HARMONIC ANALYSIS

1. Basic facts about Fourier transform

We denote the torus by $\mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z})^d$ with Lebesgue measure. For $f \in L^1(\mathbb{T}^d)$ we define the Fourier transform for $n \in \mathbb{Z}^d$

$$\hat{f}(n) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(x)e^{-ixn}\, dx.$$

The inverse Fourier transform is defined for $f_n \in \ell^1(\mathbb{Z}^d)$ by

$$\check{f}(x) := \sum_{n\in\mathbb{Z}^d} f_n e^{ixn}.$$

Theorem 1 (Parseval theorem)**.** *Let $f \in L^2(\mathbb{T}^d)$, then $\hat{f} \in \ell^2(\mathbb{Z}^d)$ and*

$$\|\hat{f}\|_{\ell^2}^2 = (2\pi)^{-d}\|f\|_{L^2}^2.$$

Moreover, the Fourier transform is bijective.

Exercise 1. *Give a proof of this result using the Stone-Weierstrass theorem.*

Basic question in Harmonic Analysis: Can we get rates on the convergence?

We claim that for any Lipschitz function $f : \mathbb{T} \to \mathbb{C}$ we have

$$|\hat{f}(n)| \le |n|^{-1}\|f\|_{\mathrm{Lip}},$$

where $\|f\|_{\mathrm{Lip}} := \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|}$. This follows from writing

$$\begin{aligned} 2\pi\hat{f}(n) &= \frac{1}{2}\int_{\mathbb{T}} e^{-inx}f(x) + e^{-in(x+\pi/n)}f(x+\pi/n)\, dx \\ &= \frac{1}{2}\int_{\mathbb{T}} (f(x) - f(x+\pi/n))e^{-inx}\, dx. \end{aligned}$$

Exercise 2. *Generalize the above argument to arbitrary dimensions!*

More generally, let f be a Hölder function, i.e.

$$\|f\|_{\Lambda_\alpha} := \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha},$$

then $\hat{f}(n) = \mathcal{O}(1/|n|^\alpha)$.

We define the continuous Fourier transform as

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-ix\xi}\, d\xi$$

and also

$$\check{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{ix\xi}\, d\xi.$$

Theorem 2 (Plancherel theorem)**.** *For any $f \in L^2(\mathbb{R}^d)$, $\|\hat{f}\|_2 = (2\pi)^{d/2}\|f\|_2$. Moreover, $f = (2\pi)^{-d}\check{\hat{f}}$.Finally, $f \mapsto (2\pi)^{-d/2}\hat{f}$ is an isometry of $L^2(\mathbb{R}^d)$ onto $L^2(\mathbb{R}^d)$.*

1.1. Schwartz functions and the space of tempered distributions.

We start by giving the definition of the Schwartz space

Definition 1.1. *$\mathcal{S}$ is the space of all $f \in C^\infty(\mathbb{R}^d)$ for which each of the following quantities*

$$\|f\|_{m,n} := \sup_{x\in\mathbb{R}^d} \langle x\rangle^n \sum_{|\alpha|\le m} |\partial^\alpha f(x)|$$

is finite.

It is complete metrizable space $d(f,g) = \sum_{m,n} 2^{-|(m,n)|} \frac{\|f-g\|_{m,n}}{1+\|f-g\|_{m,n}}$ whose locally convex topology is defined by finite intersections of open sets

$$V_{f,m,n,\varepsilon} := \{g \in \mathcal{S}; \|f-g\|_{m,n} \le \varepsilon\}.$$

Thus, if $f_n \to g$ in $\mathcal{S}$ then this is equivalent to having $\|f_n - g\|_{m,n} \to 0$ for all m, n.

Definition 1.2. *The set $\mathcal{S}'$ is the space of all continuous linear functionals from $\mathcal{S}$ to $\mathbb{C}$.*

We can find a topology such that $\varphi_j \to \varphi$ in $\mathcal{S}'$ is equivalent to $\varphi_j(f) \to \varphi(f)$ for every $f \in \mathcal{S}$.

Definition 1.3. *$\mathcal{S}'$ is a topological vector space for which a neighbourhood base of $\varphi \in \mathcal{S}'$ is the collection of all finite intersections of sets*

$$V_{\varphi,f,\varepsilon} := \{\psi \in \mathcal{S}' : |(\varphi-\psi)(f)| \le \varepsilon\}.$$

Remark 1.1. *A locally finite Borel measure μ is called* tempered *if there exist $C, N > 0$ such that*

$$\mu\{x \in \mathbb{R}^d; |x| \le R\} \le CR^N \text{ as } R \to \infty.$$

The set of tempered C^∞ functions f with measure $d\mu = |f(x)|dx$ turns out to be dense in $\mathcal{S}'$. Hence, the space $\mathcal{S}'$ got its name: The space of tempered distributions.

Lemma 1.1. *To any continuous linear transformation $T : \mathcal{S} \to \mathcal{S}$, there is an associated continuous linear transformation $T' : \mathcal{S}' \to \mathcal{S}'$ defined by*

$$(T'\varphi)(f) = \varphi(Tf).$$

Proof. By linearity, it suffices to show continuity at 0. Let V be a neighbourhood of 0 in $\mathcal{S}'$. We must show there is a neighbourhood U of $0 \in S'$ such that $T'(U) \subset V$. There is $\varepsilon > 0$ and finitely many $f_j \in \mathcal{S}$ such that $V \supset \cap_{j=1}^n V_j$ where

$$V_j = \{\varphi \in \mathcal{S}'; |\varphi(f_j)| < \varepsilon\}.$$

Define $U_j := \{\psi \in \mathcal{S}'; |\psi(Tf_j)| < \varepsilon\}$ and $U := \cap_j U_j$. If $\psi \in U$ then $|T'(\psi(f_j))| = |\psi(Tf_j)| < \varepsilon$ for each j so $T'(\psi) \in V$. □

Theorem 3. *The Fourier transform is a continuous homeomorphism from $\mathcal{S}$ onto $\mathcal{S}$ and therefore also between $\mathcal{S}'$ and $\mathcal{S}'$.*

Proof. We assume basic familiarity with the Fourier transform on $\mathcal{S}$ which imply that it is bijective. To show continuity, we notice that

$$\xi^\alpha \partial_\xi^\beta \hat{f} = (-i)^{|\alpha|}\mathcal{F}(\partial_x^\alpha((-ix)^\beta f).$$

Moreover,

$$|\xi^\alpha \partial_\xi^\beta \hat{f}| \leq \int_{\mathbb{R}^d} \langle x\rangle^{-d-1}\ dx \sup_x \langle x\rangle^{d+1} |\partial_x^\alpha (x^\beta f)(x)|$$

This shows that $\|\hat{f}\|_{k,k} \leq C\|f\|_{k,k+d+1}$. ☐

Every $f \in L^p$ can be naturally associated with a tempered distribution by defining

$$\varphi_f(g) = \int fg$$

We thus have that for $\delta_\xi(f) = f(\xi)$ that

$$\hat{\delta_\xi}(f) = \delta_\xi(\hat{f}) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\xi}\ dx$$

which justifies $\hat{\delta_\xi} = e^{-ix\xi}$.

1.2. **Convolution.** Let $f \in L^1(\mathbb{T})$ and we may ask if f is equal to $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}$, that is whether

$$S_N f(x) = \sum_{|n|\leq N} \hat{f}(n)e^{inx}$$

converges to f. We compute

$$\begin{aligned} S_N f(x) &= \sum_{|n|\leq N} e^{inx}(2\pi)^{-1}\int_0^{2\pi} f(y)e^{-iny}dy \\ &= \sum_{|n|\leq N} (2\pi)^{-1}\int_0^{2\pi} f(y)e^{-in(x-y)}dy \\ &= (2\pi)^{-1}\int_0^{2\pi} f(y)D_N(x-y)\ dy, \end{aligned}$$

where $D_N(x) = \sum_{|n|\leq N} e^{inx} = \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)}$ is the *Dirichlet kernel.*

The convergence of the Fourier series is therefore equivalent to the asymptotic properties of a certain integral operator!

Using a suitable partitioning that we leave as an exercise, one readily verifies that even though $\int D_N = 2\pi$, the L^1 norm diverges as $N \to \infty$.

We might get our hopes up that the Dirichlet kernel is just an approximate identity which would imply converges of the Fourier series in every L^p space with $p < \infty$. However,

Exercise 3. *There is $c > 0$ such that*

$$\|D_N\|_{L^1} \geq c \log(N) \textit{ for all } N.$$

On $\mathbb{R}^d$ we define the convolution by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\ dy.$$

In addition one has Young's inequality which states that for $1 + 1/r = 1/p + 1/q$

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Basic question in Harmonic Analysis: What is the sharpest constant in this inequality? What are the optimizers?[1]

Analogously, one defines a convolution on the torus by setting

$$(f * g)(x) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(x-y)g(y)\ dy.$$

If we want to convolve with a (finite) measure μ we replace $g(y)\ dy$ by $d\mu(y)$. The convolution of two measures is the measure

$$(\mu * \nu)(E) := (\mu \times \nu)\{(x,y); x + y \in E\}$$

such that

$$\int f d(\mu * \nu) = \int \int f(x+y)\ d\mu(x)\ d\nu(y).$$

One can even convolve two tempered distributions, under one constraint: indeed let first $\varphi \in \mathcal{S}'$ and $f \in \mathcal{S}$ then

$$\varphi * f(x) = \varphi(f(x - \bullet))$$

is a C^∞ function, e.g. $\delta_0^{(n)} * f(x) = f^{(n)}(x)$. However, in general this is not a Schwartz function, we therefore have to assume that ψ is compactly supported[2] and define

$$(\varphi * \psi)(f) := \psi(\varphi * f(-x)).$$

One application of convolutions are approximate identities

Definition 1.4. *A sequence $(\varphi_j)_j$ is called an approximate identity if*

- $\int \varphi_j = 1$
- $\|\varphi_j\| \leq C < \infty$
- $\int_{|x|>\delta} |\varphi_j(x)| \to 0$ *as $j \to \infty$ for all $\delta > 0$.*

An examples are centered Gaussians whose variance tends to zero. In fact, any normalized positive $\phi \in L^1$ naturally induces an approximate identity by setting $\varphi_j(x) = j^d \phi(jx)$. Approximate identities have their name because of the following property

Theorem 4.

[1]For Young's inequality this has been studied by Beckner and Brascamp-Lieb.

[2]This means that $\psi(f) = 0$ for all f that are supported away from a certain compact set

For any $f \in C_0(\mathbb{R}^d)$ we have $f * \varphi_j \to f$ uniformly.
For any $f \in L^p(\mathbb{R}^d)$ we have $f * \varphi_j \to f$ in L^p for $p < \infty$.

1.3. **Uniform convergence of Fourier series.** While we know that for $f \in C^0(\mathbb{T}^d)$ we get L^2 convergence of the Fourier series, we may ask whether for $f \in C^0(\mathbb{T}^d)$ we get uniform convergence. This is false by a very elegant observation argument using the uniform boundedness principle.

Theorem 5. *There exists $f \in C^0(\mathbb{T})$ such that $(S_N f(0))$ diverges.*

Proof. If false, then define $\ell_n f = S_n f(0) \in \mathbb{C}$ for every $f \in C^0(\mathbb{T})$. Also $\ell_n : \mathbb{C}^0 \to \mathbb{C}$ is a bounded linear functional, since

$$\ell_n f = (2\pi)^{-1} \int f(y) D_n(-y)\, dy$$

and $D_n \in L^1$. Banach-Steinhaus implies that if $\sup_n |\ell_n(f)| < C_f$ for every f, then $\sup_n \|\ell_n\| < \infty$. This is however false since $\|\ell_n\| \to \infty$ since $\|D_n\|_{L^1} \to \infty$. □

The situation improves by assuming slightly more regularity.

Theorem 6. *For any $\alpha \in (0,1)$ and every $f \in \Lambda_\alpha$, $S_N f \to f$ uniformly as $N \to \infty$.*

Moreover, there exists a constant $C_\alpha < \infty$ such that

$$\|S_N f - f\|_\infty \le C N^{-\alpha} \log(N) \|f\|_{\Lambda_\alpha}.$$

Proof. Writing

$$S_N(f)(x) - f(x) = (2\pi)^{-1} \int (f(x-y) - f(x)) D_N(y)\, dy,$$

we can decompose this integral into

$$|S_N(f)(x) - f(x)| \le C \int_{|y|<\delta} |y|^{-1} |f(x-y) - f(x)| dy + \left| \int_{|y|>\delta} \sin((N+1/2)y) g(y)\, dy \right|,$$

where $g(y) = (f(x-y) - f(x))/\sin(y/2)$. Setting $x = 0$ for simplicity, the first term is majorized by choosing $\delta = \mathcal{O}(1/N)$ by

$$\int_{|y|\le\delta} \|f\|_{\Lambda^\alpha} |y|^{\alpha-1}\, dy = \mathcal{O}(\|f\|_{\Lambda^\alpha} \delta^\alpha) = \mathcal{O}(\|f\|_{\Lambda^\alpha} N^{-\alpha}).$$

To estimate the second term, we observe that up to errors of order $\mathcal{O}(N^{-\alpha})$ it can be written using a substitution as

$$\int_{|y|>\delta} \sin((N+1/2)y)(g(y) - g(y - \pi/(N+1/2)))\, dy.$$

It remains to show that for $|y| \ge \delta$

$$|g(y) - g(y - \pi/(N+1/2))| \le C N^{-\alpha} |y|^{-1}.$$

Let $y' = y - \pi/(N+1/2)$. Then, $(C_0 - \pi)N^{-1} \le |y'| \le \pi + \pi/N$. This implies that using

$$g(y) - g(y') = \frac{f(y)-f(0)}{\sin(y/2)} + \frac{f(0)-f(y')}{\sin(y'/2)} = \frac{f(y)-f(y')}{\sin(y/2)} + \Big(\frac{f(y')-f(0)}{\sin(y/2)-\sin(y'/2)}\Big)$$

and Hölder continuity in the second line

$$\begin{aligned} |g(y)-g(y')| &\le |f(y)-f(y')||\sin(y/2)|^{-1} + |f(y')-f(0)|\frac{|\sin(y/2)-\sin(y'/2)|}{|\sin(y/2)\sin(y'/2)|} \\ &\le C\Big(|y-y'|^{\alpha}|y|^{-1} + |y'|^{\alpha}\frac{|y-y'|}{|yy'|}\Big). \end{aligned}$$

Finally, since $|y-y'| \lesssim |y-y'|^{\alpha}N^{\alpha-1}$ the second term on the right is majorized by the first term.

□

1.4. L^p **convergence of Fourier series.** We are now asking: Let $f \in L^p$ and $p \in [1,\infty]$, do we have $\|f - S_N(f)\|_p \to 0$ for all f? The convergence fails for general $f \in C(\mathbb{T})$. It also fails for $p = 1$,since convergence in L^1 at least requires that $\sup_n \|S_n f\| < \infty$, but then by Banach-Steinhaus also $\sup_n \|S_n\| < \infty$. However, the operator norm of $\|S_n\| = \|D_n\| \to \infty$ as $n \to \infty$. Indeed, recall that by using a positive approximate identity $\|S_n\varphi_j - D_n\|_{L^p} \to 0$. Thus, for $p=1$ we have $\|S_n\| \ge \|D_n\|_{L^1}$, since $\|\varphi_j\|_{L^1} = 1$, and the converse inequality follows by Young.

To see this recall that

$$S_n f(x) = \int D_n(x-y)f(y)\ dy.$$

where the first inequality follows from Young's inequality and the limit since F_n, the so-called Fejér kernel

$$F_n = (n+1)^{-1}\sum_{i=0}^{n} D_i = \frac{\sin((N+1)/2x)^2}{(n+1)\sin(x/2)^2},$$

is an approximate identity, since it is positive and integrates up to 1. In fact, the Fejér kernels form an approximate identity (show this!). In particular, this implies that

$$\sigma_N = (N+1)^{-1}\sum_{n=0}^{N} S_n f$$

satisfy

Proposition 1.2. *For any $f \in C^0(\mathbb{T})$ we have $\sigma_N f \to f$ uniformly as $N \to \infty$. For any $p \in [1,\infty)$ and any $f \in L^p$ we have $\|\sigma_N f - f\| \to 0$ as $N \to \infty$.*

Since the convergence is true for $p = 2$ (Parseval), we are left with studying $p \in (1,\infty) \setminus \{2\}$.

1.5. L^p convergence of the Fourier series.

Definition 1.5. *A Banach lattice of measurable functions is a Banach space X such that whenever $g \in X$, f is measurable and $|f| \leq |g|$ a.e., then $f \in X$ and $\|f\|_X \leq \|g\|$.*

Examples of Banach lattices are the L^p spaces, but Sobolev spaces are e.g. no Banach lattices in general.

Let $\mathcal{P}$ denote the set of trigonometric polynomials and define

Definition 1.6. *We define the operator P on $\mathcal{P}$ whose Fourier transform is the projection*

$$\widehat{Pf}(n) = \begin{cases} \hat{f}(n) & n \geq 0 \\ 0 & n < 0. \end{cases}$$

An operator T on $L^2(\mathbb{T})$ satisfying

$$\widehat{Tf}(n) = a_n \hat{f}(n)$$

with a_n a bounded sequence is called a Fourier multiplier operator and defines itself a bounded linear operator.

It is closely related to the so-called discrete Hilbert transform $\hat{H}f(n) = -i\,\mathrm{Sgn}(n)\hat{f}(n)$ such that

$$\frac{1}{2}(I + iH)f = Pf - \frac{1}{2}\hat{f}(0).$$

Thus, P extends to an L^p bounded operator if and only if H does.

We then have

Proposition 1.3. *Let $X \subset L^1(\mathbb{T})$ be a Banach lattice and suppose that $\mathcal{P}$ is dense in X. If $P : \mathcal{P} \to \mathcal{P}$ extends to a bounded linear operator $P : X \to X$ then $\|S_n f - f\|_X \to 0$ as $n \to \infty$ for every $f \in X$.*

One then shows that

Theorem 7. *P and H extend to bounded linear operators on $L^p(\mathbb{T})$ for every $p \in (1, \infty)$.*

Corollary 1.4. *Let $p \in (1, \infty)$ then the Fourier series converges for $f \in L^p(\mathbb{T})$ also in $L^p(\mathbb{T})$.*

2. Hardy-Littlewood Maximal function

We start by recalling basic properties.

The distribution function λ_f is defined as

$$\lambda_f(\alpha) := \mu\{x \in X; |f(x)| > \alpha\}.$$

One then has for any measurable $f : X \to \mathbb{C}$ and any $p \in (0, \infty)$

$$\int_X |f|^p d\mu = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha.$$

In addition one has Chebyshev's inequality for any $p \in (0, \infty)$ and $f \in L^p$

$$\lambda_f(\alpha) \leq \alpha^{-p} \|f\|_p^p.$$

Definition 2.1. *For each $p \in [1, \infty)$ we define $L^{p,\infty}$ the weak L^p space of all measurable f for which there is a constant $C > 0$ such that*

$$\lambda_f(\alpha) \leq \alpha^{-p} C^p.$$

The infimum of all such constants is defined to be $\|f\|_{p,\infty}$. An example of a function that is in the weak space but not the full space is $|x|^{-d/p}$.

The above definition defines a quasi-norm in the sense that the triangle inequality holds for some $C_p < \infty$ such that

$$\|f + g\|_{p,\infty} \leq C_p(\|f\|_{p,\infty} + \|g\|_{p,\infty}).$$

For $p \in (1, \infty)$ there actually exists a genuine norm on this space.

Definition 2.2. *An operator T is said to be of weak type (p, q) if it maps L^p to $L^{q,\infty}$ and satisfies*

$$\|Tf\|_{q,\infty}\| \leq C\|f\|_p.$$

Finally, we define a key object for our following studies

Definition 2.3. *Let $f \in L^1_{loc}(\mathbb{R}^d)$, we define the* Hardy-Littlewood Maximal Function

$$Mf(x) := \sup_{r>0} |B_r(x)|^{-1} \int_{B_r(x)} |f(y)| dy.$$

We then have the following Theorem

Theorem 8. *For each $p \in (1, \infty]$ there is $C(p, d) < \infty$ such that $\|Mf\|_p \leq C\|f\|_p$. Moreover, for any $f \in L^1$ and $\alpha > 0$ we have*

$$|\{x; Mf(x) > \alpha\}| \leq C\alpha^{-1}\|f\|_1.$$

However, M fails to map L^1 to L^1. In fact, if $Mf \in L^1$ then $f \equiv 0$. To see this, we observe that if $\int_{B_r(0)} |f| > 0$, then for any x, we find $\int_{B_{|x|+2R}(x)} |f| \leq \int_{B_r(0)} |f|$. We deduce that

$$Mf(x) \gtrsim \langle x \rangle^{-d}$$

which is not integrable.

In addition, we note that the bound $\|Mf\|_\infty \leq \|f\|_\infty$ is obvious. In fact, we even have

$$\lim_{r \to 0} |B_r(x)|^{-1} \int_{B_r(x)} f(y)\ dy = f(x)$$

by Lebesgue's differentiation theorem, which implies $Mf(x) \geq |f(x)|$. Our approach to show Theorem **??** will be to show the $L^1 \to L^{1,\infty}$ bound and then to use interpolation.

We have the Vitali covering Lemma

Lemma 2.1. *For each $d \geq 1$, there is $C_d < \infty$ such that for any measurable $E \subset \mathbb{R}^d$ of finite measure and any collection of balls $\mathcal{B}$ such that*

$$E \subset \bigcup_{B \in \mathcal{B}} B,$$

there is a collection $\mathcal{B}'$ of disjoint elements of $\mathcal{B}$ such that

$$|E| \leq C_d \sum_{B' \in \mathcal{B}'} |B'|.$$

The $L^1 \to L^{1,\infty}$ bound can then be deduced as follows.

Let $f \in L^1$ and $\alpha > 0$ be given. Define $E_\alpha := \{x; Mf(x) > \alpha\}$. Define $\mathcal{B}$ to be the balls B satisfying

$$|B|^{-1} \int_B |f| > \alpha.$$

The union of all those contains E_α. Then using the Vitali covering Lemma, we conclude

$$|E_\alpha| \leq C_d \sum_{B' \in \mathcal{B}'} |B'| \leq C_d \sum_{B' \in \mathcal{B}'} \alpha^{-1} \int_{B'} |f| \leq C_d \alpha^{-1} \|f\|_1.$$

We shall now turn to the proof of the covering Lemma

Proof. Choose $K \subset E$ compact with $|K| \geq |E|/2$. Choose a finite subcovert $\mathcal{B}'' \subset \mathcal{B}$ that covers K. Write $\mathcal{B}'' = \{B_1, B_2, ...\}$ ordering the balls so that $|B_j| \geq |B_{j+1}|$. We then define $\mathcal{B}'$ as follows: Select B_1. If B_N is disjoint from all previously selected one, we select it, otherwise we discard it. This way $\mathcal{B}'$ has only pairwise disjoint elements. We find that for any $B_m \in \mathcal{B}'' \setminus \mathcal{B}'$ there is $B' \in \mathcal{B}'$ such that $B_m \subset (B')^*$ where $(B')^*$ denotes the ball concentric with B' having three times as large a radius. Finally,

$$|K| \leq |\bigcup_{B \in \mathcal{B}''} B| \leq |\bigcup_{B' \in \mathcal{B}'} (B')^*| = 3^d \sum_{B' \in \mathcal{B}'} |B'|.$$

□

The interpolation result we need is the Marcinkiewicz Interpolation Theorem

Definition 2.4. *An operator T is said to be sublinear if it satisfies $|T(f + g)| \leq |Tf| + |Tg|$*

Let $p_\theta^{-1} = (1-\theta)p_0^{-1} + \theta p_1^{-1}$ and $q_\theta^{-1} = (1-\theta)q_0^{-1} + \theta q_1^{-1}$. The Marcinkiewiz interpolation theorem then states

Theorem 9. *Having, $\|Tf\|_{q_j,\infty} \leq C\|f\|_{p_j}$, we conclude that*

$$\|Tf\|_{q_\theta} \leq \tilde{C}\|f\|_{p_\theta}.$$

The main advantage of the Marcinkiewicz theorem is that, unlike the Riesz-Thorin theorem, it only requires weak estimates at the end-points.

Proof. To keep it simple, we will just the proof the case that we need: Let $p_0 = q_0 = 1$ and $p_1 = q_1 = \infty$. Suppose that

$$\|Tf\|_\infty \leq C_1\|f\|_\infty, \|Tf\|_{1,\infty} \leq C_0\|f\|_1$$

as well as $|T(f+g)| \leq C_2(|Tf| + |Tg|)$.

Given $\alpha > 0$ we split $f = g + h$ where $h(x) = 0$ if $|f(x)| \leq \alpha/(2C_1C_2)$ and $h(x) = f(x)$ otherwise. Then $\|g\|_\infty \leq \alpha/(2C_1C_2)$ so $\|Tg\|_\infty \leq \alpha/(2C_2)$. This implies that

$$C_2|Th| + \alpha/2 \geq C_2|Th| + C_2\|Tg\|_\infty \geq C_2(|Th| + |Tg|) \geq |Tf|$$

Hence, if $|Tf| \geq \alpha$, then $|Th| \geq \alpha/(2C_2)$.

This implies by the monotonicity of measures, since $\lambda_f(\alpha) := \mu\{x \in X; |f(x)| > \alpha\}$ we find that

$$\lambda_{Tf}(\alpha) \leq \lambda_{Th}(\alpha/(2C_2)).$$

This implies, using that $\|h\|_p = p\int_0^\infty \alpha^{p-1}\lambda_h(\alpha)\, d\alpha$ and the definition of h

$$\begin{aligned}
\lambda_{Tf}(\alpha) &\leq \lambda_{Th}(\alpha/(2C_2)) \leq 2C_2\alpha^{-1}\|Th\|_{1,\infty} \\
&\leq 2C_0C_2\alpha^{-1}\|h\|_1 \\
&= C\alpha^{-1}\int_0^\infty \lambda_h(\beta) d\beta \\
&= C\alpha^{-1}\int_0^\infty \min(\lambda_f(\beta), \lambda_f(\alpha/(2C_1C_2)))\, d\beta \\
&\leq .C\alpha^{-1}\int_{\alpha/(2C_1C_2)} \lambda_f(\beta)\, d\beta + C\lambda_f(\alpha/(2C_1C_2)).
\end{aligned}$$

Thus, for any $p \in (1,\infty)$

$$\begin{aligned}
\|Tf\|_p^p &= p\int_0^\infty \alpha^{p-1}\lambda_{Tf}(\alpha)\, d\alpha \\
&\leq C\int_0^\infty \alpha^{p-1}\left(\alpha^{-1}\int_{\alpha/(2C_1C_2)}^\infty \lambda_f(\beta)\, d\beta + C\lambda_f(\alpha/(2C_1C_2))\right)\, d\alpha \\
&= C\int_0^\infty \lambda_f(\beta)\int_0^{2C_1C_2\beta} \alpha^{p-2}\, d\alpha\, d\beta + C\int_0^\infty \alpha^{p-1}\lambda_f(\alpha/(2C_1C_2))\, d\alpha \\
&\leq C\int_0^\infty \gamma^{p-1}\lambda_f(\gamma)\, d\gamma.
\end{aligned}$$

□

3. Singular Integral Operators

Definition 3.1. *A Calderon Zygmund kernel is a continuous function on* $\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta$*, where* $\Delta = \{(x,y); x = y\}$ *is the diagonal such that*

$$|K(x,y)| \leq C|x-y|^{-d}$$

and there is $\delta \in (0,1]$ *such that whenever* $|y-y'| \leq 1/2|x-y|$ *then*

$$|K(x,y)-K(x,y')|+|K(y,x)-K(y',x)| \leq C|y-y'|^{\delta}|x-y|^{-d-\delta}.$$

Definition 3.2. *A continuous linear operator* $T:\mathcal{D}\to\mathcal{D}'$ *is associated to a kernel* $K \in L^1_{loc}(\mathbb{R}^d\times\mathbb{R}^d\setminus\Delta)$ *if for every pair* $f,g\in\mathcal{D}$ *of disjoint support, we have*

$$\langle Tf,g\rangle = \int\int K(x,y)f(x)g(y)\ dy\ dx.$$

An operator has at most one kernel but a kernel does not uniquely define an operator, e.g. $K=0$ corresponds to both the identity and the first derivative operator.

One then has

Theorem 10 (Calderon Zygmund)**.** *Suppose that for some* $q\in(1,\infty)$ T *is a bounded linear operator on* $L^q(\mathbb{R}^d)$ *and* T *is assocaited with a CZ kernel. Then* T *extends to a bounded linear operator for all* $q\in(1,\infty)$ *and is of weak* $(1,1)$ *type, i.e.*

$$\|Tf\|_{1,\infty}\leq C\|f\|_1.$$

The essential step in the proof is the following result

Proposition 3.1. *Under the assumptions of Theorem* **??**, *the operator* T *is of weak* $(1,1)$ *type.*

We can now give the proof of Theorem **??**

Proof. Using Marcinkiewicz interpolation we can conclude that T is bounded for every $p\in(1,q)$. If $q=\infty$ then we are good. If not, then we study the transpose operator $T'\in L(L^{q'})$ defined by $\int fT'g=\int Tfg$ with kernel $K'(x,y)=K(y,x)$. Applying the Proposition, T' is bounded for every $r\in(1,q')$. This however implies that T is bounded on all L^p with $p\in(1,\infty)$. □

To prove our Proposition, we need another tool that is commonly referred to as the *Calderon-Zygmund decomposition*

Proposition 3.2. *Let* $f\in L^1(\mathbb{R}^d)$ *and* $\alpha>0$. *Then* f *can be written as* $g+b$ *with* $\|g\|_\infty\leq\alpha$ *and* $b=\sum_j b_j$ *with each* b_j *supported on a dyadic cube* Q_j [3] *and*

- $Q_i\cap Q_j=\emptyset$ *for* $i\neq j$.
- $\int b_j=0$
- $\|b_j\|_1\leq 2^d\alpha|Q_j|$
- $\sum_j|Q_j|\leq\alpha^{-1}\|f\|_1$.
- $\|b\|_1+\|g\|_1\leq C\|f\|_1$.

We can now state the proof of Prop. **??**.

[3] A cube of sidelength 2^k for some $k\in\mathbb{Z}$ with vertices in $\mathbb{Z}2^k$

Proof. Let $|Tf(x)| > \alpha$ then using the same notation as in the CZ decomposition

$$|\{x; |Tf(x)| > \alpha\}| \le |\{x; |Tg(x)| > \alpha/2\}| + |\{x; |Tb(x)| > \alpha/2\}|.$$

We also have that by the CZ decomposition again.

$$\|g\|_q^q \le \|g\|_\infty^{q-1}\|g\|_1 \le C\alpha^{q-1}\|f\|_1.$$

Thus, by Chebyshev

$$|\{x; |Tg(x)| > \alpha/2\}| \le 2^q\alpha^{-q}\|Tf\|_q^q \le C\alpha^{-q}\|g\|_q^q \le C\alpha^{-1}\|f\|_1.$$

This is the weak (1,1) boundedness for g, now we also need this for q. Let Q_j^* denote the ball concentric with Q_j whose radius is twice the diameter of Q_j.

We define the exceptional set

$$E = \bigcup_j Q_j^*,$$

then using the CZ decomposition

$$|E| \le C\sum_j |Q_j| \le C\alpha^{-1}\|f\|_1.$$

This implies that

$$|\{x; |Tb(x)| > \alpha/2\}| \le |E| + |\{x \notin E; |Tb(x)| > \alpha/2\}| \le |E| + 2\alpha^{-1}\|Tb\|_{L^1(\mathbb{R}^d\setminus E)}.$$

The term $|E|$ has already been estimated two lines above. We now focus on $\|Tb\|_{L^1(\mathbb{R}^d\setminus E)}$ and use that

$$\|Tb\|_{L^1(\mathbb{R}^d\setminus E)} \le \sum_j \|Tb_j\|_{L^1(\mathbb{R}^d\setminus E)} \le \sum_j \|Tb_j\|_{L^1(\mathbb{R}^d\setminus Q_j^*)}.$$

We now need an additional Lemma that shows that

$$\|Tb_j\| \le C\|b_j\|.$$

This then allows us to show that

$$\sum_j \|Tb_j\|_{L^1(\mathbb{R}^d\setminus Q_j^*)} \le C\sum_j \|b_j\|_{L^1} = C\|b\|_1 \le C\|f\|_1.$$

□

We now show (**??**). Let y_0 denote the center of Q_j, then since $\int b_j = 0$ we have for $x \notin Q_j^*$

$$Tb_j(x) = \int (K(x,y) - K(x,y_0))b_j(y)\,dy.$$

We conclude

$$\int_{x\notin Q_j^*} |Tb_j(x)|\ dx = \int_{x\notin Q_j^*} |\int_{y\in Q_j} (K(x,y)-K(x,y_0))b_j(y)\ dy|dx$$
$$= \int_{y\in Q_j}\int_{x\notin Q_j^*} |K(x,y)-K(x,y_0)|\ dx|b_j(y)|\ dy$$
$$\le \|b_j\|_1 \sup_{y\in Q_j} \|K(\bullet,y)-K(\bullet,y_0)\|_{L^1(\mathbb{R}^d\setminus Q_j^*)}.$$

On the other hand, for ℓ the side-length of the cube Q_j

$$|Tb_j(x)| \le C|x-y_0|^{-d-\delta}\int_{Q_j} |y-y_0|^\delta |b_j(y)|\ dy \le C|x-y_0|^{-d-\delta}\ell^d\|b_j\|_1.$$

Using that for $y\in Q_j$ and $x\in Q_j^*$ we have $|x-y_0|\ge 2|y-y_0|$ we have by the properties of the CZ kernel

$$|K(x,y)-K(x,y_0)| \le C|y-y_0|^\delta |x-y_0|^{-d-\delta}.$$

Integrating then, we find

$$\int_{\mathbb{R}^d\setminus Q_j^*} |x-y_0|^{-d-\delta}\ dx \le \int_{|x-y_0|\ge 2\ell} |x-y_0|^{-d-\delta}\ dx = c\ell^{-\delta}$$

Thus, one finds

$$\|Tb_j\|_1 \le C\|b_j\|_1.$$

4. Homogeneous distributions

Let $x\in\mathbb{R}^d$ and $r>0$ then

$$\delta_r f(x) := f(rx).$$

This notation is extended to distributions by setting for $\phi\in S'$

$$(\delta_r\phi)(f) := r^{-d}\phi(\delta_{1/r}f)$$

Checking for $\phi(f)=\int gf$ we have by the change of variables that $(\delta_r\phi)(f) = \int(\delta_r g)f$.

A distribution ϕ is called homogeneous of degree γ if $\delta_r\phi = r^\gamma\phi$ for all $r>0$.

The Dirac distribution is homogeneous of degree $-d$ in $\mathbb{R}^d$. The principal value on $\mathbb{R}$ defined by

$$\phi(f) := \lim_{\varepsilon\downarrow 0}\int_{|x|>\varepsilon} f(x)x^{-1}dx$$

is homogeneous of degree -1.

It is easy to see that for any $\gamma\in\mathbb{C}$ if $\phi\in S'$ is homogeneous of degree γ then $\hat\phi$ is homogeneous of degree $-d-\gamma$. If ϕ is homogeneous of degree γ, then $\partial^\alpha\phi$ is homogeneous of degree $\gamma-|\alpha|$. However, $\log|x|$ is not homogeneous on $\mathbb{R}^2$, but $\Delta\log|x| = c\delta_0$ for some $c\in\mathbb{R}\setminus\{0\}$ and therefore homogeneous.

We start with the following result

Exercise 4. *If $\phi \in S'$ is homogeneous and belongs to $C^\infty(\mathbb{R}^d \setminus \{0\})$, then $\hat{\phi} \in C^\infty(\mathbb{R}^d \setminus \{0\})$ as well.*

The proof of this can be obtained from an approximation scheme that rests on the following Lemma

Lemma 4.1. *The Fourier transform of any compactly supported distribution belongs to C^∞.*

Proof. Let ϕ be a compactly supported distribution. We choose a function $\eta \in C_c^\infty$ that is equal to one on the support of ϕ. Thus, $\phi\eta = \phi$. By continuity there is M and C such that

$$|\phi(f)| = |\phi(\eta f)| \leq C\|\eta f\|_{C^M}.$$

This implies that

$$\hat{\phi}(f) = \phi(\hat{f}) = \phi(\eta \hat{f}).$$

Writing $e^{-ix\xi} = \sum_{n=0}^\infty \frac{(-ix\xi)^n}{n!}$ we find

$$\hat{\phi}(f) = \sum_n \frac{1}{n!} \int \phi(\eta(-ix\xi)^n) f(\xi) d\xi = \langle \psi, f\rangle$$

with $\psi(\xi) = \sum_n \phi(\eta(-ix)^n)\frac{\xi^n}{n!}$ an entire function. □

Distributions supported in a single point are particularly easy

Exercise 5. *Let $\phi \in S'(\mathbb{R}^d)$ supported in $\{0\}$, then for some m and coefficients a_α*

$$\phi = \sum_{n=1}^m a_\alpha \partial^\alpha \delta_0.$$

Hint: Use continuity to show that $\phi(f) = 0$ if $\partial^\alpha f(0) = 0$ for all $0 \leq |\alpha| \leq m$.

We find then

Theorem 11. *The Fourier transform defines a continuous bijection between all distributions that are homogeneous of degree 0 and smooth away from the origin and distributions ϕ of the form*

$$\mathbb{C}\delta_0 + \textit{ pv(K) with } K \in C^\infty(\mathbb{R} \setminus \{0\}), \int_{\mathbf{S}^{d-1}} K = 0,$$

where $pv(K)(f) := \lim_{\varepsilon \downarrow 0} \int_{|x|>\varepsilon} k(x) f(x)\ dx$.

Proof. We use that we can write our distribution as $\phi = c + m$ where c is a constant and m satisfies $\int_{\mathbf{S}^{d-1}} m = 0$. The Fourier transform of m is a distribution k which is a function h that is homogeneous of degree $-d$ and smooth on $\mathbb{R}^d \setminus \{0\}$.

Let f be radial, then $k(f) = \hat{m}(f) = m(\hat{f}) = \int m\hat{f} = 0$. This implies that h is radial.

We can now define $H = \mathrm{pv}(h)$ which is homogeneous of degree $-d$. Thus, $k - H$ is homogeneous of degree $-d$ and supported in $\{0\}$. Hence, it is a multiple of δ_0. □

This implies that

Corollary 4.2. *If $\phi \in S'$ is homogeneous of degree $-d$ and belongs to $C^\infty(\mathbb{R}^d \setminus \{0\})$, then the operator $Tf = f * \phi$ is a bounded operator on $L^2(\mathbb{R}^d)$.*

Proof. Its fourier transform $\hat{\phi}$ is homogeneous of degree 0 and belongs to $C^\infty(\mathbb{R}^d \setminus \{0\})$. Thus, $\hat{\phi} \in L^\infty$. Then, since $\widehat{(\phi * f)} = \hat{\phi}\hat{f} \in L^2$, the claim follows. □

Thus, we have shown

Theorem 12. *Let $k \in C^\infty(\mathbb{R}^d \setminus \{0\})$ be homogeneous of degree $-d$ and $\int_{\mathbf{S}^{d-1}} k = 0$ then for all $p \in (1, \infty)$*

$$\|pv(k) * f\|_p \le C\|f\|_p$$

and the weak L^1 bound holds as well.

Theorem 13. *Let $m \in C^\infty(\mathbb{R}^d \setminus \{0\})$ be homogeneous of degree zero. Then the operator $f \mapsto \mathcal{F}^{-1}(\hat{f}m)$ is a Calderon-Zygmund operator*

Proof. We have $Tf = \mathcal{F}^{-1}(\hat{f}m)$ equals $f * \mathcal{F}^{-1}m$ where $\mathcal{F}^{-1}m$ is homogeneous of degree $-d$ and belongs to $C^\infty(\mathbb{R}^d \setminus \{0\})$. Therefore $\mathcal{F}^{-1}m(x-y)$ defines a standard kernel. □