Symplectic Topology
Example Sheet 1

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Exercise 1.1. Write the elements of the configuration space $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ in the form $z = (x, y) = (x_1, \dots, x_n, y_1, \dots, y_n)$. Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function, fix two vectors $a, b \in \mathbb{R}^n$, consider the path space

$$\mathcal{P} := \left\{ z = (x, y) : [0, 1] \to \mathbb{R}^{2n} \,|\, x(0) = a,\, x(1) = b \right\},$$

and define the function $\mathcal{A}_H : \mathcal{P} \to \mathbb{R}$ by

$$\mathcal{A}_H(z) := \int_0^1 \bigl(\langle y(t), \dot{x}(t) \rangle - H(x(t), y(t)) \bigr)\, dt$$

for $z = (x, y) : [0, 1] \to \mathbb{R}^{2n}$. Prove that a path $z \in \mathcal{P}$ is a critical point of $\mathcal{A}_H$ if and only if it satisfies the Hamiltonian differential equation

$$\dot{x}_i(t) = \frac{\partial H}{\partial y_i}(x(t), y(t)), \qquad \dot{y}_i(t) = -\frac{\partial H}{\partial x_i}(x(t), y(t)), \qquad i = 1, \dots, n. \quad (1)$$

Exercise 1.2. Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function satisfying

$$\det \left(\frac{\partial^2 H}{\partial y_i \partial y_j} \right) \neq 0. \quad (2)$$

Convert the Hamiltonian system (1) locally into the Euler equation

$$\frac{d}{dt} \frac{\partial L}{\partial v_i}(x(t), \dot{x}(t)) = \frac{\partial L}{\partial x_i}(x(t), \dot{x}(t)), \qquad i = 1, \dots, n. \quad (3)$$

Hint: Solve the equation $v_i = \frac{\partial H}{\partial y_i}(x, y)$, $i = 1, \dots, n$, locally for $y = F(x, v)$, and define $L(x, v) := \langle y, v \rangle - H(x, y)$ for (x, v) in an open subset of $\mathbb{R}^n \times \mathbb{R}^n$.

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function. The **Hamiltonian vector field** of H is defined by

$$X_H(x, y) := \begin{pmatrix} \frac{\partial H}{\partial y}(x, y) \\ \frac{\partial H}{\partial x}(x, y) \end{pmatrix} = -J_0 \nabla H(x, y), \qquad J_0 := \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Here we write $\frac{\partial H}{\partial x} := (\frac{\partial H}{\partial x_1}, \ldots, \frac{\partial H}{\partial x_n})$ and $\frac{\partial H}{\partial y} := (\frac{\partial H}{\partial y_1}, \ldots, \frac{\partial H}{\partial y_n})$.

Exercise 1.3. Prove that $\psi^* X_H = X_{H \circ \psi}$ for every canonical transformation $\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ (satisfying $d\psi(\zeta)^T J_0 d\psi(\zeta) = J_0$ for every $\zeta \in \mathbb{R}^{2n}$) and every smooth function $H : \mathbb{R}^{2n} \to \mathbb{R}$.

The **symplectic linear group** is defined by

$$\mathrm{Sp}(2n) := \left\{ \Psi \in \mathbb{R}^{2n \times 2n} \,|\, \Psi^T J_0 \Psi = J_0 \right\}.$$

The elements of $\mathrm{Sp}(2n)$ are called **symplectic matrices**. Thus a **canonical transformation** is a diffeomorphism between open subsets of $\mathbb{R}^{2n}$ whose Jacobi-matrices are symplectic.

Exercise 1.4. Prove that $\mathrm{Sp}(2n)$ is a group, invariant under transposition:

$$\Phi, \Psi \in \mathrm{Sp}(2n) \qquad \Longrightarrow \qquad \Phi\Psi, \Psi^{-1}, \Psi^T \in \mathrm{Sp}(2n).$$

Prove that $\mathrm{Sp}(2n)$ is a Lie group (i.e. a submanifold of $\mathrm{GL}(2n, \mathbb{R})$ as well as a subgroup). Prove that its Lie algebra $\mathfrak{sp}(2n) := T_{\mathbb{1}}\mathrm{Sp}(2n) = \mathrm{Lie}(\mathrm{Sp}(2n))$ is given by $\mathfrak{sp}(2n) = \left\{ -J_0 S \,|\, S = S^T \in \mathbb{R}^{2n \times 2n} \right\}$.

The standard symplectic form on $\mathbb{R}^{2n}$ is the nondegenerate skew-symmetric bilinear form

$$\omega_0 := \sum_{i=1}^n dx_i \wedge dy_i : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}.$$

In explicit terms

$$\begin{aligned} \omega_0(\zeta, \zeta') &= \sum_{i=1}^n (\xi_i \eta_i' - \eta_i \xi_i') \\ &= \left\langle \xi, \eta' \right\rangle - \left\langle \eta, \xi' \right\rangle \\ &= (J_0 \zeta)^T \zeta' \end{aligned}$$

for $\zeta = (\xi, \eta), \zeta' = (\xi', \eta') \in \mathbb{R}^{2n}$. A subspace $\Lambda \subset \mathbb{R}^{2n}$ is called **Lagrangian** if it has dimension n and $\omega_0(\zeta, \zeta') = 0$ for all $\zeta, \zeta' \in \Lambda$. The **Lagrangian Grassmannian** is the set $\mathcal{L}_n$ of all Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_0)$.

Exercise 1.5. **(i)** Let $\Lambda \subset \mathbb{R}^{2n}$ be a linear subspace of the form

$$\Lambda = \{(\xi, A\xi) \,|\, \xi \in \mathbb{R}^n\}$$

with $A \in \mathbb{R}^{2n\times 2n}$. Prove that Λ is Lagrangian if and only if A is symmetric.
(ii) Prove that $\Lambda \subset \mathbb{R}^{2n}$ is a Lagrangian subspace if and only if there exists a unitary matrix $U = X + \mathbf{i}Y \in \mathrm{U}(n)$ such that

$$\Lambda = \Lambda_U := \left\{ \left(\begin{array}{c} X\xi \\ Y\xi \end{array} \right) \middle| \xi \in \mathbb{R}^n \right\}. \tag{4}$$

(iii) Let $U, V \in \mathrm{U}(n)$. Prove that $\Lambda_U = \Lambda_V$ if and only if $UU^T = VV^T$.
(iv) Prove that $\mathcal{L}_n$ is a submanifold of the real Grassmannian $\mathrm{Gr}(n, 2n)$ of dimension $\dim(\mathcal{L}_n) = \frac{n(n+1)}{2}$.
(v) Denote by $\mathrm{U}(n)/\mathrm{O}(n)$ the homogeneous space of all equivalence classes of unitary matrices $U \in \mathrm{U}(n)$ under the equivalence relation $U \sim V$ iff there exists an orhogonal matrix $O \in \mathrm{O}(n)$ such that $V = UO$. (This quotient space has naturally the structure of a manifold.) Denote by $\mathcal{S}(n) \subset \mathbb{C}^{n\times n}$ the space of symmetric complex $n \times n$-matrices. Prove that the map

$$\mathrm{U}(n)/\mathrm{O}(n) \to \mathcal{L}_n : [U] \mapsto \Lambda_U$$

is a diffeomorphism. Prove that the map

$$\mathcal{L}_n \to \mathrm{U}(n) \cap \mathcal{S}(n) : \Lambda_U \mapsto UU^T$$

is an embedding whose image is $\mathrm{U}(n) \cap \mathcal{S}(n)$. Deduce that $\mathrm{U}(n) \cap \mathcal{S}(n)$ is a submanifold of $\mathrm{U}(n)$ of dimension $\frac{n(n+1)}{2}$.
(vi) Let $U = X + \mathbf{i}Y \in \mathrm{U}(n)$ and let $\Lambda = \Lambda_U \in \mathcal{L}_n$ be given by (4). Define

$$g_\Lambda := UU^T = (XX^T - YY^T) + i(XY^T + YX^T).$$

Define $R_\Lambda \in \mathbb{R}^{2n\times 2n}$ by

$$R_\Lambda := \left(\begin{array}{cc} XX^T - YY^T & XY^T + YX^T \\ XY^T + YX^T & YY^T - XX^T \end{array} \right). \tag{5}$$

Prove that R_Λ is the unique anti-symplectic involution with fixed point set Λ, i.e. R_Λ satisfies the conditions

$$R_\Lambda^2 = \mathbb{1}, \qquad R_\Lambda^T J_0 R_\Lambda = -J_0, \qquad \ker\,(\mathbb{1} - R_\Lambda) = \Lambda, \tag{6}$$

and is uniquely determined by them. Prove that the fixed point set of every linear anti-symplectic involution of $\mathbb{R}^{2n}$ is a Lagrangian subspace.