The Viterbo–Maslov Index in Dimension Two

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Abstract

We prove a formula that expresses the Viterbo–Maslov index of a smooth strip in an oriented 2-manifold with boundary curves contained in 1-dimensional submanifolds in terms the degree function on the complement of the union of the two submanifolds.

1 Introduction

We assume throughout this paper that Σ is a connected oriented 2-manifold without boundary and $\alpha, \beta \subset \Sigma$ are connected smooth one dimensional oriented submanifolds without boundary which are closed as subsets of Σ and intersect transversally. We do not assume that Σ is compact, but when it is, α and β are embedded circles. Denote the standard half disc by

$$\mathbb{D} := \{z \in \mathbb{C} \,|\, \mathrm{Im}\, z \geq 0,\ |z| \leq 1\}.$$

Let $\mathcal{D}$ denote the space of all smooth maps $u : \mathbb{D} \to \Sigma$ satisfying the boundary conditions $u(\mathbb{D} \cap \mathbb{R}) \subset \alpha$ and $u(\mathbb{D} \cap S^1) \subset \beta$. For $x, y \in \alpha \cap \beta$ let $\mathcal{D}(x, y)$ denote the subset of all $u \in \mathcal{D}$ satisfying the endpoint conditions $u(-1) = x$ and $u(1) = y$. Each $u \in \mathcal{D}$ determines a locally constant function $\mathrm{w} : \Sigma \setminus (\alpha \cup \beta) \to \mathbb{Z}$ defined as the degree

$$\mathrm{w}(z) := \deg(u, z), \qquad z \in \Sigma \setminus (\alpha \cup \beta).$$

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When z is a regular value of u this is the algebraic number of points in the preimage $u^{-1}(z)$. The function w depends only on the homotopy class of u. We prove that the homotopy class of u is uniquely determined by its endpoints x, y and its degree function w (Theorem 2.4). The main theorem of this paper asserts that the Viterbo–Maslov index of an element $u \in \mathcal{D}(x, y)$ is given by the formula

$$\mu(u) = \frac{m_x + m_y}{2}, \tag{1}$$

where m_x denotes the sum of the four values of w encountered when walking along a small circle surrounding x, and similarly for y (Theorem 3.4). The formula (1) plays a central role in our combinatorial approach [1, 7] to Floer homology [4, 5]. An appendix contains a proof that the space of paths connecting α to β is simply connected under suitable assumptions.

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2 Chains and Traces

Define a cell complex structure on Σ by taking the set of zero-cells to be the set $\alpha \cap \beta$, the set of one-cells to be the set of connected components of $(\alpha \setminus \beta) \cup (\beta \setminus \alpha)$ with compact closure, and the set of two-cells to be the set of connected components of $\Sigma \setminus (\alpha \cup \beta)$ with compact closure. (There is an abuse of language here as the "two-cells" need not be homeomorphs of the open unit disc if the genus of Σ is positive and the "one-cells" need not be arcs if $\alpha \cap \beta = \emptyset$.) Define a boundary operator ∂ as follows. For each two-cell F let $\partial F = \sum \pm E$, where the sum is over the one-cells E which abut F and the plus sign is chosen iff the orientation of E (determined from the given orientations of α and β) agrees with the boundary orientation of F as a connected open subset of the oriented manifold Σ. For each one-cell E let $\partial E = b - a$ where a and b are the endpoints of the arc E and the orientation of E goes from a to b. (The one-cell E is either a subarc of α or a subarc of β and both α and β are oriented one-manifolds.) For $k = 0, 1, 2$ a ***k*-chain** is defined to be a formal linear combination (with integer coefficients) of k-cells, i.e. a two-chain is a locally constant map $\Sigma \setminus (\alpha \cup \beta) \to \mathbb{Z}$ (whose support has compact closure in Σ) and a one-chain is a locally constant map $(\alpha \setminus \beta) \cup (\beta \setminus \alpha) \to \mathbb{Z}$ (whose support has compact closure in $\alpha \cup \beta$). It follows directly from the definitions that $\partial^2 F = 0$ for each two-cell F.

Each $u \in \mathcal{D}$ determines a two-chain w via

$$\mathrm{w}(z) := \deg(u, z), \qquad z \in \Sigma \setminus (\alpha \cup \beta). \tag{2}$$

and a one-chain ν via

$$\nu(z) := \begin{cases} \deg(u|_{\partial\mathbb{D}\cap\mathbb{R}} : \partial\mathbb{D} \cap \mathbb{R} \to \alpha, z), & \text{for } z \in \alpha \setminus \beta, \\ -\deg(u|_{\partial\mathbb{D}\cap S^1} : \partial\mathbb{D} \cap S^1 \to \beta, z), & \text{for } z \in \beta \setminus \alpha. \end{cases} \tag{3}$$

Here we orient the one-manifolds $\mathbb{D} \cap \mathbb{R}$ and $\mathbb{D} \cap S^1$ from -1 to $+1$. For any one-chain $\nu : (\alpha \setminus \beta) \cup (\beta \setminus \alpha) \to \mathbb{Z}$ denote

$$\nu_\alpha := \nu|_{\alpha\setminus\beta} : \alpha \setminus \beta \to \mathbb{Z}, \qquad \nu_\beta := \nu|_{\alpha\setminus\beta} : \beta \setminus \alpha \to \mathbb{Z}.$$

Conversely, given locally constant functions $\nu_\alpha : \alpha\setminus\beta \to \mathbb{Z}$ and $\nu_\beta : \beta\setminus\alpha \to \mathbb{Z}$, denote by $\nu = \nu_\alpha - \nu_\beta$ the one-chain that agrees with ν_α on $\alpha \setminus \beta$ and agrees with $-\nu_\beta$ on $\beta \setminus \alpha$.

Definition 2.1 (Traces). Fix two (not necessarily distinct) intersection points $x, y \in \alpha \cap \beta$.

(i) Let $\mathrm{w} : \Sigma \setminus (\alpha \cup \beta) \to \mathbb{Z}$ be a two-chain. The triple $\Lambda = (x, y, \mathrm{w})$ is called an **(α, β)-trace** if there exists an element $u \in \mathcal{D}(x, y)$ such that w is given by (2). In this case $\Lambda =: \Lambda_u$ is also called the **(α, β)-trace of u** and we sometimes write $\mathrm{w}_u := \mathrm{w}$.

(ii) Let $\Lambda = (x, y, \mathrm{w})$ be an (α, β)-trace. The triple $\partial\Lambda := (x, y, \partial\mathrm{w})$ is called the **boundary of Λ.**

(iii) A one-chain $\nu : (\alpha\setminus\beta)\cup(\beta\setminus\alpha) \to \mathbb{Z}$ is called an **(x, y)-trace** if there exist smooth curves $\gamma_\alpha : [0, 1] \to \alpha$ and $\gamma_\beta : [0, 1] \to \beta$ such that $\gamma_\alpha(0) = \gamma_\beta(0) = x$, $\gamma_\alpha(1) = \gamma_\beta(1) = y$, γ_α and γ_β are homotopic in Σ with fixed endpoints, and

$$\nu(z) = \begin{cases} \deg(\gamma_\alpha, z), & \text{for } z \in \alpha \setminus \beta, \\ -\deg(\gamma_\beta, z), & \text{for } z \in \beta \setminus \alpha. \end{cases} \tag{4}$$

Remark 2.2. Assume Σ is simply connected. Then the condition on γ_α and γ_β to be homotopic with fixed endpoints is redundant. Moreover, if $x = y$ then a one-chain ν is an (x, y)-trace if and only if the restrictions $\nu_\alpha := \nu|_{\alpha\setminus\beta}$ and $\nu_\beta := -\nu|_{\beta\setminus\alpha}$ are constant. If $x \neq y$ and α, β are embedded circles and A, B denote the positively oriented arcs from x to y in α, β, then a one-chain ν is an (x, y)-trace if and only if $\nu_\alpha|_{\alpha\setminus(A\cup\beta)} = \nu_\alpha|_{A\setminus\beta} - 1$ and $\nu_\beta|_{\beta\setminus(B\cup\alpha)} = \nu_\beta|_{B\setminus\alpha} - 1$. In particular, when walking along α or β, the function ν only changes its value at x and y.

Lemma 2.3. *Let $x, y \in \alpha \cap \beta$ and $u \in \mathcal{D}(x, y)$. Then the boundary of the (α, β)-trace Λ_u of u is the triple $\partial\Lambda_u = (x, y, \nu)$, where ν is given by* (3). *In other words, if* w *is given by* (2) *and* ν *is given by* (3) *then* $\nu = \partial\mathrm{w}$.

Proof. Choose an embedding $\gamma : [-1, 1] \to \Sigma$ such that u is transverse to γ, $\gamma(t) \in \Sigma \setminus (\alpha \cup \beta)$ for $t \neq 0$, $\gamma(-1)$, $\gamma(1)$ are regular values of u, $\gamma(0) \in \alpha \setminus \beta$ is a regular value of $u|_{\mathbb{D}\cap\mathbb{R}}$, and γ intersects α transversally at $t = 0$ such that orientations match in

$$T_{\gamma(0)}\Sigma = T_{\gamma(0)}\alpha \oplus \mathbb{R}\dot{\gamma}(0).$$

Denote $\Gamma := \gamma([-1, 1])$. Then $u^{-1}(\Gamma) \subset \mathbb{D}$ is a 1-dimensional submanifold with boundary

$$\partial u^{-1}(\Gamma) = u^{-1}(\gamma(-1)) \cup u^{-1}(\gamma(1)) \cup \big(u^{-1}(\gamma(0)) \cap \mathbb{R}\big).$$

If $z \in u^{-1}(\Gamma)$ then

$$\operatorname{im} du(z) + T_{u(z)}\Gamma = T_{u(z)}\Sigma, \qquad T_z u^{-1}(\Gamma) = du(z)^{-1}T_{u(z)}\Gamma.$$

We orient $u^{-1}(\Gamma)$ such that the orientations match in

$$T_{u(z)}\Sigma = T_{u(z)}\Gamma \oplus du(z)\mathbf{i}T_z u^{-1}(\Gamma).$$

In other words, if $z \in u^{-1}(\Gamma)$ and $u(z) = \gamma(t)$, then a nonzero tangent vector $\zeta \in T_z u^{-1}(\Gamma)$ is positive if and only if the pair $(\dot{\gamma}(t), du(z)\mathbf{i}\zeta)$ is a positive basis of $T_{\gamma(t)}\Sigma$. Then the boundary orientation of $u^{-1}(\Gamma)$ at the elements of $u^{-1}(\gamma(1))$ agrees with the algebraic count in the definition of $\mathrm{w}(\gamma(1))$, at the elements of $u^{-1}(\gamma(-1))$ is opposite to the algebraic count in the definition of $\mathrm{w}(\gamma(-1))$, and at the elements of $u^{-1}(\gamma(0)) \cap \mathbb{R}$ is opposite to the algebraic count in the definition of $\nu(\gamma(0))$. Hence

$$\mathrm{w}(\gamma(1)) = \mathrm{w}(\gamma(-1)) + \nu(\gamma(0)).$$

In other words the value of ν at a point in $\alpha \setminus \beta$ is equal to the value of w slightly to the left of α minus the value of w slightly to the right of α. Likewise, the value of ν at a point in $\beta \setminus \alpha$ is equal to the value of w slightly to the right of β minus the value of w slightly to the left of β. This proves Lemma 2.3. $\square$

Theorem 2.4. **(i)** *Two elements of $\mathcal{D}$ belong to the same connected component of $\mathcal{D}$ if and only if they have the same (α,β)-trace.*

(ii) *Assume Σ is diffeomorphic to the two-sphere. Then $\Lambda = (x, y, \mathrm{w})$ is an (α,β)-trace if and only if $\partial\mathrm{w}$ is an (x, y)-trace.*

(iii) *Assume Σ is not diffeomorphic to the two-sphere and let $x, y \in \alpha\cap\beta$. If ν is an (x, y)-trace, then there is a unique two-chain w such that $\Lambda := (x, y, \mathrm{w})$ is an (α,β)-trace and $\partial\mathrm{w} = \nu$.*

Proof. We prove (i). "Only if" follows from the standard arguments in degree theory as in Milnor [6]. To prove "if", fix two intersection points

$$x, y \in \alpha \cap \beta$$

and, for $X = \Sigma, \alpha, \beta$, denote by $\mathcal{P}(x, y; X)$ the space of all smooth curves $\gamma : [0, 1] \to X$ satisfying $\gamma(0) = x$ and $\gamma(1) = y$. Every $u \in \mathcal{D}(x, y)$ determines smooth paths $\gamma_{u,\alpha} \in \mathcal{P}(x, y; \alpha)$ and $\gamma_{u,\beta} \in \mathcal{P}(x, y; \beta)$ via

$$\gamma_{u,\alpha}(s) := u(-\cos(\pi s), 0), \qquad \gamma_{u,\beta}(s) = u(-\cos(\pi s), \sin(\pi s)). \tag{5}$$

These paths are homotopic in Σ with fixed endpoints. An explicit homotopy is the map

$$F_u := u \circ \varphi : [0, 1]^2 \to \Sigma$$

where $\varphi : [0, 1]^2 \to \mathbb{D}$ is the map

$$\varphi(s, t) := (-\cos(\pi s), t\sin(\pi s)).$$

By Lemma 2.3, he homotopy class of $\gamma_{u,\alpha}$ in $\mathcal{P}(x, y; \alpha)$ is uniquely determined by $\nu_\alpha := \partial\mathrm{w}_u|_{\alpha\setminus\beta} : \alpha \setminus \beta \to \mathbb{Z}$ and that of $\gamma_{u,\beta}$ in $\mathcal{P}(x, y; \beta)$ is uniquely determined by $\nu_\beta := -\partial\mathrm{w}_u|_{\beta\setminus\alpha} : \beta \setminus \alpha \to \mathbb{Z}$. Hence they are both uniquely determined by the (α,β)-trace of u. If Σ is not diffeomorphic to the 2-sphere the assertion follows from the fact that each component of $\mathcal{P}(x, y; \Sigma)$ is contractible (because the universal cover of Σ is diffeomorphic to the complex plane). Now assume Σ is diffeomorphic to the 2-sphere. Then $\pi_1(\mathcal{P}(x, y; \Sigma)) = \mathbb{Z}$ acts on $\pi_0(\mathcal{D})$ because the correspondence $u \mapsto F_u$ identifies $\pi_0(\mathcal{D})$ with a space of homotopy classes of paths in $\mathcal{P}(x, y; \Sigma)$ connecting $\mathcal{P}(x, y; \alpha)$ to $\mathcal{P}(x, y; \beta)$. The induced action on the space of two-chains $\mathrm{w} : \Sigma \setminus (\alpha \cup \beta)$ is given by adding a global constant. Hence the map $u \mapsto \mathrm{w}$ induces an injective map

$$\pi_0(\mathcal{D}(x, y)) \to \{\text{2-chains}\}.$$

This proves (i).

We prove (ii) and (iii). Let w be a two-chain, suppose that

$$\nu := \partial \mathrm{w}$$

is an (x, y)-trace, and denote

$$\Lambda := (x, y, \mathrm{w}).$$

Let $\gamma_\alpha : [0, 1] \to \alpha$ and $\gamma_\beta : [0, 1] \to \beta$ be as in Definition 2.1. Then there is a $u' \in \mathcal{D}(x, y)$ such that the map $s \mapsto u'(-\cos(\pi s), 0)$ is homotopic to γ_α and $s \mapsto u'(-\cos(\pi s), \sin(\pi s))$ is homotopic to γ_β. By definition the (α, β)-trace of u' is $\Lambda' = (x, y, \mathrm{w}')$ for some two-chain w'. By Lemma 2.3, we have

$$\partial \mathrm{w}' = \nu = \partial \mathrm{w}$$

and hence $\mathrm{w} - \mathrm{w}' =: d$ is constant. If Σ is not diffeomorphic to the two-sphere and Λ is the (α, β)-trace of some element $u \in \mathcal{D}$, then u is homotopic to u' (as $\mathcal{P}(x, y; \Sigma)$ is simply connected) and hence $d = 0$ and $\Lambda = \Lambda'$. If Σ is diffeomorphic to the 2-sphere choose a smooth map $v : S^2 \to \Sigma$ of degree d and replace u' by the connected sum $u := u' \# v$. Then Λ is the (α, β)-trace of u. This proves Theorem 2.4. □

Remark 2.5. Let $\Lambda = (x, y, \mathrm{w})$ be an (α, β)-trace and define

$$\nu_\alpha := \partial \mathrm{w}|_{\alpha \setminus \beta}, \qquad \nu_\beta := -\partial \mathrm{w}|_{\beta \setminus \alpha}.$$

(i) The two-chain w is uniquely determined by the condition $\partial \mathrm{w} = \nu_\alpha - \nu_\beta$ and its value at one point. To see this, think of the embedded circles α and β as traintracks. Crossing α at a point $z \in \alpha \setminus \beta$ increases w by $\nu_\alpha(z)$ if the train comes from the left, and decreases it by $\nu_\alpha(z)$ if the train comes from the right. Crossing β at a point $z \in \beta \setminus \alpha$ decreases w by $\nu_\beta(z)$ if the train comes from the left and increases it by $\nu_\beta(z)$ if the train comes from the right. Moreover, ν_α extends continuously to $\alpha \setminus \{x, y\}$ and ν_β extends continuously to $\beta \setminus \{x, y\}$. At each intersection point $z \in (\alpha \cap \beta) \setminus \{x, y\}$ with intersection index $+1$ (respectively -1) the function w takes the values

$$k, \quad k + \nu_\alpha(z), \quad k + \nu_\alpha(z) - \nu_\beta(z), \quad k - \nu_\beta(z)$$

as we march counterclockwise (respectively clockwise) along a small circle surrounding the intersection point.

(ii) If Σ is not diffeomorphic to the 2-sphere then, by Theorem 2.4 (iii), the (α,β)-trace Λ is uniquely determined by its boundary $\partial\Lambda = (x, y, \nu_\alpha - \nu_\beta)$.

(iii) Assume Σ is not diffeomorphic to the 2-sphere and choose a universal covering $\pi : \mathbb{C} \to \Sigma$. Choose a point $\tilde{x} \in \pi^{-1}(x)$ and lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β such that $\tilde{x} \in \tilde{\alpha} \cap \tilde{\beta}$. Then Λ lifts to an $(\tilde{\alpha},\tilde{\beta})$-trace

$$\tilde{\Lambda} = (\tilde{x}, \tilde{y}, \tilde{\mathrm{w}}).$$

More precisely, the one chain $\nu := \nu_\alpha - \nu_\beta = \partial\mathrm{w}$ is an (x, y)-trace, by Lemma 2.3. The paths $\gamma_\alpha : [0, 1] \to \alpha$ and $\gamma_\beta : [0, 1] \to \beta$ in Definition 2.1 lift to unique paths $\gamma_{\tilde{\alpha}} : [0, 1] \to \tilde{\alpha}$ and $\gamma_{\tilde{\beta}} : [0, 1] \to \tilde{\beta}$ connecting $\tilde{x}$ to $\tilde{y}$. For $\tilde{z} \in \mathbb{C} \setminus (\tilde{A} \cup \tilde{B})$ the number $\tilde{\mathrm{w}}(\tilde{z})$ is the winding number of the loop $\gamma_{\tilde{\alpha}} - \gamma_{\tilde{\beta}}$ about $\tilde{z}$ (by Rouché's theorem). The two-chain w is then given by

$$\mathrm{w}(z) = \sum_{\tilde{z}\in\pi^{-1}(z)} \tilde{\mathrm{w}}(\tilde{z}), \qquad z \in \Sigma \setminus (\alpha \cup \beta).$$

To see this, lift an element $u \in \mathcal{D}(x, y)$ with (α,β)-trace Λ to the universal cover to obtain an element $\tilde{u} \in \mathcal{D}(\tilde{x}, \tilde{y})$ with $\Lambda_{\tilde{u}} = \tilde{\Lambda}$ and consider the degree.

Definition 2.6 (Catenation). Let $x, y, z \in \alpha\cap\beta$. The **catenation of two (α,β)-traces** $\Lambda = (x, y, \mathrm{w})$ **and** $\Lambda' = (y, z, \mathrm{w}')$ is defined by

$$\Lambda\#\Lambda' := (x, z, \mathrm{w} + \mathrm{w}').$$

Let $u \in \mathcal{D}(x, y)$ and $u' \in \mathcal{D}(y, z)$ and suppose that u and u' are constant near the ends $\pm 1 \in \mathbb{D}$. For $0 < \lambda < 1$ sufficiently close to one the **λ-catenation of u and u'** is the map $u\#_\lambda u' \in \mathcal{D}(x, z)$ defined by

$$(u\#_\lambda u')(\zeta) := \begin{cases} u\left(\frac{\zeta+\lambda}{1+\lambda\zeta}\right), & \text{for } \mathrm{Re}\,\zeta \le 0, \\ u'\left(\frac{\zeta-\lambda}{1-\lambda\zeta}\right), & \text{for } \mathrm{Re}\,\zeta \ge 0. \end{cases}$$

Lemma 2.7. *If $u \in \mathcal{D}(x, y)$ and $u' \in \mathcal{D}(y, z)$ are as in Definition 2.6 then*

$$\Lambda_{u\#_\lambda u'} = \Lambda_u\#\Lambda_{u'}.$$

Thus the catenation of two (α,β)-traces is again an (α,β)-trace.

Proof. This follows directly from the definitions. □

3 The Maslov Index

Definition 3.1. Let $x, y \in \alpha \cap \beta$ and $u \in \mathcal{D}(x, y)$. Choose an orientation preserving trivialization

$$\mathbb{D} \times \mathbb{R}^2 \to u^*T\Sigma : (z, \zeta) \mapsto \Phi(z)\zeta$$

and consider the Lagrangian paths

$$\lambda_0, \lambda_1 : [0, 1] \to \mathbb{R}\mathrm{P}^1$$

given by

$$\begin{aligned}
\lambda_0(s) &:= \Phi(-\cos(\pi s), 0)^{-1} T_{u(-\cos(\pi s), 0)} \alpha, \\
\lambda_1(s) &:= \Phi(-\cos(\pi s), \sin(\pi s))^{-1} T_{u(-\cos(\pi s), \sin(\pi s))} \beta.
\end{aligned}$$

The **Viterbo–Maslov index of** u is defined as the relative Maslov index of the pair of Lagrangian paths (λ_0, λ_1) and will be denoted by

$$\mu(u) := \mu(\Lambda_u) := \mu(\lambda_0, \lambda_1).$$

By the naturality and homotopy axioms for the relative Maslov index (see for example [8]), the number $\mu(u)$ is independent of the choice of the trivialization and depends only on the homotopy class of u; hence it depends only on the (α, β)-trace of u, by Theorem 2.4. The relative Maslov index $\mu(\lambda_0, \lambda_1)$ is the degree of the loop in $\mathbb{R}\mathrm{P}^1$ obtained by traversing λ_0, followed by a counterclockwise turn from $\lambda_0(1)$ to $\lambda_1(1)$, followed by traversing λ_1 in reverse time, followed by a clockwise turn from $\lambda_1(0)$ to $\lambda_0(0)$. This index was first defined by Viterbo [9] (in all dimensions). Another exposition is contained in [8].

Remark 3.2. The Viterbo–Maslov index is additive under catenation, i.e. if

$$\Lambda = (x, y, \mathrm{w}), \qquad \Lambda' = (y, z, \mathrm{w}')$$

are (α, β)-traces then

$$\mu(\Lambda \# \Lambda') = \mu(\Lambda) + \mu(\Lambda').$$

For a proof of this formula see [9, 8].

Definition 3.3. Let $\Lambda = (x, y, \mathrm{w})$ be an (α, β)-trace and

$$\nu_\alpha := \partial \mathrm{w}|_{\alpha \setminus \beta}, \qquad \nu_\beta := -\partial \mathrm{w}|_{\beta \setminus \alpha}.$$

Λ is said to satisfy the **arc condition** if

$$x \neq y, \qquad \min |\nu_\alpha| = \min |\nu_\beta| = 0. \tag{6}$$

When Λ satisfies the arc condition there are arcs $A \subset \alpha$ and $B \subset \beta$ from x to y such that

$$\nu_\alpha(z) = \begin{cases} \pm 1, & \text{if } z \in A, \\ 0, & \text{if } z \in \alpha \setminus \overline{A}, \end{cases} \quad \nu_\beta(z) = \begin{cases} \pm 1, & \text{if } z \in B, \\ 0, & \text{if } z \in \beta \setminus \overline{B}. \end{cases} \tag{7}$$

Here the plus sign is chosen iff the orientation of A from x to y agrees with that of α, respectively the orientation of B from x to y agrees with that of β. In this situation the quadruple (x, y, A, B) and the triple $(x, y, \partial \mathrm{w})$ determine one another and we also write

$$\partial \Lambda = (x, y, A, B)$$

for the boundary of Λ. When $u \in \mathcal{D}$ and $\Lambda_u = (x, y, \mathrm{w})$ satisfies the arc condition and $\partial \Lambda_u = (x, y, A, B)$ then

$$s \mapsto u(-\cos(\pi s), 0)$$

is homotopic in α to a path traversing A and the path

$$s \mapsto u(-\cos(\pi s), \sin(\pi s))$$

is homotopic in β to a path traversing B.

Theorem 3.4. *Let $\Lambda = (x, y, \mathrm{w})$ be an (α, β)-trace. For $z \in \alpha \cap \beta$ denote by $m_z(\Lambda)$ the sum of the four values of w encountered when walking along a small circle surrounding z. Then the Viterbo–Maslov index of Λ is given by*

$$\mu(\Lambda) = \frac{m_x(\Lambda) + m_y(\Lambda)}{2}. \tag{8}$$

We first prove the result for the 2-plane and the 2-sphere (Section 4). When Σ is not simply connected we reduce the result to the case of the 2-plane (Section 5). The key is the identity

$$m_{g\tilde{x}}(\tilde{\Lambda}) + m_{g^{-1}\tilde{y}}(\tilde{\Lambda}) = 0 \tag{9}$$

for every lift $\tilde{\Lambda}$ to the universal cover and every deck transformation $g \neq \mathrm{id}$.

4 The Simply Connected Case

A connected oriented 2-manifold Σ is called **planar** if it admits an (orientation preserving) embedding into the complex plane.

Proposition 4.1. *Equation* (8) *holds when* Σ *is planar.*

Proof. Assume first that $\Sigma = \mathbb{C}$ and $\Lambda = (x, y, \mathrm{w})$ satisfies the arc condition. Thus the boundary of Λ has the form

$$\partial\Lambda = (x, y, A, B),$$

where $A \subset \alpha$ and $B \subset \beta$ are arcs from x to y and $\mathrm{w}(z)$ is the winding number of the loop $A - B$ about the point $z \in \Sigma \setminus (A \cup B)$ (see Remark 2.5). Hence the formula (8) can be written in the form

$$\mu(\Lambda) = 2k_x + 2k_y + \frac{\varepsilon_x - \varepsilon_y}{2}. \tag{10}$$

Here $\varepsilon_z = \varepsilon_z(\Lambda) \in \{+1, -1\}$ denotes the intersection index of A and B at a point $z \in A \cap B$, $k_x = k_x(\Lambda)$ denotes the value of the winding number w at a point in $\alpha \setminus A$ close to x, and $k_y = k_y(\Lambda)$ denotes the value of w at a point in $\alpha \setminus A$ close to y. We now prove (10) under the assumption that Λ satisfies the arc condition. The proof is by induction on the number of intersection points of B and α and has seven steps.

Step 1. *We may assume without loss of generality that*

$$\Sigma = \mathbb{C}, \qquad \alpha = \mathbb{R}, \qquad A = [x, y], \qquad x < y, \tag{11}$$

and $B \subset \mathbb{C}$ *is an embedded arc from* x *to* y *that is transverse to* $\mathbb{R}$*.*

Choose a diffeomorphism from Σ to $\mathbb{C}$ that maps A to a bounded closed interval and maps x to the left endpoint of A. If α is not compact the diffeomorphism can be chosen such that it also maps α to $\mathbb{R}$. If α is an embedded circle the diffeomorphism can be chosen such that its restriction to B is transverse to $\mathbb{R}$; now replace the image of α by $\mathbb{R}$. This proves Step 1.

Step 2. *Assume* (11) *and let* $\bar{\Lambda} := (x, y, z \mapsto -\mathrm{w}(\bar{z}))$ *be the* $(\alpha, \bar{\beta})$*-trace obtained from* Λ *by complex conjugation. Then* Λ *satisfies* (10) *if and only if* $\bar{\Lambda}$ *satisfies* (10)*.*

Step 2 follows from the fact that the numbers $\mu, k_x, k_y, \varepsilon_x, \varepsilon_y$ change sign under complex conjugation.

Step 3. *Assume* (11)*. If $B \cap \mathbb{R} = \{x, y\}$ then Λ satisfies* (10)*.*

In this case B is contained in the upper or lower closed half plane and the loop $A \cup B$ bounds a disc contained in the same half plane. By Step 1 we may assume that B is contained in the upper half space. Then $\varepsilon_x = 1$, $\varepsilon_y = -1$, and $\mu(\Lambda) = 1$. Moreover, the winding number w is one in the disc encircled by A and B and is zero in the complement of its closure. Since the intervals $(-\infty, 0)$ and $(0, \infty)$ are contained in this complement, we have $k_x = k_y = 0$. This proves Step 3.

Step 4. *Assume* (11) *and $\#(B \cap \mathbb{R}) > 2$, follow the arc of B, starting at x, and let x' be the next intersection point with $\mathbb{R}$. Assume $x' < x$, denote by B' the arc in B from x' to y, and let $A' := [x', y]$ (see Figure 1). If the (α, β)-trace Λ' with boundary $\partial\Lambda' = (x', y, A', B')$ satisfies* (10) *so does Λ.*

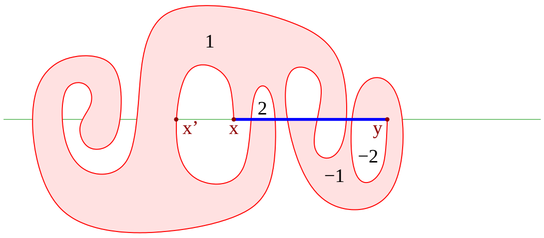

Figure 1: Maslov index and catenation: $x' < x < y$.

By Step 2 we may assume $\varepsilon_x(\Lambda) = 1$. Orient B from x to y. The Viterbo–Maslov index of Λ is minus the Maslov index of the path $B \to \mathbb{R}\mathrm{P}^1 : z \mapsto T_z B$, relative to the Lagrangian subspace $\mathbb{R} \subset \mathbb{C}$. Since the Maslov index of the arc in B from x to x' is $+1$ we have

$$\mu(\Lambda) = \mu(\Lambda') - 1. \tag{12}$$

Since the orientations of A' and B' agree with those of A and B we have

$$\varepsilon_{x'}(\Lambda') = \varepsilon_{x'}(\Lambda) = -1, \qquad \varepsilon_y(\Lambda') = \varepsilon_y(\Lambda). \tag{13}$$

Now let $x_1 < x_2 < \cdots < x_m < x$ be the intersection points of $\mathbb{R}$ and B in the interval $(-\infty, x)$ and let $\varepsilon_i \in \{-1, +1\}$ be the intersection index of $\mathbb{R}$ and B at x_i. Then there is an integer $\ell \in \{1, \ldots, m\}$ such that $x_\ell = x'$ and $\varepsilon_\ell = -1$. Moreover, the winding number w slightly to the left of x is

$$k_x(\Lambda) = \sum_{i=1}^{m} \varepsilon_i.$$

It agrees with the value of w slightly to the right of $x' = x_\ell$. Hence

$$k_x(\Lambda) = \sum_{i=1}^{\ell} \varepsilon_i = \sum_{i=1}^{\ell-1} \varepsilon_i - 1 = k_{x'}(\Lambda') - 1, \qquad k_y(\Lambda') = k_y(\Lambda). \tag{14}$$

It follows from equation (10) for Λ' and equations (12), (13), and (14) that

$$\begin{aligned}
\mu(\Lambda) &= \mu(\Lambda') - 1 \\
&= 2k_{x'}(\Lambda') + 2k_y(\Lambda') + \frac{\varepsilon_{x'}(\Lambda') - \varepsilon_y(\Lambda')}{2} - 1 \\
&= 2k_{x'}(\Lambda') + 2k_y(\Lambda') + \frac{-1 - \varepsilon_y(\Lambda)}{2} - 1 \\
&= 2k_{x'}(\Lambda') + 2k_y(\Lambda') + \frac{1 - \varepsilon_y(\Lambda)}{2} - 2 \\
&= 2k_x(\Lambda) + 2k_y(\Lambda) + \frac{\varepsilon_x(\Lambda) - \varepsilon_y(\Lambda)}{2}.
\end{aligned}$$

This proves Step 4.

Step 5. *Assume* (11) *and* $\#(B \cap \mathbb{R}) > 2$*, follow the arc of* B*, starting at* x*, and let* x' *be the next intersection point with* $\mathbb{R}$*. Assume* $x < x' < y$*, denote by* B' *the arc in* B *from* x' *to* y*, and let* $A' := [x', y]$ *(see Figure 2). If the* (α, β)*-trace* Λ' *with boundary* $\partial\Lambda' = (x', y, A', B')$ *satisfies* (10) *so does* Λ*.*

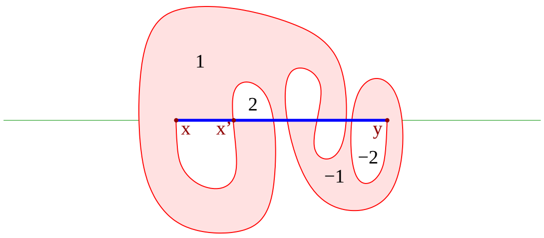

Figure 2: Maslov index and catenation: $x < x' < y$.

By Step 2 we may assume $\varepsilon_x(\Lambda) = 1$. Since the Maslov index of the arc in B from x to x' is -1, we have

$$\mu(\Lambda) = \mu(\Lambda') + 1. \tag{15}$$

Since the orientations of A' and B' agree with those of A and B we have

$$\varepsilon_{x'}(\Lambda') = \varepsilon_{x'}(\Lambda) = -1, \qquad \varepsilon_y(\Lambda') = \varepsilon_y(\Lambda). \tag{16}$$

Now let $x < x_1 < x_2 < \cdots < x_m < x'$ be the intersection points of $\mathbb{R}$ and B in the interval (x, x') and let $\varepsilon_i \in \{-1, +1\}$ be the intersection index of $\mathbb{R}$ and B at x_i. Since the value of w slightly to the left of x' agrees with the value of w slightly to the right of x we have

$$\sum_{i=1}^{m} \varepsilon_i = 0.$$

Since $k_{x'}(\Lambda')$ is the sum of the intersection indices of $\mathbb{R}$ and B' at all points to the left of x' we obtain

$$k_{x'}(\Lambda') = k_x(\Lambda) + \sum_{i=1}^{m} \varepsilon_i = k_x(\Lambda), \qquad k_y(\Lambda') = k_y(\Lambda). \tag{17}$$

It follows from equation (10) for Λ' and equations (15), (16), and (17) that

$$\begin{aligned}
\mu(\Lambda) &= \mu(\Lambda') + 1 \\
&= 2k_{x'}(\Lambda') + 2k_y(\Lambda') + \frac{\varepsilon_{x'}(\Lambda') - \varepsilon_y(\Lambda')}{2} + 1 \\
&= 2k_x(\Lambda) + 2k_y(\Lambda) + \frac{-1 - \varepsilon_y(\Lambda)}{2} + 1 \\
&= 2k_x(\Lambda) + 2k_y(\Lambda) + \frac{\varepsilon_x(\Lambda) - \varepsilon_y(\Lambda)}{2}.
\end{aligned}$$

This proves Step 5.

Step 6. *Assume* (11) *and* $\#(B \cap \mathbb{R}) > 2$, *follow the arc of* B, *starting at* x, *and let* y' *be the next intersection point with* $\mathbb{R}$. *Assume* $y' > y$. *Denote by* B' *the arc in* B *from* y *to* y', *and let* $A' := [y, y']$ *(see Figure 3). If the* (α, β)*-trace* Λ' *with boundary* $\partial\Lambda' = (y, y', A', B')$ *satisfies* (10) *so does* Λ.

By Step 2 we may assume $\varepsilon_x(\Lambda) = 1$. Since the orientation of B' from y to y' is opposite to the orientation of B and the Maslov index of the arc in B from x to y' is -1, we have

$$\mu(\Lambda) = 1 - \mu(\Lambda'). \tag{18}$$

Using again the fact that the orientation of B' is opposite to the orientation of B we have

$$\varepsilon_y(\Lambda') = -\varepsilon_y(\Lambda), \qquad \varepsilon_{y'}(\Lambda') = -\varepsilon_{y'}(\Lambda) = 1. \tag{19}$$

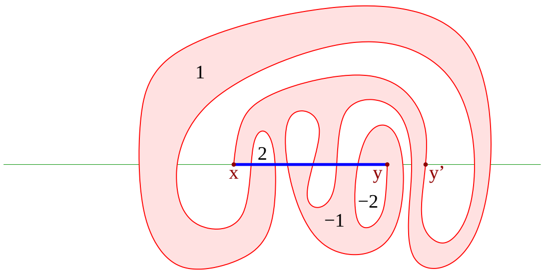

Figure 3: Maslov index and catenation: $x < y < y'$.

Now let $x_1 < x_2 < \cdots < x_m$ be all intersection points of $\mathbb{R}$ and B and let $\varepsilon_i \in \{-1, +1\}$ be the intersection index of $\mathbb{R}$ and B at x_i. Choose

$$j < k < \ell$$

such that

$$x_j = x, \qquad x_k = y, \qquad x_\ell = y'.$$

Then

$$\varepsilon_j = \varepsilon_x(\Lambda) = 1, \qquad \varepsilon_k = \varepsilon_y(\Lambda), \qquad \varepsilon_\ell = \varepsilon_{y'}(\Lambda) = -1,$$

and

$$k_x(\Lambda) = \sum_{i<j} \varepsilon_i, \qquad k_y(\Lambda) = -\sum_{i>k} \varepsilon_i.$$

For $i \neq j$ the intersection index of $\mathbb{R}$ and B' at x_i is $-\varepsilon_i$. Moreover, $k_y(\Lambda')$ is the sum of the intersection indices of $\mathbb{R}$ and B' at all points to the left of y and $k_{y'}(\Lambda')$ is minus the sum of the intersection indices of $\mathbb{R}$ and B' at all points to the right of y'. Hence

$$k_y(\Lambda') = -\sum_{i<j} \varepsilon_i - \sum_{j<i<k} \varepsilon_i, \qquad k_{y'}(\Lambda') = \sum_{i>\ell} \varepsilon_i.$$

We claim that

$$k_{y'}(\Lambda') + k_x(\Lambda) = 0, \qquad k_y(\Lambda') + k_y(\Lambda) = \frac{1 + \varepsilon_y(\Lambda)}{2}. \tag{20}$$

To see this, note that the value of the winding number w slightly to the left of x agrees with the value of w slightly to the right of y', and hence

$$0 = \sum_{i<j} \varepsilon_i + \sum_{i>\ell} \varepsilon_i = k_x(\Lambda) + k_{y'}(\Lambda').$$

This proves the first equation in (20). To prove the second equation in (20) we observe that

$$\sum_{i=1}^m \varepsilon_i = \frac{\varepsilon_x(\Lambda)+\varepsilon_y(\Lambda)}{2}$$

and hence

$$\begin{aligned}
k_y(\Lambda')+k_y(\Lambda) &= -\sum_{i<j}\varepsilon_i - \sum_{j<i<k}\varepsilon_i - \sum_{i>k}\varepsilon_i \\
&= \varepsilon_j+\varepsilon_k - \sum_{i=1}^m \varepsilon_i \\
&= \varepsilon_x(\Lambda)+\varepsilon_y(\Lambda) - \sum_{i=1}^m \varepsilon_i \\
&= \frac{\varepsilon_x(\Lambda)+\varepsilon_y(\Lambda)}{2} \\
&= \frac{1+\varepsilon_y(\Lambda)}{2}.
\end{aligned}$$

This proves the second equation in (20).

It follows from equation (10) for Λ' and equations (18), (19), and (20) that

$$\begin{aligned}
\mu(\Lambda) &= 1-\mu(\Lambda') \\
&= 1-2k_y(\Lambda')-2k_{y'}(\Lambda') - \frac{\varepsilon_y(\Lambda')-\varepsilon_{y'}(\Lambda')}{2} \\
&= 1-2k_y(\Lambda')-2k_{y'}(\Lambda') - \frac{-\varepsilon_y(\Lambda)-1}{2} \\
&= 2k_y(\Lambda)-\varepsilon_y(\Lambda)+2k_x(\Lambda)+\frac{1+\varepsilon_y(\Lambda)}{2} \\
&= 2k_x(\Lambda)+2k_y(\Lambda)+\frac{1-\varepsilon_y(\Lambda)}{2}.
\end{aligned}$$

Here the first equality follows from (18), the second equality follows from (10) for Λ', the third equality follows from (19), and the fourth equality follows from (20). This proves Step 6.

Step 7. *Equation* (8) *holds when* $\Sigma=\mathbb{C}$ *and* Λ *satisfies the arc condition.*

It follows from Steps 3-6 by induction that equation (10) holds for every (α,β)-trace $\Lambda=(x,y,\mathrm{w})$ whose boundary $\partial\Lambda=(x,y,A,B)$ satisfies (11). Hence Step 7 follows from Step 1.

Next we drop the assumption that Λ satisfies the arc condition and extend the result to planar surfaces. This requires a further three steps.

Step 8. *Equation* (8) *holds when* $\Sigma = \mathbb{C}$ *and* $x = y$.

Under these assumptions $\nu_\alpha := \partial \mathrm{w}|_{\alpha\setminus\beta}$ and $\nu_\beta := -\partial \mathrm{w}|_{\beta\setminus\alpha}$ are constant. There are four cases.

Case 1. *α is an embedded circle and β is not an embedded circle.* In this case we have $\nu_\beta \equiv 0$ and $B = \{x\}$. Moroeover, α is the boundary of a unique disc Δ_α and we assume that α is oriented as the boundary of Δ_α. Then the path $\gamma_\alpha : [0,1] \to \Sigma$ in Definition 2.1 satisfies $\gamma_\alpha(0) = \gamma_\alpha(1) = x$ and is homotopic to $\nu_\alpha \alpha$. Hence

$$m_x(\Lambda) = m_y(\Lambda) = 2\nu_\alpha = \mu(\Lambda).$$

Here the last equation follows from the fact that Λ can be obtained as the catenation of ν_α copies of the disc Δ_α.

Case 2. *α is not an embedded circle and β is an embedded circle.* This follows from Case 1 by interchanging α and β.

Case 3. *α and β are embedded circles.* In this case there is a unique pair of embedded discs Δ_α and Δ_β with boundaries α and β, respectively. Orient α and β as the boundaries of these discs. Then, for every $z \in \Sigma \setminus \alpha \cup \beta$, we have

$$\mathrm{w}(z) = \begin{cases} \nu_\alpha - \nu_\beta, & \text{for } z \in \Delta_\alpha \cap \Delta_\beta, \\ \nu_\alpha, & \text{for } z \in \Delta_\alpha \setminus \overline{\Delta}_\beta, \\ -\nu_\beta, & \text{for } z \in \Delta_\beta \setminus \overline{\Delta}_\alpha, \\ 0, & \text{for } z \in \Sigma \setminus \overline{\Delta}_\alpha \cup \overline{\Delta}_\beta. \end{cases}$$

Hence

$$m_x(\Lambda) = m_y(\Lambda) = 2\nu_\alpha - 2\nu_\beta = \mu(\Lambda).$$

Here the last equation follows from the fact Λ can be obtained as the catenation of ν_α copies of the disc Δ_α (with the orientation inherited from Σ) and ν_β copies of $-\Delta_\beta$ (with the opposite orientation).

Case 4. *Neither α nor β is an embedded circle.* Under this assumption we have $\nu_\alpha = \nu_\beta = 0$. Hence it follows from Theorem 2.4 that $\mathrm{w} = 0$ and $\Lambda = \Lambda_u$ for the constant map $u \equiv x \in \mathcal{D}(x,x)$. Thus

$$m_x(\Lambda) = m_y(\Lambda) = \mu(\Lambda) = 0.$$

This proves Step 8.

Step 9. *Equation* (8) *holds when* $\Sigma = \mathbb{C}$.

By Step 8, it suffices to assume $x \neq y$. It follows from Theorem 2.4 that every $u \in \mathcal{D}(x,y)$ is homotopic to a catentation $u = u_0 \# v$, where $u_0 \in \mathcal{D}(x,y)$ satisfies the arc condition and $v \in \mathcal{D}(y,y)$. Hence it follows from Steps 7 and 8 that

$$\begin{aligned}\mu(\Lambda_u) &= \mu(\Lambda_{u_0}) + \mu(\Lambda_v) \\ &= \frac{m_x(\Lambda_{u_0}) + m_y(\Lambda_{u_0})}{2} + m_y(\Lambda_v) \\ &= \frac{m_x(\Lambda_u) + m_y(\Lambda_u)}{2}.\end{aligned}$$

Here the last equation follows from the fact that $\mathrm{w}_u = \mathrm{w}_{u_0} + \mathrm{w}_v$ and hence $m_z(\Lambda_u) = m_z(\Lambda_{u_0}) + m_z(\Lambda_v)$ for every $z \in \alpha \cap \beta$. This proves Step 9.

Step 10. *Equation* (8) *holds when* Σ *is planar.*

Choose an element $u \in \mathcal{D}(x,y)$ such that $\Lambda_u = \Lambda$. Modifying α and β on the complement of $u(\mathbb{D})$, if necessary, we may assume without loss of generality that α and β are mebedded circles. Let $\iota : \Sigma \to \mathbb{C}$ be an orientation preserving embedding. Then $\iota_*\Lambda := \Lambda_{\iota \circ u}$ is an $(\iota(\alpha), \iota(\beta))$-trace in $\mathbb{C}$ and hence satisfies (8) by Step 9. Since $m_{\iota(x)}(\iota_*\Lambda) = m_x(\Lambda)$, $m_{\iota(y)}(\iota_*\Lambda) = m_y(\Lambda)$, and $\mu(\iota_*\Lambda) = \mu(\Lambda)$ it follows that Λ also satisfies (8). This proves Step 10 and Proposition 4.1 $\square$

Remark 4.2. Let $\Lambda = (x, y, A, B)$ be an (α, β)-trace in $\mathbb{C}$ as in Step 1 in the proof of Theorem 3.4. Thus $x < y$ are real numbers, A is the interval $[x, y]$, and B is an embedded arc with endpoints x, y which is oriented from x to y and is transverse to $\mathbb{R}$. Thus $Z := B \cap \mathbb{R}$ is a finite set. Define a map

$$f : Z \setminus \{y\} \to Z \setminus \{x\}$$

as follows. Given $z \in Z \setminus \{y\}$ walk along B towards y and let $f(z)$ be the next intersection point with $\mathbb{R}$. This map is bijective. Now let I be any of the three open intervals $(-\infty, x)$, (x, y), (y, ∞). Any arc in B from z to $f(z)$ with both endpoints in the same interval I can be removed by an isotopy of B which does not pass through x, y. Call Λ a **reduced** (α, β)**-trace** if $z \in I$ implies $f(z) \notin I$ for each of the three intervals. Then every (α, β)-trace is isotopic to a reduced (α, β')-trace and the isotopy does not effect the numbers $\mu, k_x, k_y, \varepsilon_x, \varepsilon_y$.

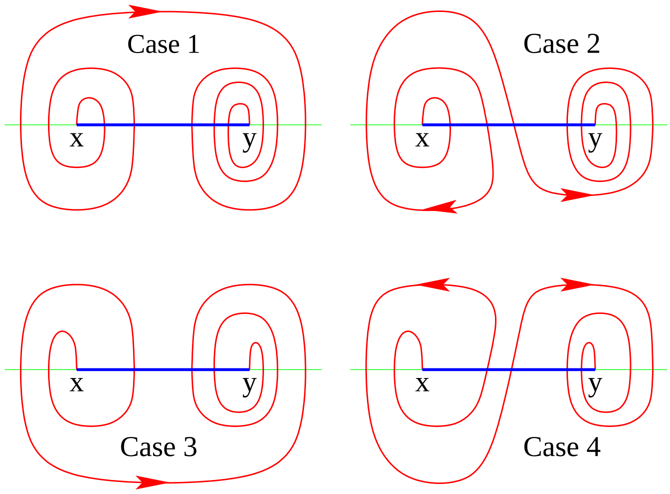

Figure 4: Reduced (α, β)-traces in $\mathbb{C}$.

Let Z^+ (respectively Z^-) denote the set of all points $z \in Z = B \cap \mathbb{R}$ where the positive tangent vectors in T_zB point up (respectively down). One can prove that every reduced (α, β)-trace satisfies one of the following conditions.

Case 1: If $z \in Z^+ \setminus \{y\}$ then $f(z) > z$. **Case 2:** $Z^- \subset [x, y]$.
Case 3: If $z \in Z^- \setminus \{y\}$ then $f(z) > z$. **Case 4:** $Z^+ \subset [x, y]$.

(Examples with $\varepsilon_x = 1$ and $\varepsilon_y = -1$ are depicted in Figure 4.) One can then show directly that the reduced (α, β)-traces satisfy equation (10). This gives rise to an alternative proof of Proposition 4.1 via case distinction.

Proof of Theorem 3.4 in the Simply Connected Case. If Σ is diffeomorphic to the 2-plane the result has been established in Proposition 4.1. Hence assume

$$\Sigma = S^2.$$

Let $u \in \mathcal{D}(x, y)$. If u is not surjective the assertion follows from the case of the complex plane (Proposition 4.1) via stereographic projection. Hence assume u is surjective and choose a regular value $z \in S^2 \setminus (\alpha \cup \beta)$ of u. Denote

$$u^{-1}(z) = \{z_1, \ldots, z_k\}.$$

For $i = 1, \ldots, k$ let $\varepsilon_i = \pm 1$ according to whether or not the differential $du(z_i) : \mathbb{C} \to T_z\Sigma$ is orientation preserving. Choose an open disc $\Delta \subset S^2$ centered at z such that

$$\bar{\Delta} \cap (\alpha \cup \beta) = \emptyset$$

and $u^{-1}(\Delta)$ is a union of open neighborhoods $U_i \subset \mathbb{D}$ of z_i with disjoint closures such that

$$u|_{U_i} : U_i \to \Delta$$

is a diffeomorphism for each i which extends to a neighborhood of $\bar{U}_i$. Now choose a continuous map $u' : \mathbb{D} \to S^2$ which agrees with u on $\mathbb{D} \setminus \bigcup_i U_i$ and restricts to a diffeomorphism from $\bar{U}_i$ to $S^2 \setminus \Delta$ for each i. Then z does not belong to the image of u' and hence equation (8) holds for u' (after smoothing along the boundaries ∂U_i). Moreover, the diffeomorphism

$$u'|_{\bar{U}_i} : \bar{U}_i \to S^2 \setminus \Delta$$

is orientation preserving if and only if $\varepsilon_i = -1$. Hence

$$\begin{aligned}
\mu(\Lambda_u) &= \mu(\Lambda_{u'}) + 4\sum_{i=1}^{k} \varepsilon_i, \\
m_x(\Lambda_u) &= m_x(\Lambda_{u'}) + 4\sum_{i=1}^{k} \varepsilon_i, \\
m_y(\Lambda_u) &= m_y(\Lambda_{u'}) + 4\sum_{i=1}^{k} \varepsilon_i.
\end{aligned}$$

By Proposition 4.1 equation (8) holds for $\Lambda_{u'}$ and hence it also holds for Λ_u. This proves Theorem 3.4 when Σ is simply connected. □

5 The Non Simply Connected Case

The key step for extending Proposition 4.1 to non-simply connected two-manifolds is the next result about lifts to the universal cover.

Proposition 5.1. *Suppose Σ is not diffeomorphic to the 2-sphere. Let $\Lambda = (x, y, \mathrm{w})$ be an (α, β)-trace and $\pi : \mathbb{C} \to \Sigma$ be a universal covering. Denote by $\Gamma \subset \mathrm{Diff}(\mathbb{C})$ the group of deck transformations. Choose an element $\tilde{x} \in \pi^{-1}(x)$ and let $\tilde{\alpha}$ and $\tilde{\beta}$ be the lifts of α and β through $\tilde{x}$. Let $\tilde{\Lambda} = (\tilde{x}, \tilde{y}, \tilde{\mathrm{w}})$ be the lift of Λ with left endpoint $\tilde{x}$. Then*

$$m_{g\tilde{x}}(\tilde{\Lambda}) + m_{g^{-1}\tilde{y}}(\tilde{\Lambda}) = 0 \tag{21}$$

for every $g \in \Gamma \setminus \{\mathrm{id}\}$.

Lemma 5.2 (**Annulus Reduction**). *Suppose Σ is not diffeomorphic to the 2-sphere. Let Λ, π, Γ, $\tilde{\Lambda}$ be as in Proposition 5.1. If*

$$m_{g\tilde{x}}(\tilde{\Lambda}) - m_{g\tilde{y}}(\tilde{\Lambda}) = m_{g^{-1}\tilde{y}}(\tilde{\Lambda}) - m_{g^{-1}\tilde{x}}(\tilde{\Lambda}) \tag{22}$$

for every $g \in \Gamma \setminus \{\mathrm{id}\}$ then equation (21) *holds for every $g \in \Gamma \setminus \{\mathrm{id}\}$.*

Proof. If (21) does not hold then there is a deck transformation $h \in \Gamma \setminus \{\mathrm{id}\}$ such that $m_{h\tilde{x}}(\tilde{\Lambda}) + m_{h^{-1}\tilde{y}}(\tilde{\Lambda}) \neq 0$. Since there can only be finitely many such $h \in \Gamma \setminus \{\mathrm{id}\}$, there is an integer $k \geq 1$ such that $m_{h^k\tilde{x}}(\tilde{\Lambda}) + m_{h^{-k}\tilde{y}}(\tilde{\Lambda}) \neq 0$ and $m_{h^\ell\tilde{x}}(\tilde{\Lambda}) + m_{h^{-\ell}\tilde{y}}(\tilde{\Lambda}) = 0$ for every integer $\ell > k$. Define $g := h^k$. Then

$$m_{g\tilde{x}}(\tilde{\Lambda}) + m_{g^{-1}\tilde{y}}(\tilde{\Lambda}) \neq 0 \tag{23}$$

and $m_{g^k\tilde{x}}(\tilde{\Lambda}) + m_{g^{-k}\tilde{y}}(\tilde{\Lambda}) = 0$ for every integer $k \in \mathbb{Z} \setminus \{-1, 0, 1\}$. Define

$$\Sigma_0 := \mathbb{C}/\Gamma_0, \qquad \Gamma_0 := \left\{ g^k \,|\, k \in \mathbb{Z} \right\}.$$

Then Σ_0 is diffeomorphic to the annulus. Let $\pi_0 : \mathbb{C} \to \Sigma_0$ be the obvious projection, define $\alpha_0 := \pi_0(\tilde{\alpha})$, $\beta_0 := \pi_0(\tilde{\beta})$, and let $\Lambda_0 := (x_0, y_0, \mathrm{w}_0)$ be the (α_0, β_0)-trace in Σ_0 with $x_0 := \pi_0(\tilde{x})$, $y_0 := \pi_0(\tilde{y})$, and

$$\mathrm{w}_0(z_0) := \sum_{\tilde{z} \in \pi_0^{-1}(z_0)} \tilde{\mathrm{w}}(\tilde{z}), \qquad z_0 \in \Sigma_0 \setminus (\alpha_0 \cup \beta_0).$$

Then

$$m_{x_0}(\Lambda_0) = m_{\tilde{x}}(\tilde{\Lambda}) + \sum_{k \in \mathbb{Z} \setminus \{0\}} m_{g^k\tilde{x}}(\tilde{\Lambda}),$$
$$m_{y_0}(\Lambda_0) = m_{\tilde{y}}(\tilde{\Lambda}) + \sum_{k \in \mathbb{Z} \setminus \{0\}} m_{g^{-k}\tilde{y}}(\tilde{\Lambda}).$$

By Proposition 4.1 both $\tilde{\Lambda}$ and Λ_0 satisfy equation (8) and they have the same Viterbo–Maslov index. Hence

$$\begin{aligned} 0 &= \mu(\Lambda_0) - \mu(\tilde{\Lambda}) \\ &= \frac{m_{x_0}(\Lambda_0) + m_{y_0}(\Lambda_0)}{2} - \frac{m_{\tilde{x}}(\tilde{\Lambda}) + m_{\tilde{y}}(\tilde{\Lambda})}{2} \\ &= \frac{1}{2} \sum_{k \neq 0} \Big(m_{g^k\tilde{x}}(\tilde{\Lambda}) + m_{g^{-k}\tilde{y}}(\tilde{\Lambda}) \Big) \\ &= m_{g\tilde{x}}(\tilde{\Lambda}) + m_{g^{-1}\tilde{y}}(\tilde{\Lambda}). \end{aligned}$$

Here the last equation follows from (22). This contradicts (23) and proves Lemma 5.2. $\square$

Lemma 5.3. *Suppose Σ is not diffeomorphic to the 2-sphere. Let Λ, π, Γ, $\tilde{\Lambda}$ be as in Proposition 5.1 and denote $\nu_{\tilde{\alpha}} := \partial\tilde{\mathrm{w}}|_{\tilde{\alpha}\setminus\tilde{\beta}}$ and $\nu_{\tilde{\beta}} := -\partial\tilde{\mathrm{w}}|_{\tilde{\beta}\setminus\tilde{\alpha}}$. Choose smooth paths*

$$\gamma_{\tilde{\alpha}} : [0,1] \to \tilde{\alpha}, \qquad \gamma_{\tilde{\beta}} : [0,1] \to \tilde{\beta}$$

from $\gamma_{\tilde{\alpha}}(0) = \gamma_{\tilde{\beta}}(0) = \tilde{x}$ to $\gamma_{\tilde{\alpha}}(1) = \gamma_{\tilde{\beta}}(1) = \tilde{y}$ such that $\gamma_{\tilde{\alpha}}$ is an immersion when $\nu_{\tilde{\alpha}} \not\equiv 0$ and constant when $\nu_{\tilde{\alpha}} \equiv 0$, the same holds for $\gamma_{\tilde{\beta}}$, and

$$\nu_{\tilde{\alpha}}(\tilde{z}) = \deg(\gamma_{\tilde{\alpha}}, \tilde{z}) \quad \text{for} \quad \tilde{z} \in \tilde{\alpha} \setminus \{\tilde{x}, \tilde{y}\},$$
$$\nu_{\tilde{\beta}}(\tilde{z}) = \deg(\gamma_{\tilde{\beta}}, \tilde{z}) \quad \text{for} \quad \tilde{z} \in \tilde{\beta} \setminus \{\tilde{x}, \tilde{y}\}.$$

Define

$$\tilde{A} := \gamma_{\tilde{\alpha}}([0,1]), \qquad \tilde{B} := \gamma_{\tilde{\beta}}([0,1]).$$

Then, for every $g \in \Gamma$, we have

$$g\tilde{x} \in \tilde{A} \qquad \Longleftrightarrow \qquad g^{-1}\tilde{y} \in \tilde{A}, \tag{24}$$
$$g\tilde{x} \notin \tilde{A} \quad \text{and} \quad g\tilde{y} \notin \tilde{A} \qquad \Longleftrightarrow \qquad \tilde{A} \cap g\tilde{A} = \emptyset, \tag{25}$$
$$g\tilde{x} \in \tilde{A} \quad \text{and} \quad g\tilde{y} \in \tilde{A} \qquad \Longleftrightarrow \qquad g = \mathrm{id}. \tag{26}$$

The same holds with $\tilde{A}$ replaced by $\tilde{B}$.

Proof. If α is a contractible embedded circle or not an embedded circle at all we have $\tilde{A} \cap g\tilde{A} = \emptyset$ whenever $g \neq \mathrm{id}$ and this implies (24), (25) and (26). Hence assume α is a noncontractible embedded circle. Then we may also assume, without loss of generality, that $\pi(\mathbb{R}) = \alpha$, the map $\tilde{z} \mapsto \tilde{z} + 1$ is a deck transformation, π maps the interval $[0,1)$ bijectively onto α, and $\tilde{x}, \tilde{y} \in \mathbb{R} = \tilde{\alpha}$ with $\tilde{x} < \tilde{y}$. Thus $\tilde{A} = [\tilde{x}, \tilde{y}]$ and, for every $k \in \mathbb{Z}$,

$$\tilde{x} + k \in [\tilde{x}, \tilde{y}] \qquad \Longleftrightarrow \qquad 0 \leq k \leq \tilde{y} - \tilde{x} \qquad \Longleftrightarrow \qquad \tilde{y} - k \in [\tilde{x}, \tilde{y}].$$

Similarly, we have

$$\tilde{x} + k, \tilde{y} + k \notin [\tilde{x}, \tilde{y}] \qquad \Longleftrightarrow \qquad [\tilde{x} + k, \tilde{y} + k] \cap [\tilde{x}, \tilde{y}] = \emptyset$$

and

$$\tilde{x} + k, \tilde{y} + k \in [\tilde{x}, \tilde{y}] \qquad \Longleftrightarrow \qquad [\tilde{x} + k, \tilde{y} + k] \subset [\tilde{x}, \tilde{y}] \qquad \Longleftrightarrow \qquad k = 0.$$

This proves (24), (25), and (26) for the deck transformation $\tilde{z} \mapsto \tilde{z} + k$. If g is any other deck transformation, then we have

$$\tilde{\alpha} \cap g\tilde{\alpha} = \emptyset$$

and so (24), (25), and (26) are trivially satisfied. This proves Lemma 5.3. □

Lemma 5.4 (**Winding Number Comparison**). *Suppose Σ is not diffeomorphic to the 2-sphere. Let Λ, π, Γ, $\tilde{\Lambda}$ be as in Proposition 5.1, and let $\tilde{A}, \tilde{B} \subset \mathbb{C}$ be as in Lemma 5.3. Then the following holds.*

(i) *Equation (22) holds for every $g \in \Gamma$ that satisfies $g\tilde{x}, g\tilde{y} \notin \tilde{A} \cup \tilde{B}$.*

(ii) *If Λ satisfies the arc condition then (21) holds for every $g \in \Gamma \setminus \{\mathrm{id}\}$.*

Proof. We prove (i). Let $g \in \Gamma$ such that $g\tilde{x}, g\tilde{y} \notin \tilde{A} \cup \tilde{B}$ and let $\gamma_{\tilde{\alpha}}, \gamma_{\tilde{\beta}}$ be as in Lemma 5.3. Then $\tilde{\mathrm{w}}(\tilde{z})$ is the winding number of the loop $\gamma_{\tilde{\alpha}} - \gamma_{\tilde{\beta}}$ about the point $\tilde{z} \in \mathbb{C} \setminus (\tilde{A} \cup \tilde{B})$. Moreover, the paths

$$g\gamma_{\tilde{\alpha}} : [0,1] \to \mathbb{C}, \qquad g\gamma_{\tilde{\beta}} : [0,1] \to \mathbb{C}$$

connect the points $g\tilde{x}, g\tilde{y} \in \mathbb{C} \setminus (\tilde{A} \cup \tilde{B})$. Hence

$$\tilde{\mathrm{w}}(g\tilde{y}) - \tilde{\mathrm{w}}(g\tilde{x}) = (\gamma_{\tilde{\alpha}} - \gamma_{\tilde{\beta}}) \cdot g\gamma_{\tilde{\alpha}} = (\gamma_{\tilde{\alpha}} - \gamma_{\tilde{\beta}}) \cdot g\gamma_{\tilde{\beta}}.$$

Similarly with g replaced by g^{-1}. Moreover, it follows from Lemma 5.3, that

$$\tilde{A} \cap g\tilde{A} = \emptyset, \qquad \tilde{B} \cap g^{-1}\tilde{B} = \emptyset.$$

Hence

$$\begin{aligned}
\tilde{\mathrm{w}}(g\tilde{y}) - \tilde{\mathrm{w}}(g\tilde{x}) &= \big(\gamma_{\tilde{\alpha}} - \gamma_{\tilde{\beta}}\big) \cdot g\gamma_{\tilde{\alpha}} \\
&= g\gamma_{\tilde{\alpha}} \cdot \gamma_{\tilde{\beta}} \\
&= \gamma_{\tilde{\alpha}} \cdot g^{-1}\gamma_{\tilde{\beta}} \\
&= \big(\gamma_{\tilde{\alpha}} - \gamma_{\tilde{\beta}}\big) \cdot g^{-1}\gamma_{\tilde{\beta}} \\
&= \tilde{\mathrm{w}}(g^{-1}\tilde{y}) - \tilde{\mathrm{w}}(g^{-1}\tilde{x})
\end{aligned}$$

Here we have used the fact that every $g \in \Gamma$ is an orientation preserving diffeomorphism of $\mathbb{C}$. Thus we have proved that

$$\tilde{\mathrm{w}}(g\tilde{x}) + \tilde{\mathrm{w}}(g^{-1}\tilde{y}) = \tilde{\mathrm{w}}(g\tilde{y}) + \tilde{\mathrm{w}}(g^{-1}\tilde{x}).$$

Since $g\tilde{x}, g\tilde{y} \notin \tilde{A} \cup \tilde{B}$, we have

$$m_{g\tilde{x}}(\tilde{\Lambda}) = 4\tilde{\mathrm{w}}(g\tilde{x}), \qquad m_{g^{-1}\tilde{y}}(\tilde{\Lambda}) = 4\tilde{\mathrm{w}}(g^{-1}\tilde{y}),$$

and the same identities hold with g replaced by g^{-1}. This proves (i).

We prove (ii). If Λ satisfies the arc condition then $g\tilde{A} \cap \tilde{A} = \emptyset$ and $g\tilde{B} \cap \tilde{B} = \emptyset$ for every $g \in \Gamma \setminus \{\mathrm{id}\}$. In particular, for every $g \in \Gamma \setminus \{\mathrm{id}\}$, we have $g\tilde{x}, g\tilde{y} \notin \tilde{A} \cup \tilde{B}$ and hence (22) holds by (i). Hence it follows from Lemma 5.2 that (21) holds for every $g \in \Gamma \setminus \{\mathrm{id}\}$. This proves Lemma 5.4. □

The next lemma deals with (α, β)-traces connecting a point $x \in \alpha \cap \beta$ to itself. An example on the annulus is depicted in Figure 5.

Lemma 5.5 (**Isotopy Argument**). *Suppose Σ is not diffeomorphic to the 2-sphere. Let Λ, π, Γ, $\tilde{\Lambda}$ be as in Proposition 5.1. Suppose that there is a deck transformation $g_0 \in \Gamma \setminus \{\mathrm{id}\}$ such that $\tilde{y} = g_0\tilde{x}$. Then Λ has Viterbo–Maslov index zero and $m_{g\tilde{x}}(\tilde{\Lambda}) = 0$ for every $g \in \Gamma \setminus \{\mathrm{id}, g_0\}$.*

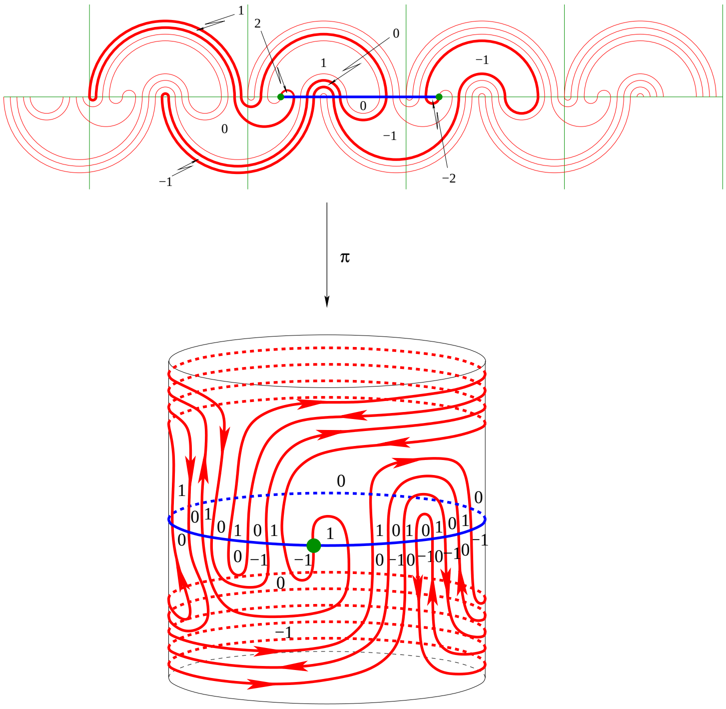

Figure 5: An (α, β)-trace on the annulus with $x = y$.

Proof. By assumption, we have $\tilde{\alpha} = g_0\tilde{\alpha}$ and $\tilde{\beta} = g_0\tilde{\beta}$. Hence α and β are noncontractible embedded circles and some iterate of α is homotopic to some iterate of β. Hence, by Lemma A.4, α must be homotopic to β (with some orientation). Hence we may assume, without loss of generality, that $\pi(\mathbb{R}) = \alpha$, the map $\tilde{z} \mapsto \tilde{z} + 1$ is a deck transformation, π maps the interval $[0, 1)$ bijectively onto α, $\mathbb{R} = \tilde{\alpha}$, $\tilde{x} = 0 \in \tilde{\alpha} \cap \tilde{\beta}$, $\tilde{\beta} = \tilde{\beta} + 1$, and that $\tilde{y} = \ell > 0$ is an integer. Then g_0 is the translation

$$g_0(\tilde{z}) = \tilde{z} + \ell.$$

Let $\tilde{A} := [0, \ell] \subset \tilde{\alpha}$ and let $\tilde{B} \subset \tilde{\beta}$ be the arc connecting 0 to ℓ. Then, for $\tilde{z} \in \mathbb{C} \setminus (\tilde{A} \cup \tilde{B})$, the integer $\tilde{\mathrm{w}}(\tilde{z})$ is the winding number of $\tilde{A} - \tilde{B}$ about $\tilde{z}$. Define the projection $\pi_0 : \mathbb{C} \to \mathbb{C}$ by

$$\pi_0(\tilde{z}) := e^{2\pi\mathbf{i}\tilde{z}/k},$$

denote $\alpha_0 := \pi_0(\tilde{\alpha}) = S^1$ and $\beta_0 := \pi(\tilde{\beta})$, and let $\Lambda_0 = (1, 1, \mathrm{w}_0)$ be the induced (α_0, β_0)-trace in $\mathbb{C}$ with $\mathrm{w}_0(z) := \sum_{\tilde{z} \in \pi^{-1}(z)} \tilde{\mathrm{w}}(\tilde{z})$. Then α_0 and β_0 are embedded circles and have the winding number ℓ about zero. Hence it follows from Step 8, Case 3 in the proof of Proposition 4.1 that Λ_0 has Viterbo–Maslov index zero and satisfies $m_{x_0}(\Lambda_0) + m_{y_0}(\Lambda_0) = 2\mu(\Lambda_0) = 0$. Hence $\tilde{\Lambda}$ also has Viterbo–Maslov index zero.

It remains to prove that $m_{g\tilde{x}}(\tilde{\Lambda}) = 0$ for every $g \in \Gamma \setminus \{\mathrm{id}, g_0\}$. To see this we use the fact that the embedded loops α and β are homotopic with fixed endpoint x. Hence, by a Theorem of Epstein, they are isotopic with fixed basepoint x (see [2, Theorem 4.1]). Thus there exists a smooth map $f : \mathbb{R}/\mathbb{Z} \times [0, 1] \to \Sigma$ such that

$$f(s, 0) \in \alpha, \qquad f(s, 1) \in \beta, \qquad f(0, t) = x,$$

for all $s \in \mathbb{R}/\mathbb{Z}$ and $t \in [0, 1]$, and the map $\mathbb{R}/\mathbb{Z} \to \Sigma : s \mapsto f(s, t)$ is an embedding for every $s \in [0, 1]$. Lift this homotopy to the universal cover to obtain a map $\tilde{f} : \mathbb{R} \times [0, 1] \to \mathbb{C}$ such that $\pi \circ \tilde{f} = f$ and

$$\tilde{f}(s, 0) \in [0, 1], \quad \tilde{f}(s, 1) \in \tilde{B}_1, \quad \tilde{f}(0, t) = \tilde{x}, \quad \tilde{f}(s + 1, t) = \tilde{f}(s, t) + 1$$

for all $s \in \mathbb{R}$ and $t \in [0, 1]$. Here $\tilde{B}_1 \subset \tilde{B}$ denotes the arc in $\tilde{B}$ from 0 to 1. Since the map $\mathbb{R}/\mathbb{Z} \to \Sigma : s \mapsto f(s, t)$ is injective for every t, we have

$$g\tilde{x} \notin \{\tilde{x}, \tilde{x} + 1, \ldots, \tilde{x} + \ell\} \qquad \Longrightarrow \qquad g\tilde{x} \notin \tilde{f}([0, \ell] \times [0, 1])$$

for every every $g \in \Gamma$. Now choose a smooth map $\tilde{u} : \mathbb{D} \to \mathbb{C}$ with $\Lambda_{\tilde{u}} = \tilde{\Lambda}$ (see Theorem 2.4). Define the homotopy $F_{\tilde{u}} : [0, \ell] \times [0, 1] \to \mathbb{C}$ by $F_{\tilde{u}}(s, t) := \tilde{u}(-\cos(\pi s/\ell), t \sin(\pi s/\ell))$. Then, by Theorem 2.4, $F_{\tilde{u}}$ is homotopic to $\tilde{f}|_{[0,\ell]\times[0,1]}$ subject to the boundary conditions $\tilde{f}(s, 0) \in \tilde{\alpha} = \mathbb{R}$, $\tilde{f}(s, 1) \in \tilde{\beta}$, $\tilde{f}(0, t) = \tilde{x}$, $\tilde{f}(\ell, t) = \tilde{y}$. Hence, for every $\tilde{z} \in \mathbb{C} \setminus (\tilde{\alpha} \cup \tilde{\beta})$, we have

$$\tilde{\mathrm{w}}(\tilde{z}) = \deg(\tilde{u}, z) = \deg(F_{\tilde{u}}, \tilde{z}) = \deg(\tilde{f}, \tilde{z}).$$

In particular, choosing $\tilde{z}$ near $g\tilde{x}$, we find $m_{g\tilde{x}}(\tilde{\Lambda}) = 4 \deg(\tilde{f}, g\tilde{x}) = 0$ for every $g \in \Gamma$ that is not one of the translations $\tilde{z} \mapsto \tilde{z} + k$ for $k = 0, 1, \ldots, \ell$. This proves the assertion in the case $\ell = 1$.

If $\ell > 1$ it remains to prove $m_k(\tilde{\Lambda}) = 0$ for $k = 1, \ldots, \ell - 1$. To see this, let $\tilde{A}_1 := [0, 1]$, $\tilde{B}_1 \subset \tilde{B}$ be the arc from 0 to 1, $\tilde{\mathrm{w}}_1(\tilde{z})$ be the winding number of $\tilde{A}_1 - \tilde{B}_1$ about $\tilde{z} \in \mathbb{C} \setminus (\tilde{A}_1 \cup \tilde{B}_1)$, and define $\tilde{\Lambda}_1 := (0, 1, \tilde{\mathrm{w}}_1)$. Then, by what we have already proved, the $(\tilde{\alpha}, \tilde{\beta})$-trace $\tilde{\Lambda}_1$ satisfies $m_{g\tilde{x}}(\tilde{\Lambda}_1) = 0$ for every $g \in \Gamma$ other than the translations by 0 or 1. In particular, we have $m_j(\tilde{\Lambda}_1) = 0$ for every $j \in \mathbb{Z} \setminus \{0, 1\}$ and also $m_0(\tilde{\Lambda}_1) + m_1(\tilde{\Lambda}_1) = 2\mu(\tilde{\Lambda}_1) = 0$. Since $\tilde{\mathrm{w}}(\tilde{z}) = \sum_{j=0}^{\ell-1} \tilde{\mathrm{w}}_1(\tilde{z} - j)$ for $\tilde{z} \in \mathbb{C} \setminus (\tilde{A} \cup \tilde{B})$, we obtain

$$m_k(\tilde{\Lambda}) = \sum_{j=0}^{\ell-1} m_{k-j}(\tilde{\Lambda}_1) = 0$$

for every $k \in \mathbb{Z} \setminus \{0, \ell\}$. This proves Lemma 5.5. □

The next example shows that Lemma 5.4 cannot be strengthened to assert the identity $m_{g\tilde{x}}(\tilde{\Lambda}) = 0$ for every $g \in \Gamma$ with $g\tilde{x}, g\tilde{y} \notin \tilde{A} \cup \tilde{B}$.

Example 5.6. Figure 6 depicts an (α, β)-trace $\Lambda = (x, y, \mathrm{w})$ on the annulus $\Sigma = \mathbb{C}/\mathbb{Z}$ that has Viterbo–Maslov index one and satisfies the arc condition. The lift satisfies $m_{\tilde{x}}(\tilde{\Lambda}) = -3$, $m_{\tilde{x}+1}(\tilde{\Lambda}) = 4$, $m_{\tilde{y}}(\tilde{\Lambda}) = 5$, and $m_{\tilde{y}-1}(\tilde{\Lambda}) = -4$. Thus $m_x(\Lambda) = m_y(\Lambda) = 1$.

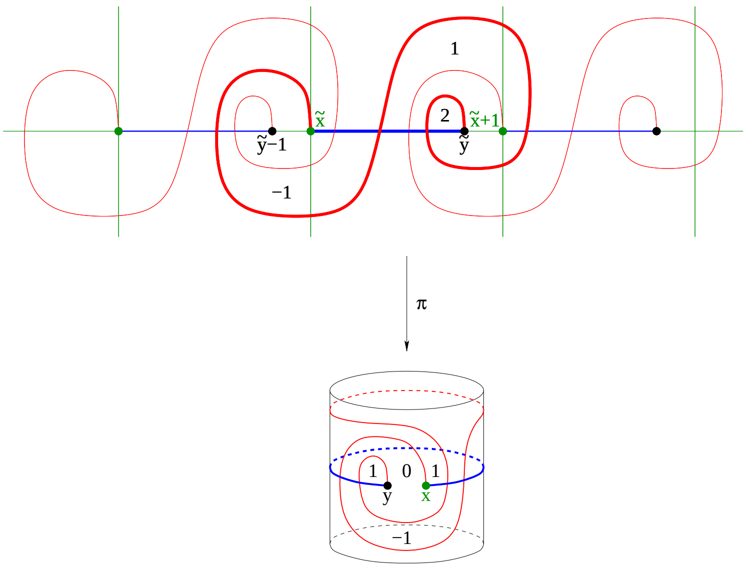

Figure 6: An (α, β)-trace on the annulus satisfying the arc condition.

Proof of Proposition 5.1. The proof has five steps.

Step 1. *Let $\tilde{A}, \tilde{B} \subset \mathbb{C}$ be as in Lemma 5.3 and let $g \in \Gamma$ such that*

$$g\tilde{x} \in \tilde{A} \setminus \tilde{B}, \qquad g\tilde{y} \notin \tilde{A} \cup \tilde{B}.$$

(An example is depicted in Figure 7.) Then (22) *holds.*

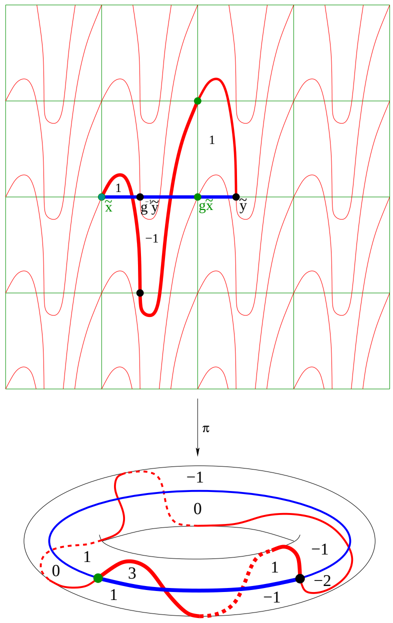

Figure 7: An (α, β)-trace on the torus not satisfying the arc condition.

The proof is a refinement of the winding number comparison argument in Lemma 5.4. Since $g\tilde{x} \notin \tilde{B}$ we have $g \neq \mathrm{id}$ and, since $\tilde{x}, g\tilde{x} \in \tilde{A} \subset \tilde{\alpha}$, it follows that α is a noncontractible embedded circle. Hence we may choose the universal covering $\pi : \mathbb{C} \to \Sigma$ and the lifts $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\Lambda}$ such that $\pi(\mathbb{R}) = \alpha$, the map $\tilde{z} \mapsto \tilde{z} + 1$ is a deck transformation, the projection π maps the interval $[0, 1)$ bijectively onto α, and

$$\tilde{\alpha} = \mathbb{R}, \qquad \tilde{x} = 0 \in \tilde{\alpha} \cap \tilde{\beta}, \qquad \tilde{y} > 0.$$

By assumption and Lemma 5.3 there is an integer k such that

$$0 < k < \tilde{y}, \qquad g\tilde{x} = k, \qquad g^{-1}\tilde{y} = \tilde{y} - k.$$

Thus g is the deck transformation $\tilde{z} \mapsto \tilde{z} + k$.

Since $g\tilde{x} \notin \tilde{B}$ and $g\tilde{y} \notin \tilde{B}$ it follows from Lemma 5.3 that $g^{-1}\tilde{y} \notin \tilde{B}$ and $g^{-1}\tilde{x} \notin \tilde{B}$ and hence, again by Lemma 5.3, we have

$$\tilde{B} \cap g\tilde{B} = \tilde{B} \cap g^{-1}\tilde{B} = \emptyset.$$

With $\gamma_{\tilde{\alpha}}$ and $\gamma_{\tilde{\beta}}$ chosen as in Lemma 5.3, this implies

$$\gamma_{\tilde{\beta}} \cdot (\gamma_{\tilde{\beta}} - k) = (\gamma_{\tilde{\beta}} + k) \cdot \gamma_{\tilde{\beta}} = 0. \tag{27}$$

Since $k, -k, \tilde{y}+k, \tilde{y}-k \notin \tilde{B}$, there exists a constant $\varepsilon > 0$ such that

$$-\varepsilon \le t \le \varepsilon \qquad \Longrightarrow \qquad k + \mathbf{i}t, \;\; -k + \mathbf{i}t, \;\; \tilde{y} - k + \mathbf{i}t, \;\; \tilde{y} + k + \mathbf{i}t \notin \tilde{B}.$$

The paths $g\gamma_{\tilde{\alpha}} \pm \mathbf{i}\varepsilon$ and $g\gamma_{\tilde{\beta}} \pm \mathbf{i}\varepsilon$ both connect the point $g\tilde{x} \pm \mathbf{i}\varepsilon$ to $g\tilde{y} \pm \mathbf{i}\varepsilon$. Likewise, the paths $g^{-1}\gamma_{\tilde{\alpha}} \pm \mathbf{i}\varepsilon$ and $g^{-1}\gamma_{\tilde{\beta}} \pm \mathbf{i}\varepsilon$ both connect the point $g^{-1}\tilde{x} \pm \mathbf{i}\varepsilon$ to $g^{-1}\tilde{y} \pm \mathbf{i}\varepsilon$. Hence

$$\begin{aligned}
\tilde{\mathrm{w}}(g\tilde{y} \pm \mathbf{i}\varepsilon) - \tilde{\mathrm{w}}(g\tilde{x} \pm \mathbf{i}\varepsilon) &= (\gamma_{\tilde{\alpha}} - \gamma_{\tilde{\beta}}) \cdot (g\gamma_{\tilde{\alpha}} \pm \mathbf{i}\varepsilon) \\
&= (\gamma_{\tilde{\alpha}} - \gamma_{\tilde{\beta}}) \cdot (\gamma_{\tilde{\alpha}} + k \pm \mathbf{i}\varepsilon) \\
&= (\gamma_{\tilde{\alpha}} + k \pm \mathbf{i}\varepsilon) \cdot \gamma_{\tilde{\beta}} \\
&= \gamma_{\tilde{\alpha}} \cdot (\gamma_{\tilde{\beta}} - k \mp \mathbf{i}\varepsilon) \\
&= (\gamma_{\tilde{\alpha}} - \gamma_{\tilde{\beta}}) \cdot (\gamma_{\tilde{\beta}} - k \mp \mathbf{i}\varepsilon) \\
&= (\gamma_{\tilde{\alpha}} - \gamma_{\tilde{\beta}}) \cdot (g^{-1}\gamma_{\tilde{\beta}} \mp \mathbf{i}\varepsilon) \\
&= \tilde{\mathrm{w}}(g^{-1}\tilde{y} \mp \mathbf{i}\varepsilon) - \tilde{\mathrm{w}}(g^{-1}\tilde{x} \mp \mathbf{i}\varepsilon).
\end{aligned}$$

Here the last but one equation follows from (27). Thus we have proved

$$\begin{aligned}
\tilde{\mathrm{w}}(g\tilde{x} + \mathbf{i}\varepsilon) + \tilde{\mathrm{w}}(g^{-1}\tilde{y} - \mathbf{i}\varepsilon) = \tilde{\mathrm{w}}(g^{-1}\tilde{x} - \mathbf{i}\varepsilon) + \tilde{\mathrm{w}}(g\tilde{y} + \mathbf{i}\varepsilon), \\
\tilde{\mathrm{w}}(g\tilde{x} - \mathbf{i}\varepsilon) + \tilde{\mathrm{w}}(g^{-1}\tilde{y} + \mathbf{i}\varepsilon) = \tilde{\mathrm{w}}(g^{-1}\tilde{x} + \mathbf{i}\varepsilon) + \tilde{\mathrm{w}}(g\tilde{y} - \mathbf{i}\varepsilon).
\end{aligned} \tag{28}$$

Since

$$\begin{aligned}
m_{g\tilde{x}}(\tilde{\Lambda}) &= 2\tilde{\mathrm{w}}(g\tilde{x} + \mathbf{i}\varepsilon) + 2\tilde{\mathrm{w}}(g\tilde{x} - \mathbf{i}\varepsilon), \\
m_{g\tilde{y}}(\tilde{\Lambda}) &= 2\tilde{\mathrm{w}}(g\tilde{y} + \mathbf{i}\varepsilon) + 2\tilde{\mathrm{w}}(g\tilde{y} - \mathbf{i}\varepsilon), \\
m_{g^{-1}\tilde{x}}(\tilde{\Lambda}) &= 2\tilde{\mathrm{w}}(g^{-1}\tilde{x} + \mathbf{i}\varepsilon) + 2\tilde{\mathrm{w}}(g^{-1}\tilde{x} - \mathbf{i}\varepsilon), \\
m_{g^{-1}\tilde{y}}(\tilde{\Lambda}) &= 2\tilde{\mathrm{w}}(g^{-1}\tilde{y} + \mathbf{i}\varepsilon) + 2\tilde{\mathrm{w}}(g^{-1}\tilde{y} - \mathbf{i}\varepsilon),
\end{aligned}$$

Step 1 follows by taking the sum of the two equations in (28).

Step 2. *Let $\tilde{A}, \tilde{B} \subset \mathbb{C}$ be as in Lemma 5.3 and let $g \in \Gamma$. Suppose that either $g\tilde{x}, g\tilde{y} \notin \tilde{A}$ or $g\tilde{x}, g\tilde{y} \notin \tilde{B}$. Then* (22) *holds.*

If $g\tilde{x}, g\tilde{y} \notin \tilde{A} \cup \tilde{B}$ the assertion follows from Lemma 5.4. If $g\tilde{x} \in \tilde{A} \setminus \tilde{B}$ and $g\tilde{y} \notin \tilde{A} \cup \tilde{B}$ the assertion follows from Step 1. If $g\tilde{x} \notin \tilde{A} \cup \tilde{B}$ and $g\tilde{y} \in \tilde{A} \setminus \tilde{B}$ the assertion follows from Step 1 by interchanging $\tilde{x}$ and $\tilde{y}$. Namely, (22) holds for $\tilde{\Lambda}$ if and only if it holds for the $(\tilde{\alpha}, \tilde{\beta})$-trace $-\tilde{\Lambda} := (\tilde{y}, \tilde{x}, -\tilde{\mathrm{w}})$. This covers the case $g\tilde{x}, g\tilde{y} \notin \tilde{B}$. If $g\tilde{x}, g\tilde{y} \notin \tilde{A}$ the assertion follows by interchanging $\tilde{A}$ and $\tilde{B}$. Namely, (22) holds for $\tilde{\Lambda}$ if and only if it holds for the $(\tilde{\beta}, \tilde{\alpha})$-trace $\tilde{\Lambda}^* := (\tilde{x}, \tilde{y}, -\tilde{\mathrm{w}})$. This proves Step 2.

Step 3. *Let $\tilde{A}, \tilde{B} \subset \mathbb{C}$ be as in Lemma 5.3 and let $g \in \Gamma$ such that*

$$g\tilde{x} \in \tilde{A} \setminus \tilde{B}, \qquad g\tilde{y} \in \tilde{B} \setminus \tilde{A}.$$

(An example is depicted in Figure 8.) Then (21) *holds for g and g^{-1}.*

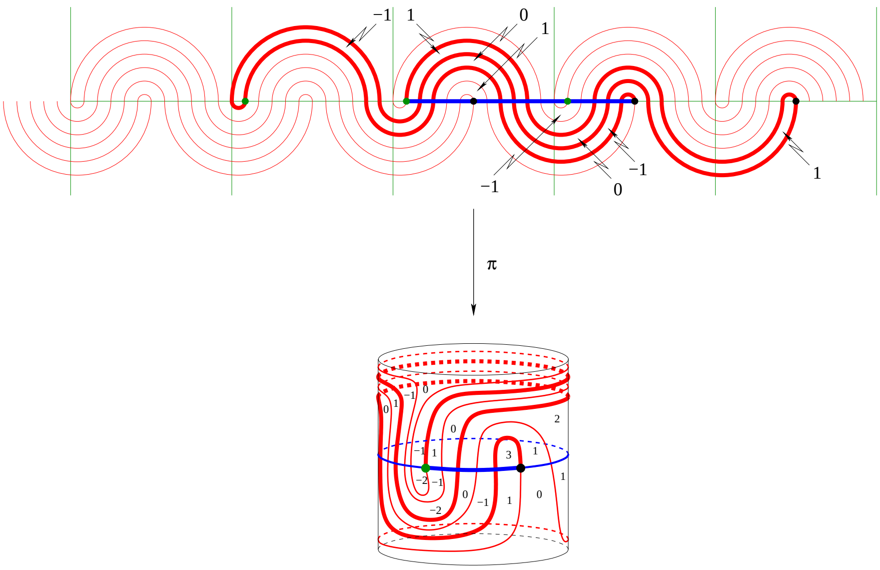

Figure 8: An (α, β)-trace on the annulus with $g\tilde{x} \in \tilde{A}$ and $g\tilde{y} \in \tilde{B}$.

Since $g\tilde{x} \notin \tilde{B}$ (and $g\tilde{y} \notin \tilde{A}$) we have $g \neq \mathrm{id}$ and, since $\tilde{x}, g\tilde{x} \in \tilde{A} \subset \tilde{\alpha}$ and $\tilde{y}, g\tilde{y} \in \tilde{B} \subset \tilde{\beta}$, it follows that $g\tilde{\alpha} = \tilde{\alpha}$ and $g\tilde{\beta} = \tilde{\beta}$. Hence α and β are noncontractible embedded circles and some iterate of α is homotopic to some iterate of β. So α is homotopic to β (with some orientation), by Lemma A.4. Hence we may choose the universal covering $\pi : \mathbb{C} \to \Sigma$ and the lifts $\tilde{\alpha}, \tilde{\beta}, \tilde{\Lambda}$

such that $\pi(\mathbb{R}) = \alpha$, the map $\tilde{z} \mapsto \tilde{z} + 1$ is a deck transformation, π maps the interval $[0, 1)$ bijectively onto α, and $\tilde{\alpha} = \mathbb{R}$, $\tilde{x} = 0 \in \tilde{\alpha} \cap \tilde{\beta}$, $\tilde{y} > 0$. Thus $\tilde{A} = [0, \tilde{y}]$ is the arc in $\tilde{\alpha}$ from 0 to $\tilde{y}$ and $\tilde{B}$ is the arc in $\tilde{\beta}$ from 0 to $\tilde{y}$. Moreover, $\tilde{\beta} = \tilde{\beta} + 1$ and the arc in $\tilde{\beta}$ from 0 to 1 is a fundamental domain for β. By assumption and Lemma 5.3 there is an integer k such that $k \in \tilde{A}$ and $-k \in \tilde{B}$. Hence $\tilde{A}$ does not contain any negative integers and $\tilde{B}$ does not contain any positive integers. Choose $k_{\tilde{A}}, k_{\tilde{B}} \in \mathbb{N}$ such that

$$\tilde{A} \cap \mathbb{Z} = \{0, 1, 2, \cdots, k_{\tilde{A}}\}, \qquad \tilde{B} \cap \mathbb{Z} = \{0, -1, -2, \cdots, -k_{\tilde{B}}\}.$$

For $0 \le k \le k_{\tilde{A}}$ let $\tilde{A}_k \subset \tilde{\alpha}$ and $\tilde{B}_k \subset \tilde{\beta}$ be the arcs from 0 to $\tilde{y} - k$ and consider the $(\tilde{\alpha}, \tilde{\beta})$-trace

$$\tilde{\Lambda}_k := (0, \tilde{y} - k, \tilde{\mathrm{w}}_k), \qquad \partial\tilde{\Lambda}_k := (0, \tilde{y} - k, \tilde{A}_k, \tilde{B}_k),$$

where $\tilde{\mathrm{w}}_k(\tilde{z})$ is the winding number of $\tilde{A}_k - \tilde{B}_k$ about $\tilde{z} \in \mathbb{C} \setminus (\tilde{A}_k \cup \tilde{B}_k)$. Note that $\tilde{\Lambda}_0 = \tilde{\Lambda}$ and

$$\tilde{B}_k \cap \mathbb{Z} = \{0, -1, -2, \cdots, -k_{\tilde{B}} - k\}.$$

We prove that, for each k, the $(\tilde{\alpha}, \tilde{\beta})$-trace $\tilde{\Lambda}_k$ satisfies

$$m_j(\tilde{\Lambda}_k) + m_{\tilde{y}-k-j}(\tilde{\Lambda}_k) = 0 \qquad \forall\, j \in \mathbb{Z} \setminus \{0\}. \tag{29}$$

If $\tilde{y}$ is an integer, then (29) follows from Lemma 5.5. Hence we may assume that $\tilde{y}$ is not an integer.

We prove equation (29) by reverse induction on k. First let $k = k_{\tilde{A}}$. Then we have $j, \tilde{y} + j \notin \tilde{A}_k$ for every $j \in \mathbb{N}$. Hence it follows from Step 2 that

$$m_j(\tilde{\Lambda}_k) + m_{\tilde{y}-k-j}(\tilde{\Lambda}_k) = m_{-j}(\tilde{\Lambda}_k) + m_{\tilde{y}-k+j}(\tilde{\Lambda}) \qquad \forall\, j \in \mathbb{N}. \tag{30}$$

Thus we can apply Lemma 5.2 to the projection of $\tilde{\Lambda}_k$ to the quotient $\mathbb{C}/\mathbb{Z}$. Hence $\tilde{\Lambda}_k$ satisfies (29).

Now fix an integer $k \in \{0, 1, \ldots, k_{\tilde{A}} - 1\}$ and suppose, by induction, that $\tilde{\Lambda}_{k+1}$ satisfies (29). Denote by $\tilde{A}' \subset \tilde{\alpha}$ and $\tilde{B}' \subset \tilde{\beta}$ the arcs from $\tilde{y} - k - 1$ to 1, and by $\tilde{A}'' \subset \tilde{\alpha}$ and $\tilde{B}'' \subset \tilde{\beta}$ the arcs from 1 to $\tilde{y} - k$. Then $\tilde{\Lambda}_k$ is the catenation of the $(\tilde{\alpha}, \tilde{\beta})$-traces

$$\begin{aligned}
\tilde{\Lambda}_{k+1} &:= (0, \tilde{y} - k - 1, \tilde{\mathrm{w}}_{k+1}), & \partial\tilde{\Lambda}_{k+1} &= (0, \tilde{y} - k - 1, \tilde{A}_{k+1}, \tilde{B}_{k+1}), \\
\tilde{\Lambda}' &:= (\tilde{y} - k - 1, 1, \tilde{\mathrm{w}}'), & \partial\tilde{\Lambda}' &= (\tilde{y} - k - 1, 1, \tilde{A}', \tilde{B}'), \\
\tilde{\Lambda}'' &:= (1, \tilde{y} - k, \tilde{\mathrm{w}}''), & \partial\tilde{\Lambda}'' &= (1, \tilde{y} - k, \tilde{A}'', \tilde{B}'').
\end{aligned}$$

Here $\tilde{w}'(\tilde{z})$ is the winding number of the loop $\tilde{A}' - \tilde{B}'$ about $\tilde{z} \in \mathbb{C} \setminus (\tilde{A}' \cup \tilde{B}')$ and simiarly for $\tilde{w}''$. Note that $\tilde{\Lambda}''$ is the shift of $\tilde{\Lambda}_{k+1}$ by 1. The catenation of $\tilde{\Lambda}_{k+1}$ and $\tilde{\Lambda}'$ is the $(\tilde{\alpha}, \tilde{\beta})$-trace from 0 to 1. Hence it has Viterbo–Maslov index zero, by Lemma 5.5. and satisfies

$$m_j(\tilde{\Lambda}_{k+1}) + m_j(\tilde{\Lambda}') = 0 \qquad \forall j \in \mathbb{Z} \setminus \{0,1\}. \tag{31}$$

Since the catenation of $\tilde{\Lambda}'$ and $\tilde{\Lambda}''$ is the $(\tilde{\alpha}, \tilde{\beta})$-trace from $\tilde{y} - k - 1$ to $\tilde{y} - k$, it also has Viterbo–Maslov index zero and satisfies

$$m_{\tilde{y}-k-j}(\tilde{\Lambda}') + m_{\tilde{y}-k-j}(\tilde{\Lambda}'') = 0 \qquad \forall j \in \mathbb{Z} \setminus \{0,1\}. \tag{32}$$

Moreover, by the induction hypothesis, we have

$$m_j(\tilde{\Lambda}_{k+1}) + m_{\tilde{y}-k-1-j}(\tilde{\Lambda}_{k+1}) = 0 \qquad \forall j \in \mathbb{Z} \setminus \{0\}. \tag{33}$$

Combining the equations (31), (32), and (33) we find

$$\begin{aligned}
m_j(\tilde{\Lambda}_k) + m_{\tilde{y}-k-j}(\tilde{\Lambda}_k) &= m_j(\tilde{\Lambda}_{k+1}) + m_j(\tilde{\Lambda}') + m_j(\tilde{\Lambda}'') \\
&\quad + m_{\tilde{y}-k-j}(\tilde{\Lambda}_{k+1}) + m_{\tilde{y}-k-j}(\tilde{\Lambda}') + m_{\tilde{y}-k-j}(\tilde{\Lambda}'') \\
&= m_j(\tilde{\Lambda}_{k+1}) + m_j(\tilde{\Lambda}') \\
&\quad + m_{\tilde{y}-k-j}(\tilde{\Lambda}') + m_{\tilde{y}-k-j}(\tilde{\Lambda}'') \\
&\quad + m_{j-1}(\tilde{\Lambda}_{k+1}) + m_{\tilde{y}-k-j}(\tilde{\Lambda}_{k+1}) \\
&= 0
\end{aligned}$$

for $j \in \mathbb{Z} \setminus \{0,1\}$. For $j = 1$ we obtain

$$\begin{aligned}
m_1(\tilde{\Lambda}_k) + m_{\tilde{y}-k-1}(\tilde{\Lambda}_k) &= m_1(\tilde{\Lambda}_{k+1}) + m_1(\tilde{\Lambda}') + m_1(\tilde{\Lambda}'') \\
&\quad + m_{\tilde{y}-k-1}(\tilde{\Lambda}_{k+1}) + m_{\tilde{y}-k-1}(\tilde{\Lambda}') + m_{\tilde{y}-k-1}(\tilde{\Lambda}'') \\
&= m_1(\tilde{\Lambda}_{k+1}) + m_{\tilde{y}-k-2}(\tilde{\Lambda}_{k+1}) \\
&\quad + m_0(\tilde{\Lambda}_{k+1}) + m_{\tilde{y}-k-1}(\tilde{\Lambda}_{k+1}) \\
&\quad + m_{\tilde{y}-k-1}(\tilde{\Lambda}') + m_1(\tilde{\Lambda}') \\
&= 2\mu(\tilde{\Lambda}_{k+1}) + 2\mu(\tilde{\Lambda}') \\
&= 0.
\end{aligned}$$

Here the last but one equation follows from equation (33) and Proposition 4.1, and the last equation follows from Lemma 5.5. Hence $\tilde{\Lambda}_k$ satisfies (29). This completes the induction argument for the proof of Step 3.

Step 4. *Let $\tilde{A}, \tilde{B} \subset \mathbb{C}$ be as in Lemma 5.3 and let $g \in \Gamma$ such that*

$$g\tilde{x} \in \tilde{A} \cap \tilde{B}, \qquad g\tilde{y} \notin \tilde{A} \cup \tilde{B}.$$

Then (21) *holds for g and g^{-1}.*

Since $g\tilde{y} \notin \tilde{A} \cup \tilde{B}$ we have $g \neq \mathrm{id}$. Since $g\tilde{x} \in \tilde{A} \cap \tilde{B}$ we have $\tilde{\alpha} = g\tilde{\alpha}$ and $\tilde{\beta} = g\tilde{\beta}$. Hence α and β are noncontractible embedded circles, and they are homotopic (with some orientation) by Lemma A.4. Thus we may choose $\pi : \mathbb{C} \to \Sigma$, $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\Lambda}$ as in Step 3. By assumption there is an integer $k \in \tilde{A} \cap \tilde{B}$. Hence $\tilde{A}$ and $\tilde{B}$ do not contain any negative integers. Choose $k_{\tilde{A}}, k_{\tilde{B}} \in \mathbb{N}$ such that

$$\tilde{A} \cap \mathbb{Z} = \{0, 1, \ldots, k_{\tilde{A}}\}, \qquad \tilde{B} \cap \mathbb{Z} = \{0, 1, \ldots, k_{\tilde{B}}\}.$$

Assume without loss of generality that $k_{\tilde{A}} \leq k_{\tilde{B}}$. For $0 \leq k \leq k_{\tilde{A}}$ denote by $\tilde{A}_k \subset \tilde{A}$ and $\tilde{B}_k \subset \tilde{B}$ the arcs from 0 to $\tilde{y} - k$ and consider the $(\tilde{\alpha}, \tilde{\beta})$-trace

$$\tilde{\Lambda}_k := (0, \tilde{y} - k, \tilde{\mathrm{w}}_k), \qquad \partial\tilde{\Lambda}_k := (0, \tilde{y} - k, \tilde{A}_k, \tilde{B}_k).$$

In this case

$$\tilde{B}_k \cap \mathbb{Z} = \{0, 1, \ldots, k_{\tilde{B}} - k\}.$$

As in Step 3, it follows by reverse induction on k that $\tilde{\Lambda}_k$ satisfies (29) for every k. We assume again that $\tilde{y}$ is not an integer. (Otherwise (29) follows from Lemma 5.5). If $k = k_{\tilde{A}}$ then $j, \tilde{y} - j \notin \tilde{A}_k$ for every $j \in \mathbb{N}$, hence it follows from Step 2 that $\tilde{\Lambda}_k$ satisfies (30), and hence it follows from Lemma 5.2 for the projection of $\tilde{\Lambda}_k$ to the annulus $\mathbb{C}/\mathbb{Z}$ that $\tilde{\Lambda}_k$ also satisfies (29). The induction step is verbatim the same as in Step 3 and will be omitted. This proves Step 4.

Step 5. *We prove the proposition.*

If both points $g\tilde{x}, g\tilde{y}$ are contained in $\tilde{A}$ (or in $\tilde{B}$) then $g = \mathrm{id}$ by Lemma 5.3, and in this case equation (22) is a tautology. If both points $g\tilde{x}, g\tilde{y}$ are not contained in $\tilde{A} \cup \tilde{B}$, equation (22) has been established in Lemma 5.4. Moreover, we can interchange $\tilde{x}$ and $\tilde{y}$ or $\tilde{A}$ and $\tilde{B}$ as in the proof of Step 2. Thus Steps 1 and 4 cover the case where precisely one of the points $g\tilde{x}, g\tilde{y}$ is contained in $\tilde{A} \cup \tilde{B}$ while Step 3 covers the case where $g \neq \mathrm{id}$ and both points $g\tilde{x}, g\tilde{y}$ are contained in $\tilde{A} \cup \tilde{B}$. This shows that equation (22) holds for every $g \in \Gamma \setminus \{\mathrm{id}\}$. Hence, by Lemma 5.2, equation (21) holds for every $g \in \Gamma \setminus \{\mathrm{id}\}$. This proves Proposition 5.1. $\square$

Proof of Theorem 3.4 in the Non Simply Connected Case. Choose a universal covering $\pi : \mathbb{C} \to \Sigma$ and let Γ, $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\Lambda} = (\tilde{x}, \tilde{y}, \tilde{\mathrm{w}})$ be as in Proposition 5.1. Then

$$m_x(\Lambda) + m_y(\Lambda) - m_{\tilde{x}}(\tilde{\Lambda}) - m_{\tilde{y}}(\tilde{\Lambda}) = \sum_{g \neq \mathrm{id}} \Big(m_{g\tilde{x}}(\tilde{\Lambda}) + m_{g^{-1}\tilde{y}}(\tilde{\Lambda}) \Big) = 0.$$

Here the last equation follows from Proposition 5.1. Hence, by Proposition 4.1, we have

$$\mu(\Lambda) = \mu(\tilde{\Lambda}) = \frac{m_{\tilde{x}}(\tilde{\Lambda}) + m_{\tilde{y}}(\tilde{\Lambda})}{2} = \frac{m_x(\Lambda) + m_y(\Lambda)}{2}.$$

This proves (8) in the case where Σ is not simply connected. □

A The Space of Paths

We assume throughout that Σ is a connected oriented smooth 2-manifold without boundary and $\alpha, \beta \subset \Sigma$ are two embedded loops. Let

$$\Omega_{\alpha,\beta} := \{x \in C^\infty([0,1], \Sigma) \,|\, x(0) \in \alpha,\ x(1) \in \beta\}$$

denote the space of paths connecting α to β.

Proposition A.1. *Assume that α and β are not contractible and that α is not isotopic to β. Then each component of $\Omega_{\alpha,\beta}$ is simply connected and hence $H^1(\Omega_{\alpha,\beta}; \mathbb{R}) = 0$.*

The proof was explained to us by David Epstein [3]. It is based on the following three lemmas. We identify $S^1 \cong \mathbb{R}/\mathbb{Z}$.

Lemma A.2. *Let $\gamma : S^1 \to \Sigma$ be a noncontractible loop and denote by*

$$\pi : \tilde{\Sigma} \to \Sigma$$

the covering generated by γ. Then $\tilde{\Sigma}$ is diffeomorphic to the cylinder.

Proof. By assumption, Σ is oriented and has a nontrivial fundamental group. By the uniformization theorem, choose a metric of constant curvature. Then the universal cover of Σ is isometric to either $\mathbb{R}^2$ with the flat metric or to the upper half space $\mathbb{H}^2$ with the hyperbolic metric. The 2-manifold $\tilde{\Sigma}$ is a quotient of the universal cover of Σ by the subgroup of the group of covering transformations generated by a single element (a translation in the case of $\mathbb{R}^2$ and a hyperbolic element of $\mathrm{PSL}(2,\mathbb{R})$ in the case of $\mathbb{H}^2$). Since γ is not contractible, this element is not the identity. Hence $\tilde{\Sigma}$ is diffeomorphic to the cylinder. □

Lemma A.3. *Let $\gamma : S^1 \to \Sigma$ be a noncontractible loop and, for $k \in \mathbb{Z}$, define $\gamma^k : S^1 \to \Sigma$ by*

$$\gamma^k(s) := \gamma(ks).$$

Then γ^k is contractible if and only if $k = 0$.

Proof. Let $\pi : \tilde{\Sigma} \to \Sigma$ be as in Lemma A.2. Then, for $k \neq 0$, the loop $\gamma^k : S^1 \to \Sigma$ lifts to a noncontractible loop in $\tilde{\Sigma}$. □

Lemma A.4. *Let $\gamma_0, \gamma_1 : S^1 \to \Sigma$ be noncontractible embedded loops and suppose that k_0, k_1 are nonzero integers such that $\gamma_0^{k_0}$ is homotopic to $\gamma_1^{k_1}$. Then either γ_1 is homotopic to γ_0 and $k_1 = k_0$ or γ_1 is homotopic to ${\gamma_0}^{-1}$ and $k_1 = -k_0$.*

Proof. Let $\pi : \tilde{\Sigma} \to \Sigma$ be the covering generated by γ_0. Then ${\gamma_0}^{k_0}$ lifts to a closed curve in $\tilde{\Sigma}$ and is homotopic to ${\gamma_1}^{k_1}$. Hence ${\gamma_1}^{k_1}$ lifts to a closed immersed curve in $\tilde{\Sigma}$. Hence there exists a nonzero integer j_1 such that ${\gamma_1}^{j_1}$ lifts to an embedding $S^1 \to \tilde{\Sigma}$. Any embedded curve in the cylinder is either contractible or is homotopic to a generator. If the lift of ${\gamma_1}^{j_1}$ were contractible it would follow that ${\gamma_0}^{k_0}$ is contractible, hence, by Lemma A.3, $k_0 = 0$ in contradiction to our assumption. Hence the lift of ${\gamma_1}^{j_1}$ to $\tilde{\Sigma}$ is not contractible. With an appropriate sign of j_1 it follows that the lift of ${\gamma_1}^{j_1}$ is homotopic to the lift of γ_0. Interchanging the roles of γ_0 and γ_1, we find that there exist nonzero integers j_0, j_1 such that

$$\gamma_0 \sim {\gamma_1}^{j_1}, \qquad \gamma_1 \sim {\gamma_0}^{j_0}$$

in $\tilde{\Sigma}$. Hence γ_0 is homotopic to ${\gamma_0}^{j_0 j_1}$ in the free loop space of $\tilde{\Sigma}$. Since the homotopy lifts to the cylinder $\tilde{\Sigma}$ and the fundamental group of $\tilde{\Sigma}$ is abelian, it follows that

$$j_0 j_1 = 1.$$

If $j_0 = j_1 = 1$ then γ_1 is homotopic to γ_0, hence $\gamma_0^{k_1}$ is homotopic to ${\gamma_0}^{k_0}$, hence ${\gamma_0}^{k_0 - k_1}$ is contractible, and hence $k_0 - k_1 = 0$, by Lemma A.3. If $j_0 = j_1 = -1$ then γ_1 is homotopic to ${\gamma_0}^{-1}$, hence $\gamma_0^{-k_1}$ is homotopic to ${\gamma_0}^{k_0}$, hence ${\gamma_0}^{k_0+k_1}$ is contractible, and hence $k_0 + k_1 = 0$, by Lemma A.3. This proves Lemma A.4. □

Proof of Proposition A.1. Orient α and β and and choose orientation preserving diffeomorphisms

$$\gamma_0 : S^1 \to \alpha, \qquad \gamma_1 : S^1 \to \beta.$$

A closed loop in $\Omega_{\alpha,\beta}$ gives rise to a map $u : S^1 \times [0,1] \to \Sigma$ such that

$$u(S^1 \times \{0\}) \subset \alpha, \qquad u(S^1 \times \{1\}) \subset \beta.$$

Let k_0 denote the degree of $u(\cdot, 0) : S^1 \to \alpha$ and k_1 denote the degree of $u(\cdot, 1) : S^1 \to \beta$. Since the homotopy class of a map $S^1 \to \alpha$ or a map $S^1 \to \beta$ is determined by the degree we may assume, without loss of generality, that

$$u(s, 0) = \gamma_0(k_0 s), \qquad u(s, 1) = \gamma_1(k_1 s).$$

If one of the integers k_0, k_1 vanishes, so does the other, by Lemma A.3. If they are both nonzero then γ_1 is homotopic to either γ_0 or γ_0^{-1}, by Lemma A.4. Hence γ_1 is isotopic to either γ_0 or γ_0^{-1}, by [2, Theorem 4.1]. Hence α is isotopic to β, in contradiction to our assumption. This shows that

$$k_0 = k_1 = 0.$$

With this established it follows that the map $u : S^1 \times [0,1] \to \Sigma$ factors through a map $v : S^2 \to \Sigma$ that maps the south pole to α and the north pole to β. Since $\pi_2(\Sigma) = 0$ it follows that v is homotopic, via maps with fixed north and south pole, to one of its meridians. This proves Proposition A.1. □

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