C^* and W^* dinamical systems

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1 C^*-dynamical systems (Gaia Torresani)

[Pag 159-160, [2]] Physical theories consist essentially of two elements, a **kinematical structure** describing the instantaneous **states** and **observables** of the system, and a **dynamical rule** describing the change of these states and observables with time. In the classical mechanics of point particles a state is represented by a point in a differentiable manifold and the observables by functions over the manifold. In the **quantum mechanics** of systems with a **finite number of degrees of freedom** the states are given by **rays in a Hilbert space** and the observables by **operators** acting on the space. For particle systems with an **infinite number of degrees of freedom** we intend to identify the states with **states over** appropriate **algebras of** fields, or **operators**. In each of these examples the **dynamical description** of the system is given by a flow, a **one-parameter group of automorphisms of the underlying kinematical structure**, which represents the motion of the system with time. In classical mechanics one has a group of **diffeomorphisms**, in quantum mechanics a group of **unitary operators on the Hilbert space**, and for systems with an infinite number of degrees of freedom a group of **automorphisms of the algebra of observables**.
One-parameter semigroups will be used in the study the time evolution of Open Quantum Systems.
The general problem is to study the **differential equation**

$$\frac{dA_t}{dt} = SA_t.$$

The A corresponds to an observable, or state, of the system and will be represented by an element of some suitable space X. The function $t \mapsto A_t \in X$ describes the motion of A and **S** is an operator on X, which generates the **infinitesimal change** of A. Formally, the solution of the differential equation is $A_t = U_t A$, where $U_t = \exp tS$ and the problem is to give a meaning to the exponential. Independently of the manner in which this is done one expects U_t to have the property that U_0 is the identity and that $U_t U_s = U_{t+s}$ and so we seek solutions of this nature. There are, however, many **different possible types of continuity** of $t \mapsto U_t$ and this leads to a structural hierarchy. We examine uniform, strong, and σ weak continuity.

Note: [Page 228 [2]] the concept of a σ weakly continuous group, of an algebra $\mathfrak{U}$ is only defined when $\mathfrak{U}$: has a predual. But in this case $\mathfrak{U}$: is automatically a von Neumann algebra by Sakai's theorem. Moreover, one may demonstrate (see Example 3.2.36 in [2]) that a strongly continuous group, of $*$-automorphisms of a von Neumann algebra $\mathfrak{M}$ is automatically uniformly continuous. Then strongly continuous groups are appropriate to C^*-algebras, σ-weakly continuous to von Neumann algebras, and uniformly continuous groups to both structures.

On this talk we will focus on strongly continuous groups of C^*-algebras. We start by stating some general facts of the theory of strongly continuous one-parameter semigroups.

Definition 1.1. *Let $\{A(t)\}_{t\geqslant 0}$ be a family of bounded linear operators defined on a Banach space $\mathcal{B}$. We say that $\{A(t)\}_{t\geqslant 0}$ is a strongly continuous semigroup or C_0 semigroup if*

1. $A(0) = I$
2. $A(t+s) = A(t)A(s)$ *for any* $s, t \geqslant 0$.
3. $A(t)\varphi$ *is continuous as a function of* t *on* $[0, \infty)$*, with respect to the norm of* $\mathcal{B}$*, for all* $\varphi \in \mathcal{B}$.

Note: A semigroup $A(t)$ is defined only for $t \geqslant 0$, a group is defined for $t \in \mathbb{R}$.

Remark: The third property is equivalent to the continuity in 0^+ with respect to the norm of $\mathcal{B}$, that is $\|A(t)\varphi - \varphi\| \to 0$ for $t \to 0^+$.

Proof. One direction is obvious. We prove that the continuity in 0^+ implies (3). Let $t, h \geqslant 0$

$$\|A(t+h)\varphi - A(t)\varphi\| \leqslant \|A(t)\|\|A(h)x - x\| \leqslant Me^{\omega t}\|A(h)x - x\|$$

where we have used property (a) of Proposition 1, (that we will present later). If $h \leqslant 0$ the proof is analogous. ☐

We are now ready to give the definition of C^* dynamical system.

Recall: A $*$-morphism between two C^*-algebras $\mathfrak{U}_1$ and $\mathfrak{U}_2$ is a mapping $\pi : \mathfrak{U}_1 \to \mathfrak{U}_2$ such that, for $A, B \in \mathfrak{U}_1$ and $\alpha, \beta \in \mathbb{C}$:

- $\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)$,
- $\pi(AB) = \pi(A)\pi(B)$,
- $\pi(A^*) = \pi(A)^*$.

A $*$-automorphism of a C^*-algebra $\mathfrak{U}$ is a $*$-morphism $\pi : \mathfrak{U} \to \mathfrak{U}$, that is bijective.

Definition 1.2. *A C^*-dynamical system is a pair $(\mathfrak{U}, \tau^t)$ where $\mathfrak{U}$ is a C^*-algebra with a unit and τ^t a strongly continuous group of $*$-automorphisms of $\mathfrak{U}$.*

Remark: Strong continuity means that $t \mapsto \tau^t(A)$ is continuous with respect to the norm topology of $\mathfrak{U}$.

Remark: Since $\tau^t((z-A)^{-1}) = (z-\tau^t(A))^{-1}$, a $*$-automorphism τ^t preserves the spectrum. Furthermore, we recall that it is norm continuous and since it is also invertible, it is isometric, i.e. $\|\tau_t(A)\| = \|A\| \; \forall A \in \mathfrak{U}$ [Corollary 2.3.4 in [2]].

We investigate some properties of one parameter semigorup and introduce the concept of infinitesimal generator. We will later apply this results to C^* dynamical systems.

Definition 1.3. *The infinitesimal generator of the C_0 semigroup $\{A(t)\}_{t\geqslant 0}$ on Banach space $\mathcal{B}$, is the linear operator (S,D) defined by*

$$D = \left\{ \varphi \in \mathcal{B} | \lim_{t\to 0^+} \frac{A(t)-I}{t}\varphi \text{ exists in } \mathcal{B} \right\} \tag{1}$$

$$S\varphi = \lim_{t\to 0^+} \frac{A(t)-I}{t}\varphi, \;\; \varphi \in D. \tag{2}$$

Proposition 1.4 (Theorems 2.2-2.6 in [3] and proposition 6.4 in [1])**.** *Let $\{A(t)\}_{t\geqslant 0}$ be a C_0 semigroup on Banach space $\mathcal{B}$ of generator A. Then*

a) There exist $\omega \in \mathbb{R}$ and $M \geqslant 1$ such that $A(t) \leqslant Me^{\omega t}$, for all $t \geqslant 0$.

b) For any $t \geqslant 0, \varphi \in \mathcal{B}$, we have $\lim_{h\to 0^+} 1/t \int_t^{t+h} A(\tau)\varphi d\tau = A(t)\varphi$.

c) For any $t \geqslant 0, \varphi \in \mathcal{B}$, we have $\int_0^t A(\tau)\varphi d\tau \in D$ and

$$S\left(\int_0^t A(\tau)d\tau\right) = A(t)\varphi - \varphi \tag{3}$$

d) For any $t \geqslant 0$, $A(t) : D \to D$ and if $\varphi \in D, t \mapsto A(t)\varphi$ is in $C^1([0,\infty))$ and

$$\frac{d}{dt}A(t)\varphi = SA(t)\varphi = A(t)S\varphi, \quad t \geqslant 0 \tag{4}$$

e) The generator S is closed with dense domain D.

f) If $\{A_1(t)\}_{t\geqslant 0}$ and $\{A_2(t)\}_{t\geqslant 0}$ are two C_0 semigroups with the same generator S, then $A_1(t) \equiv A_2(t)$.

Proof. a) Recall that the Bancah Steinhaus theorem says that if X is a Banach space, Y a normed vector space and $B(X,Y)$ the space of continuous linear operators between X and Y and $F \subset B(X,Y)$ then $(\sup_{T\in F} \|T(x)\|_Y \leqslant \infty, \; \forall x \in$

$X) \Rightarrow (\sup_{T\in F, \|x\|\leqslant 1} \|T(x)\|_Y = \sup_{T\in F} \|T\|_{B(X,Y)} \leqslant \infty)$. By the right continuity at 0 and the Banach Steinhaus theorem we have that there exists $\varepsilon > 0$, $M \geqslant 1$ such that $\|A(t)\| \leqslant M$ if $t \in [0, \varepsilon]$. For every $t \geqslant$ there exists $n \in \mathbb{N}$ and $0 < \delta \leqslant \varepsilon$ such that $t = \delta + n\varepsilon$, then by property 2 of C_0 semigroups,

$$\|A(t)\| = \|A(\delta)A(\varepsilon)^n\| \leqslant M^{n+1} \leqslant MM^{t/\varepsilon} = Me^{\omega t}$$

where $\omega = \ln M/\varepsilon \geqslant 0$.
b) Follows from continuity of $t \mapsto A(t)\varphi$.
c) For $\varphi \in \mathcal{B}, t \geqslant 0$ and any $\varepsilon > 0$, by applying the definition of infinitesiaml generator, we have

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} \frac{A(\varepsilon)-I}{\varepsilon}\int_0^t A(\tau)\varphi d\tau = \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_0^t (A(\varepsilon)A(\tau)-A(\tau))\varphi d\tau \\ &= \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_0^t (A(\varepsilon+\tau)-A(\tau))\varphi d\tau = \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_\varepsilon^{t+\varepsilon} A(\tau)\varphi d\tau - \frac{1}{\varepsilon}\int_0^t A(\tau)\varphi d\tau \\ &= \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_t^{t+\varepsilon} A(\tau)\varphi d\tau - \frac{1}{\varepsilon}\int_0^\varepsilon A(\tau))\varphi d\tau = A(t)\varphi - \varphi \end{aligned}$$

where we have used property (2) and (3) of the C_0 semigroup and point (b).
d) Let $t \geqslant 0, \varphi \in D$, we have that:

$$\lim_{\varepsilon\to 0^+}\frac{A(\varepsilon)-I}{\varepsilon}A(t)\varphi = \lim_{\varepsilon\to 0^+} A(t)\frac{A(\varepsilon)-I}{\varepsilon}\varphi = A(t)S\varphi$$

which proves $A(t)\varphi \in D$ and $SA(t) = A(t)S$. By property (2) of C_0 semigroup and the above consideration, the function $t \mapsto A(t)\varphi$ has a right derivative that is continuous on $[0, \infty)$. This implies continuous differentiability on $[0, \infty)$.
e) Let $\varphi \in \mathcal{B}$, let $\varphi_\varepsilon = \frac{1}{\varepsilon}\int_0^\varepsilon A(\tau)\varphi d\tau$. By point (c) $\varphi_\varepsilon \in D$, furthermore $\lim_{\varepsilon\to 0^+}\varphi_\varepsilon = \varphi$ by point (b) which proves that D is dense.
Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence in D such that $\varphi_n \to \varphi$ and $S\varphi_n \to \psi$, for some $\varphi, \psi \in \mathcal{B}$. For any $n \in \mathbb{N}$, integrating the expression in (d) implies that

$$A(t)\varphi_n - \varphi_n = \int_0^t A(\tau)S\varphi_n d\tau$$

by letting $n \to \infty$, we get $A(t)\varphi - \varphi = \int_0^t A(\tau)\psi d\tau$, therefore by part (b)

$$\lim_{t\to 0^+}\frac{1}{t}(A(t)\varphi - \varphi) = \psi$$

then $\varphi \in D$, and $\psi = S\varphi$.
f) For $\varphi \in D$, $t > 0$, we define $\psi(\tau) = A_1(t-\tau)A_2(\tau)\varphi$ for $\tau \in [0, T]$. Thanks to (d) we can differentiate and we get:

$$\frac{d}{d\tau}\psi(\tau) = -A_1(t-\tau)SA_2(\tau)\varphi + A_1(t-\tau)SA_2(\tau)\varphi = 0$$

then $\psi(t) = \psi(0)$, which says that $A_1(t)\varphi = A_2(t)\varphi$, then by density of D and due to the fact that $A_1(t), A_2(t)$ are bounded, we have that $A_1(t) \equiv A_2(t)$. $\square$

Proposition 1.5. *Let $\{A(t)\}_{t\geqslant 0}$ be a one parameter semigroup of bounded linear operators $S_t \in \mathcal{L}(\mathcal{B})$ of a Banach space $\mathcal{B}$. The following conditions are equivalent:*

1. *A_t is uniformly continuous i.e.,* $\lim_{t\to 0} \|A_t - I\| = 0$*;*
2. *it has a bounded generator S, i.e.,* $\lim_{t\to 0} \|(A_t - I)/t - S\| = 0$*;*
3. *there is a bounded operator $S \in \mathcal{L}(\mathcal{B})$ such that $A_t = \sum_{n\geqslant 0} \frac{t^n}{n!} S^n$.*

If these conditions are fulfilled then A_t extends to a uniformly continuous one parameter group satisfying $\|A_t\| \leqslant e^{|t|\|S\|}$.

Proof. See proposition 3.1.1. in [2] □

We now give a characterization of the generator of strongly continuous one parameter groups of $*-$automorphisms of a C^*-algebra. We begin by introducing the concept of $*-$derivation.

Definition 1.6. *Let $\mathfrak{U}$ be a $*$- algebra and $\mathfrak{D} \subset \mathfrak{U}$. A linear operator $\delta : \mathfrak{D} \to \mathfrak{U}$ is called a $*$-derivation if*

a) $\mathfrak{D}$ is a $*$-subalgebra of $\mathfrak{U}$.

b) $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathfrak{D}$.

c) $\delta(A^*) = \delta(A)^*$ for all $A \in \mathfrak{D}$.

Observation: The unit $\mathbb{1} \in D(\delta)$ (Corollary 3.2.30 in [2]) then $\delta(\mathbb{1}) = 0$ because $\delta(\mathbb{1}) = \delta(\mathbb{1}^2) = 2\delta(\mathbb{1})$. Derivations arise as infinitesimal generators of strongly continuous groups $\{\tau_t\}_{t\in\mathbb{R}}$ of $*$-automorphisms of a C^* algebra $\mathfrak{U}$. The two defining properties originate by differentiation of the relations $\tau_t(A^*) = \tau_t(A)^*$ and $\tau_t(AB) = \tau_t(A)\tau_t(B)$ for $A, B \in \mathfrak{U}$:

$$\begin{aligned} \lim_{t\to 0} \frac{\tau_t(A^*) - A^*}{t} &= \lim_{t\to 0} \frac{\tau_t(A)^* - A^*}{t} \\ \Rightarrow \lim_{t\to 0} \frac{\tau_t - I}{t} A^* = \lim_{t\to 0} \left(\frac{\tau_t - I}{t}\right)^* &A \Rightarrow \delta(A^*) = \delta(A)^* \end{aligned}$$

and

$$\begin{aligned} \lim_{t\to 0} \frac{\tau_t(AB) - AB}{t} &= \lim_{t\to 0} \frac{\tau_t(A)\tau_t(B) - AB}{t} \\ \Rightarrow \lim_{t\to 0} \frac{\tau_t - I}{t} AB &= \lim_{t\to 0} \frac{\tau_t(A)\tau_t(B) - \tau_t(A)B + \tau_t(A)B - AB}{t} \\ \Rightarrow \delta(AB) &= \lim_{t\to 0} \frac{\tau_t(A)(\tau_t(B) - B)}{t} + \frac{\tau_t(A)B - AB}{t} \\ &= \lim_{t\to 0} \tau_t(A) \frac{\tau_t - I}{t} B + \left(\frac{\tau_t - I}{t} A\right) B = A\delta(B) + \delta(A)B. \end{aligned}$$

The domain $\mathfrak{D}$ of the infinitesimal generator δ is contained in $\mathfrak{U}$, hence it is a $*-$subalgebra of $\mathfrak{U}$. We have proved that the generator of a dynamical group is a derivation.

We can characterize infinitesiaml generators by exponentiating them in a suitable form. We begin with a result which characterizes the generator S of a semigroup of contractions by properties of its resolvent. The algorithm $e^{tx} = \lim_{n\to\infty}(1 - tx/n)^{-n}$ for the numerical exponential can be extended to an operator relation if the "resolvent" $(I - tS/n)^{-n}$ has suitable properties. The definition of the resolvent of a closed operator S requires two pieces of information. Firstly, one must know that the range of $(1 - tS/n)$ is equal to the whole space in order that $(1 - tS/n)^{-l}$ should be everywhere defined and, secondly, one needs a bound on $\|(I - tS/n)^{-n}\|$.

The Hille-Yosida theorem characterizes generators by properties of their resolvents.

Theorem 1.7 (Hille Yosida)**.** *Let S be an operator on the Banach space $\mathcal{B}$. The following conditions are equivalent:*

1. *S is the infinitesimal generator of a strongly continuous semigroup of contractions $U(t)_{t\geqslant 0}$;*
2. *S is densely defined in $\mathcal{B}$ and closed. For $\alpha \geqslant 0$*
$$\|(I - \alpha S)\varphi\| \geqslant \|\varphi\|, \text{ for } \varphi \in D(S) \tag{5}$$
and for some $\alpha > 0$,
$$Ran(I - \alpha S) = \mathcal{B}. \tag{6}$$

If these conditions are satisfied then the semigroup is defined in terms of S by either of the limits

$$U_t\varphi = \lim_{\varepsilon\to 0} \exp\{tS(I - \varepsilon S)^{-1}\}\varphi = \lim_{n\to\infty}(I - tS/n)^{-n}\varphi.$$

where the exponential of the bounded operator is defined by power series expansion, $\varphi \in \mathcal{B}$.

Proof. (1) $\Rightarrow$ (2) If S is the infinitesimal generator of a strongly continuous semigroup then it is closed and its domain $D(S)$ is dense in $\mathcal{B}$. Let $\lambda > 0$ and $x \in \mathcal{B}$, let

$$R_\lambda x = \int_0^\infty e^{-\lambda t} U(t) x dt.$$

by continuity of $t \mapsto U(t)x$ and uniformly boundedness (we are dealing with contractions) the integral exists as an improper Rienmann integral and defines a bounded linear operator R_λ that satisfies:

$$\|R_\lambda x\| \leqslant \int_0^\infty e^{-\lambda t}\|U(t)x\|dt \leqslant \frac{1}{\lambda}\|x\|.$$

Let $h > 0$, then

$$\frac{U(h)-I}{h}R_\lambda x = \frac{1}{h}\int_0^\infty e^{-\lambda t}(U(t+h)x - U(t)x)dt$$
$$= \frac{1}{h}\int_h^\infty e^{-\lambda(t-h)}U(t)xdt - \frac{1}{h}\int_0^\infty e^{-\lambda t}U(t)xdt$$
$$= \frac{e^{\lambda h}}{h}\int_0^\infty e^{-\lambda t}U(t)xdt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t}U(t)xdt - \frac{1}{h}\int_0^\infty e^{-\lambda t}U(t)xdt$$
$$= \frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda t}U(t)xdt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t}U(t)xdt$$

As $h \to 0^+$, the right hand side converges to $\lambda R_\lambda x - x$. This implies that for every $x \in \mathcal{B}$ and $\lambda > 0$ $R_\lambda x \in D(S)$ and $SR_\lambda = \lambda R_\lambda - I$, which is the same as $(\lambda I - S)R_\lambda = I$.
For $x \in D(S)$ we have

$$R_\lambda Sx = \int_0^\infty e^{-\lambda t}U(t)Sxdt = \int_0^\infty e^{-\lambda t}SU(t)xdt$$
$$= S\left(\int_0^\infty e^{-\lambda t}U(t)xdt\right) = SR_\lambda x$$

where we used that the infinitesimal generator commutes with the elements of the semigroup and closeness of S. Finally we have that $R_\lambda(\lambda I - S)x = x$ for $x \in D(S)$. Thus $R_\lambda = (\lambda I - S)^{-1}$ and is therefore well defined for every $\lambda > 0$ and satisfies $\|(\lambda I - S)^{-1}x\| = \|R_\lambda x\| \leqslant \frac{1}{\lambda}\|x\| \Rightarrow \|(\lambda I - S)x\| \geqslant \lambda\|x\|$. Now if we let $\alpha = 1/\lambda$ we get that $Ran(I - \alpha S) = \mathcal{B}$ because it is invertible and $\|(I - \alpha S)x\| \geqslant \|x\|$.
(2) $\Rightarrow$ (1) Condition 2 implies that $(I - \varepsilon S)^{-1}$ is bounded strongly continuous and $\|(I - \varepsilon S)^{-1}\| \leqslant 1$ (by putting $\varphi = \mathbb{1}$ in (5)) for $\varepsilon = \alpha_0$ such that $Ran(I - \varepsilon S) = \mathcal{B}$. The series $(I - \alpha S)^{-1} = \frac{\alpha_0}{\alpha}\sum_{n\geqslant 0}\left(\frac{\alpha-\alpha_0}{\alpha}\right)^n (I - \alpha_0 S)^{-n-1}$ establishes that $Ran(I - \varepsilon S) = \mathcal{B}$ for all $\varepsilon > 0$.
We define $S_\varepsilon = S(I - \varepsilon S)^{-1}$ and notice that $S_\varepsilon = -\varepsilon^{-1}(I - (I - \varepsilon S)^{-1})$, then

$$\|\exp\{tS_\varepsilon\}\| \leqslant \exp\{-t\varepsilon^{-1}\}\sum_{n\geqslant 0}\frac{(t\varepsilon^{-1})^n}{n!}\|(I - \varepsilon S)^{-n}\| \leqslant 1.$$

for $t \geqslant 0$. Thus $U_t^\varepsilon = \exp\{tS_\varepsilon\}$ are uniformly continuous contraction semigroup. Moreover the bounded operators S_ε and S_δ commute, and for $\varphi \in D(S)$

$$\|U_t^\varepsilon\varphi - U_t^\delta\varphi\| = \left\|\int_0^1 \frac{d}{ds}e^{t(sS_\varepsilon+(1-s)S_\delta)}\varphi ds\right\|$$
$$= \left\|t\int_0^1 e^{tsS_\varepsilon}e^{t(1-s)S_\delta}(S_\varepsilon - S_\delta)\varphi ds\right\| \leqslant t\|(S_\varepsilon - S_\delta)\varphi\|$$

for all $t \geqslant 0$. Note that if $\varphi \in D(S)$ then $\|(I-\varepsilon S)^{-1}\varphi - \varphi\| = \varepsilon\|(I-\varepsilon S)^{-1}S\varphi\| \leqslant \varepsilon\|S\varphi\|$. Then the uniformly bounded family of operators $(I - \varepsilon S)^{-1}$ converges strongly to the identity on the dense set $D(S)$. Then from the relation $(S_\varepsilon - S)\varphi = ((I-\varepsilon S)^{-1} - I)S\varphi$ we have that $S_\varepsilon \varphi$ converges in norm to $S\varphi$ for all $\varphi \in D(S)$. By previous inequality, we have that $\{U_t^\varepsilon \varphi\}_{\varepsilon \geqslant 0}$ is uniformly norm convergent for t in compacts and for $\varphi \in D(S)$. By uniform boundedness $\|U_t^\varepsilon\| \leqslant 1$, we conclude that $\{U_t^\varepsilon\}_{\varepsilon\geqslant 0}$ converges strongly on $\overline{D(S)}$ uniformly for t in compacts. If we denote $U = \{U_t\}_{t\geqslant 0}$ the strong limit it immediately follows that U is a strongly continuous semigroup of contractions (by triangular inequality). It is left to prove that the infinitesimal generator of $\{U_t\}_{t\geqslant 0}$ is in fact S. We see that

$$\frac{(U_t^\varepsilon - I)x}{t} = \frac{1}{t}\int_0^t U_s^\varepsilon S_\varepsilon x ds$$

for all $x \in \mathcal{B}$. But if $\varphi \in D(S)$ we obtain the relation

$$\frac{(U_t - I)\varphi}{t} = \frac{1}{t}\int_0^t U_s S\varphi ds$$

by strong limits. Therefore

$$\left\| \frac{(U_t - I)\varphi}{t} - S\varphi \right\| \leqslant \sup_{0\leqslant s\leqslant t} \|(U_s - I)S\varphi\|$$

and it follows from the strong continuity of U that its generator S' is an extension of S. But this implies that $(I-\alpha S')^{-1}$ is an extension of $(I-\alpha S)^{-1}$ for all α. However, the latter operator is everywhere defined than we must have that $S' = S$. □

Proposition 1.8. *Let $\mathfrak{U}$ be a C^*-algebra with a unit. A densely defined, closed operator δ on $\mathfrak{U}$ generates a strongly continuous group of $*$-automorphism of $\mathfrak{U}$ if and only if:*

1. *δ is a $*$-derivation, and*
2. *$Ran(Id + \lambda\delta) = \mathfrak{U}$ for all $\lambda \in \mathbb{R}\backslash\{0\}$, and*
3. *$\|A + \lambda\delta(A)\| \geqslant \|A\|$ for all $\lambda \in \mathbb{R}$ and $A \in D(\delta)$.*

Proof. $\Rightarrow$ Assume that δ is the generator of a strongly continuous one parameter group τ_t of $*$-automorphisms of $\mathfrak{U}$. We already proved that δ is a $*$-derivation. Since τ_t are isometries we can apply Hille-Yosida theorem to $\pm\delta$ and conditions 2) and 3) are implied.
$\Leftarrow$ Now assume that $(1), (2), (3)$ hold and we prove that δ generates a strongly continuous group of $*$-automorphisms of $\mathfrak{U}$. The group generated by δ is a strongly continuous one-parameter group τ_t of isometries by Hille-Yosida. We know that (1) implies that $\mathbb{1} \in D(\delta)$ and $\delta(\mathbb{1}) = 0$, then $\tau_t(\mathbb{1}) = \mathbb{1}$, τ_t is an $*$-automorphism (by Corollary 3.2.12 in [2]) .

□

If the C^*-algebra acts on a Hilbert space $\mathcal{H}$ then a dynamical group τ_t can be constructed (see Example 3.2.14) from a group of unitary operators U_t on $\mathcal{H}$:

$$\tau_t(A) = U_t A U_t^*.$$

Such $*$-automorphisms are called spatial.

Theorem 1.9 (Stone's theorem)**.** *Let $\{U_t\}_{t\in\mathbb{R}}$ a strongly continuous unitary one parameter group. Then there exists a unique (not necessarily bounded) operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ that is self adjoint on $D(A)$ and such that*

$$U_t = e^{itA} \quad \forall t \in \mathbb{R}.$$

Example: (Finite quantum system).
Consider the quantum system with a finite number of degrees of freedom determined by the Hilbert space $\mathcal{H}$ and by the self-adjoint operator H. On the C^*-algebra $\mathfrak{U} = \mathfrak{B}(\mathcal{H})$ the dynamics is given by

$$\tau_t(A) = e^{itH} A e^{-itH}. \tag{7}$$

The group τ_t is strongly continuous if and only if H is bounded.
In fact, let H be bounded, then there exists $M \geqslant \|H\|$, we have that

$$\|e^{itH}\| = \left\| \sum_{n\geqslant 0} \frac{(itH)^n}{n!} \right\| \leqslant \sum_{n\geqslant 0} \left\| \frac{(itH)^n}{n!} \right\| \leqslant e^{tM}$$

and analogously with the minus. Then for every $A \in \mathcal{H}$,

$$\begin{aligned} \|\tau_t(A) - IA\| = \|e^{itH} A e^{-itH} - A\| &= \|e^{itH} A e^{-itH} - e^{itH} A + e^{itH} A - A\| \\ &\leqslant \|e^{itH}\| \|A e^{-itH} - A\| + \|e^{itH} A - A\| \\ &\leqslant e^{tM} \|A e^{-itH} - A\| + \|e^{itH} A - A\| \end{aligned}$$

by taking the limit $t \to 0$ we have that $\|e^{itH} A - A\|, \|A e^{-itH} - A\| \to 0$ and then we have proven strong continuity.
Conversely, assume that H is unbounded (which means that its spectrum is unbounded). Denote with E_H the spectral family of H (a spectral family is a family $\{E_\lambda\}$ of orthogonal projectors onto the space generated by the eigenvectors corresponding to eigenvalues that are less or equal than λ). Let $\varepsilon > 0$, $\delta > 0$; we can choose a real number a and a sequence $\{a_n\}_{n\geqslant 0} \subset \mathbb{R}$ such that the intervals $I_n = [a_n, a_n + a]$ are disjoint and $E_H(I_n)\mathcal{H}$ is non empty, and

$$\sup_n |e^{i(a_n - a_{n+1})t} - 1| \geqslant 2 - \delta, \quad |e^{ita} - 1| \leqslant 1/2$$

for all $|t| \leqslant \varepsilon$. Now choose unit vectors $\psi_n \in E_H(I_n)\mathcal{H}$ and define V by

$$V\psi = \sum_{n\geqslant 0} \psi_n (\psi_{n+1}, \psi).$$

One has $\|V\| = 1$ and

$$\begin{aligned}(e^{i(a_n - a_{n+1})t} - 1)\psi_n \\ = (\tau_t(V) - V)\psi_{n+1} - (e^{itH} - e^{ia_n t})\psi_n(\psi_{n+1}, e^{-itH}\psi_{n+1}) \\ -e^{ia_n t}\psi_n(\psi_{n+1}, (e^{-itH} - e^{-a_{n+1}t})\psi_{n+1})\end{aligned}$$

and hence

$$2 \leqslant \|\tau_t(V) - V\| + 1 + \delta$$

for all $|t| \leqslant \varepsilon$, which is a contradiction. Then H must be bounded.
In general, an everywhere defined derivation of a C^*-dynamical system is bounded [Corollary 3.2.23 in [2]].

A consequence of this is that if τ_t is strongly continuous it is also uniformly continuous and its generator is the bounded $*$-derivation $\delta(A) = i[H, A]$ ($[,]$ is the commutator).

2 W^*-dynamical systems (Luca Giudici)

This part about W^*-dynamical systems follows [1].

Definition 2.1 (W^*-dynamical system)**.** *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ a von Neumann algebra. A **σ-weakly continuous group of $*$-automorphism** of $\mathcal{M}$ is a group homomorphism*

$$\tau : t \in \mathbb{R} \mapsto \tau^t \in \mathrm{Aut}(\mathcal{M})$$

such that for any $A \in \mathcal{M}$ the map

$$t \in \mathbb{R} \mapsto \tau^t(A) \in \mathcal{M}$$

is continuous w.r.t. the σ-weak topology on $\mathcal{M}$. This is equivalent to the map

$$t \in \mathbb{R} \mapsto \mathrm{tr}(\tau^t(A)T) \in \mathbb{R}$$

*being continuous for any trace class operator $T \in \mathcal{T}(\mathcal{H})$. We call $(\mathcal{M}, \tau^t)$ a **W^*-dynamical system**.*

There is a similar proposition charaterising σ-weakly continuous groups of $*$-automorphisms of a von Neumann algebra in terms of a $*$-derivations as for strongly continuous groups of $*$-automorphisms of a C^*-algebra, for which the proof is analogous.

Proposition 2.2. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $\delta : D(\delta) \to \mathcal{M}$ be a σ-weakly densely define closed linear operator. Then $\tau^t = e^{t\delta}$, $t \in \mathbb{R}$, defines a σ-weakly continuous group $*$-automorphisms of $\mathcal{M}$ if and onyl if the the following three statements hold* *(i)* δ *is* $*$*-derivation and* $1 \in D(\delta)$.

(ii) $R(id + \lambda\delta) = \mathcal{M}$ *for all* $\lambda \in \mathbb{R}\backslash\{0\}$.

(iii) $\|A + \lambda\delta(A)\| \geqslant \|A\|$ *for all* $\lambda \in \mathbb{R}$ *and* $A \in D(\delta)$.

Let's turn our attention to a first simple example.

Example 2.3 (Heisenberg equation)**.** *Consider the von Neumann algebra* $\mathcal{M} = \mathcal{B}(\mathcal{H})$ *and consider the evolution of a observable* $A \in \mathcal{B}(\mathcal{H})$ *under the Heisenberg equation*

$$\begin{cases} \partial_t A_t = i\,[H, A_t]\,, \quad H = H^* : \mathcal{H} \to \mathcal{H} \text{ linear (possibly unbounded)} \\ A_0 = A. \end{cases}$$

with the solution $A_t = e^{itH} A e^{-itH}$. *Define*

$$\tau^t(A) = e^{itH} A e^{-itH}, \; A \in \mathcal{M}.$$

We now claim that τ^t *is a* σ*-weakly continuous group of* $*$*-automorphisms of* $\mathcal{M}$. *We check:*

- τ^t ***is*** $*$***-automorphism:*** *Fix* $A, B \in \mathcal{M}$ *and* $\lambda \in \mathbb{C}$ *then we have*
  - $\tau^t(A + \lambda B) = e^{itH}(A + \lambda B)e^{-itH} = e^{itH}Ae^{-itH} + \lambda e^{itH}Be^{-itH} = \tau^t(A) + \lambda\tau^t(B)$.
  - $\tau^t(AB) = e^{itH}ABe^{-itH} = e^{itH}Ae^{-itH}e^{itH}Be^{-itH} = \tau^t(A)\tau^t(B)$.
  - *Since* $(e^{itH})^* = e^{-itH^*} = e^{-itH}$ *as* H *is self adjoint, we find:*

    $$\tau^t(A^*) = e^{itH}A^*e^{-itH} = (e^{-itH})^*A^*(e^{itH})^* = (e^{itH}Ae^{-itH})^* = \tau^t(A)^*.$$

  - *Note that* $\tau^{-t}\tau^t(A) = \tau^t\tau^{-t}(A) = A$. *So* τ^t *is a* $*$*-automorphism.*
- $t \mapsto \tau^t$ ***is a group homomorphism:*** *Clearly we have* $\tau^0(A) = A$ *and so* $\tau^0 = \mathrm{id}$. *Moreover we can check that*

  $$\tau^{t+s}(A) = e^{i(t+s)H}Ae^{-i(t+s)H} = e^{itH}e^{isH}Ae^{-isH}e^{-itH} = e^{itH}\tau^s(A)e^{-itH} = \tau^t\tau^s(A).$$

  Thus $\tau^{t+s} = \tau^t\tau^s$. *So* $t \mapsto \tau^t$ *is a group homomorphism.*
- *Note that for unit vectors* $\Phi, \Psi \in \mathcal{H}$ *we have by Cauchy-Schwarz inequality*

  $$\begin{aligned} |\langle \Phi, \tau^t(A)\Psi\rangle| &= |\langle \Phi, e^{itH}Ae^{-itH}\Psi\rangle| \\ &= |\langle e^{-itH}\Phi, Ae^{-itH}\Psi\rangle| \\ &\leqslant \|e^{-itH}\Phi\|\|Ae^{-itH}\Psi\| \\ &\leqslant \|e^{-itH}\|\|\Phi\|\|A\|\|e^{-itH}\|\|\Psi\| \\ &\leqslant \|A\|. \end{aligned}$$

This shows that $t \mapsto \langle \Phi, \tau^t(A)\Psi \rangle$ is uniformly bounded by $\|A\|$ and therefor is a continuous map. Let $T \in \mathcal{T}(\mathcal{H})$ be a trace class operator. We can write $T = \sum_n \lambda_n \langle \Phi_n, \cdot \rangle \Psi_n$ for some unit vectors $\Phi_n, \Psi_n \in \mathcal{H}$ and $\sum_n |\lambda_n| < \infty$. Then by the above the map

$$t \mapsto \mathrm{tr}(\tau^t(A)T) = \sum_n \lambda_n \langle \Phi_n, \tau^t(A)\Psi_n \rangle$$

is continuous, as it is the limt of a uniform convergent seqeunce of continuous functions.

2.1 Interlude to spectral theory of bounded derivations

We will follow the exposition presented in [2]. For τ^t a σ-weakly continuous group of $\star$-automorphisms of a von Neumann algebra, we define for any $f \in L^1(\mathbb{R})$

$$\tau(f) = \int_{\mathbb{R}} dt f(t)\tau^t.$$

It can be checked that

$$\tau(f * g) = \tau(f)\tau(g).$$

Definition 2.4 (Spectrum of one-parameter families)**.** *Let $\mathcal{M}$ be a von Neumann algebra and τ^t a σ-weakly continuous group of $*$-automorphisms of $\mathcal{M}$ such that $\|\tau^t\| \leqslant M$ for all $t \in \mathbb{R}$.*

- *For any subset $\mathcal{Y} \subset \mathcal{M}$ we define*

$$\mathcal{J}^\tau_{\mathcal{Y}} = \left\{ f \in L^1(\mathbb{R}) | \tau(f)A = 0 \ \forall A \in \mathcal{Y} \right\}.$$

  This is a $*$-ideal in $L^1(\mathbb{R})$.

- *The **spectrum** of $\mathcal{Y}$ is given by*

$$\sigma_\tau(\mathcal{Y}) = \{k \in \mathbb{R} | \hat{f}(k) = 0 \ \forall f \in \mathcal{J}^\tau_{\mathcal{Y}}\},$$

  where $\hat{f}$ is the Fourier transform of f.

- *The **spectrum of** τ is given by*

$$\sigma(\tau) = \sigma_\tau(\mathcal{M})$$

  *and the **spectral subspace** to a subset $E \subseteq \mathbb{R}$ is defined by*

$$\mathcal{M}^\tau(E) = \overline{\{A \in \mathcal{M} | \sigma_\tau(A) \subset E\}}.$$

We now consider some elementary properties of the spectrum of a element.

Lemma 2.5. *Let τ^t be a uniformly bounded σ-weakly continuous group of $*$-automorphisms of $\mathcal{M}$. Then for all $A, B \in \mathcal{M}$, $f \in L^1(\mathbb{R})$ it holds that:*

(i) $\sigma_\tau(\tau^t A) = \sigma_\tau(A)$ *for all* $t \in \mathbb{R}$.

(ii) $\sigma_\tau(\alpha A + B) \subset \sigma_\tau(A) \cup \sigma_\tau(B)$.

(iii) $\sigma_\tau(\tau(f)A) \subseteq \text{supp}(\hat{f}) \cap \sigma_\tau(A)$.

(iv) *If* $f_1, f_2 \in L^1(\mathbb{R})$ *and* $\hat{f}_1 = \hat{f}_2$ *in a neighbourhood of* $\sigma_\tau(A)$*, then*

$$\tau(f_1)A = \tau(f_2)A.$$

Proof. (i) We have for $f_t(s) = f(s-t)$

$$\begin{aligned} \tau(f)\tau^t A &= \int ds f(s)\tau^s\tau^t A \\ &= \int ds f(s)\tau^{s+t} A \\ &= \int ds f_t(s)\tau^s A \\ &= \tau(f_t)A. \end{aligned}$$

Now, the Fourier transform of f_t is given by $\hat{f}_t(k) = e^{-kt}\hat{f}(k)$. Hence we find:

$$\begin{aligned} k \in \sigma_\tau(\tau^t A) &\iff \forall f \in L^1(\mathbb{R}) \text{ with } \tau(f)\tau^t A = 0 : \ \hat{f}(k) = 0 \\ &\iff \forall f \in L^1(\mathbb{R}) \text{ with } \tau(f_t)A = 0 : \ \hat{f}_t(k) = 0 \\ &\iff k \in \sigma_\tau(A). \end{aligned}$$

(ii) By linearity of the Fourier transform and linearity of τ we find $\sigma_\tau(\alpha A) = \sigma_\tau(A)$. So assume $\alpha = 1$. If now $k \notin \sigma_\tau(A) \cup \sigma_\tau(B)$, we find $f, g \in L^1(\mathbb{R})$ with $\tau(f)A = 0$ and $\tau(g)B = 0$ such that $\hat{f}(k) = \hat{g}(k) = 1$. Now for $f * g$ we have that $\tau(f * g)(A + B) = 0$ and thus by the convolution theorem[1] that $(f * g)^\wedge(k) = \hat{f}(k)\hat{g}(k) = 1$. Hence, $k \notin \sigma_\tau(A + B)$.

(iii) If $\tau(g)A = 0$ we have $\tau(g)\tau(f)A = \tau(f)\tau(g)A = 0$ and hence $\sigma_\tau(\tau(f)A) \subseteq \sigma_\tau(A)$. On the other hand, if $\hat{g}$ vanishes on $\text{supp}(\hat{f})$ the $f * g = 0$, thus we have $\tau(g)\tau(f)A = \tau(f * g)A = 0$. Hence, $\sigma_\tau(\tau(f)A) \subseteq \text{supp}(\hat{f})$.

(iv) Set $g = f_1 - f_2$. We must show $\tau(g)A = 0$. Since g vanishes on a neighbourhood of $\sigma_\tau(A)$ we have by (iii) that

$$\sigma_\tau(\tau(g)A) \subseteq \text{supp}(\hat{g}) \cap \sigma_\tau(A) = \varnothing.$$

Hence, $\tau(g)A = 0$.

□

Now let us turn to some properties of the spectral subspaces.

[1] We have by the convolution theorem that $(f * g)^\wedge = \hat{f}\hat{g}$.

Lemma 2.6. *Let τ^t be a uniformly bounded σ-weakly continuous group of $*$-automorphisms of $\mathcal{M}$. Then for all $E \subset \mathbb{R}$ it holds that*

(i) $\mathcal{M}_0^\tau(E) \subseteq \mathcal{M}^\tau(E)$ *where* $\mathcal{M}_0^\tau(E)$ *is the σ-weakly closed linear span of elements of the form $\tau(f)A$ with $\operatorname{supp}(\hat{f}) \subseteq E$.*

(ii) $\tau^t\mathcal{M}_0^\tau(E) = \mathcal{M}_0^\tau(E)$ *and* $\tau^t\mathcal{M}^\tau(E) = \mathcal{M}^\tau(E)$.

(iii) If E is closed, then

$$\mathcal{M}^\tau(E) = \{A \in \mathcal{M} | \sigma_\tau(A \subseteq E\}.$$

(iv) If E is open, then

$$\mathcal{M}^\tau(E) = \mathcal{M}_0^\tau(E) = \bigvee\{\mathcal{M}^\tau(K) | K \subseteq E \textit{ compact}\},$$

where $\bigvee$ denotes the σ-closed linear span.

(v) If E is closed and N ranges over the open neigbourhoods of $0 \in \mathbb{R}$ then

$$\mathcal{M}^\tau(E) = \bigcap_N \mathcal{M}_0^\tau(E + N).$$

The next proposition charaterises the spectrum of σ-weakly continuous group of $*$-automorphisms in terms of the spectrum of its generator δ, i.e.

$$\sigma(\delta) = \mathbb{C} \backslash \rho(\delta),$$

where $\rho(\delta) = \{\lambda \in \mathbb{C} | \lambda \mathrm{id} - \delta \text{ is invertible.}\}$

Proposition 2.7. *Let τ^t be a σ-weakly continuous uniformly bounded group of $*$-automorphism in a von Neumann algebra $\mathcal{M}$ with generator δ, i.e. $\tau^t = e^{t\delta}$. Then TFAE:*

(i) $k_0 \in \sigma(\tau)$.

(ii) For any neigbourhood $E \ni k_0$ it holds that $\mathcal{M}_0^\tau(E) \neq \{0\}$.

(iii) For all $\varepsilon > 0$ and all compacts sets $K \subseteq \mathbb{R}$ there is a compact neighbourhood $E \ni k_0$ such that $\mathcal{M}^\tau(E) \neq \{0\}$ and

$$\|\tau^t A - e^{-k_0 t} A\| \leqslant \varepsilon \|A\|$$

for all $A \in \mathcal{M}^\tau(E)$ and $t \in K$.

(iv) There is a sequence of elements $A_\alpha \in \mathcal{M}$ such that $\|A_\alpha\| = 1$ and uniformly for t in compacts

$$\lim_{\alpha \to \infty} \|\tau^t A_\alpha - e^{-k_0 t} A_\alpha\| = 0.$$

(v) For all $f \in L^1(\mathbb{R})$ it holds that $|\hat{f}(k_0)| \leqslant \|\tau(f)\|$.

(vi) $-ik_0 \in \sigma(\delta)$*, i.e.* $\sigma(\delta) = -i\sigma(\tau)$*.*

Recall that a operator $U \in \mathcal{B}(\mathcal{H})$ is **unitary** if $U^*U = UU^* = \text{id}$. Recall again Stone's theorem.

Theorem 2.8 (Stone)**.** *Let* $t \mapsto U_t \in \mathcal{B}(\mathcal{H})$ *be a strongly continuous group of unitary operators. Then exists a unique (possibly unbounded) operator* $H : D(H) \to \mathcal{H}$*, which is self-adjoint on* $D(H)$ *and such that*

$$\forall t \geqslant 0 : U_t = e^{itH}.$$

The domain of H *is given by*

$$D(H) = \left\{ \Psi \in \mathcal{H} \middle| \lim_{t\to 0} \frac{-i(U_t(\Psi) - \Psi)}{t} \textit{ exists.} \right\}.$$

We can reformulate this in terms of the projection-valued measure P associated to the self-adjoint operator H by the spectral theorem, i.e.

$$H = \int_{\mathbb{R}} k dP(k).$$

Recall that for a measurable-function $g : \mathbb{R} \to \mathbb{C}$ we can write

$$g(A) = \int_{\mathbb{R}} g(k) dP(k).$$

And so by Stone's theorem we have the strongly continuous group of unitary operators $t \mapsto U_t \in \mathcal{B}(\mathcal{H})$ given by

$$U_t = \int_{\mathbb{R}} e^{itk} dP(k).$$

The next proposition gives a result on the spectrum of so-called spatial groups of $*$-automorphisms.

Proposition 2.9. *Let* $t \mapsto U_t \in \mathcal{B}(\mathcal{H})$ *be a strongly continuous group of unitary operators and* $\tau^t(A) = U_t A U_t^*$ *the* σ*-weakly continuous group of* $*$*-automorphism generated by* U*. Let*

$$U_t = \int_{\mathbb{R}} e^{itk} dP(k),$$

be the spectral decomposition of U*. Then TFAE for any* $A \in \mathcal{B}(\mathcal{H})$ *and* $k \in \mathbb{R}$

(i) $\sigma_\tau(A) \subseteq [k, \infty)$

(ii) $AP([k, \infty))\mathcal{H} \subset P([k + j, \infty))\mathcal{H}$ *for any* $j \in \mathbb{R}$*.*

In particular, we find for von Neumann algebra $\mathcal{M}$ *and all* $t \in \mathbb{R}$*.*

$$\mathcal{M}^\tau([t, \infty))\mathcal{H} = \mathcal{M}^\tau([t, \infty))P([0, \infty))\mathcal{H} \subseteq P([t, \infty))\mathcal{H}.$$

Theorem 2.10 (Borchers-Arveson)**.** *Let τ^t be a σ-weakly continuous group of $*$-automorphisms of a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$. The following conditions are equivalent:*

(i) There is a strongly continuous group of unitary operators $U_t \in \mathcal{B}(\mathcal{H})$ with nonnegative spectrum (that is in Stone's theorem H is positive) such that $\tau^t(A) = U_t A U_t^*$.

(ii) There is a strongly continuous group of unitary operators $U_t \in \mathcal{M}$ with nonnegative spectrum (that is in Stone's theorem H is positive) such that $\tau^t(A) = U_t A U_t^*$.

(iii) $\bigcap_{t\in\mathbb{R}} \mathcal{M}^\tau([t,\infty))\mathcal{H} = \{0\}$.

Proof.

- $(ii) \implies (i)$ is clear.
- For $(i) \implies (iii)$ we leverage that if P is the projection-valued measure associated to U then $P([t,\infty)) = 1$ for all $t \leqslant 0$ and so clearly

$$\bigcap_{t\in\mathbb{R}} P([t,\infty))\mathcal{H} = \{0\}.$$

  By the proposition above we have that

$$\mathcal{M}^\tau([t,\infty))\mathcal{H} = \mathcal{M}^\tau([t,\infty))P([0,\infty))\mathcal{H} \subseteq P([t,\infty))\mathcal{H}$$

  and so (iii) follows.

- For $(iii) \implies (ii)$ The idea is to set $Q_t = \bigcap_{s<t} \mathcal{M}^\tau([s,\infty))\mathcal{H}$ and show that there is a unique projection-valued measure P on $\mathbb{R}$ such that $P([t,\infty)) = Q_t$ for all t and then define

$$U_t = \int e^{-itk} dP(k).$$

  This then gives us the searched for unitary group.

□

The intersting corllary to this theorem will give a characterisation of bounded derivations of a von Neumann algebra in terms of a "Hamiltonian".

Corollary 2.11. *Let δ be an everywhere defined bounded derivation of a von Neumann algebra $\mathcal{M}$. Then exists $H = H^* \in \mathcal{M}$ with $\|\mathcal{H}\| \leqslant \frac{\|\delta\|}{2}$ and such that $\delta(A) = i[H,A]$.*

I will give the idea of the proof.

Proof. We define

$$\tau^t(A) = e^{t\delta}(A) = \sum_{t\geqslant 0} \frac{t^n}{n!}\delta^n(A).$$

This is then a continuous group of $*$-automorphisms. The next step is to show that $\mathcal{M}^{\tau}([t,\infty)) = \{0\}$ for all $t \geqslant \|\delta\|$. So by the Borchers-Arveson theorem the result follows that we have a strongly continuous unitary group U_t in $\mathcal{M}$ such that

$$\tau^t(A) = U_t A U_t^*.$$

By Stone's theorem there is a self-adjoint operator H such that

$$U_t = e^{itH}$$

and hence, by differentiating with respect to t we get the result. □

References

[1] Stéphane Attal, Alain Joye, and Claude-Alain Pillet. *Open Quantum Systems I: The Hamiltonian Approach*. Springer, 2006.

[2] Ola Bratteli and Derek William Robinson. *Operator algebras and quantum statistical mechanics: Volume 1: C*-and W*-Algebras. Symmetry Groups. Decomposition of States*. Springer Science & Business Media, 2012.

[3] Amnon Pazy. *Semigroups of linear operators and applications to partial differential equations*. Vol. 44. Springer Science & Business Media, 2012.