The Spectral flow and the Maslov index

Joel Robbin[†] and Dietmar Salamon[*]
Mathematics Institute[*]
University of Warwick
Coventry CV4 7AL
Great Britain
and
Mathematics Department[†]
University of Wisconsin
Madison, WI 53706
USA

September 1992

1 Introduction

Atiyah, Patodi and Singer [3] studied operators of form

$$D_A = \frac{d}{dt} - A(t)$$

when $A(t)$ is a first order elliptic operator on a closed odd-dimensional manifold and the limits

$$A^{\pm} = \lim_{t \to \pm\infty} A^{\pm}(t)$$

exist and have no zero eigenvalue. A typical example for $A(t)$ is the div-grad-curl operator on a 3-manifold twisted by a connection which depends on t. Atiyah et al proved that the Fredholm index of such an operator D_A is equal to minus the "spectral flow" of the family $\{A(t)\}_{t \in \mathbb{R}}$. This spectral flow represents the net change in the number of negative eigenvalues of $A(t)$ as t runs from $-\infty$ to ∞. This "Fredholm index = spectral flow" theorem holds for rather general families $\{A(t)\}_{t \in \mathbb{R}}$ of self-adjoint operators on Hilbert spaces. This is a folk theorem that has been used many times in the literature, but no adequate exposition has yet appeared. We give such an exposition here as well as several applications. More precisely, we shall prove the following theorem.

Theorem A *Assume that, for each t, $A(t)$ is an unbeounded, self-adjoint operator on a Hilbert space H with time independent domain $W = \operatorname{dom}(A(t))$. Assume, moreover, that W is a Hilbert space in its own right with a compact dense injection $W \hookrightarrow H$ and that the norm of W is equivalent to the graph norm of $A(t)$ for every t. Assume further that the map $\mathbb{R} \to \mathcal{L}(W, H) : t \mapsto A(t)$ is continuously differentiable with respect to the weak operator topology. Assume finally that $A(t)$ converges in the norm topology to invertible operators $A^\pm \in \mathcal{L}(W, H)$ as t tends to $\pm\infty$. Then the operator*

$$D_A : W^{1,2}(\mathbb{R}, H) \cap L^2(\mathbb{R}, W) \to L^2(\mathbb{R}, H)$$

is Fredholm and its index is given by the spectral flow of the operator family $A(t)$.

The assumptions of Theorem A imply that the spectrum of $A(t)$ is discrete and consists only of eigenvalues. Hence the "spectral flow" is well defined and we shall give the precise definition in Section 4. We shall also prove that Theorem A remains valid if $A(t)$ is perturbed by a family of compact linear operators $C(t) : W \to H$ which is continuous in t with respect to the norm topology and converges to zero as t tends to $\pm\infty$. The operators $C(t)$ in this perturbation are not required to be self-adjoint when regarded as unbounded operators on H.

A drawback of Theorem A is the assumption that the domain of $A(t)$ be independent of t. This assumption will in general exclude the case of differential operators on manifolds with boundary. Theorem A will remain valid for a suitable class of operator families with time dependent domains but the precise conditions on how the domain is required to vary with time will be a topic for future research.

The "Fredholm index = spectral flow" theorem is used in a generalization of Morse theory known as "Floer Homology". Both Morse theory and Floer's theory are used to prove the existence of critical points of a nonlinear functional f via topological arguments. In either case the operators $A^\pm$ appear as the Hessian of f at critical points $x^\pm$ and the operators $A(t)$ represent the covariant Hessian of f along a gradient flow line $x(t)$, i.e. a solution of

$$\dot{x} = \nabla f(x),$$

which connects $x^- = \lim_{t \to -\infty} x(t)$ with $x^+ = \lim_{t \to +\infty} x(t)$. Here both the gradient $\nabla f(x)$ and the Hessian $A = \nabla^2 f(x)$ are taken with respect to a suitable metric on the underlying (finite or infinite dimensional) manifold. The operator D_A arises from linearizing the gradient flow equation.

In the case of Morse Theory, the function f is bounded below and each operator $A(t)$ has only finitely many negative eigenvalues and hence has a well defined index, namely the dimension of the negative eigenspace. In this case the spectral flow can be expressed as the index difference so the "Fredholm index

2

= spectral flow" formula is

$$\text{index}\, D_A = \nu^-(A^+) - \nu^-(A^-) \qquad (*)$$

where ν^- denotes the number of negative eigenvalues counted with multiplicity. In the finite dimensional case it turns out that the operator D_A is onto (for each connecting orbit) if and only if the unstable manifold of x^- intersects the stable manifold of x^+ transversally (cf. [24]). So in this case the space $\mathcal{M}(x^-, x^+) = W^u(x^-) \cap W^s(x^+)$ of connecting orbits is a finite dimensional manifold whose dimension is the difference of the Morse indices. On the one hand this follows from finite dimensional transversality arguments. On the other hand this can be proved by using an infinite dimensional implicit function theorem in a suitable path space where D_A appears as the linearized operator and its kernel as the tangent space to the manifold of connecting orbits. Full details of this second approach can be found in [26].

In Floer's theory (see [10] and [11] for example) the operators $A^\pm$ can have both infinite index and infinite coindex so the right hand side of equation $(*)$ is undefined. The spectral flow can still be defined as the number of eigenvalues of $A(t)$ which cross zero as t runs from $-\infty$ to $+\infty$. The counting is done so that each negative eigenvalue which becomes positive contributes $+1$ and each positive eigenvalue which becomes negative contributes -1. We make this precise in section 4.

In Floer's theory the initial value problem for the gradient flow equation is not wellposed and hence there are no stable or unstable manifolds. However, one can still prove that the operator D_A is onto for suitable "generic perturbations" of the function f. It then follows from an infinite dimensional implicit function theorem that the space of connecting orbits is a manifold. Its dimension is the Fredhom index of D_A and hence, by Theorem A, agrees with the spectral flow of the 1-parameter family of the Hessians $A(t)$. Even in cases where the index and coindex of $A^\pm$ are both infinite, this spectral flow can still be viewed as an index difference and this leads to Floer's "relative Morse index". Floer then proceeds to analyse the properties of these manifolds of connecting orbits to construct a chain complex generated by the critical points of f and graded by the relative Morse index. The boundary operator is defined by counting the connecting orbits (with appropriate signs) when the index difference is 1. The homology groups of this chain complex are called "Floer homology". Details of this construction can be found in [10], [11], [24], [25], and, for the finite dimensional analogue, in [26].

For example, Floer homology can be used to study closed orbits of Hamiltonian systems. These closed orbits are critical points of the symplectic action. Like Morse theory, Floer's theory constructs a gradient flow for this action and studies orbits connecting critical points. A connecting orbit joining two critical points in this infinite dimensional gradient flow is a cylinder joining two closed orbits in a finite dimensional symplectic manifold. The cylinder satisfies a certain nonlinear partial differential equation. Linearization along such a

connecting orbit gives an operator D_A. In this case the spectral flow along the connecting orbit is the difference of the Maslov indices of the two critical points at the ends. See [24] and [25], for example. This linearization is an example of the Cauchy-Riemann operators studied in section 7.

The fact that the spectral flow is sometimes the difference of two Maslov indices is not surprising, since the spectral flow can be thought of as an infinite dimensional analogue of the Maslov index for Lagrangian paths. The graph of a path of symmetric matrices $A : [-T, T] \to \mathbb{R}^{n \times n}$ is a path of Lagrangian subspaces in $\mathbb{R}^{2n}$. Its endpoints are transverse to $\mathbb{R}^n \times 0$ if and only if the matrices $A^\pm = A(\pm T)$ are nonsingular. In this case the Maslov index μ is the intersection number of the path $\mathrm{Gr}(A)$ with the Maslov cycle Σ of those Lagrangian subspaces which intersect the horizontal $\mathbb{R}^n \times 0$ in a nonzero subspace. The Maslov index can be expressed in the form

$$\mu(\mathrm{Gr}(A)) = \tfrac{1}{2}\mathrm{sign}(A^+) - \tfrac{1}{2}\mathrm{sign}(A^-)$$

and agrees with the spectral flow of the matrix family $A(t)$.

Our main application is an index theorem for the Cauchy-Riemann operator

$$\bar{\partial}_{S,\Lambda}\zeta = \frac{\partial \zeta}{\partial t} - J_0 \frac{\partial \zeta}{\partial t} + S\zeta$$

on the infinite cylinder $[0, 1] \times \mathbb{R}$ with general non-local boundary conditions

$$(\zeta(0, t), \zeta(1, t)) \in \Lambda(t)$$

where $S(s, t) = S(s, t)^T \in \mathbb{R}^{2n \times 2n}$ is symmetric and $\Lambda(t)$ is a Lagrangian path in $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, (-\omega_0) \times \omega_0)$. We prove that $\bar{\partial}_{S,\Lambda}$ is a Fredholm operator between suitable Sobolev spaces and express the Fredholm index in terms of the relative Maslov index for a pair of Lagrangian paths.

Theorem B   index $\bar{\partial}_{S,\Lambda} = -\mu(\mathrm{Gr}(\Psi_1), \Lambda)$.

Here $\Psi_1(t) = \Psi(1, t)$ is a path of symplectic matrices determined by S via $J_0 \partial_s \Psi = S\Psi$ with $\Psi(0, t) = \mathbb{1}$. This generalizes a theorem of Floer [8] for $S = 0$ and local boundary conditions $\Lambda = \Lambda_0 \oplus \Lambda_1$ and a theorem in [25] for periodic boundary conditions. Both theorems play an important role in Floer homology for Lagrangian intersections [9] and for symplectic fixed points [11].

In section 2 we discuss the finite dimensional case as a warm-up. In section 3 we prove that D_A is a Fredholm operator. In section 4 we characterize the spectral flow axiomatically and prove that the Fredholm index satisfies these axioms. In section 5 we review the properties of the Maslov index. In section 6 we use the spectral flow and the Maslov index to give a proof of the Morse index theorem. A special case of this is Sturm oscillation. Finally, in section 7 we discuss the aforementionned Cauchy-Riemann operators.

4

2 The finite dimensional case

Linearization along a connecting orbit of a gradient flow of a Morse function leads to a differential operator

$$(D_A\xi)(t) = \dot{\xi}(t) - A(t)\xi(t). \tag{1}$$

The index of this operator is the difference of the Morse indices of the critical points at the two ends. In this example the matrices $A(t)$ may be chosen to be symmetric. In this section we shall prove a more general fact. The hypothesis that the vector field is a gradient field is dropped. We linearize along an orbit connecting two hyperbolic critical points. As a result the matrices $A(t)$ will no longer be symmetric but the limit matrices

$$A^\pm = \lim_{t\to\pm\infty} A(t)$$

exist and are hyperbolic (no eigenvalues on the imaginary axis). For any matrix $B \in \mathbb{R}^{n\times n}$ we define

$$
\begin{aligned}
E^s(B) &= \left\{ v \in \mathbb{R}^n : \lim_{t\to\infty} e^{Bt}v = 0 \right\}, \\
E^u(B) &= \left\{ v \in \mathbb{R}^n : \lim_{t\to-\infty} e^{Bt}v = 0 \right\}.
\end{aligned}
$$

Then $E^s(B)$ is the direct sum of the generalized eigenspaces corresponding to eigenvalues with negative real parts and similarly for $E^u(B)$ with positive real parts. Hence the matrix B is hyperbolic if and only if

$$\mathbb{R}^n = E^s(B) \oplus E^u(B).$$

Theorem 2.1 *Assume that $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ is continuous and that the limit matrices $A^\pm$ exist and are hyperbolic. Then formula (1) defines a Fredholm operator*

$$D_A : W^{1,2}(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n)$$

with index

$$\mathrm{index}\, D_A = \dim E^u(A^-) - \dim E^u(A^+).$$

Proof: That the operator is Fredholm follows from the inequality

$$\|\xi\|_{W^{1,2}(\mathbb{R})} \le c \left(\|\xi\|_{L^2(I)} + \|D_A\xi\|_{L^2(\mathbb{R})} \right) \tag{2}$$

for a sufficiently large interval $I = [-T, T]$. This estimate is proved in three steps. *Firstly*, since $\dot{\xi} = D_A\xi + A\xi$, the estimate is obvious for $I = \mathbb{R}$:

$$\|\xi\|_{W^{1,2}(\mathbb{R})} \le c \left(\|\xi\|_{L^2(\mathbb{R})} + \|D_A\xi\|_{L^2(\mathbb{R})} \right). \tag{3}$$

Secondly, if $A(t) \equiv A_0$ is a constant hyperbolic matrix, then the associated operator D_{A_0} satisfies

$$\|\xi\|_{W^{1,2}(\mathbb{R})} \le c\|D_{A_0}\xi\|_{L^2(\mathbb{R})}. \qquad (4)$$

To see this decompose $\mathbb{R}^n$ as a direct sum where each of the summands has all its eigenvalues in one of the two halfplanes. Hence it suffices to treat the special case where all the eigenvalues of A_0 have negative real part. For $\eta \in L^2(\mathbb{R}, \mathbb{R}^n)$ the unique solution of $\dot\xi - A_0\xi = \eta$ with $\xi \in L^2(\mathbb{R}, \mathbb{R}^n)$ is given by

$$\xi(t) = \int_{-\infty}^{t} e^{A_0(t-s)}\eta(s)\,ds = \Phi * \eta(t)$$

where $\Phi(t) = e^{A_0 t}$ for $t \ge 0$ and $\Phi(t) = 0$ for $t < 0$. By Young's inequality,

$$\|\xi\|_{L^2} \le \|\Phi\|_{L^1}\|\eta\|_{L^2}.$$

Since $\dot\xi = A_0\xi + \eta$ we also have

$$\|\dot\xi\|_{L^2} \le \|A_0\xi\|_{L^2} + \|\eta\|_{L^2} \le (\|A_0\|\,\|\Phi\|_{L^1} + 1)\,\|\eta\|_{L^2}.$$

Note in fact that the operator D_{A_0} is bijective since any function in its kernel is an exponential and can lie in L^2 only if it vanishes identically. This proves (4). Alternatively, (4) can be proved with Fourier transforms as in Lemma 3.7 below.

Finally, the estimate is proved by a patching argument. It follows from (4) that there exist constants $T > 0$ and $c > 0$ such that for every $\xi \in W^{1,2}(\mathbb{R}, \mathbb{R}^n)$

$$\xi(t) = 0 \quad \text{for} \quad |t| \le T-1 \qquad \Longrightarrow \qquad \|\xi\|_{W^{1,2}(\mathbb{R})} \le c\|D_A\xi\|_{L^2(\mathbb{R})}. \qquad (5)$$

Now choose a smooth cutoff function $\beta : \mathbb{R} \to [0,1]$ such that $\beta(t) = 0$ for $|t| \ge T$ and $\beta(t) = 1$ for $|t| \le T-1$. Using the estimate (3) for $\beta\xi$ and (5) for $(1-\beta)\xi$ we obtain

$$
\begin{aligned}
\|\xi\|_{W^{1,2}} &\le& \|\beta\xi\|_{W^{1,2}} + \|(1-\beta)\xi\|_{W^{1,2}} \\
&\le& c_1\left(\|\beta\xi\|_{L^2} + \|D_A(\beta\xi)\|_{L^2} + \|D_A((1-\beta)\xi)\|_{L^2}\right) \\
&\le& c_2\left(\|\xi\|_{L^2[-T,T]} + \|D_A\xi\|_{L^2}\right).
\end{aligned}
$$

This proves (2). Since the restriction operator $W^{1,2}(\mathbb{R}, \mathbb{R}^n) \to L^2([-T,T], \mathbb{R}^n)$ is compact it follows from Lemma 3.5 below that D_A has a finite dimensional kernel and a closed range.

We examine the kernel of D_A. It consists of those solutions of the differential equation $\dot\xi = A\xi$ which converge to zero as t tends to $+\infty$ and $-\infty$. Consider the fundamental solution $\Phi(t,s) \in \mathbb{R}^{n \times n}$ defined by

$$\frac{\partial}{\partial t}\Phi(t,s) = A(t)\Phi(t,s), \qquad \Phi(s,s) = \mathbb{1},$$

and note that $\Phi(t,s)\Phi(s,r) = \Phi(t,r)$. Define the stable and unstable subspaces

$$E^s(t_0) = \left\{ \xi_0 \in \mathbb{R}^n : \lim_{t \to +\infty} \Phi(t,t_0)\xi_0 = 0 \right\},$$

$$E^u(t_0) = \left\{ \xi_0 \in \mathbb{R}^n : \lim_{t \to -\infty} \Phi(t,t_0)\xi_0 = 0 \right\}.$$

Both subspaces define invariant vector bundles over $\mathbb{R}$. This means that $E^s(t) = \Phi(t,s)E^s(s)$ and $E^u(t) = \Phi(t,s)E^u(s)$. Moreover, $\lim_{t \to +\infty} E^s(t) = E^s(A^+)$ and $\lim_{t \to -\infty} E^u(t) = E^u(A^-)$. Hence

$$\dim E^s(t) = n - \dim E^u(A^+), \qquad \dim E^u(t) = \dim E^u(A^-).$$

Now let $\xi(t)$ be any solution of the differential equation $\dot{\xi} = A\xi$. Then $\xi(t) = \Phi(t,s)\xi(s)$ for all t and s. Moreover, $|\xi(t)|$ converges to 0 exponentially for $t \to +\infty$ whenever $\xi(t) \in E^s(t)$ and $|\xi(t)|$ converges to ∞ exponentially for $t \to +\infty$ whenever $\xi(t) \notin E^s(t)$. Similarly for $t \to -\infty$. Hence

$$\xi \in \ker D_A \qquad \Longleftrightarrow \qquad \xi(t) = \Phi(t,s)\xi(s) \quad \text{and} \quad \xi(t) \in E^s(t) \cap E^u(t).$$

We examine the cokernel of D_A. Assume that $\eta \in L^2(\mathbb{R}, \mathbb{R}^n)$ is orthogonal to the range of D_A. Then

$$\int_{-\infty}^{\infty} \langle \eta(t), \dot{\xi}(t) \rangle \, dt - \int_{-\infty}^{\infty} \langle \eta(t), A(t)\xi(t) \rangle \, dt = 0$$

for every $\xi \in W^{1,2}(\mathbb{R}, \mathbb{R}^n)$. If $\xi(t) = 0$ for $|t| \geq T$ then this implies

$$\int_{-T}^{T} \langle \eta(t) - \int_t^T A(s)^T \eta(s) \, ds, \dot{\xi}(s) \rangle \, ds = 0$$

and hence

$$\dot{\eta}(t) + A(t)^T \eta(t) = 0.$$

The fundamental solution of this equation is $\widetilde{\Phi}(t,s) = \Phi(s,t)^T$ and it is easy to see that the associated stable and unstable bundles are given by $\widetilde{E}^s(t) = E^s(t)^\perp$ and $\widetilde{E}^u(t) = E^u(t)^\perp$. Hence

$$\eta \perp \operatorname{range} D_A \qquad \Longleftrightarrow \qquad \eta(t) = \Phi(s,t)^T \eta(s) \quad \text{and} \quad \eta(t) \perp E^s(t) + E^u(t).$$

In particular the cokernel of D_A is finite dimensional. Moreover,

$$
\begin{aligned}
\operatorname{index} D_A &= \dim(E^s \cap E^u) - \dim(E^s + E^u)^\perp \\
&= \dim(E^s \cap E^u) + \dim(E^s + E^u) - n \\
&= \dim E^s + \dim E^u - n \\
&= \dim E^u(A^-) - \dim E^u(A^+).
\end{aligned}
$$

This proves the theorem. □

Remark 2.2 Assume that the matrices $A^{\pm}$ are symmetric. Then the index formula of Theorem 2.1 can be expressed in terms of the signature as

$$\text{index}\, D_A = \tfrac{1}{2}\text{sign}\, A^- - \tfrac{1}{2}\text{sign}\, A^+.$$

3 Fredholm theory

Assume that W and H are separable real Hilbert spaces with

$$W \subset H = H^* \subset W^*.$$

Here the inclusion $W \hookrightarrow H$ is compact with a dense range. Throughout we identify H with its dual space. We shall not use the inner product on W but only the norm. Hence we distinguish W from its dual space W^*. The notation $\langle \eta, \xi \rangle$ will denote the inner product in H when $\xi, \eta \in H$ and the pairing of W with W^* when $\xi \in W$ and $\eta \in W^*$.

Fix a family of bounded linear operators

$$A(t) : W \to H$$

indexed by $t \in \mathbb{R}$. Given a differentiable curve $\xi : \mathbb{R} \to W$, define $D_A\xi : \mathbb{R} \to H$ by

$$(D_A\xi)(t) = \dot{\xi}(t) - A(t)\xi(t) \tag{6}$$

for $t \in \mathbb{R}$. In the intended application $W = W^{1,2}$ and $H = L^2$ and $A(t)$ is a first order linear elliptic differential operator whose coefficients depend smoothly on t. We impose the following conditions:

(A-1) The map $A : \mathbb{R} \to \mathcal{L}(W, H)$ is BC^1. This means that it is continuously differentiable in the weak operator topology and there exists a constant $c_0 > 0$ such that

$$\|A(t)\xi\|_H + \|\dot{A}(t)\xi\|_H \leq c_0\|\xi\|_W.$$

for every $t \in \mathbb{R}$ and every $\xi \in W$.

(A-2) The operators $A(t)$ are uniformly self-adjoint. This means that for each t the operator $A(t)$ when considered as an unbounded operator on H with $\text{dom}\, A(t) = W$ is self-adjoint and that there is a constant c_1 such that

$$\|\xi\|_W^2 \leq c_1 \left(\|A(t)\xi\|_H^2 + \|\xi\|_H^2 \right). \tag{7}$$

for every $t \in \mathbb{R}$ and every $\xi \in W$.

(A-3) There are invertible operators $A^{\pm} \in \mathcal{L}(W, H)$ such that

$$\lim_{t \to \pm\infty} \|A(t) - A^{\pm}\|_{\mathcal{L}(W,H)} = 0.$$

8

Remark 3.1 Condition (A-1) implies that the map $t \mapsto A(t)$ is continuous in the norm topology but $t \mapsto \dot{A}(t)$ is only weakly continuous. We shall use the fact that $\xi \in L^2(\mathbb{R}, W)$ implies $\dot{A}\xi \in L^2(\mathbb{R}, H)$.

Remark 3.2 Let A be a self-adjoint operator on H with dense domain $W = \mathrm{dom}\, A$. Then W is a Hilbert space in its own right with respect to the graph norm of A and the estimate (7) holds trivially. The inclusion $W \hookrightarrow H$ is compact if and only if the resolvent operator $(\lambda \mathbb{1} - A)^{-1} : H \to H$ is compact for every $\lambda \notin \sigma(A)$. In this case the spectrum of A is discrete and consists of real eigenvalues of finite multiplicity.

Remark 3.3 Let A be a closed symmetric operator on H with dense domain $W = \mathrm{dom}\, A$. When A is regarded as a bounded operator from W to H its adjoint is a bounded operator from H to W^*. Since A is symmetric the restriction of this adjoint to W agrees with A. Thus the adjoint is an extension of A which we still denote by A. With this notation we have

$$A \in \mathcal{L}(W, H) \cap \mathcal{L}(H, W^*).$$

The condition that A be self-adjoint now means that $A\xi \in H$ implies $\xi \in W$.

Remark 3.4 A symmetric operator $A : W \to H$ which satisfies (7) is necessarily closed but need not be self-adjoint.

We define Hilbert spaces $\mathcal{H}$ and $\mathcal{W}$ by

$$
\begin{aligned}
\mathcal{H} &= L^2(\mathbb{R}, H), \\
\mathcal{W} &= L^2(\mathbb{R}, W) \cap W^{1,2}(\mathbb{R}, H)
\end{aligned}
$$

with norms

$$
\begin{aligned}
\|\xi\|_{\mathcal{H}}^2 &= \int_{-\infty}^{\infty} \|\xi(t)\|_H^2 dt, \\
\|\xi\|_{\mathcal{W}}^2 &= \int_{-\infty}^{\infty} \left(\|\xi(t)\|_W^2 + \|\dot{\xi}(t)\|_H^2 \right) dt.
\end{aligned}
$$

The inclusion $\mathcal{W} \hookrightarrow \mathcal{H}$ is a bounded linear injection with a dense range since $C_0^{\infty}(\mathbb{R}, W)$ is dense in both spaces. The uniform bound on $A(t)$ from condition (A-1) means that $\xi \mapsto D_A \xi$ defines a a bounded linear operator

$$D_A : \mathcal{W} \to \mathcal{H}.$$

Our first aim is to show that it is Fredholm. The proof relies on the following

Lemma 3.5 (Abstract Closed Range Lemma) *Suppose that X, Y, and Z are Banach spaces, that $D : X \to Y$ is a bounded linear operator, and that $K : X \to Z$ is a compact linear operator. Assume that*

$$\|x\|_X \leq c \left(\|Dx\|_Y + \|Kx\|_Z \right)$$

for $x \in X$. Then D has a closed range and a finite dimesional kernel.

For each $T > 0$ define Hilbert spaces $\mathcal{W}(T)$ and $\mathcal{H}(T)$ by

$$
\begin{aligned}
\mathcal{H}(T) &= L^2([-T,T], H), \\
\mathcal{W}(T) &= L^2([-T,T], W) \cap W^{1,2}([-T,T], H)
\end{aligned}
$$

with norms as above.

Lemma 3.6 *For every $T > 0$ the inclusion $\mathcal{W}(T) \hookrightarrow \mathcal{H}(T)$ is a compact operator.*

Proof: Choose an orthonormal basis for H and denote by $\pi_n : H \to \mathbb{R}^n$ the orthogonal projection determined by the first n elements. Since $\pi_n^* \pi_n \in \mathcal{L}(H)$ converges strongly to the identity of H and the inclusion $W \to H$ is compact, the operator

$$
\pi_n^* \pi_n|_W : W \to H
$$

converges to the inclusion $W \hookrightarrow H$ in the norm topology. The induced operator $\mathcal{W}(T) \to \mathcal{H}(T) : \xi \mapsto \pi_n^* \pi_n \xi$ can be decomposed as

$$
\mathcal{W}(T) \to W^{1,2}([-T,T], \mathbb{R}^n) \to C([-T,T], \mathbb{R}^n) \to \mathcal{H}(T).
$$

Here the first operator is induced by π_n, the second is compact by the Arzela-Ascoli theorem, and the last is induced by π_n^*. Now

$$
\|\xi - \pi_n^* \pi_n \xi\|_{\mathcal{H}(T)} \leq \|\mathbb{1} - \pi_n^* \pi_n\|_{\mathcal{L}(W,H)} \|\xi\|_{\mathcal{W}(T)}.
$$

Hence the inclusion $\mathcal{W}(T) \hookrightarrow \mathcal{H}(T)$ is a uniform limit of compact operators and is therefore compact. $\qquad\square$

Lemma 3.7 *There exist constants $c > 0$ and $T > 0$ such that*

$$
\|\xi\|_{\mathcal{W}} \leq c \left(\|\xi\|_{\mathcal{H}(T)} + \|D_A \xi\|_{\mathcal{H}} \right)
$$

for every $\xi \in \mathcal{W}$.

Proof: The proof is analogous to the proof of (2). The *first step* is to prove the estimate with $T = \infty$. For every $\xi \in C_0^\infty(\mathbb{R}, W)$

$$
\begin{aligned}
\|D_A \xi\|_{\mathcal{H}}^2 &= \|\dot{\xi}\|_{\mathcal{H}}^2 + \|A\xi\|_{\mathcal{H}}^2 - 2\int_{-\infty}^{\infty} \langle \dot{\xi}, A\xi \rangle \, dt \\
&= \|\dot{\xi}\|_{\mathcal{H}}^2 + \|A\xi\|_{\mathcal{H}}^2 + \int_{-\infty}^{\infty} \langle \xi, \dot{A}\xi \rangle \, dt \\
&\geq \|\dot{\xi}\|_{\mathcal{H}}^2 + \|A\xi\|_{\mathcal{H}}^2 - c_0 \|\xi\|_{\mathcal{H}} \|\xi\|_{L^2(\mathbb{R},W)} \\
&\geq \|\dot{\xi}\|_{\mathcal{H}}^2 + \frac{1}{c_1} \|\xi\|_{L^2(\mathbb{R},W)}^2 - \|\xi\|_{\mathcal{H}}^2 - c_0 \|\xi\|_{\mathcal{H}} \|\xi\|_{L^2(\mathbb{R},W)} \\
&\geq \|\dot{\xi}\|_{\mathcal{H}}^2 + \frac{1}{2c_1} \|\xi\|_{L^2(\mathbb{R},W)}^2 - \left(1 + \frac{c_0 c_1}{2}\right) \|\xi\|_{\mathcal{H}}^2 \\
&\geq \frac{1}{2c_1} \|\xi\|_{\mathcal{W}}^2 - c\|\xi\|_{\mathcal{H}}^2.
\end{aligned}
$$

10

The *second step* is to prove the estimate with A replaced by a constant bijective operator $A(t) \equiv A_0$. The associated operator D_{A_0} satisfies

$$\|\xi\|_{\mathcal{W}} \leq c \|D_{A_0}\xi\|_{\mathcal{H}} \tag{8}$$

In terms of the Fourier transform the equation $D_{A_0}\xi = \eta$ can be rewritten as

$$i\omega\hat{\xi}(i\omega) - A_0\hat{\xi}(i\omega) = \hat{\eta}(i\omega).$$

Since the operator A_0 is symmetric we have

$$|\omega| \|\xi\|_H^2 \leq |\langle i\omega\xi - A_0\xi, \xi\rangle| \leq \|i\omega\xi - A_0\xi\|_H \|\xi\|_H$$

and hence

$$|\omega| \|\xi\|_H \leq \|i\omega\xi - A_0\xi\|_H.$$

With $c = \|A_0^{-1}\|_{\mathcal{L}(H,W)}$ we obtain

$$
\begin{aligned}
\|\xi\|_W &\leq c\|A_0\xi\|_H \\
&\leq c\|i\omega\xi - A_0\xi\|_H + c|\omega| \|\xi\|_H \\
&\leq 2c\|i\omega\xi - A_0\xi\|_H.
\end{aligned}
$$

Hence it follows from the Fourier-Plancherel theorem that for every $\xi \in \mathcal{W}$

$$
\begin{aligned}
\|\xi\|_{\mathcal{W}}^2 &= \int_{-\infty}^{\infty} \left(\|\hat{\xi}(i\omega)\|_W^2 + \omega^2\|\hat{\xi}(i\omega)\|_H^2 \right) d\omega \\
&\leq (1 + 4c^2) \int_{-\infty}^{\infty} \|i\omega\hat{\xi}(i\omega) - A_0\hat{\xi}(i\omega)\|_H^2 d\omega \\
&= (1 + 4c^2)\|D_{A_0}\xi\|_{\mathcal{H}}^2.
\end{aligned}
$$

This proves (8).

The *final step* uses a patching argument and is analogous to the patching argument in the proof of (2). $\qquad\square$

Assume $A : W \to H$ is self-adjoint and $\xi, \eta \in H$ such that

$$\langle A\phi, \xi\rangle = \langle\phi, \eta\rangle$$

for every $\phi \in W$. Then $\xi \in W$ (see Remark 3.3). In other words every 'weak solution' $\xi \in H$ of $A\xi = \eta$ with $\eta \in H$ is a 'strong solution'. The following theorem says that a similar result holds for the operator D_A. Since D_A is not not self-adjoint we must use the formal adjoint operator of D_A to define the notion of a weak solution. The formula

$$\langle\dot{\phi} + A\phi, \xi\rangle_{\mathcal{H}} + \langle\phi, D_A\xi\rangle_{\mathcal{H}} = 0$$

for $\phi, \xi \in \mathcal{W}$ shows that $-D_{-A}$ is the formal adjoint operator of D_A.

11

Theorem 3.8 (Elliptic regularity) *Assume $\xi, \eta \in \mathcal{H}$ satisfy*

$$\langle \dot{\phi} + A\phi, \xi \rangle_{\mathcal{H}} + \langle \phi, \eta \rangle_{\mathcal{H}} = 0$$

for every $\phi \in C_0^\infty(\mathbb{R}, W)$. Then $\xi \in \mathcal{W}$ and $D_A \xi = \eta$.

Proof: We first prove (in four steps) that the theorem holds under the assumption that ξ and η are supported in an interval I such that $A(t) : W \to H$ is bijective with

$$\|A(t)^{-1}\|_{\mathcal{L}(H,W)} \leq c$$

for $t \in I$.

Step 1: *$\xi \in W^{1,2}(\mathbb{R}, W^*)$ and*

$$\dot{\xi}(t) = A(t)\xi(t) + \eta(t) \tag{9}$$

where $A(t) \in \mathcal{L}(H, W^*)$ as in Remark 3.3.

For $\phi \in C_0^\infty(\mathbb{R}, W)$

$$
\begin{aligned}
\int_{-\infty}^{\infty} \left\langle \dot{\phi}(t), \xi(t) \right\rangle_H dt &= -\int_{-\infty}^{\infty} \langle \phi(s), A(s)\xi(s) + \eta(s) \rangle_{W,W^*} ds \\
&= -\int_{-\infty}^{\infty} \left\langle \dot{\phi}(t), \int_t^{\infty} \left(A(s)\xi(s) + \eta(s) \right) ds \right\rangle_{W,W^*} dt.
\end{aligned}
$$

Since the derivatives of test functions ϕ are dense in $L^2(\mathbb{R}, W)$ this implies Step 1.

Now choose a smooth cutoff function $\rho : \mathbb{R} \to \mathbb{R}$ such that $\rho(t) = 0$ for $|t| \geq 1$, $\rho(t) \geq 0$ and $\int \rho = 1$. For $\delta > 0$ define $\rho_\delta(t) = \delta^{-1}\rho(\delta^{-1}t)$.

Step 2: *For $\delta > 0$ sufficiently small we have $\xi_\delta = \rho_\delta * \xi \in \mathcal{W}$.*

Multiply equation (9) by A^{-1} to obtain $\xi = A^{-1}\dot{\xi} - A^{-1}\eta$ and convolve with ρ_δ:

$$
\begin{aligned}
\xi_\delta &= \rho_\delta * \left(A^{-1}\dot{\xi} \right) - \rho_\delta * \left(A^{-1}\eta \right) \\
&= \dot{\rho}_\delta * \left(A^{-1}\xi \right) + \rho_\delta * \left(A^{-1}\dot{A}A^{-1}\xi \right) - \rho_\delta * \left(A^{-1}\eta \right) \\
&= \dot{\rho}_\delta * \left(A^{-1}\xi \right) + \rho_\delta * \left(A^{-1}\zeta \right)
\end{aligned}
$$

where $\zeta = \dot{A}A^{-1}\xi - \eta \in \mathcal{H}$. We have used $\rho * (u\dot{v}) = \dot{\rho} * (uv) - \rho * (\dot{u}v)$.

Step 3: *There exists a constant $c > 0$ such that*

$$\|D_A(\rho_\delta * \xi)\|_{\mathcal{H}} \leq c$$

for every sufficiently small δ.

Use step 2 and the identity $\dot{\xi}_\delta = \dot{\rho}_\delta * \xi$ to obtain

$$
\begin{aligned}
D_A \xi_\delta &= \dot{\xi}_\delta - A \xi_\delta \\
&= \dot{\rho}_\delta * \xi - A\dot{\rho}_\delta * (A^{-1}\xi) + A\rho_\delta * (A^{-1}\zeta) \\
&= A(A^{-1}\dot{\rho}_\delta * \xi - \dot{\rho}_\delta * (A^{-1}\xi)) + A\rho_\delta * (A^{-1}\zeta).
\end{aligned}
$$

The second term on the right is bounded in $\mathcal{H}$, uniformly in δ. For the other term we have

$$
\begin{aligned}
&\left\| (A^{-1}\dot{\rho}_\delta * \xi(t) - \dot{\rho}_\delta * (A^{-1}\xi))(t) \right\|_W \\
&= \left\| \int_{t-\delta}^{t+\delta} \frac{1}{\delta}\dot{\rho}\left(\frac{t-s}{\delta}\right) \frac{A^{-1}(t) - A^{-1}(s)}{\delta}\xi(s)\,ds \right\|_W \\
&\leq c \int_{-\infty}^{\infty} \left| \frac{1}{\delta}\dot{\rho}\left(\frac{t-s}{\delta}\right) \right| \, \|\xi(s)\|_H \, ds.
\end{aligned}
$$

Here c is a uniform bound for the derivative of A^{-1} on I. By Young's inequality

$$
\left\| A^{-1}\dot{\rho}_\delta * \xi - \dot{\rho}_\delta * (A^{-1}\xi) \right\|_{L^2(\mathbb{R},W)} \leq c\|\dot{\rho}\|_{L^1(\mathbb{R})}\|\xi\|_{\mathcal{H}}.
$$

This proves Step 3.

Step 4: $\xi \in \mathcal{W}$ and $D_A \xi = \eta$.

It follows from Step 2 and Lemma 3.7 that $\|\xi_\delta\|_{\mathcal{W}} \leq c$ for some constant c independent of δ. Choose a sequence $\delta_\nu \to 0$ such that ξ_{δ_ν} converges weakly in $\mathcal{W}$. Let $\xi_0 \in \mathcal{W}$ denote the weak limit. Then ξ_{δ_ν} converges weakly to ξ_0 in $\mathcal{H}$. On the other hand $\xi_{\delta_\nu} = \rho_{\delta_\nu} * \xi$ converges strongly to ξ in $\mathcal{H}$ and hence $\xi = \xi_0 \in \mathcal{W}$. Now it follows from (9) that $D_A \xi = \dot{\xi} - A\xi = \eta$.

This proves the theorem under the assumption that ξ and η are supported in an interval on which A is bijective. Cover the real axis by finitely many open intervals I_j such that $\lambda_j \mathbb{1} - A(t) : W \to H$ is bijective with

$$
\|(\lambda_j \mathbb{1} - A(t))^{-1}\|_{\mathcal{L}(H,W)} \leq c_j
$$

for $t \in I_j$. Now choose a partition of unity β_j subordinate to the cover. Then the function $\xi_j = \beta_j \xi$ is a weak solution of

$$
\dot{\xi}_j - A_j \xi_j = \eta_j
$$

where

$$
A_j = A - \lambda_j \mathbb{1}, \qquad \eta_j = \beta_j \eta + (\dot{\beta}_j + \lambda_j \beta_j)\xi.
$$

By the special case $\xi_j \in \mathcal{W}$ and hence $\xi = \sum_j \xi_j \in \mathcal{W}$. $\qquad\square$

Remark 3.9 The previous theorem only requires the estimate of Lemma 3.7 with $I = \mathbb{R}$. Hence it continues to hold if the limits $A^\pm$ do not exist. Moreover, local regularity does not require bounds on the function $A : \mathbb{R} \to \mathcal{L}_{\mathrm{sym}}(W, H)$. However, we cannot dispense with the assumption that $A(t)$ be self-adjoint.

Theorem 3.10 *The operator D_A is Fredholm.*

Proof: By Lemma 3.7 and Lemma 3.6 the operator D_A has a finite dimensional kernel and a closed range. By Theorem 3.8 the cokernel of D_A is the kernel of the operator $D_{-A} : \mathcal{W} \to \mathcal{H}$. Hence the cokernel of D_A is finite dimensional. $\square$

Theorem 3.11 *Assume $A : \mathbb{R} \to \mathcal{L}(W, H)$ is of class C^{k-1} with $d^k A/dt^k$ weakly continuous and $d^j A/dt^j$ uniformly bounded for $0 \leq j \leq k$. If $\xi \in \mathcal{W}$ and*

$$D_A \xi = \eta \in W^{k,2}(\mathbb{R}, H)$$

then

$$\xi \in W^{k,2}(\mathbb{R}, W) \cap W^{k+1,2}(\mathbb{R}, H).$$

Proof: The proof is by induction on k. Assume $k = 1$. Then $\xi_1 = \dot{\xi}$ is a 'weak solution' of $D_A \xi_1 = \eta_1$ with $\eta_1 = \dot{A}\xi + \dot{\eta} \in L^2(\mathbb{R}, H)$. To see this note that

$$\xi_1 = A\xi + \eta \in W^{1,2}(\mathbb{R}, W^*)$$

with

$$\dot{\xi}_1 = \dot{A}\xi + A\dot{\xi} + \dot{\eta} = A\xi_1 + \eta_1.$$

By Theorem 3.8 this implies that $\xi_1 \in \mathcal{W}$ and hence hence $\xi \in W^{1,2}(\mathbb{R}, W) \cap W^{2,2}(\mathbb{R}, H)$.

Suppose by induction that the statement has been proved for $k \geq 1$. Let $\eta \in W^{k+1,2}(\mathbb{R}, H)$ and $\xi \in \mathcal{W}$ with $D_A \xi = \eta$. By what we just proved $\dot{\xi} \in \mathcal{W}$ and

$$D_A \dot{\xi} = \dot{A}\xi + \dot{\eta} \in W^{k,2}(\mathbb{R}, H).$$

Hence it follows from the induction hypothesis that

$$\dot{\xi} \in W^{k,2}(\mathbb{R}, W) \cap W^{k+1,2}(\mathbb{R}, H).$$

This proves the theorem. $\square$

Proposition 3.12 *Suppose $A(t)$ is bijective for all t and*

$$\|A(t)^{-1}\|_{\mathcal{L}(H,W)} \|\dot{A}(t)\|_{\mathcal{L}(W,H)} \|A(t)^{-1}\|_{\mathcal{L}(H,H)} < 2. \tag{10}$$

Then D_A is bijective.

Proof: Let $\xi \in \ker D_A$. Then it follows from Theorem 3.11 that

$$\xi \in W^{1,2}(\mathbb{R}, W) \cap W^{2,2}(\mathbb{R}, H).$$

Hence

$$\frac{d^2}{dt^2}\|\xi\|_H^2 = 4\|A\xi\|_H^2 + 2\langle \xi, \dot{A}\xi \rangle. \tag{11}$$

The right hand side is continuous and hence the function $t \mapsto \|\xi(t)\|_H^2$ is C^2. By (10) this function is convex. Since it is integrable on $\mathbb{R}$ it must vanish. Hence D_A is injective. The same argument with A replaced by $-A$ shows that D_A is onto. $\qquad\square$

Corollary 3.13 *If $A(t)$ is bijective for all t then D_A has Fredholm index 0.*

Proof: The operator family $A(\varepsilon t)$ satisfies the hypotheses of Proposition 3.12 for small ε. $\qquad\square$

Example 3.14 If we drop the hypothesis that each $A(t)$ is self-adjoint then D_A need not be Fredholm. For example, let $H = W^{1,2}(S^1) \times L^2(S^1)$, $W = W^{2,2}(S^1) \times W^{1,2}(S^1)$, and let $A_0 : W \to H$ be defined by

$$A_0(u, v) = (v, u'').$$

This operator $A_0 : W \to H$ has a compact resolvent and is the infinitesimal generator of a strongly continuous one parameter group $U(t) \in \mathcal{L}(H)$ of unitary operators on H. Define

$$A(t) = A_0 - b(t)\mathbb{1}$$

where $b(t) = 1$ for $t \leq -1$ and $b(t) = -1$ for $t \geq 1$. Then the Cauchy problem

$$\dot{\xi}(t) = A(t)\xi(t), \qquad \xi(0) = \xi_0 \in W,$$

is well-posed and all solutions converge to zero exponentially as t tends to $\pm\infty$. Hence the kernel of D_A is infinite dimensional. Hence the operator D_A is not Fredholm.

In contrast to the previous example 'lower order perturbations' of A always produce Fredholm operators. The perturbation is a multiplication operator induced by $C(t) : W \to H$. We assume that the function $C : \mathbb{R} \to \mathcal{L}(W, H)$ is continuous in the norm topology such that $C(t) : W \to H$ is a compact operator for every t and

$$\lim_{|t| \to \infty} \|C(t)\|_{\mathcal{L}(W,H)} = 0.$$

Remark 3.15 If $B : \mathbb{R} \to \mathcal{L}(H, H)$ is strongly continuous and converges to 0 in the norm topology as t tends to $\pm\infty$ then the operator family $C(t) = B(t)|_W$ satisfies the above requirements.

15

Lemma 3.16 *The operator*

$$\mathcal{W} \to \mathcal{H} : \xi \mapsto C\xi$$

is compact.

Proof: First assume that C is compactly supported in an interval I. Choose projection operators $\pi_n : H \to \mathbb{R}^n$ as in Lemma 3.6. Then the operator

$$C_n(t) = \pi_n^* \pi_n C(t) : W \to H$$

converges in the norm topology to the operator $C(t) \in \mathcal{L}(W, H)$ and the convergence is uniform in t since $C : \mathbb{R} \to \mathcal{L}(W, H)$ is continuous in the norm topology. The multiplication operator induced by C_n can be decomposed as

$$\mathcal{W} \to W^{1,2}(I, \mathbb{R}^n) \to L^2(I, \mathbb{R}^n) \to \mathcal{H}.$$

Here the first operator is induced by $\pi_n \circ C$, the second is compact, and the last is induced by π_n^*. Since the operator $C_n : \mathcal{W} \to \mathcal{H}$ converges to $C : \mathcal{W} \to \mathcal{H}$ in the norm topology it follows that the operator C is compact. In the general case use a cutoff function to approximate C in the norm topology by operators with compact support. $\square$

Corollary 3.17 *The operator $D_{A+C} : \mathcal{W} \to \mathcal{H}$ defined by*

$$D_{A+C}\xi = \dot{\xi} - A\xi - C\xi$$

is Fredholm. It has the same index as D_A:

$$\operatorname{index} D_{A+C} = \operatorname{index} D_A.$$

Remark 3.18 Assume that the curve $A : \mathbb{R} \to \mathcal{L}(W, H)$ is continuous in the norm topology and satisfies (A-2) and (A-3). We were not able to prove under these assumptions that D_A is Fredholm.

4 The spectral flow

We continue the notation of the previous section. For $A : \mathbb{R} \to \mathcal{L}(W, H)$ and $t \in \mathbb{R}$ we define the **crossing operator** by

$$\Gamma(A, t) = P\dot{A}(t)P|_{\ker A}$$

where $P : H \to H$ denotes the orthogonal projection onto the kernel of A. A **crossing** for A is a number $t \in \mathbb{R}$ for which $A(t)$ is not injective. The set of crossings is compact. A crossing t is called **regular** if the crossing operator $\Gamma(A, t)$ is nonsingular. It is called **simple** if it is regular and in addition

$\dim \ker A = 1$. If t_0 is a simple crossing then there is a unique real valued C^1 function $\lambda = \lambda(t)$ defined near t_0 such that $\lambda(t)$ is an eigenvalue of $A(t)$ and $\lambda(t_0) = 0$. This function is called the **crossing eigenvalue**. For a simple crossing the crossing operator $\Gamma(A, t_0)$ is given by multiplication with $\dot{\lambda}(t_0)$ and hence $\dot{\lambda}(t_0) \neq 0$.

Theorem 4.1 *Assume A satisfies (A-1), (A-2), (A-3) and has only regular crossings. Let D_A be defined by (6). Then the set of crossings is finite and*

$$\text{index } D_A = -\sum_t \text{sign } \Gamma(A, t) \tag{12}$$

where the sum is over all crossings t and sign denotes the signature (the number of positive eigenvalues minus the number of negative eigenvalues). Hence for a curve having only simple crossings

$$\text{index } D_A = -\sum_t \text{sign } \dot{\lambda}(t) \tag{13}$$

where λ denotes the crossing eigenvalue at t.

Note that the direct sum of two curves having only regular crossings again has only regular crossings. The analogous result fails for simple crossings. Indeed $A \oplus A$ has no simple crossings. We will see in this section that the property of having only simple crossings is generic so that (13) suffices. On the other hand the following theorem shows how to use formula (12) without perturbing A.

Theorem 4.2 *The curve $A - \delta \mathbb{1}$ has only regular crossings for almost every $\delta \in \mathbb{R}$.*

To prove these results we characterize the Fredholm index axiomatically and show that the right hand sides of the formulae (12) and (13) satisfy these axioms. It is convenient to introduce some notation.

Denote the Banach space of bounded symmetric operators from W to H by

$$\mathcal{L}_{\text{sym}}(W, H) = \{A \in \mathcal{L}(W, H) : A^*|_W = A\}$$

and let $\mathcal{S}(W, H) \subset \mathcal{L}_{\text{sym}}(W, H)$ be the open subset consisting of those operators with nonempty resolvent set, i.e. the operator

$$\lambda \mathbb{1} - A : W \to H$$

is bijective for some real number λ. This means that the operator A, when regarded as an unbounded operator on H with dense domain $W = \text{dom } A$ is self-adjoint with compact resolvent. (See remark 3.2.)

Denote by $\mathcal{B} = \mathcal{B}(\mathbb{R}, W, H)$ the Banach space of all continuous[1] maps $A : \mathbb{R} \to \mathcal{L}_{\mathrm{sym}}(W, H)$ which have limits

$$A^{\pm} = \lim_{t \to \pm\infty} A(t)$$

in the norm topology. Denote by $\mathcal{B}^1 = \mathcal{B}^1(\mathbb{R}, W, H) \subset \mathcal{B}$ the Banach space of those $A \in \mathcal{B}$ which are continuously differentiable in the norm topology and satisfy

$$\|A\|_{\mathcal{B}^1} = \sup_{t \in \mathbb{R}} \left(\|A(t)\| + \|\dot{A}(t)\| \right) < \infty.$$

Define an open subset

$$\mathcal{A} = \mathcal{A}(\mathbb{R}, W, H) \subset \mathcal{B}(\mathbb{R}, W, H)$$

consisting of those $A \in \mathcal{B}$ for which the limit operators $A^{\pm} : W \to H$ are bijective and $A(t) \in \mathcal{S}(W, H)$ for each $t \in \mathbb{R}$. The set

$$\mathcal{A}^1 = \mathcal{A}^1(\mathbb{R}, W, H) = \mathcal{A} \cap \mathcal{B}^1$$

is open in $\mathcal{B}^1$. The set $\mathcal{A}$ consists of all maps $A : \mathbb{R} \to \mathcal{L}(W, H)$ which are continuous in the norm topology and satisfy (A-2), and (A-3). The set $\mathcal{A}^1$ consists of all maps $A : \mathbb{R} \to \mathcal{L}(W, H)$ which satisfy (A-1), (A-2), (A-3) and in addition are continuously differentiable in the norm topology. Theorem 3.10 implies that D_A is a Fredholm operator for every $A \in \mathcal{A}^1$.

Given $A_i \in \mathcal{A}(\mathbb{R}, W_i, H_i)$, $i = 1, 2$, the **direct sum**

$$A_1 \oplus A_2 \in \mathcal{A}(\mathbb{R}, W_1 \oplus W_2, H_1 \oplus H_2)$$

is defined pointwise. Given $A, A_\ell, A_r \in \mathcal{A}(\mathbb{R}, W, H)$ we say that A is the **catenation** of A_ℓ and A_r if

$$A(t) = \begin{cases} A_\ell(t) & \text{for } t \leq 0 \\ A_r(t) & \text{for } t \geq 0 \end{cases}$$

and $A_\ell(t) = A(0) = A_r(-t)$ for $t > 0$. In this case we write

$$A = A_\ell \# A_r.$$

Note that the operation $(A_\ell, A_r) \mapsto A_\ell \# A_r$ is only partially defined.

Theorem 4.3 *There exist unique maps $\mu : \mathcal{A}(\mathbb{R}, W, H) \to \mathbb{Z}$, one for every compact dense injection of Hilbert spaces $W \hookrightarrow H$, satisfying the following axioms.*

(homotopy) *μ is constant on the connected components of $\mathcal{A}(\mathbb{R}, W, H)$.*

---

[1] *Continuous* means continuous in the norm topology unless otherwise mentioned.

(constant) *If A is constant, then $\mu(A) = 0$.*

(direct sum) $\mu(A_1 \oplus A_2) = \mu(A_1) + \mu(A_2)$.

(catenation) *If $A = A_\ell \# A_r$, then $\mu(A) = \mu(A_\ell) + \mu(A_r)$.*

(normalization) *For $W = H = \mathbb{R}$ and $A(t) = \arctan(t)$, we have $\mu(A) = 1$.*

The number $\mu(A)$ is called the **spectral flow** *of A.*

Remark 4.4 Choose $A \in \mathcal{A}(\mathbb{R}, W, H)$. Then there exists a constant $\varepsilon > 0$ such that if $A_0, A_1 \in \mathcal{A}^1(\mathbb{R}, W, H)$ satisfy

$$\sup_{t \in \mathbb{R}} \|A(t) - A_j(t)\|_{\mathcal{L}(W,H)} \leq \varepsilon$$

then the path $A_\tau = (1 - \tau)A_0 + \tau A_1$ lies in $\mathcal{A}^1(\mathbb{R}, W, H)$ for $0 \leq \tau \leq 1$. Moreover, if $A_0, A_1 \in \mathcal{A}^1(\mathbb{R}, W, H)$ are homotopic by a continuous homotopy in $\mathcal{A}(\mathbb{R}, W, H)$ then they are homotopic by a C^1-homotopy in $\mathcal{A}^1(\mathbb{R}, W, H)$. Hence any homotopy invariant on $\mathcal{A}^1(\mathbb{R}, W, H)$ extends uniquely to a homotopy invariant on $\mathcal{A}(\mathbb{R}, W, H)$.

We may write the set $\mathcal{S} = \mathcal{S}(W, H)$ as an infinite disjoint union

$$\mathcal{S}(W, H) = \bigcup_{k=0}^{\infty} \mathcal{S}_k(W, H)$$

where $\mathcal{S}_k = \mathcal{S}_k(W, H)$ denotes the set of operators $L \in \mathcal{S}(W, H)$ with k-dimensional kernel. The set $\mathcal{S}_k$ is a submanifold of $\mathcal{S}$ of codimension $k(k+1)/2$. The tangent space to $\mathcal{S}_k$ at a point $L \in \mathcal{S}_k$ is given by

$$T_L \mathcal{S}_k = \left\{ \hat{L} \in \mathcal{L}_{\mathrm{sym}}(W, H) : P\hat{L}P = 0 \right\}$$

where $P : H \to H$ denotes the orthogonal projection onto the kernel of L. In other words a curve $A \in \mathcal{A}$ is tangent to $\mathcal{S}_k$ at $t = 0$ if and only if $A(0) \in \mathcal{S}_k$ and the crossing operator $\Gamma(A, 0) = 0$. Since $\mathcal{S}_1$ has codimension 1 a curve has only simple crossings if and only it is transverse to $\mathcal{S}_k$ for every k. (Recall that $\mathcal{S}_k$ has codimension greater than 1 for $k \geq 2$ and hence a curve is transverse to $\mathcal{S}_k$ if and only if it does not intersect $\mathcal{S}_k$.)

Proof of Theorem 4.3: For $A \in \mathcal{A}^1(\mathbb{R}, W, H)$ the number $\mu(A)$ can be defined as the intersection number of the curve A with the cycle $\mathcal{S}_1$ (appropriately oriented) as in [18]. This requires showing that the set of curves transverse to all $\mathcal{S}_k$ is open and dense in $\mathcal{A}^1$ and that the set of homotopies which are transverse to all $\mathcal{S}_k$ is dense. The assertion about curves can be proved using the transversality theory in [1]. In the notation of [1] ne uses the representation of maps $\rho : \mathcal{A} \to C^1(X, Y)$ with $\mathcal{A} = \mathcal{A}^1(\mathbb{R}, W, H)$, $X = \mathbb{R}$, $Y = \mathcal{S}(W, H)$,

19

$W = \mathcal{S}_k(W, H)$ and ρ the inclusion. Homotopies are handled similarly but a preliminary smoothing argument is required. By Remark 4.4 the definition of $\mu(A)$ as the intersection number with $\mathcal{S}_1$ extends to $A \in \mathcal{A}(\mathbb{R}, W, H)$.

If the curve $A \in \mathcal{A}^1(\mathbb{R}, W, H)$ is transverse to each $\mathcal{S}_k$ then the intersection number of A with $\mathcal{S}_1$ is given by the explicit formula

$$\mu(A) = \sum_t \operatorname{sign} \dot\lambda(t)$$

where the right hand side is as in (13). The transversality argument shows that this intersection number satisfies the homotopy axiom. The other axioms are obvious. The proof that the axioms characterize the spectral flow requires the following

Theorem 4.5 *For every $A \in \mathcal{A}(\mathbb{R}, W, H)$ there exists an integer m and a path of matrices $B \in \mathcal{A}(\mathbb{R}, \mathbb{R}^m, \mathbb{R}^m)$ such that $A \oplus B$ is homotopic to a constant path.*

Proof: We prove the theorem in three steps.

Step 1: *If $A(t)$ is bijective for every t then the theorem holds with $m = 0$.*

Homotop A to a constant $A(0)$ using the formula $A_\tau(t) = A(\tan(\tau \arctan t))$.

Step 2: *If $A \in \mathcal{A}^1(\mathbb{R}, W, H)$ has m simple crossings then there exists a curve $b \in \mathcal{A}^1(\mathbb{R}, \mathbb{R}, \mathbb{R})$ such that $A \oplus b$ is homotopic to a curve with $m-1$ simple crossings.*

Assume A has a simple crossing at $t = t_0$ and let $\lambda(t) \in \sigma(A(t))$ be the crossing eigenvalue for t near t_0. For $\varepsilon > 0$ sufficiently small choose a C^1-curve of eigenvectors $\zeta : (t_0 - \varepsilon, t_0 + \varepsilon) \to H$ such that $\lambda(t)\zeta(t) = A(t)\zeta(t)$ and $\|\zeta(t)\|_H = 1$. Define $\pi(t) : H \to \mathbb{R}$ by

$$\pi(t)\xi = \langle \zeta(t), \xi \rangle.$$

Moreover, choose a smooth cutoff function $\beta : \mathbb{R} \to [0, 1]$ such that $\beta(t) = 1$ for $|t - t_0| \le \varepsilon/2$ and $\beta(t) = 0$ for $|t - t_0| \ge \varepsilon$. Finally, choose a C^1-function $b : \mathbb{R} \to \mathbb{R}$ such that

$$b(t) = -\lambda(t) \quad \text{for} \quad |t - t_0| \le \varepsilon/2,$$

$b(t) \ne 0$ for $t \ne t_0$, and $b(t)$ is constant for $|t| \ge \varepsilon$. Consider the curve $\tilde{A} \in \mathcal{A}^1(\mathbb{R}, W \oplus \mathbb{R}, H \oplus \mathbb{R})$ defined by

$$\tilde{A}_\delta(t) = \begin{pmatrix} A(t) & \delta\beta(t)\pi(t)^* \\ \delta\beta(t)\pi(t) & b(t) \end{pmatrix}.$$

Then $\tilde{A}_0 = A \oplus b$ and for $\delta > 0$ the curve $\tilde{A}_\delta$ has no crossing at $t = t_0$.

Step 3: *The general case.*

20

By transversality homotop A to a curve in $\mathcal{A}^1(\mathbb{R}, W, H)$ with only simple crossings. Now use step 2 inductively to construct a curve $B \in \mathcal{A}^1(\mathbb{R}, \mathbb{R}^m, \mathbb{R}^m)$ such that $A \oplus B$ is homotopic to a curve without crossings. Finally use step 1. $\square$

Proof of Theorem 4.3 continued: Denote by $\mu(A)$ the spectral flow as defined by intersection numbers. Let $\tilde{\mu} : \mathcal{A}(\mathbb{R}, W, H) \to \mathbb{Z}$ be any putative spectral flow which satisfies the axioms of Theorem 4.3. We prove that $\tilde{\mu} = \mu$. To see this note that a curve of matrices $B \in \mathcal{A}(\mathbb{R}, \mathbb{R}^m, \mathbb{R}^m)$ is homotopic to a curve of diagonal matrices. Hence it follows from the homotopy, direct sum, and normalization axioms that

$$\tilde{\mu}(B) = \mu(B) = \tfrac{1}{2}\mathrm{sign}\, B^+ - \tfrac{1}{2}\mathrm{sign}\, B^-.$$

Now let $A \in \mathcal{A}(\mathbb{R}, W, H)$ be any curve and choose $B \in \mathcal{A}(\mathbb{R}, \mathbb{R}^m, \mathbb{R}^m)$ as in Theorem 4.5. Then it follows from the homotopy and constant axioms that $\tilde{\mu}(A \oplus B) = 0$. Hence $\tilde{\mu}(A) = -\tilde{\mu}(B) = -\mu(B) = \mu(A)$. This proves the theorem. $\square$

We have not used the catenation axiom to prove uniqueness of the spectral flow. Hence the catenation axiom follows from the other axioms. Here is a direct proof of this observation.

Proposition 4.6 *The catenation axiom follows from the homotopy, direct sum, and constant axioms.*

Proof: Let $A_\ell, A_r \in \mathcal{A}(\mathbb{R}, W, H)$ such that $A_\ell(t) = A_r(-t)$ for $t \geq -1$. Call this constant operator L. We construct a homotopy $A_\tau \in \mathcal{A}(\mathbb{R}, W \oplus W, H \oplus H)$ such that

$$A_0 = A_\ell \oplus A_r, \qquad A_1 = A_\ell \# A_r \oplus L.$$

This homotopy is given by $A_\tau(t) = A_\ell(t) \oplus A_r(t)$ for $t \leq 0$ and

$$A_\tau(t) = R\left(-\frac{\pi\tau}{2}\right) \left(\begin{array}{cc} A_\ell(t) & 0 \\ 0 & A_r(t) \end{array} \right) R\left(\frac{\pi\tau}{2}\right)$$

for $t \geq 0$. Here $R(\theta)$ is the Hilbert space isomorphism of $H \oplus H$ defined by

$$R(\theta) = \left(\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right).$$

Note in particular that $R(\theta)$ commutes with $L \oplus L$. Thus we have proved that $A_0 = A_\ell \oplus A_r$ and $A_1 = A_\ell \# A_r \oplus L$ are homotopic. Hence

$$\begin{aligned}
\mu(A_\ell) + \mu(A_r) &= \mu(A_\ell \oplus A_r) \\
&= \mu(A_\ell \# A_r \oplus L) \\
&= \mu(A_\ell \# A_r) + \mu(L) \\
&= \mu(A_\ell \# A_r).
\end{aligned}$$

21

The first equality follows from the direct sum axiom, the second from the homotopy axiom, and the last from the constant axiom. □

Lemma 4.7 *Assume $A : \mathbb{R} \to \mathcal{L}(W, H)$ satisfies (A-1),(A-2), (A-3) and has only regular crossings. Then the number of crossings is finite and the spectral flow of A is given by*

$$\mu(A) = \sum_t \operatorname{sign} \Gamma(A, t)$$

where the right hand side is as in (12).

To prove this result we require Kato's selection theorem for the eigenvalues of a one parameter family of self-adjoint operators.

Theorem 4.8 (Kato Selection Theorem) *For $A \in \mathcal{A}^1(\mathbb{R}, \mathbb{R}^n, \mathbb{R}^n)$ there exists a C^1 curve of diagonal matrices*

$$\mathbb{R} \to \mathbb{R}^{n \times n} : t \mapsto \Lambda(t) = \operatorname{diag}(\lambda_1(t), \dots, \lambda_n(t))$$

such that $\Lambda(t) \sim A(t)$ for every t. Here the sign $\sim$ denotes similarity. Moreover,

$$\Gamma(\Lambda - \lambda, t) \sim \Gamma(A - \lambda, t)$$

for all t and λ.

This is a reformulation of Theorem II.5.4 and Theorem II.6.8 in [14]. The existence of a continuous family $\Lambda(t)$ which is pointwise similar to $A(t)$ is easy to prove. Simply use the ordering of the real line to select the diagonal entries. In general one cannot choose the similarity continuously. There might not exist a continuous family of bijective matrices $Q(t)$ with $A(t) = Q(t)\Lambda(t)Q(t)^{-1}$. To find a differentiable function $\Lambda(t)$ is much harder.

The functions $\lambda_j(t)$ are the eigenvalues of $A(t)$ counted with multiplicity. The theorem also asserts that the derivatives $\dot{\lambda}_j(t)$ for those j with $\lambda_j(t) = \lambda$ are the eigenvalues of the crossing operator $\Gamma(A - \lambda, t)$, again counted with multiplicity.

Corollary 4.9 *Assume $A : \mathbb{R} \to \mathcal{L}(W, H)$ satisfies (A-1) and (A-2). Let $t_0 \in \mathbb{R}$ and $c > 0$ such that $\pm c \notin \sigma(A(t_0))$. Then there exists a constant $\varepsilon > 0$ and a C^1 function $\Lambda(t)$ of diagonal matrices defined for $t_0 - \varepsilon < t < t_0 + \varepsilon$ such that*

$$\Gamma(\Lambda - \lambda, t) \sim \Gamma(A - \lambda, t)$$

for $t_0 - \varepsilon < t < t_0 + \varepsilon$ and $-c < \lambda < c$. (This implies that the diagonal entries of $\Lambda(t)$ are the eigenvalues of $A(t)$ between $-c$ and c, counted with multiplicity.)

Proof: We prove this by reducing it to the finite dimensional situation. First choose $\varepsilon > 0$ such that $\pm c \notin \sigma(A(t))$ for $t_0 - \varepsilon < t < t_0 + \varepsilon$. Let $E(t)$ denote the sum of the eigenspace of $A(t)$ corresponding to eigenvalues between $-c$ and c. Choose C^1-functions $\xi_j : (t_0 - \varepsilon, t_0 + \varepsilon) \to H$ such that the vectors $\xi_1(t), \ldots, \xi_N(t)$ form an orthonormal basis of $E(t)$ for every t. Define $\pi(t) : H \to \mathbb{R}^N$ by the formula

$$\pi(t)^* x = \sum_{j=1}^{N} x_j \xi_j(t).$$

By the finite dimensional case the theorem holds for the symmetric matrix

$$B(t) = \pi(t) A(t) \pi(t)^*.$$

Differentiating the definition of B gives

$$\dot{B} = \pi \dot{A} \pi^* + \dot{\pi} A \pi^* + \pi A \dot{\pi}^*$$

and hence

$$Q \dot{B} Q = \pi P \dot{A} P \pi^*$$

where P and Q denote the spectral projections for A and B corresponding to the same eigenvalue λ. $\square$

Proof of Lemma 4.7: A curve $A : \mathbb{R} \to \mathcal{L}(W, H)$ which satisfies (A-1), (A-2), and (A-3) has only regular crossings if and only if it is transverse to

$$\bar{\mathcal{S}}_1 = \bigcup_{k \geq 1} \mathcal{S}_k.$$

in the sense that its derivative $\dot{A}(t)$ at a crossing t does not lie in the tangent cone at $L = A(t)$

$$T_L \bar{\mathcal{S}}_1 = \left\{ \hat{L} \in \mathcal{L}_{\text{sym}}(W, H) : 0 \in \sigma(P \hat{L} P|_{\ker L}) \right\}.$$

Choose $t_0 \in \mathbb{R}$ with $A(t_0) \in \bar{\mathcal{S}}_1$. Then 0 is an eigenvalue of $A(t_0)$ with finite multiplicity m. Choose $c > 0$ such that there is no other eigenvalue of $A(t_0)$ in the interval $-c \leq \lambda \leq c$. Now choose $\varepsilon > 0$ such that $\pm c \notin \sigma(A(t))$ for $t_0 - \varepsilon \leq t \leq t_0 + \varepsilon$. By Corollary 4.9 there exist m continuously differentiable curves

$$\lambda_1, \ldots, \lambda_m : [t_0 - \varepsilon, t_0 + \varepsilon] \to (-c, c)$$

representing the eigenvalues of $A(t)$ in $(-c, c)$. Since t_0 is a regular crossing of A it follows that $\dot{\lambda}_j(t_0) \neq 0$ for every j. This proves that the crossings are isolated.

Shrinking ε if necessary we may assume that $\lambda_j(t) \neq 0$ for $0 < |t - t_0| \leq \varepsilon$. This proves that

$$\text{sign } \lambda_j(t_0 + \varepsilon) = -\text{sign } \lambda_j(t_0 - \varepsilon) = \text{sign } \dot{\lambda}_j(t_0).$$

Hence

$$\operatorname{sign}\left(\Gamma(A, t_0)\right) = \#\left\{j : 0 < \lambda_j(t_0 + \varepsilon) < \delta\right\} - \#\left\{j : 0 < \lambda_j(t_0 - \varepsilon) < \delta\right\}.$$

The right hand side is unchanged by small perturbation and agrees with the spectral flow across the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$ for a nearby curve in $\mathcal{A}$ with simple crossings. This proves that the intersection number of A with $\bar{\mathcal{S}}_1$ at t_0 is the signature of the crossing operator. $\square$

Proof of Theorem 4.2: By Corollary 4.9 cover the set $\{(t, \lambda) : \lambda \in \sigma(A(t))\}$ by countably many graphs of smooth curves $t \mapsto \lambda_j(t)$ each defined on an interval $[a_j, b_j]$. By Sard's theorem the complement of the set of common regular values has measure zero. By Corollary 4.9 $\delta \in \mathbb{R}$ is a common regular value of the functions λ_j if an only if $A - \delta \mathbb{1}$ has only regular crossings. $\square$

Proof of Theorem 4.1: We prove that minus the Fredholm index satisfies the axioms of Theorem 4.3:

$$\mathcal{A}^1(\mathbb{R}, W, H) \to \mathbb{Z} : A \mapsto -\operatorname{index} D_A.$$

The homotopy and direct sum axioms are obvious. The constant axiom follows from Corollary 3.13, the normalization axiom from Theorem 2.1, and the catenation axiom from Proposition 4.6. Hence

$$\operatorname{index} D_A = -\mu(A)$$

for $A \in \mathcal{A}^1(\mathbb{R}, W, H)$. To prove this formula in general approximate a curve which satisfy (A-1), (A-2), and (A-3) (but is only continuously differentiable in the weak operator topology) by a curve in $\mathcal{A}^1(\mathbb{R}, W, H)$. Finally use Lemma 4.7. $\square$

We include here some observations about catenation. Assume $A_\ell, A_r \in \mathcal{A}$ such that $A_\ell(t) = A_r(-t)$ for $t \geq 0$. Form the shifted catenation

$$A = A_\ell \#_\tau A_r$$

by $A(t) = A_\ell(t + \tau)$ for $t \leq 0$ and $A(t) = A_r(t - \tau)$ for $t \geq 0$.

Proposition 4.10 *If the operators D_{A_ℓ} and D_{A_r} are onto (resp injective), then the operator D_A is onto (resp injective) for τ sufficiently large.*

Proof: We consider the injective case; the onto case follows by duality. By assumption there exist constants $c_\ell > 0$ and $c_r > 0$ such that

$$\|\xi\|_{\mathcal{W}} \leq c_\ell \|D_{A_\ell} \xi\|_{\mathcal{H}}, \qquad \|\xi\|_{\mathcal{W}} \leq c_r \|D_{A_r} \xi\|_{\mathcal{H}},$$

for every $\xi \in \mathcal{W}$. Choose a nondecreasing cutoff function $\beta : \mathbb{R} \to \mathbb{R}$ such that $\beta(t) = 1$ for $t \geq T$, $\beta(t) = 0$ for $t \leq -T$ and $\dot{\beta}(t) \leq 1/T$. Then for $\tau \geq T$

$$
\begin{aligned}
\|\xi\|_{\mathcal{W}} &\leq \|(1-\beta)\xi\|_{\mathcal{W}} + \|\beta\xi\|_{\mathcal{W}} \\
&\leq c_\ell \|D_{A_\ell}((1-\beta)\xi)\|_{\mathcal{H}} + c_r \|D_{A_r}(\beta\xi)\|_{\mathcal{H}} \\
&\leq c_\ell \|(1-\beta)D_{A_\ell}\xi\|_{\mathcal{H}} + c_r \|\beta D_{A_r}\xi\|_{\mathcal{H}} + (c_\ell + c_r)\frac{1}{T}\|\xi\|_{\mathcal{H}} \\
&\leq (c_\ell + c_r)\left(\|D_A\xi\|_{\mathcal{H}} + \frac{1}{T}\|\xi\|_{\mathcal{H}}\right).
\end{aligned}
$$

This estimate shows that D_A is injective for T sufficiently large. Note that in the estimate

$$\|\xi\|_{\mathcal{W}} \leq c\|D_A\xi\|_{\mathcal{H}}$$

the constant c is independent of $\tau > T$. □

The operator $A = A_\ell \#_\tau A_r$ is constant and bijective on the time interval $-\tau \leq t \leq \tau$. The proof of Proposition 3.12 shows that every ξ in the kernel of D_A satisfies an estimate

$$\frac{d^2}{dt^2}\|\xi\|_H^2 \geq \delta^2\|\xi\|_H^2$$

on this interval. Hence for large τ, $\xi(0)$ must be small. This shows that elements of the kernel of D_A are roughly of the form $\xi = \xi_\ell \#_\tau \xi_r$ where $\xi_\ell \in \ker D_{A_\ell}$ and $\xi_r \in \ker D_{A_r}$. Conversely the catenation of two such elements ξ_ℓ and ξ_r can be approximated by an element in the kernel of D_A. The argument uses Proposition 4.10. This provides an alternative proof for the catenation axiom.

5 The Maslov index

Let (E, ω) be a symplectic vector space and denote by $\mathcal{L} = \mathcal{L}(E, \omega)$ the manifold of Lagrangian subspaces of E. The Maslov index as defined in [21] assigns to every pair of Lagrangian paths $\Lambda, \Lambda' : [a, b] \to \mathcal{L}(E, \omega)$ a half integer $\mu(\Lambda, \Lambda')$. In this section we enumerate the properties of the Maslov index that will be needed in the sequel.

Any two symplectic vector spaces of the same dimension are symplectomorphic. The Maslov satisfies the naturality property

$$\mu(\Psi\Lambda, \Psi\Lambda') = \mu(\Lambda, \Lambda') \tag{14}$$

for a symplectomorphism $\Psi : (E, \omega) \to (E', \omega')$. Hence we will give our definitions for the standard symplectic vector space

$$E = \mathbb{R}^{2n}, \qquad \omega = \omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

The only other example which we need is

$$E = \mathbb{R}^{2n} \times \mathbb{R}^{2n}, \qquad \omega = (-\omega_0) \times \omega_0.$$

In the latter case the graph of a symplectomorphism of $(\mathbb{R}^{2n}, \omega_0)$ is an element of $\mathcal{L}$. The Maslov index has the following properties.

(naturality) Equation (14) holds when Ψ is time dependent.

(homotopy) The Maslov index is invariant under fixed endpoint homotopies

(zero) If $\Lambda(t) \cap \Lambda'(t)$ is of constant dimension then $\mu(\Lambda, \Lambda') = 0$.

(direct sum) If $E = E_1 \oplus E_2$ then

$$\mu(\Lambda_1 \oplus \Lambda_2, \Lambda_1' \oplus \Lambda_2') = \mu(\Lambda_1, \Lambda_1') + \mu(\Lambda_2, \Lambda_2').$$

(catenation) For $a < c < b$

$$\mu(\Lambda, \Lambda') = \mu(\Lambda|_{[a,c]}, \Lambda'|_{[a,c]}) + \mu(\Lambda|_{[c,b]}, \Lambda'|_{[c,b]}).$$

(localization) If $(E, \omega) = (\mathbb{R}^{2n}, \omega_0)$, $\Lambda'(t) = \mathbb{R}^n \times 0$ and $\Lambda(t) = \mathrm{Gr}(A(t))$ for a path $A : [a,b] \to \mathbb{R}^{n \times n}$ of symmetric matrices then the Maslov index of Λ is given by the spectral flow

$$\mu(\Lambda, \Lambda') = \tfrac{1}{2}\mathrm{sign}A(b) - \tfrac{1}{2}\mathrm{sign}A(a). \tag{15}$$

Remark 5.1 These axioms characterize the Maslov index (see [21]). By the localization property the spectral flow of a path of finite dimensional symmetric matrices is a special case of the Maslov index. However, we only define the spectral flow in the case where the matrices $A(a)$ and $A(b)$ are invertible whereas the Maslov index is defined for any path. The reason for the former is that the operator D_A is not Fredholm unless $A^{\pm}$ are invertible. The reason for the latter is that it is often necessary to consider Lagrangian pairs with $\Lambda(a) = \Lambda'(a)$.

For $t \in [a,b]$ and $\Lambda'(t) = V$ constant the **crossing form** $\Gamma(\Lambda, V, t)$ is a quadratic form on $\Lambda(t) \cap V$ defined as follows. Let W be a fixed Lagrangian complement of $\Lambda(t)$. For $v \in \Lambda(t) \cap V$ and $s - t$ small define $w(s) \in W$ by $v + w(s) \in \Lambda(s)$. The form

$$\Gamma(\Lambda, V, t)(v) = \frac{d}{ds}\bigg|_{s=t} \omega(v, w(s))$$

is independent of the choice of W. In general the crossing form $\Gamma(\Lambda, \Lambda', t)$ is defined on $\Lambda(t) \cap \Lambda'(t)$ and is given by

$$\Gamma(\Lambda, \Lambda', t) = \Gamma(\Lambda, \Lambda'(t), t) - \Gamma(\Lambda', \Lambda(t), t).$$

A **crossing** is a time $t \in [a, b]$ such that $\Lambda(t) \cap \Lambda'(t) \neq \{0\}$. A crossing is called **regular** if $\Gamma(\Lambda, \Lambda', t)$ is nondegenerate. It is called **simple** if in addition $\Lambda(t) \cap \Lambda'(t)$ is one dimensional. For a pair with only regular crossings the **Maslov index** is defined by

$$\mu(\Lambda, \Lambda') = \tfrac{1}{2}\mathrm{sign}\,\Gamma(\Lambda, \Lambda', a) + \sum_{a < t < b} \mathrm{sign}\,\Gamma(\Lambda, \Lambda', t) + \tfrac{1}{2}\mathrm{sign}\,\Gamma(\Lambda, \Lambda', b).$$

Since regular crossings are isolated this is a finite sum.

Remark 5.2 If $V = 0 \times \mathbb{R}^n$ and $Z(t) = (X(t), Y(t))$ is a frame for $\Lambda(t)$ then

$$\Gamma(\Lambda, V, t)(v) = -\langle Y(t)u, \dot{X}(t)u \rangle, \qquad X(t)u = 0,$$

where $v = (0, Y(t)u)$.

Remark 5.3 Consider the symplectic vector space $E = \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ with $\omega = (-\omega_0) \times \omega_0$. For a path of a symplectomorphisms $\Psi : [a, b] \to \mathrm{Sp}(2n)$ we have

$$\mu(\Psi\Lambda, \Lambda') = \mu(\mathrm{Gr}(\Psi), \Lambda \times \Lambda').$$

When $\Psi(t) = \mathbb{1}$ this means

$$\mu(\Lambda, \Lambda') = \mu(\Delta, \Lambda \times \Lambda')$$

where $\Delta \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ denotes the diagonal.

Remark 5.4 Let $V = 0 \times \mathbb{R}^n$ denote the vertical. The **Maslov index** of a symplectic path $\Psi : [a, b] \to \mathrm{Sp}(2n)$ is defined by

$$\mu(\Psi) = \mu(\Psi V, V) = \mu(\mathrm{Gr}(\Psi), V \times V).$$

If $\Psi(a) = \mathbb{1}$ and $\Psi(b)V \cap V = 0$ then $\mu(\Psi) + n/2 \in \mathbb{Z}$. The condition $\Psi(t)V \cap V = 0$ holds if and only if $\Psi(t)$ admits a generating function as in [23]. If Ψ is written in block matrix form

$$\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{16}$$

then this is equivalent to $\det B(t) \neq 0$. By Remark 5.2 the crossing form $\Gamma(\Psi, t)$ is given by

$$\Gamma(\Psi, t)(y) = -\langle D(t)y, \dot{B}(t)y \rangle.$$

for $y \in \mathbb{R}^n$ with $B(t)y = 0$.

Remark 5.5 The **Conley-Zehnder** index of a symplectic path is defined by

$$\mu_{CZ}(\Psi) = \mu(\mathrm{Gr}(\Psi), \Delta).$$

This index was introduced in [5] for paths $\Psi : [a, b] \to \mathrm{Sp}(2n)$ such that $\Psi(a) = \mathbb{1}$ and $\mathbb{1} - \Psi(b)$ is invertible. For such paths the Conley-Zehnder index is an integer.

6 The Morse index

6.1 Sturm oscillation

Consider the operator family $A(t) : W \to H$ defined by

$$A(t) = -\frac{d^2}{ds^2} - q(s,t)$$

with

$$H = L^2([0,1]), \qquad W = W^{2,2}([0,1]) \cap W_0^{1,2}([0,1]).$$

Here s is the coordinate on $[0,1]$. Assume that q is C^1 on the closed strip $[0,1] \times \mathbb{R}$ and independent of t for $|t| \geq T$. Let $\phi = \phi(s,t)$ be the solution of the initial value problem

$$\frac{\partial^2 \phi}{\partial s^2} + q\phi = 0, \qquad \phi(0,t) = 0, \qquad \frac{\partial \phi}{\partial s}(0,t) = 1.$$

Define $q^{\pm}(s) = q(s, \pm T)$ and $\phi^{\pm}(s) = \phi(s, \pm T)$ and assume that

$$\phi^{\pm}(1) \neq 0.$$

This means that 0 is not in the spectrum of $A^{\pm}$.

Proposition 6.1 *The spectral flow of $A(t)$ is given by*

$$\mu(A) = \nu(\phi^-) - \nu(\phi^+)$$

where $\nu(\phi^{\pm})$ denotes the number of zeros of the function $\phi^{\pm}(s)$ in the interval $0 < s \leq 1$.

Proof: First note that all eigenvalues of $A(t)$ have multiplicity 1. By Theorem 4.2 we may assume that all crossings are simple. We investigate the behaviour of the quotient $\phi/\partial_s \phi$ along the boundary $s = 1$. Differentiating the identity

$$\int_0^1 \left((\partial_s \phi)^2 - q\phi^2 \right) ds = \phi(1,t)\partial_s\phi(1,t)$$

with respect to t and integrating by parts we obtain

$$\int_0^1 (\partial_t q)\phi^2 \, ds = \partial_t\phi(1,t)\partial_s\phi(1,t) - \phi(1,t)\partial_t\partial_s\phi(1,t).$$

At a crossing t the left hand side is the crossing operator (up to a scalar multiple):

$$-\frac{\int_0^1 (\partial_t q)\phi^2 \, ds}{\int_0^1 \phi^2 \, ds} = \operatorname{tr}(P\dot{A}P) = \dot{\lambda}(t).$$

Here $\lambda(t)$ is the crossing eigenvalue. Hence

$$\text{sign}\,\dot\lambda(t) = -\text{sign}\,\frac{d}{dt}\frac{\phi(1,t)}{\partial_s\phi(1,t)}.$$

Since $s \mapsto \phi(s,t)$ solves a second order equation ϕ and $\partial_s\phi$ cannot vanish simultaneously. Hence the loop of lines $\mathbb{R}(\phi(s,t),\partial_s\phi(s,t)) \subset \mathbb{R}^2$ around the boundary of $[-T,T] \times [0,1]$ is contractible. Set $t = \pm T$ and let s run from 0 to 1: The intersections of the line $\mathbb{R}(\phi(s,\pm T),\partial_s\phi(s,\pm T))$ with the vertical $\mathbb{R}(0,1)$ count the zeros of $\phi^{\pm}$

$$\nu(\phi^{\pm}) = \sum_{\phi(s,\pm T)=0} \text{sign}\,\frac{d}{ds}\frac{\phi(s,\pm T)}{\partial_s\phi(s,\pm T)} = \sum_{\phi(s,\pm T)=0} 1.$$

The intersections of the line $\mathbb{R}(\phi(1,t),\partial_s\phi(1,t))$ with the vertical $\mathbb{R}(0,1)$ count the crossings of $A(t)$

$$\begin{aligned}
\mu(A) &= \sum_{\phi(1,t)=0} \text{sign}\,\dot\lambda(t) \\
&= -\sum_{\phi(1,t)=0} \text{sign}\,\frac{d}{dt}\frac{\phi(1,t)}{\partial_s\phi(1,t)} \\
&= \sum_{\phi(s,-T)=0} \text{sign}\,\frac{d}{ds}\frac{\phi(s,-T)}{\partial_s\phi(s,-T)} - \sum_{\phi(s,T)=0} \text{sign}\,\frac{d}{ds}\frac{\phi(s,T)}{\partial_s\phi(s,T)} \\
&= \nu(\phi^-) - \nu(\phi^+).
\end{aligned}$$

$\square$

Corollary 6.2 (The Sturm oscillation theorem) *The n-th eigenfunction of the problem*

$$\frac{\partial^2 u}{\partial s^2} + qu + \lambda u = 0, \qquad u(0) = u(1) = 0$$

has $n-1$ interior zeros.

Proof: Consider the spectral flow for the operator family

$$A(t) = -\frac{d^2}{ds^2} - q - b(t)$$

where $b : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $b(t) = b^- < \lambda_1$ for $t \leq -1$ and $\lambda_{n-1} < b(t) = b^+ < \lambda_n$ for $t \geq 1$. $\square$

This proof also shows that if the operator $A : W \to H$ defined by $Au = -d^2u/ds^2 - qu$ is invertible then its Morse index (the number of negative eigenvalues) is the number of zeros of the fundamental solution $\phi(s)$ in the interval $0 < s < 1$. This is a special case of the Morse index theorem proved below.

29

6.2   The Morse index theorem

In suitable coordinates the Jacobi equation in differential geometry has the form

$$Au = -\frac{d^2u}{ds^2} - Q(s)u = 0. \tag{17}$$

Here $s \in [0,1]$, $u(s) \in \mathbb{R}^n$, and $Q(s)$ is a symmetric matrix representationg the curvature. This generalizes the previous example from 1 dimension to n. We call $s_0 \in (0,1]$ a **conjugate point** of A iff there is a non-trivial solution u of equation (17) satisfying $u(0) = u(s_0) = 0$. The dimension of the vector space of all solutions u of (17) satisfying $u(0) = u(s_0) = 0$ is called the **multiplicity** of the conjugate point. Denote by $\nu(A)$ the number of conjugate points of A in the interval $0 < s \le 1$ counted with multiplicity. Let $\Phi(s) \in \mathbb{R}^{n \times n}$ be the fundamental solution of (17) defined by

$$\frac{d^2\Phi}{ds^2} + Q\Phi = 0, \qquad \Phi(0) = 0, \qquad \frac{d\Phi}{ds}(0) = \mathbb{1},$$

for $0 \le s \le 1$. Then

$$\Lambda(s) = \mathrm{range}\left(\begin{array}{c} \Phi(s) \\ \dot{\Phi}(s) \end{array}\right)$$

is a Lagrangian plane for every s.

Proposition 6.3 *Assume* $\det \Psi(1) \ne 0$. *Then the number* $\nu(A)$ *of conjugate points is related to the Maslov index of* Λ *by*

$$\mu(\Lambda, V) = -\nu(A) - \frac{n}{2}$$

where $V = 0 \times \mathbb{R}^n$.

Proof: Suppose s_0 is a crossing of multiplicity m_0. By Remark 5.2 the crossing form is given by

$$\Gamma(\Lambda, V, s_0)(v) = -\langle \dot{\Phi}(s_0)u_0, \dot{\Phi}(s_0)u_0 \rangle, \qquad v = (0, \dot{\Phi}(s_0)u_0), \qquad \Phi(s_0)u_0 = 0.$$

Since $\dot{\Phi}(s_0)$ is injective on the kernel of $\Phi(s_0)$ the crossing form is negative definite and of rank m_0. This shows that all crossings are regular. Moreover, $s_0 = 0$ is a crossing with crossing index $m_0 = n$. The Proposition is proved by summing over the crossings:

$$
\begin{aligned}
\mu(\Lambda, V) &= -\frac{1}{2}\dim\ker\Phi(0) - \sum_{\substack{0<s<1 \\ \det \Phi(s)=0}} \dim\ker\Phi(s) \\
&= -\frac{n}{2} - \nu(A).
\end{aligned}
$$

$\square$

Let H and W be as before but tensored with $\mathbb{R}^n$:

$$H = L^2([0,1], \mathbb{R}^n), \qquad W = W^{2,2}([0,1], \mathbb{R}^n) \cap W_0^{1,2}([0,1], \mathbb{R}^n),$$

and replace the operator of (17) by a one-parameter family

$$A(t) = -\frac{d^2}{ds^2} - Q(s,t).$$

Assume that Q is C^1 on the closed strip $[0,1] \times \mathbb{R}$ and independent of t for $|t| \geq T$. Assume that 1 is not a conjugate point for either of the operators $A^\pm$. This says that these operators are invertible.

Proposition 6.4 *The spectral flow of $A(t)$ is given by*

$$\mu(A) = \nu(A^-) - \nu(A^+).$$

Proof: Let $\Phi = \Phi(s,t) \in \mathbb{R}^{n \times n}$ be the fundamental solution defined by

$$\frac{\partial^2 \Phi}{\partial s^2} + Q\Phi = 0, \qquad \Phi(0,t) = 0, \qquad \frac{\partial \Phi}{\partial s}(0,t) = \mathbb{1}.$$

Then the operator $A(t)$ is injective if and only if $\det \Phi(1,t) \neq 0$. The kernel of $A(t)$ consists of all functions of the form

$$u(s) = \Phi(s,t)u_0, \qquad \Phi(1,t)u_0 = 0.$$

Think of the crossing operator $\Gamma(A,t)$ as a quadratic form on the kernel of $A(t)$:

$$\Gamma(A,t)(u) = -\int_0^1 \langle u(s), \partial_t Q(s,t) u(s) \rangle \, ds.$$

The next lemma shows that that this agrees with the crossing form of the Lagrangian path $t \mapsto \Lambda(1,t)$ with the vertical $V = 0 \times \mathbb{R}^n$ evaluated at u_0 where

$$\Lambda(s,t) = \mathrm{range} \begin{pmatrix} \Phi(s,t) \\ \partial_s \Phi(s,t) \end{pmatrix}.$$

Hence

$$\begin{aligned} \mu(A) &= \mu(\Lambda(1,\cdot), V) \\ &= \mu(\Lambda^+, V) - \mu(\Lambda^-, V) \\ &= \nu(A^-) - \nu(A^+). \end{aligned}$$

The second equality follows from the fact that the loop of Lagrangian subspaces $\Lambda(s,t)$ around the boundary of the square $[0,1] \times [-T,T]$ is contractible. The last equality follows from Proposition 6.3. $\qquad \square$

Lemma 6.5 *Let $V = 0 \times \mathbb{R}^n$. Then*

$$\Gamma(A, t)(u) = \Gamma(\Lambda(1, \cdot), V, t)(v)$$

for $u(s) = \Phi(s, t)u_0$ and $v = (0, \partial_s\Phi(1, t)u_0)$ with $\Phi(1, t)u_0 = 0$.

Proof: Differentiate the identity

$$\int_0^1 \left((\partial_s\Phi)^T\partial_s\Phi - \Phi^TQ\Phi\right) dx = \Phi(1, t)^T\partial_s\Phi(1, t)$$

with respect to t and integrate by parts to obtain

$$\int_0^1 \Phi^T(\partial_t Q)\Phi \, ds = \partial_t\Phi(1, t)^T\partial_s\Phi(1, t) - \partial_t\partial_s\Phi(1, t)^T\Phi(1, t).$$

Now multiply on the left an right with u_0 where $\Phi(1, t)u_0 = 0$. The result is

$$\int_0^1 \langle u(s), \partial_t Q(s, t)u(s)\rangle \, ds = \langle \partial_s\Phi(1, t)u_0, \partial_t\Phi(1, t)u_0\rangle.$$

The left hand side is $-\Gamma(A, t)(u)$ and the right hand side is $-\Gamma(\Lambda(1, \cdot), V, t)(v)$.
□

Corollary 6.6 (The Morse index theorem) *Assume that the operator $A : W \to H$ defined by (17) is invertible. Then its Morse index (the number of negative eigenvalues) is the number $\nu(A)$ of conjugate points.*

Proof: Consider the spectral flow for the operator family

$$A(t) = -\frac{d^2}{ds^2} - Q - b(t)$$

where $b : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $b(t) = b^- < \lambda_1$ for $t \leq -1$ and $b(t) = 0$ for $t \geq 1$. □

7 Cauchy-Riemann operators

Denote by

$$J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

the standard complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$. Consider the perturbed Cauchy Riemann operator

$$\bar{\partial}_{S,\Lambda}\zeta = \frac{\partial\zeta}{\partial t} - J_0\frac{\partial\zeta}{\partial s} + S\zeta \tag{18}$$

where $\zeta : [0,1] \times \mathbb{R} \to \mathbb{R}^{2n}$ satisfies the non-local boundary condition

$$(\zeta(0,t), \zeta(1,t)) \in \Lambda(t). \tag{19}$$

Here $\Lambda(t) \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ is a path of Lagrangian subspaces and $S(s,t) \in \mathbb{R}^{2n \times 2n}$ is a family of matrices. We impose the following conditions:

(CR-1) The function $\Lambda : \mathbb{R} \to \mathcal{L}(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, (-\omega_0) \times \omega_0)$ is of class C^1. Moreover, there exist Lagrangian subspaces $\Lambda^{\pm}$ and a constant $T > 0$ with $\Lambda(t) = \Lambda^+$ for $t \geq T$ and $\Lambda(t) = \Lambda^-$ for $t \leq -T$.

(CR-2) The function $S : [0,1] \times \mathbb{R} \to \mathbb{R}^{2n \times 2n}$ is continuous. Moreover, there exist symmetric matrix valued functions $S^{\pm} : [0,1] \to \mathbb{R}^{2n \times 2n}$ such that

$$\lim_{t \to \pm\infty} \sup_{0 \leq s \leq 1} \|S(s,t) - S^{\pm}(s)\| = 0.$$

(CR-3) Let $\Psi^{\pm} : [0,1] \to \mathrm{Sp}(2n)$ be defined by

$$\frac{\partial \Psi^{\pm}}{\partial s} + J_0 S^{\pm} \Psi^{\pm} = 0, \qquad \Psi^{\pm}(0) = \mathbb{1}.$$

Then the graph of $\Psi^{\pm}(1)$ is transverse to $\Lambda^{\pm}$.

The operator $\bar{\partial}_{S,\Lambda}$ has the form $D_A = d/dt - A(t)$ but in contrast to section 3 the domain of the operator $A(t) : W(t) \to H$ depends on t so Theorem 3.10 does not apply directly. We overcome this difficulty below by changing coordinates. Condition (CR-1) asserts that the domain of the operator $A(t)$ is independent of t for $|t| \geq T$. The Lagrangian boundary condition and the symmetry of $S^{\pm}$ imply that the limit operators $A^{\pm}$ are self-adjoint. Condition (CR-3) asserts that these operators are invertible. Abbreviate

$$
\begin{aligned}
L^2 &= L^2([0,1] \times \mathbb{R}, \mathbb{R}^{2n}), \\
W_{\Lambda}^{1,2} &= \left\{ \zeta \in W^{1,2}([0,1] \times \mathbb{R}, \mathbb{R}^{2n}) : (\zeta(0,t), \zeta(1,t)) \in \Lambda(t) \right\}.
\end{aligned}
$$

In the case $\Lambda(t) \equiv \Lambda$ of constant boundary conditions these are the spaces $\mathcal{H} = L^2$ and $\mathcal{W} = W_{\Lambda}^{1,2}$ of section 3.

Theorem 7.1 *The operator $\bar{\partial}_{S,\Lambda} : W_{\Lambda}^{1,2} \to L^2$ is Fredholm. Its index is given by*

$$\mathrm{index}\, \bar{\partial}_{S,\Lambda} = \mu(\mathrm{Gr}(\Psi^-), \Lambda^-) - \mu(\mathrm{Gr}(\Psi^+), \Lambda^+) - \mu(\Delta, \Lambda) \tag{20}$$

Proof: We prove the theorem in five steps.

Step 1: *Let $\Psi(s,t) \in \mathrm{Sp}(2n)$ be defined by*

$$\frac{\partial \Psi}{\partial s} + J_0 \frac{S + S^T}{2} \Psi = 0, \qquad \Psi(0,t) = \mathbb{1}. \tag{21}$$

Then

$$\mu(\mathrm{Gr}(\Psi(1,\cdot)),\Lambda) = \mu(\Delta,\Lambda) + \mu(\mathrm{Gr}(\Psi^+),\Lambda^+) - \mu(\mathrm{Gr}(\Psi^-),\Lambda^-). \qquad (22)$$

By condition (CR-2) we have

$$\Psi^{\pm}(s) = \lim_{t\to\pm\infty} \Psi(s,t).$$

By condition (CR-3) the path $s \mapsto \Psi(s,\pm T)$ has the same Maslov index as $\Psi^{\pm}$ for T sufficiently large. Hence step 1 follows by considering the loops of Lagrangrangian subspaces $\Lambda(s,t) = \Lambda(t)$ and $\Lambda'(s,t) = \mathrm{Gr}(\Psi(s,t))$ around the boundary of the rectangle $[0,1]\times[-T,T]$. In view of step 1 it suffices to prove that

$$\mathrm{index}\,\bar\partial_{S,\Lambda} = -\mu(\mathrm{Gr}(\Psi(1,\cdot)),\Lambda). \qquad (23)$$

Step 2: *The theorem holds when $\Lambda(t) = V \oplus V$, $S(s,t) = S(s,t)^T$ is symmetric and continuouly differentiable, and the path $t \mapsto \Psi(1,t)$ has only simple crossings.*

By Theorem 3.10 the operator is Fredholm and by Theorem 4.1 the Fredholm index is given by the spectral flow for the self-adjoint operator family

$$A(t) = J_0\frac{d}{ds} - S(s,t)$$

on $H = L^2([0,1],\mathbb{R}^{2n})$ with dense domain

$$W = W_0^{1,2}([0,1],\mathbb{R}^n) \times W^{1,2}([0,1],\mathbb{R}^n).$$

We examine the crossing operator $\Gamma(A,t)$ at acrossing t. The operator $A(t)$ is injective if and only if $\Psi(1,t)V \cap V = 0$ where $\Psi(s,t) \in \mathrm{Sp}(2n)$ is defined as in step 1. The kernel of $A(t)$ consists of all functions of the form

$$\zeta(s) = \Psi(s,t)v, \qquad v = (0,y), \qquad B(1,t)y = 0.$$

Here $B(1,t)$ is the right upper block in the block decomposition (16) of $\Psi(1,t)$. Think of the crossing operator $\Gamma(A,t)$ as a quadratic form on the kernel of A:

$$\Gamma(A,t)(\zeta) = -\int_0^1 \langle \zeta(s), \partial_t S(s,t)\zeta(s)\rangle\,ds.$$

We shall prove in the lemma below that this form agrees with $\Gamma(\Psi(1,\cdot),t)$. Hence the operator family $A(t)$ has only regular crossings and

$$\mathrm{index}\,\bar\partial_{S,V\oplus V} = -\mu(A) = -\mu(\Psi(1,\cdot)) = -\mu(\mathrm{Gr}(\Psi(1,\cdot)),V\oplus V).$$

The last equality follows from Remark 5.4. Now use step 1.

34

Lemma 7.2 *We have*

$$\Gamma(A, t)(\zeta) = \Gamma(\Psi(1, \cdot), t)(y)$$

for $\zeta(s) = \Psi(s, t)v$ with $v = (0, y) \in V$ and $B(1, t)y = 0$.

Proof: Differentiate the identity

$$S\Psi = J_0 \partial_s \Psi$$

with respect to t, multiply on the left by Ψ^T and integrate by parts to obtain

$$
\begin{aligned}
\int_0^1 \Psi^T (\partial_t S) \Psi \, ds &= \int_0^1 \Psi^T J_0 \partial_s \partial_t \Psi \, ds - \int_0^1 \Psi^T S \partial_t \Psi \, ds \\
&= \int_0^1 \Psi^T J_0 \partial_s \partial_t \Psi \, ds + \int_0^1 (\partial_s \Psi)^T J_0 \partial_t \Psi \, ds \\
&= \Psi(1, t)^T J_0 \partial_t \Psi(1, t).
\end{aligned}
$$

Now multiply on the left and right by $v = (0, y)$ with $B(1, t)y = 0$ to obtain

$$
\begin{aligned}
\Gamma(A, t)(\zeta) &= -\langle \Psi(1, t)v, J_0 \partial_t \Psi(1, t)v \rangle \\
&= -\langle D(1, t)y, \partial_t B(1, t)y \rangle \\
&= \Gamma(\Psi(1, \cdot), t)(y).
\end{aligned}
$$

Here $D(1, t)$ denote the lower right block in the decomposition (16) of $\Psi(1, t)$. The last equality follows from Remark 5.4. This proves the lemma.

Step 3: *The theorem holds when $\Lambda(t) = V \oplus V$.*

Choose any smooth cutoff function $\beta : \mathbb{R} \to [0, 1]$ such that $\beta(t) = 0$ for $t \leq -T$ and $\beta(t) = 1$ for $t \geq T$ and replace S by

$$S'(s, t) = \beta(t)S^+(s) + (1 - \beta(t))S^-(s).$$

Then the right hand side of (18) is unchanged. Moreover, the multiplication operator induced by $S - S'$ satisfies the assumptions of Lemma 3.16. Hence, by Corollary 3.17, the operator $\bar{\partial}_{S, V \oplus V}$ is Fredholm and has the same index as $\bar{\partial}_{S', V \oplus V}$. Now choose a small perturbation to obtain a symmetric C^1-function S'' such that the associated symplectic path $t \mapsto \Psi''(1, t)$ has only simple crossings. Finally use step 2.

Step 4: *The theorem holds in the case of local boundary conditions*

$$\Lambda(t) = \Lambda_0(t) \oplus \Lambda_1(t)$$

with $\Lambda_j(t) \in \mathcal{L}(\mathbb{R}^{2n}, \omega_0)$.

Identify $\mathbb{R}^{2n} = \mathbb{C}^n$ and choose a unitary transformation $\Phi(s,t) \in U(n) = O(2n) \cap GL(n, \mathbb{C})$ of class C^1 such that $\Phi(s,t)$ is independent of t for $|t| \geq T$. Then

$$\bar{\partial}_{S,\Lambda} \circ \Phi = \Phi \circ \bar{\partial}_{S',\Lambda'}$$

where

$$S' = \Phi^{-1}\frac{\partial\Phi}{\partial t} - \Phi^{-1}J_0\frac{\partial\Phi}{\partial s} + \Phi^{-1}S\Phi$$

and $\Lambda' = \Lambda'_0 \oplus \Lambda'_1$ with

$$\Lambda'_0(t) = \Phi(0,t)^{-1}\Lambda_0(t), \qquad \Lambda'_1(t) = \Phi(1,t)^{-1}\Lambda_1(t).$$

Since $\Phi^{-1} = \Phi^T$ the matrix $S'(s,t)$ is symmetric for $|t| \geq T$. The corresponding symplectic matrices $\Psi'(s,t) \in Sp(2n)$ defined by (21) with S replace by S' are given by

$$\Psi'(s,t) = \Phi(s,t)^{-1}\Psi(s,t)\Phi(0,t).$$

Now denote $\overline{\Phi}(t) = \Phi(0,t) \oplus \Phi(1,t) \in Sp(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, (-\omega_0) \times \omega_0)$. Then

$$Gr(\Psi'(1,t)) = \overline{\Phi}(t)^{-1}Gr(\Psi(1,t)), \qquad \Lambda'(t) = \overline{\Phi}(t)^{-1}\Lambda(t).$$

and, by the naturality axiom for the Maslov index,

$$\mu(Gr(\Psi(1,\cdot)), \Lambda) = \mu(Gr(\Psi'(1,\cdot)), \Lambda').$$

Now choose Φ such that $\Lambda'(t) = V \times V$ and use step 3.

Step 5: *The general case.*

Define the operator $\mathcal{T} : L^2([0,1] \times \mathbb{R}, \mathbb{R}^{2n}) \to L^2([0,1] \times \mathbb{R}, \mathbb{R}^{2n} \times \mathbb{R}^{2n})$ which sends ζ to the pair $\mathcal{T}\zeta = \eta = (\eta_0, \eta_1)$ where

$$\eta_0(s,t) = \zeta((1-s)/2, t/2), \qquad \eta_1(s,t) = \zeta((1+s)/2, t/2).$$

If $\zeta \in W^{1,2}_\Lambda$ then η satisfies the local boundary conditions

$$\eta(0,t) \in \Delta = \overline{\Lambda}_0(t), \qquad \eta(1,t) \in \Lambda(t) = \overline{\Lambda}_1(t).$$

Moreover, the operator $\mathcal{T} \circ \bar{\partial}_{S,\Lambda} \circ \mathcal{T}^{-1}$ is given by

$$(\eta_0, \eta_1) \mapsto (\partial_t\eta_0 + J_0\partial_t + S_0\eta_0, \partial_t\eta_1 - J_0\partial_t + S_1\eta_1)$$

where $S_0(s,t) = S((1-s)/2, t/2)/2$ and $S_1(s,t) = S((1+s)/2, t/2)/2$. This is a Cauchy-Riemann operator with respect to the complex structure $\overline{J} = (-J_0) \oplus J_0$ which is compatible with $\bar{\omega} = (-\omega_0) \oplus \omega_0$. The corresponding fundamental solution $\overline{\Psi}(s,t) = \Psi_0(s,t) \oplus \Psi_1(s,t)$ is given by

$$\Psi_0(s,t) = \Psi((1-s)/2, t/2)\Psi(1/2, t/2)^{-1},$$

$$\Psi_1(s,t) = \Psi((1+s)/2, t/2)\Psi(1/2, t/2)^{-1}.$$

Hence $\overline{\Psi}(1,t)\Delta = \mathrm{Gr}(\Psi(1, t/2))$ and it follows that

$$\mu(\mathrm{Gr}(\Psi(1, \cdot)), \Lambda) = \mu(\overline{\Psi}(1, \cdot)\overline{\Lambda}_0, \overline{\Lambda}_1)$$

By step 4 the operator $\mathcal{T} \circ \bar{\partial}_{S,\Lambda} \circ \mathcal{T}^{-1}$ is Fredholm and its index is given by

$$\mathrm{index}\, \mathcal{T} \circ \bar{\partial}_{S,\Lambda} \circ \mathcal{T}^{-1} = -\mu(\overline{\Psi}(1, \cdot)\overline{\Lambda}_0, \overline{\Lambda}_1).$$

Hence $\bar{\partial}_{S,\Lambda}$ is a Fredholm operator and

$$\mathrm{index}\, \bar{\partial}_{S,\Lambda} = -\mu(\mathrm{Gr}(\Psi(1, \cdot)), \Lambda).$$

By step 1 this proves the theorem. $\qquad\qquad\qquad\qquad\qquad\square$

Remark 7.3 (Periodic boundary conditions) Assume $\Lambda(t) = \Delta$ for all t. Then condition (5) above means that

$$\det(\mathbb{1} - \Psi^{\pm}(1)) \neq 0.$$

In this case the Fredholm index is related to the Conley-Zehnder index by

$$\mathrm{index}\, \bar{\partial}_{S,\Delta} = \mu_{CZ}(\Psi^-) - \mu_{CZ}(\Psi^+).$$

This result was proved in [25]. With these boundary conditions the operator $\bar{\partial}_S$ plays a central role in Floer homology for symplectomorphisms. The mod 2 index is the **relative fixed point index** $\varepsilon(\Psi) = \mathrm{sign}\det(\mathbb{1} - \Psi^+(1)) \cdot \mathrm{sign}\det(\mathbb{1} - \Psi^-(1))$:

$$(-1)^{\mathrm{index}\,\bar{\partial}_S} = \varepsilon(\Psi).$$

As a result the Euler characteristic of Floer homology for a symplectomorphism is the Lefschetz number [11], [6].

Remark 7.4 (Local boundary conditions) Assume $S = 0$ and

$$\Lambda(t) = \Lambda_0(t) \oplus \Lambda_1(t)$$

where $\Lambda_0(t), \Lambda_1(t) \in \mathcal{L}(\mathbb{R}^{2n}, \omega_0)$. Then condition (5) above means that

$$\Lambda_0(\pm T) \cap \Lambda_1(\pm T) = 0.$$

By Remark 5.3 the Fredholm index is given by

$$\mathrm{index}\, \bar{\partial}_\Lambda = -\mu(\Lambda_0, \Lambda_1).$$

This was proved by Floer [8] using results by Viterbo [28]. With these boundary conditions the operator $\bar{\partial}_\Lambda$ plays a central role in Floer homology for Lagrangian intersections. The mod 2 index is the **relative intersection number**

$\varepsilon(\Lambda_0, \Lambda_1)$. Choose orientations of Λ_0 and Λ_1 and define $\varepsilon(\Lambda_0, \Lambda_1) = \pm 1$ according to whether the induced orientations on $\mathbb{R}^{2n} = \Lambda_0(\pm T) \oplus \Lambda_1(\pm T)$ agree. Then

$$(-1)^{\text{index}\, \bar{\partial}_\Lambda} = \varepsilon(\Lambda_0, \Lambda_1).$$

As a result the Euler characteristic of Floer homology for a pair of Lagrangian submanifolds is the intersection number [9].

Remark 7.5 (Dirichlet boundary conditions) Assume $S = S^T$ is symmetric and $\Lambda(t) = V \oplus V$ where $V = 0 \times \mathbb{R}^n$ is the vertical. Then condition (5) above means that $\Psi^\pm(1)V \cap V = 0$. By Remark 5.4 the Fredholm index is given by

$$\text{index}\, \bar{\partial}_{S,\Lambda} = \mu(\Psi^-) - \mu(\Psi^+).$$

In this case the results of section 3 apply and the operator $A = A(t)$ is given by

$$A\zeta = J_0 \dot{\zeta} - S\zeta$$

for $\zeta = (\xi, \eta)$ with boundary condition $\xi(0) = \xi(1) = 0$. This operator appears as the second variation of the symplectic action on phase space. The signature of A is undefined since the index and the coindex are both infinite. In [23] we interpret the Maslov index as the signature of A via finite dimensional approximation:

$$\text{sign}\, A^\pm = 2\mu(\Psi^\pm).$$

Hence the index theorem can be written in the form

$$\text{index}\, \bar{\partial}_{S,\Lambda} = \tfrac{1}{2}\text{sign}\, A^- - \tfrac{1}{2}\text{sign}\, A^+.$$

This is consistent with the finite dimensional formula in Remark 2.2.

Remark 7.6 (Totally real boundary conditions) The operator $\bar{\partial}_{S,\Lambda}$ continues to be Fredholm when $\Lambda(t)$ is only totally real with respect to the complex structure $(-J_0) \oplus J_0$ on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. To see this in the case of local boundary conditions $\Lambda = \Lambda_0 \oplus \Lambda_1$ choose a family of symplectic forms $\omega(s,t)$ on $\mathbb{R}^{2n}$ which are compatible with J_0 and satisfy

$$\Lambda_0(t) \in \mathcal{L}(\mathbb{R}^{2n}, \omega(0,t)), \qquad \Lambda_1(t) \in \mathcal{L}(\mathbb{R}^{2n}, \omega(1,t)).$$

Now choose a unitary frame

$$\Phi(s,t) : (\mathbb{R}^{2n}, J_0, \omega_0) \to (\mathbb{R}^{2n}, J_0, \omega(s,t))$$

and consider the operator $\bar{\partial}_{S,\Lambda}$ in the new coordinates $\zeta' = \Phi^{-1}\zeta$. Then the operator has the above form with Lagrangian boundary conditions. Its index is independent of the choice of Φ since the space of all symplectic forms on $\mathbb{R}^{2n}$ which are compatible with J_0 is contractible. The general case can be reduced to that of local boundary conditions as in Step 5 of the proof of Theorem 7.1.

References

[1] R. Abraham and J. Robbin, *Transversal Flows and Maps*, Benjamin, 1970.

[2] V.I. Arnold, On a characteristic class entering into conditions of quantization, *Functional analysis* **1** (1967) 1-8.

[3] M.F. Atiyah, V.K. Patodi and I.M. Singer, Spectral asymmetry and Riemannian geometry, *Math. Proc. Camb. Phil. Soc.* **79** (1976), 79–99.

[4] S.E. Cappell, R. Lee and E.Y. Miller, Self-adjoint elliptic operators and manifold decomposition I: general techniques and applications to Casson's invariant, Preprint, 1990.

[5] C. Conley and E. Zehnder, The Birkhoff-Lewis fixed point theorem and a conjecture of V.I. Arnold, *Invent. Math.* **73** (1983), 33–49.

[6] S. Dostoglou and D.A. Salamon, Instanton homology and symplectic fixed points, *Proceedings of the Conference on Symplectic Geometry*, edited by D. Salamon, Warwick, 1990.

[7] J.J. Duistermaat, On the Morse index in variational calculus, *Advances in Mathematics* **21** (1976), 173–195.

[8] A. Floer, A relative Morse index for the symplectic action, *Comm. Pure Appl. Math.* **41** (1988), 393–407.

[9] A. Floer, Morse theory for Lagrangian intersections *J. Diff. Geom.* **28** (1988), 513–547.

[10] A. Floer, An instanton invariant for 3-manifolds, *Commun. Math. Phys.* **118** (1988), 215–240.

[11] A. Floer, Symplectic fixed points and holomorphic spheres, *Commun. Math. Phys.* **120** (1989), 575–611.

[12] M. Gromov, Pseudo holomorphic curves in symplectic manifolds, *Invent. Math.* **82** (1985), 307–347.

[13] L. Hörmander, Fourier integral operators I, *Acta Math.* **127** (1971), 79–183.

[14] T. Kato, *Perturbation Theory for Linear Operators* Springer-Verlag, 1976.

[15] J. Leray, *Lagrangian Analysis and Quantum Mechanics*, MIT press, 1981.

[16] R.B. Lockhardt and R.C. McOwen, Elliptic operators on noncompact manifolds *Ann. Sci. Normale Sup. Pisa* **IV-12** (1985), 409–446.

[17] D. McDuff, Elliptic methods in symplectic geometry, *Bull. A.M.S.* **23** (1990), 311–358.

[18] J. Milnor, *Topology from the Differentiable Viewpoint*, The University Press of Virginia, Charlottesville, 1969.

[19] Reed and Simon, *Methods of Mathematical Physics II, Fourier Analysis and Self-Adjointness*, Academic Press, 1975.

[20] J.W. Robbin, Self-adjoint operators and symplectic geometry, in preparation.

[21] J.W. Robbin and D.A. Salamon, The Maslov index for paths, Preprint 1992.

[22] J.W. Robbin and D.S. Salamon, Phase functions and path integrals, *Proceedings of the Conference on Symplectic Geometry*, edited by D. Salamon, Warwick, 1990.

[23] J.W.Robbin and D.A. Salamon, Path integrals on phase space and the metaplectic representation. Preprint 1992.

[24] D. Salamon, Morse theory, the Conley index and Floer homology, *Bull. L.M.S.* **22** (1990), 113–140.

[25] D. Salamon and E. Zehnder, Morse theory for periodic solutions of Hamiltonian systems and the Maslov index, to appear in *Comm. Pure Appl. Math.*

[26] M. Schwarz, *Morse Homology*, Birkhäuser Verlag, Basel, 1993.

[27] S. Smale, An infinite dimensional version of Sard's theorem, *Am. J. Math.* **87** (1973), 213–221.

[28] C. Viterbo, Intersections de sous-variétés Lagrangiennes, fonctionelles d'action et indice des systèmes Hamiltoniens, *Bull. Soc. Math. France* **115** (1987), 361–390.

[29] J. Wloka, *Partial Differential Equations* Cambrisge University Press, 1987.

[30] T. Yoshida, Floer homology and splittings of manifolds, *J. Diff. Geom.* **23** (1988), 383–405.