\mathscr{D}(A)=\{\phi\in M^p\,|\,\phi^1\in W^{1,p},\ \phi^1(0)=\phi^0\},$$
$$A\phi=(L\phi^1,\dot{\phi}^1).$$

Here $W^{1,p}$ denotes the Sobolev space $W^{1,p}[-h,0,\mathbb{R}^n]$.

We will consider the evolution of the state (3.3) of system (3.1) in the dense subspace $\{(\phi(0),\phi)|\phi\in W^{1,p}\}\subset M^p$ which we shall identify with $W^{1,p}$. Then B becomes an "unbounded" operator ranging in the larger space M^p. However, it follows from the existence, uniqueness and continuous dependence result for the solutions of (3.1) and (3.2) that the state $\hat{x}(t)$ of (3.1), (3.2) defines a continuous function in $W^{1,p}$ provided that $\phi\in W^{1,p}$ and $u(\cdot)\in L^p_{\rm loc}[0,\infty;\mathbb{R}^l]$. Hence, the operators A and B satisfy hypothesis (H1) with $Z=M^p$ and $X=W^{1,p}$. This implies that the state $\hat{x}(t)\in W^{1,p}$ of (3.1), (3.2) with $\phi\in W^{1,p}$ satisfies the Cauchy problem

$$\begin{aligned}&\frac{d}{dt}\hat{x}(t)=A\hat{x}(t)+Bu(t),\qquad \hat{x}(0)=\phi\in W^{1,p},\\ &v(t)=C\hat{x}(t),\end{aligned} \tag{3.5}$$

in the sense of Remark 2.2(ii). Of course, the output operator $C\colon W^{1,p}\to\mathbb{R}^m$ is given by

$$C\phi=\int_{-h}^0 d\gamma(\tau)\phi^1(\tau),\qquad \phi\in W^{1,p}.$$

On the state space $W^{1,p}$ this operator is bounded.

Remarks 3.1. (i) If the equation

$$\eta(\tau)=\eta(-h)+A_1,\qquad -h<\tau\leqq\varepsilon-h, \tag{3.6}$$

holds for some $\varepsilon>0$, then the generalized eigenfunctions of A are complete in M^p and in $W^{1,p}$ if and only if

$$\det A_1\neq 0 \tag{3.7}$$

(see Manitius [9], Salamon [14, Chap. 3]).

(ii) It is well known that the operator A satisfies (H3).

(iii) The exponential growth of the semigroup $S(t)$ on the complementary subspace X^Λ corresponding to $\Lambda=\{\lambda\in\sigma(A)|\operatorname{Re}\lambda\geqq 0\}$ is determined by $\sup\{\operatorname{Re}\lambda|\lambda\in\sigma(A),\operatorname{Re}\lambda<0\}<0$.

(iv) Let (2.15) denote the reduced finite-dimensional system obtained by spectral projection of the solutions of (3.5) on the generalized eigenspace X_Λ. Then (2.15) is controllable iff

$$\operatorname{rank}[\lambda I - L(e^{\lambda\cdot}), B_0] = n \quad \forall \lambda \in \Lambda \tag{3.8}$$

(Pandolfi [10]) and observable iff

$$\operatorname{rank}\begin{bmatrix} \lambda I - L(e^{\lambda\cdot}) \\ C(e^{\lambda\cdot}) \end{bmatrix} = n \quad \forall \lambda \in \Lambda \tag{3.9}$$

(Bhat-Koivo [1], Salamon [13], [14]).

Combining these facts with Proposition 2.6 and Theorem 2.5, we obtain the following existence result for a finite dimensional compensator for system (3.1).

THEOREM 3.2. *If* (3.6)-(3.9) *are satisfied, then there exists a finite dimensional compensator of the form* (2.9) *such that the closed loop system* (3.1), (2.9) *is exponentially stable.*

3.2. Retarded systems with input delays. In this section we consider the RFDE

$$\begin{aligned} \dot{x}(t) &= Lx_t + Bu_t, \\ y(t) &= C_0 x(t), \end{aligned} \tag{3.10}$$

with general delays in the state and input and no delays in the output variable. This time C_0 is a real $m \times n$-matrix and B a bounded linear functional on $\mathscr{C}[-h, 0, \mathbb{R}^l]$ with values in $\mathbb{R}^n$ given by

$$B\xi = \int_{-h}^{0} d\beta(\tau)\xi(\tau), \qquad \xi \in \mathscr{C}[-h, 0, \mathbb{R}^l],$$

where $\beta(\tau)$ is an $n \times l$-matrix valued, normalized function of bounded variation. Of course, we can immediately get an existence result for a finite dimensional compensator for system (3.10) by dualizing Theorem 3.2. However, for reasons to become clear later, we make use of a direct approach for system (3.10), following the ideas of Vinter and Kwong [18] (see also Delfour [6], Salamon [14]).

First note that (3.10) admits a unique solution $x(\cdot) \in L^p_{\text{loc}}[-h, \infty, \mathbb{R}^n] \cap W^{1,p}_{\text{loc}}[0, \infty, \mathbb{R}^n]$ for every input $u(\cdot) \in L^p_{\text{loc}}[0, \infty; \mathbb{R}^l]$ and every initial condition of the form

$$\begin{aligned} x(0) &= \phi^0, \qquad x(\tau) = \phi^1(\tau), \\ u(\tau) &= \xi(\tau), \qquad -h \leqq \tau < 0, \end{aligned} \tag{3.11}$$

where $\phi \in M^p$ and $\xi \in L^p[-h, 0; \mathbb{R}^l]$. In order to reformulate system (3.10) as an evolution equation in a product space, we rewrite (3.10)-(3.11) as

$$\begin{aligned} \dot{x}(t) &= \int_{-t}^{0} d\eta(\tau)x(t+\tau) + \int_{-t}^{0} d\beta(\tau)u(t+\tau) + f^1(-t), \\ y(t) &= C_0 x(t), \qquad x(0) = f^0, \end{aligned} \tag{3.12}$$

where the pair $f = (f^0, f^1) \in M^p$, given by

$$\begin{aligned} f^0 &= \phi^0, \\ f^1(\sigma) &= \int_{-h}^{\sigma} d\eta(\tau)\phi^1(\tau-\sigma) + \int_{-h}^{\sigma} d\beta(\tau)\xi(\tau-\sigma), \qquad -h \leqq \sigma \leqq 0, \end{aligned} \tag{3.13}$$

is regarded as the initial state of system (3.12). The corresponding state at time $t \geqq 0$ is given by

$$\hat{\hat{x}}(t) = (x(t), x^t) \in M^p,$$
(3.14)
$$x^t(\sigma) = \int_{\sigma-t}^{\sigma} d\eta(\tau)x(t+\tau-\sigma) + \int_{\sigma-t}^{\sigma} d\beta(\tau)u(t+\tau-\sigma) + f^1(\sigma-t).$$

It has been shown in [6], [14], [18] that the evolution of this state can be described by the variation-of-constants formula

$$\hat{\hat{x}}(t) = S^{T^*}(t)f + \int_0^t S^{T^*}(t-s)B^{T^*}u(s)\, ds, \qquad t \geqq 0. \tag{3.15}$$

Here $S^{T^*}(t) \in \mathscr{L}(M^p)$ is the adjoint semigroup of $S^T(t) \in \mathscr{L}(M^q)$, $1/p + 1/q = 1$, which corresponds to the transposed equation $\dot{x}(t) = L^T x_t$ in the sense of § 3.1. Since $S^T(t)$ restricts to a semigroup on the dense subspace $W^{1,q} \subset M^q$, the adjoint semigroup $S^{T^*}(t)$ extends to the dual space $W^{-1,p} = (W^{1,q})^*$ which contains M^p as a dense subspace in a natural way. The input operator $B^{T^*} \in \mathscr{L}(\mathbb{R}^l, W^{-1,p})$ is the adjoint operator of $B^T \in \mathscr{L}(W^{1,q}, \mathbb{R}^l)$ given by

$$B^T\psi = \int_{-h}^0 d\beta(\tau)\psi^1(\tau) \in \mathbb{R}^l, \qquad \psi \in W^{1,q} \subset M^q.$$

Since the infinitesimal generator A^{T^*} of $S^{T^*}(t)$ and the input operator B^{T^*} satisfy the hypothesis (H1) of § 2 with $X = M^p$ and $Z = W^{-1,p}$ (see Salamon [14]), the state $\hat{\hat{x}}(t) \in M^p$ of system (3.12), given by (3.14), defines the unique solution of the abstract Cauchy problem

$$\frac{d}{dt}\hat{\hat{x}}(t) = A^{T^*}\hat{\hat{x}}(t) + B^{T^*}u(t), \qquad \hat{\hat{x}}(0) = f \in M^p,$$
(3.16)
$$y(t) = C\hat{\hat{x}}(t),$$

in the sense of Remark 2.2(ii). Of course, the output operator $C: M^p \to \mathbb{R}^m$ is now given by

$$Cf = C_0 f^0 \in \mathbb{R}^m, \qquad f \in M^p.$$

In order to make the results of this section more precise, we briefly outline the construction of the reduced system (2.15). For this purpose let $X_\Lambda \subset W^{1,p}$ and $X_\Lambda^T \subset W^{1,q}$ denote the generalized eigenspaces of A and A^T, respectively, correspondig to $\Lambda = \{\lambda \in \sigma(A) | \text{Re}\, \lambda \geqq 0\}$. Since Λ is a symmetric set, we can choose real bases $\{\phi_1, \cdots, \phi_{N_\Lambda}\}$ or X_Λ and $\{\psi_1, \cdots, \psi_{N_\Lambda}\}$ of X_Λ^T such that the matrices

$$\Phi = [\phi_1 \cdots \phi_{N_\Lambda}] \in W^{1,p}[-h, 0; \mathbb{R}^{n\times N_\Lambda}],$$
$$\Psi = [\psi_1 \cdots \psi_{N_\Lambda}] \in W^{1,q}[-h, 0; \mathbb{R}^{n\times N_\Lambda}],$$

satisfy

$$\Psi^T(0)\Phi(0) + \int_{-h}^0 \int_\tau^0 \Psi^T(\sigma)\, d\eta(\tau)\Phi(\tau-\sigma)\, d\sigma = I \in \mathbb{R}^{N_\Lambda \times N_\Lambda}.$$

Then $\iota_\Lambda: \mathbb{R}^{N_\Lambda} \to M^p$ and $\pi_\Lambda: M^p \to \mathbb{R}^{N_\Lambda}$ may be defined by

$$[\iota_\Lambda x_\Lambda]^0 = \Phi(0)x_\Lambda, \quad [\iota_\Lambda x_\Lambda]^1(\sigma) = \int_{-h}^\sigma d\eta(\tau)\Phi(\tau-\sigma)x_\Lambda, \quad -h \leqq \sigma \leqq 0,$$

$$\pi_\Lambda f = \Psi^T(0)f^0 + \int_{-h}^0 \Psi^T(\sigma)f^1(\sigma)\, d\sigma$$

for $x_\Lambda \in \mathbb{R}^{N_\Lambda}$ and $f \in M^p$, and the matrices $A_\Lambda \in \mathbb{R}^{N_\Lambda \times N_\Lambda}$, $B_\Lambda \in \mathbb{R}^{N_\Lambda \times l}$, $C_\Lambda \in \mathbb{R}^{m \times N_\Lambda}$ are given by

$$A(\Phi(0),\Phi) = (\Phi(0), \Phi)A_\Lambda, \quad B_\Lambda = \int_{-h}^{0} \Psi^T(\tau)\, d\beta(\tau), \quad C_\Lambda = C_0\Phi(0)$$

(see Salamon [14, § 2.4]).

Remarks 3.3. (i) If (3.6) is satisfied for some $\varepsilon > 0$, then the eigenfunctions of A^{T^*} are complete in M^P and in $W^{-1,p}$ if and only if (3.7) holds (see Manitius [9], Salamon [14, Chapter 3]).

(ii) If A_Λ, B_Λ, C_Λ are defined as above, then system (2.15) is controllable iff

$$\operatorname{rank}[\lambda I - L(e^{\lambda \cdot}), B(e^{\lambda \cdot})] = n \quad \forall \lambda \in \Lambda \tag{3.17}$$

and observable iff

$$\operatorname{rank}\begin{bmatrix} \lambda I - L(e^{\lambda \cdot}) \\ C_0 \end{bmatrix} = n \quad \forall \lambda \in \Lambda \tag{3.18}$$

(see Salamon [14]).

THEOREM 3.4. *If* (3.6)-(3.7) *and* (3.17)-(3.18) *are satisfied, then there exists a finite-dimensional compensator of the form* (2.9), *such that the closed loop system* (3.10), (2.9) *is exponentially stable.*

Remark 3.1(i) and Remark 3.3(i) show that the completeness property of A and A^{T^*} can be destroyed by arbitrarily small perturbations in the delays (compare Manitius [9]). However such perturbations would not affect the stability of the closed loop system (3.1), (2.9) respectively (3.10), (2.9). This indicates that the completeness assumption is somewhat artifical for the purpose of stabilization by a finite-dimensional compensator. This assumption can be weakened slightly in the special case of the RFDE

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + A_1x(t-h) + B_0u(t), \\ y(t) &= C_0x(t), \end{aligned} \tag{3.19}$$

with a single point delay in the state variable if the state space is chosen to be

$$X = (f \in M^p | f^1(\tau) \in \operatorname{range} A_1, \ -h \leqq \tau \leqq 0\}.$$

It has been shown by Manitius [9] that the completeness property for the operator A^{T^*} in this space is equivalent to the rank condition

$$\operatorname{rank}\begin{bmatrix} A_0 - \lambda I & A_1 \\ A_1 & 0 \end{bmatrix} = n + \operatorname{rank} A_1 \tag{3.20}$$

for some $\lambda \in \mathbb{C}$. Therefore we have the following result.

COROLLARY 3.5. *If* (3.8), (3.18) *and* (3.20) *are satisfied then there exists a finite dimensional compensator of the form* (2.9) *such that the closed loop system* (3.19), (2.9) *is exponentially stable.*

Remark 3.6. This result suggests that it should be possible to weaken the completeness assumption for the general RFDE which would be an important improvement. Another extension would be an existence result for RFDEs with delays in simultaneously the control and the observation. However, it is not obvious how this can be achieved with the present approach, the main difficulty being the completeness property.

Remark 3.7. Although the main results of this section, Theorems 3.2 and 3.4 are stated as existence results, we emphasize that the stabilizing compensator can in fact be constructed using exactly the same procedure as it is explained in detail in [16] for

RFDEs without delays in the external variables. The construction, as outlined in the proof of Proposition 2.6, involves calculating finitely many eigenvalues and eigenvectors of A and hence the projected system A_Λ, B_Λ, C_Λ as it is described in § 3.2. Then the matrices F_Λ and G_Λ can be calculated by standard finite dimensional procedures. The most difficult part of the design lies in finding eigenvectors of $A+BF$ to generate the subspace W. The approximation of $G=\iota_\Lambda G_\Lambda$ by an operator $\hat{G}$ with range in W then reduces to a finite-dimensional linear optimization procedure. This procedure has to be repeated—while increasing W—until $\hat{G}$ is close enough to G. The numerical example for a retarded system examined in [16] gives insight into the details of the design procedure.

4. Boundary control systems. The purpose of this section is to show how abstract boundary control systems in Hilbert spaces fit into the framework of § 2. When these results are applied to obtain finite-dimensional compensators for particular classes of partial differential equations (PDE), there is a considerable overlap with results of Curtain in [2], [3], [4]. The relation between both approaches will be discussed in detail at the end of this section.

Let W, X, U, Y be Hilbert spaces and suppose that

$$W\subset X$$

with a continuous, dense injection. Furthermore, let $\Delta\in\mathscr{L}(W, X)$, $\Gamma\in\mathscr{L}(W, U)$, $C\in\mathscr{L}(X, Y)$ be given. Then we consider the boundary control system

$$\begin{aligned}&\frac{d}{dt}x(t)=\Delta x(t), \qquad x(0)=x_0\in W,\\ &\Gamma x(t)=u(t), \qquad t\geqq 0,\end{aligned} \tag{4.1}$$

with the output

$$y(t)=Cx(t), \qquad t\geqq 0. \tag{4.2}$$

DEFINITION 4.1 (*strong solution, well-posedness*).

(i) Let $u(\cdot)\in\mathscr{C}[0, T; U]$ and $x_0\in W$ satisfy $\Gamma x_0=u(0)$. Then a function $x(\cdot)\in\mathscr{C}[0, T; W]$ is said to be a (strong) solution of (4.1) if $x(\cdot)\in\mathscr{C}^1[0, T; X]$ and if (4.1) is satisfied for every $t\in[0, T]$.

(ii) The boundary control system (4.1) is said to be well-posed if the subspace $\{x\in W|\Gamma x=0\}$ is dense in X, if the restriction of Λ to this subspace is a closed operator on X, and if for all $x_0\in W$ and $u(\cdot)\in W^{1,2}[0, T; U]$ with $\Gamma x_0=u(0)$ there exists a unique solution $x(\cdot)\in\mathscr{C}[0, t; W]\cap\mathscr{C}^1[0, T; X]$ of (4.1) depending continuously on x_0 and $u(\cdot)$. This means that there exists a constant $K>0$ such that the inequality

$$\sup_{0\leqq t\leqq T}\|x(t)\|_W+\sup_{0\leqq t\leqq T}\|\dot{x}(t)\|_X\leqq K\left\{\|x_0\|_W+\left[\int_0^T\|\dot{u}(t)\|_U^2\,dt\right]^{1/2}\right\}$$

holds for every solution $x(t)$ of (4.1).

Remarks 4.2. Let system (4.1) be well posed.

(i) Taking $u(t)\equiv 0$, it follows from a classical result in semigroup theory (Phillips [11]) that the operator

$$Ax=\Delta x, \qquad \mathscr{D}(A)=\{x\in W|\Gamma x=0\} \tag{4.3}$$

is the infinitesimal generator of a strongly continuous semigroup $S(t)$ on X and that, for every $x_0\in\mathscr{D}(A)$, the function $x(t)=S(t)x_0$ is the solution of (4.1) with $u(t)\equiv 0$.

(ii) As in § 2 we introduce the dense subspace $Z^* = \mathscr{D}_{X^*}(A^*) \subset X^*$. Then

$$X \subset Z$$

with a continuous, dense injection, A extends to a bounded operator from X to Z and $S(t)$ to a strongly continuous semigroup on Z.

(iii) It follows from [14, Lemma 1.3.2(i)] that

$$\{x \in W | \Gamma x = 0\} = \mathscr{D}_X(A) = \{x \in X = \mathscr{D}_Z(A) | Ax \in X\}$$

or in other words, if $x \in X$ and $Ax \in X$, then $x \in W$ and $\Gamma x = 0$. Furthermore, the W-norm on $\mathscr{D}_X(A)$ is equivalent to the graph norm of A [14, Remark 1.3.1(iii)]. This means that there exists a constant $K_1 > 0$ such that the inequality

$$\|x\|_W \leqq K_1[\|x\|_X + \|Ax\|_X]$$

holds for all $x \in W$ with $\Gamma x = 0$.

(iv) Γ is onto. Hence there exists a constant $K_0 > 0$ such that for every $u \in U$ there exists a $w \in W$ such that

$$\Gamma w = u, \qquad \|w\|_W \leqq K_0 \|u\|_U. \tag{4.4}$$

Let us now construct the input operator $B \in \mathscr{L}(U, Z)$.

1. Given $u \in U$ we may choose $w \in W$ such that $\Gamma w = u$ since Γ is onto (Remark 4.2(iv)). For this $w \in W$ we define $Bu := \Delta w - Aw \in Z$. This expression is well defined since $\Gamma w = 0$ if and only if $Aw = \Delta w$ (Remark 4.2(iii)). Hence the map $B: U \to Z$ satisfies, by definition, the equation

$$B\Gamma x = \Delta x - Ax, \qquad x \in W. \tag{4.5}$$

2. It is easy to see that B is a linear map.

3. Let $u \in U$ be given and choose $w \in W$ such that (4.4) holds. Then

$$\|Bu\|_Z \leqq \|\Delta - A\|_{\mathscr{L}(W,Z)} \|w\|_W \leqq K_0 \|\Delta - A\|_{\mathscr{L}(W,Z)} \|u\|_U$$

and therefore $B: U \to Z$ is bounded.

LEMMA 4.3. *Let the operators A and B be defined by* (4.3) *and* (4.5), *respectively. Furthermore, let $x \in X$ and $u \in U$ satisfy $Ax + Bu \in X$. Then*

$$x \in W, \quad \Gamma x = u, \quad \Delta x = Ax + Bu.$$

Furthermore there exists a constant $K > 0$ such that

$$\|x\|_W \leqq K[\|x\|_X + \|u\|_U + \|Ax + Bu\|_X]$$

for all $x \in X$, $u \in U$ with $Ax + Bu \in X$.

Proof. Let $x \in X$ and $u \in U$ satisfy $Ax + Bu \in X$ and choose $w \in W$ such that (4.4) holds. Then

$$A(x - w) = Ax + Bu - (A + B\Gamma)w = Ax + Bu - \Delta w \in X.$$

By Remark 4.2(iii), this implies that $x \in W$ and

$$\begin{aligned}
\|x\|_W &\leqq \|w\|_W + \|x - w\|_W \\
&\leqq \|w\|_W + K_1[\|x - w\|_X + \|A(x - w)\|_X] \\
&\leqq [1 + K_1 \|\mathrm{id}\|_{\mathscr{L}(W,X)} + K_1 \|\Delta\|_{\mathscr{L}(W,X)}] \|w\|_W \\
&\quad + K_1 \|x\|_X + K_1 \|Ax + Bu\|_X \\
&\leqq K[\|u\|_U + \|x\|_X + \|Ax + Bu\|_X].
\end{aligned}$$

Finally, we obtain again from Remark 4.2(iii) that $\Gamma x = \Gamma w = u$ and from (4.5) that $\Delta x = Ax + B\Gamma x = Ax + Bu$. □

The above operators A and B allow us to reformulate the boundary control system (4.1) as a Cauchy problem of the type (3.1). More precisely, we introduce the following concept of a weak solution for (4.1).

DEFINITION 4.4 (*weak solution*). Let the operators $A \in \mathscr{L}(X, Z)$ and $B \in \mathscr{L}(U, Z)$ be defined as above. Moreover, let $x_0 \in X$ and $u(\cdot) \in L^2[0, T, U]$ be given. Then $x(\cdot) \in \mathscr{C}[0, T; X] \cap W^{1,2}[0, T; Z]$ is said to be a weak solution of (4.1) if

$$\tag{4.6} \begin{aligned} &\frac{d}{dt} x(t) = Ax(t) + Bu(t), \qquad 0 \leqq t \leqq T, \\ &x(0) = x_0 \end{aligned}$$

is satisfied in Z (almost everywhere).

It follows from the definition of the operator B (Remark 4.2(iv)) that every strong solution $x(\cdot) \in \mathscr{C}[0, T; W] \cap \mathscr{C}^1[0, T; X]$ of (4.1) is a weak solution in the sense of Definition 4.4. Moreover we have the following result.

PROPOSITION 4.5. *Suppose that the operator A defined by* (4.3) *is the infinitesimal generator of a strongly continuous semigroup $S(t)$ on X and that $\Gamma \in \mathscr{L}(W, U)$ is onto. Furthermore let $B \in \mathscr{L}(U, Z)$ be defined by* (4.5). *Then the following statements are equivalent.*

(i) *System* (4.1) *is well posed.*

(ii) *The operators A and B satisfy hypothesis* (H1) *of* § 2 *with $p = 2$.*

(iii) *For every $x_0 \in X$ and every $u(\cdot) \in L^2[0, T; U]$ there exists a unique weak solution $x(\cdot) \in \mathscr{C}[0, T; X] \cap W^{1,2}[0, T; Z]$ of* (4.1) *depending continuously on x_0 and $u(\cdot)$. Moreover, the weak* (*and in particular the strong*) *solutions of* (4.1) *are given by*

$$\tag{4.7} x(t) = S(t)x_0 + \int_0^t S(t-s)Bu(s)\, ds \in X, \qquad 0 \leqq t \leqq T.$$

Proof. It is a well-known semigroup theoretic result that the solutions of (4.6), and therefore the weak solutions of (4.1), are given by (4.7). Furthermore, it follows from Remark 2.2(ii) that (ii) is equivalent to (iii).

In order to prove that (ii) implies (i), suppose that (H1) is satisfied and let $x(t)$ be given by (4.7) with $x_0 \in W$ and $u(\cdot) \in \mathscr{C}^1[0, T; U]$ satisfying $\Gamma x_0 = u(0)$. Then it is a well-known result from semigroup theory that $x(\cdot) \in \mathscr{C}[0, T; X] \cap \mathscr{C}^1[0, T; Z]$ satisfies

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ &= S(t)[Ax_0 + Bu(0)] + \int_0^t S(t-s)B\dot{u}(s)\, ds \\ &= S(t)\Delta x_0 + \int_0^t S(t-s)B\dot{u}(s)\, ds, \qquad 0 \leqq t \leqq T. \end{aligned}$$

By (H1) and Remark 2.1(ii), this implies that $\dot{x}(\cdot) \in \mathscr{C}[0, T; X]$ and

$$\sup_{0 \leqq t \leqq T} \|\dot{x}(t)\|_X < \sup_{0 \leqq t \leqq T} \|S(t)\|_{\mathscr{L}(X)} \|\Delta\|_{\mathscr{L}(W,X)} \|x_0\|_W + b_T \|\dot{u}(\cdot)\|_{L^2[0,T;U]}.$$

Applying Lemma 4.3 to the term $Ax(t) + Bu(t) = \dot{x}(t) \in X$, we obtain that $x(\cdot) \in \mathscr{C}[0, T; W] \cap \mathscr{C}^1[0, T; X]$ satisfies (4.1). Since every strong solution of (4.1) is given by (4.7), $x(t)$ is in fact the unique solution. Furthermore, we obtain from Lemma 4.3 that

$$\|x(t)\|_W \leqq K[\|x(t)\|_X + \|u(t)\|_U + \|\dot{x}(t)\|_X].$$

Since

$$\sup_{0\leqq t\leqq T} \|x(t)\|_X \leqq \sup_{0\leqq t\leqq T} \|S(t)\|_{\mathscr{L}(X)}\|\mathrm{id}\|_{\mathscr{L}(W,X)}\|x_0\|_W + b_T\|u(\cdot)\|_{L^2[0,T;U]}$$

and

$$\sup_{0\leqq t\leqq T} \|u(t)\|_U \leqq \|\Gamma\|_{\mathscr{L}(W,U)}\|x_0\|_W + \sqrt{T}\|\dot{u}(\cdot)\|_{\mathscr{L}^2[0,T;U]}$$

for $u(\cdot)\in \mathscr{C}^1[0, T; U]$ with $u(0)=\Gamma x_0$, this shows that system (4.1) is well posed.

Conversely, suppose that system (4.1) is well posed in the sense of Definition 4.1, let $v(\cdot)\in\mathscr{C}^1[0, T; U]$ and define

$$x(t)=\int_0^t S(t-s)B\int_0^s v(\tau)\,d\tau\,ds, \qquad u(t)=\int_0^t v(\tau)\,d\tau, \qquad 0\leqq t\leqq T.$$

Then $x(\cdot)\in C^1[0, T; Z]$ and

$$Ax(t)+Bu(t)=\dot{x}(t)=\int_0^t S(t-s)Bv(s)\,ds, \qquad 0\leqq t\leqq T.$$

Hence $x(\cdot)\in\mathscr{C}^1[0, T; X]$ and we obtain from Lemma 4.3 that $x(\cdot)\in\mathscr{C}[0, T; W]$, $\Gamma x(t)=u(t)$ and $\Delta x(t)=Ax(t)+Bu(t)$. Hence $x(t)$ is a strong solution of (4.1) in the sense of Definition 4.1(i) and satisfies the inequality

$$\left\|\int_0^T S(T-s)Bv(s)\,ds\right\|_X = \|\dot{x}(T)\|_X \leqq K\|v(\cdot)\|_{L^2[0,T;U]}.$$

This shows that the operators A and B satisfy the hypothesis (H1). □

Having established hypothesis (H1) we are now in a position to apply the perturbation result of § 2 (Theorem 2.2) to the boundary control system (4.1).

COROLLARY 4.6. *If system* (4.1) *is well posed, then the following statements hold.*

(i) *For every $F\in\mathscr{L}(X, U)$ the operator*

$$A_F x=\Delta x, \qquad \mathscr{D}(A_F)=\{x\in W|\Gamma x=Fx\} \tag{4.8}$$

is the infinitesimal generator of a strongly continuous semigroup $S_F(t)$ on X.

(ii) *For every $x_0\in\mathscr{D}(A_F)$ the function $x(t)=S_F(t)x_0$, $t\geqq 0$, is continuous in W, continuously differentiable in X, and satisfies the closed loop boundary control equations*

$$\begin{aligned} &\frac{d}{dt}x(t)=\Delta x(t), \qquad x(0)=x_0,\\ &\Gamma x(t)=Fx(t), \qquad t\geqq 0, \end{aligned} \tag{4.9}$$

where the derivative has to be understood in the space X.

(iii) *If U is finite dimensional, then $S_F(t)$ extends to a strongly continuous semigroup on Z whose infinitesimal generator is given by the extended operator $A+BF: X\to Z$.*

Proof. By Proposition 4.5, the operators A and B defined by (4.3) and (4.5), respectively, satisfy hypothesis (H1) of § 2. Hence it follows from Theorem 2.3(i) that the operator

$$A_F x=Ax+BFx, \qquad \mathscr{D}(A_F)=\{x\in X|Ax+BFx\in X\}$$

generates a strongly continuous semigroup of X (note that the proof of this result in

[14, Thm. 1.3.7] does not require U to be finite dimensional). Lemma 4.3 shows that this operator A_F coincides with the one defined by (4.8). This proves statement (i).

In order to prove statement (ii), let $x_0 \in \mathcal{D}(A_F)$ be given and define $u(t) = FS_F(t)x_0$, $t \geqq 0$. Then $u(t)$ is continuously differentiable for $t \geqq 0$ and satisfies $u(0) = Fx_0 = \Gamma x_0$. Hence (4.1) admits a unique strong solution $x(t)$, $t \geqq 0$, which by definition of the operators A and B also satisfies (4.6) and is therefore given by (4.7). This implies

$$x(t) = S(t)x_0 + \int_0^t S(t-s)BFS_F(s)x_0\,ds = S_F(t)x_0,$$

by definition of the semigroup $S_F(t)$.

Statement (iii) is an immediate consequence of Theorem 2.3(iii). □

So far we have shown that the general theory of § 2 also covers abstract boundary control systems. In particular, we have reformulated the boundary control system (4.1) in the semigroup theoretic framework with an unbounded input operator. A very similar approach has been developed by Ho and Russell in [8] under only slightly more restrictive assumptions. However, [8] does not contain any feedback results and also the above Proposition 4.5 seems to be new. Furthermore we point out that earlier results in this direction for various classes of partial differential equations can be found, e.g., in the classical work by Lions-Magenes [22], in the more recent papers by Washburn [19], Lasiecka-Triggiani [20], [21] and in the book by Curtain-Pritchard [5] (this list is by no means complete). Another general approach has been presented by Fattorini [7]. In [7] the input operator is bounded, however, there are derivatives in the input function which do not appear in our approach.

In [2] and [3] Curtain has used Fattorini's results for the construction of finite dimensional compensators which leads to integral terms in the loop. These integral terms will disappear if we apply the approach of this section to obtain existence results for finite dimensional compensators. More precisely, we have to assume that the operators A, B and C, introduced in this section, satisfy hypothesis (H2) of § 2, or respectively, hypothesis (H3) and the assumptions of Proposition 2.6. Under these conditions it follows readily from Theorem 2.5 that there exists a finite dimensional compensator of the form (2.9) such that the closed loop system (4.1), (4.2), (2.9) is exponentially stable.

Starting from (4.1), (4.2), the following problems have to be solved for the construction of the compensator.

1. Find the operators A and B.
2. Determine the spectrum of the operator A and the reduced subsystem (2.15).
3. Find the stabilizing operators $F: X \to U$ and $G: Y \to X$.
4. Determine the eigenvalues and eigenvectors of A_F to approximate G.

To illustrate this procedure, we consider the heat equation with Neumann boundary conditions and boundary control which has also been treated in [3] with different methods.

Example 4.7. Consider the parabolic PDE

(4.10.1) $$z_t = \pi^{-2} z_{\xi\xi}, \qquad 0 < \xi < 1, \quad t > 0,$$

(4.10.2) $$z_\xi(0, t) = u(t), \qquad z_\xi(1, t) = 0, \qquad t > 0,$$

(4.10.3) $$z(\xi, 0) = z_0(\xi), \qquad 0 < \xi < 1,$$

(4.10.4) $$y(t) = \int_0^1 c(\xi) z(\xi, t)\,d\xi, \qquad t > 0.$$

This system can be written in the abstract form (4.1) with

$$X = L^2[0,1], \quad W = \{\phi \in H^2[0,1] \mid \dot\phi(1) = 0\}, \quad U = \mathbb{R},$$
$$\Delta\phi = \pi^{-2}\ddot\phi, \qquad \Gamma\phi = \dot\phi(0),$$
$$Z^* = \mathcal{D}(A^*) = \mathcal{D}(A) = \{\psi \in H^2[0,1] \mid \dot\psi(0) = 0 = \dot\psi(1)\}.$$

The operator A satisfies (H3) and has a complete set of eigenvectors $\phi_0(\xi) \equiv 1$, $\phi_n(\xi) = \sqrt{2}\cos n\pi\xi$, corresponding to the eigenvalues $\lambda_0 = 0$, $\lambda_n = -n^2$, $n \in \mathbb{N}$. In order to determine the operator $B: \mathbb{R} \to Z$, let us choose any $\phi \in W$ such that $\Gamma\phi = 1$, e.g. $\phi(\xi) = -(\xi - 1)^2/2$. Then, for every $\psi \in Z^*$, the following equation holds

$$\begin{aligned} B^*\psi &= \langle B^*\psi, \Gamma\phi\rangle = \langle \psi, B\Gamma\phi\rangle = \langle \psi, \Delta\phi - A\phi\rangle_{Z^*,Z} \\ &= \langle \psi, \Delta\phi\rangle_H - \langle A^*\psi, \phi\rangle_H \\ &= \frac{1}{\pi^2}\int_0^1 \psi(\xi)\ddot\phi(\xi)\,d\xi - \frac{1}{\pi^2}\int_0^1 \ddot\psi(\xi)\phi(\xi)\,d\xi \\ &= \frac{1}{\pi^2}\int_0^1 [\psi(\xi)\ddot\phi(\xi) + \dot\psi(\xi)\dot\phi(\xi)]\,d\xi \\ &= \frac{1}{\pi^2}[\psi(1)\dot\phi(1) - \psi(0)\dot\phi(0)] \\ &= -\frac{1}{\pi^2}\psi(0). \end{aligned}$$

It has been shown in Pritchard-Salamon [12] that these operators A and B satisfy hypothesis (H1) and therefore system (4.10) is well posed in $X = L^2[0,1]$ in the sense of Definition 4.1 (see Proposition 4.5). The spectral projection of $L^2[0,1]$ onto the eigenspace $X_\Lambda = \{\alpha\phi_0 \mid \alpha \in \mathbb{R}\}$ of A corresponding to the unstable part $\Lambda = \{0\}$ of the spectrum is given by

$$P_\Lambda\phi(\xi) = \int_0^1 \phi(\tau)\,d\tau, \qquad 0 < \xi < 1.$$

With the choice of $\{\phi_0\}$ as a basis of X_Λ, this operator splits into $P_\Lambda = \iota_\Lambda\pi_\Lambda$, where $\pi_\Lambda: L^2[0,1] \to R$ and $\iota_\Lambda: \mathbb{R} \to L^2[0,1]$ are given by

$$\pi_\Lambda\phi = \int_0^1 \phi(\tau)\,d\tau, \qquad \iota_\Lambda x_\Lambda(\xi) = x_\Lambda, \qquad 0 \leqq \xi \leqq 1.$$

Then the reduced finite dimensional system (2.15) is described by the "matrices"

$$A_\Lambda = 0, \quad B_\Lambda = -\pi^{-2}, \quad C_\Lambda = \int_0^1 c(\xi)\,d\xi.$$

This sytem is controllable and obervable if and only if

$$C_\Lambda = \int_0^1 c(\xi)\,d\xi \neq 0. \tag{4.11}$$

Stabilizing matrices are given e.g. by

$$F_\Lambda = \frac{\pi^2}{4}, \qquad G_\Lambda = -C_\Lambda^{-1}$$

so that $A_\Lambda + B_\Lambda F_\Lambda = -\frac{1}{4}$, $A_\Lambda + G_\Lambda C_\Lambda = -1$. Then the operator A_F with $F = F_\Lambda \pi_\Lambda$: $L^2[0,1] \to \mathbb{R}$ given by

$$\mathscr{D}(A_F) = \left\{ \phi \in H^2[0,1] \,|\, \dot{\phi}(1) = 0,\ \dot{\phi}(0) = \frac{\pi^2}{4} \int_0^1 \phi(\xi)\, d\xi \right\},$$

$$A_F \phi = \frac{1}{\pi^2} \ddot{\phi}.$$

The eigenvectors and eigenvalues of A_F coincide with those of A except for $\lambda_0 = 0$ which is now replaced by $\lambda_F = -\frac{1}{4}$. The corresponding normalized eigenfunction is

$$\phi_F(\xi) = \sqrt{2} \sin \frac{\pi}{2} \xi.$$

We will choose $W = \operatorname{span}\{\phi_F\}$ and the maps

$$(\iota_F w)(\xi) = \phi_F(\xi) w, \qquad 0 < \xi < 1, \quad w \in \mathbb{R},$$

$$(\pi_F \phi)(\xi) = \int_0^1 \phi_F(\xi) \phi(\xi)\, d\xi, \qquad \phi \in L^2[0,1],$$

So that $\iota_F \pi_F: L^2[0,1] \to W$ is the orthogonal projection onto W and $\pi_F \iota_F = 1$.

Let us now consider the case that $c(\xi) = \xi$ for $0 \leqq \xi \leqq 1$. Then $C_\Lambda = \frac{1}{2}$ and we choose $G_\Lambda = -\pi g / 2\sqrt{2}$, $g > 0$. With this choice the operator $G: \mathbb{R} \to L^2[0,1]$ is given by

$$[Gy](\xi) = [\iota_\Lambda G_\Lambda y](\xi) = -\frac{\pi g}{2\sqrt{2}} y, \qquad 0 \leqq \xi \leqq 1.$$

We replace this operator by

$$[\hat{G}y](\xi) = [\iota_F \pi_F G y](\xi) = -g y \sqrt{2} \sin \frac{\pi}{2} \xi, \qquad 0 \leqq \xi \leqq 1,$$

whose range is in W. Since the perturbed operator $A + \hat{G}C$ generates an analytic semigroup it satisfies the "spectrum determined growth" assumption. Furthermore, its spectrum is given by

$$\sigma(A + \hat{G}C) = \{ -\omega^2 | g[1 - \cos \omega\pi] = \omega^3 [8K + \pi^2 / 2\sqrt{2} - \sqrt{2}\pi^2 \omega^2] \sin \omega\pi,\ \omega \neq 0 \}$$

if $g > 0$. In the case $g < 0$ there is an additional positive eigenvalue $\lambda_0 = \omega^2 > 0$ where

$$-g \frac{e^{\omega\pi} + e^{-\omega\pi} - 2}{e^{\omega\pi} - e^{-\omega\pi}} = \omega^3 [8K + \pi^2 / 2\sqrt{2} + \sqrt{2}\pi^2 \omega^2].$$

We conclude that $A + \hat{G}C$ generates an exponentially stable semigroup if and only if $g > 0$. Hence the operators F and $\hat{G}$ satisfy the hypothesis (H2) with the one-dimensional subspace $W = \operatorname{span}\{\phi_F\}$. In this case the compensator (2.9) is described by the "matrices"

$$M = \pi_F (A_F + \hat{G}C) \iota_F = -\frac{1}{4} - \frac{4\sqrt{2}}{\pi^2} g,$$

$$H = \pi_F \hat{G} = -g,$$

$$K = F_\Lambda \pi_\Lambda \iota_F = \frac{\pi}{\sqrt{2}}.$$

Hence the first order system

$$\begin{aligned} \dot{w} &= -\frac{1}{4} w - g \left[\frac{4\sqrt{2}}{\pi^2} w - y \right], \\ u &= \frac{\pi}{\sqrt{2}} w, \end{aligned} \tag{4.12}$$

defines a stablizing compensator for the parabolic PDE (4.10) with $c(\xi) \equiv \xi$ if and only if $g > 0$.

Remark 4.8. The results of this section show that the abstract framework of § 2 is general enough to cover both FDEs and PDEs. We mention that the approach of this section can also be applied to damped hyperbolic systems. Hence this paper represents a complete generalization of the compensator design of Schumacher [16] to infinite dimensional systems with unbounded control action. However, the degree of unboundedness which we can allow for the input/output operators is not as general as one would desire. For example, for the parabolic PDE (4.10) we cannot allow simultaneously Neumann boundary control and point observation. Also we cannot allow Dirichlet boundary control when the output operator is an arbitrary functional on $L^2[0, 1]$. A general theory which covers these cases would require the consideration of *unbounded output operators* as well. The extension of our theory to this case seems to involve some further difficulties and would be an interesting problem for future investigations.

Remark 4.9. Using the abstract approach outlined in § 2, it is possible to directly extend the results of Schumacher [15], [17] on tracking and regulation in infinite dimensions to unbounded control action. A different approach is to use the extended system formulation discussed in [2], which results in integral control action and this can be found in Curtain [4].

Note. Stronger results on finite dimensional compensators for some classes of functional differential equations have recently been developed by Kamen-Khargonekar-Tannenbaum [23], Nett [24], Logemann [25] using frequency domain methods.

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