Spectral Analysis

Puthuparambil Sharon, Okuyama Chihiro

1 Spectral analysis

Let $\mathcal{A}$ be a C^*-algebra with unit I.

Definition 1.1. *One calls **resolvant** set of A the set*

$$\rho(A) = \{\lambda \in \mathbb{C};\ \lambda I - A \text{ is invertible}\}.$$

We put

$$\sigma(A) = \mathbb{C} \backslash \rho(A)$$

*and call it the **spectrum** of A.*

If $|\lambda| > \|A\|$ then the series

$$\frac{1}{\lambda} \sum_n \left(\frac{A}{\lambda}\right)^n$$

is normally convergent and its sum is equal to $(\lambda I - A)^{-1}$ (this is the well known Neumann series). This implies that $\sigma(A)$ is included in $B(0, \|A\|)$.

Furthermore, if λ_0 belongs to $\rho(A)$ and if $\lambda \in \mathbb{C}$ is such that $|\lambda - \lambda_0| < \|\lambda_0 I - A\|$, then the series

$$(\lambda_0 I - A)^{-1} \sum_n \left(\frac{\lambda_0 - \lambda}{\lambda_0 I - A}\right)^n$$

normally converges to $(\lambda I - A)^{-1}$. In particular we have proved that:

1. the set $\rho(A)$ is open (the second point makes sure the existence of an open ball around $\lambda_0 \in \rho(A)$, in which $\lambda I - A$ is invertible).
2. the mapping $\lambda \mapsto (\lambda I - A)^{-1}$ is analytic on $\rho(A)$.
3. the set $\sigma(A)$ is compact (as it is a closed and bounded set).

Definition 1.2. *We define*

$$r(A) = \sup\{|\lambda|; \lambda \in \sigma(A)\},$$

*the **spectral radius** of A.*

Theorem 1.3. *We have for all $A \in \mathcal{A}$*

$$r(A) = \lim_n \|A^n\|^{1/n} = \inf_n \|A^n\|^{1/n} \leq \|A\|.$$

In particular the above limit always exists and $\sigma(A)$ is never empty.

Corollary 1.4. *A C^*-algebra $\mathcal{A}$ with unit and all of which elements, except 0, are invertible is isomorphic to $\mathbb{C}$.*

All the above results made use of the fact that we considered a C^*-algebra with unit. If $\mathcal{A}$ is a C^*-algebra without unit and if $\widetilde{\mathcal{A}}$ is its natural exetension with unit, then the notion of spectrum and resolvent set are extended as follows. The spectrum of $A \in \mathcal{A}$ is its spectrum as an element of $\widetilde{\mathcal{A}}$. We extend the notion of resolvent set in the same way.

Definition 1.5. *An element A of a C^*-algebra $\mathcal{A}$ is*

- ***normal*** *if $A^*A = AA^*$,*
- ***self-adjoint*** *if $A = A^*$.*

If $\mathcal{A}$ contains a unit, then an element $A \in \mathcal{A}$ is

- ***isometric*** *if $A^*A = I$,*
- ***unitary*** *if $A^*A = AA^* = I$.*

Theorem 1.6. *Let $\mathcal{A}$ be a C^*-algebra with unit.*

a) If A is normal then $r(A) = \|A\|$.

b) If A is self-adjoint then $\sigma(A) \subset [-\|A\|, \|A\|]$.

c) If A is isometric then $r(A) = 1$.

d) If A is unitary then $\sigma(A) \subset \{\lambda \in \mathbb{C}; |\lambda| = 1\}$.

e) For all $A \in \mathcal{A}$ we have $\sigma(A^*) = \overline{\sigma(A)}$ and $\sigma\left(A^{-1}\right) = \sigma(A)^{-1}$.

f) For every polynomial function P we have

$$\sigma(P(A)) = P(\sigma(A)).$$

g) For any two $A, B \in \mathcal{A}$ we have

$$\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}.$$

Theorem 1.7. *The norm which makes a *-algebra being a C^*-algebra, when it exists, is unique.*

Proposition 1.8. *The set of invertible elements of a C^*-algebra $\mathcal{A}$ with unit is open and the mapping $A \mapsto A^{-1}$ is continuous on this set.*

Proof. If A is invertible and if B is such that $\|B - A\| < \left\|A^{-1}\right\|^{-1}$ then $B = A\left(I - A^{-1}(A - B)\right)$ is invertible for

$$r\left(A^{-1}(A - B)\right) \leq \left\|A^{-1}(A - B)\right\| < 1$$

and thus $I - A^{-1}(A-B)$ is invertible. The open character is proved. Let us now show the continuity. If $\|B-A\| < 1/2 \left\|A^{-1}\right\|^{-1}$ then

$$\begin{aligned}
\left\|B^{-1} - A^{-1}\right\| &= \left\| \sum_{n=0}^{\infty} \left(A^{-1}(A-B)\right)^n A^{-1} - A^{-1} \right\| \\
&\leq \sum_{n=1}^{\infty} \left\|A^{-1}(A-B)\right\|^n \left\|A^{-1}\right\| \\
&\leq \frac{\left\|A^{-1}\right\|^2 \|A-B\|}{1 - \|A^{-1}(A-B)\|} \\
&\leq 2\left\|A^{-1}\right\|^2 \|A-B\|
\end{aligned}$$

This proves the continuity. □

In the following, we denote by $\mathbb{1}$ the constant function equal to 1 on $\mathbb{C}$ and by id_E the function $\lambda \mapsto \lambda$ on $E \subset \mathbb{C}$.

A $*$-algebra morphism is a linear mapping $\Pi : \mathcal{A} \to \mathcal{B}$, between two $*$ algebras $\mathcal{A}$ and $\mathcal{B}$, such that $\Pi(A^*B) = \Pi(A)^*\Pi(B)$ for all $A, B \in \mathcal{A}$. A C^*-algebra morphism is *-algebra morphism Π between two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, such that $\|\Pi(A)\|_{\mathcal{B}} = \|A\|_{\mathcal{A}}$, for all $A \in \mathcal{A}$.

Theorem 1.9 (Functional calculus)**.** *Let $\mathcal{A}$ be a C^*-algebra with unit. Let A be a self-adjoint element in $\mathcal{A}$. Let $C(\sigma(A))$ be the C^*-algebra of continuous functions on $\sigma(A)$. Then there is a unique morphism of C^*-algebra*

$$\begin{aligned}
C(\sigma(A)) &\longrightarrow \mathcal{A} \\
f &\longmapsto f(A)
\end{aligned}$$

which sends the function $\mathbb{1}$ on I and the function $id_{\sigma(A)}$ on A.

Furthermore we have

$$\sigma(f(A)) = f(\sigma(A))$$

for all $f \in C(\sigma(A))$.

An element A of a C^*-algebra $\mathcal{A}$ is positive if it is self-adjoint and its spectrum is included in $\mathbb{R}^+$.

Theorem 1.10. *Let A be an element of $\mathcal{A}$. The following assertions are equivalent.*

1. *A is positive.*
2. *(if $\mathcal{A}$ contains a unit) A is self-adjoint and $\|tI - A\| \leq t$ for some $t \geq \|A\|$. $\|A\|$.*
3. *(if $\mathcal{A}$ contains a unit) A is self-adjoint and $\|tI - A\| \leq t$ for all $t \geq$*
4. *$A = B^*B$ for a $B \in \mathcal{A}$.*
5. *$A = C^2$ for a self-adjoint $C \in \mathcal{A}$.*

This notion of positivity defines an order on elements of $\mathcal{A}$, by saying that $U \geq V$ in $\mathcal{A}$ if $U - V$ is a positive element of $\mathcal{A}$.

Proposition 1.11. *Let U, V be self-adjoint elements of $\mathcal{A}$ such that $U \geq V \geq 0$. Then*

1. *$W^*UW \geq W^*VW \geq 0$ for all $W \in \mathcal{A}$;*
2. *$(V + \lambda I)^{-1} \geq (U + \lambda I)^{-1}$ for all $\lambda \geq 0$.*