Completely bounded maps

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1 Notation

We will always assume our C^* algebras to unitary and we will denote them by capital frak letters, *i.e.* $\mathfrak{U}, \mathfrak{V}, \ldots$ Similarly, Hilbert spaces will be denoted by $\mathfrak{h}, \mathfrak{g}, \ldots$ The space of bounded operators on a Hilbert space is denoted $\mathcal{B}(\mathfrak{h})$. Since we will introduce a concept of tensor product that does not coincide with the algebraic one, we will denote the algebraic tensor product of two vector spaces as $U \odot V$. In preparing for this seminar, I have mostly used the sources [2] and [1]. Where not mentioned otherwise, the propositions and theorems come from [1].

2 Introduction

Before starting our study of completely bounded maps, we need to develop the notion of tensor product for C^* algebras. As for Hilbert spaces, the issue is that while the algebraic definition works without problems, the resulting space does not have (a priori) a C^*algebra structure; to construct it, we will need to choose the right norm on the algebraic tensor product and complete it accordingly.

***Definition* 1** (Tensor product)**.** Let $\mathfrak{U}, \mathfrak{V}$ be two C^* algebras and assume they are realised as subalgebras of $\mathcal{B}(\mathfrak{h}), \mathcal{B}(\mathfrak{g})$ respectively. We define the tensor product $\mathcal{B}(\mathfrak{h}) \otimes \mathcal{B}(\mathfrak{g}) := \mathcal{B}(\mathfrak{h} \otimes \mathfrak{g})$ (with the associated norm) and the tensor product of $\mathfrak{U}, \mathfrak{V}$ as

$$\mathfrak{U} \otimes_{min} \mathfrak{V} := \left(\overline{\mathfrak{U} \odot \mathfrak{V}}^{\|\cdot\|_{\mathcal{B}(\mathfrak{g} \oplus \mathfrak{h})}}, \|\cdot\|_{\mathcal{B}(\mathfrak{g} \oplus \mathfrak{h})} \right)$$

Now that we have the appropriate concept of tensor product, we would like to prove the standard result concerning tensor products of maps but as we will see, this cannot work for general linear maps. To define it, we need to focus on completely bounded maps, which we now introduct

Remark 1. It is easy to see that the above definition can be equivalently stated as:

$$\|x \in \mathfrak{U} \odot \mathfrak{V}\|_{min} := \sup \|\pi(x)\|,$$

where the supremum runs over the family of morphisms of $\mathfrak{U} \odot \mathfrak{V}$ into some $\mathcal{B}(\mathfrak{f})$ of the form $\pi = \pi_1 \otimes \pi_2$ where π_1 is a representation of $\mathfrak{U}$ and π_2 one of $\mathfrak{V}$. If we instead take the supremum over all the morphisms (this norm is denoted by $\|x\|_max)$ we obtain a new tensor product of the two C^* algebras:

$$\mathfrak{U} \otimes_{max} \mathfrak{V} := \left(\overline{\mathfrak{U} \odot \mathfrak{V}}^{\|\cdot\|_{max}}, \|\cdot\|_{max} \right)$$

As in the case of topological vector spaces, there are many other equally worthy definitions of completed tensor product. As in that case, all such possible norms are (on $\mathfrak{U} \odot \mathfrak{V}$), stronger than $\|\cdot\|_{min}$ and weaker than $\|\cdot\|_{max}$. By analogy to the TVS case, the C^* algebras for which the min norm and the max norm coincide are called nuclear. For more information, see [2].

Remark 2. One can easily see that $M_n(\mathfrak{U}) = M_n(\mathbb{C}) \otimes_{\min} \mathfrak{U}$.

***Definition* 2.** Let $\mathfrak{U}, \mathfrak{V}$ be unitary C^* algebra. We say that a linear map $\phi : \mathfrak{U} \to \mathfrak{V}$ is completely bounded if

$$\sup_n \|\varphi \otimes I\|_{M_n(\mathfrak{U},\mathfrak{V})} < \infty.$$

Such supremum, if finite, is denoted by $\|\varphi\|_{cb}$. The operator $\varphi \otimes I_{M_n}$ is denoted by φ_n.

While the definition might not seem particularly useful at a first glance, the following proposition shows the usefulness of completely bounded maps inn the study of C^* algebras:

Proposition 2.1. *Let $\mathfrak{U}_{1,2}$, $\mathfrak{V}_{1,2}$ be C^* algebras and let $\varphi_{1,2} : \mathfrak{U}_{1,2} \to \mathfrak{V}_{1,2}$ be cb maps. Then there exists a unique linear map $\tilde{v}arphi$ from $\mathfrak{U}_1 \otimes_{min} \mathfrak{U}_2$ to $\mathfrak{V}_1 \otimes_{min} \mathfrak{V}_2$ that extends the algebraic tensor product of the two maps. The map also satisfies*

$$\|\varphi_1 \otimes \varphi_2\|_{cb} \leq \|\varphi_1\|_{cb} \|\varphi_2\|_{cb}.$$

Moreover, we have

$$\|\varphi\|_{cb} = \sup_{\mathfrak{V}} \|\varphi \otimes Id\|$$

Proof. We begin proving the last equation. By taking $\mathfrak{V} = M_n$, it is clear that RHS$\geq$ LHS. To prove the other direction, assume $\mathfrak{V}$ is a subalgebra of $\mathcal{B}(\mathfrak{f})$ and $\mathfrak{U}$ of $\mathcal{B}(\mathfrak{h})$. For simplicity, assume $\mathfrak{f} = \ell_2$ (in general, we have $\mathfrak{h} = \ell_2(k)$ for some cardinal k but this complicates working with indices) equipped with the standard basis (e_i). Denoting by P_n the standard projection of ℓ_2 on ℓ_2^n and by abuse of notation also the projection $P_n \ell_2 \otimes \mathfrak{h} \to \ell_2^n \otimes \mathfrak{h}$ it is clear that, given $x \in \mathfrak{U} \odot \mathfrak{V}$, we have

$$\|x\|_{min} = \sup_n \|P_n(x)\|,$$

since $\bigcup \ell_2^n \otimes \mathfrak{h}$ is dense in $\ell_2 \otimes \mathfrak{h}$. Writing explicitly $x = \sum_{i=1}^m h_i \otimes f_i$, it follows that using standard identifications we have

$$t_n(i,j) = \sum_k \langle e_i, f_k e_j \rangle h_k,$$

an element of $M_n(\mathfrak{U})$. The equation is thus proved. To prove the first half of the proposition, it suffices to notice that $\varphi_1 \otimes \varphi_2 = (\varphi_1 \otimes Id_{\mathfrak{V}_2}) \otimes (Id_{\mathfrak{U}_1} \otimes \varphi_2)$ and we are done, since

$$\|\varphi_1 \otimes \varphi_2\| \leq \|\varphi_1 \otimes Id\| \|Id \otimes \varphi_2\| \leq \|\varphi_1\|_{cb} \|\varphi_2\|_{cb}$$

and the same argument can be easily adapted to $(\varphi_1 \otimes \varphi_2) \otimes Id_{M_n}$. ■

Remark 3. It is clear that every positive map and every linear combination thereof is completely bounded. Since there is a strong similarity between the definition of cbms and cpms, it is logical to expect that the converse be true as well. This is indeed the case, as we will see.

Example 1. While one might be tempted, given the above remark, to believe that most results true for completely positive maps generalize to completely bounded ones, we urge caution in doing so and present a cautionary counterexample to a natural conjecture:

Proposition 2.2. *There exists a bounded morphism $\varphi : \mathfrak{U} \to \mathfrak{V}$ between two unital C^* algebras, with $\mathfrak{V}$ commutative, that is not completely bounded.*

See [2] for details. Consider the space ℓ_∞^n (i.e. $\mathbb{R}^n$ with the maximum norm), equipped with pointwise multiplication. There exists $N(n)$ such that, on $M_N(\mathbb{R}^n)$, we can construct a spin system, i.e. a family of matrices $\{U_i\}$ satisfying

1. $U_j U_i = -U_i U_j$ whenever $i \neq j$,

2. U_j is unitary and self adjoint for all j.

One can then prove that the map $\varphi : \ell_\infty^n \to M_N(\mathbb{R}^n)$ defined as

$$\varphi(e_i) = U_j \frac{1}{\sqrt{2n}}$$

is not completely bounded. ■

However, if the codomain is commutative, the result is true:

Proposition 2.3. *Let $\varphi : \mathfrak{U} \to \mathfrak{V}$ be a linear map and assume $\mathfrak{V}$ to be commutative. Then $\|\varphi\| = \|\varphi\|_{cb}$.*

Proof. Without loss of generality, we can assume that $\mathfrak{V} = C_0(X)$ for some locally compact X. It is then easy to see that $M_n(\mathfrak{V}) = C_0(X, M_n(\mathbb{C}))$, hence

$$\begin{aligned}
\|\varphi \otimes I(a_{ij})\|_{M_n(\mathfrak{V})} &= \left\| \left(\varphi(a_{i,j})(x)\right) \right\|_\infty \\
&= \sup_x \sup_{\xi,\eta \in \mathbb{C}^n : \|\xi\|, \|\eta\| \leq 1} |\langle \varphi(a_{i,j})(x)\xi, \nu \rangle| \\
&= \sup_x \sup_{\xi,\eta \in \mathbb{C}^n : \|\xi\|, \|\eta\| \leq 1} |\varphi(\eta^t a \zeta)(x)| \\
&\leq \|\varphi\| \|(a_{i,j})\|.
\end{aligned}$$

Since the other inequality is trivial, it follows that $\|\varphi\| = \|\varphi_n\|$ and we are done. ■

Moreover, the result is true even if the codomain is finite dimensional:

Proposition 2.4 ([2])**.** *Let $\varphi : \mathfrak{U} \to \mathfrak{V}$ be a linear map and assume $\mathfrak{V}$ is finite-dimensional of dimension n. Then there exists a constant C such that $\|\varphi\|_{cb} \leq C(n)\|\varphi\|$.*

Remark 4. The problem of finding an explicit expression for the best value of $C(n)$ is still open. One easy bound is $C(n) \leq n$, which can be proven as follows: since the space is n-dimensional it admits an Auerbach basis, i.e. n pairs (x_i, y_i) with $x_i \in \mathfrak{V}$ and $y_i \in \mathfrak{V}^*$ such that $y_i(x_j) = \delta_i^j$, we can decompose $Id_{\mathfrak{V}}$ as

$$Id = \sum y_i(\cdot)x_i.$$

Since each of the maps $y_i(\cdot)x_i$ has values in the commutative C^* algebra $\mathbb{C}$ and has unit norm. it follows that $\|Id\|_{cb} \leq n$. Similarly, given φ a general linear map, we have

$$\varphi = \sum y_i(\cdot)\varphi(x_i)$$

and hence $C(n) \leq n$.

3 Decomposition of cb maps

As mentioned before, one of the cornerstone results in the theory of cb maps is the decomposition as a linear combination of positive maps. This will be the main goal of this section. To prove the result, we take an indirect road: we will first prove an extension of Stinespring's representation theorem to cb maps, from which the promised decomposition will follow.

Theorem 3.1 (Stinespringer decomposition for cb maps, (Wittstock, Haagerup, and Paulsen))**.** *Let φ be a completely bounded map from $\mathfrak{A}$ to $\mathcal{B}(\mathfrak{h})$. Then there exist a representation $(\pi, \mathfrak{k})$ of $\mathfrak{A}$ and bounded operators $V_{1,2} : \mathfrak{h} \to \mathfrak{k}$ such that*

$$\|V_1\|\|V_2\| = \|\varphi\|_{cb}$$
$$\varphi = V_1^* \pi V_2.$$

Moreover, if φ is a completely bounded contraction (i.e. $\|\varphi\|_{cb} \leq 1$) we can choose $V_{1,2}$ to be isometries.

The main ingredients for this proof (and for many generalizations of statements for cp maps to cb maps) are: a uniqueness result for the stinespringer representation and a lemma (due to Paulsen), that is the bread and butter of extending results from cp to cb maps.

Recall that, given a completely bounded map $\varphi : \mathfrak{A} \to \mathcal{B}(\mathfrak{h})$, the Stinespring factorization theorem implies the existence of a representation $(\pi, \mathfrak{k})$ of $\mathfrak{A}$ and a map $V : \mathfrak{h} \to \mathfrak{k}$ such that $\varphi = V^* \pi V$. The couple (π, V) is called a Stinespring representation of φ.

Proposition 3.1 (Minimal Stinespring representations)**.** *Let (π, V) be a Stinespring representation of a completely positive map $\varphi : \mathfrak{A} \to \mathcal{B}(\mathfrak{h})$ and call the representation be minimal if $\{\pi(x)Vu\}$ is dense in $\mathfrak{k}$. Then two minimal representations (π_1, V_1), (π_2, V_2) are necessarily unitarily equivalent, i.e. there exists a unitary operator $U : \mathfrak{k}_1 \to \mathfrak{k}_2$ such that*

$$UV_1 = V_2$$
$$U\pi_1 U^* = \pi_2$$

Proof. We begin by constructing U on $A = \{\pi_1(x)V_1 u\}$. If we want U to satisfy the above conditions, it follows that we must define it as

$$U\left(\sum \pi_1(x_i)V_1 u_i\right) = \sum \pi_2(x_i)V_2 u_i.$$

Then it suffices to prove that U is indeed an isometry and onto, from which we will extend it by continuity to all $\mathfrak{k}_1$. The surjectivity of U follows from the density of $\{\pi_2(x)V_2 u\}$ in $\mathfrak{k}_2$, while the isometricity from simple calculations- ∎

Lemma 3.2 (Paulsen, see [2] chapter 1)**.** *Let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be a morphism. Then φ is a completely bounded contraction if and only if the map*

$$\Phi \begin{pmatrix} \lambda 1 & a \\ b^* & \mu 1 \end{pmatrix} := \begin{pmatrix} \lambda 1 & \varphi(a) \\ \varphi(b)^* & \mu 1 \end{pmatrix}$$

from the operator system

$$\mathcal{S} := \left\{ \begin{pmatrix} \lambda 1 & a \\ b^* & \mu 1 \end{pmatrix} : \mu, \lambda \in \mathbb{C}, a, b \in \mathfrak{A} \right\}$$

to $\mathfrak{B}$ is a unital cp map.

For the proof of the lemma, which is rather long and involves somewhat lengthy computations, we refer the reader to chapter 1 of [2].

Proof. Without loss of generality, let us assume φ is cbc (since it suffices to divide it by a scalar to get this additional hypothesis). Define Φ as in the lemma. By the Arveson extension theorem, Φ can be extended to $M_2(\mathfrak{U}) \supset \mathcal{S}$ and let $(\tilde{\pi}, \mathfrak{t}, V)$ the minimal stinespringer representation (with associated operator). Since Φ is a contraction, we can assume V to be an isometry and $\tilde{\pi}$ unital. We can additionally find a decomposition of $\mathfrak{t}$ as $\mathfrak{t} = \mathfrak{t} \oplus \mathfrak{t}$ such that

$$\tilde{\pi}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \pi(a) & \pi(b) \\ \pi(c) & \pi(d) \end{pmatrix}$$

with $\pi : \mathfrak{U} \to \mathcal{B}(\mathfrak{t})$ a unital morphism. Identifying $\mathfrak{h} \simeq \mathfrak{h} \oplus \mathfrak{h}$, we can see V as an isometry from $\mathfrak{h} \oplus \mathfrak{h}$ to $\mathfrak{t} \oplus \mathfrak{t}$ satisfying

$$\begin{pmatrix} a & \varphi(b) \\ \varphi(c)^* & \mu d \end{pmatrix} = \Phi\begin{pmatrix} a & b \\ c^* & d \end{pmatrix} = V^*\begin{pmatrix} \pi(a) & \pi(b) \\ \pi(c) & \pi(d) \end{pmatrix}.$$

It is now a matter of straightforward calculations to verify that the maps V_1, V_2 defined through

$$V_{1,2} \circ p_{1,2} = q_{1,2} \circ V,$$

where $p_{1,2}$ denote the projections associated to the aforementioned decomposition of $\mathfrak{h}$ and similarly $q_{1,2}$ for $\mathfrak{t}$ satisfy, together with π, the conclusion of the theorem. ∎

We obtain, as a direct corollary of the theorem, the promised decomposition of cb maps:

Corollary 3.2.1. *Let $\varphi : \mathfrak{U} \to \mathcal{B}(\mathfrak{h})$ be a completely bounded map. Then there exist 4 completely positive maps $\varphi_{1,2,3,4}$ such that*

$$\varphi = (\varphi_1 - \varphi_2) + i(\varphi_3 - \varphi_4).$$

Moreover, we have

$$\begin{aligned} \|\varphi_i\| &\leq \|\varphi\| \\ \|\varphi_1 + \varphi_2\| &\leq \|\varphi| \\ \|\varphi_3 + \varphi_4\| &\leq \|\varphi| \end{aligned}$$

Proof. We assume, without loss of generality, that $\|\varphi\|_{cb} = 1$, which allows us to assume V_1, V_2 to be isometries. The result is proved in the same spirit as the polarization formula:

$$\begin{aligned} \varphi_1 &:= \frac{1}{4}(V_1 + V_2)^* \pi (V_1 + V_2) \\ \varphi_2 &:= \frac{1}{4}(V_1 - V_2)^* \pi (V_1 - V_2) \\ \varphi_3 &:= \frac{1}{4}(V_1 + iV_2)^* \pi (V_1 + iV_2) \\ \varphi_3 &:= \frac{1}{4}(V_1 - iV_2)^* \pi (V_1 - iV_2). \end{aligned}$$

It is then clear from the definition that $\|\varphi_i\| \leq \frac{1}{4} 2 \cdot 2 = 1$, so it remains only to prove that $\|\varphi_1 + \varphi_2\| \leq \|\varphi\|$ (since the proof of the similar inequality for $(3, 4)$ is virtually identical): by construction, we have

$$\varphi_1 + \varphi_2 = \frac{1}{2}\left(V_1^* \pi V_1 + V_2^* \pi V_2\right),$$

which implies

$$\|\varphi_1 + \varphi_2\| \leq \frac{1}{2}\sqrt{\|V_1\|^2 + \|V_2\|^2} = 1.$$

∎

The theorem has many other applications; we briefly mention two of them: first, it directly implies the GNS factorization[1]: it suffices to take $\varphi = f$ and $\mathfrak{V} = \mathbb{C}$ in the theorem (where f is completely bounded since $\mathbb{C}$ is commutative). The second result we mention is the following characterization of Schur Multipliers (due to Grothendieck):

Theorem 3.3 (see [2])**.** *The following are equivalent:*

- *There exists a completely bounded map* $u : \varphi : \mathcal{B}(\ell_2) \to \mathcal{B}(\ell_2)$ *such that* $u(e_{ij}) = f(i,j)e_{ij}$[2]*,*
- *There exist a Hilbert space* $\mathfrak{h}$ *and two bounded functions* $x, y : \mathbb{N} \to \mathfrak{h}$
- *The Schur multiplier* $u : \mathcal{B}(\ell_2) \to \mathcal{B}(\ell_2)$ *that sends* (a_{ij}) *to* $(f(i,j)a_{ij})$ *is bounded.*

References

[1] Stéphane Attal, Alain Joye, and Claude-Alain Pillet, eds. *Open quantum systems II.* Springer Berlin Heidelberg, Aug. 2006.

[2] Gilles Pisier. *Tensor Products of C*-Algebras and Operator Spaces: The Connes–Kirchberg Problem.* London Mathematical Society Student Texts. Cambridge University Press, 2020.

[1] while in this course this results in a circular proof, this is not necessarily the case, since the factorization theorem can be proven directly by means of of the theory of injective spaces, see [2], chapter 1

[2] e_ij is the map that sends e_i to e_j