Quantum Dynamical Systems, Sec. 4.3-4.4

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4.3 Invariant States

Definition 1. *If τ^t is a group of $*$-autmorphisms of the C^*-algebra $\mathfrak{A}$, a state μ on $\mathfrak{A}$ is callte τ^t**-invariant** if $\mu \circ \tau^t = \mu$ for all $t \in \mathbb{R}$. The set of such invarianct sets is denoted as $E(\mathfrak{A}, \tau^t) \subset E(\mathfrak{A})$ the set of all states on $\mathfrak{A}$.*

Theorem 1. *Let τ^t be a group of $*$-autmorphisms of the C^*-algebra $\mathfrak{A}$. If there exist a state ω on $\mathfrak{A}$ such that the function $t \mapsto \omega(\tau^t(A))$ is continuous for all $A \in \mathfrak{A}$ then $E(\mathfrak{A}, \tau^t)$ is a non-empty, convex and weak-$*$-compact subset of $\mathfrak{A}^*$. In particular, this holds if $(\mathfrak{A}, \tau^t)$ is a C^*-dynamical system.*

Proof. To prove existance we want to construct a net that has a converging subnet with a τ^t-invariant limit.
So for all $A \in \mathfrak{A}$ consider the expresseion:

$$\omega_T(A) = \frac{1}{T} \int_0^T \omega \circ \tau^s(A) ds. \tag{1}$$

By assumption, the function $s \mapsto \omega(\tau^s(A))$ is continuous, thus the integral is well defined and we have $\omega_T \in E(\mathfrak{A})$ for all $T > 0$. Since $E(\mathfrak{A})$ is weak-$*$-compact, the net $(\omega_T)_{T>0}$ has a weak-$*$-convergent subnet. The formula

$$\omega_T(\tau^t(A)) = \frac{1}{T} \int_0^T \omega \circ \tau^s(\tau^t(A)) ds \tag{2}$$

$$= \frac{1}{T} \int_0^T \omega \circ \tau^{s+t}(A) ds \tag{3}$$

$$= \frac{1}{T} \int_t^{T+t} \omega \circ \tau^{s'}(A) ds' \tag{4}$$

$$= \omega_T(A) - \frac{1}{T} \int_0^t \omega \circ \tau^{s'}(A) ds' + \frac{1}{T} \int_T^{T+t} \omega \circ \tau^{s'}(A) ds' \tag{5}$$

is used for an estimate. Using this we can estimate

$$|\omega_T(\tau^t(A)) - \omega_T(A)| \leq 2\|A\| \frac{|t|}{T}, \tag{6}$$

from which it follows that the limit of an convergent subnet of $(\omega_T)T > 0$ is τ^t-invariant. Let μ, ω be states in $E(\mathfrak{A}, \tau)$ and $\lambda \in (0, 1)$. Then $\gamma(A) = \mu(A) + \lambda\omega(A)$ is a state and also τ^t-invariant since

$$\gamma(\tau^t(A)) = \mu(\tau^t(A)) + \lambda\omega(\tau^t(A)) = \mu(A) + \lambda\omega(A) = \gamma(A) \tag{7}$$

and weak-$*$-closedness of the set of invariant state is clear. □

Definition 2. *If τ^t is a group of $*$-autmorphisms of the von Neumann algebra $\mathfrak{M}$ we denote by $N(\mathfrak{M}, \tau^t) \equiv E(\mathfrak{M}, \tau^t) \cap N(\mathfrak{M})$ the set of normal τ^t-invariant states.*

Note that for a W^*-dynamical system the compactness argument used in this proof breaks down. There is no general existence result for *normal invariant states.* In fact

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Definition 3. *If $\mathfrak{C}$ is a C^*-algebra and τ^t a group of $*$-automorphisms of $\mathfrak{C}$ we define*

$$\mathcal{E}(\mathfrak{C}, \tau^t) \equiv \{\mu \in E(\mathfrak{C}, \tau^t) | t \mapsto \mu(A^*\tau^t(A)) \text{ is continuous for all } A \text{ in } \mathfrak{C}\} \tag{8}$$

If $\mu \in \mathcal{E}(\mathfrak{C}, \tau)$ we say that $(\mathfrak{C}, \tau^t, \mu)$ is a ***quantum dynamical system****.*

Example 1. *If $(\mathfrak{A}, \tau^t)$ is a C^*-dynamical system then $\mathcal{E}(\mathfrak{A}, \tau) = E(\mathfrak{A}, \tau)$ and $(\mathfrak{A}, \tau^t, \mu)$ is a quantum dynamical system for any τ^t-invariant state μ.*

Example 2. *If $(\mathfrak{M}, \tau^t)$ is a W^*-dynamical system then $N(\mathfrak{M}, \tau) \subset E(\mathfrak{M}, \tau)$ and $(\mathfrak{M}, \tau^t, \mu)$ is a quantum dynamical system for any τ^t-invariant state μ.*

Lemma 1. *Let $(\mathfrak{C}, \tau^t, \mu)$ be a quantum dynamical system and denote the GNS representation of $\mathfrak{C}$ associated to μ by $(\mathcal{H}_\mu, \pi_\mu, \Omega_\mu)$. Then there exists a unique self-adjoint operator L_μ on $\mathcal{H}_\mu$ such that*

1. $\pi_\mu(\tau^t(A)) = e^{itL_\mu}\pi_\mu e^{-itL_\mu}$ *for all $A \in \mathfrak{C}$ and $t \in \mathbb{R}$*
2. $L_\mu\Omega_\mu = 0$

Proof. For a fixed $t \in \mathbb{R}$ one easily checks that $(\mathcal{H}_\mu, \pi_\mu \circ \tau^t, \Omega_\mu)$ is a GNS representation of $\mathfrak{C}$ associated to μ. By unicity of the GNS construction there exists a unitary operator U_μ^t on $\mathcal{H}_\mu$ such that, for any $A \in \mathfrak{C}$, one has

$$U_\mu^t \pi_\mu(A)\Omega_\mu = \pi_\mu(\tau^t(A))\Omega_\mu \tag{9}$$

and in particular

$$U_\mu^t \Omega_\mu = \Omega_\mu \tag{10}$$

For $s, t \in \mathbb{R}$ we have

$$U_\mu^t U_\mu^s \pi_\mu(A)\Omega_\mu = U_\mu^t \pi_\mu(\tau^s(A))\Omega_\mu = \pi_\mu(\tau^{t+s}(A))\Omega_\mu = U_\mu^{t+s}\pi_\mu(A)\Omega_\mu, \tag{11}$$

and the cyclic property of Ω_μ yields that U_μ^t si a unitary group on $\mathcal{H}_\mu$. Using an earlier result one can show that U_μ^t is also strongly continuous. By Stone theorem $U_\mu^t = e^{itL_\mu}$ for some self-adjoint operator L_μ and property 2 follows from Equation (9). Finally for $A, B \in \mathfrak{C}$ we get

$$U_\mu^t \pi_\mu(A)\pi_\mu(B)\Omega_\mu = \pi_\mu(\tau^t(A))\pi_\mu(\tau^t(B))\Omega_\mu = \pi_\mu(\tau^t(A))U_\mu^t\pi_\mu(B)\Omega_\mu, \quad (12)$$

and property (1) follows from the cyclic property of Ω_μ. □

Definition 4. *Given a quantum dynamical system $(\mathfrak{C}, \tau^t, \mu)$, we denote by*

- *$(\mathcal{H}_\mu, \pi_\mu, \Omega_\mu)$ its GNS-representation*
- *$\mathfrak{C}_\mu = \pi_\mu(\mathfrak{C})''$ the enveloping von Neumann Algebra*
- *$(\pi_\mu, \mathfrak{C}_\mu, \mathcal{H}_\mu, L_\mu, \Omega_\mu)$ its Normal Form (which exists by the Lemma 1).*

Definition 5. *Two quantum dynamical systems $(\mathfrak{C}, \tau^t, \mu)$ and $(\mathfrak{D}, \sigma^t, \nu)$ are isomorphic if there exists a $*$-isomorphism $\phi : \mathfrak{C} \to \mathfrak{D}$ such that $\phi \circ \tau^t = \sigma^t \circ \phi$ for all $t \in \mathbb{R}$ and $\mu = \nu \circ \phi$.*

Definition 6. *Let ω, μ be states on $\mathfrak{C}$. μ is called ω-normal if $\mu = \tilde{\mu} \circ \pi_\omega$ for some $\mu \in N(\mathfrak{C}_\omega)$. The set of ω-normal states on $\mathfrak{C}$ is denoted by $N(\mathfrak{C}, \omega)$.*

Theorem 2. *(Simplified Version of Thm. 2.30 in the Book) Let ω, μ be states on $\mathfrak{C}$. Then $\mu \in N(\mathfrak{C}, \omega)$ if and only if there exists a σ-weakly continuous $*$-morphism $\pi_{\mu|\omega} : \mathfrak{C}_\omega \to \mathfrak{C}_\mu$ such that $\pi_\mu = \pi_{\mu|\omega} \circ \pi_\omega$. If this is the case, then there exists a subalgebra $z_{\mu|\omega}\mathfrak{C}_\omega \subseteq \mathfrak{C}_\omega$ such that the restriction of $\pi_{\mu|\omega}$ to $z_{\mu|\omega}\mathfrak{C}_\omega$ is a $*$-isomorphism.*

Proof. We only show the first part. Assume such a $*$-morphism exists. Write $\hat{\mu}(A) = \langle \Omega_\mu, A\Omega_\mu \rangle$ for the extension of μ to $\mathfrak{C}_\mu$. We get

$$\mu = \hat{\mu} \circ \pi_\mu = \hat{\mu} \circ \pi_{\mu|\omega} \circ \pi_\omega$$

Since $\tilde{\mu} := \hat{\mu} \circ \pi_{\mu|\omega}$ defines a normal state on $\mathfrak{C}_\omega$ we can conclude that $\mu \in N(\mathfrak{C}, \omega)$ is ω-normal.

For the other direction, assume we have $\mu = \tilde{\mu} \circ \pi_\omega$ for some $\mu \in N(\mathfrak{C}_\omega)$. Let $(\mathcal{K}, \Phi, \Psi)$ be the GNS-representation of $\mathfrak{C}_\omega$ corresponding to $\tilde{\mu}$. Then $(\mathcal{K}, \Phi \circ \pi_\omega, \Psi)$ is a GNS-representation of $\mathfrak{C}$ corresponding to μ. Indeed, we have

$$\mu(A) = \tilde{\mu}(\pi_\omega(A)) = \langle \Psi, \Phi(\pi_\omega(A))\Psi \rangle$$

for all $A \in \mathfrak{C}$ which, by density of $\pi_\omega(\mathfrak{C})$ in $\mathfrak{C}_\omega$ proves $\mu(A) = \langle \Psi, \Phi(A)\Psi \rangle$ for all $A \in \mathfrak{C}_\omega$. Furthermore, $\mathcal{K} = \overline{\Phi(\mathfrak{C}_\omega)\Psi} = \overline{\Phi \circ \pi_\omega(\mathfrak{C})\Psi}$ also since $\pi_\omega(\mathfrak{C})$ is dense in $\mathfrak{C}_\omega$. By the uniqueness of the GNS-representation, there exists a unitary map $U : \mathcal{K} \to \mathcal{H}_\mu$ such that

$$\pi_\mu(A) = U\Phi(\pi_\omega(A))U^*$$

for all $A \in \mathfrak{C}$. Thus, for $X \in \mathfrak{C}_\omega$ we can simply define

$$\pi_{\mu|\omega}(X) := U\Phi(X)U^*$$

and we get $\pi_\mu = \pi_{\mu|\omega} \circ \pi_\omega$. □

Lemma 2. *Let* $\omega \in \mathcal{E}(\mathfrak{C}, \tau^t)$ *and* $\mu \in N(\mathfrak{C}, \omega) \cap E(\mathfrak{C}, \tau^t)$. *Then we have* $\mu \in \mathcal{E}(\mathfrak{C}, \tau^t)$ *and, the map* $\pi_{\mu|\omega}$ *from Thm 2 is an isomorphism between the quantum dynamical systems* $(z_{\mu|\omega}\mathfrak{C}_\omega, \hat{\tau}^t_\omega, \tilde{\mu})$ *and* $(\mathfrak{C}_\mu, \hat{\tau}^t_\mu, \hat{\mu})$*, where* $\hat{\mu}$ *and* $\tilde{\mu}$ *are defined the same way as in the previous proof and* $\hat{\tau}^t_\mu$ *is defined as*

$$\hat{\tau}^t_\mu(A) := e^{itL_\mu} A e^{-itL_\mu}$$

for $A \in \mathfrak{C}_\mu$. $\hat{\tau}^t_\omega$ *is defined analogously.*

Proof. Notice that for all $A \in \mathfrak{C}$ we have

$$\mu(A^* \tau^t(A)) = \tilde{\mu}(\pi_\omega(A)^* \pi_\omega(\tau^t(A))) = \tilde{\mu}(\pi_\omega(A)^* e^{itL_\omega} \pi_\omega(A) e^{-itL_\omega})$$

which is continuous in t since $\tilde{\mu}$ is normal.

We still need to show that $\hat{\mu} \circ \pi_{\mu|\omega} = \tilde{\mu}$ and that $\hat{\tau}^t_\mu \circ \pi_{\mu|\omega} = \pi_{\mu|\omega} \circ \hat{\tau}^t_\omega$. By density, it suffices to show that these equalities holds on $\pi_\omega(\mathfrak{C})$. We have

$$\hat{\mu} \circ \pi_{\mu|\omega} \circ \pi_\omega(A) = \hat{\mu}(\pi_\mu(A)) = \mu(A) = \tilde{\mu} \circ \pi_\omega(A)$$

where we have used (in this order) Thm 2, a property of the GNS representation and the definition of $\tilde{\mu}$. For the other equation, Thm 2 implies

$$\hat{\tau}^t_\mu \circ \pi_{\mu|\omega}(\pi_\omega(A)) = \hat{\tau}^t_\mu \circ \pi_\mu(A)$$

and the first property from Thm 1 gives

$$\hat{\tau}^t_\mu \circ \pi_\mu(A) = \pi_\mu \circ \tau^t(A)$$

Applying these two facts again, we get

$$\pi_\mu \circ \tau^t(A) = \pi_{\mu|\omega} \circ \pi_\omega \circ \tau^t(A) = \pi_{\mu|\omega} \circ \hat{\tau}^t_\omega \circ \pi_\omega(A)$$

□

References

[1] Stéphane Attal, Alain Joye and Claude-Alain Pillet. *Open Quantum Systems I - The Hamiltonian Approach* Springer, 2006.