Symplectic Topology
Example Sheet 5

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Generating Functions

Exercise 5.1. Let $A = A^T \in \mathbb{R}^{n\times n}$ and $C = C^T \in \mathbb{R}^{N\times N}$ be symmetric matrices and let and $B \in \mathbb{R}^{n\times N}$. Prove that the following set is a Lagrangian subspace of $\mathbb{R}^{2n}$:

$$\Lambda := \left\{ (x,y) \in \mathbb{R}^{2n} \,\middle|\, \begin{array}{l} \exists\, \xi \in \mathbb{R}^N \text{ such that } B^T x + C\xi = 0 \\ \text{and } Ax + B\xi = y \end{array} \right\}.$$

Exercise 5.2 (Generating Functions). Let $\pi : E \to L$ be a submersion between smooth manifolds and let $f : E \to \mathbb{R}$ be a smooth function. Denote the fiber over $q \in L$ by $E_q := \pi^{-1}(q)$, the restriction of f to the fiber by $f_q := f|_{E_q} : E_q \to \mathbb{R}$, and the set of **fiber critical points** by

$$\mathcal{C} := \mathcal{C}(E,f) := \{c \in E \mid \ker d\pi(c) \subset \ker df(c)\}.$$

Define the map $\iota_f : \mathcal{C} \to T^*L$ by $\iota_f(c) := (q, v^*)$, where $q := \pi(c)$ and $v^* \in T_q^*L$ is the unique **Lagrange multiplier** given by

$$df(c) = v^* \circ d\pi(c). \tag{1}$$

Assume that the graph of df in T^*E intersects the fiber normal bundle $N_E := \{(c,\eta) \in T^*E \mid \ker d\pi(c) \subset \ker \eta\}$ transversally. Prove that $\mathcal{C}$ is an n-dimensional submanifold of E and that $\iota_f : \mathcal{C} \to T^*L$ is a Lagrangian immersion. Thus the immersed submanifold $\Lambda := \iota_f(\mathcal{C}) \subset T^*L$ of Lagrange multipliers is a Lagrangian submanifold. **Hint:** Assume first that $L = \mathbb{R}^n$ and $E = \mathbb{R}^n \times \mathbb{R}^N$. Use Exercise 5.1.

Exercise 5.3. Let M be a manifold and let $f, g_1, \dots, g_n : M \to \mathbb{R}$ be smooth functions. For $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ define

$$f_y := f - \sum_{i=1}^{n} y_i g_i : M \to \mathbb{R}.$$

Consider the set

$$\mathcal{C} := \left\{ (p, y) \in M \times \mathbb{R}^n \,\middle|\, df_y(p) = 0 \right\}.$$

Assume that

$$\ker d^2 f_y(p) \cap \bigcap_{i=1}^{n} \ker dg_i(p) = \{0\}$$

for all $(p, y) \in \mathcal{C}$. Prove that $\mathcal{C}$ is an n-dimensional submanifold of $M \times \mathbb{R}^n$. Prove that the map

$$\mathcal{C} \to \mathbb{R}^{2n} : (p, y) \mapsto (g(p), y)$$

is a Lagrangian immersion. **Hint:** This is a special case of Exercise 5.2. If the map $g = (g_1, \dots, g_n) : M \to \mathbb{R}^n$ is a submersion, take $E := M$, $L := \mathbb{R}^n$, and $\pi := g$. Alternatively, take $E := M \times \mathbb{R}^n$, $L := \mathbb{R}^n$, $\pi(p, y) := y$, and $f(p, y) := f_y(p)$. In this case the roles of x and y are reversed, and $x = g(p)$ is now the Lagrange multiplier.

Energy

Exercise 5.4. Let (M, ω) be a symplectic manifold, let $L \subset M$ be a Lagrangian submanifold, let $J \in \mathcal{J}_\tau(M, \omega)$ be an ω-tame almost complex structure, and denote by $g_J := \frac{1}{2}(\omega(\cdot, J\cdot) - \omega(J\cdot, \cdot))$ the Riemannian metric determined by ω and J. Let (Σ, j) be a compact Riemann surface with boundary and let $u : (\Sigma, \partial\Sigma) \to (M, L)$ be a J-holomorphic curve with boundary values in L. Prove that the energy

$$E(u) := \frac{1}{2} \int_\Sigma |du|_J \, \mathrm{dvol}_\Sigma$$

of u depends only on the homotopy class of u subject to the boundary condition $u(\partial\Sigma) \subset L$.

Holomorphic equivalence relations

Let $\Gamma \subset \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ be an equivalence relation and write $z \sim \zeta$ when $(z, \zeta) \in \Gamma$. Denote the equivalence class of $z \in \mathbb{C}\mathrm{P}^1$ by

$$[z] := \left\{ \zeta \in \mathbb{C}\mathrm{P}^1 \,|\, \zeta \sim z \right\}.$$

The equivalence relation is called **holomorphic** if there exists a finite set $X \subset \mathbb{C}\mathrm{P}^1$ such that Γ intersects the dense open set $(\mathbb{C}\mathrm{P}^1 \setminus X) \times (\mathbb{C}\mathrm{P}^1 \setminus X)$ in a one-dimensional complex submanifold whose projection onto the first factor is a proper holomorphic covering, and Γ is the closure of its intersection with $(\mathbb{C}\mathrm{P}^1 \setminus X) \times (\mathbb{C}\mathrm{P}^1 \setminus X)$. Associated to such a holomorphic equivalence relation Γ is a **multiplicity function** $m_\Gamma : \mathbb{C}\mathrm{P}^1 \to \mathbb{N}$ defined by

$$m_\Gamma(z) := \#([w] \cap U)$$

for a sufficently small neighborhood $U \subset \mathbb{C}\mathrm{P}^1$ of z and for $w \in \mathbb{C}\mathrm{P}^1 \setminus \{z\}$ sufficiently close to z. In particular $m_\Gamma(z) = 1$ for $z \in \mathbb{C}\mathrm{P}^1 \setminus X$. The number

$$d := \sum_{\zeta \sim z} m_\Gamma(z)$$

is independent of the choice of z and is called the **degree of** Γ.

Exercise 5.5. Let $\Gamma \subset \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ be a holomorphic equivalence relation of degree d. Prove that there is a rational function $\phi_\Gamma : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ of degree d such that

$$\phi_\Gamma(z) = \phi_\Gamma(\zeta) \qquad \Longleftrightarrow \qquad z \sim \zeta. \tag{2}$$

Hint: Choose an identification of $\mathbb{C}\mathrm{P}^1$ with the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ such that $0 \not\sim \infty$ and define

$$P_0 := \{ z \in \mathbb{C} \,|\, z \sim 0 \}, \qquad P_\infty := \{ z \in \mathbb{C} \,|\, z \sim \infty \}.$$

For $z \in \mathbb{C} \setminus P_\infty$ define

$$\phi_\Gamma(z) := \prod_{\zeta \sim z} \zeta^{m_\Gamma(\zeta)}. \tag{3}$$

Prove that ϕ_Γ is holomorphic, extends to a rational function of degree d from $\overline{\mathbb{C}}$ to itself, has a zero of order $m_\Gamma(z)$ at $z \in P_0$, and has a pole of order $m_\Gamma(z)$ at $z \in P_\infty$. Prove that ϕ_Γ satisfies (2).

Exercise 5.6. Let $u : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a nonconstant rational function such that $u(0) \neq u(\infty)$. Prove that the set $\Gamma := \{(z, \zeta) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}} \,|\, u(z) = u(\zeta)\}$ is a holomorphic equivalence relation. Prove that $m_\Gamma(z)$ is the order of z as a pole of u when $u(z) = \infty$, and that $m_\Gamma(z)$ is the order of z as a zero of $u - u(z)$ when $u(z) \neq \infty$. Define $\phi_\Gamma : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by (3) as in Exercise 5.5. Prove that there exists a Möbius transformation $u' : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $u = u' \circ \phi_\Gamma$.

Exercise 5.7 (Simple J-Holomorphic Curves). Let J be a C^2 almost complex structure on a manifold M and let (Σ_0, j_0), (Σ_1, j_1) be closed connected Riemann surfaces. Let $u_0 : \Sigma_0 \to M$, $u_1 : \Sigma_1 \to M$ be simple J-holomorphic curves of class C^2.

(i) Assume $u_0(\Sigma_0) = u_1(\Sigma_1)$. Prove that there exists a unique holomorphic diffeomorphism $\phi : (\Sigma_1, j_1) \to (\Sigma_0, j_0)$ such that $u_1 = u_0 \circ \phi$.

(ii) Assume $u_0(\Sigma_0) \neq u_1(\Sigma_1)$. Prove that the set $u_0^{-1}(u_1(\Sigma_1)) \subset \Sigma_0$ is at most countable and can only accumulate at the critical points of u_0.

Positivity of Intersections

Let $\mathbb{D} \subset \mathbb{C}$ denote the closed unit disc and let $v_0, v_1 : \mathbb{D} \to \mathbb{R}^4$ be smooth maps such that

$$v_0(\partial\mathbb{D}) \cap v_1(\mathbb{D}) = \emptyset, \qquad v_0(\mathbb{D}) \cap v_1(\partial\mathbb{D}) = \emptyset \tag{4}$$

and v_0 and v_1 intersect transversally, i.e.

$$\mathbb{R}^4 = \operatorname{im} dv_0(w_0) \oplus \operatorname{im} dv_1(w_1)$$

for every pair $(w_0, w_1) \in \mathbb{D} \times \mathbb{D}$ such that $v_0(w_0) = v_1(w_1)$. The **intersection number of v_0 and v_1** is defined

$$v_0 \cdot v_1 := \sum_{v_0(w_0) = v_1(w_1)} \varepsilon(w_0, w_1),$$

where the sum runs over all $(w_0, w_1) \in \mathbb{D} \times \mathbb{D}$ such that $v_0(w_0) = v_1(w_1)$ and the sign $\varepsilon(w_0, w_1) \in \{\pm 1\}$ is chosen according to whether or not orientations match in the direct sum decomposition $\mathbb{R}^4 = \operatorname{im} dv_0(w_0) \oplus \operatorname{im} dv_1(w_1)$. Standard intersection theory asserts that the intersection number $v_0 \cdot v_1$ is invariant under homotopies preserving condition (4) and hence is well defined for any pair of smooth maps $v_0, v_1 : \mathbb{D} \to \mathbb{R}^4$ satisfying (4).

Now let Σ_0, Σ_1 be closed oriented 2-manifold, M be an oriented 4-manifold and $u_0 : \Sigma_0 \to M$ and $u_1 : \Sigma_1 \to M$ be smooth maps such that

$$Z := \{(z_0, z_1) \in \Sigma_0 \times \Sigma_1 \,|\, u_0(z_0) = u_1(z_1)\}$$

is a finite set. The **intersection index** of u_0 and u_1 at a pair $(z_0, z_1) \in Z$ is the integer

$$\iota(u_0, u_1; z_0, z_1) := v_0 \cdot v_1,$$

where $\phi_i : (U_i, z_i) \to (\mathbb{C}, 0)$ is an orientation preserving coordinate chart on Σ_i for $i = 0, 1$, $\psi : (V, p) \to (\mathbb{R}^4, 0)$ is an orientation preserving coordinate chart on M centered at $p := u_0(z_0) = u_1(z_1)$, and $v_i : \mathbb{D} \to \mathbb{R}^4$ is defined by $v_i(z) := \psi \circ u_i \circ \phi_i^{-1}(\varepsilon z)$ for $i = 0, 1$, and $\varepsilon > 0$ sufficiently small. The **intersection number** of u_0 and u_1 is the integer defined as the sum of the intersection indices

$$u_0 \cdot u_1 := \sum_{(z_0,z_1)\in Z} \iota(u_0, u_1; z_0, z_1).$$

This is a homotopy invariant.

Exercise 5.8 (Transversality). Let $v_0, v_1 : \mathbb{C} \to \mathbb{C}^2$ be smooth maps. Let A_{reg} be the set of vectors $a \in \mathbb{C}^2$ such that $v_0 + a$ and v_1 intersect transversally. Prove that the complement $\mathbb{C}^2 \setminus A_{\mathrm{reg}}$ has Lebesque measure zero. **Hint:** Prove that the set

$$\mathcal{Z} := \left\{(w_0, w_1, a) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^2 \,|\, v_1(w_1) - v_0(w_0) = a\right\}$$

is a smooth 4-dimensional submanifold of $\mathbb{C} \times \mathbb{C} \times \mathbb{C}^2$. Prove that $a \in A_{\mathrm{reg}}$ if and only if a is a regular value of the projection $\mathcal{Z} \to \mathbb{C}^2 : (w_0, w_1, a) \mapsto a$.

Exercise 5.9 (Positivity of Intersections, Part 1). Let $v_0, v_1 : \mathbb{C} \to \mathbb{C}^2$ be polynomials of the form

$$v_0(w_0) = w_0^{k_0}(1, p_0(w_0)), \qquad v_1(w_1) = w^{k_1}(p_1(w_1), 1)$$

where $k_0, k_1 \in \mathbb{N}$ and $p_0, p_1 : \mathbb{C} \to \mathbb{C}$ are polynomials that vanish at the origin. Prove that $w_0 = w_1 = 0$ is an isolated intersection of v_0 and v_1 and that the intersection index is $\iota(v_0, v_1; 0, 0) = k_0 k_1$. **Hint:** Assume first that $p_0 = p_1 = 0$.

Exercise 5.10 (Positivity of Intersections, Part 2). Let $v_0, v_1 : \mathbb{C} \to \mathbb{C}^2$ be polynomials of the form

$$v_0(w_0) = w_0^k(1, p_0(w_0)), \qquad v_1(w_1) = w_1^k(1, p_1(w_1))$$

where $k \in \mathbb{N}$ and $p_0, p_1 : \mathbb{C} \to \mathbb{C}$ are polynomials that vanish at the origin. Assume that $w_0 = w_1 = 0$ is an isolated intersection point of v_0 and v_1. Prove that the intersection index satisfies the inequality

$$\iota(v_0, v_1; 0, 0) \geq k(k+1)$$

and hence is at least two. **Hint:** Consider the intersections of the curves

$$v_{0,a}(w_0) := (w_0^k, w_0^k p_0(w_0) + a), \qquad v_1(w_1) = (w_1^k, w_1^k p_1(w_1))$$

near the origin for $a \neq 0$ sufficiently small.

Exercise 5.11 (Positivity of Intersections, Part 3). Let $v_0, v_1 : \mathbb{C} \to \mathbb{C}^2$ be polynomials of the form

$$v_0(w_0) = w_0^{k_0}(1, p_0(w_0)), \qquad v_1(w_1) = w_1^{k_1}(1, p_1(w_1)), \qquad 0 < k_0 < k_1,$$

where $p_0, p_1 : \mathbb{C} \to \mathbb{C}$ are polynomials that vanish at the origin. Assume that $w_0 = w_1 = 0$ is an isolated intersection of v_0 and v_1. Prove that the intersection index satisfies the inequality

$$\iota(v_0, v_1; 0, 0) \geq k_0 + 1$$

and hence is at least two. **Hint:** Consider the intersections of the curves

$$v_{0,a}(w_0) := (w_0^{k_0}, w_0^{k_0} p_0(w_0) + a), \qquad v_1(w_1) = (w_1^{k_1}, w_1^{k_1} p_1(w_1))$$

near the origin for $a \neq 0$ sufficiently small. Take $w_0 = z^{k_1}$ and $w_1 = \lambda z^{k_0}$ where $\lambda^{k_1} = 1$.