Notes on complex Lie groups

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Abstract

The purpose of these notes is to give a self-contained proof of the assertion that every compact Lie group G admits a complexification G^c that is unique up to canonical isomorphism. The notes include a proof of the Cartan Decomposition Theorem and of Cartan's Theorem that every compact subgroup of G^c is conjugate in G^c to a subgroup of G. Both proofs are based on the observation that the homogeneous space G^c/G is a Hadamard manifold.

Contents

1 Complex Lie groups

This section introduces the concept of a complex Lie group and states the main results proved in these notes. We begin with some comments on the sign conventions for Lie brackets used throughout. These conventions are based on the following two fundamental principles.

(L1) The Lie bracket of two endomorphisms A, B of a vector space V is

$$[A, B] = AB - BA. \tag{1}$$

(L2) A Lie group homomorphism induces a Lie algebra homomorphism and a Lie group anti-homomorphism induces a Lie algebra anti-homomorphism.

These principles include infinite-dimensional settings. For example, if M is a closed manifold, then the standard group operation on the group $\mathrm{Diff}(M)$ of diffeomorphism of M is the composition

$$\mathrm{Diff}(M) \times \mathrm{Diff}(M) \to \mathrm{Diff}(M) : (\phi, \psi) \mapsto \phi \circ \psi.$$

Thus the pullback action $\phi \mapsto \phi^*$ of $\mathrm{Diff}(M)$ on the space of functions, or differential form, or vector fields is an infinite-dimensional analogue of a Lie group anti-homomorphism. The space $\mathrm{Vect}(M)$ of vector fields on M is understood as the Lie algebra of $\mathrm{Diff}(M)$, and the induced map on the Lie algebra $\mathrm{Vect}(M)$ assigns to every vector field X the Lie derivative $\mathcal{L}_X$, an endomorphism of the respective vector space. Thus the Lie derivative is a Lie algebra anti-homomorphism, and hence the Lie bracket of two vector fields $X, Y \in \mathrm{Vect}(M)$ is determined by each of the two equations

$$\mathcal{L}_{[X,Y]} + [\mathcal{L}_X, \mathcal{L}_Y] = 0, \qquad [X, Y] = \mathcal{L}_Y X. \tag{2}$$

This is consistent with the standard conventions in Lie group theory. More precisely, let G be a Lie group with the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$. Then the map $\mathrm{G} \to \mathrm{Diff}(\mathrm{G}) : h \mapsto \phi_h$, defined by $\phi_h(g) := hg$ for $g, h \in \mathrm{G}$, is a Lie group homomorphism. Accordingly, the induced map $\mathfrak{g} \to \mathrm{Vect}(\mathrm{G}) : \xi \mapsto X_\xi$, which assigns to every $\xi \in \mathfrak{g}$ the right-invariant vector field $X_\xi(g) = \xi g$, is a Lie algebra homomorphism. Likewise, the map $\mathrm{G} \to \mathrm{Diff}(\mathrm{G}) : h \mapsto \psi_h$, defined by $\psi_h(g) := gh$ for $g, h \in \mathrm{G}$, is a Lie group anti-homomorphism. Accordingly, the induced map $\mathfrak{g} \to \mathrm{Vect}(\mathrm{G}) : \xi \mapsto Y_\xi$, which assigns to every $\xi \in \mathfrak{g}$ the left-invariant vector field $Y_\xi(g) = g\xi$, is a Lie algebra anti-homomorphism. The reader may verify that for matrix groups $\mathrm{G} \subset \mathrm{GL}(V)$ this is consistent with the principle (L1).

Definition 1.1. A **complex Lie group** is a Lie group G equipped with the structure of a complex manifold such that the structure maps

$$\mathrm{G} \times \mathrm{G} \to \mathrm{G} : (g, h) \mapsto gh, \qquad \mathrm{G} \to \mathrm{G} : g \mapsto g^{-1}$$

are holomorphic.

Proposition 1.2. *Let* G *be a Lie group, assume that its Lie algebra* $\mathfrak{g}$ *is equipped with a linear complex structure* $\mathfrak{g} \to \mathfrak{g} : \xi \mapsto \mathbf{i}\xi$*, and define an almost complex structure* J *on* G *by*

$$J_g \widehat{g} := \big(\mathbf{i}(\widehat{g}g^{-1})\big)g \tag{3}$$

for $\widehat{g} \in T_g\mathrm{G}$*. Consider the following assertions.*

(i) (G, J) *is a complex Lie group.*

(ii) *The linear complex structure on* $\mathfrak{g}$ *is invariant under conjugation, i.e.*

$$\mathbf{i}(g\xi g^{-1}) = g(\mathbf{i}\xi)g^{-1} \qquad \text{for all } g \in \mathrm{G} \text{ and all } \xi \in \mathfrak{g}. \tag{4}$$

(iii) *The Lie bracket* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (\xi, \eta) \mapsto [\xi, \eta]$ *is complex bilinear, i.e.*

$$[\mathbf{i}\xi, \eta] = [\xi, \mathbf{i}\eta] = \mathbf{i}[\xi, \eta] \qquad \text{for all } \xi, \eta \in \mathfrak{g}. \tag{5}$$

Then (i) is equivalent to (ii), and (ii) implies (iii). If G *is connected, then all three conditions are equivalent.*

Lemma 1.3. *Let* G *be a Lie group and let* $A : \mathfrak{g} \to \mathfrak{g}$ *be a linear map on its Lie algebra* $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$*. Consider the following assertions.*

(i) *For all* $\xi \in \mathfrak{g}$ *and* $g \in \mathrm{G}$ *we have* $A(g\xi g^{-1}) = g(A\xi)g^{-1}$*.*

(ii) *For all* $\xi, \eta \in \mathfrak{g}$ *we have* $A[\xi, \eta] = [A\xi, \eta] = [\xi, A\eta]$*.*

Then (i) implies (ii). If G *is connected, then (i) and (ii) are equivalent.*

Proof. To prove that (i) implies (ii), differentiate the identity

$$A(\exp(t\xi)\eta\exp(-t\xi)) = \exp(t\xi)(A\eta)\exp(-t\xi)$$

with respect to t at $t = 0$. To prove the converse when G is connected, choose a curve $g : [0, 1] \to \mathrm{G}$ such that $g(0) = \mathbb{1}$ and an element $\xi \in \mathfrak{g}$. Define the curves $\eta, \zeta : [0, 1] \to \mathfrak{g}$ by $\eta(t) := g(t)^{-1}\xi g(t)$ and $\zeta(t) := g(t)^{-1}(A\xi)g(t)$. Then, by part (ii),

$$\partial_t(A\eta) + [g^{-1}\dot{g}, A\eta] = 0, \qquad \partial_t\zeta + [g^{-1}\dot{g}, \zeta] = 0, \qquad A\eta(0) = A\xi = \zeta(0).$$

It follows that $A\eta(t) = \zeta(t)$ for all t. This proves Lemma 1.3. □

Proof of Proposition 1.2. The proof has four steps.

Step 1. *The diffeomorphism*

$$\psi_h : \mathrm{G} \to \mathrm{G}, \qquad \psi_h(g) := gh,$$

is holomorphic for every $h \in \mathrm{G}$.

For all $g, h \in \mathrm{G}$ and $\widehat{g} \in T_g\mathrm{G}$ it follows from (3) that $J_{gh}(\widehat{g}h) = (J_g\widehat{g})h$ and hence $J_{\psi_h(g)} \circ d\psi_h(g) = d\psi_h(g) \circ J_g$. Thus $\psi_h^*J = J$ for all $h \in \mathrm{G}$.

Step 2. *Let $g \in \mathrm{G}$. Then the diffeomorphism*

$$\phi_g : \mathrm{G} \to \mathrm{G}, \qquad \phi_g(h) := gh,$$

is holomorphic if and only if $\mathbf{i}(g\xi g^{-1}) = g(\mathbf{i}\xi)g^{-1}$ for all $\xi \in \mathfrak{g}$.

We have $\phi_g^*J = J$ if and only if $J_{gh}(g\widehat{h}) = gJ_h\widehat{h}$ for all $\widehat{h} \in T_h\mathrm{G}$. Take $\widehat{h} = \xi h$ and use (3) to write this equation in the form $(\mathbf{i}(g\xi g^{-1}))gh = g(\mathbf{i}\xi)h \in T_{gh}\mathrm{G}$, or equivalently in the form $\mathbf{i}(g\xi g^{-1}) = g(\mathbf{i}\xi)g^{-1}$ for all $\xi \in \mathfrak{g}$.

Step 3. *Define $X_\xi \in \mathrm{Vect}(\mathrm{G})$ by $X_\xi(g) := \xi g$ for $g \in \mathrm{G}$ and $\xi \in \mathfrak{g}$. Then*

$$JX_\xi = X_{\mathbf{i}\xi}, \qquad [X_\xi, X_\eta] = X_{[\xi,\eta]} \qquad \text{for all } \xi, \eta \in \mathfrak{g}. \tag{6}$$

The first equation in (6) follows from the definition of J in (3). The second equation follows from our sign conventions (see the beginning of this section).

Step 4. *If (iii) holds, then J is integrable.*

By Step 3 the Nijenhuis tensor of J is given by

$$\begin{aligned} N_J(X_\xi, X_\eta) &= [X_\xi, X_\eta] + J[JX_\xi, X_\eta] + J[X_\xi, JX_\eta] - [JX_\xi, JX_\eta] \\ &= [X_\xi, X_\eta] + J[X_{\mathbf{i}\xi}, X_\eta] + J[X_\xi, X_{i\eta}] - [X_{\mathbf{i}\xi}, X_{\mathbf{i}\eta}] \\ &= X_{[\xi,\eta]} + JX_{[\mathbf{i}\xi,\eta]} + JX_{[\xi,\mathbf{i}\eta]} - X_{[\mathbf{i}\xi,\mathbf{i}\eta]} \\ &= X_{[\xi,\eta]+\mathbf{i}[\mathbf{i}\xi,\eta]+\mathbf{i}[\xi,\mathbf{i}\eta]-[\mathbf{i}\xi,\mathbf{i}\eta]} \end{aligned}$$

for all $\xi, \eta \in \mathfrak{g}$. Since the vector fields X_ξ span the tangent bundle, this shows that $N_J = 0$ and so J is integrable whenever (iii) holds.

It follows from Step 2 that (i) implies (ii). Conversely assume (ii). Then also (iii) holds by Lemma 1.3, and hence J is integrable by Step 4. Moreover, the multiplication map $\mathrm{G} \times \mathrm{G} \to \mathrm{G} : (g, h) \mapsto gh$ is holomorphic by Step 1 and Step 2. Since the the multiplication map is a submersion, the preimage of the neutral element $1 \in \mathrm{G}$ is a complex submanifold of $\mathrm{G} \times \mathrm{G}$ and it is the graph of the map $g \mapsto g^{-1}$. Hence this map is holomorphic as well. This shows that (i) is equivalent to (ii). That (ii) implies (iii), and that (iii) implies (ii) whenever G is connected, was shown in Lemma 1.3. This proves Proposition 1.2. □

Theorem 1.4. *Let* G *be a compact Lie group with the Lie algebra* $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$*, let* G^c *be a complex Lie group with the Lie algebra* $\mathfrak{g}^c = \mathrm{Lie}(\mathrm{G}^c)$*, and let*

$$\iota : \mathrm{G} \to \mathrm{G}^c$$

be an injective Lie group homomorphism. Then the following are equivalent.

(i) *The image* $\iota(\mathrm{G})$ *is a maximal ccompact subgroup of* G^c*, the homogeneous space* $\mathrm{G}^c/\iota(\mathrm{G})$ *is connected, and the image of* $d\iota(1) : \mathfrak{g} \to \mathfrak{g}^c$ *is a totally real subspace of* $\mathfrak{g}^c$*, i.e.* $\mathfrak{g}^c = d\iota(1)\mathfrak{g} \oplus \mathbf{i}d\iota(1)\mathfrak{g}$.

(ii) *The homogeneous space* $\mathrm{G}^c/\iota(\mathrm{G})$ *is connected and simply connected and the image of* $d\iota(1) : \mathfrak{g} \to \mathfrak{g}^c$ *is a totally real subspace of* $\mathfrak{g}^c$.

(iii) *For every complex Lie group* H *and every Lie group homomorphism*

$$\rho : \mathrm{G} \to \mathrm{H}$$

there exists a unique holomorphic Lie group homomorphism $\rho^c : \mathrm{G}^c \to \mathrm{H}$ *such that* $\rho = \rho^c \circ \iota$.

Definition 1.5. *A* **complexification** *of a compact Lie group* G *is an injective Lie group homomorphism* $\iota : \mathrm{G} \to \mathrm{G}^c$ *to a complex Lie group* G^c *that satisfies the universality condition in part (iii) of Theorem 1.4.*

Any two complexifications of a compact Lie group G are unique up to canonical isomorphism: If $\iota_j : \mathrm{G} \to \mathrm{G}^c_j$ for $j = 1, 2$ are two complexifications, then there exist unique holomorphic Lie group homomorphisms $\phi : \mathrm{G}^c_1 \to \mathrm{G}^c_2$ and $\psi : \mathrm{G}^c_2 \to \mathrm{G}^c_1$ such that $\phi \circ \iota_1 = \iota_2$ and $\psi \circ \iota_2 = \iota_1$, hence $\psi \circ \phi = \mathrm{id}_{\mathrm{G}^c_1}$ and $\phi \circ \psi = \mathrm{id}_{\mathrm{G}^c_2}$ by uniqueness, and hence ϕ is a Lie group isomorphism.

Example 1.6. None of the conditions in part (i) follows from the others.

(a) The complex torus $\mathbb{T}^2 := \mathbb{C}/(\mathbb{Z} \oplus \mathbf{i}\mathbb{Z})$ is not a complexification of the real circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ because $\mathbb{T}^1$ is not a maximal compact subgroup of $\mathbb{T}^2$. Since the homomorphism $\mathbb{T}^1 \to \mathbb{C}^* : [s] \mapsto \exp(2\pi\mathbf{i}s)$ does not extend to a holomorphic homomorphism from $\mathbb{T}^2$ to $\mathbb{C}^*$, existence fails in (iii).

(b) The complex Lie group $\mathbb{C}^* \times \mathbb{Z}$ is not a complexification of the unit circle $S^1 \subset \mathbb{C}$ because $(\mathbb{C}^* \times \mathbb{Z})/S^1$ is not connected. Since the holomorphic homomorphism $\mathbb{C}^* \times \mathbb{Z} \to \mathbb{C}^* : (z, k) \mapsto \lambda^k z$ restricts to the inclusion of S^1 into $\mathbb{C}^*$ for every $\lambda \in \mathbb{C}^*$, uniqueness fails in (iii).

(c) The group $\mathrm{A} := \mathbb{C}^* \times \mathbb{C}$ with $(a_1, b_1) \cdot (a_2, b_2) := (a_1 a_2, b_1 + a_1 b_2)$ is not a complexification of S^1 because $\mathrm{Lie}(S^1)$ is not a totally real subspace of $\mathrm{Lie}(\mathrm{A})$. Since the holomorphic homomorphism $\mathrm{A} \to \mathrm{A} : (a, b) \mapsto (a, \lambda b)$ restricts to the inclusion of S^1 into A for every $\lambda \in \mathbb{C}$, uniqueness fails in (iii).

In the following theorems we use the notations

$$\mathfrak{g} := \mathrm{Lie}(\mathrm{G}), \qquad \mathfrak{g}^c := \mathrm{Lie}(\mathrm{G}^c)$$

wherever they are needed.

Theorem 1.7. *Every compact Lie group admits a complexification, unique up to canonical isomorphism.*

Theorem 1.8 (Cartan). *Let* G^c *be a complex Lie group and let* $\mathrm{G} \subset \mathrm{G}^c$ *be a maximal compact subgroup such that* G^c/G *is connected and* $\mathfrak{g}^c = \mathfrak{g} \oplus \mathbf{i}\mathfrak{g}$. *Then every compact subgroup of* G^c *is conjugate in* G^c *to a subgroup of* G.

Theorem 1.9 (Cartan Decomposition Theorem). *Let* G^c *be a complex Lie group and let* $\mathrm{G} \subset \mathrm{G}^c$ *be a maximal compact subgroup such that* G^c/G *is connected and* $\mathfrak{g}^c = \mathfrak{g} \oplus \mathbf{i}\mathfrak{g}$. *Then the map*

$$\mathrm{G} \times \mathfrak{g} \to \mathrm{G}^c : (u, \eta) \mapsto \exp(\mathbf{i}\eta)u \tag{7}$$

is a diffeomorphism.

Theorem 1.10 (Mumford). *Let*

$$\mathrm{G}^c \subset \mathrm{GL}(n, \mathbb{C})$$

be a complex Lie subgroup and let $\mathrm{G} \subset \mathrm{G}^c$ *be a maximal compact subgroup of* G^c *such that* G^c/G *is connected and* $\mathfrak{g}^c = \mathfrak{g} \oplus \mathbf{i}\mathfrak{g}$. *Let* $\zeta \in \mathfrak{g}^c$ *such that*

$$\exp(\zeta) = 1.$$

Then there exist elements $p, p^+ \in \mathrm{G}^c$ *such that*

$$p^{-1}\zeta p \in \mathfrak{g}, \qquad \lim_{t\to\infty} \exp(\mathbf{i}t\zeta)p\exp(-\mathbf{i}t\zeta) = p^+.$$

Next we prove that (ii) implies (iii) in Theorem 1.4. The remaining assertions of Theorem 1.4 are proved in §6. Theorem 1.8 and Theorem 1.9 are also proved in §6. Theorem 1.7 is proved in §2 for Lie subgroups $\mathrm{G} \subset \mathrm{U}(n)$ and in §3 for compact Lie groups in the intrinsic setting. The proof of Theorem 1.10 is deferred to §7. In §4 we introduce Hadamard manifolds and prove some of their key properties, and in §5 we give a proof of the Cartan Fixed Point Theorem, which is needed in the proof of Theorem 1.8.

Proof of Theorem 1.4 "(ii) $\Longrightarrow$ (iii)". Let G^c be a complex Lie group and let $\mathrm{G} \subset \mathrm{G}^c$ be a compact subgroup such that G^c/G is connected and simply connected and

$$\mathfrak{g}^c = \mathfrak{g} \oplus \mathbf{i}\mathfrak{g}, \qquad \mathfrak{g} := \mathrm{Lie}(\mathrm{G}), \qquad \mathfrak{g}^c := \mathrm{Lie}(\mathrm{G}^c).$$

Let H be a complex Lie group with the Lie algebra $\mathfrak{h} := \mathrm{Lie}(\mathrm{H})$, let

$$\rho : \mathrm{G} \to \mathrm{H}$$

be a Lie group homomorphism, and define $\Phi^c : \mathfrak{g}^c \to \mathfrak{h}$ by

$$\Phi^c(\xi + \mathbf{i}\eta) := \Phi\xi + \mathbf{i}\Phi\eta, \qquad \Phi := d\rho(1) : \mathfrak{g} \to \mathfrak{h}, \tag{8}$$

for $\xi, \eta \in \mathfrak{g}$. Then Φ^c is a Lie algebra homomorphism. We prove in five steps that there exists a unique holomorphic Lie group homomorphism $\rho^c : \mathrm{G}^c \to \mathrm{H}$ such that $\rho^c|_{\mathrm{G}} = \rho$. The first step establishes uniqueness and explains how to define the extension ρ^c.

Step 1. *Let $\rho^c : \mathrm{G}^c \to \mathrm{H}$ be a holomorphic Lie group homomorphism such that $\rho^c|_{\mathrm{G}} = \rho$ and let $\gamma : [0,1] \to \mathrm{G}^c$ be a smooth curve such that $\gamma(0) \in \mathrm{G}$. Then the curve*

$$\gamma' := \rho^c \circ \gamma : [0,1] \to \mathrm{H}$$

satisfies the initial value problem

$$\gamma'^{-1}\dot{\gamma}' = \Phi^c(\gamma^{-1}\dot{\gamma}), \qquad \gamma'(0) = \rho(\gamma(0)). \tag{9}$$

This implies uniqueness in part (iii).

Since ρ is holomorphic, its derivative $d\rho^c(1) : \mathfrak{g}^c \to \mathfrak{h}$ is complex linear. Since $\rho^c|_{\mathrm{G}} = \rho$, this implies $d\rho^c(1) = \Phi^c : \mathfrak{g}^c \to \mathfrak{h}$ and hence

$$d\rho^c(g)g\zeta = \rho^c(g)\Phi^c\zeta \tag{10}$$

for all $g \in \mathrm{G}^c$ and all $\zeta \in \mathfrak{g}^c$. Thus, for all $t \in [0,1]$,

$$\dot{\gamma}'(t) = d\rho^c(\gamma(t))\dot{\gamma}(t) = \rho^c(\gamma(t))\Phi^c\big(\gamma(t)^{-1}\dot{\gamma}(t)\big) = \gamma'(t)\Phi^c\big(\gamma(t)^{-1}\dot{\gamma}(t)\big).$$

Moreover, $\gamma'(0) = \rho(\gamma(0))$ because $\gamma(0) \in \mathrm{G}$ and $\rho^c|_{\mathrm{G}} = \rho$. This proves (9). Hence, since G^c/G is connected, there exists at most one holomorphic Lie group homomorphism $\rho^c : \mathrm{G}^c \to \mathrm{H}$ such that $\rho^c|_{\mathrm{G}} = \rho$. This proves Step 1.

Step 2. *There exists a unique map $\rho^c : \mathrm{G}^c \to \mathrm{H}$ satisfying $\rho^c(\gamma(1)) = \gamma'(1)$ for every pair of smooth smooth curves $\gamma : [0,1] \to \mathrm{G}^c$ and $\gamma' : [0,1] \to \mathrm{H}$ such that $\gamma(0) \in \mathrm{G}$ and γ' satisfies* (9).

Fix an element $g \in \mathrm{G}^c$. Since G^c/G is connected, there exists a smooth curve $\gamma : [0,1] \to \mathrm{G}^c$ such that $\gamma(0) \in \mathrm{G}$ and $\gamma(1) = g$. Then by Corollary A.2 the initial value problem (9) has a unique solution $\gamma' : [0,1] \to \mathrm{H}$. We must prove that the endpoint $\gamma'(1)$ of this solution depends only on g and not on the choice of the curve γ from G to g. Thus assume that $\gamma_0, \gamma_1 : [0,1] \to \mathrm{G}^c$ are two smooth curves such that $\gamma_0(0), \gamma_1(0) \in \mathrm{G}$ and $\gamma_0(1) = \gamma_1(1) = g$, and for $j = 0,1$ let $\gamma_j' : [0,1] \to \mathrm{H}$ be the unique solution of (9) with $\gamma = \gamma_j$. We must prove that $\gamma_0'(1) = \gamma_1'(1)$. To see this, note first that both curves γ_0 and γ_1 take values in the same connected component of G^c, because they have the same endpoint. Second, since G^c/G is simply connected, the intersection of G with this connected component of G^c is connected and there exists a smooth homotopy $\gamma : [0,1]^2 \to \mathrm{G}^c$ from $\gamma(0,\cdot) = \gamma_0$ to $\gamma(1,\cdot) = \gamma_1$ such that

$$\gamma(s,0) \in \mathrm{G}, \qquad \gamma(s,1) = g \qquad \text{for } 0 \le s \le 1.$$

Define $\gamma' : [0,1]^2 \to \mathrm{H}$ by

$$\gamma'^{-1}\partial_t\gamma' = \Phi^c(\gamma^{-1}\partial_t\gamma), \qquad \gamma'(s,0) = \rho(\gamma(s,0)). \tag{11}$$

Then γ' is smooth and $\gamma'(0,t) = \gamma_0'(t)$ and $\gamma'(1,t) = \gamma_1'(t)$. We claim that

$$\gamma'^{-1}\partial_s\gamma' = \Phi^c(\gamma^{-1}\partial_s\gamma). \tag{12}$$

To see this, abbreviate

$$\xi' := \gamma'^{-1}\partial_s\gamma', \qquad \eta' := \gamma'^{-1}\partial_t\gamma'. \qquad \xi := \gamma^{-1}\partial_s\gamma, \qquad \eta := \gamma^{-1}\partial_t\gamma.$$

Then, since Φ^c is a Lie algebra homomorphism, we obtain

$$\partial_t\xi' = \partial_s\eta' + [\xi',\eta'], \qquad \partial_t\Phi^c(\xi) = \partial_s\Phi^c(\eta) + [\Phi^c(\xi),\Phi^c(\eta)].$$

Moreover, it follows from (11) that $\eta' = \Phi^c(\eta)$ and $\gamma'(s,0) = \rho(\gamma(s,0))$. When $t = 0$ we also have $d\rho(\gamma)\gamma\xi = \rho(\gamma)\Phi\xi$ and hence

$$\begin{aligned}\xi'(s,0) &= \gamma'(s,0)^{-1}\partial_s\gamma'(s,0) = \rho(\gamma(s,0))^{-1}d\rho(\gamma(s,0))\partial_s\gamma(s,0)\\ &= \Phi(\gamma(s,0)^{-1}\partial_s\gamma(s,0)) = \Phi(\xi(s,0)).\end{aligned}$$

Hence both curves $t \mapsto \xi'(s,t)$ and $t \mapsto \Phi^c(\xi(s,t))$ satisfy the same initial value problem and so agree. This proves (12). By (12) we have $\partial_s\gamma'(s,1) = 0$ for all s and hence $\gamma_0'(1) = \gamma'(0,1) = \gamma'(1,1) = \gamma_1'(1)$. This proves Step 2.

Step 3. *The map $\rho^c : \mathrm{G}^c \to \mathrm{H}$ in Step 2 satisfies the equation*

$$\Phi^c(g^{-1}\zeta g) = \rho^c(g)^{-1}\Phi^c(\zeta)\rho^c(g) \tag{13}$$

for all $g \in \mathrm{G}^c$ and all $\zeta \in \mathfrak{g}^c$.

Let $g \in \mathrm{G}^c$, choose a smooth curve $\gamma : [0,1] \to \mathrm{G}^c$ such that $\gamma(0) \in \mathrm{G}$ and $\gamma(1) = g$, and let $\gamma' : [0,1] \to \mathrm{H}$ be the unique solution of (9). Let $\zeta \in \mathfrak{g}^c$ and for $0 \le t \le 1$ define

$$\eta(t) := \gamma(t)^{-1}\zeta\gamma(t), \qquad \eta'(t) := \gamma'(t)^{-1}\Phi^c(\zeta)\gamma'(t).$$

Then the curves η' and $\Phi^c(\eta)$ satsify the same inital value problem

$$\dot\eta' + [\gamma'^{-1}\dot\gamma', \eta'] = 0,$$

$$\eta'(0) = \gamma'(0)^{-1}\Phi^c(\zeta)\gamma'(0) = \rho(\gamma(0))^{-1}\Phi^c(\zeta)\rho(\gamma(0)) = \Phi^c(\eta(0)).$$

Hence $\eta'(t) = \Phi^c(\eta(t))$ for all t. For $t = 1$ this proves (13) and Step 3.

Step 4. *The map $\rho^c : \mathrm{G}^c \to \mathrm{H}$ in Step 2 is a group homomorphism.*

Let $g_1, g_2 \in \mathrm{G}^c$, for $j = 1,2$ choose a smooth curve $\gamma_j : [0,1] \to \mathrm{G}^c$ such that $\gamma_j(0) \in \mathrm{G}$ and $\gamma_j(1) = g_j$, and let $\gamma'_j : [0,1] \to \mathrm{H}$ be the unique solution of (9) with $\gamma = \gamma_j$. Define $\gamma := \gamma_1\gamma_2$ and $\gamma' := \gamma'_1\gamma'_2$. Then, by Step 3,

$$\begin{aligned}
\gamma'^{-1}\dot\gamma' &= \gamma_2'^{-1}\dot\gamma'_2 + \gamma_2'^{-1}\gamma_1'^{-1}\dot\gamma'_1\gamma'_2 \\
&= \Phi^c(\gamma_2^{-1}\dot\gamma_2) + \rho^c(\gamma_2)^{-1}\Phi^c(\gamma_1^{-1}\dot\gamma_1)\rho^c(\gamma_2) \\
&= \Phi^c(\gamma_2^{-1}\dot\gamma_2 + \gamma_2^{-1}\gamma_1^{-1}\dot\gamma_1\gamma_2) = \Phi^c(\gamma^{-1}\dot\gamma).
\end{aligned}$$

Hence $\rho^c(g_1g_2) = \gamma'(1) = \rho^c(g_1)\rho^c(g_2)$. This proves Step 4.

Step 5. *The map $\rho^c : \mathrm{G}^c \to \mathrm{H}$ in Step 2 satisfies the equation*

$$\rho^c(\exp(\zeta)) = \exp(\Phi^c(\zeta))$$

for all $\zeta \in \mathfrak{g}^c$.

Let $\zeta \in \mathfrak{g}^c$ and define $\gamma(t) := \exp(t\zeta)$ and $\gamma'(\zeta) := \exp(t\Phi^c(\zeta))$ for $0 \le t \le 1$. Then $\gamma(0) = 1 \in \mathrm{G}$, $\gamma'(0) = 1 = \rho(\gamma(0))$, and

$$\gamma'(t)^{-1}\dot\gamma'(t) = \Phi^c(\zeta) = \Phi^c(\gamma(t)^{-1}\dot\gamma(t)) \qquad \text{for } 0 \le t \le 1.$$

Thus γ' satisfies (9) and so $\exp(\Phi^c(\zeta)) = \gamma'(1) = \rho^c(\gamma(1)) = \rho^c(\exp(\zeta))$. This proves Step 5.

By Step 4 the map $\rho^c : \mathrm{G}^c \to \mathrm{H}$ constructed in Step 2 is a group homomorphism. By Step 5 the group homomorphism ρ^c is smooth near the identity element $g = 1$ and its derivative $d\rho^c(1) = \Phi^c : \mathfrak{g}^c \to \mathfrak{h}$ at $g = 1$ is complex linear. Hence ρ^c is smooth everywhere and is holomorphic. This proves that (ii) implies (iii) in Theorem 1.4. □

2 First existence proof

The archetypal example of a complexification is the inclusion of the unitary group $\mathrm{U}(n)$ into the general linear group $\mathrm{GL}(n,\mathbb{C})$. Polar decomposition gives rise to a diffeomorphism

$$\phi : \mathrm{U}(n) \times \mathfrak{u}(n) \to \mathrm{GL}(n,\mathbb{C}), \qquad \phi(u,\eta) := \exp(\mathbf{i}\eta)u. \tag{14}$$

This example extends to every Lie subgroup of $\mathrm{U}(n)$.

Theorem 2.1. *Let $\mathrm{G} \subset \mathrm{U}(n)$ be a Lie subgroup with Lie algebra $\mathfrak{g} \subset \mathfrak{u}(n)$. Then the set*

$$\mathrm{G}^c := \{\exp(\mathbf{i}\eta)u \,|\, u \in \mathrm{G},\, \eta \in \mathfrak{g}\} \subset \mathrm{GL}(n,\mathbb{C}) \tag{15}$$

is a complex Lie subgroup of $\mathrm{GL}(n,\mathbb{C})$ and the inclusion of G into G^c satisfies the conditions (i), (ii), and (iii) in Theorem 1.4.

Corollary 2.2. *Theorem 1.7 holds for Lie subgroups $\mathrm{G} \subset \mathrm{U}(n)$.*

Proof of Theorem 2.1. The proof has nine steps.

Step 1. *G^c is a closed submanifold of $\mathrm{GL}(n,\mathbb{C})$, the group G acts freely on G^c, and the map $\mathfrak{g} \to \mathrm{G}^c/\mathrm{G} : \eta \mapsto [\exp(\mathbf{i}\eta)]$ is a diffeomorphism.*

This follows from the fact that (14) is a diffeomorphism.

Step 2. *$\mathbb{1} \in \mathrm{G}^c$ and $\mathfrak{g}^c := \mathrm{Lie}(\mathrm{G}^c) = T_{\mathbb{1}}\mathrm{G} = \mathfrak{g} \oplus \mathbf{i}\mathfrak{g}$.*

For $\xi, \eta \in \mathfrak{g}$ the curve $\gamma(t) := \exp(\mathbf{i}t\eta)\exp(t\xi) \in \mathrm{G}^c$ satisfies $\dot{\gamma}(0) = \xi + \mathbf{i}\eta$. Hence $\mathfrak{g}^c \subset T_{\mathbb{1}}\mathrm{G}^c$ and both spaces have the same dimension.

Step 3. *Let $g \in \mathrm{G}^c$. Then $T_g\mathrm{G}^c = g\mathfrak{g}^c$ is a complex subspace of $\mathbb{C}^{n\times n}$. Hence G^c is a complex submanifold of $\mathrm{GL}(n,\mathbb{C})$.*

Let $(u,\eta) \in \mathrm{G} \times \mathfrak{g}$ and $g := \phi(u,\eta) = \exp(\mathbf{i}\eta)u \in \mathrm{G}^c$. Then, for $\xi \in \mathfrak{g}$, we have $d\phi(u,\eta)(u\xi, 0) = \exp(\mathbf{i}\eta)u\xi \in g\mathfrak{g}^c$. Let $\widehat{\eta} \in \mathfrak{g}$, define $\gamma : \mathbb{R}^2 \to \mathrm{G}^c$ by

$$\gamma(s,t) := \phi(u, s(\eta + t\widehat{\eta})) = \exp(\mathbf{i}s(\eta + t\widehat{\eta}))u \qquad \text{for } s,t \in \mathbb{R},$$

and define $\xi_s := \gamma^{-1}\partial_s\gamma$ and $\xi_t := \gamma^{-1}\partial_t\gamma$. Then $\xi_s(s,t) = u^{-1}\mathbf{i}(\eta + t\widehat{\eta})u \in \mathfrak{g}^c$ for all s,t and $\partial_s\xi_t + [\xi_s, \xi_t] = \partial_t\xi_s$, $\xi_t(0,t) = 0$. Since $\xi_s(s,t) \in \mathfrak{g}^c$ for all s,t, this implies $\xi_t(s,t) \in \mathfrak{g}^c$ for all s,t. Take $s = 1$ and $t = 0$ to obtain

$$d\phi(u,\eta)(0,\widehat{\eta}) = \partial_t\gamma(1,0) = \gamma(1,0)\xi_t(1,0) = g\xi_t(1,0) \in g\mathfrak{g}^c.$$

Thus we have proved that $T_g\mathrm{G}^c \subset g\mathfrak{g}^c$. Since both spaces have the same dimension we deduce that $T_g\mathrm{G}^c = g\mathfrak{g}^c$.

Step 4. *Let $g \in \mathrm{GL}(n,\mathbb{C})$. Then $g \in \mathrm{G}^c$ if and only if there exists a smooth curve $\gamma : [0,1] \to \mathrm{GL}(n,\mathbb{C})$ satisfying $\gamma(0) \in \mathrm{G}$, $\gamma(1) = g$, and $\gamma(t)^{-1}\dot\gamma(t) \in \mathfrak{g}^c$ for every t.*

To prove that the condition is necessary let $g = \exp(i\eta)u \in \mathrm{G}^c$ be given. Then the path $\gamma(t) := \exp(\mathbf{i}t\eta)u$ satisfies the requirements of Step 4. To prove the converse, suppose that $\gamma : [0,1] \to \mathrm{GL}(n,\mathbb{C})$ is a smooth curve satisfying $\gamma(0) \in \mathrm{G}$, $\gamma(1) = g$, and $\gamma(t)^{-1}\dot\gamma(t) \in \mathfrak{g}^c$ for all t. Consider the set

$$I := \{t \in [0,1] \,|\, \gamma(t) \in \mathrm{G}^c\}.$$

This set is nonempty, because $0 \in I$. It is closed because G^c is a closed subset of $\mathrm{GL}(n,\mathbb{C})$ by Step 1. To prove it is open, define $\xi(t) := \gamma(t)^{-1}\dot\gamma(t) \in \mathfrak{g}^c$ and consider the vector fields X_t on $\mathbb{C}^{n\times n}$ given by $X_t(A) := A\xi(t)$. By Step 3, these vector fields are all tangent to G^c. Hence every solution of the differential equation $\dot A(t) = A(t)\xi(t)$ that starts in G^c remains in G^c on a sufficiently small time interval. In particular, this holds for the curve $A(t) = \gamma(t)$ and so I is open. Thus $I = [0,1]$ and hence $g = \gamma(1) \in \mathrm{G}^c$.

Step 5. *Let $g \in \mathrm{G}^c$ and $\zeta \in \mathfrak{g}^c$. Then $g^{-1}\zeta g \in \mathfrak{g}^c$ and $g\zeta g^{-1} \in \mathfrak{g}^c$.*

Choose $\gamma : [0,1] \to \mathrm{G}^c$ as in Step 4 so that

$$\gamma(0) \in \mathrm{G}, \qquad \gamma(1) = g, \qquad \xi(t) := \gamma(t)^{-1}\dot\gamma(t) \in \mathfrak{g}^c,$$

and define $\eta(t) := \gamma(t)^{-1}\zeta\gamma(t)$ for $0 \le t \le 1$. Then

$$\dot\eta + [\xi, \eta] = 0, \qquad \eta(0) = \gamma(0)^{-1}\zeta\gamma(0) \in \mathfrak{g}^c.$$

Here the second assertion holds because $\gamma(0) \in \mathrm{G}$. Since $\xi(t) \in \mathfrak{g}^c$ for all t this implies that $\eta(t) \in \mathfrak{g}^c$ for all t and, in particular, $g^{-1}\zeta g = \eta(1) \in \mathfrak{g}^c$.

This shows that the linear map $\mathfrak{g}^c \to \mathfrak{g}^c : \zeta \mapsto g^{-1}\zeta g$ is a vector space isomorphism for every $g \in \mathrm{G}^c$. Hence, for every $\zeta' \in \mathfrak{g}^c$ and every $g \in \mathrm{G}^c$, there exists a $\zeta \in \mathfrak{g}^c$ such that $g^{-1}\zeta g = \zeta'$, and so $g\zeta' g^{-1} = \zeta \in \mathfrak{g}^c$.

Step 6. *If $g_1, g_2 \in \mathrm{G}^c$, then $g_1 g_2 \in \mathrm{G}^c$.*

By Step 4 choose curves $\gamma_j : [0,1] \to \mathrm{G}^c$ such that $\gamma_j(0) \in \mathrm{G}$ and $\gamma_j(1) = g_j$ and $g_j(t)^{-1}\dot g_j(t) \in \mathfrak{g}^c$ for $j = 1,2$ and $0 \le t \le 1$. Then the curve

$$\gamma := \gamma_1\gamma_2 : [0,1] \to \mathrm{GL}(n,\mathbb{C})$$

satisfies $\gamma(0) \in \mathrm{G}$ and $\gamma^{-1}\dot\gamma = \gamma_2^{-1}\dot\gamma_2 + \gamma_2^{-1}(\gamma_1^{-1}\dot\gamma_1)\gamma_2$. Hence it follows from Step 5 that $\gamma(t)^{-1}\dot\gamma(t) \in \mathfrak{g}^c$ for all t and hence, by Step 4, $g_1 g_2 = \gamma(1) \in \mathrm{G}^c$.

Step 7. *If* $g \in \mathrm{G}^c$*, then* $g^{-1} \in \mathrm{G}^c$*.*

Let γ be as in Step 4 and define $\gamma'(t) := \gamma(t)^{-1}$. Then $\gamma'(0) \in \mathrm{G}$ and

$$\gamma'^{-1}\dot{\gamma}' = \gamma \frac{d}{dt}\gamma^{-1} = -\dot{\gamma}\gamma^{-1} = \gamma(-\gamma^{-1}\dot{\gamma})\gamma^{-1}.$$

By Step 5, $\gamma'(t)^{-1}\dot{\gamma}'(t) \in \mathfrak{g}^c$ for all t and hence, by Step 4, $g^{-1} = \gamma'(1) \in \mathrm{G}^c$.

Step 8. G^c *is a complex Lie subgroup of* $\mathrm{GL}(n, \mathbb{C})$*.*

By Step 3 the set G^c in (15) is a complex submanifold of $\mathrm{GL}(n, \mathbb{C})$, and by Step 6 and Step 7 it is a subgroup of $\mathrm{GL}(n, \mathbb{C})$.

Step 9. G *is a maximal compact subgroup of* G^c*.*

Let $\mathrm{H} \subset \mathrm{G}^c$ be a subgroup such that $\mathrm{G} \subsetneq \mathrm{H}$. Choose an element $h \in \mathrm{H} \setminus \mathrm{G}$. Then there exists a pair $(u, \eta) \in \mathrm{G} \times \mathfrak{g}$ such that $h = \exp(\mathbf{i}\eta)u$. Since $\mathrm{G} \subset \mathrm{H}$ and H is a subgroup of G^c, we have

$$p := \exp(\mathbf{i}\eta) = hu^{-1} \in \mathrm{H}.$$

The matrix p is Hermitian and positive definite. Since $h \notin \mathrm{G}$, we have $\eta \neq 0$ and hence at least one eigenvalue of p is not equal to 1. This implies that the sequence $p^k = \exp(\mathbf{i}k\eta) \in \mathrm{H}$ has no subsequence that converges to an element of $\mathrm{GL}(n, \mathbb{C})$. Thus H is not compact and this proves Step 9.

The inclusion of G into G^c satisfies condition (i) in Theorem 1.4 by Step 1, Step 2, Step 8, and Step 9. Step 1 also asserts that G^c/G is diffeomorphic to $\mathfrak{g}$ and hence is simply connected. Thus the inclusion also satisfies (ii). Since we have already shown that (ii) implies (iii), this proves Theorem 2.1. □

The tangent space of the submanifold $\mathrm{G}^c \subset \mathrm{GL}(n, \mathbb{C})$ in Theorem 2.1 at the identity element is given by $T_{\mathbb{1}}\mathrm{G}^c = \mathfrak{g} \oplus \mathbf{i}\mathfrak{g} = \mathfrak{g}^c$. Since G^c is a Lie subgroup of $\mathrm{GL}(n, \mathbb{C})$, the curve $t \mapsto \exp(-\mathbf{i}\eta)\exp(\mathbf{i}\eta + t\mathbf{i}\widehat{\eta})$ lies in G^c, for every pair $\eta, \widehat{\eta} \in \mathfrak{g}$, and hence

$$B_\eta(\widehat{\eta}) := \frac{d}{dt}\bigg|_{t=0} \exp(-\mathbf{i}\eta)\exp(\mathbf{i}\eta + t\mathbf{i}\widehat{\eta}) \in \mathfrak{g}^c.$$

It turns out that $B \in \Omega^1(\mathfrak{g}, \mathfrak{g}^c)$ is a flat connection 1-form. Moreover, it satisfies $B_\eta(\widehat{\eta}) = \mathbf{i}\widehat{\eta}$ whenever η and $\widehat{\eta}$ commute. Conversely, it is shown in Lemma 3.8 that the connection B is uniquely determined by these two conditions and in Theorem 3.10 that the group multiplication on $\mathrm{G} \times \mathfrak{g} \cong \mathrm{G}^c$ can be reconstructed from B. This gives rise to an intrinsic construction of a complexified Lie group for any compact connected Lie group G that does not rely on an embedding into the unitary group (see §3).

3 Second existence proof

This section deals with compact Lie groups in the intrinsic setting.

Definition 3.1. Let X be a connected smooth manifold and let $\mathfrak{g}$ be a Lie algebra of the same dimension as X. An **infinitesimal group law** is a flat connection $A \in \Omega^1(X, \mathfrak{g})$ that satisfies the following three conditions.

(Monodromy) The monodromy representation of A is trivial, i.e. for any two smooth paths $\gamma : [0,1] \to X$ and $\zeta : [0,1] \to \mathfrak{g}$ we have

$$\dot{\zeta} + [A_\gamma(\dot{\gamma}), \zeta] = 0, \quad \gamma(0) = \gamma(1) \qquad \Longrightarrow \qquad \zeta(0) = \zeta(1).$$

(Inverse) $A_x : T_xX \to \mathfrak{g}$ is a vector space isomorphism for every $x \in X$.

(Complete) The vector fields $Y_\xi \in \mathrm{Vect}(X)$, defined by

$$A_x(Y_\xi(x)) = \xi \qquad \text{for } \xi \in \mathfrak{g} \text{ and } x \in X, \tag{16}$$

are complete, i.e. for every smooth path $\mathbb{R} \to \mathfrak{g} : t \mapsto \xi(t)$ the solutions of the differential equation $\dot{\gamma}(t) = Y_{\xi(t)}(\gamma(t))$ exist for all time.

Lemma 3.2. *Let X be a smooth manifold and let $A \in \Omega^1(X, \mathfrak{g})$ be a Lie algebra valued 1-form such that $A_z : T_xX \to \mathfrak{g}$ is a vector space isomorphism for every $x \in X$. For $\xi \in \mathfrak{g}$ define $Y_\xi \in \mathrm{Vect}(X)$ by (16). Then A is flat if and only if the map $\mathfrak{g} \to \mathrm{Vect}(X) : \xi \mapsto Y_\xi$ is a Lie algebra anti-homomorphism.*

Proof. The curvature $F_A \in \Omega^2(X, \mathfrak{g})$ is given by

$$\begin{aligned} F_A(Y_\xi, Y_\eta) &= dA(Y_\xi, Y_\eta) + [A(Y_\xi), A(Y_\eta)] \\ &= \mathcal{L}_{Y_\xi}(A(Y_\eta)) - \mathcal{L}_{Y_\eta}(A(Y_\xi)) + A([Y_\xi, Y_\eta]) + [\xi, \eta] \\ &= A([Y_\xi, Y_\eta]) + [\xi, \eta] = A\big([Y_\xi, Y_\eta] + Y_{[\xi,\eta]}\big) \end{aligned}$$

for $\xi, \eta \in \mathfrak{g}$. This proves Lemma 3.2. □

Example 3.3. Let G be a connected Lie group with the Lie algebra $\mathfrak{g}$. Then the connection 1-form $A \in \Omega^1(\mathrm{G}, \mathfrak{g})$ defined by

$$A_g(\widehat{g}) := g^{-1}\widehat{g} \qquad \text{for } \widehat{g} \in T_g\mathrm{G}$$

is an infinitesimal group law. The vector fields in (16) are given by $Y_\xi(g) = g\xi$ for $g \in \mathrm{G}$ and $\xi \in \mathfrak{g}$, so Lemma 3.2 implies that A is flat. The *(Monodromy)* condition holds because, for every smooth curve $g : [0,1] \to \mathrm{G}$, the solutions of the equation $\dot{\xi} + [g^{-1}\dot{g}, \xi] = 0$ have the form $\xi(t) = g(t)^{-1}\xi_0 g(t)$. The *(Complete)* condition holds because, for every smooth curve $\xi : \mathbb{R} \to \mathfrak{g}$, the solutions of the equation $g^{-1}\dot{g} = \xi$ exist for all time (Corollary A.2).

Lemma 3.4. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then there exists a unique flat connection $A \in \Omega^1(\mathfrak{g}, \mathfrak{g})$ such that*

$$[\xi, \widehat{\xi}] = 0 \qquad \Longrightarrow \qquad A_\xi\big(\widehat{\xi}\big) = \widehat{\xi} \tag{17}$$

for all $\xi, \widehat{\xi} \in \mathfrak{g}$.

In general, the connection in Lemma 3.4 is not an infinitesimal group law. The idea behind this example is as follows. If G is a Lie group with the Lie algebra $\mathfrak{g}$, one might attempt to reconstruct the group multiplication locally as an operation $m : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that $\exp(\xi)\exp(\eta) = \exp(m(\xi, \eta))$. While this is not possible globally in most cases, the associated connection 1-form $A_\xi(\widehat{\xi}) := \exp(-\xi) d\exp(\xi)\widehat{\xi}$ does exist globally and satisfying (17).

Proof of Lemma 3.4. A connection $A \in \Omega^1(\mathfrak{g}, \mathfrak{g})$ is flat if and only if every smooth map $\gamma : \mathbb{R}^2 \to \mathfrak{g}$ satisfies the equation

$$\partial_s(A_\gamma(\partial_t\gamma)) - \partial_t(A_\gamma(\partial_s\gamma)) + [A_\gamma(\partial_s\gamma), A_\gamma(\partial_t\gamma)] = 0. \tag{18}$$

If in addition the connection satisfies (17) and $\gamma(s, t) := t(\xi + s\widehat{\xi})$, then

$$A_\gamma(\partial_t\gamma) = \xi + s\widehat{\xi}, \qquad A_\gamma(\partial_s\gamma) = A_{t(\xi+s\widehat{\xi})}(t\widehat{\xi}),$$

and so, for $s = 0$, the curve $\zeta(t) := A_{t\xi}(t\widehat{\xi})$ satisfies the differential equation

$$\dot{\zeta} + [\xi, \zeta] = \widehat{\xi}, \qquad \zeta(0) = 0. \tag{19}$$

Thus

$$A_\xi\big(\widehat{\xi}\big) = \zeta(1) = \int_0^1 \exp(-t\mathrm{ad}(\xi))\widehat{\xi}\,dt = \sum_{k=0}^\infty \frac{(-1)^k}{(k+1)!}\mathrm{ad}(\xi)^k\widehat{\xi}, \tag{20}$$

where $\mathrm{ad}(\xi) := [\xi, \cdot]$. This proves uniqueness.

Conversely, let $A \in \Omega^1(\mathfrak{g}, \mathfrak{g})$ be defined by (20). If $[\xi, \widehat{\xi}] = 0$, then $\zeta(t) = t\widehat{\xi}$ is the unique solution of (19) and so $A_\xi\big(\widehat{\xi}\big) = \zeta(1) = \widehat{\xi}$. Now fix three elements $\xi, \widehat{\xi}_1, \widehat{\xi}_2 \in \mathfrak{g}$, define $\zeta_j : [0, 1] \to \mathfrak{g}$ as the solutions of (19) with $\widehat{\xi} = \widehat{\xi}_j$, and define $\zeta_{ij} : [0, 1] \to \mathfrak{g}$ as the solution of the linearized equation

$$\dot{\zeta}_{ij} + [\xi, \zeta_{ij}] + [\widehat{\xi}_i, \zeta_j] = 0, \qquad \zeta_{ij}(0) = 0.$$

Then $A_\xi(\widehat{\xi}_j) = \zeta_j(1)$ and $(dA)_\xi(\widehat{\xi}_1, \widehat{\xi}_2) = \zeta_{12}(1) - \zeta_{21}(1)$. Moreover,

$$\dot{\zeta} + [\xi, \zeta] = 0, \qquad \zeta := \zeta_{12} - \zeta_{21} + [\zeta_1, \zeta_2],$$

so $\zeta \equiv 0$ and thus A is flat. This proves Lemma 3.4. □

Theorem 3.5. *Let X be a connected smooth manifold, let $\mathfrak{g}$ be a Lie algebra, let $A \in \Omega^1(X, \mathfrak{g})$ be an infinitesimal group law, and fix an element $1 \in X$. Then there exists a unique Lie group structure on X with the unit 1 such that*

$$A_x(\widehat{x}) = A_1(x^{-1}\widehat{x}) \qquad \text{for all } \widehat{x} \in T_xX.$$

Moreover, the map $A_1 : T_1X = \mathrm{Lie}(X) \to \mathfrak{g}$ is a Lie algebra isomorphism.

Proof. Assume first that X is a Lie group with the unit $1 \in X$, that $\mathfrak{g} = T_1X$ is its Lie algebra, and that the 1-form A is defined by $A_1(\widehat{x}) := x^{-1}\widehat{x}$ for $\widehat{x} \in T_xX$, so in particular $A_1 = \mathrm{id}$. For each $x \in X$ define the Lie algebra automorphism $\Phi(x) : \mathfrak{g} \to \mathfrak{g}$ by the adjoint action of x^{-1} so that

$$\Phi(x)\xi := x^{-1}\xi x \qquad \text{for } \xi \in \mathfrak{g}.$$

Then Φ and A are related by the equation

$$\Phi(1) = \mathrm{id}, \qquad (d\Phi(x)\widehat{x})\xi + [A_x(\widehat{x}), \Phi(x)\xi] = 0$$

for $\widehat{x} \in T_xX$ and $\xi \in \mathfrak{g}$. Conversely, this equation can be used to define Φ. The group multiplication is a collection of diffeomorphisms $\phi_x : X \to X$, defined by $\phi_x(y) := xy$ for $x, y \in X$. For each x, ϕ_x is related to A by

$$\phi_x(1) = x, \qquad A_{\phi_x(y)} \circ d\phi_x(y) = A_y \qquad \text{for all } y \in X, \tag{21}$$

or equivalently $\phi_x^*A = A$. The conditions in (21) determine ϕ_x uniquely and hence can be used to recover the multiplication from the connection 1-form A. With this understood, we prove the theorem in seven steps.

Step 1. *There exists a unique smooth map $\Phi : X \to \mathrm{Aut}(\mathfrak{g})$ satisfying*

$$\Phi(1) = \mathrm{id}, \qquad (d\Phi(x)\widehat{x})\xi + [A_x(\widehat{x}), \Phi(x)\xi] = 0 \tag{22}$$

for all $x \in X$, $\widehat{x} \in T_xX$, and $\xi \in \mathfrak{g}$.

Given $x \in X$ choose a smooth curve $\gamma : [0, 1] \to X$ such that $\gamma(0) = 1$ and $\gamma(1) = x$, and define the linear map $\Phi(x) : \mathfrak{g} \to \mathfrak{g}$ by $\Phi(x)\xi(0) := \xi(1)$ for every solution $\xi : [0, 1] \to \mathfrak{g}$ of the differential equation

$$\dot{\xi} + [A_\gamma(\dot{\gamma}), \xi] = 0. \tag{23}$$

The *(Monodromy)* axiom guarantees that $\Phi(x)$ is independent of the choice of γ. Moreover, if ξ, η satisfy (23), so does $[\xi, \eta]$, and so $\Phi(x) \in \mathrm{Aut}(\mathfrak{g})$. The resulting map $\Phi : X \to \mathrm{Aut}(\mathfrak{g})$ is obviously smooth and satisfies (22).

Step 2. *Let $\gamma, \gamma' : [0,1] \to X$ be smooth curves such that*

$$A_\gamma(\dot\gamma) = A_{\gamma'}(\dot\gamma'). \tag{24}$$

Then $\gamma(0) = \gamma(1)$ if and only if $\gamma'(0) = \gamma'(1)$.

Assume $\gamma(0) = \gamma(1) = 1$ and choose a smooth curve $\alpha : [0,1] \to X$ such that

$$\alpha(0) = 1, \qquad \alpha(1) = \gamma'(0).$$

Define the map $\beta : [0,1]^2 \to X$ by the condition that, for each s, the curve $\beta_s(t) := \beta(s,t)$ is the unique solution of the initial value problem

$$A_{\beta_s(t)}\big(\dot\beta_s(t)\big) = A_{\gamma(t)}\big(\dot\gamma(t)\big), \qquad \beta_s(0) = \alpha(s).$$

Then β is smooth and satisfies

$$\beta(0,t) = \gamma(t), \qquad \beta(1,t) = \gamma'(t)$$

for all t. Since A is flat, we have

$$\partial_s\big(A_\beta(\partial_t\beta)\big) - \partial_t\big(A_\beta(\partial_s\beta)\big) + \big[A_\beta(\partial_s\beta), A_\beta(\partial_t\beta)\big] = 0.$$

Since $A_{\beta(s,t)}(\partial_t\beta(s,t)) = A_{\gamma(t)}(\dot\gamma(t)) =: \eta(t)$ is independent of s, it follows that the curve $t \mapsto \xi(s,t) := A_{\beta(s,t)}(\partial_s\beta(s,t))$ satisfies the differential equation

$$\partial_t\xi + [\eta, \xi] = 0, \qquad \xi(s,0) = A_{\alpha(s)}(\dot\alpha(s)).$$

By Step 1 the curve $t \mapsto \xi'(s,t) := \Phi(\gamma(t))A_{\alpha(s)}(\dot\alpha(s))$ satisfies the same initial value problem and hence

$$A_{\beta(s,t)}\big(\partial_s\beta(s,t)\big) = \Phi\big(\gamma(t)\big)A_{\alpha(s)}\big(\dot\alpha(s)\big) \qquad \text{for } 0 \le s,t \le 1.$$

Take $t = 1$ and use the equations $\beta(0,1) = \gamma(1) = 1$ and $\Phi(1) = \mathrm{id}$ to obtain

$$A_{\beta(s,1))}\big(\partial_s\beta(s,1)\big) = A_{\alpha(s)}\big(\dot\alpha(s)\big), \qquad \beta(0,1) = \alpha(0),$$

for $0 \le s \le 1$. Thus the curves $s \mapsto \alpha(s)$ and $s \mapsto \beta(s,1)$ satisfy the same initial value problem and hence $\beta(s,1) = \alpha(s)$ for all s. Take $s = 1$ to obtain the equation $\gamma'(1) = \beta(1,1) = \alpha(1) = \gamma'(0)$.

Since the element $1 \in X$ has been arbitrarily chosen, we have proved that $\gamma(0) = \gamma(1)$ implies $\gamma'(0) = \gamma'(1)$. The converse implication follows by the symmetry of the assertion in γ and γ'. This proves Step 2.

Step 3. *Let $\gamma_0, \gamma_1, \gamma_0', \gamma_1' : [0,1] \to X$ be smooth curves such that*

$$\begin{array}{ll} A_{\gamma_0}(\dot\gamma_0) = A_{\gamma_0'}(\dot\gamma_0'), & A_{\gamma_1}(\dot\gamma_1) = A_{\gamma_1'}(\dot\gamma_1'), \\ \gamma_0(0) = \gamma_1(0), & \gamma_0'(0) = \gamma_1'(0). \end{array} \tag{25}$$

Then $\gamma_0(1) = \gamma_1(1)$ if and only if $\gamma_0'(1) = \gamma_1'(1)$.

It suffices to prove that $\gamma_0(1) = \gamma_1(1)$ implies $\gamma_0'(1) = \gamma_1'(1)$. Assume first that the curve γ_1 is a reparametrization of γ_0, i.e. there exists a smooth function $\rho : [0,1] \to [0,1]$ such that $\rho(0) = 0$ and $\rho(1) = 1$ and $\gamma_1 = \gamma_0 \circ \rho$. Then it follows from (25) that $\gamma_1' = \gamma_0' \circ \rho$ and hence $\gamma_0'(1) = \gamma_1'(1)$.

Now assume that $\gamma_0(1) = \gamma_1(1)$ and that γ_0 and γ_1 are constant near the endpoints. Then the curve $\gamma : [0,1] \to X$, defined by

$$\gamma(t) := \left\{ \begin{array}{rl} \gamma_0(2t), & \text{if } 0 \le t \le 1/2, \\ \gamma_1(2-2t), & \text{if } 1/2 \le t \le 1, \end{array} \right. \tag{26}$$

is smooth and satisfies $\gamma(0) = \gamma_0(0) = \gamma_1(0) = \gamma(1)$. Let $\gamma' : [0,1] \to X$ be the unique solution of equation (24) satisfying $\gamma'(0) = \gamma_0'(0) = \gamma_1'(0)$. Then Step 2 asserts that $\gamma'(0) = \gamma'(1)$. Hence it follows from (24), (25), and (26) that $\gamma_0'(t) = \gamma'(t/2)$ and $\gamma_1'(t) = \gamma'(1-t/2)$ for $0 \le t \le 1$. Take $t = 1$ to obtain $\gamma_0'(1) = \gamma'(1/2) = \gamma_1'(1)$. This proves Step 3.

Step 4. *For every $x \in X$ there exists a unique diffeomorphism $\phi_x : X \to X$ such that*

$$\phi_x(1) = x, \qquad \phi_x^* A = A. \tag{27}$$

If the map $X \to \mathrm{Diff}(X) : x \mapsto \phi_x$ is determined by (27), then the map

$$X \times X \to X : (x,y) \mapsto \phi_x(y) =: xy \tag{28}$$

is smooth.

The proof of Step 4 has six parts labeled (a), (b), (c), (d), (e), (f).

(a) *Let $x \in X$ and let $\phi_x : X \to X$ be a diffeomorphism that satisfies (27). Then ϕ_x has the following property.*

> **(P)** *If $\gamma, \gamma' : [0,1] \to X$ are smooth curves such that*
>
> $$A_{\gamma'(t)}\big(\dot\gamma'(t)\big) = A_{\gamma(t)}\big(\dot\gamma(t)\big), \quad \gamma(0) = 1, \quad \gamma'(0) = x, \tag{29}$$
>
> *then $\phi_x(\gamma(1)) = \gamma'(1)$.*

If $\gamma : [0,1] \to X$ is a smooth curve such that $\gamma(0) = 1$, then $\gamma' := \phi_x \circ \gamma$ is the unique solution of (29) and $\gamma'(1) = \phi_x(\gamma(1))$. Thus (27) implies (P).

(b) *Let $x \in X$. Then there exists a unique map $\phi_x : X \to X$ with the property* (P). Step 3 shows that, given any smooth curve $\gamma : [0,1] \to X$ such that $\gamma(0) = 1$, the endpoint $\gamma'(1)$ of the unique solution $\gamma' : [0,1] \to X$ of the intial value problem (29) depends only on the endpoint $y := \gamma(1)$ and not on the choice of the curve γ from 1 to y. This proves (b).

(c) *Let $x \in X$ and let ϕ_x be as in* (b). *Then $\phi_x(1) = x$ and ϕ_x is bijective.* That $\phi_x(1) = x$ follows by taking the constant curve $\gamma \equiv 1$ in (29). That ϕ_x is bijective follows by reversing the roles of γ and γ' in property (P). Namely, if $\gamma' : [0,1] \to X$ is any smooth curve such that $\gamma'(0) = x$, and $\gamma : [0,1] \to X$ is the unique solution of (29), then by Step 3 the endpoint $\gamma(1)$ depends only on $z := \gamma'(1)$ and not on the choice of the curve γ' from x to z. Hence there exists a unique map $\phi'_x : X \to X$ such that (29) implies $\phi'_x(\gamma'(1)) = \gamma(1)$. This map satisfies $\phi'_x \circ \phi_x = \phi_x \circ \phi'_x = \mathrm{id}$. Thus we have proved (c).

(d) *For each $x \in X$ let ϕ_x be as in* (b). *Then the map* (28) *is smooth.* Each point in X has an open neighborhood U equipped with a smooth map

$$U \times [0,1] \to X : (y,t) \mapsto \gamma_y(t)$$

such that $\gamma_y(0) = 1$ and $\gamma_y(1) = y$ for all $y \in U$. For $x \in X$ and $y \in U$ let $\gamma'_{x,y} : [0,1] \to X$ be the unique solution of (29) with $\gamma = \gamma_y$. Then the map

$$X \times U \to X : (x,y) \mapsto \gamma'_{x,y}(1) = \phi_x(y)$$

is smooth both in the initial condition $x \in X$ and in the parameter $y \in U$. Since X can be covered by such open sets U, this proves (d).

(e) *Let $x \in X$. Then the map ϕ_x in* (b) *is a diffeomorphism.* The map ϕ_x is bijective by (c) and is smooth by (d). That its inverse is smooth is proved by the same argument. Each point in X has an open neighborhood V equipped with a smooth map $V \times [0,1] \to X : (z,t) \mapsto \gamma'_z(t)$ such that $\gamma'_z(0) = x$ and $\gamma'_z(1) = z$ for all $z \in V$. For each $z \in V$ let $\gamma_z : [0,1] \to X$ be the solution of (29) with $\gamma' = \gamma'_z$. Then γ_z depends smoothly on $z \in V$. Thus the map $V \to X : z \mapsto \gamma_z(1) = \phi_x^{-1}(z)$ is smooth, and this proves (e).

(f) *Let $x \in X$ and let ϕ_x be as in* (b). *Then ϕ_x is a diffeomorphism and it satisfies* (27). That ϕ_x is a diffeomorphism was proved in (e) and that $\phi_x(1) = x$ was proved in (c). That $\phi_x^* A = A$ follows from the observation that, under the assumption (29), we have $\gamma'(t) = \phi_x(\gamma(t))$ for all t, and hence $(\phi^* A)_\gamma(\dot\gamma) = A_{\gamma'}(\dot\gamma') = A_\gamma(\dot\gamma)$ for every smooth curve $\gamma : [0,1] \to X$ such that $\gamma(0) = 1$. Thus $\phi_x^* A = A$. This proves (f) and Step 4.

Step 5. *For each $x \in X$ let $\phi_x : X \to X$ be the diffeomorphism in Step 4. Then, for every $y \in X$, the map $\psi_y : X \to X$, defined by*

$$\psi_y(x) := \phi_x(y) \qquad \text{for } x \in X, \tag{30}$$

is a diffeomorphism. These diffeomorphisms satisfy the equation

$$\psi_{\psi_z(y)} = \psi_z \circ \psi_y \qquad \text{for all } y, z \in X. \tag{31}$$

Fix an element $y \in X$ and choose a smooth curve $\gamma_y : [0,1] \to X$ such that

$$\gamma_y(0) = 1, \qquad \gamma_y(1) = y.$$

Define the vector fields $Y_t \in \mathrm{Vect}(X)$ by

$$A_x(Y_t(x)) = A_{\gamma_y(t)}(\dot{\gamma}_y(t))$$

for $x \in X$ and $0 \le t \le 1$, and let $[0,1] \to \mathrm{Diff}(X) : t \mapsto \psi_{y,t}$ be the smooth isotopy determined by the vector fields Y_t via

$$\partial_t \psi_{y,t} = Y_t \circ \psi_{y,t}, \qquad \psi_{y,0} = \mathrm{id}.$$

Then, for each $x \in X$, the curve $\gamma'_{x,y}(t) := \psi_{y,t}(x)$ is the unique solution of the initial value problem (29) with $\gamma = \gamma_y$. Hence, for every $x \in X$,

$$\psi_{y,1}(x) = \gamma'_{x,y}(1) = \phi_x(\gamma_y(1)) = \phi_x(y) = \psi_y(x).$$

Thus $\psi_y = \psi_{y,1}$ is a diffeomorphism.

Now let $z \in X$ and define the triple $\gamma_z, Z_t, \psi_{z,t}$ exactly like $\gamma_y, Y_t, \psi_{y,t}$. Assume without loss of generality that the curves γ_y, γ_z are constant near the endpoints and define the curve $\gamma : [0,1] \to X$ by

$$\gamma(t) := \begin{cases} \gamma_y(2t), & \text{if } 0 \le t \le 1/2, \\ \psi_{z,2t-1}(y), & \text{if } 1/2 \le t \le 1. \end{cases}$$

Then γ is smooth and $\gamma(0) = 1$, $\gamma(1) = \psi_z(y)$. Define the vector fields W_t by $A_x(W_t(x)) = A_{\gamma(t)}(\dot{\gamma}(t))$ for $x \in X$ and $0 \le t \le 1$. Then

$$W_t = \begin{cases} 2Y_{2t}, & \text{if } 0 \le t \le 1/2, \\ 2Z_{2t-1}, & \text{if } 1/2 \le t \le 1. \end{cases}$$

The isotopy $[0,1] \to \mathrm{Diff}(M) : t \mapsto \chi_t$ generated by these vector fields is given by $\chi_t = \psi_{y,2t}$ for $0 \le t \le 1/2$ and by $\chi_t = \psi_{z,2t-1} \circ \psi_y$ for $1/2 \le t \le 1$. Hence $\psi_{\psi_z(y)} = \psi_{\gamma(1)} = \chi_1 = \psi_z \circ \psi_y$. This proves (31) and Step 5.

Step 6. *X is a Lie group with the unit* 1 *and the product* (28) *in Step 4.* The product (28) is defined by $xy := \phi_x(y) = \psi_y(x)$ for $x, y \in X$. Hence it follows from equation (31) in Step 5 that, for all $x, y, z \in X$,

$$(xy)z = \psi_z(\psi_y(x)) = \psi_{\psi_z(y)}(x) = \psi_{yz}(x) = x(yz).$$

Thus the product is associative. That 1 is the unit follows from the identities

$$x1 = \phi_x(1) = x = \phi_1(x) = 1x.$$

Here the second equality follows from (27) and the third equality follows from uniqueness in Step 4. Moreover, every element $x \in X$ has an inverse

$$x^{-1} := \phi_x^{-1}(1) = \psi_x^{-1}(1) \tag{32}$$

that satisfies $xx^{-1} = \phi_x(x^{-1}) = 1 = \psi_x(x^{-1}) = x^{-1}x$. The second equality in (32) follows from the standard argument in elementary group theory, which here takes the form that $x \in X$ and $y := \phi_x^{-1}(1)$ satisfy

$$\phi_x(\psi_x(y)) = \psi_{\psi_x(y)}(x) = \psi_x(\psi_y(x)) = \psi_x(\phi_x(y)) = \psi_x(1) = \phi_1(x) = x$$

and hence $\psi_x(y) = \phi_x^{-1}(x) = 1$. Thus (28) defines a group operation on X. The group operation is smooth by Step 4, and the inverse map $x \mapsto \phi_x^{-1}(1)$ is smooth by the Implicit Function Theorem. This proves Step 6.

Step 7. *The map $A_1 : T_1X = \mathrm{Lie}(X) \to \mathfrak{g}$ is a Lie algebra homomorphism and satisfies $A_x(x\widehat{x}) = A_1(\widehat{x})$ for all $x \in X$ and all $\widehat{x} \in T_1X$.*

The formula $A_x(x\widehat{x}) = A_1(\widehat{x})$ with $x\widehat{x} = d\phi_x(1)\widehat{x}$ follows directly from the fact that $\phi_x^*A = A$ and $\phi_x(1) = x$. This formula shows that the vector fields $Y_\xi \in \mathrm{Vect}(X)$, defined by equation (16) in Definition 3.1, satisfy

$$A_1(x^{-1}Y_\xi(x)) = A_x(Y_\xi(x)) = \xi$$

for all $x \in X$ and all $\xi \in \mathfrak{g}$. Take $\xi = A_1(\widehat{x})$ to deduce that the left invariant vector field associated to $\widehat{x} \in \mathrm{Lie}(X) = T_1X$ is given by

$$Y_{A_1(\widehat{x})}(x) = x\widehat{x} \qquad \text{for } x \in X.$$

Hence the linear map $\mathrm{Lie}(X) \to \mathrm{Vect}(X) : \widehat{x} \mapsto Y_{A_1(\widehat{x})}$ is a Lie algebra anti-homomorphism. By Lemma 3.2 the linear map $\mathfrak{g} \to \mathrm{Vect}(X) : \xi \mapsto Y_\xi$ is also a Lie algebra anti-homomorphism. Thus the linear map $A_1 : \mathrm{Lie}(X) \to \mathfrak{g}$ is a Lie algebra homomorphism and this proves Step 7.

Step 7 completes the proof of the existence statement. That the Lie group structure on X is uniquely determined by A and the choice of $1 \in X$ follows from uniqueness in Step 4. This proves Theorem 3.5. $\square$

Definition 3.6 (Invariant Inner Product). *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. An inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ is called* **invariant** *iff it satisfies*

$$\langle \xi, [\eta, \zeta]\rangle = \langle [\xi, \eta], \zeta\rangle \tag{33}$$

for all $\xi, \eta, \zeta \in \mathfrak{g}$. When an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ is given, denote the corresponding norm of an element $\xi \in \mathfrak{g}$ by

$$|\xi| := \sqrt{\langle \xi, \xi\rangle}. \tag{34}$$

The Lie algebra of any compact Lie group admits an invariant inner product.

Lemma 3.7. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and assume that $\mathfrak{g}$ admits an invariant inner product $\langle\cdot,\cdot\rangle$. Let $\eta \in \mathfrak{g}$ and let $\xi : \mathbb{R} \to \mathfrak{g}$ be a solution of the linear second order differential equation*

$$\ddot{\xi} + [\eta, [\eta, \xi]] = 0, \qquad \xi(0) = 0. \tag{35}$$

Then $|\xi(t)| \geq |t|\,|\dot{\xi}(0)|$ for every $t \in \mathbb{R}$.

Proof. By (33) and (35) we have

$$\frac{d}{dt}\left(|\dot{\xi}|^2 - |[\xi, \eta]|^2\right) = 2\langle \dot{\xi}, \ddot{\xi}\rangle + 2\langle [\dot{\xi}, \eta], [\eta, \xi]\rangle = 0.$$

Since $\xi(0) = 0$, this implies

$$|\dot{\xi}(t)|^2 = |\dot{\xi}(0)|^2 + |[\xi(t), \eta]|^2 \geq |\dot{\xi}(0)|^2 \tag{36}$$

for all $t \in \mathbb{R}$. For $t \geq 0$ it follows from (35) and (36) that

$$\begin{aligned} 0 &= \int_0^t \langle \xi(s), \ddot{\xi}(s) + [\eta, [\eta, \xi(s)]]\rangle \, ds \\ &= \langle \xi(t), \dot{\xi}(t)\rangle - \int_0^t \left(|\dot{\xi}(s)|^2 + |[\xi(s), \eta]|^2\right) ds \\ &\leq \langle \xi(t), \dot{\xi}(t)\rangle - t|\dot{\xi}(0)|^2. \end{aligned}$$

Hence, for all $t \geq 0$,

$$|\xi(t)|^2 = 2\int_0^t \langle \xi(s), \dot{\xi}(s)\rangle \, ds \geq 2\int_0^t s|\dot{\xi}(0)|^2 \, ds = t^2|\dot{\xi}(0)|^2.$$

Since equation (35) is time reversible, this proves Lemma 3.7. □

Lemma 3.8. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and define $\mathfrak{g}^c := \mathfrak{g} \oplus \mathbf{i}\mathfrak{g}$. Then there exists a unique flat connection $B \in \Omega^1(\mathfrak{g}, \mathfrak{g}^c)$ such that*

$$[\eta, \widehat{\eta}] = 0 \qquad \Longrightarrow \qquad B_\eta(\widehat{\eta}) = \mathbf{i}\widehat{\eta} \tag{37}$$

for all $\eta, \widehat{\eta} \in \mathfrak{g}$. If $\mathfrak{g}$ admits an invariant inner product $\langle \cdot, \cdot \rangle$, then

$$|\widehat{\eta}| \leq |\mathrm{Im}(B_\eta(\widehat{\eta}))| \tag{38}$$

for all $\eta, \widehat{\eta} \in \mathfrak{g}$.

Proof. Let $A^c \in \Omega^1(\mathfrak{g}^c, \mathfrak{g}^c)$ be the flat connection in Lemma 3.4 on $\mathfrak{g}^c$, and define B as the pullback of this connection under the map $\mathfrak{g} \to \mathfrak{g}^c : \eta \mapsto \mathbf{i}\eta$, i.e. $B_\eta(\widehat{\eta}) := A^c_{\mathbf{i}\eta}(\mathbf{i}\widehat{\eta})$ for $\eta, \widehat{\eta} \in \mathfrak{g}$. This connection is flat and satisfies (37). Moreover, for each pair $\eta, \widehat{\eta} \in \mathfrak{g}$ it follows from (19) in the proof of Lemma 3.4 that the curve $\zeta(t) := B_{t\eta}(t\widehat{\eta})$ satisfies the initial value problem

$$\dot{\zeta} + [\mathbf{i}\eta, \zeta] = \mathbf{i}\widehat{\eta}, \qquad \zeta(0) = 0. \tag{39}$$

Hence B is uniquely determined by (37) and the condition that it is flat.

To prove (38), let $\zeta : \mathbb{R} \to \mathfrak{g}^c$ be the unique solution of (39). Then the curve $\xi(t) := \mathrm{Im}(\zeta(t))$ satisfies $\ddot{\xi} + [\eta, [\eta, \xi]] = 0$, $\xi(0) = 0$, and $\dot{\xi}(0) = \widehat{\eta}$. Thus, by Lemma 3.7, $|\mathrm{Im}(B_\eta(\widehat{\eta}))| = |\xi(1)| \geq |\widehat{\eta}|$. This proves Lemma 3.8. □

Lemma 3.9. *Let G be a compact connected Lie group, denote its Lie algebra by $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$, and define*

$$\mathrm{G}^c := \mathrm{G} \times \mathfrak{g}, \qquad \mathfrak{g}^c := \mathfrak{g} \oplus \mathbf{i}\mathfrak{g}.$$

Let $B \in \Omega^1(\mathfrak{g}, \mathfrak{g}^c)$ be as in Lemma 3.8 and define $A \in \Omega^1(\mathrm{G}^c, \mathfrak{g}^c)$ by

$$A_{(u,\eta)}(\widehat{u}, \widehat{\eta}) := u^{-1}\widehat{u} + u^{-1}\big(B_\eta(\widehat{\eta})\big)u \tag{40}$$

for $u \in \mathrm{G}$, $\widehat{u} \in T_u\mathrm{G}$, and $\eta, \widehat{\eta} \in \mathfrak{g}$. Then A is an infinitesimal group law.

Proof. We prove that A is flat. By Example 3.3 the $\mathfrak{g}$-connection A_0 on G defined by $A_{0,u}(\widehat{u}) := u^{-1}\widehat{u}$ is flat and by Lemma 3.8 the 1-form B is flat. Hence, for $g = (u, \eta) \in \mathrm{G} \times \mathfrak{g}$ and $\widehat{g}_j = (\widehat{u}_j, \widehat{\eta}_j) \in T_g(\mathrm{G} \times \mathfrak{g})$, we have

$$\begin{aligned}
F_{A,g}(\widehat{g}_1, \widehat{g}_2) &= (dA)_g(\widehat{g}_1, \widehat{g}_2) + \big[A_g(\widehat{g}_1), A_g(\widehat{g}_2)\big] \\
&= (dA_0)_u(\widehat{u}_1, \widehat{u}_2) + u^{-1}(dB)_\eta(\widehat{\eta}_1, \widehat{\eta}_2)u \\
&\quad + \big[u^{-1}\big(B_\eta(\widehat{\eta}_2)\big)u, u^{-1}\widehat{u}_1\big] - \big[u^{-1}\big(B_\eta(\widehat{\eta}_1)\big)u, u^{-1}\widehat{u}_2\big] \\
&\quad + \big[u^{-1}\widehat{u}_1 + u^{-1}\big(B_\eta(\widehat{\eta}_1)\big)u, u^{-1}\widehat{u}_2 + u^{-1}\big(B_\eta(\widehat{\eta}_2)\big)u\big] \\
&= F_{A_0,u}(\widehat{u}_1, \widehat{u}_2) + u^{-1}F_{B,\eta}(\widehat{\eta}_1, \widehat{\eta}_2)u = 0.
\end{aligned}$$

This shows that A is a flat connection.

For the *(Monodromy)* axiom it suffices to consider loops based at $(1,0)$. It is obviously satisfied for loops in $\mathrm{G}\times\{0\}$ and hence follows from the fact that A is flat and that every based loop in G^c is homotopic to one in $\mathrm{G}\times\{0\}$.

To verify the *(Inverse)* and *(Complete)* axioms, choose an invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. Then it follows from the estimate (38) that the linear map $A_{(u,\eta)}:T_{(u,\eta)}\mathrm{G}^c\to\mathfrak{g}^c$ is invertible for every pair $(u,\eta)\in\mathrm{G}\times\mathfrak{g}$. It also follows from (38) that, for every smooth curve $\zeta:\mathbb{R}\to\mathfrak{g}^c$ and every $T>0$, the solutions $[0,T]\to\mathrm{G}^c:t\mapsto(u(t),\eta(t))$ of the differential equation

$$u(t)^{-1}\dot{u}(t)+u(t)^{-1}\big(B_{\eta(t)}(\dot{\eta}(t))\big)u(t)=\zeta(t)$$

satisfy $\sup_{0\le t\le T}|\eta(t)-\eta(0)|\le cT$, where $c:=\sup_{0\le t\le T}|\mathrm{Im}(\zeta(t))|$. Hence the solutions exist for all time. This proves Lemma 3.9. □

Theorem 3.10. *Let* G *be a compact connected Lie group, define*

$$\mathfrak{g}:=\mathrm{Lie}(\mathrm{G}),\qquad \mathrm{G}^c:=\mathrm{G}\times\mathfrak{g},\qquad \mathfrak{g}^c:=\mathfrak{g}\oplus\mathbf{i}\mathfrak{g},$$

and equip G^c *with the Lie group structure with the unit* $(1,0)$*, associated to the infinitesimal group law* $A\in\Omega^1(\mathrm{G}^c,\mathfrak{g}^c)$ *in Lemma 3.9 via Theorem 3.5. Then* G^c *is a complex Lie group with the standard complex structure on its Lie algebra* $T_{(1,0)}\mathrm{G}^c=\mathfrak{g}\times\mathfrak{g}\cong\mathfrak{g}^c$*. Moreover, for* $u,v\in\mathrm{G}$ *and* $\xi,\eta\in\mathfrak{g}$*,*

$$[\xi,u\eta u^{-1}]=0\qquad\Longrightarrow\qquad(u,\xi)\cdot(v,\eta)=(uv,\xi+u\eta u^{-1}).\tag{41}$$

The inclusion $\mathrm{G}\to\mathrm{G}^c:u\mapsto(u,0)$ *satisfies (i), (ii), (iii) in Theorem 1.4.*

Proof. The isomorphism

$$A_{(1,0)}:T_{(1,0)}(\mathrm{G}\times\mathfrak{g})=\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}^c$$

in (40) is given by $A_{(1,0)}(\widehat{u},\widehat{\eta})=\widehat{u}+\mathbf{i}\widehat{\eta}$ for $\widehat{u},\widehat{\eta}\in\mathfrak{g}=T_1\mathrm{G}$. Hence it follows from Proposition 1.2 that G^c is a complex Lie group with the standard complex structure on its Lie algebra. Now let $u,v\in\mathrm{G}$ and $\xi,\eta\in\mathfrak{g}$ such that $[\xi,u\eta u^{-1}]=0$. Choose a smooth curve $\alpha:[0,1]\to\mathrm{G}$ such that $\alpha(0)=1$ and $\alpha(1)=v$ and define the curves $\gamma,\gamma':[0,1]\to\mathrm{G}\times\mathfrak{g}$ by

$$\gamma(t):=(\alpha(t),t\eta),\qquad\gamma'(t):=(u\alpha(t),\xi+tu\eta u^{-1}).$$

Then $\gamma(0)=(1,0)$, $\gamma'(0)=(u,\xi)$, and $A_\gamma(\dot{\gamma})=\alpha^{-1}\dot{\alpha}+\mathbf{i}\alpha^{-1}\eta\alpha=A_{\gamma'}(\dot{\gamma}')$. Hence $\gamma'(t)=(u,\xi)\cdot\gamma(t)$ for all t (property (P) in the proof of Theorem 3.5). Take $t=1$ with $\gamma(1)=(v,\eta)$ to obtain $(uv,\xi+u\eta u^{-1})=\gamma'(1)=(u,\xi)\cdot(v,\eta)$. This proves (41).

By (41) the embedding $\mathrm{G} \to \mathrm{G}^c : u \mapsto \iota(u) := (u, 0)$ is a Lie group homomorphism and the image of the differential $d\iota(1) : T_1\mathrm{G} \to T_{(1,0)}\mathrm{G}^c$ is the totally real subspace $\mathfrak{g} \times \{0\}$ of the Lie algebra $T_{(1,0)}\mathrm{G}^c = \mathfrak{g} \times \mathfrak{g} \cong \mathfrak{g}^c$.

Second, $\iota(\mathrm{G}) = \mathrm{G} \times \{0\}$ is a maximal compact subgroup of G^c. Namely, let $\mathrm{H} \subset \mathrm{G}^c$ be a subgroup such that $\mathrm{G} \subsetneq \mathrm{H}$. Then H contains an element of the form (u, ξ) with $\xi \neq 0$. Hence, by (41), the pair $(1, \xi) = (u, \xi) \cdot (u^{-1}, 0)$ is also an element of H and hence, so is $(1, k\xi)$ for every integer $k \geq 1$. This sequence has no convergent subsequence, so H is not compact.

Third, the homogeneous space $\mathrm{G}^c/\iota(\mathrm{G})$ is diffeomorphic to $\mathfrak{g}$. To see this, note first that for $(0, \eta) \in \mathrm{Lie}(\mathrm{G}^c) = \mathfrak{g} \times \mathfrak{g}$ we have

$$\exp(0, \eta) = (1, \eta). \tag{42}$$

Indeed, by (41) the curve $\gamma(t) := (1, t\eta) \in \mathrm{G}^c$ satisfies $\gamma(s) \cdot \gamma(t) = \gamma(s + t)$ for all $s, t \in \mathbb{R}$ and $\dot\gamma(0) = (0, \eta) \in \mathrm{Lie}(\mathrm{G}^c)$. It follows fron (41) and (42) that

$$\exp(0, \eta) \cdot \iota(u) = (1, \eta) \cdot (u, 0) = (u, \eta).$$

Hence the map $\mathrm{G} \times \mathfrak{g} \to \mathrm{G}^c : (u, \eta) \mapsto \exp(0, \eta) \cdot \iota(u)$ is the tautological diffeomorphism and the set $\{(1, \eta) \,|\, \eta \in \mathfrak{g}\} \subset \mathrm{G}^c$ is a slice of the G-action. Hence $\mathrm{G}^c/\iota(\mathrm{G})$ is diffeomorphic to $\mathfrak{g}$, and hence the inclusion of G into G^c satisfies (i) and (ii) in Theorem 1.4. Since we have already shown in §1 that (ii) implies (iii), this proves Theorem 3.10. □

Corollary 3.11. *Theorem 1.7 holds for every compact Lie group.*

Proof (sketch). Let G be a compact Lie group with the Lie algebra $\mathfrak{g}$ and the identity component $\mathrm{G}_0 \subset \mathrm{G}$. By Theorem 3.10 there exists a complex Lie group G_0^c such that $\mathrm{G}_0 \subset \mathrm{G}_0^c$ satisfies (i), (ii), (iii) in Theorem 1.4.

Step 1. Extend the adjoint action of G on G_0 to an adjoint action

$$\mathrm{G} \times \mathrm{G}_0^c \to \mathrm{G}_0^c : (u, g) \mapsto ugu^{-1}$$

by integrating the adjoint action of G on the Lie algebra $\mathfrak{g}^c = \mathrm{Lie}(\mathrm{G}_0^c) = \mathfrak{g} \oplus \mathbf{i}\mathfrak{g}$ along paths in G_0^c starting in G_0. This step uses Corollary A.2.

Step 2. The group G_0 acts on $\mathrm{G}_0^c \times \mathrm{G}$ on the right by $u_0^*(g, u) := (gu_0, u_0^{-1}u)$. Define $\mathrm{G}^c := (\mathrm{G}_0^c \times \mathrm{G})/\mathrm{G}_0$ and denote by $\pi : \mathrm{G}_0^c \times \mathrm{G} \to \mathrm{G}^c$ the canonical projection. Then G^c is a complex Lie group with the unit $\pi(1, 1)$, the Lie algebra $\mathrm{Lie}(\mathrm{G}^c) = T_{\pi(1,1)}\mathrm{G}^c \cong \mathfrak{g}^c$, and the group multiplication

$$\pi(g, u) \cdot \pi(g', u') := \pi\big(g(ug'u^{-1}), uu'\big).$$

Step 3. The inclusion $\iota : \mathrm{G} \to \mathrm{G}^c$, defined by $\iota(u) := \pi(1, u)$ for $u \in \mathrm{G}$, satisfies the conditions (i), (ii), (iii) in Theorem 1.4. □

4 Hadamard's theorem

The following theorem characterizes complete manifolds. The Hopf–Rinow Theorem is the assertion that (i) implies (ii).

Theorem 4.1 (Hopf–Rinow). *Let M be a nonempty connected Riemannian manifold, denote by $d : M \times M \to \mathbb{R}$ the distance function associated to the Riemannian metric, and let $p_0 \in M$. Then the following are equivalent.*

(i) *The geodesics starting at p_0 exist for all time.*

(ii) *For every $p_1 \in M$ there exists a geodesic $\gamma : [0,1] \to M$ such that*

$$\gamma(0) = p_0, \qquad \gamma(1) = p_1, \qquad L(\gamma) := \int_0^1 |\dot{\gamma}(t)|\, dt = d(p_0, p_1).$$

(iii) *Every closed and bounded subset of (M, d) is compact.*

(iv) *(M, d) is a complete metric space.*

Proof. See [12, Theorems 4.6.5 and 4.6.6]. □

A connected Rimannian manifold satisfying the conditions of Theorem 4.1 is called **complete**. In a complete manifold any two points are joined by a geodesic. A **Hadamard manifold** is a nonempty, complete, connected, simply connected Riemannian manifold with nonpositive sectional curvature. In a Hadamard manifold any two points are joined by a unique geodesic.

Theorem 4.2 (Hadamard). *Let M be a Hadamard manifold and let $p \in M$. Then the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism and*

$$\left| d\exp_p(v)\widehat{v} \right| \geq |\widehat{v}| \qquad \text{for all } v, \widehat{v} \in T_pM, \tag{43}$$

$$d\big(\exp_p(v_0), \exp_p(v_1)\big) \geq |v_0 - v_1| \qquad \text{for all } v_0, v_1 \in T_pM. \tag{44}$$

Proof. The proof has four steps. Let ∇ be the Levi-Civita connection on TM and denote by $R \in \Omega^2(M, \mathrm{End}(TM))$ the Riemann curvature tensor.

Step 1. *The exponential map $\exp_p : T_pM \to M$ is surjective and is a local diffeomorphism that satisfies* (43)*.*

The exponential map is surjective by the Hopf–Rinow Theorem 4.1. Now let $v, \widehat{v} \in T_pM$ such that $\widehat{v} \neq 0$. Define the curve $\gamma : [0,1] \to M$ and the vector field $X : [0,1] \to TM$ along γ by

$$\gamma(t) := \exp_p(tv), \qquad X(t) := d\exp_p(tv)t\widehat{v} \qquad \text{for } 0 \leq t \leq 1. \tag{45}$$

We prove that X satisfies the **Jacobi equation**

$$\nabla_t \nabla_t X + R(X, \dot{\gamma})\dot{\gamma} = 0. \tag{46}$$

To see this, define $\gamma_s(t) := \exp_p(t(v + s\widehat{v}))$ for $s, t \in \mathbb{R}$. Then $\gamma_s : \mathbb{R} \to M$ is a geodesic for every s and

$$\gamma_0(t) = \gamma(t), \qquad \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_s(t) = X(t) \qquad \text{for } 0 \le t \le 1.$$

Moreover,

$$\begin{aligned}\nabla_t \nabla_t \partial_s \gamma_s &= \nabla_t \nabla_s \partial_t \gamma_s \\ &= \nabla_s \nabla_t \partial_t \gamma_s - R(\partial_s \gamma_s, \partial_t \gamma_s)\partial_t \gamma_s \\ &= -R(\partial_s \gamma_s, \partial_t \gamma_s)\partial_t \gamma_s.\end{aligned}$$

Take $s = 0$ and $0 \le t \le 1$ to obtain (46).

Next we observe that the function $[0,1] \to \mathbb{R} : t \mapsto |X(t)|$ is differentiable at $t = 0$ and that its derivative is given by

$$\left.\frac{d}{dt}\right|_{t=0} |X(t)| = \lim_{t \searrow 0} \frac{|X(t)|}{t} = \lim_{t \searrow 0} \left| d\exp_p(tv)\widehat{v} \right| = \left| d\exp_p(0)\widehat{v} \right| = |\widehat{v}|. \tag{47}$$

Thus $X(t) \ne 0$ for $t > 0$ sufficiently small. Moreover, at each point t where $X(t) \ne 0$, the Jacobi equation (46) and the hypothesis of nonpositive sectional curvature yield the inequality

$$\begin{aligned}\frac{d^2}{dt^2}|X| &= \frac{d}{dt}\frac{\langle \nabla_t X, X\rangle}{|X|} \\ &= \frac{|\nabla_t X|^2 + \langle \nabla_t \nabla_t X, X\rangle}{|X|} - \frac{\langle \nabla_t X, X\rangle^2}{|X|^3} \\ &= \frac{|\nabla_t X|^2 |X|^2 - \langle \nabla_t X, X\rangle^2}{|X|^3} - \frac{\langle R(X, \dot{\gamma})\dot{\gamma}, X\rangle}{|X|} \ge 0.\end{aligned}$$

This implies that the function $[0,1] \to \mathbb{R} : t \mapsto |X(t)|$ is positive for all $t > 0$ and that its derivative is nondecreasing. Hence it follows from (47) that

$$|X(t)| \ge t|\widehat{v}| \qquad \text{for } 0 \le t \le 1.$$

Take $t = 1$ to obtain $\left| d\exp_p(v)\widehat{v} \right| = |X(1)| \ge |\widehat{v}|$. This proves (43). By (43) the derivative $d\exp_p(v) : T_pM \to T_{\exp_p(v)}M$ is bijective for every $v \in T_pM$. Thus the exponential map is a local diffeomorphism and this proves Step 1.

Step 2. *Let $\gamma : [0,1] \to M$ be a smooth curve and let $v_0 \in T_pM$ be a tangent vector such that $\exp_p(v_0) = \gamma(0)$. Then there exists a unique smooth curve $v : [0,1] \to T_pM$ such that*

$$v(0) = v_0, \qquad \gamma(t) = \exp_p(v(t)) \qquad \text{for } 0 \le t \le 1. \tag{48}$$

We prove uniqueness. Assume that $v, v' : [0,1] \to T_pM$ are two smooth curves satisfying (48). Then the set $I := \{t \in [0,1] \,|\, v(t) = v'(t)\}$ is nonempty and closed. Let $t \in I$ and choose an open neighborhood $U \subset T_pM$ of $v(t)$ such that $\exp_p|_U$ is injective. Then $v(t+s) = v'(t+s)$ for all s such that $\gamma(t+s)$ belongs to the open set $\exp_p(U)$. Hence I is open and so $I = [0,1]$. This proves uniqueness.

We prove existence. Let $\gamma : [0,1] \to M$ be a smooth curve and choose a tangent vector $v_0 \in T_pM$ such that $\exp_p(v_0) = \gamma(0)$. Let $E \subset [0,1]$ be the set of all t such that there exists a smooth curve $v : [0,t] \to T_pM$ that satisfies

$$v(0) = v_0, \qquad \exp_p(v(s)) = \gamma(s) \qquad \text{for } 0 \le s \le t. \tag{49}$$

For every $t \in E$ the curve v satisfying (49) is unique, by the above uniqueness argument. The set E is nonempty because $0 \in E$. Moreover, E is open because the restriction of $\exp_p$ to a sufficiently small neighbourhood of $v(t)$ is a diffeomorphism onto its image, where $v : [0,t] \to T_pM$ is the lift in (49).

We prove that E is closed. Let $t_i \in E$ be a sequence converging to t_*. Then, by uniqueness, there exists a smooth curve $v : [0, t_*) \to T_pM$ such that $v(0) = v_0$ and $\exp_p(v(t)) = \gamma(t)$ for $0 \le t < t_*$. Hence, by Step 1,

$$|\dot{v}(t)| \le |d\exp_p(v(t))\dot{v}(t)| = |\dot{\gamma}(t)|$$

and so

$$|v(t) - v(t')| \le \int_t^{t'} |\dot{v}(s)|\, ds \le \int_t^{t'} |\dot{\gamma}(s)| ds \le |t - t'| \max_{0 \le s \le 1} |\dot{\gamma}(s)|$$

for $0 \le t \le t' < t_*$. Thus the limit

$$v_* := \lim_{t \nearrow t_*} v(t) \in T_pM$$

exists. Since the restriction of $\exp_p$ to a sufficiently small neighborhood of v_* is a diffeomorphism onto its image, the extended curve $v : [0, t_*] \to T_pM$ defined by $v(t_*) := v_*$ is smooth. Hence $t_* \in E$. Thus we have proved that the set E is closed and so $E = [0,1]$. This proves existence and Step 2.

Step 3. *The exponential map* $\exp_p : T_pM \to M$ *is a diffeomorphism.*

By Step 1 it remains to prove that $\exp_p$ is injective. Let $v_0, v_1 \in T_pM$ such that $p_0 := \exp_p(v_0) = \exp_p(v_1)$. Since M is simply connected, there exists a smooth map $\gamma : [0,1]^2 \to M$ such that, for all $s,t \in [0,1]$,

$$\gamma(0,t) = p_0, \qquad \gamma(1,t) = \exp_p(tv_1 + (1-t)v_0), \qquad \gamma(s,0) = \gamma(s,1) = p_0.$$

For each $s \in [0,1]$ it follows from Step 2 that there exists a unique smooth curve $[0,1] \to T_pM : t \mapsto v(s,t)$ such that

$$v(s,0) = v_0, \qquad \exp_p(v(s,t)) = \gamma(s,t) \qquad \text{for } 0 \le t \le 1.$$

For $s = 0,1$ it follows from uniqueness that

$$v(0,t) = v_0, \qquad v(1,t) = tv_1 + (1-t)v_0 \qquad \text{for } 0 \le t \le 1.$$

Define

$$C := \{t \in [0,1] \,|\, \text{the curve } s \mapsto v(s,t) \text{ is continuous}\}, \quad c := \sup|\partial_s\gamma|.$$

Then $0 \in C$. Also, if $t \in C$, then the curve $v(\cdot,t)$ is smooth and by Step 1

$$|v(s_0,t) - v(s_1,t)| \le \int_{s_0}^{s_1} |\partial_s v(s,t)|\,ds \le \int_{s_0}^{s_1} |\partial_s\gamma(s,t)|\,ds \le c|s_0 - s_1|$$

for $0 \le s_0 \le s_1 \le 1$. Hence C is closed. Now let $(s_0,t_0) \in [0,1] \times C$, choose an open neighborhood $V \subset T_pM$ of $v(s_0,t_0)$ such that $\exp_p|_V$ is injective, then choose a convex relatively open neighborhood $U \subset [0,1]^2$ of (s_0,t_0) such that $\gamma(\overline{U}) \subset \exp_p(V)$, and observe that $v|_U = (\exp_p|_V)^{-1} \circ \gamma|_U$. Thus every pair $(s_0,t_0) \in [0,1] \times C$ has a relatively open neighborhood $U \subset [0,1]^2$ such that $v|_U$ is smooth. Hence C is open, hence $C = [0,1]$, and so $1 \in C$. Since $\exp_p(v(s,1)) = \gamma(s,1) = p_0$ for all s and $v(0,1) = v_0$, we conclude that $v(s,1) = v_0$ for all s. Hence $v_1 = v(1,1) = v_0$ and this proves Step 3.

Step 4. *The exponential map* $\exp_p : T_pM \to M$ *satisfies* (44).

Let $v_0, v_1 \in T_pM$. Then by Theorem 4.1 there exists a geodesic $\gamma : [0,1] \to M$ such that $\gamma(0) = p_0 := \exp_p(v_0)$, $\gamma(1) = p_1 := \exp_p(v_1)$, and $L(\gamma) = d(p_0,p_1)$. Thus by Step 3 there exists a unique smooth curve $v : [0,1] \to T_pM$ such that $\exp_p(v(t)) = \gamma(t)$ for $0 \le t \le 1$ and hence $v(0) = v_0$ and $v(1) = v_1$. By Step 1 this curve satisfies $|\dot v(t)| \le |d\exp_p(v(t))\dot v(t)| = |\dot\gamma(t)|$ for all t. Hence

$$|v_0 - v_1| = \left|\int_0^1 \dot v(t)\,dt\right| \le \int_0^1 |\dot v(t)|\,dt \le \int_0^1 |\dot\gamma(t)|\,dt = d(p_0,p_1).$$

This proves Step 4 and Theorem 4.2. □

5 Cartan's fixed point theorem

Theorem 5.1 (Cartan). *Let M be a Hadamard manifold and let* G *be a compact Lie group that acts on M by isometries. Then there exists an element $p \in M$ such that $up = p$ for every $u \in$* G.

Proof. The proof has three steps and follows the argument given by Bill Casselmann in [1]. The second step is Serre's uniqueness theorem for the *circumcentre* of a bounded set in a *semi-hyperbolic space*.

Step 1. *Let $m \in M$ and $v \in T_mM$ and define*

$$p_0 := \exp_m(-v), \qquad p_1 := \exp_m(v).$$

Then

$$2d(m,q)^2 + \frac{d(p_0,p_1)^2}{2} \leq d(p_0,q)^2 + d(p_1,q)^2$$

for every $q \in M$.

By Theorem 4.2 the exponential map

$$\exp_m : T_mM \to M$$

is a diffeomorphism. Hence

$$d(p_0,p_1) = 2|v|.$$

Now let $q \in M$. Then there is a unique tangent vector $w \in T_mM$ such that

$$q = \exp_m(w), \qquad d(m,q) = |w|.$$

Moreover, the exponential map is expanding by (44) in Theorem 4.2. Thus

$$d(p_0,q) \geq |w+v|, \qquad d(p_1,q) \geq |w-v|.$$

Hence

$$\begin{aligned} d(m,q)^2 &= |w|^2 = \frac{|w+v|^2 + |w-v|^2}{2} - |v|^2 \\ &\leq \frac{d(p_0,q)^2 + d(p_1,q)^2}{2} - \frac{d(p_0,p_1)^2}{4}. \end{aligned}$$

This proves Step 1.

Step 2. *For $p \in M$ and $r \geq 0$ denote by $B(p,r) \subset M$ the closed ball of radius r centered at p. Let $\Omega \subset M$ be a nonempty bounded set and define*

$$R_\Omega := \left\{ r \geq 0 \,\middle|\, \begin{array}{l} \text{there exists a } p \in M \\ \text{such that } \Omega \subset B(p,r) \end{array} \right\}, \qquad r_\Omega := \inf R_\Omega \geq 0.$$

Then there exists a unique element $p_\Omega \in M$ such that $\Omega \subset B(p_\Omega, r_\Omega)$.

We prove existence. By assumption R_Ω is a nonempty interval and so $r_\Omega < \infty$. Hence there exists a sequence $r_i > r_\Omega$ and a sequence $p_i \in M$ such that

$$\lim_{i\to\infty} r_i = r_\Omega, \qquad \Omega \subset B(p_i, r_i).$$

Choose $q \in \Omega$. Then $d(q, p_i) \leq r_i$ for every i. Since the sequence r_i is bounded and M is complete, Theorem 4.1 asserts that p_i has a convergent subsequence $(p_{i_k})_{k\in\mathbb{N}}$. Its limit $p_\Omega := \lim_{k\to\infty} p_{i_k}$ satisfies $\Omega \subset B(p_\Omega, r_\Omega)$.

We prove uniqueness. Let $p_0, p_1 \in M$ such that $\Omega \subset B(p_0, r_\Omega) \cap B(p_1, r_\Omega)$. By Theorem 4.2, there exists a unique vector $v_0 \in T_{p_0}M$ such that

$$p_1 = \exp_{p_0}(v_0).$$

Denote the midpoint between p_0 and p_1 by

$$m := \exp_{p_0}\left(\tfrac{1}{2}v_0\right).$$

Then by Step 1 every $q \in \Omega$ satisfies the inequality

$$d(m,q)^2 \leq \frac{d(p_0,q)^2 + d(p_1,q)^2}{2} - \frac{d(p_0,p_1)^2}{4} \leq r_\Omega^2 - \frac{d(p_0,p_1)^2}{4}.$$

Since Ω is nonempty, it then follows from the definition of r_Ω that

$$r_\Omega^2 \leq \sup_{q\in\Omega} d(m,q)^2 \leq r_\Omega^2 - \frac{d(p_0,p_1)^2}{4}.$$

Thus $d(p_0,p_1) = 0$ and so $p_0 = p_1$. This proves Step 2.

Step 3. *We prove Theorem 5.1.*

Let $q \in M$. Then, since G is compact, the group orbit $\Omega := \{uq \,|\, u \in \mathrm{G}\}$ is a nonempty bounded subset of M. Define $r_\Omega \geq 0$ and $p_\Omega \in M$ as in Step 2. Then $\Omega \subset B(p_\Omega, r_\Omega)$. Since G acts on M by isometries, this implies

$$\Omega = u\Omega \subset B(up_\Omega, r_\Omega)$$

for every $u \in \mathrm{G}$. Hence it follows from the uniqueness statement in Step 2 that $up_\Omega = p_\Omega$ for every $u \in \mathrm{G}$. This proves Step 3 and Theorem 5.1. □

6 The homogeneous space G^c/G

In this section we prove Theorem 1.4 characterizing the complexified group, Cartan's Theorem 1.8 about the uniqueness of maximal compact subgroups of the complexified group, and the Cartan Decomposition Theroem 1.9. Throughout we assume the following.

> **(C)** G^c *is a complex Lie group with the Lie algebra* $\mathfrak{g}^c := \mathrm{Lie}(\mathrm{G}^c)$ *and* $\mathrm{G} \subset \mathrm{G}^c$ *is a maximal compact subgroup such that* G^c/G *is connected and the Lie algebra* $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ *is a totally real subspace of* $\mathfrak{g}^c$*, i.e.* $\mathfrak{g}^c = \mathfrak{g} \oplus \mathbf{i}\mathfrak{g}$.

If G is a Lie subgroup of $\mathrm{U}(n)$ for some $n \in \mathbb{N}$, then the complex Lie group G^c in Theorem 2.1 satisfies (C). If G is a compact connected Lie group, and $\iota : \mathrm{G} \to \mathrm{G}^c$ is the Lie group homomorphism constructed in Theorem 3.10 in the intrinsic setting, then the pair $(\mathrm{G}^c, \iota(\mathrm{G}))$ satisfies (C).

A key step in the proofs of the aforemantioned theorems is the observation that under the assumption (C) the homogeneous space G^c/G is a manifold of nonpositive sectional curvature (Theorem 6.2).

Definition 6.1. *Assume (C). Define the* **homogeneous space** G^c/G *by*

$$\mathrm{G}^c/\mathrm{G} := \{gG \,|\, g \in \mathrm{G}^c\}, \qquad g\mathrm{G} := \{gu \,|\, u \in \mathrm{G}\},$$

and denote the canonical projection by $\pi : \mathrm{G}^c \to \mathrm{G}^c/\mathrm{G}$*. The tangent space of* G^c/G *at an element* $\pi(g)$ *is the quotient of the tangent spaces*

$$T_{\pi(g)}(\mathrm{G}^c/\mathrm{G}) \cong \frac{T_g\mathrm{G}^c}{T_g g\mathrm{G}} = \frac{T_g\mathrm{G}^c}{\{g\xi \,|\, \xi \in \mathfrak{g}\}}.$$

Throughout we use the notation

$$\mathrm{Re}(\zeta) := \xi, \qquad \mathrm{Im}(\zeta) := \eta, \qquad \zeta = \xi + \mathbf{i}\eta \in \mathfrak{g}^c,$$

for $\xi, \eta \in \mathfrak{g}$*. Thus the tangent vector* $d\pi(g)\widehat{g} \in T_{\pi(g)}(\mathrm{G}^c/\mathrm{G})$ *is uniquely determined by* $\mathrm{Im}(g^{-1}\widehat{g})$*. Choose an invariant inner product* $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ *on* $\mathfrak{g}$ *and define a* **Riemannian metric on** G^c/G *by*

$$\langle d\pi(g)\widehat{g}_1, d\pi(g)\widehat{g}_2\rangle := \langle \eta_1, \eta_2\rangle_{\mathfrak{g}}, \qquad \eta_j := \mathrm{Im}(g^{-1}\widehat{g}_j),$$

for $g \in \mathrm{G}^c$ *and* $\widehat{g}_1, \widehat{g}_2 \in T_g\mathrm{G}^c$.

The next theorem shows that the homogeneous space G^c/G with the metric in Definition 6.1 is complete and has nonpositive sectional curvature.

Theorem 6.2. *Assume (C). Then the following holds.*
(i) *The geodesics in* G^c/G *have the form*

$$\gamma(t) = \pi(g_0 \exp(\mathbf{i}t\eta))$$

for $g_0 \in \mathrm{G}^c$ *and* $\eta \in \mathfrak{g}$. *Thus* G^c/G *is complete.*
(ii) *The Riemann curvature tensor on* G^c/G *is given by*

$$R(X,Y)Z = d\pi(g)g\mathbf{i}[[\xi,\eta],\zeta]$$

for $g \in \mathrm{G}^c$, $\xi,\eta,\zeta \in \mathfrak{g}$, *and* $X := d\pi(g)g\mathbf{i}\xi$, $Y := d\pi(g)g\mathbf{i}\eta$, $Z := d\pi(g)g\mathbf{i}\zeta$. *Thus* G^c/G *has nonpositive sectional curvature.*

Proof. The proof has three steps. The first step gives a formula for the Levi-Civita connection on G^c/G.

Step 1. *Let* $g : \mathbb{R} \to \mathrm{G}^c$ *and* $\xi : \mathbb{R} \to \mathfrak{g}$ *be smooth curves and define*

$$\gamma(t) := \pi(g(t)) \in \mathrm{G}^c, \qquad X(t) := d\pi(g(t))g(t)\mathbf{i}\xi(t) \in T_{\gamma(t)}(\mathrm{G}^c/\mathrm{G}).$$

Then the covariant derivative of the vector field X *along* γ *is given by*

$$\nabla X(t) = d\pi(g(t))g(t)\mathbf{i}\eta(t), \qquad \eta(t) := \dot\xi(t) + [\mathrm{Re}(g(t)^{-1}\dot g(t)), \xi(t)]. \quad (50)$$

To prove that the formula is well defined, choose a smooth curve $u : \mathbb{R} \to \mathrm{G}$ and replace g, ξ, η by

$$g' := gu, \qquad \xi' := u^{-1}\xi u, \qquad \eta' := u^{-1}\eta u.$$

Then ${g'}^{-1}\dot g' = u^{-1}\dot u + u^{-1}g^{-1}\dot g u$ and $\dot\xi' + [u^{-1}\dot u, \xi'] = u^{-1}\dot\xi u$, and hence

$$\eta' = u^{-1}\dot\xi u + [\mathrm{Re}(u^{-1}g^{-1}\dot g u), \xi'] = \dot\xi' + [\mathrm{Re}({g'}^{-1}\dot g'), \xi'].$$

The formula $\langle[\zeta,\xi_1],\xi_2\rangle + \langle\xi_1,[\zeta,\xi_2]\rangle = 0$ for $\xi_1,\xi_2,\zeta \in \mathfrak{g}$ shows that (50) is a Riemannian connection, i.e. for any two vector fields X_1, X_2 along a curve γ

$$\frac{d}{dt}\langle X_1, X_2\rangle = \langle \nabla X_1, X_2\rangle + \langle X_1, \nabla X_2\rangle.$$

The formula $\partial_s(g^{-1}\partial_t g) - \partial_t(g^{-1}\partial_s g) + [g^{-1}\partial_s g, g^{-1}\partial_t g] = 0$ for every smooth map $g : \mathbb{R}^2 \to \mathrm{G}^c$ shows that $\gamma(s,t) := \pi(g(s,t)) \in \mathrm{G}^c/\mathrm{G}$ satisfies

$$\begin{aligned}\nabla_t\partial_s\gamma &= d\pi(g)g\mathbf{i}\big(\partial_t \mathrm{Im}(g^{-1}\partial_s g) + [\mathrm{Re}(g^{-1}\partial_t g), \mathrm{Im}(g^{-1}\partial_s g)]\big)\\ &= d\pi(g)g\mathbf{i}\big(\partial_s \mathrm{Im}(g^{-1}\partial_t g) + [\mathrm{Re}(g^{-1}\partial_s g), \mathrm{Im}(g^{-1}\partial_t g)]\big) = \nabla_s\partial_t\gamma.\end{aligned}$$

Thus the connection is torsion-free and this proves Step 1.

Step 2. *We prove part (i).*

A smooth curve $\gamma(t) = \pi(g(t))$ is a geodesic in G^c/G if and only if $\nabla_t\dot\gamma \equiv 0$. By Step 1 this is equivalent to the differential equation

$$\frac{d}{dt}\mathrm{Im}(g^{-1}\dot g) + [\mathrm{Re}(g^{-1}\dot g), \mathrm{Im}(g^{-1}\dot g)] = 0. \tag{51}$$

A smooth curve $g : \mathbb{R} \to \mathrm{G}^c$ satisfies this equation if and only if there exist elements $g_0 \in \mathrm{G}^c$ and $\eta \in \mathfrak{g}$ and a smooth curve $u : \mathbb{R} \to \mathrm{G}$ such that

$$g(t) = g_0 \exp(\mathbf{i}t\eta)u(t) \qquad \text{for all } t \in \mathbb{R}.$$

Hence by Theorem 4.1 the manifold G^c/G is complete. This proves part (i).

Step 3. *We prove part (ii).*

Choose smooth maps $g : \mathbb{R}^2 \to \mathrm{G}^c$ and $\zeta : \mathbb{R}^2 \to \mathfrak{g}$ and define

$$\gamma := \pi \circ g, \qquad X := \partial_s\gamma, \qquad Y := \partial_t\gamma, \qquad Z := d\pi(g)g\mathbf{i}\zeta.$$

Then $X = d\pi(g)g\xi$ and $Y = d\pi(g)g\eta$, where

$$\xi := g^{-1}\partial_s g, \qquad \eta := g^{-1}\partial_t g. \qquad \partial_s\eta - \partial_t\xi + [\xi, \eta] = 0.$$

Moreover,

$$\begin{aligned} \nabla_s Z &= d\pi(g)g\mathbf{i}\zeta_s, \qquad \zeta_s := \partial_s\zeta + [\mathrm{Re}(\xi), \zeta], \\ \nabla_t Z &= d\pi(g)g\mathbf{i}\zeta_t, \qquad \zeta_t := \partial_t\zeta + [\mathrm{Re}(\eta), \zeta]. \end{aligned}$$

Hence $R(X, Y)Z = \nabla_s\nabla_t Z - \nabla_t\nabla_s Z = d\pi(g)g\mathbf{i}\rho$, where

$$\begin{aligned} \rho &= \partial_s\zeta_t + [\mathrm{Re}(\xi), \zeta_t] - \partial_t\zeta_s - [\mathrm{Re}(\eta), \zeta_s] \\ &= \partial_s\big(\partial_t\zeta + [\mathrm{Re}(\eta), \zeta]\big) + [\mathrm{Re}(\xi), \partial_t\zeta + [\mathrm{Re}(\eta), \zeta]] \\ &\quad - \partial_t\big(\partial_s\zeta + [\mathrm{Re}(\xi), \zeta]\big) - [\mathrm{Re}(\eta), \partial_s\zeta + [\mathrm{Re}(\xi), \zeta]] \\ &= [\mathrm{Re}(\partial_s\eta), \zeta] + [\mathrm{Re}(\xi), [\mathrm{Re}(\eta), \zeta]] \\ &\quad - [\mathrm{Re}(\partial_t\xi), \zeta] - [\mathrm{Re}(\eta), [\mathrm{Re}(\xi), \zeta]] \\ &= -[\mathrm{Re}([\xi, \eta]), \zeta] - [\zeta, [\mathrm{Re}(\xi), \mathrm{Re}(\eta)]] \\ &= [[\mathrm{Im}(\xi), \mathrm{Im}(\eta)], \zeta]. \end{aligned}$$

Thus $X = d\pi(g)g\mathbf{i}\mathrm{Im}(\xi)$, $Y = d\pi(g)g\mathbf{i}\mathrm{Im}(\eta)$, $Z = d\pi(g)g\mathbf{i}\zeta$, and

$$R(X, Y)Z = d\pi(g)g\mathbf{i}[[\mathrm{Im}(\xi), \mathrm{Im}(\eta)], \zeta].$$

The sectional curvature is nonpositive because

$$\langle R(X, Y)Y, X\rangle = -\left|[\mathrm{Im}(\xi), \mathrm{Im}(\eta)]\right|^2 \le 0.$$

This proves part (ii) and Theorem 6.2. □

Theorem 6.3. *Assume (C). Then* G^c/G *is simply connected.*

Proof. The proof has three steps.

Step 1. *Let* $\eta \in \mathfrak{g}$ *such that* $\exp(\mathbf{i}\eta) \in \mathrm{G}$*. Then* $[\xi, \eta] = 0$ *for every* $\xi \in \mathfrak{g}$*.*

Define $\gamma : \mathbb{R}^2 \to \mathrm{G}^c/\mathrm{G}$ by $\gamma(s,t) := \pi(\exp(\mathbf{i}s\xi)\exp(\mathbf{i}t\eta))$. By Theorem 6.2 the curve $t \mapsto \gamma(s,t)$ is a geodesic for every s, and by assumption it is periodic with period 1. Hence, for every $s \in \mathbb{R}$,

$$\begin{aligned} 0 &= \int_0^1 \partial_t \langle \nabla_t \partial_s \gamma, \partial_s \gamma \rangle \, dt = \int_0^1 \left(|\nabla_t \partial_s \gamma|^2 + \langle \nabla_t \nabla_t \partial_s \gamma, \partial_s \gamma \rangle \right) \, dt \\ &= \int_0^1 \left(|\nabla_t \partial_s \gamma|^2 + \langle \nabla_t \nabla_s \partial_t \gamma, \partial_s \gamma \rangle \right) \, dt \\ &= \int_0^1 \left(|\nabla_t \partial_s \gamma|^2 - \langle R(\partial_s \gamma, \partial_t \gamma) \partial_t \gamma, \partial_s \gamma \rangle \right) \, dt. \end{aligned}$$

Since G^c/G has nonpositive sectional curvature (Theorem 6.2), this implies that the function $\langle R(\partial_s\gamma, \partial_t\gamma)\partial_t\gamma, \partial_s\gamma\rangle$ vanishes identically. Moreover, we have $\partial_s\gamma(0,0) = d\pi(1)\mathbf{i}\xi$ and $\partial_t\gamma(0,0) = d\pi(1)\mathbf{i}\eta$, so it follows from part (ii) of Theorem 6.2 that $0 = \langle R(\mathbf{i}\xi, \mathbf{i}\eta)\mathbf{i}\eta, \mathbf{i}\xi\rangle = -\left|[\xi,\eta]\right|^2$. This proves Step 1.

Step 2. *Let* $\eta \in \mathfrak{g}$ *such that* $\exp(\mathbf{i}\eta) \in \mathrm{G}$*. Then* $\eta = 0$*.*

By Step 1, we have $[\xi, \eta] = 0$ for all $\xi \in \mathfrak{g}$. This implies that the map

$$\mathrm{G} \to \mathfrak{g} : u \mapsto u\eta u^{-1} \tag{52}$$

is locally constant. Since the group G is compact, it has only finitely many connected components and hence the image of the map (52) is a finite set. Choose pairwise distinct elements $\eta_1, \dots, \eta_N \in \mathfrak{g}$ such that

$$\left\{\eta_1, \dots, \eta_N\right\} = \left\{u\eta u^{-1} \,\middle|\, u \in \mathrm{G}\right\}. \tag{53}$$

Then, for all $j \in \{1, \dots, N\}$ and all $\xi \in \mathfrak{g}$, we have

$$\exp(\mathbf{i}\eta_j) \in \mathrm{G}, \qquad [\eta_j, \xi] = 0. \tag{54}$$

We claim that the set

$$\mathrm{H} := \left\{ \exp\left(\mathbf{i}\sum_{j=1}^N t_j \eta_j\right) u \,\middle|\, t_j \in \mathbb{R},\, u \in \mathrm{G} \right\} \tag{55}$$

is a compact subgroup of G^c.

To see that H is a subgroup, choose real numbers s_j, t_j for $j = 1, \dots, N$ and elements $u, v \in \mathrm{G}$. Then by (53) there exists a permutation σ of the set $\{1, \dots, N\}$ such that $u\eta_j u^{-1} = \eta_{\sigma(j)}$ for $j = 1, \dots, N$. Hence, by (54),

$$\begin{aligned}
\exp\left(\mathbf{i}\sum_{j=1}^N s_j\eta_j\right) u \exp\left(\mathbf{i}\sum_{j=1}^N t_j\eta_j\right) v &= \exp\left(\mathbf{i}\sum_{j=1}^N \big(s_j\eta_j + t_j u\eta_j u^{-1}\big)\right) uv \\
= \exp\left(\mathbf{i}\sum_{j=1}^N \big(s_j\eta_j + t_j\eta_{\sigma(j)}\big)\right) uv &= \exp\left(\mathbf{i}\sum_{j=1}^N \big(s_j + t_{\sigma^{-1}(j)}\big)\eta_j\right) uv \in \mathrm{H}.
\end{aligned}$$

A similar argument shows that $h^{-1} \in \mathrm{H}$ for every $h \in \mathrm{H}$. Thus H is a subgroup of G^c. It follows also from (54) that

$$\mathrm{H} = \left\{ \exp\left(\mathbf{i}\sum_{j=1}^N t_j\eta_j\right) u \,\middle|\, 0 \le t_j \le 1,\, u \in \mathrm{G} \right\}. \tag{56}$$

Hence H is compact, hence $\mathrm{H} = \mathrm{G}$ because G is maximal, hence $\exp(\mathbf{i}t\eta) \in \mathrm{G}$ for all $t \in \mathbb{R}$, hence $\mathbf{i}\eta \in \mathfrak{g}$, and hence $\eta = 0$. This proves Step 2.

Step 3. *The homogeneous space* G^c/G *is simply connected.*

Suppose not. Then, by using the completeness of G^c/G (Theorem 6.2) and applying the Hopf–Rinow Theeorem 4.1 to the universal cover of G^c/G, we find a nonconstant geodesic $\gamma : [0, 1] \to \mathrm{G}^c/\mathrm{G}$ based at $\gamma(0) = \gamma(1) = \pi(1)$. By Theorem 6.2 the geodesic has the form $\gamma(t) = \pi(\exp(\mathbf{i}t\eta))$ for some $\eta \in \mathfrak{g}$. Since $\gamma(1) = \pi(1)$ we have $\exp(\mathbf{i}\eta) \in \mathrm{G}$, and hence $\eta = 0$ by Step 2. Thus the geodesic is constant, a contradiction. This proves Step 3 and Theorem 6.3 □

Proof of Theorem 1.4. We prove that (i) implies (ii). If ι is an injective Lie group homomorphism from a compact Lie group G to a complex Lie group G^c that satisfies (i), then the pair $(\mathrm{G}^c, \iota(\mathrm{G}))$ satisfies (C), hence Theorem 6.3 asserts that $\mathrm{G}^c/\iota(\mathrm{G})$ is simply connected, and so ι satisfies (ii). Thus (i) implies (ii). That (ii) implies (iii) was shown in §1.

We prove that (iii) implies (i). Let $\iota : \mathrm{G} \to \mathrm{G}^c$ be the injective Lie group homomorphism to a complex Lie group G^c constructed in Theorem 3.10 and the proof of Corollary 3.11. Then ι satisfies the conditions (i), (ii), and (iii). Now let $\iota' : \mathrm{G} \to \mathrm{G}'^c$ be any injective Lie group homomorphism to a complex Lie group that satisfies (iii). Since ι and ι' both have the universality property (iii), they are canonically isomorphic, and hence, since ι satisfies (i), so does ι'. Thus (iii) implies (i) and this proves Theorem 1.4. □

Proof of Theorem 1.8. Assume (C). Then the manifold G^c/G is connected, by Theorem 6.2 it is complete and has nonpositive sectional curvature, and by Theorem 6.3 it is simply connected. Hence G^c/G is a Hadamard manifold. Now let $\mathrm{K} \subset \mathrm{G}^c$ be a compact subgroup and consider the group action

$$\mathrm{K} \times \mathrm{G}^c/\mathrm{G} \to \mathrm{G}^c/\mathrm{G} : (k, \pi(g)) \mapsto \pi(kg).$$

By definition of the Riemannian metric on G^c/G (Definition 6.1), the group K acts by isometries. Hence Theorem 5.1 asserts that the action of K on G^c/G has a fixed point. Choose $g \in \mathrm{G}^c$ such that $\pi(g) \in \mathrm{G}^c/\mathrm{G}$ is a fixed point of the K-action on G^c/G. Then $\pi(kg) = \pi(g)$ for all $k \in \mathrm{K}$. Thus, for each element $k \in \mathrm{K}$ there exists a $u \in \mathrm{G}$ such that $kg = gu$. Hence $g^{-1}kg \in \mathrm{G}$ for every $k \in \mathrm{K}$, i.e. $g^{-1}\mathrm{K}g \subset \mathrm{G}$. This proves Theorem 1.8. □

Proof of Theorem 1.9. Assume (C). Then, by Theorem 6.2 and Theorem 6.3, the homogeneous space G^c/G is a Hadamard manifold, and the Riemannian exponential map of G^c/G at the point $\pi(1)$ is given by

$$T_{\pi(1)}\mathrm{G}^c/\mathrm{G} \to \mathrm{G}^c/\mathrm{G} : d\pi(1)\mathbf{i}\eta \mapsto \pi(\exp(\mathbf{i}\eta)). \tag{57}$$

By Theorem 4.2 this map is a diffeomorphism.

We prove that the map (7) is a local diffeomorphism. Let $(u, \eta) \in \mathrm{G} \times \mathfrak{g}$. Then the derivative of the map (57) at the point $d\pi(1)\mathbf{i}\eta$ is injective. Hence, for every element $\widehat{\eta} \in \mathfrak{g}$,

$$\exp(-\mathbf{i}\eta)\big(d\exp(\mathbf{i}\eta)\mathbf{i}\widehat{\eta}\big) \in \mathfrak{g} \qquad \Longrightarrow \qquad \widehat{\eta} = 0.$$

This implies that the linear map

$$T_u\mathrm{G} \times \mathfrak{g} \to T_{\exp(\mathbf{i}\eta)u}\mathrm{G}^c : (\widehat{u}, \widehat{\eta}) \mapsto \exp(\mathbf{i}\eta)\widehat{u} + \big(d\exp(\mathbf{i}\eta)\mathbf{i}\widehat{\eta}\big)u \tag{58}$$

is injective, and hence also bijective for dimensional reasons. Since (58) is the derivative of the map (7) at the point $(u, \eta) \in \mathrm{G} \times \mathfrak{g}$, this shows that the map (7) is a local diffeomorphism.

We prove that (7) is surjective. If $g \in \mathrm{G}^c$, then by surjectivity of (57), there exists an element $\eta \in \mathfrak{g}$ such that $\pi(\exp(\mathbf{i}\eta)) = \pi(g) \in \mathrm{G}^c/\mathrm{G}$. Hence there exists an element $u \in \mathrm{G}$ such that $\exp(\mathbf{i}\eta)u = g$.

We prove that the map (7) in injective. Let $(u_0, \eta_0), (u_1, \eta_1) \in \mathrm{G} \times \mathfrak{g}$ such that $\exp(\mathbf{i}\eta_0)u_0 = \exp(\mathbf{i}\eta_1)u_1$. Then $\pi(\exp(\mathbf{i}\eta_0)) = \pi(\exp(\mathbf{i}\eta_1)) \in \mathrm{G}^c/\mathrm{G}$, hence $\eta_0 = \eta_1$ by injectivity of the map (57), and hence $u_0 = u_1$. This proves Theorem 1.9. □

Remark 6.4. **(a)** The theorems proved in this section extend to many Lie groups that are not **reductive**, i.e. that are not complexifications of compact Lie groups. To explain this, we recall some basic definitions. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ is called an **ideal** iff $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$. The Lie algebra $\mathfrak{g}$ is called **simple** iff it is not abelian and contains no proper ideals. It is called **semisimple** iff it is a direct sum of simple ideals. A **derivation** on $\mathfrak{g}$ is a linear map $\delta : \mathfrak{g} \to \mathfrak{g}$ that satisfies the equation $\delta[\xi, \eta] = [\delta\xi, \eta] + [\xi, \delta\eta]$ for all $\xi, \eta \in \mathfrak{g}$. The space of derivations on $\mathfrak{g}$ is denoted by $\mathrm{Der}(\mathfrak{g})$. An inner product on $\mathfrak{g}$ is called **symmetric** iff the space of derivations on $\mathfrak{g}$ is invariamt under the adjoint map, i.e.

$$\delta \in \mathrm{Der}(\mathfrak{g}) \qquad \Longrightarrow \qquad \delta^* \in \mathrm{Der}(\mathfrak{g}). \tag{59}$$

(b) In [2] Donaldson proved with geometric arguments, circumventing Lie algebra theory, that every finite-dimensional simple Lie algebra $\mathfrak{g}$ admits a symmetric inner product, and this result readily extends to the semisimple case (see also [12, Theorem 7.6.16]). For a Lie algebra $\mathfrak{g}$ with a trivial center it follows from the existence of a symmetric inner product that the Killing form is nondegenerate and that the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is bijective (see [12, Lemma 7.6.8]). Hence, for each symmetric inner product on $\mathfrak{g}$, there exists a unique involution $\mathfrak{g} \to \mathfrak{g} : \xi \mapsto \xi^*$ such that

$$\mathrm{ad}(\xi^*) = \mathrm{ad}(\xi)^*, \qquad [\xi, \eta]^* = [\eta^*, \xi^*] \qquad \text{for all } \xi, \eta \in \mathfrak{g}. \tag{60}$$

The **Cartan involution** $\mathfrak{g} \to \mathfrak{g} : \xi \mapsto -\xi^*$ is a Lie algebra automorphism, and it gives rise to a decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \qquad \mathfrak{k} := \{\xi \in \mathfrak{g} \,|\, \xi = -\xi^*\}, \qquad \mathfrak{p} := \{\eta \in \mathfrak{g} \,|\, \eta = \eta^*\}. \tag{61}$$

(c) Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra equipped with a symmetric inner product, let $\mathrm{Aut}_0(\mathfrak{g})$ be the identity component in the group of automorphisms of $\mathfrak{g}$, let $\mathrm{SO}(\mathfrak{g})$ be the special orthogonal group, and define

$$\mathrm{K}_0 := \mathrm{Aut}_0(\mathfrak{g}) \cap \mathrm{SO}(\mathfrak{g}), \qquad \mathrm{Lie}(\mathrm{K}_0) = \mathrm{ad}(\mathfrak{k}). \tag{62}$$

Then K_0 is connected and is a maximal compact subgroup of $\mathrm{Aut}_0(\mathfrak{g})$, every compact subgroup of $\mathrm{Aut}_0(\mathfrak{g})$ is conjugate in $\mathrm{Aut}_0(\mathfrak{g})$ to a subgroup of K_0, the homogeneous space $\mathrm{Aut}_0(\mathfrak{g})/\mathrm{K}_0$ is a Hadamard manifold, and the map

$$\mathrm{K}_0 \times \mathfrak{p} \to \mathrm{Aut}_0(\mathfrak{g}) : (u, \eta) \mapsto \exp\big(\mathrm{ad}(\eta)\big)u \tag{63}$$

is a diffeomorphism. These are the analogues of Theorems 1.8, 1.9, and 6.2 for semisimple Lie algebras, and in [2] they are derived as consequences of the existence of symmetric inner products (see also [12, Corollary 7.6.21]).

(d) Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and denote the Killing form by $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Then a symmetric inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ can be adjusted by scaling on each simple ideal so that it satisfies

$$\langle \xi, \eta \rangle = \kappa(\xi^*, \eta) \qquad \text{for all } \xi, \eta \in \mathfrak{g}. \tag{64}$$

Denote the manifold of symmetric inner products satisfying (64) by $\mathcal{M}_{\mathfrak{g}}$. They mimimize the norm of the Lie bracket among all inner products with a fixed determinant. Moreover, if one fixes an inner product $\langle \cdot, \cdot \rangle$ in $\mathcal{M}_{\mathfrak{g}}$ and defines K_0 by (62), then the map $g \mapsto \langle \cdot, (gg^*)^{-1/2} \cdot \rangle$ descends to a diffeomorphism from the homogeneous space $\mathrm{Aut}_0(\mathfrak{g})/\mathrm{K}_0$ to the Hadamard manifold $\mathcal{M}_{\mathfrak{g}}$. These observations lie at the heart of the approach in [2].

(e) As explained in [2], reductive groups fit into this setup as follows. Let $\mathfrak{g}$ be a finite-dimensional semisimple complex Lie algebra, i.e. $\mathfrak{g}$ is a direct sum of simple complex ideals. By [2, Lemma 7] every simple complex Lie algebra is also simple as a real Lie algebra. Hence $\mathfrak{g}$ is semisimple as a real Lie algebra and so admits a symmetric inner product. By [2, Lemma 8] this implies

$$\mathfrak{p} = \mathbf{i}\mathfrak{k} \tag{65}$$

in the decomposition (61). This is the theorem of Cartan which asserts that every semisimple complex Lie algebra $\mathfrak{g}$ has a compact real form, i.e. $\mathfrak{g}$ is the complexification of the Lie algebra of a compact Lie group.

Let G be a connected complex Lie group whose Lie algebra $\mathfrak{g}$ is semisimple and equipped with a symmetric inner product. By (65) the Lie subalgebra $\mathfrak{k}$ in (61) has a trivial center, and hence the compact Lie group K_0 in (62) has a finite fundamental group (see [13, Theorem 6.2]). By (63) the group $\mathrm{Aut}_0(\mathfrak{g})$ is homotopy equivalent to K_0 and so also has a finite fundamental group. Thus, by Corollary A.3 the homomorphism $\mathrm{Ad} : \mathrm{G} \to \mathrm{Aut}_0(\mathfrak{g})$, defined by $\mathrm{Ad}(g)\zeta := g\zeta g^{-1}$, is a finite cover of $\mathrm{Aut}_0(\mathfrak{g})$. Hence the subgroup

$$\mathrm{K} := \left\{ g \in \mathrm{G} \,\middle|\, \mathrm{Ad}(g) \in \mathrm{SO}(\mathfrak{g}) \right\} = \mathrm{Ad}^{-1}(\mathrm{K}_0) \tag{66}$$

with the Lie algebra $\mathrm{Lie}(\mathrm{K}) = \{\xi \in \mathfrak{g} \,|\, \mathrm{ad}(\xi) \in \mathfrak{so}(\mathfrak{g})\} = \mathfrak{k}$ is a maximal compact subgroup of G. Hence G is a complexification of K. It also follows directly from (c) and (d) that the map $\mathrm{K} \times \mathfrak{k} \to \mathrm{G} : (u, \eta) \mapsto \exp(\mathbf{i}\eta)u$ is a diffeomorphism, that the homogeneous space $\mathrm{G}/\mathrm{K} \cong \mathrm{Aut}_0(\mathfrak{g})/\mathrm{K}_0$ is a Hadamard manifold, and that every compact subgroup of G is conjugate in G to a subgroup of K. This shows that the methods developed by Donaldson in [2] also give rise to an alternative approach to the results in these notes for compact connected Lie groups with a finite center.

7 Matrix factorization

Theorem 7.1. *Let* $\mathrm{G} \subset \mathrm{U}(n)$ *be a Lie subgroup, let* $\mathrm{G}^c \subset \mathrm{GL}(n, \mathbb{C})$ *be the subgroup in Theorem 2.1, and denote their Lie algebras by* $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ *and* $\mathfrak{g}^c := \mathrm{Lie}(\mathrm{G}^c)$. *Let* $\xi \in \mathfrak{g}$ *such that* $\exp(\xi) = 1$ *and let* $g \in \mathrm{G}^c$. *Then there exists a pair* $p, p^+ \in \mathrm{G}^c$ *such that*

$$\lim_{t\to\infty} \exp(\mathbf{i}t\xi) p \exp(-\mathbf{i}t\xi) = p^+, \qquad pg^{-1} \in \mathrm{G}.$$

Proof. See page 44. □

The proof relies on the next four lemmas.

Lemma 7.2. *Let* $\mathrm{G}, \mathrm{G}^c, \xi$ *be as in Theorem 7.1. Then the set*

$$\mathrm{P} := \left\{ p \in \mathrm{G}^c \,|\, \textit{the limit } \lim_{t\to\infty} \exp(\mathbf{i}t\xi) p \exp(-\mathbf{i}t\xi) \textit{ exists in } \mathrm{G}^c \right\} \tag{67}$$

is a Lie subgroup of G^c *with Lie algebra*

$$\mathfrak{p} := \left\{ \zeta \in \mathfrak{g}^c \,|\, \textit{the limit } \lim_{t\to\infty} \exp(\mathbf{i}t\xi) \zeta \exp(-\mathbf{i}t\xi) \textit{ exists in } \mathfrak{g}^c \right\}. \tag{68}$$

Proof. Since $\mathrm{G} \subset \mathrm{U}(n)$ and $\xi \in \mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ satisfies $\exp(\xi) = 1$, it follows that $\mathbf{i}\xi \in \mathbb{C}^{n\times n}$ is a Hermitian matrix with eigenvalues in $2\pi\mathbb{Z}$. Consider the decomposition $\mathbb{C}^n = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ into the eigenspaces $E_j \subset \mathbb{C}^n$ of $\mathbf{i}\xi$ with the eigenvalues λ_j and the ordering chosen such that $\lambda_1 < \lambda_2 < \cdots < \lambda_k$. Write a matrix $\zeta \in \mathfrak{g}^c = \mathrm{Lie}(\mathrm{G}^c) \subset \mathfrak{gl}(n, \mathbb{C})$ in the block form

$$\zeta = \begin{pmatrix} \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1k} \\ \zeta_{21} & \zeta_{22} & \cdots & \zeta_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{k1} & \zeta_{k2} & \cdots & \zeta_{kk} \end{pmatrix}, \qquad \zeta_{ij} \in \mathrm{Hom}(E_j, E_i).$$

Then

$$\exp(\mathbf{i}t\xi)\zeta\exp(-\mathbf{i}t\xi) = \begin{pmatrix} \zeta_{11} & e^{(\lambda_1-\lambda_2)t}\zeta_{12} & \cdots & e^{(\lambda_1-\lambda_k)t}\zeta_{1k} \\ e^{(\lambda_2-\lambda_1)t}\zeta_{21} & \zeta_{22} & \cdots & e^{(\lambda_2-\lambda_k)t}\zeta_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ e^{(\lambda_k-\lambda_1)t}\zeta_{k1} & e^{(\lambda_k-\lambda_2)t}\zeta_{k2} & \cdots & \zeta_{kk} \end{pmatrix}.$$

Thus $\zeta \in \mathfrak{p}$ if and only if $\zeta \in \mathfrak{g}^c$ and $\zeta_{ij} = 0$ for $i > j$. Likewise, $g \in \mathrm{P}$ if and only if $g \in \mathrm{G}^c$ and $g_{ij} = 0$ for $i > j$. Hence P is a closed subset of G^c. Since every closed subgroup of a Lie group is a Lie subgroup, this proves Lemma 7.2. □

Lemma 7.3. *Let $N \in \mathbb{N}$. There exist real numbers*

$$\beta_0(N), \beta_1(N), \dots, \beta_{2N-1}(N)$$

such that $\beta_\nu(N) = 0$ when ν is even and, for $k = 1, 3, 5, \dots, 4N-1$,

$$\sum_{\nu=0}^{2N-1} \beta_\nu(N) \exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right) = \begin{cases} \mathbf{i}, & \text{if } 0 < k < 2N, \\ -\mathbf{i}, & \text{if } 2N < k < 4N. \end{cases} \tag{69}$$

Proof. Define $\lambda := \exp(\frac{\pi\mathbf{i}}{2N})$ and consider the Vandermonde matrix

$$\Lambda := \begin{pmatrix} \lambda & \lambda^3 & \lambda^5 & \dots & \lambda^{2N-1} \\ \lambda^3 & \lambda^9 & \lambda^{15} & \dots & \lambda^{6N-3} \\ \lambda^5 & \lambda^{15} & \lambda^{25} & \dots & \lambda^{10N-5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda^{2N-1} & \lambda^{6N-3} & \lambda^{10N-5} & \dots & \lambda^{(2N-1)^2} \end{pmatrix} \in \mathbb{C}^{N\times N}.$$

Its complex determinant is

$$\det{}^c(\Lambda) = \lambda^{(2N-1)(N-1)} \prod_{0\le i<j\le N-1} \left(\lambda^{4j} - \lambda^{4i}\right).$$

Since λ is a primitive $4N$th root of unity, the numbers λ^{4i}, $i = 0, \dots, N-1$, are pairwise distinct. Hence Λ is nonsingular. Hence there exists a unique vector $z = (z_1, z_3, \dots, z_{2N-1}) \in \mathbb{C}^N$ such that

$$\sum_{\substack{0<\nu<2N \\ \nu \text{ odd}}} \exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right) z_\nu = \mathbf{i}, \qquad k = 1, 3, \dots, 2N-1. \tag{70}$$

The numbers z_ν also satisfy the equation

$$\sum_{\substack{0<\nu<2N \\ \nu \text{ odd}}} \exp\left(\frac{(2N-k)\nu\pi\mathbf{i}}{2N}\right) \overline{z}_\nu = -\sum_{\substack{0<\nu<2N \\ \nu \text{ odd}}} \overline{\exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right) z_\nu} = \mathbf{i}$$

for $k = 1, 3, \dots, 2N-1$ and hence they are real.

Define $\beta_\nu(N) := z_\nu$ for $\nu = 1, 3, \dots, 2N-1$ and $\beta_\nu(N) := 0$ for ν even. By (70) these numbers satisfy equation (69) for $k = 1, 3, \dots, 2N-1$. That equation (69) also holds for $k = 2N+1, 2N+3, \dots, 4N-1$ follows from the fact that $\exp(k\pi\mathbf{i}) = -1$ whenever k is odd. This proves Lemma 7.3. $\square$

Lemma 7.4. *Let $m \in \mathbb{N}$ and $N := 2^m$. There exist real numbers*

$$\alpha_0(N), \alpha_1(N), \ldots, \alpha_{2N-1}(N)$$

such that $\alpha_\nu(N) = 0$ when ν is even and, for every $k \in \{0, 1, \ldots, 2N-1\}$,

$$\sum_{\nu=0}^{2N-1} \alpha_\nu(N) \exp\left(\frac{k\nu\pi\mathbf{i}}{N}\right) = \begin{cases} \mathbf{i}, & \text{if } 1 \leq k \leq N-1, \\ -\mathbf{i}, & \text{if } N+1 \leq k \leq 2N-1, \\ 0, & \text{if } k = 0 \text{ or } k = N. \end{cases} \tag{71}$$

Proof. The proof is by induction on m. For $m = 1$ and $N = 2^m = 2$ define the numbers $\alpha_1(2) := 1/2$ and $\alpha_3(2) := -1/2$. Then

$$\sum_{\nu=0}^{3} \alpha_\nu(2) \exp\left(\frac{k\nu\pi\mathbf{i}}{2}\right) = \frac{\mathbf{i}^k - (-\mathbf{i})^k}{2} = \begin{cases} \mathbf{i}, & \text{for } k = 1, \\ -\mathbf{i}, & \text{for } k = 3, \\ 0, & \text{for } k = 0, 2. \end{cases}$$

Now fix an integer $m \in \mathbb{N}$ and define $N := 2^m$. Assume, by induction, that the numbers $\alpha_\nu(N)$, $\nu = 0, 1, \ldots, 2N-1$, have been found such that (71) holds for $k = 0, 1, \ldots, 2N-1$. Let $\beta_\nu(N)$ for $\nu = 0, 1, \ldots, 2N-1$ be the constants of Lemma 7.3. Define

$$\alpha_{2N+\nu}(N) := \alpha_\nu(N), \qquad \beta_{2N+\nu}(N) := -\beta_\nu(N),$$

for $\nu = 0, 1, 2, \ldots, 2N-1$ and

$$\alpha_\nu(2N) := \frac{\alpha_\nu(N) + \beta_\nu(N)}{2}, \qquad \nu = 0, 1, 2, \ldots, 4N-1. \tag{72}$$

Then

$$\sum_{\nu=0}^{4N-1} \alpha_\nu(2N) \exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right) = A_k + B_k,$$

where

$$A_k := \frac{1}{2} \sum_{\nu=0}^{4N-1} \alpha_\nu(N) \exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right),$$

$$B_k := \frac{1}{2} \sum_{\nu=0}^{4N-1} \beta_\nu(N) \exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right).$$

Since $\alpha_{2N+\nu}(N) = \alpha_\nu(N)$, we have

$$\begin{aligned} A_k &= \frac{1}{2} \sum_{\nu=0}^{4N-1} \alpha_\nu(N) \exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right) \\ &= \frac{1+\exp(k\pi\mathbf{i})}{2} \sum_{\nu=0}^{2N-1} \alpha_\nu(N) \exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right) \\ &= \frac{1+(-1)^k}{2} \sum_{\nu=0}^{2N-1} \alpha_\nu(N) \exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right). \end{aligned}$$

If k is odd, then the right hand side vanishes. If k is even, then it follows from the induction hypothesis that

$$A_k = \sum_{\nu=0}^{2N-1} \alpha_\nu(N) \exp\left(\frac{(k/2)\nu\pi\mathbf{i}}{N}\right) = \begin{cases} \mathbf{i}, & \text{for } k = 2, 4, \ldots, 2N-2, \\ -\mathbf{i}, & \text{for } k = 2N+2, \ldots, 4N-2, \\ 0, & \text{for } k = 0, 2N. \end{cases}$$

Since $\beta_{2N+\nu}(N) = -\beta_\nu(N)$, we have

$$\begin{aligned} B_k &= \frac{1}{2} \sum_{\nu=0}^{2N-1} \beta_\nu(N) \left(\exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right) - \exp\left(\frac{k(2N+\nu)\pi\mathbf{i}}{2N}\right)\right) \\ &= \frac{1-\exp(k\pi\mathbf{i})}{2} \sum_{\nu=0}^{2N-1} \beta_\nu(N) \exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right) \\ &= \frac{1-(-1)^k}{2} \sum_{\nu=0}^{2N-1} \beta_\nu(N) \exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right). \end{aligned}$$

If k is even, then the right hand side vanishes. If k is odd, then it follows from Lemma 7.3 that

$$B_k = \sum_{\nu=0}^{2N-1} \beta_\nu(N) \exp\left(\frac{k\nu\pi\mathbf{i}}{2N}\right) = \begin{cases} \mathbf{i}, & \text{if } k = 1, 3, \ldots, 2N-1, \\ -\mathbf{i}, & \text{if } k = 2N+1, \ldots, 4N-1. \end{cases}$$

Combining the formulas for A_k and B_k we find that

$$A_k + B_k = \begin{cases} \mathbf{i}, & \text{for } k = 1, 2, 3, \ldots, 2N-1, \\ -\mathbf{i}, & \text{for } k = 2N+1, 2N+2, \ldots, 4N-1, \\ 0, & \text{for } k = 0, 2N. \end{cases}$$

This proves Lemma 7.4. □

Lemma 7.5. *Let* $\mathrm{G}, \mathrm{G}^c, \xi$ *be as in Theorem 7.1.and let* $\eta \in \mathfrak{g}$. *Then there exists a pair* $\zeta, \zeta^+ \in \mathfrak{g}^c$ *such that*

$$\lim_{t\to\infty} \exp(\mathbf{i}t\xi)\zeta\exp(-\mathbf{i}t\xi) = \zeta^+, \qquad \zeta - \mathbf{i}\eta \in \mathfrak{g}.$$

Proof. Let $\mathbb{C}^n = E_1 \oplus \cdots \oplus E_k$ and $\lambda_1 < \cdots < \lambda_k$ be as in Lemma 7.2. Then

$$\lambda_i - \lambda_j = 2\pi m_{ij}, \qquad m_{ij} \in \mathbb{Z},$$

with $m_{ij} > 0$ for $i > j$ and $m_{ij} < 0$ for $i < j$. Choose $m \in \mathbb{N}$ such that

$$N := 2^m > m_{k1} = \frac{\lambda_k - \lambda_1}{2\pi}.$$

Choose $\alpha_0, \ldots, \alpha_{2N-1} \in \mathbb{R}$ as in Lemma 7.4. Let

$$\eta = \begin{pmatrix} \eta_{11} & \eta_{12} & \cdots & \eta_{1k} \\ \eta_{21} & \eta_{22} & \cdots & \eta_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{k1} & \eta_{k2} & \cdots & \eta_{kk} \end{pmatrix} \in \mathfrak{g},$$

where $\eta_{ij} \in \mathrm{Hom}(E_j, E_i)$. Define

$$\zeta := \mathbf{i}\eta - \sum_{\nu=0}^{2N-1} \alpha_\nu \exp\left(-\frac{\nu}{2N}\xi\right)\eta\exp\left(\frac{\nu}{2N}\xi\right) \in \mathfrak{g}^c.$$

Then, for $i > j$, we have

$$\begin{aligned} \zeta_{ij} &= \mathbf{i}\eta_{ij} - \sum_{\nu=0}^{2N-1} \alpha_\nu \exp\left(\frac{\nu}{2N}\mathbf{i}(\lambda_i - \lambda_j)\right)\eta_{ij} \\ &= \left(\mathbf{i} - \sum_{\nu=0}^{2N-1} \alpha_\nu \exp\left(\frac{m_{ij}\nu\pi\mathbf{i}}{N}\right)\right)\eta_{ij} \\ &= 0. \end{aligned}$$

The last equality follows from Lemma 7.4 and the fact that $1 \le m_{ij} \le N-1$ for $i > j$. Since $\zeta_{ij} = 0$ for $i > j$, it follows from the proof of Lemma 7.2 that $\zeta \in \mathfrak{p}$. Moreover, by construction $\mathbf{i}\eta - \zeta \in \mathfrak{g}$. This proves Lemma 7.5. □

Proof of Theorem 7.1. Let $\mathrm{P} \subset \mathrm{G}^c$ and $\mathfrak{p} \subset \mathfrak{g}^c$ be defined by (67) and (68) (see Lemma 7.2). We prove in three steps that, for every element $g \in \mathrm{G}^c$, there exists an element $p \in \mathrm{P}$ such that $pg^{-1} \in \mathrm{G}$.

Step 1. *The set*

$$A := \left\{ g \in \mathrm{G}^c \,|\, \exists\, p \in \mathrm{P} \text{ such that } pg^{-1} \in \mathrm{G} \right\}$$

is (relatively) closed in G^c*.*

Let $g_i \in A$ be a sequence which converges to an element $g \in \mathrm{G}^c$. Then there exists a sequence $p_i \in \mathrm{P}$ such that $u_i := p_i g_i^{-1} \in \mathrm{G}$. Since G is compact there exists a subsequence (still denoted by u_i) which converges to an element $u \in \mathrm{G}$. Since P is a closed subset of G^c, by Step 1 in the proof of Lemma 7.4, we have $p := ug = \lim_{i\to\infty} u_i g_i = \lim_{i\to\infty} p_i \in \mathrm{P}$. Hence $pg^{-1} = u \in \mathrm{G}$ and hence $g \in A$. This proves Step 1.

Step 2. *The map* $f : P \times \mathrm{G} \to \mathrm{G}^c$*, defined by*

$$f(p, u) := pu$$

for $p \in \mathrm{P}$ *and* $u \in \mathrm{G}$*, is a submersion.*

Let $p \in \mathrm{P}$ and $u \in \mathrm{G}$ and denote $g := f(p, u) = pu$. Let $\widehat{g} \in T_g\mathrm{G}^c$ and denote

$$\widetilde{\zeta} := p^{-1}\widehat{g}u^{-1} = u(g^{-1}\widehat{g})u^{-1} \in \mathfrak{g}^c. \tag{73}$$

Let $\eta \in \mathfrak{g}$ be the imginary part of $\widetilde{\zeta}$ so that $\widetilde{\zeta} - \mathbf{i}\eta \in \mathfrak{g}$. By Lemma 7.5, there exists an element $\zeta \in \mathfrak{p}$ such that $\zeta - \mathbf{i}\eta \in \mathfrak{g}$ and hence $\widetilde{\zeta} - \zeta \in \mathfrak{g}$. Define

$$\widehat{p} := p\zeta, \qquad \widehat{u} := \left(\widetilde{\zeta} - \zeta\right) u.$$

Then $\widehat{p} \in T_p\mathrm{P}$, $\widehat{u} \in T_u\mathrm{G}$, and

$$df(p, u)(\widehat{p}, \widehat{u}) = \widehat{p}u + p\widehat{u} = p\widetilde{\zeta}u = \widehat{g}.$$

Here the last equality follows from (73). Thus we have proved that the differential $df(p, u) : T_p\mathrm{P} \times T_u\mathrm{G} \to T_{pu}\mathrm{G}^c$ is surjective for every $p \in \mathrm{P}$ and every $u \in \mathrm{G}$. Hence f is a submersion and this proves Step 2.

Step 3. $A = \mathrm{G}^c$.

The set A contains G by definition. By Step 1 it is closed and by Step 2 it is the image of a submersion and hence is open. Since G^c is homeomorphic to $\mathrm{G} \times \mathfrak{g}$ (see Theorem 1.9) and A contains $\mathrm{G} \cong \mathrm{G} \times \{0\}$, it follows that A intersects each connected component of G^c in a nonempty open and closed set. Hence $A = \mathrm{G}^c$. This proves Step 3 and Theorem 7.1. □

Proof of Theorem 1.10. By Theorem 1.8 we may assume, without loss of generality, that $\mathrm{G} \subset \mathrm{U}(n)$. Then $\mathrm{G}^c \subset \mathrm{GL}(n, \mathbb{C})$ is the complexification of G in Theorem 2.1. Denote their Lie algebras by $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ and $\mathfrak{g}^c := \mathrm{Lie}(\mathrm{G}^c)$. Let $\zeta \in \mathfrak{g}^c$ such that $\exp(\zeta) = 1$. We prove in three steps that there exist two elements $p, p^+ \in \mathrm{G}^c$ such that

$$p^{-1}\zeta p \in \mathfrak{g}, \qquad \lim_{t\to\infty} \exp(\mathbf{i}t\zeta)p\exp(-\mathbf{i}t\zeta) = p^+. \tag{74}$$

Step 1. *There exists an element $g \in \mathrm{G}^c$ such that $g^{-1}\zeta g \in \mathfrak{g}$.*

The set $S := \{\exp(s\zeta) \,|\, s \in \mathbb{R}\}$ is a compact subgroup of G^c. Hence it follows from Theorem 1.8 that there exists an element $g \in \mathrm{G}^c$ such that $g^{-1}Sg \subset \mathrm{G}$. Thus $g^{-1}\zeta g = \frac{d}{ds}\big|_{s=0} g^{-1}\exp(s\zeta)g \in \mathfrak{g}$ and this proves Step 1.

Step 2. *Let $g \in \mathrm{G}^c$ and $\xi \in \mathfrak{g}$ such that $\exp(\xi) = 1$. Then there exist two elements $q, q^+ \in \mathrm{G}^c$ such that*

$$qg^{-1} \in \mathrm{G}, \qquad \lim_{t\to\infty} \exp(\mathbf{i}t\xi)q\exp(-\mathbf{i}t\xi) = q^+.$$

This is the assertion of Theorem 7.1.

Step 3. *There exist two elements $p, p^+ \in \mathrm{G}^c$ that satisfy* (74)*.*

Let $g \in \mathrm{G}^c$ be as in Step 1 and define

$$\xi := g^{-1}\zeta g \in \mathfrak{g}.$$

Now choose $q, q^+ \in \mathrm{G}^c$ as in Step 2 and define

$$p := gqg^{-1}, \qquad p^+ := gq^+g^{-1}.$$

Then

$$\begin{aligned} p^+ &= gq^+g^{-1} \\ &= \lim_{t\to\infty} g\exp(\mathbf{i}t\xi)q\exp(-\mathbf{i}t\xi)g^{-1} \\ &= \lim_{t\to\infty} \exp(\mathbf{i}tg\xi g^{-1})(gqg^{-1})\exp(-\mathbf{i}tg\xi g^{-1}) \\ &= \lim_{t\to\infty} \exp(\mathbf{i}t\zeta)p\exp(-\mathbf{i}t\zeta). \end{aligned}$$

Moreover $g^{-1}p = qg^{-1} \in \mathrm{G}$ and hence

$$p^{-1}\zeta p = (g^{-1}p)^{-1}\xi(g^{-1}p) \in \mathfrak{g}.$$

Thus p satisfies the requirements of Step 3 and this proves Theorem 1.10. $\square$

Comments on the literature

The group $P(\xi)$ in Lemma 7.2 was introduced by Mumford. In [9, Proposition 2.6] he proved that it is a *parabolic subgroup* of G^c. If $T \subset G$ is a maximal torus whose Lie algebra contains ξ then there exists a *Borel subgroup* $B \subset G^c$ such that $T \subset B \subset P(\xi)$. In this situation $B \cap G = T$ and it then follows that the inclusion of G into G^c descends to a diffeomorphism $G/T \cong G^c/B$ (see Schmid [14, Lemma 2.4.6]). This implies Theorem 7.1. The proof of Theorem 7.1 given above uses direct arguments, and does not rely on the structure theory for Lie groups.

The proof on page 45 shows that Theorem 1.10 is an easy consequence of Theorem 7.1 and Cartan's uniqueness theorem for maximal compact subgroups of G^c (Theorem 1.8). Theorem 1.10 is mentioned in the work of Ness [11, page 1292] as a consequence of Mumford's result that $P(\xi)$ is parabolic. It plays a central role in the study by Kempf and Ness of *Mumford's numerical function* and of the *Hilbert–Mumford stability criterion* for linear G^c-actions (see [5, 6, 10, 11]). Specifically, Theorem 1.10 is needed in the proof of the *moment-weight* inequality (see Ness [11, Lemma 3.1 (iv)] and Szekelyhidi [15, Theorem 1.3.6]). The moment-weight inequality implies the necessity of the Hilbert–Mumford criterion for semistability. It also implies the *Kirwan–Ness inequality* in [7] and [11, Theorem 1.2], which asserts that the restriction of the moment map squared to the complexified group orbit of a critical point attains its minimum at that critical point. (See [3] for an exposition of these results.)

Remark. The Cartan–Iwasawa–Malcev Theorem in [4, Theorem 14.1.3] is a general result about maximal compact subgroups. It is not restricted to reductive groups and contains Theorem 1.8 as a special case.

Cartan–Iwasawa–Malcev Theorem. *Let* G *ba any Lie group with finitely many connected components. Then it admits a compact subgroup* K *with the following properties.*

(C) *Every compact subgroup of* G *is conjugate in* G *to a subgroup of* K.

(I) K *intersects each connected component of* G *in a connected set.*

(M) *If* $K' \subset G$ *is a compact subgroup such that* $K \subset K'$, *then* $K = K'$.

This implies that every maximal compact subgroup of G is conjugate to K and hence also satisfies (I) and (C). It also implies that, if K is a maximal compact subgroup of G, then its identity component K_0 is a maximal compact subgroup of the identity component G_0 of G.

A Right invariant metrics

Let G be a connected Lie group with the Lie algebra $\mathfrak{g} = T_1\mathrm{G} = \mathrm{Lie}(\mathrm{G})$. Fix any inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ and define a Riemannian metric on G by

$$\langle \widehat{g}_1, \widehat{g}_2\rangle := \langle \widehat{g}_1 g^{-1}, \widehat{g}_2 g^{-1}\rangle \qquad \text{for } \widehat{g}_1, \widehat{g}_2 \in T_g\mathrm{G}. \tag{75}$$

Theorem A.1. *The Riemannian metric* (75) *is complete. Moreover, for every smooth curve $\xi : \mathbb{R} \to \mathfrak{g}$ and every $g_0 \in \mathrm{G}$ the initial value problem*

$$\dot{g}g^{-1} = \xi, \qquad g(0) = g_0 \tag{76}$$

has a unique solution $g : \mathbb{R} \to \mathrm{G}$.

Corollary A.2. *For every smooth curve $\xi : \mathbb{R} \to \mathfrak{g}$ and every $g_0 \in \mathrm{G}$ the initial value problem $g^{-1}\dot{g} = \xi$, $g(0) = g_0$, has a unique solution $g : \mathbb{R} \to \mathrm{G}$.*

Proof. By Theorem A.1 there exists a unique solution $h : \mathbb{R} \to \mathrm{G}$ of the differential equation $\dot{h}h^{-1} = -\xi$ with the initial condition $h(0) = g_0^{-1}$. Hence the curve $g := h^{-1}$ satisfies $\dot{g} = -h^{-1}\dot{h}h^{-1} = g\xi$ and $g(0) = g_0$. □

Corollary A.3. *Let G' be a connected Lie group and let $\rho : \mathrm{G} \to \mathrm{G}'$ be a Lie group homomorphism such that $d\rho(1) : \mathfrak{g} \to \mathfrak{g}' := \mathrm{Lie}(\mathrm{G}')$ is bijective. Then ρ is a covering map and, in particular, ρ is surjective.*

Proof. By Corollary A.2 ρ has the path lifting property, i,e, for every smooth path $\gamma' : [0,1] \to \mathrm{G}'$ and every $g_0 \in \rho^{-1}(\gamma'(0))$, there exists a unique smooth path $\gamma : [0,1] \to \mathrm{G}$ such that $\gamma(0) = g_0$ and $\rho(\gamma(t)) = \gamma'(t)$ for all t. □

Lemma A.4. *There exists a unique linear map $A : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ such that*

$$A(\xi) + A(\xi)^* = 0, \qquad A(\xi)\eta - A(\eta)\xi + [\xi,\eta] = 0 \tag{77}$$

for all $\xi, \eta \in \mathfrak{g}$.

Proof. Let $A : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ be a linear map that satisfies (77). Then

$$\begin{aligned}
\langle A(\xi)\eta, \zeta\rangle &= \tfrac{1}{2}\langle A(\xi)\eta, \zeta\rangle - \tfrac{1}{2}\langle A(\xi)\zeta, \eta\rangle \\
&= \tfrac{1}{2}\langle -[\xi,\eta] + A(\eta)\xi, \zeta\rangle + \tfrac{1}{2}\langle [\xi,\zeta] - A(\zeta)\xi, \eta\rangle \\
&= -\tfrac{1}{2}\langle [\xi,\eta], \zeta\rangle + \tfrac{1}{2}\langle [\xi,\zeta], \eta\rangle + \tfrac{1}{2}\langle A(\zeta)\eta - A(\eta)\zeta, \xi\rangle \\
&= -\tfrac{1}{2}\langle [\xi,\eta], \zeta\rangle + \tfrac{1}{2}\langle [\xi,\zeta], \eta\rangle + \tfrac{1}{2}\langle [\eta,\zeta], \xi\rangle
\end{aligned}$$

for all $\xi, \eta, \zeta \in \mathfrak{g}$. This proves uniquenes. Conversely, if A is defined by

$$\langle A(\xi)\eta, \zeta\rangle := -\tfrac{1}{2}\langle [\xi,\eta], \zeta\rangle + \tfrac{1}{2}\langle [\xi,\zeta], \eta\rangle + \tfrac{1}{2}\langle [\eta,\zeta], \xi\rangle \tag{78}$$

for $\xi, \eta, \zeta \in \mathfrak{g}$, then A satisfies (77). This proves Lemma A.4. □

Lemma A.5. *The Levi-Civita conection of the metric* (75) *assigns to each vector field* $X(t) \in T_{g(t)}\mathrm{G}$ *along a curve* $g : \mathbb{R} \to \mathrm{G}$ *the covariant derivative*

$$\nabla X = \big(\dot{\xi} + A(\dot{g}g^{-1})\xi\big)g, \qquad \xi := Xg^{-1} : \mathbb{R} \to \mathfrak{g}. \tag{79}$$

Proof. Let $g : \mathbb{R} \to \mathrm{G}$ and $\xi, \eta : \mathbb{R} \to \mathfrak{g}$ be smooth curves and define the vector fields X, Y along g by $X := \xi g$ and $Y = \eta g$. Then, by (77),

$$\begin{aligned}\frac{d}{dt}\langle X, Y\rangle &= \langle\dot{\xi}, \eta\rangle + \langle\xi, \dot{\eta}\rangle = \langle\dot{\xi} + A(\dot{g}g^{-1})\xi, \eta\rangle + \langle\xi, \dot{\eta} + A(\dot{g}g^{-1})\eta\rangle \\ &= \langle\nabla X, Y\rangle + \langle X, \nabla Y\rangle.\end{aligned}$$

Thus the connection (79) is Riemannian. Now let $\mathbb{R}^2 \to \mathrm{G} : (s,t) \mapsto g(s,t)$ be a smooth map and abbreviate $\xi := (\partial_s g)g^{-1}$ and $\eta := (\partial_t g)g^{-1}$. Then

$$\partial_s\eta - \partial_t\xi = [\xi, \eta]$$

and hence

$$\begin{aligned}\big(\nabla_s\partial_t g - \nabla_t\partial_s g\big)g^{-1} &= \partial_s\eta + A((\partial_s g)g^{-1})\eta - \partial_t\xi - A((\partial_t g)g^{-1})\xi \\ &= \partial_s\eta - \partial_t\xi + A(\xi)\eta - A(\eta)\xi \\ &= \partial_s\eta - \partial_t\xi - [\xi, \eta] = 0.\end{aligned}$$

Thus the connection (79) is torsion-free and this proves Lemma A.5. □

Proof of Theorem A.1. The proof has five steps.

Step 1. *A smooth curve* $g : I \to \mathrm{G}$ *on an interval* $I \subset \mathbb{R}$ *is a geodesic if and only if it satisfies the differential equation*

$$\frac{d}{dt}(\dot{g}g^{-1}) + A(\dot{g}g^{-1})\dot{g}g^{-1} = 0. \tag{80}$$

A curve $g : I \to \mathrm{G}$ is a geodesic if and only if $\nabla\dot{g} = 0$. Hence Step 1 follows from Lemma A.5.

Step 2. *There exists a constant* $\varepsilon > 0$ *such that, for every* $\xi_0 \in \mathfrak{g}$ *such that* $|\xi_0| = 1$*, the geodesic initial value problem*

$$\frac{d}{dt}(\dot{g}g^{-1}) + A(\dot{g}g^{-1})\dot{g}g^{-1} = 0, \qquad g(0) = 1, \qquad \dot{g}(0) = \xi_0 \tag{81}$$

has a unique solution $g : (-\varepsilon, \varepsilon) \to \mathrm{G}$.

A general theorem about differential equations guarantees the existence and uniqueness of solutions on a uniform time interval for all initial conditions in a given compact set.

Step 3. *For every $\xi_0 \in \mathfrak{g}$ with $|\xi_0| = 1$ the geodesic initial value problem* (81) *has a unique solution $g : \mathbb{R} \to \mathrm{G}$ on the entire real axis.*

Let $I \subset (0, \infty)$ be the set of all real numbers $T > 0$ such that the initial value problem (81) has a solution $g : (-T, T) \to \mathrm{G}$. By Step 2 the set I is nonempty and by defintion I is a relatively closed subset of the interval $(0, \infty)$.

We prove that I is open. Let $T \in I$ and let $g : (-T, T) \to \mathrm{G}$ be the unique solution of (81). Let $\varepsilon > 0$ be the constant in Step 2 and choose T' such that $T - \varepsilon < T' < T$. Then, by Step 2, the initial value problem

$$\frac{d}{dt}(\dot{g}'g'^{-1}) + A(\dot{g}'g'^{-1})\dot{g}'g'^{-1} = 0, \qquad g'(0) = 1, \qquad \dot{g}'(0) = \dot{g}(T')g(T')^{-1}$$

has a unique solution on the interval $-\varepsilon < t < \varepsilon$. Moreover, by uniqueness this solution satisfies $g'(t)g(T') = g(t + T')$ for $-\varepsilon < t < T - T'$. Hence the extended curve $g : (-T, T' + \varepsilon) \to \mathrm{G}$, defined by

$$g(t) := g'(t - T')g(T') \qquad \text{for } T' - \varepsilon < t < T' + \varepsilon,$$

satisfies (81) on the larger interval $-T < t < T' + \varepsilon$. Reverse time to deduce that the solution of (81) exists on the interval $-T' - \varepsilon' < t < T' + \varepsilon$ and thus $T' + \varepsilon \in I$. Hence $(0, T + \varepsilon) \subset I$ for every $T \in I$, hence I is open, and hence $I = (0, \infty)$. This proves Step 3.

Step 4. *The Riemannian metric* (75) *on* G *is complete.*

This follows from Step 3 and the Hopf–Rinow Theorem 4.1.

Step 5. *Let $\xi : \mathbb{R} \to \mathfrak{g}$ be a smooth curve and let $g_0 \in \mathrm{G}$. Then the initial value problem* (76) *has a unique solution $g : \mathbb{R} \to \mathrm{G}$ on the entire real axis.*

Denote by $d : \mathrm{G} \times \mathrm{G} \to \mathbb{R}$ the distance function associated to the Riemannian metric (75), choose any real number $T > 0$, and define

$$c_T := \int_{-T}^{T} |\xi(t)|\, dt, \qquad K_T := \{ g \in \mathrm{G} \,|\, d(g_0, g) \le c_T \}.$$

By Step 4 the set K_T is compact. Moreover, if a solution $g : (-T, T) \to \mathrm{G}$ of (76) exists on the interval $-T < t < T$, then it satisfies

$$d(g_0, g(t)) \le \left| \int_0^t |\dot{g}(s)|\, ds \right| = \left| \int_0^t |\xi(s)|\, ds \right| \le c_T$$

for $-T < t < T$. Thus the solution $g : (-T, T) \to \mathrm{G}$ of (76) does not leave the compact set K_T and hence extends to a solution on the interval $(-T', T')$ for some $T' > T$. This proves Step 5 and Theorem A.1. □

References

[1] B. Casselmann, Symmetric spaces of semi-simple groups. *Essays on representations of real groups*, August, 2012. `http://www.math.ubc.ca/~cass/research/pdf/Cartan.pdf`

[2] S.K. Donaldson, Lie algebra theory without algebra. In *"Algebra, Arithmetic, and Geometry, In honour of Yu. I. Manin"*, edited by Yuri Tschinkel and Yuri Zarhin, Progress in Mathematics, Birkhäuser **269**, 2009, pp. 249–266.

[3] V. Georgoulas & J.!W. Robbin & D.A. Salamon, *The Moment-Weight Inequality and the Hilbert–Mumford Criterion.* Lecture Notes in Mathematics **2297**, Springer 2021. `https://people.math.ethz.ch/~salamon/PREPRINTS/momentweight-book2.pdf`

[4] J. Hilgert & K.-H. Neeb, *Structure and Geometry of Lie groups.* Springer Monographs in Mathematics, Springer-Verlag, 2012.

[5] G. Kempf, Instability in invariant theory. *Annals of Mathematics* **108** (1978), 299–317.

[6] G. Kempf, and L. Ness, The length of vectors in representation spaces. Springer Lecture Notes **732**, *Algebraic Geometry, Proceedings*, Copenhagen, 1978, pp 233–244.

[7] F. Kirwan, *Cohomology of Quotients in Symplectic and Algebraic Geometry.* Princeton UP, 1984.

[8] A.I. Malcev, On the theory of the Lie groups in the large. *Rec. Math. (Matematichevskii Sbornik)* **16(58)** (1945), 163–190.

[9] D. Mumford, *Geometric Invariant Theory.* Springer Verlag, New York, 1965.

[10] L. Ness, Mumford's numerical function and stable projective hypersurfaces. Springer Lecture Notes **732**, *Algebraic Geometry, Proceedings*, Copenhagen, 1978, pp 417–454.

[11] L. Ness, A stratification of the null cone by the moment map. *Am. J. Math.* **106** (1984), 1281–1329.

[12] J.W. Robbin, D.A. Salamon, *Introduction to Differential Geometry.* Springer Studium Mathematik (Master), Springer Verlag, 2022. `http://www.math.ethz.ch/~salamon/PREPRINTS/diffgeo.pdf`

[13] D.A. Salamon, Notes on compact Lie groups. Unpublishes Notes, ETH Zürich, 2022. `http://www.math.ethz.ch/~salamon/PREPRINTS/liegroup.pdf`

[14] W. Schmid, *Geometric Methods in Representation Theory.* Part 4 of *Poisson Geometry, Deformation Quantisation and Group Representations.* LMS Lecture Note Series **323**. CUP 2005. `http://www.math.harvard.edu/~schmid/articles/brussels_1_30_04.pdf`

[15] G. Szekelyhidi, Extremal metrics and K-stability. PhD thesis, Imperial College, London, 2006. `http://arxiv.org/pdf/math.DG/0611002.pdf`