More in depth statements are given in Section 4.5 of [3] and more detailed proofs can be found in [2].

Let (X, μ) be a probability space. Then we can view $L^\infty(X, \mu)$ as a subspace of $\mathcal{B}(L^2(X, \mu))$: Indeed, we can identify any $f \in L^\infty(X, \mu)$ with the multiplication operator $M_f : L^2(X, \mu) \to L^2(X, \mu)$, defined via $\psi \mapsto f\psi$. Since the identity operator corresponds to the constant function $1 \in L^\infty(X, \mu)$, this gives (together with weak closedness) a von Neumann algebra.

In the following, $L^\infty(X, \mu)$ will serve as a model for other von Neumann algebras.

Definition 1. *A subset $C \subset \mathcal{H}$ of some Hilbert space is called* ***cone*** *if for all $t \geq 0$ and $\psi \in C$, we have $t\psi \in C$.*

Example. In $\mathbb{R}^n$, such a set is just an infinite cone with its apex being the origin.
Of course, our model space $\mathcal{H} = L^2(X, \mu)$ also admits many cones. One of special interest is the cone $C = \{\psi \in L^2 : \psi \geq 0 \text{ a.e.}\}$.

Remark. In general, Hilbert spaces are complex, so in the example above, we look at the subset of $L^2(X, \mu)$ that contains (almost everywhere) real-valued functions that are non-negative. Notice also for the next definition that a scalar product can be complex valued in general.

Definition 2. *The* ***dual*** *of a cone $C \subset \mathcal{H}$ is defined as the set $\widehat{C} = \{\phi \in \mathcal{H} : \langle \phi, \psi \rangle \geq 0$ for all $\psi \in C\}$. A cone is called* ***self-dual*** *if $C = \widehat{C}$.*

It is immediate from the definition of a cone that the dual of a cone is a non-empty, closed and convex cone. Thus it makes sense to compare a cone to its dual and to look at the class of self-dual cones.

Example. The cone $C = \{\psi \in L^2 : \psi \geq 0 \text{ a.e.}\}$ is actually a self-dual cone:

- We have $C \subset \widehat{C}$, as for any $\psi, \phi \in C$, $\langle \phi, \psi \rangle = \int_X \phi \cdot \psi \, \mathrm{d}\mu \geq 0$.
- Conversely, suppose there is $\phi \in \widehat{C} \setminus C$. Then there exists some subset $E \subset X$ with $\mu(E) > 0$ s.t. $\int_X \overline{\phi} \, \mathrm{d}\mu \ngeq 0$. Take now $\psi = \chi_E \in C$ and we have a contradiction, as $\langle \phi, \psi \rangle \ngeq 0$.

Definition 3. *A map $J : \mathcal{H} \to \mathcal{H}$ is called* ***anti-unitary*** *if it is anti-linear (i.e. additive and $J(\lambda\psi) = \bar{\lambda}J(\psi)$) and satisfies $\langle J\phi, J\psi \rangle = \overline{\langle \phi, \psi \rangle}$. Additionally, J is an* ***involution*** *if $J^2 = \mathrm{id}$.*

Example. The pointwise complex conjugation $J : L^2(X,\mu) \to L^2(X,\mu), \psi \mapsto \overline{\psi}$ is an anti-unitary involution.

Definition 4. *A von Neumann algebra $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ is in* ***standard form*** *if there are an anti-unitary involution $J : \mathcal{H} \to \mathcal{H}$ and a self-dual cone $C \subset \mathcal{H}$ s.t.*

1) $J\mathfrak{M}J = \mathfrak{M}'$

2) $J\psi = \psi$ *for all* $\psi \in C$

3) $AJAC \subset C$ *for all* $A \in \mathfrak{M}$

4) $JAJ = J^*$ *for all* $A \in \mathfrak{M} \cap \mathfrak{M}'$

We write $(\mathfrak{M}, \mathcal{H}, J, C)$ for a von Neumann algebra in standard form.

Example. As we have seen above, $\mathfrak{M} = L^\infty(X,\mu)$ admits a self-dual cone C and an anti-unitary involution J. Items *1)* and *4)* follow from the equality $JfJ(\psi) = \overline{f}\psi$ for all $f \in \mathfrak{M}, \psi \in \mathcal{H}$ and the observation that $\mathfrak{M} = \mathfrak{M}'$ as multiplication is commutative. *2)* is true because $J\psi = \psi$ for all $\psi \in C$, as elements of the cone are real-valued and thus immune to complex conjugation. Since fJf just corresponds to multiplication by $|f|^2$, leaving C invariant, *3)* is also fulfilled.

This example shows that we know at least one von Neumann algebra that is in standard form and we have (at least some) hope that the definition of standard forms is justified. The following theorem solidifies this hope further:

Theorem 5. *Any von Neumann algebra $\mathfrak{M}$ has a faithful representation $(\mathcal{H}, \pi)$, i.e. a $*$-homomorphism $\pi : \mathfrak{M} \to \mathcal{B}(\mathcal{H})$ that is bijective onto the image $\pi(\mathfrak{M})$, such that $\pi(\mathfrak{M})$ is in standard form.*
Moreover, if $(\mathfrak{M}_1, \mathcal{H}_1, J_1, C_1)$ and $(\mathfrak{M}_2, \mathcal{H}_2, J_2, C_2)$ are two von Neumann algebras in standard form and we have a $*$-isomorphism $\Phi : \mathfrak{M}_1 \to \mathfrak{M}_2$, then there exists a uniqe unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that:

- $\Phi(A) = UAU^*$ *for all* $A \in \mathfrak{M}_1$
- $J_2 = UJ_1U^*$
- $C_2 = U(C_1)$

A von Neumann algebra $\mathfrak{M}$ in standard form has two special properties that allow us to relate intrinsic information about $\mathfrak{M}$ with properties of the corresponding Hilbert space $\mathcal{H}$.
The first such property treats normal states on $\mathfrak{M}$:

Theorem 6. *Let $(\mathfrak{M}, \mathcal{H}, J, C)$ be a von Neumann algebra in standard form and $\psi \in C$ a unit vector. Consider the normal state $\omega_\psi \in N(\mathfrak{M})$, defined via $\omega_\psi(A) = \langle \psi, A\psi \rangle$ on $\mathfrak{M}$. Then the map*

$$\begin{aligned} \{\psi \in C : ||\psi|| = 1\} &\to N(\mathfrak{M}), \\ \psi &\mapsto \omega_\psi \end{aligned}$$

is a homeomorphism.

Thus, for any $\mu \in N(\mathfrak{M})$, there is a unique unit vector $\psi_\mu \in C$ such that $\mu = \omega_{\psi_\mu}$. We call ψ_μ the **standard representative** of μ.

Example. Let $\omega \in N(L^\infty(X, \mu))$. The theorem above tells us that there is a unique $\psi_\omega \in L^2(X, \mu)$ with $||\psi_\omega||_{L^2} = 1$ and $\psi_\omega \geq 0$ almost everywhere such that

$$\omega(f) = \langle \psi_\omega, f\psi_\omega \rangle = \int_X \overline{\psi_\omega} f \psi_\omega \, \mathrm{d}\mu = \int_X |\psi_\omega|^2 \, f \, \mathrm{d}\mu \quad \text{for any } f \in L^\infty(X, \mu).$$

This classifies all the normal states on $L^\infty(X, \mu)$.

The second result is of very similar flavour as Theorem 6. It relates the $*$-automorphism group of $\mathfrak{M}$ to the *standard unitaries* of $\mathcal{H}$.

Definition 7. *Let $(\mathfrak{M}, \mathcal{H}, J, C)$ be a von Neumann algebra in standard form. A unitary $U : \mathcal{H} \to \mathcal{H}$ is called **standard unitary** if*

- $UC \subset C$
- $U\mathfrak{M}U^* = \mathfrak{M}$

Proposition 8. *$\mathcal{U}_s = \{U : \mathcal{H} \to \mathcal{H} \,|\, U \text{ is standard unitary}\}$ is a subgroup of $\mathcal{B}(\mathcal{H})$ that is closed with respect to the strong operator topology on $\mathcal{B}(\mathcal{H})$.*

Theorem 9. *Let $(\mathfrak{M}, \mathcal{H}, J, C)$ be a von Neumann algebra in standard form and $U \in \mathcal{U}_s$. Consider the $*$-automorphism τ_U on $\mathfrak{M}$, given by $\tau_U(A) = UAU^*$. Then the group homomorphism*

$$\begin{aligned} (\mathcal{U}_s, \textit{strong topology}) &\to (\mathrm{Aut}(\mathfrak{M}), \sigma\textit{-weak topology}) \\ U &\mapsto \tau_U \end{aligned}$$

is a homeomorphism.

As before, for any $\sigma \in \mathrm{Aut}(\mathfrak{M})$, there exists a unique standard unitary operator $U_\sigma \in \mathcal{U}_s$ such that $\sigma = \tau_{U_\sigma}$. We call U_σ the standard implementation of σ.

Note that for any $\sigma \in \mathrm{Aut}(\mathfrak{M})$ and any $\omega \in N(\mathfrak{M})$, $\omega \circ \sigma$ is a normal state as well. We have two standard representatives $\psi_\omega, \psi_{\omega\circ\tau} \in C$. These are related by the standard implementation of σ via

$$U_\sigma^* \psi_\omega = \psi_{\omega\circ\sigma}$$

Example.[1] The $*$-automorphism group of $L^\infty(X,\mu)$ is given by

$$\mathrm{Aut}(L^\infty(X,\mu)) \cong \{\sigma : X \to X \ \mid \sigma \text{ is an automorphism and } \sigma_*\mu = \mu\}.$$

The corresponding standard implementation U_σ of such an automorphism is given by $U_\sigma : L^2(X,\mu) \to L^2(X,\mu), \psi \mapsto \psi \circ \sigma$. This is a unitary, as

$$\langle U_\sigma \phi, \psi\rangle = \int_X \overline{\phi(\sigma(x))}\, \psi(x)\ \mathrm{d}\mu = \overline{\phi(x)}\, \psi(\sigma^{-1}(x))\ \mathrm{d}\sigma_*\mu = \langle \phi, U_{\sigma^{-1}}\psi\rangle$$

We can also check that it is standard unitary, as

- precomposition with a measure preserving automorphism preserves functions that are almost everywhere non-negative.
- for any $f \in L^\infty(X,\mu), \psi \in L^2(X,\mu)$, we have $U_\sigma f U_\sigma^* \psi = f \circ \sigma \cdot \psi$, that is, $U_\sigma f U_\sigma^* = f \circ \sigma \in L^\infty(X,\mu)$.

Theorem 9 tells us now that every standard unitary on $L^2(X,\mu)$ takes this shape.

References

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