October 8, 2023

Seminar Operator Algebra

2.3 Representations
We note that *-Algebra morphisms are indeed positive, since $\Pi(A^*A) = \Pi(A)^*\Pi(A)$.

Theorem 2.10
Let $\mathcal{A}, \mathcal{B}$ be a C^*-Algebra morphism. Then:

1. Π is continuous.
2. $ran(\Pi) \subset \mathcal{B}$ is a sub C^*-Algebra.

proof

1. In the first part we first show the statement for selfadjoint operators. Then, using the trick $\|A^*A\| = \|A\|^2$ twice, the result for general $A \in \mathcal{A}$ follows.
2. For the second part we assume WLOG $ker(\Pi) = \{0\}$. Else we can look at the quotient C^*-Algebra where we divide through the kernel of Π. Since Π is now injective, there exists an inverse map Π^{-1} from $ran(\Pi)$ to $\mathcal{P}\rangle$. This map is also a C^*-Algebra morphism, requiring $\|A\| = \|\Pi(A)\|$ (using (a) on Π^{-1}). But this type of inequality is typical in Functional Analysis and immediately implies that $ran(\Pi)$ is closed and thus complete. The other properties all hold trivially or transfer easily. Thus, we can conclude that $ran(\Pi)$ is a sub C^*-Algebra.

□

Def A *representation* of a C^*-Algebra $\mathcal{A}$ is a pair of (H, Π) with Hilbertspace H and C^*-Algebra morphism $\Pi : \mathcal{A} \longrightarrow \mathcal{B}(H)$, where $\mathcal{B}(H)$ is the set of all bounded linear operators on the Hilbert space H. A representation is *faithful*, if $ker(\Pi) = \{0\}$.

Proposition 2.11
Let (H, Π) be a representation of a C^*-Algebra $\mathcal{A}$. Then TFAE:

1. Π is faithful.
2. $\|\Pi(A)\| = \|A\|$
3. $A > 0 \Longrightarrow \Pi(A) > 0$

proof
The argument from (i) to (ii) is Theorem 2.10. One then can prove directly (ii) implies (iii). Finally, to show (iii) implies (i) one argues by contradiction.

□

Def A linear form w on a C^*-Algebra $\mathcal{A}$ is called *positive* if $w(A^*A) \geq 0$ for all $A \in \mathcal{A}$.

Rmk This gives us a Cauchy-Schwartz like inequality: $|w(B^*A)| \leq |w(B^*B)| \cdot |w(A^*A)|$

Proposition 2.12
Let w be a Linear form on $(\mathcal{A}, I)$, i.e. a C^*-Algebra with unit. Then TFAE:

1. w is positive.
2. w is continuous.

Def A linear form w on a C^*-Algebra $\mathcal{A}$ is called a *state* if $\|w\| = 1$.

Theorem 2.13 [Existence of states]
Let $A \in \mathcal{A}$. Then there exists a state w on $\mathcal{A}$ such that $w(A^*A) = \|A\|^2$.

proof
Let $\mathcal{B} := \{\alpha \cdot I + \beta \cdot A^*A | \ \alpha, \beta \in \mathbb{C}\}$. We set

$$f(\alpha \cdot I + \beta \cdot A^*A) := \alpha + \beta \cdot \|A\|^2$$

Using that A^*A is selfadjoint, we find the following bound:

$$|f(\alpha \cdot I + \beta \cdot A^*A)| \leq \|\alpha \cdot I + \beta \cdot A^*A\|, \text{ i.e. } \|f\| \leq 1$$

But setting $\alpha = 1, \beta = 0$ gives us $f(I) = 1$ which implies $\|f\| \geq 1$, thus $\|f\| = 1$.

So far f defines a state on the C^*-Algebra $\mathcal{B}$. Using the Hahn-Banach Theorem, we can extend the linear form f to a linear form w on the C^*-Algebra $\mathcal{A}$ with $w(B) = f(B)$ for all $B \in \mathcal{B}$. As a consequence of the Hahn-Banach Theorem, we also get $\|w\| = 1$, thus w is a state on $\mathcal{A}$. Indeed by construction it holds that:

$$w(A^*A) = f(A^*A) = \|A\|^2$$

□