Symplectic Topology
Example Sheet 3

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Exercise 3.1. Let L be an n-dimensional manifold, equipped with and atlas $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ and define an atlas on the cotangent bundle T^*L by

$$\psi_\alpha : T^*U_\alpha \to \mathbb{R}^n \times (\mathbb{R}^n)^* = \mathbb{R}^{2n}, \qquad \psi_\alpha(q, v^*) := \left(\phi_\alpha(q), (d\phi_\alpha(q)^*)^{-1} v^*\right),$$

for $q \in U_\alpha \subset L$ and $v \in T_q^*L$.

(i) Prove that there is a unique 1-form $\lambda \in \Omega^1(T^*L)$ such that

$$\lambda|_{T^*U_\alpha} = \psi_\alpha^* \left(\sum_{i=1}^n y_i dx_i\right)$$

for every α.

(ii) Show that the 1-form $\lambda \in \Omega^1(T^*L)$ in (i) can be written in the form

$$\lambda_p(\widehat{p}) = \left\langle v^*, d\pi(p)\widehat{p}\right\rangle, \qquad p = (q, v^*) \in T^*L, \quad \widehat{p} \in T_p(T^*L).$$

Here $\pi : T^*L \to L$ denotes the obvious projection.

(iii) A 1-form $\alpha \in \Omega^1(L)$ is a section of the cotangent bundle of L and hence can be interpreted as a smooth map from L to T^*L. Show that the pullback of the 1-form $\lambda \in \Omega^1(T^*L)$ under the smooth map $\alpha : L \to T^*L$ is the 1-form α itself, i.e.

$$\alpha^*\lambda = \alpha.$$

Prove that the 1-form λ on T^*L is uniquely determined by this property. It is also called the **canonical 1-form** on T^*L and denoted by λ_{can}.

(iv) Prove that

$$\omega_{\rm can} := -d\lambda_{\rm can}$$

is a symplectic form on T^*L and that the canonical coordinates

$$\psi_\alpha : T^*U_\alpha \to \mathbb{R}^{2n}$$

are Darboux charts for $\omega_{\rm can}$.

Warning: Many authors, inspired by physics, use the notation q instead of x for the position variable and p instead of y for the momentum variable; in this notation the canonical 1-form is $\lambda_{\rm can} = pdq$ and $d\lambda_{\rm can} = dp \wedge dq$ is then often used as the canonical symplectic form on the cotangent bundle. In the (x, y)-notation this would be $dy \wedge dx$. This is a key source for different choices of sign conventions in symplectic topology. (In short $dy \wedge dx$ for physicists and $dx \wedge dy$ for complex geometers.)

Consider the coordinates $z = (z_1, \dots, z_n)$ on $\mathbb{C}^n$ with $z_j = x_j + \mathbf{i}y_j \in \mathbb{C}$ for $j = 1, \dots, n$. The next exercise uses the complex-valued 1-forms

$$dz_j := dx_j + \mathbf{i}dy_j, \qquad d\bar{z}_j := dx_j - \mathbf{i}dy_j$$

and the differential operators

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \mathbf{i}\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \mathbf{i}\frac{\partial}{\partial y_j}\right)$$

on the space of complex valued smooth functions on $\mathbb{C}^n$. Thus dz_j is complex linear, $d\bar{z}_j$ is complex anti-linear, and every complex-valued 1-form on $\mathbb{C}^n$ is, at each point, a linear combination of the dz_j and the $d\bar{z}_j$ with complex coefficients. For example the differential of a smooth function $f : \mathbb{C}^n \to \mathbb{C}$ can be written in the form $df = \partial f + \bar{\partial} f$, where

$$\partial f := \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j, \qquad \bar{\partial} f := \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

For each $z \in \mathbb{C}^n$ the complex-linear functional $\partial f_z : \mathbb{C}^n \to \mathbb{C}$ is the complex-linear part of $df_z = df(z) : \mathbb{C}^n \to \mathbb{C}$, and the complex-anti-linear functional $\bar{\partial} f_z : \mathbb{C}^n \to \mathbb{C}$ is the complex-anti-linear part of df_z. Thus f is holomorphic if and only if $\bar{\partial} f = 0$. We also need the 2-form

$$\partial\bar{\partial} f := \sum_{j,k=1}^n \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k.$$

This 2-form is imaginary-valued when f is real valued. Also $\partial\bar{\partial} + \bar{\partial}\partial = 0$.

Exercise 3.2. The Fubini-Study form $\omega_{\mathrm{FS}} \in \Omega^2(\mathbb{C}\mathrm{P}^n)$ is defined by

$$\omega_{\mathrm{FS}} := \frac{\mathbf{i}}{2\,|z|^4} \sum_{j,k=1}^{n} \Big(|z_j|^2 \, dz_k \wedge d\bar{z}_k - \bar{z}_j z_k dz_j \wedge d\bar{z}_k \Big). \tag{1}$$

(Strictly speaking this is a real valued 2-form on $\mathbb{C}^{n+1} \setminus \{0\}$ and ω_{FS} is its pullback the the quotient manifold $\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$.)

(i) Prove that

$$\omega_{\mathrm{FS}} = \frac{\mathbf{i}}{2} \partial\bar{\partial} \log\left(|z|^2\right).$$

(ii) Let $\omega_0 := \sum_{j=0}^{n} dx_j \wedge dy_j = \frac{\mathbf{i}}{2} \sum_{j=0}^{n} dz_j \wedge d\bar{z}_j$ be the standard symplectic form on $\mathbb{C}^{n+1}$. Define $\phi : \mathbb{C}^{n+1} \setminus \{0\} \to S^{2n+1}$ by $\phi(z) := |z|^{-1} z$. Prove that

$$\omega_{\mathrm{FS}} = \phi^* \omega_0.$$

(iii) Let $B \subset \mathbb{C}^n$ be the open unit ball and define the coordinate charts $\phi_j : U_j \to B$, $j = 0, \ldots, n$, on $\mathbb{C}\mathrm{P}^n$ by $U_j := \{[z_0 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^n \,|\, z_j \neq 0\}$ and

$$\phi_j([z_0 : \cdots : z_n]) := \frac{(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)}{|z|}. \tag{2}$$

Show that

$$\phi^{-1}(\zeta) = \left(\frac{\zeta_1}{\sqrt{1 - |\zeta|^2}}, \ldots \frac{\zeta_{j-1}}{\sqrt{1 - |\zeta|^2}}, 1, \frac{\zeta_j}{\sqrt{1 - |\zeta|^2}}, \ldots \frac{\zeta_n}{\sqrt{1 - |\zeta|^2}} \right)$$

for $\zeta = (\zeta_1, \ldots, \zeta_n) \in B$. Show that the ϕ_j are Darboux charts on $\mathbb{C}\mathrm{P}^n$. Deduce that $\mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^n$ has area π. **Warning:** $\mathbb{C}\mathrm{P}^n$ has a natural complex structure. However, the coordinate charts ϕ_j are not holomorphic.

(iv) Consider the case $n = 1$, $x_0 = 1$, $z_1 = z = x + \mathbf{i}y \in \mathbb{C}^n$. Show that

$$\omega_{\mathrm{FS}} = \frac{\mathbf{i}}{2} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2} = \frac{dx \wedge dy}{(1 + x^2 + y^2)^2}, \qquad \int_{\mathbb{C}\mathrm{P}^1} \omega_{\mathrm{FS}} = \pi.$$

(v) Let $S^2 := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere with the volume form $\mathrm{dvol}_{S^2} := x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2$. Let $\phi : S^2 \to \mathbb{C}\mathrm{P}^1$ be the stereographic projection $\phi(x) := [1 - x_3 : x_1 + \mathbf{i}x_2]$. Prove that

$$\phi^* \omega_{\mathrm{FS}} = \frac{1}{4} \mathrm{dvol}_{S^2}.$$

Exercise 3.3. Show that the **real projective space**

$$\mathbb{RP}^n := \{[z_0 : \cdots : z_n] \in \mathbb{CP}^n \mid z_0, \dots, z_n \in \mathbb{R}\}$$

and the **Clifford torus**

$$\mathbb{T}^n := \{[z_0 : \cdots : z_n] \in \mathbb{CP}^n \mid |z_0| = |z_1| = \cdots = |z_n|\}$$

are Lagrangian submanifolds of $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$.

Exercise 3.4. Let (M, ω) be closed (i.e. compact without boundary) symplectic manifold and let $F, G : M \to \mathbb{R}$ be smooth functions. Show that the Poisson bracket of F and G satisfies the identity

$$\{F, G\} \frac{\omega^n}{n!} = dF \wedge dG \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

Deduce that the Poisson bracket has mean value zero, i.e.

$$\int_M \{F, G\} \frac{\omega^n}{n!} = 0.$$

Exercise 3.5. Let (M, ω) be closed symplectic manifold and let $\phi : M \to M$ be a symplectomorphism. Prove that, if ϕ is sufficiently close to the identity in the C^1-topology, then ϕ is **symplectically isotopic** to the identity, i.e. there exists a smooth map

$$[0, 1] \times M \to M : (t, p) \mapsto \phi_t(p)$$

such that $\phi_t^* \omega = \omega$ for every t and

$$\phi_0 = \mathrm{id}, \qquad \phi_1 = \phi.$$

Hint: Use Weinstein's Lagrangian neighborhood theorem for the diagonal in

$$\widetilde{M} := M \times M, \qquad \widetilde{\omega} := (-\omega) \times \omega = \mathrm{pr}_1^* \omega - \mathrm{pr}_0^* \omega,$$

where $\mathrm{pr}_0, \mathrm{pr}_1 : M \times M \to M$ are defined by $\mathrm{pr}_0(p, q) := p$ and $\mathrm{pr}_1(p, q) := q$. Show that every Lagrangian submanifold of T^*M sufficiently close to the zero section in the C^1-topology is the graph of a closed 1-form. Show that every Lagrangian submanifold of $\widetilde{M}$ sufficently close to the diagonal in the C^1-topology is the graph of a symplectomorphism.

Exercise 3.6. Let J be an almost complex structure on a manifold M. The **Nijenhuis tensor of** J is the 2-form N_J on M with values in the tangent bundle TM, defined by

$$N_J(X,Y) := [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$$

for $X, Y \in \mathrm{Vect}(M)$.

(i) Verify that N_J is a **tensor**, i.e.

$$N_J(fX, gY) = fgN_J(X,Y)$$

for every pair of smooth functions $f, g : M \to \mathbb{R}$. Equivalently, for every $p \in M$, the tangent vector $N_J(X,Y)(p) \in T_pM$ depends only on $X(p)$ and $Y(p)$, but not on the derivatives of X and Y at p.

(ii) Show that

$$N_J(JX,Y) = N_J(X,JY) = -JN_J(X,Y)$$

for all $X, Y \in \mathrm{Vect}(M)$.

(iii) Denote by $TM^c := TM \otimes_{\mathbb{R}} \mathbb{C}$ the complexified tangent bundle and by

$$E^{\pm} := \{(p, v^c) \in TM^c \,|\, J_p v^c = \pm \mathbf{i} v^c\}$$

the subbundles determines by the eigenspaces of J. Thus $TM^c = E^+ \oplus E^-$. Prove that the Nijenhuis tensor vanishes if and only if the subbundles $E^{\pm}$ are **involutive** (i.e. invariant under Lie brackets).

(iv) Let $\phi : M' \to M$ be a diffeomorphism. Prove that

$$N_{\phi^* J}(\phi^* X, \phi^* Y) = \phi^* N_J(X,Y)$$

for all $X, Y \in \mathrm{Vect}(M)$.

(v) Assume $\dim(M) = 2$. Prove that $N_J = 0$ for every $J \in \mathcal{J}(M)$.

Exercise 3.7. Let (M, ω) be a symplectic manifold. An almost complex structure $J \in \mathcal{J}(M)$ is said to be **tamed by** ω if

$$\omega(v, Jv) > 0$$

for every nonzero tangent vector v. Let $\mathcal{J}_\tau(M,\omega)$ denote the space of ω-tame almost complex structures J on M. Prove that $\mathcal{J}_\tau(M,\omega)$ is contractible.

Hint: Fix an ω-compatible almost complex structure $J_0 \in \mathcal{J}(M,\omega)$ and denote by $g_0 := \omega(\cdot, J_0\cdot)$ the associated Riemannian metric. Let

$$|v|_0 := \sqrt{g_0(v,v)}$$

be the norm of a tangent vector $v \in T_pM$ with respect to this metric and let

$$\|A\|_0 := \sup_{v \in T_pM \setminus \{0\}} \frac{|Av|_0}{|v|_0}$$

be the corresponding operator norm of an endomorphism $A : T_pM \to T_pM$. Define

$$\mathcal{A} := \left\{ A \in \Omega^0(M, \mathrm{End}(TM)) \,\middle|\, AJ_0 + J_0A = 0,\ \|A(p)\|_0 < 1\ \forall\, p \in M \right\}.$$

Prove that the formula

$$\mathcal{F}(J) := (\mathbb{1} + J_0J)(\mathbb{1} - J_0J)^{-1}$$

defines a homeomorphism $\mathcal{F} : \mathcal{J}_\tau(M,\omega) \to \mathcal{A}$ with inverse

$$\mathcal{F}^{-1}(A) = J_0(\mathbb{1} + A)^{-1}(\mathbb{1} - A).$$

Use the fact that $\mathcal{A}$ is convex.