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Realization Theory in Hilbert Space*

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Abstract. A representation theorem for infinite-dimensional, linear control systems is proved in the context of strongly continuous semigroups in Hilbert spaces. The result allows for unbounded input and output operators and is used to derive necessary and sufficient conditions for the realizability in a Hilbert space of a time-invariant, causal input–output operator $\mathcal{T}$. The relation between input–output stability and stability of the realization is discussed. In the case of finite-dimensional input and output spaces the boundedness of the output operator is related to the existence of a convolution kernel representing the operator $\mathcal{T}$.

1. Introduction

In this paper we prove a representation theorem for infinite-dimensional, time-invariant, linear systems with continuous time. We show that a well-posed control system in the sense of Kalman [12] with square integrable inputs and outputs and a Hilbert space H as a state space can always be represented by a differential equation on H with unbounded input and output operators as described by the author in [22]. Using shift semigroups as in [9] and [11] this result allows us to prove that a causal, time-invariant, linear input–output operator $\mathcal{T}$ admits a realization in a Hilbert space if and only if it satisfies an exponential bound.

The latter result extends the theory by Helton [11] who worked with a slightly more restricted class of input–output operators and the work by Yamamoto [27] who did not assume the aforementioned exponential bound and arrived at a state-space description in a Frechét space. Without attempting to give a complete overview over the literature on infinite-dimensional realization theory we mention

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the papers [1]-[4], [6], [8]-[11], [13], [14], [17], and [23] including results on delay systems, systems over rings, and general infinite-dimensional systems in the context of semigroup theory.

In a preliminary section we briefly review the state-space theory as developed in [22] and describe a few examples. The representation theorem is proved in Section 3. We then derive the realization theorem in Section 4. Moreover, in Section 5 we discuss the relationship between the smoothing properties of the Hankel operator and the boundedness of the input and output operators.

The publication of this paper has been considerably delayed as the original manuscript reached the editor in 1985. Meanwhile, part of the main result of this paper (Theorem 3.1) has been independently proved and extended by Weiss [24] and [25]. We also mention the recent work by Curtain [5] and by Ober and Montgomery-Smith [16] which is closely related to results presented here.

2. State-Space Theory

In this section we briefly review the functional analytic approach to infinite-dimensional control systems with unbounded input and output operators developed in [22]. Let U, H, Y be Hilbert spaces and consider the linear differential equation

$$\begin{aligned}\frac{dz}{dt} &= Az + Bu, \qquad z(0) = \varphi \in H,\\ y &= C(z - (\mu I - A)^{-1}Bu) + G(\mu)u,\end{aligned} \tag{2.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup $S(t) \in \mathrm{L}(H)$. We consider $W = \mathrm{D}(A) \subset H$ and $V^* = \mathrm{D}(A^*) \subset H^*$ as Hilbert spaces with the respective graph norms so that

$$W \subset H \subset V$$

with continuous dense injections. Then $S(t)$ defines a semigroup on all three spaces W, H, V, and A can be regarded as a bounded operator in $\mathrm{L}(W, H) \cap \mathrm{L}(H, V)$. We assume that $B \in \mathrm{L}(U, V)$ and $C \in \mathrm{L}(W, Y)$. The formula for the output in (2.1) is motivated by the fact that the weak solution

$$z(t; \varphi, u) = S(t)\varphi + \int_0^t S(t-s)Bu(s)\, ds \tag{2.2}$$

of (2.1) will in general not be in W unless $Bu(t) \in H$, whereas the expression $z - (\mu I - A)^{-1}Bu = (\mu I - A)^{-1}(\mu z - \dot{z})$ will indeed be in W whenever $u \in W^{2,2}[0, T; U]$ and $A\varphi + Bu(0) \in H$ [18]. In the output equation we have chosen a particular value of $\mu \notin \sigma(A)$. If for any other value $\lambda \notin \sigma(A)$ the operator $G(\lambda) \in \mathrm{L}(U, Y)$ is defined by the identity

$$G(\lambda) - G(\mu) = (\mu - \lambda)C(\lambda I - A)^{-1}(\mu I - A)^{-1}B, \tag{2.3}$$

then in (2.1) μ can be replaced by λ without changing y. Moreover, a simple calculation shows that the analytic operator family $G(\lambda)$ thus defined describes the input-output behavior of (2.1) in the frequency domain. Hence $G(\lambda)$ is to be understood as a generalization of the expression "$C(\lambda I - A)^{-1}B$" which does not make sense in general. If it does make sense (for example if $B \in L(U, H)$ or $C \in L(H, Y)$), then (2.3) follows immediately from the resolved identity for the operator A

$$(\lambda I - A)^{-1} - (\mu I - A)^{-1} = (\mu - \lambda)(\lambda I - A)^{-1}(\mu I - A)^{-1}. \tag{2.4}$$

System (2.1) is said to be *well-posed* if for every $T > 0$ there exists a constant $c > 0$ such that

$$\|z(T; \varphi, u)\|_H^2 + \int_0^T \|y(t; \varphi, u)\|_Y^2 \, dt \le c\left(\|\varphi\|_H^2 + \int_0^T \|u(t)\|_U^2 \, dt\right) \tag{2.5}$$

for $\varphi \in H$ and $u \in W^{2,2}[0, T; U]$ with $A\varphi + Bu(0) \in H$ where $y(t; \varphi, u)$ denotes the output defined by the second equation in (2.1) with $z(t) = z(t; \varphi, u)$ given by (2.2).

If system (2.1) is well-posed, then for every $\varphi \in H$ and every $u \in L^2[0, T; U]$ equation (2.2) defines a weak solution of (2.1) of class $z \in C[0, T; H] \cap W^{1,2}[0, T; V]$. If in addition $u \in W^{1,2}[0, T; U]$ and $A\varphi + Bu(0) \in H$, then $z \in C^1[0, T; H]$, $y \in W^{1,2}[0, T; Y]$, and $\dot{z}(t; \varphi, u) = z(t; A\varphi + Bu(0), \dot{u})$, $\dot{y}(t; \varphi, u) = y(t; A\varphi + Bu(0), \dot{u})$ [22]. Moreover, a well-posed system defines (by continuous extension) a time-invariant, causal input-output operator

$$\mathcal{T}: L^2_{\text{loc}}[0, \infty; U] \to L^2_{\text{loc}}[0, \infty; Y], \qquad \mathcal{T}u(t) = y(t; 0, u).$$

In [5] Curtain proved that the extendability of the input-output operator to locally square integrable inputs and outputs (*input-output well-posedness*) is equivalent to

$$\sup_{\operatorname{Re}\lambda > \omega} \|G(\lambda)\|_{L(U,Y)} < \infty \tag{2.6}$$

for any $\omega > \omega_0$ where $\omega_0 = \lim_{t \to \infty} t^{-1} \log\|S(t)\|$ denotes the exponential growth rate of the semigroup. Estimate (2.6) also follows from Plancherel's theorem and the next lemma.

Lemma 2.1. *If system* (2.1) *is well-posed, then for every* $\omega > \omega_0$ *there exists a constant* $c > 0$ *such that*

$$\int_0^\infty e^{-2\omega t}\|y(t; \varphi, u)\|_Y^2 \, dt \le c\left(\|\varphi\|_H^2 + \int_0^\infty e^{-2\omega t}\|u(t)\|_U^2 \, dt\right),$$

$$\left\|\int_0^\infty S(t)Bu(-t) \, dt\right\|_H^2 \le c \int_0^\infty e^{2\omega t}\|u(-t)\|_U^2 \, dt.$$

Proof. With $T > 0$ sufficiently large we have $\|S(T)\| < e^{\omega T}$ and hence it follows from (2.5) that

$$\begin{aligned}\int_0^\infty e^{-2\omega t}\|y(t;\varphi,u)\|_Y^2\,dt &\le \sum_{k=0}^\infty e^{-2k\omega T}\int_0^T \|y(t;S(kT)\varphi,\sigma_{kT}u)\|_Y^2\,dt\\ &\le c\sum_{k=0}^\infty e^{-2k\omega T}\left(\|S(T)\|^{2k}\|\varphi\|_H^2+\int_0^T\|u(t+kT)\|_U^2\,dt\right)\\ &\le c_1\left(\|\varphi\|_H^2+\int_0^\infty e^{-2\omega t}\|u(t)\|_U^2\,dt\right)\end{aligned}$$

for $\varphi\in W$ and $u\in W^{1,2}[0,\infty;U]$ with compact support and $A\varphi+Bu(0)\in H$. This proves the first assertion in Lemma 2.1 and the second follows by duality. □

For any interval $I\subset\mathbb{R}$ and any Hilbert space X we define the weighted function spaces

$$L_\omega^2[I;X]=\left\{f\in L_{\mathrm{loc}}^2[I;X];\int_I e^{-2\omega t}\|f(t)\|_X^2\,dt<\infty\right\},$$

$$W_\omega^{1,2}[I,X]=\left\{f\in L_\omega^2[I;X];\frac{df}{dt}\in L_\omega^2[I;X]\right\}.$$

Then Lemma 2.1 allows us to introduce for $\omega>\omega_0$ the extended input-state operator $\mathscr{B}\colon L_\omega^2[-\infty,0;U]\to H$ and the extended state-output operator $\mathscr{C}\colon H\to L_\omega^2[0,\infty;Y]$ by

$$\mathscr{B}u=\int_0^\infty S(t)Bu(-t)\,dt,\qquad \mathscr{C}\varphi(t)=CS(t)\varphi. \tag{2.7}$$

Moreover, the *Hankel operator* of system (2.1) is given by the composition

$$\mathscr{H}=\mathscr{C}\mathscr{B}\colon L_\omega^2[-\infty,0;U]\to L_\omega^2[0,\infty;Y].$$

It follows from the above-mentioned properties of well-posed systems that if $u\in W_\omega^{1,2}[-\infty,0;U]$ with $u(0)=0$, then $\mathscr{B}u\in W=D(A)$ and $A\mathscr{B}u=\mathscr{B}\dot u$. Likewise, $\varphi\in W$ implies $\mathscr{C}\varphi\in W_\omega^{1,2}[0,\infty;Y]$ and $d\mathscr{C}\varphi/dt=\mathscr{C}A\varphi$.

As an example consider the one-dimensional heat equation with Neumann boundary control and point observation in the derivative:

$$\begin{aligned}&\frac{\partial z}{\partial t}=\frac{\partial^2 z}{\partial x^2},\qquad 0<x<\pi,\quad t>0,\\ &\frac{\partial z}{\partial x}(0,t)=0,\qquad \frac{\partial z}{\partial x}(\pi,t)=u(t),\\ &y(t)=\frac{\partial z}{\partial x}(x_0,t).\end{aligned} \tag{2.8}$$

This system is well-posed in the state space $H = H^{1/2}[0, \pi]$ of all functions $\varphi \in L^2[0, \pi]$ for which the norm

$$\|\varphi\|_{1/2}^2 = \sum_{n=1}^{\infty} n\frac{2}{\pi}\left(\int_0^{\pi} \varphi(x)\cos nx\,dx\right)^2$$

is finite or equivalently

$$\int_0^{\pi}\int_0^{\pi}\left(\frac{\varphi(x)-\varphi(y)}{x-y}\right)^2 dx\,dy < \infty.$$

More precisely, system (2.8) can be represented in the form (2.1) with

$$W = V^* = \left\{\varphi \in H^2[0, \pi];\ \frac{\partial\varphi}{\partial x}(0) = \frac{\partial\varphi}{\partial x}(\pi) = 0,\ \frac{\partial^2\varphi}{\partial x^2} \in H^{1/2}[0, \pi]\right\}$$

and $A\varphi = \partial^2\varphi/\partial x^2$, $C\varphi = \varphi(x_0)$, $B^*\varphi = \varphi(\pi)$ for $\varphi \in W$. The transfer function is given by

$$G(\lambda) = \frac{\sinh(\sqrt{\lambda}\,x_0)}{\sinh(\sqrt{\lambda}\,\pi)}$$

and satisfies the identity (2.3). It is bounded in $\operatorname{Re}\lambda \geq 0$ and actually converges to zero as $\operatorname{Im}\lambda \to \infty$ unless $x_0 = \pi$. We point out that $s = \frac{1}{2}$ is the only value of s for which this system is well-posed in the state space $H^s[0, \pi]$. A similar example was discussed by Curtain [5]. The proof of well-posedness in our case is analogous and is omitted. (See also [20].)

Next we consider the one-dimensional wave equation

$$\begin{aligned} &\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}, \qquad 0 < x < L, \quad t > 0, \\ &z(0, t) = 0, \qquad \frac{\partial z}{\partial x}(L, t) = u(t), \\ &y(t) = \frac{\partial z}{\partial t}(L, t). \end{aligned} \tag{2.9}$$

This system is well-posed and exactly controllable (in time $T > 2L/c$) in the state space

$$\left(z, \frac{\partial z}{\partial t}\right) \in H = \{\varphi = (\varphi^0, \varphi^1) \in H^1[0, L] \times L^2[0, L];\ \varphi^0(0) = 0\}.$$

Its transfer function is given by

$$G(\lambda) = c\frac{\sinh(\lambda L/c)}{\cosh(\lambda L/c)}$$

and satisfies condition (2.6) for $\omega > 0$.

As a third example we mention the linear retarded functional differential equation

$$\dot{x}(t) = L(x_t) + B(u_t), \qquad y(t) = C(x_t) + D(u_t), \tag{2.10}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$. Here x_t denotes the solution segment given by $x_t(\tau) = x(t+\tau)$ for $-h < \tau < 0$ and L, B, C, D are bounded linear functionals on the appropriate spaces of continuous functions. This system can again be represented as a well-posed Cauchy problem (2.1) in the product space

$$(x(t), x_t, u_t) \in H = \mathbb{R}^n \times L^2[-h, 0; \mathbb{R}^n] \times L^2[-h, 0; \mathbb{R}^m]$$

and in this case the transfer function is given by

$$G(\lambda) = C(e^{\lambda \cdot})(\lambda I - L(e^{\lambda \cdot}))^{-1} B(e^{\lambda \cdot}) + D(e^{\lambda \cdot}).$$

More details on the state-space representation of systems (2.9) and (2.10) can be found in [5] and [22] and the literature cited therein.

The above examples illustrate the significance of our state-space formulation since despite their well-posedness neither of these systems can be formulated, e.g., in the framework developed by Lions [15] nor the one in [20]. This is due to the large degree of unboundedness in the input and output operators. We shall now prove that every well-posed, time-invariant, linear system in the sense of Kalman [12] can be represented in the form (2.1).

3. A Representation Theorem

A *time-invariant, linear control system* consists of three Hilbert spaces U, H, Y, and two continuous, linear maps

$$H \times L^2_{\mathrm{loc}}[0, \infty; U] \to C_{\mathrm{loc}}[0, \infty; H], \qquad (\varphi, u) \mapsto z(\cdot\,; \varphi, u),$$

$$H \times L^2_{\mathrm{loc}}[0, \infty; U] \to L^2_{\mathrm{loc}}[0, \infty; Y], \qquad (\varphi, u) \mapsto y(\cdot\,; \varphi, u),$$

which satisfy the following conditions:

(i) *Initial condition*:

$$z(0; \varphi, 0) = \varphi, \qquad \forall \varphi \in H, \quad \forall u \in L^2_{\mathrm{loc}}[0, \infty; U].$$

(ii) *Causality*:

$$u(t) = 0, \quad \forall t \le T \quad \Rightarrow \quad z(t; 0, u) = 0, \quad y(t; 0, u) = 0, \quad \forall t \le T.$$

(iii) *Time invariance*:

$$z(t+s; \varphi, u) = z(t; z(s; \varphi, u), \sigma_s u),$$

$$y(t+s; \varphi, u) = y(t; z(s; \varphi, u), \sigma_s u),$$

$$\forall \varphi \in H, \quad \forall u \in L^2_{\mathrm{loc}}[0, \infty; U].$$

Here σ_s denotes the shift operator defined by $\sigma_s u(t) = u(t+s)$ for $t+s \ge 0$ and $\sigma_s u(t) = 0$ for $t+s < 0$. The above definition is similar to the one given by Kalman *et al.* [12] and every well-posed semigroup control system of the form (2.1) satisfies its requirements. The converse statement is formulated next.

Theorem 3.1. *Every time-invariant, linear control system can be represented in a unique way by operators* A, B, C, $G(\lambda)$ *via* (2.1) *and* (2.2) *with* $G(\lambda)$ *satisfying* (2.3).

The remainder of this section is devoted to the proof of this result. The main tool is the next lemma.

Lemma 3.2. *Let $\varphi \in H$ be the initial state, let $u \in W^{1,2}[0, T; U]$ be the input of a time-invariant linear control system, and assume that $z = z(\cdot\,; \varphi, u) \in C^1[0, \infty, H]$. Then $y(\cdot\,; \varphi, u) \in W^{1,2}[0, T; Y]$ and $\dot{y}(t; \varphi, u) = y(t; \dot{z}(0), \dot{u})$ for almost every $t \in [0, T]$.*

Proof. Note that $h^{-1}(\sigma_h u - u)$ converges to $\dot{u}$ in $L^2[0, T; U]$. Therefore the function

$$\frac{y(t+h; \varphi, u) - y(t; \varphi, u)}{h} = y\left(t; \frac{z(h; \varphi, u) - \varphi}{h}, \frac{\sigma_h u - u}{h}\right)$$

converges to $y(t; \dot{z}(0), \dot{u})$ in $L^2[0, T; Y]$ as $h \to 0$. □

Proof of Theorem 3.1. The operators $S(t) \in \mathrm{L}(H)$ defined by $S(t)\varphi = z(t; \varphi, 0)$ for $\varphi \in H$ and $t \geq 0$ form a strongly continuous semigroup and we define A to be its infinitesimal generator. As in Section 2 we regard $W = \mathrm{D}(A) \subset H$ and $V^* = \mathrm{D}(A^*) \subset H^*$ as Hilbert spaces with the respective graph norms so that $W \subset H \subset V$ with continuous dense injections and $S(t) \in \mathrm{L}(V) \cap \mathrm{L}(W)$ and $A \in \mathrm{L}(W, H) \cap \mathrm{L}(H, V)$.

If $\varphi \in \mathrm{D}(A)$ then it follows from Lemma 3.2 with $u \equiv 0$ that $y(\cdot\,; \varphi, 0) \in W^{1,2}[0, T; Y]$ with $\dot{y}(t; \varphi, 0) = y(t; A\varphi, 0)$. In particular $y(t; \varphi, 0)$ is continuous and its sup norm can be estimated by the $W^{1,2}$ norm (see, e.g., [26]) and thus by $(\|\varphi\|_H^2 + \|A\varphi\|_H^2)^{1/2} = \|\varphi\|_W$. It follows that the operator C: $W \to Y$ defined by

$$C\varphi = y(0; \varphi, 0), \qquad \varphi \in W,$$

is bounded.

In order to determine the input operator $B \in \mathrm{L}(U, V)$ we define $w(\cdot\,; \psi) \in L^2_{\mathrm{loc}}[0, T; U]$ for $\psi \in H$ by the identity

$$\int_0^T \langle w(T-s; \psi), u(s)\rangle_U \, ds = \langle \psi, z(T; 0, u)\rangle_H,$$
$$\forall T > 0, \quad \forall u \in L^2[0, T; U]. \tag{3.1}$$

It follows from the causality and time invariance that $w(t; \psi)$ is well defined. Moreover, the following equation holds for every $\psi \in H$ and all $t, s \geq 0$:

$$w(t+s; \psi) = w(t; S^*(s)\psi). \tag{3.2}$$

This is a consequence of the identity

$$\begin{aligned}\int_0^T \langle w(t; S^*(s)\psi), u(T-t)\rangle_U \, dt &= \langle S^*(s)\psi, z(T; 0, u)\rangle_H \\ &= \langle \psi, z(T+s; 0, u \cdot \chi_{[0,T]})\rangle_H \\ &= \int_0^T \langle w(T+s-t; \psi), u(t)\rangle_U \, dt \\ &= \int_0^T \langle w(t+s; \psi), u(T-t)\rangle_H \, dt.\end{aligned}$$

(Here χ_I denotes the characteristic function of the interval I.) If $\psi \in D(A^*)$ then it follows from (3.2) and Lemma 3.2 that $w(\cdot\,;\psi) \in W^{1,2}[0, T; U]$ with $\dot{w}(t; \psi) = w(t; A^*\psi)$. This allows us to define the operator $B \in L(U, V)$ by

$$B^*\psi = w(0; \psi), \qquad \psi \in V^*. \tag{3.3}$$

Now let $u \in L^2[0, T; U]$ and $\psi \in V^*$ be given. Then we obtain from (3.1) to (3.3) that

$$\langle \psi, z(T; 0, u)\rangle_H = \int_0^T \langle w(T-s; \psi), u(s)\rangle_U \, ds = \int_0^T \langle B^*S^*(T-s)\psi, u(s)\rangle_U \, ds$$

so that $z(T; \varphi, u) \in H$ is given by (2.2) for every $\varphi \in H$ and every $u \in L^2[0, T; U]$. In particular it follows that $z(\cdot\,; \varphi, u) \in C^1[0, T; H]$ whenever $u \in W^{1,2}[0, T; U]$ and $A\varphi + Bu(0) \in H$.

For every $u_0 \in U$ we have the identity

$$A(\lambda I - A)^{-1}Bu_0 + Bu_0 = \lambda(\lambda I - A)^{-1}Bu_0 \in H$$

and this implies $z(\cdot\,; (\lambda I - A)^{-1}Bu_0, u_0) \in C^1[0, T; H]$ where u_0 also denotes the constant function $u_0(t) \equiv u_0$. Then Lemma 3.2 shows that

$$y(\cdot\,; (\lambda I - A)^{-1}Bu_0, u_0) \in W^{1,2}[0, T; Y]$$

with $\dot{y}(t; (\lambda I - A)^{-1}Bu_0, u_0) = y(t; \lambda(\lambda I - A)^{-1}Bu_0, 0)$. This allows us to define the bounded linear operator $G(\lambda) \in L(U, Y)$ for $\lambda \notin \sigma(A)$ by

$$G(\lambda)u_0 = y(0; (\lambda I - A)^{-1}Bu_0, u_0), \qquad u_0 \in U. \tag{3.4}$$

Then the resolvent identity (2.4) implies the compatibility condition (2.3). It remains to be shown that $y(t; \varphi, u)$ is indeed given by (2.1) provided that $u \in W^{1,2}[0, T; U]$ and $A\varphi + Bu(0) \in H$. To this end we first observe that in the case $u(0) = 0$ we have $y(0; 0, u) = y(\varepsilon; 0, \sigma_{-\varepsilon}u) = 0$ since $y(\cdot\,; 0, \sigma_{-\varepsilon}u)$ is continuous (Lemma 3.2) and vanishes almost everywhere in the interval $[0, \varepsilon]$. In the general case we make use of the fact that

$$\varphi - (\mu I - A)^{-1}Bu_0 = (\mu I - A)^{-1}(\mu\varphi - A\varphi - Bu_0) \in W$$

with $u_0 = u(0)$ and hence

$$\begin{aligned} y(0; \varphi, u) &= y(0; \varphi - (\mu I - A)^{-1}Bu_0, 0) \\ &\quad + y(0; (\mu I - A)^{-1}Bu_0, u_0) + y(0; 0, u - u_0) \\ &= C(\mu I - A)^{-1}(\mu\varphi - A\varphi - Bu_0) + G(\mu)u_0. \end{aligned}$$

This proves the output identity in (2.1) for $t = 0$. In general (2.1) follows from the time invariance of the control system. □

4. Realization

We begin by introducing some notation. For any Hilbert space X let $L^2_{0,\mathrm{loc}}[\mathbb{R}; X]$ denote the space of locally square integrable functions $\mathbb{R} \to X$ whose support is bounded to the left. This space is self-dual via the pairing

$$\langle\psi, \varphi\rangle = \int_{-\infty}^{\infty} \langle\psi(-t), \varphi(t)\rangle_X \, dt.$$

Moreover, note that via the same pairing the dual space of $L^2_{\text{loc}}[0, \infty; X]$ is the space $L^2_0[-\infty, 0; X]$ of compactly supported, square integrable functions $(-\infty, 0] \to X$. For any interval $I \subset \mathbb{R}$ and any $s \in \mathbb{R}$ we denote by σ_s the shift operator acting on functions $f: I \to X$ by $\sigma_s f(t) = f(t+s)$ for $t+s \in I$ and $\sigma_s f(t) = 0$ for $t+s \notin I$.

Let U and Y be Hilbert spaces. A continuous linear input-output operator

$$\mathcal{T}: L^2_{0,\text{loc}}[\mathbb{R}; U] \to L^2_{0,\text{loc}}[\mathbb{R}; Y]$$

is called *time invariant* if $\sigma_t \mathcal{T} = \mathcal{T} \sigma_t$ for every $t \in \mathbb{R}$ and *causal* if

$$u(t) = 0, \quad \forall t \leq T \quad \Rightarrow \quad \mathcal{T}u(t) = 0, \quad \forall t \leq T.$$

These properties imply that the operator $\mathcal{T}$ is uniquely determined by its restriction to the interval $[0, \infty)$ which we still denote by

$$\mathcal{T}: L^2_{\text{loc}}[0, \infty; U] \to L^2_{\text{loc}}[0, \infty; Y].$$

The *Hankel operator*

$$\mathcal{H}: L^2_0[-\infty, 0; U] \to L^2_{\text{loc}}[0, \infty; Y]$$

associated to $\mathcal{T}$ is defined by first extending u to all of $\mathbb{R}$ via $u(t) = 0$ for $t \geq 0$ and then restricting $y = \mathcal{T}u$ to the interval $[0, \infty)$. Of course, the Hankel operator $\mathcal{H}$ only determines the input-output operator $\mathcal{T}$ up to an additive constant operator of the form $u \mapsto Du$ with $D \in \mathrm{L}(U, Y)$.

A *realization* of a time-invariant, causal input-output operator $\mathcal{T}$ is a well-posed semigroup control system of the form (2.1) whose input-output behavior is described by the given operator $\mathcal{T}$. If $\mathcal{T}$ admits such a realization, then it follows from Lemma 2.1 and the time invariance that $\mathcal{T}$ is *ω-stable* meaning that it extends to a bounded linear operator

$$\mathcal{T}: L^2_\omega[\mathbb{R}; U] \to L^2_\omega[\mathbb{R}; Y]$$

for some $\omega \in \mathbb{R}$. We shall prove that this condition is necessary and sufficient for the existence of a realization of the form (2.1).

Using the same argument as in the proof of Lemma 3.2 we can show that if $u \in L^2_{0,\text{loc}}[\mathbb{R}; U]$ is absolutely continuous with a locally square integrable derivative, then so is $y = \mathcal{T}u$ and moreover $\dot{y} = \mathcal{T}\dot{u}$. This shows that

$$u \in W^{1,2}_\omega[\mathbb{R}; U] \quad \Rightarrow \quad \mathcal{T}u \in W^{1,2}_\omega[\mathbb{R}; Y]$$

provided that $\mathcal{T}$ is ω-stable. We conclude that for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \omega$ there is a bounded linear operator $G(\lambda) \in \mathrm{L}(U, Y)$ defined by

$$G(\lambda)u = (\mathcal{T}(e^{\lambda \cdot} u))(0). \tag{4.1}$$

(Observe that $G(\lambda)u = (\mathcal{T}(\beta e^{\lambda \cdot} u))(0)$ for any cutoff function $\beta: \mathbb{R} \to [0, 1]$ with $\beta(t) = 1$ for $t \leq 0$ and $\beta(t) = 0$ for $t \geq \varepsilon$.) If $\mathcal{T}$ happens to be the input-output operator of a well-posed semigroup control system, then the operator $G(\lambda)$ in (2.1) is indeed given by (4.1) (as formula (3.4) shows). We can also prove directly that the operator $G(\lambda)$ defined by (4.1) is the transfer function of $\mathcal{T}$. More precisely, denoting by $\hat{u}$ the Laplace transform of u we obtain the following result.

Proposition 4.1. *If $\mathcal{T}$ is ω-stable and $u \in L^2[\mathbb{R}; U]$ has compact support, then $y \in \mathcal{T}u$ is given by $\hat{y}(\lambda) = G(\lambda)\hat{u}(\lambda)$.*

Proof. We only sketch the proof since the statement also follows from the realization theorem below. If $\mathcal{T}u$ is of class C^1 for every compactly supported $u \in L^2[\mathbb{R}, U]$, then there exists a strongly continuous operator family $K(t) \in \mathrm{L}(U, Y)$ and a constant operator $D \in \mathrm{L}(U, Y)$ such that

$$\mathcal{T}u(t) = \int_{-\infty}^{t} K(t-s)u(s)\,ds + Du(t).$$

In this case the result follows from the fact that $G(\lambda)$ is the Laplace transform of K. Now every time-invariant, causal input-output operator $\mathcal{T}$ can be approximated by a sequence of operators $\mathcal{T}_\varepsilon = \rho_\varepsilon * \mathcal{T}$ satisfying the above requirement. (Here $\rho: \mathbb{R} \to \mathbb{R}$ is a smooth function of mean value 1 vanishing outside the interval $[0, 1]$ and $\rho_\varepsilon(t) = \rho(t/\varepsilon)/\varepsilon$.) This proves Proposition 4.1. □

If $\mathcal{T}$ is ω-stable and $G(\lambda)$ is defined by (4.1), then it follows from Plancherel's theorem and Proposition 4.1 that

$$\sup_{\mathrm{Re}\,\lambda > \omega} \|G(\lambda)\|_{\mathrm{L}(U,Y)} < \infty. \tag{4.2}$$

Conversely, every real holomorphic operator-valued function $G(\lambda) \in \mathrm{L}(U, Y)$ satisfying (4.2) defines an ω-stable, time-invariant, causal input-output operator. Taking $\omega = 0$ and $U = Y = \mathbb{R}$ we have recovered the well-known fact that there is a one-to-one correspondence between time-invariant, causal operators $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and H^∞.

Let us now assume that $A \in \mathrm{L}(W, H) \cap \mathrm{L}(H, V)$, $B \in \mathrm{L}(U, V)$, and $C \in \mathrm{L}(W, Y)$ satisfy the requirements of Section 2, namely that A is the generator of a semigroup $S(t) \in \mathrm{L}(H)$ with $W = \mathrm{D}(A)$ and $V^* = \mathrm{D}(A^*)$ and that (2.7) defines bounded linear operators $\mathcal{B}: L^2_\omega[-\infty, 0; U] \to H$ and $\mathcal{C}: H \to L^2_\omega[0, \infty; Y]$.

Proposition 4.2. *Suppose that $\mathcal{T}$ is ω-stable, let $G(\lambda)$ be defined by* (4.1) *and let the operators A, B, C be given as above. Then $G(\lambda)$ satisfies the identity* (2.3) *if and only if the Hankel operator $\mathcal{H}$ of $\mathcal{T}$ is given by $\mathcal{H} = \mathcal{C}\mathcal{B}$. If these conditions are satisfied, then the associated system* (2.1) *defines a well-posed realization of $\mathcal{T}$.*

Proof. If $\mathcal{H} = \mathcal{C}\mathcal{B}$ then $G(\lambda)u - G(\mu)u = (\mathcal{C}\varphi)(0) = C\varphi$ where

$$\varphi = \mathcal{B}(e^{\lambda\cdot}u - e^{\mu\cdot}u) = (\lambda I - A)^{-1}Bu - (\mu I - A)^{-1}Bu \in W$$

so that (2.3) follows from (2.4). Conversely, (2.3) implies that the operators A, B, C, and $G(\lambda)$ form a well-posed semigroup control system and Proposition 4.1 shows that its input-output behavior is given by $\mathcal{T}$ which proves $\mathcal{H} = \mathcal{C}\mathcal{B}$. □

The previous result suggests that we may regard a triple A, B, C with the above properties as a realization of the input-output operator $\mathcal{T}$ if they satisfy the identity (2.3) with $G(\lambda)$ given by (4.1).

Theorem 4.3. *Let $\omega_1 \in \mathbb{R}$ be given. Then a time-invariant, causal input-output operator $\mathcal{T}$ has a well-posed realization of the form* (2.1) *with a semigroup of exponential growth rate $\omega_0 = \lim_{t\to\infty} t^{-1} \log\|S(t)\| < \omega_1$ if and only if it is ω-stable for some $\omega < \omega_1$.*

We construct two state-space systems in the sense of Section 3 whose input-output behavior is described by $\mathcal{T}$. The statement of Theorem 4.3 then follows from Lemma 2.1 and Theorem 3.1. We first choose the state space $H_U = L^2_\omega[-\infty, 0; U]$ and define

$$z_U(t; \varphi, u)(s) = \begin{cases} u(t+s), & -t < s < 0, \\ \varphi(t+s), & s < -t, \end{cases} \tag{4.3}$$
$$y_U(t; \varphi, u) = (\mathcal{H}\varphi)(t) + (\mathcal{T}u)(t),$$

for $\varphi \in H_U$ and $u \in L^2_{\text{loc}}[0, \infty; U]$. This system satisfies $y_U(\cdot\,; 0, u) = \mathcal{T}u$ and we easily check that it is time invariant and causal. An alternative system in the state space $H_Y = L^2_\omega[0, \infty; Y]$ is given by

$$z_Y(t; \psi, u) = \sigma_t\psi + \sigma_t\mathcal{T}(u \cdot \chi_{[0,t]}), \qquad y_Y(t; \psi, u) = \psi(t) + (\mathcal{T}u)(t), \tag{4.4}$$

for $\psi \in H_Y$ and $u \in L^2_{\text{loc}}[0, \infty; U]$ and has the same input-output behavior. (Here χ_I denotes the characteristic function of the interval I.) We also point out that the Hankel operator defines a state-space homomorphism between systems (4.3) and (4.4) meaning that

$$z_Y(t; \mathcal{H}\varphi, u) = \mathcal{H}z_U(t; \varphi, u), \qquad y_Y(t; \mathcal{H}\varphi, u) = y_U(t; \varphi, u). \tag{4.5}$$

More explicitly, the semigroup control system associated to system (4.3) via Theorem 3.1 is described by the following spaces and operators:

$$H_U = L^2_\omega[-\infty, 0; U], \qquad H_U^* = L^2_\omega[0, \infty; U],$$
$$W_U = \{\varphi \in W^{1,2}_\omega[-\infty, 0; U];\ \varphi(0) = 0\}, \qquad V_U^* = W^{1,2}_\omega[0, \infty; U],$$
$$A_U\varphi = \dot\varphi, \qquad A_U^*\psi = \dot\psi,$$
$$S_U(t)\varphi = \sigma_t\varphi, \qquad S_U^*(t)\psi = \sigma_t\psi,$$
$$C_U\varphi = (\mathcal{H}\varphi)(0), \qquad B_U^*\psi = \psi(0).$$

We denote this system by Σ_U. Likewise, system (4.4) is represented by the spaces and operators

$$H_Y = L^2_\omega[0,\infty;\ Y], \qquad H^*_Y = L^2_\omega[-\infty, 0;\ Y],$$

$$W_Y = W^{1,2}_\omega[0,\infty;\ Y], \qquad V^*_Y = \{\psi \in W^{1,2}_\omega[-\infty,0;\ Y];\ \psi(0)=0\},$$

$$A_Y\psi = \dot\psi, \qquad A^*_Y\varphi = \dot\varphi,$$

$$S_Y(t)\psi = \sigma_t\psi, \qquad S^*_Y(t)\varphi = \sigma_t\varphi,$$

$$C_Y\psi = \psi(0), \qquad B^*_Y\varphi = (\mathcal{H}\varphi)(0),$$

and this system is denoted by Σ_Y. In both cases the operator $G(\lambda)$ is given by (4.1) and we can check directly that (2.3) is satisfied. We also point out that

$$\|S_U(t)\| = \|S_Y(t)\| = e^{\omega t}$$

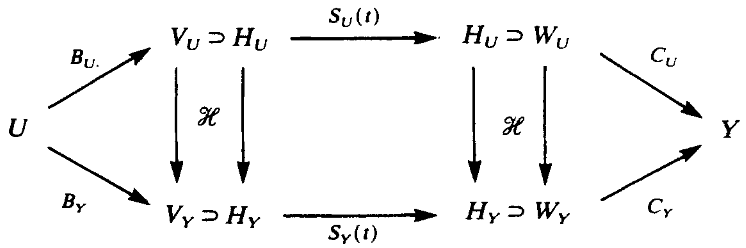
and that in the case $\omega = 0$ the semigroups $S_Y(t)$ and $S^*_U(t)$ are strongly stable whereas $S^*_Y(t)$ and $S_U(t)$ are not strongly stable.

In order to construct the realization Σ_U it suffices to assume that the Hankel operator $\mathcal{H}$ extends to a continuous operator from $L^2_\omega[-\infty,0;\ U]$ to $L^2_{loc}[0,\infty;\ Y]$ and likewise the realization Σ_Y can be constructed if $\mathcal{H}$ maps $L^2_0[-\infty,0;\ U]$ continuously into $L^2_\omega[0,\infty;\ Y]$. In both cases Lemma 2.1 shows that $\mathcal{T}$ is $(\omega+\varepsilon)$-stable for every $\varepsilon > 0$.

If $\mathcal{T}$ is ω-stable then the arguments preceding equation (4.1) show that

$$\mathcal{H} \in \mathrm{L}(H_U, H_Y) \cap \mathrm{L}(W_U, W_Y) \cap \mathrm{L}(V_U, V_Y)$$

and it follows from the identity (4.5) or by direct verification that the following diagram commutes:



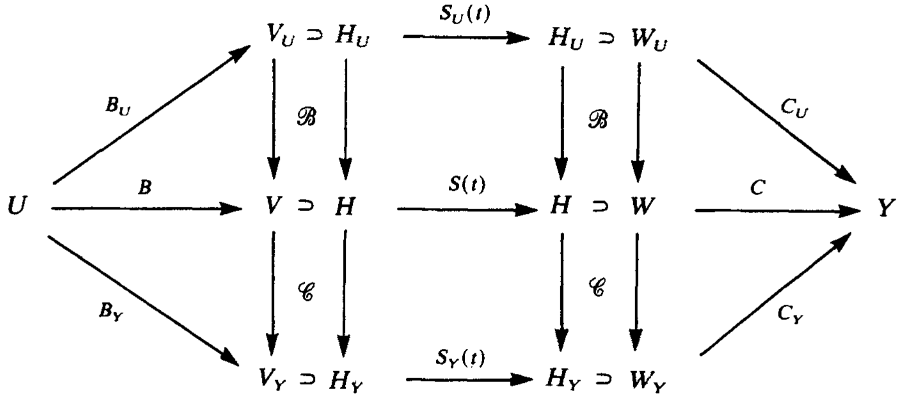
Note that the role of the Hankel operator in this context is quite analogous to the role of the structural operator F in the theory of retarded functional differential equations with the output playing the role of the forcing function and the input the role of the initial function [7], [19], and [21].

Now suppose that the input-output operator $\mathcal{T}$ is realized by a third semigroup control system $\Sigma = (A, B, C, G(\lambda))$ of the form (2.1) where A generates a semigroup $S(t) \in \mathrm{L}(H)$ of exponential type $\omega_0 < \omega$. Then the operators

$$\mathcal{B} \in \mathrm{L}(H_U, H) \cap \mathrm{L}(W_U, W) \cap \mathrm{L}(V_U, V),$$

$$\mathcal{C} \in \mathrm{L}(H, H_Y) \cap \mathrm{L}(W, W_Y) \cap \mathrm{L}(V, V_Y)$$

defined by (2.7) are state-space homomorphisms $\Sigma_U \to \Sigma$ and $\Sigma \to \Sigma_Y$ so that there is a commuting diagram as follows:



Let $\mathscr{U} \supset L^2_0[-\infty, 0; U]$ and $\mathscr{Y} \subset L^2_{\text{loc}}[0, \infty, Y]$ be complete topological vector spaces such that $\mathscr{B}$ extends to a continuous linear operator from $\mathscr{U}$ into H and $\mathscr{C}$ maps H continuously into $\mathscr{Y}$. Then system (2.1) is called *exactly $\mathscr{U}$-controllable* if $\mathscr{B}\mathscr{U} = H$ and *continuously $\mathscr{Y}$-observable* if $\mathscr{C}$ is injective and has a closed range in $\mathscr{Y}$. It is called *observable* if the unobservable subspace

$$\mathscr{N} = \{\varphi \in H;\, y(t;\, \varphi, 0) = 0,\ \forall t \geq 0\}$$

is zero (i.e., $\mathscr{C}$ is injective) and *approximately controllable* if the reachable subspace

$$\mathscr{R} = \{z(T;\, 0, u);\ T > 0,\ u \in L^2_{\text{loc}}[0, \infty;\, U]\}$$

is dense in H (i.e., $\mathscr{B}$ has a dense range). Every semigroup control system $\Sigma = (A, B, C, G(\lambda))$ can be made approximately controllable and observable by restricting the statespace to $\text{cl}(\mathscr{R}) \subset H$ and then factorizing through the subspace $\mathscr{N} \cap \text{cl}(\mathscr{R})$.

The realization Σ_U is exactly $\mathscr{U}$-controllable with $\mathscr{U} = H_U = L^2_\omega[-\infty, 0;\, U]$ and we denote by $\tilde{\Sigma}_U$ the system obtained by factorizing the state space H_U through the unobservable subspace $\mathscr{N}_U = \ker \mathscr{H}$. If $\Sigma = (A, B, C, G(\lambda))$ is any other realization of $\mathscr{T}$ which is exactly $\mathscr{U}$-controllable and observable, then the associated operator $\mathscr{B}\colon \mathscr{U} \to H$ is onto and $\mathscr{C}$ is injective so that $\ker \mathscr{B} = \ker \mathscr{C}\mathscr{B} = \mathscr{N}_U$. It follows that $\mathscr{B}$ induces a state-space isomorphism $H_U/\mathscr{N}_U \to H$ between $\tilde{\Sigma}_U$ and Σ. Likewise, any realization Σ of $\mathscr{T}$ of the form (2.1) which is approximately controllable and continuously $\mathscr{Y}$-observable with $\mathscr{Y} = L^2_\omega[0, \infty;\, Y]$ is isomorphic to the restriction $\tilde{\Sigma}_Y$ of system Σ_Y to the closure of the reachable subspace $\mathscr{R}_Y = \mathscr{H}(L^2_0[-\infty, 0;\, U])$. System (2.9) in Section 2 is indeed exactly controllable and continuously observable in finite time and its Hankel operator has a closed range. Therefore both shift-realizations $\tilde{\Sigma}_U$ and $\tilde{\Sigma}_Y$ of its transfer function are isomorphic to the given state-space system for any choice of $\omega > 0$. In this respect system (2.9) is quite exceptional. System (2.8) for example is neither exactly controllable nor continuously observable.

The above procedure of constructing a state-space isomorphism is, of course, well known. In particular, the concept of continuous $\mathscr{Y}$-observability was used in [27] with $\mathscr{Y} = L^2_{\text{loc}}[0, \infty]$.

We point out that system Σ_Y will in general not be continuously $\mathcal{Y}$-observable with $\mathcal{Y} = L^2_{\omega+\varepsilon}[0, \infty; Y]$ for any $\varepsilon > 0$. In order for the concept of $\mathcal{Y}$-observability to be independent of $\mathcal{Y}$ with $L^2_0[0, \infty; Y] \subset \mathcal{Y} \subset L^2_{\text{loc}}[0, \infty; Y]$ we need system Σ_Y to be *continuously L^2-observable in finite time* meaning that the operator

$$H \ni \varphi \to y(\cdot; \varphi, 0) \in L^2[0, T; Y]$$

is injective and has a closed range for some $T > 0$. The existence of a realization of $\mathcal{T}$ with this property requires that there exist constants $T > 0$, $M \geq 1$, $\omega > 0$ such that for every $t \geq 0$ and every $\varphi \in L^2_0[-\infty, 0; U]$

$$\int_0^T \|\mathcal{H}\varphi(t+s)\|^2 \, ds \leq M e^{\omega t} \int_0^T \|\mathcal{H}\varphi(s)\|^2 \, ds. \tag{4.6}$$

This condition is also sufficient, as was shown in [27], for the case $U = Y = \mathbb{R}$.

Example 4.4. Let $\alpha_n > 0$ be a summable sequence such that

$$\operatorname{rank}\begin{bmatrix} \alpha_0 & \cdots & \alpha_n \\ \vdots & \iddots & \vdots \\ \alpha_n & \cdots & \alpha_{2n} \end{bmatrix} = n+1, \qquad \forall n \in \mathbb{N}, \tag{4.7}$$

and consider the input-output operator $\mathcal{T}$ on $L^2(\mathbb{R})$ defined by

$$\mathcal{T}u(t) = \sum_{n=0}^{\infty} \alpha_n u(t-n). \tag{4.8}$$

(This operator is ω-stable with $\omega = 0$.) Then for any $T > 0$ and any output $\psi \in L^2[0, T]$ there exists a (unique) input $\varphi \in L^2[-T, 0]$ such that $\mathcal{H}\varphi(t) = \psi(t)$ for $0 \leq t \leq T$. This shows that there is no inequality of the form (4.6) for the Hankel operator $\mathcal{H}$ associated to (4.8). Therefore the realization constructed by Yamamoto [27] does not have a Hilbert space as a state space in contrast to our result (Theorem 4.3). This is a consequence of his concept of continuous observability in the output space $\mathcal{Y} = L^2_{\text{loc}}[0, \infty]$. In other words, the state space of the realization is chosen to be the closure of range $\mathcal{H}$ in $L^2_{\text{loc}}[0, \infty; Y]$ rather than $L^2[0, \infty]$ and is therefore not a Hilbert space. The construction in [27] leads to a Hilbert space if and only if (4.6) is satisfied.

5. Continuity and Boundedness

For a well-posed semigroup control system $\Sigma = (A, B, C, G(\lambda))$ of the form (2.1) there is an obvious relationship between the smoothing properties of the operator $\mathcal{C}: H \to L^2_{\text{loc}}[0, \infty; Y]$ defined by (2.7) and the boundedness of the output operator $C \in \mathrm{L}(W, Y)$. We first observe that C extends to a bounded linear operator $H \to Y$ if and only if $\mathcal{C}$ is a continuous operator $H \to C_{\text{loc}}[0, \infty; Y]$ and $C \in \mathrm{L}(V, Y)$ if and only if $\mathcal{C}$ maps H into $C^1_{\text{loc}}[0, \infty; Y]$. Furthermore, $C \in \mathrm{L}(H, Y)$ satisfies an estimate

$$\int_0^T \|CS(t)\varphi\|_Y^2 \, dt \leq c\|\varphi\|_V^2, \qquad \varphi \in H, \tag{5.1}$$

for some constants $T>0$, $c>0$ if and only if $\mathscr{C}$ is a continuous operator $H\to W^{1,2}_{\text{loc}}[0,\infty; Y]$. If $\mathscr{T}$ is an ω-stable, time-invariant, causal input-output operator, then the operator $\mathscr{C}_U: H_U\to L^2_{\text{loc}}[0,\infty; Y]$ associated as above to the realization Σ_U in the state space $H_U=L^2_\omega[-\infty,0; U]$ agrees with the Hankel operator $\mathscr{H}$. Thus we have proved the following result.

Proposition 5.1. *The Hankel operator* $\mathscr{H}$ *maps* $L^2_\omega[-\infty,0; U]$ *continuously into* $C_{\text{loc}}[0,\infty; Y]$ *(respectively* $C^1_{\text{loc}}[0,\infty; Y]$*) for every* $\omega<\omega_1$ *if and only if* $\mathscr{T}$ *admits a well-posed realization* $\Sigma=(A, B, C, G(\lambda))$ *of the form* (2.1) *such that*

$$\lim_{t\to\infty} t^{-1}\log\|S(t)\|<\omega_1 \tag{5.2}$$

and $C\in L(H, Y)$ *(respectively* $C\in L(V, Y)$*). Moreover,* $\mathscr{H}$ *is a continuous linear operator* $L^2_\omega[-\infty,0; U]\to W^{1,2}_\omega[0,\infty; Y]$ *for every* $\omega<\omega_1$ *if and only if* $\mathscr{T}$ *admits a realization* Σ *such that* (5.2) *holds and* $C\in L(H, Y)$ *satisfies* (5.1).

The obvious dual result relates the smoothing properties of the Hankel operator $\mathscr{H}^*$ to the boundedness of the input operator $B\in L(U, V)$. In particular $\mathscr{H}^*$ is a continuous linear operator $L^2_\omega[-\infty,0; U]\to W^{1,2}_\omega[0,\infty; Y]$ for every $\omega<\omega_1$ if and only if $\mathscr{T}$ admits a realization Σ such that (5.2) holds and $B\in L(U, H)$ is well posed in the state space W meaning

$$\left\|\int_0^T S(T-s)Bu(s)\,ds\right\|_W\le c\|u\|_{L^2[0,T;U]}.$$

In connection with Proposition 5.1 this shows that $\mathscr{H}\in L(L^2_\omega[-\infty,0; U], W^{1,2}_\omega[0,\infty; Y])$ for every $\omega<\omega_1$ if and only if $\mathscr{H}^*$ has the same property.

In the case of finite-dimensional input and output spaces the input-output operator $\mathscr{T}$ can be represented as a convolution operator

$$\mathscr{T}u(t)=\int_{-\infty}^t K(t-s)u(s)\,ds+Du(t) \tag{5.3}$$

if one of the conditions in Proposition 5.1 is satisfied. More precisely, we prove the following theorem, the second statement of which is a modified version of a result due to Yamamoto [27].

Theorem 5.2. *If* $U=\mathbb{R}^m$ *and* $Y=\mathbb{R}^p$ *then the following statements hold*:

(i) $\mathscr{T}$ *admits a well-posed realization* $\Sigma=(A, B, C, G(\lambda))$ *with either* $B\in L(U, H)$ or $C\in L(H, Y)$ *if and only if* $\mathscr{T}$ *is given by* (5.3) *with* $K\in L^2_\omega[0,\infty; \mathbb{R}^{p\times m}]$ *for some* $\omega\in\mathbb{R}$.

(ii) $\mathscr{T}$ *admits a well-posed realization* $\Sigma=(A, B, C, G(\lambda))$ *with either* $B\in L(U, W)$ *or* $C\in L(V, Y)$ *if and only if* $\mathscr{T}$ *is given by* (5.3) *with* $K\in W^{1,2}_\omega[0,\infty; \mathbb{R}^{p\times m}]$ *for some* $\omega\in\mathbb{R}$.

Proof. By Proposition 5.1 the input-output operator $\mathscr{T}$ admits a well-posed realization $\Sigma=(A, B, C, G(\lambda))$ with $C\in L(H, Y)$ if and only if the Hankel operator $\mathscr{H}$ is continuous from $L^2_\omega[-\infty,0; \mathbb{R}^m]$ into $C_{\text{loc}}[0,\infty; \mathbb{R}^p]$ for some $\omega\in\mathbb{R}$.

It follows from the Riesz representation theorem and the time invariance that this is equivalent to the existence of a kernel $K \in L^2_\omega[0, \infty; \mathbb{R}^{p\times m}]$ such that

$$\mathcal{H}\varphi(t) = \int_{-\infty}^{0} K(t-s)\varphi(s)\,ds, \qquad t > 0. \tag{5.4}$$

It remains to be shown that $\mathcal{T}$ is given by (5.3) whenever $\mathcal{H}$ is given by (5.4). Consider therefore the input-output operator $\mathcal{T}'$ defined by (5.3) with $D = 0$ and let $G'(\lambda)$ be the associated transfer function defined by (4.1). Then $\mathcal{T}$ and $\mathcal{T}'$ have the same Hankel operator $\mathcal{H}$ and hence Proposition 4.2 shows that $\Sigma' = (A, B, C, G'(\lambda))$ is a realization of $\mathcal{T}'$ whenever $\Sigma = (A, B, C, G(\lambda))$ is a realization of $\mathcal{T}$. It then follows from (2.3) that $D = G(\lambda) - G'(\lambda) \in \mathbb{R}^{p\times m}$ is independent of λ and hence $\mathcal{T}$ is given by (5.3).

We also point out that if $B \in \mathrm{L}(U, H)$ in (2.1) then $\mathcal{H}$ is given by (5.4) with $K(t)u = y(t; Bu, 0)$ for $u \in \mathbb{R}^m$ and Lemma 2.1 implies that $K \in L^2_\omega[0, \infty; \mathbb{R}^{p\times m}]$.

If $\mathcal{T}$ is given by (5.3) with $K \in W^{1,2}_\omega[0, \infty; \mathbb{R}^{p\times m}]$, then it follows from (5.4) that $\psi = \mathcal{H}\varphi \in C^1_{\mathrm{loc}}[0, \infty; Y]$ for every $\varphi \in L^2_\omega[-\infty, 0; \mathbb{R}^m]$ and $\dot{\psi} = \dot{K} * \varphi$. Hence Proposition 5.1 shows that $\mathcal{T}$ admits a realization $\Sigma = (A, B, C, G(\lambda))$ with $C \in \mathrm{L}(V, Y)$. Conversely, suppose that $\mathcal{H}$ maps $L^2_\omega[-\infty, 0; \mathbb{R}^m]$ continuously into $C^1_{\mathrm{loc}}[0, \infty; Y]$. Then $\mathcal{T}$ is given by (5.3) for some matrix $D \in \mathbb{R}^{p\times m}$ and some function $K \in L^2_\omega[0, \infty; \mathbb{R}^{p\times m}]$. Furthermore, it follows again from the Riesz representation theorem and time invariance that

$$\frac{d}{dt}\mathcal{H}\varphi(t) = \int_{-\infty}^{0} L(t-s)\varphi(s)\,ds$$

for some function $L \in L^2_\omega[0, \infty; \mathbb{R}^{p\times m}]$. We conclude that

$$\begin{aligned}\int_{-\infty}^{0} \left(K(T-s) - K(-s) - \int_0^T L(t-s)\,dt\right)\varphi(s)\,ds \\ = \mathcal{H}\varphi(T) - \mathcal{H}\varphi(0) - \int_0^T \frac{d}{dt}\mathcal{H}\varphi(t)\,dt = 0\end{aligned}$$

for all compactly supported $\varphi \in L^2[-\infty, 0; \mathbb{R}^m]$ and hence K is absolutely continuous with $dK/dt = L$. This proves Theorem 5.2. □

We have characterized those time-invariant, causal input-output operators $\mathcal{T}$ which admit a well-posed realization $\Sigma = (A, B, C, G(\lambda))$ of the form (2.1) with either $B \in \mathrm{L}(U, H)$ or $C \in \mathrm{L}(H, Y)$. It seems to be a much more subtle problem to find necessary and sufficient conditions under which there exists a realization with both $B \in \mathrm{L}(U, H)$ and $C \in \mathrm{L}(H, Y)$. A necessary condition is, of course, that $\mathcal{T}$ can be represented in the form (5.3) with $K(t) = CS(t)B$ strongly continuous and satisfying an exponential bound. A sufficient condition is $K \in W^{1,2}_\omega[0, \infty; \mathbb{R}^{p\times m}]$ for some $\omega \in \mathbb{R}$ where $U = \mathbb{R}^m$ and $Y = \mathbb{R}^p$ (Theorem 5.2). We finally point out that if $U = \mathbb{R}^m$, $Y = \mathbb{R}^p$, and $\mathcal{T}$ is given by (5.3) with K locally of bounded variation and

$$\operatorname*{Var}_{[0,t]} K \le M e^{\omega t}$$

then there exists a realization $\Sigma=(A, B, C, G(\lambda))$ with $C \in L(H, Y)$ satisfying (5.1) (Proposition 5.1).

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