Von Neumann Algebras

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The following handout is a summary of subsection 3.1 and 3.2 of Stéphane Attal's notes "Elements of Operator Algebras and Modular Theory". We closely follow the original text, and use the same numbering.

3.1 Topologies on $\mathcal{B}(\mathcal{H})$

Let $\mathcal{B}(\mathcal{H})$ be the C*-algebra of bounded linear operators over a complex Hilbert space $\mathcal{H}$. We start by defining a von Neumann algebra.

Definition. *A* ***von Neumann algebra*** *(or* ***W*-algebra****) is a weakly closed C*-algebra acting on $\mathcal{H}$ that contains the identity $\mathrm{I} \in \mathcal{B}(\mathcal{H})$.*

To understand this definition in full detail, we define the different topologies on $\mathcal{B}(\mathcal{H})$.

Definition. *The* ***strong topology*** *on $\mathcal{B}(\mathcal{H})$ is the locally convex topology defined by the family of semi norms $\mathrm{P_x}(\mathrm{A}) = \|Ax\|$ where $x \in \mathcal{H}$.*

Definition. *The* ***weak topology*** *on $\mathcal{B}(\mathcal{H})$ is the locally convex topology defined by the family of semi norms $\mathrm{P_{x,y}}(\mathrm{A}) = |\langle x, Ay\rangle|$ where $x, y \in \mathcal{H}$.*

The obvious first example of a von Neumann algebra is $\mathcal{B}(\mathcal{H})$ itself.

Proposition 3.1.

1. *The weak topology is weaker than the strong topology which is itself weaker than the uniform topology. Once $\mathcal{H}$ is infinite dimensional then these comparisons are strict.*
2. *A linear form $\varphi \in \mathcal{B}_*(\mathcal{H})$ is strongly continuous $\Leftrightarrow$ it is weakly continuous.*
3. *The strong and the weak closure of any convex subset of $\mathcal{B}(\mathcal{H})$ coincide.*

Proof. This is just a sketch of the proof.

1. Let $\mathcal{H}$ be an infinite dimensional, separable Hilbert space with an ONB $(e_n)_n$.
   Consider the orthogonal projection $\mathrm{P_n} : \mathcal{H} \to \mathrm{span}(e_1, \dots e_n)$. Then we have that $\forall x \in \mathcal{H}$
   $$\lim_{n\to\infty} \|\,\mathrm{P}(x) - x\| = 0$$

Therefore we have strong but not uniform convergence to the identity operator.
Consider the unilateral shift $\mathrm{S} : \mathcal{H} \to \mathcal{H}, e_i \mapsto e_{i+1}$ and $x = (1, 1, \dots) \in \mathcal{H}$. Then we have $\forall y \in \mathcal{H}$

$$\lim_{k\to\infty} \langle y, \mathrm{S}^{\mathrm{k}}(x)\rangle = 0$$

where one can define S^{k} inductively. On the other hand we have that $\|Sx\| = \|x\|$ therefore

$$\lim_{k\to\infty} \|S^k(X)\| \ = \infty.$$

We just found an example that converges w.r.t. the weak topology but not the strong.

2. See Stéphane Attal's notes.

3. Assuming 2 and with the geometric form of Hahn-Banach we get the desired result.

□

Note that this proposition allows us to define a von Neumann algebra as a strongly closed C*-algebra instead of weakly closed.

Definition. *The **σ-weak topology** on $\mathcal{B}(\mathcal{H})$ is the locally convex topology defined by the family of semi norms* $\mathrm{P}_{(\mathrm{x_n})_\mathrm{n},(\mathrm{y_n})_\mathrm{n}}(\mathrm{A}) = \sum_n |\langle x_n, Ay_n\rangle|$ *where* $(x_n)_n, (y_n)_n \subset \mathcal{H}$ *and* $\sum_n |x_n|^2 < \infty, \sum_n |y_n|^2 < \infty$.

To understand the following theorem better, we introduce an alternative definition of the mathematician Shoichiro Sakai for a von Neumann algebra. Note that this definition doesn't rely on an underlying space.

Definition. *A **von Neumann algebra** is a C*-algebra $\mathcal{A}$ if it is a dual space as a Banach space. ($\exists$ a Banach space $\mathcal{M}$ s.t. $\mathcal{M}^* = \mathcal{A}$)*

Definition. $\mathcal{T}(\mathcal{H})$*:= The Banach space of the trace class operators of* $\mathcal{H}$ *equipped with the norm* $\|H\|_1 = \mathrm{tr}(|H|)$ *where* $|H| = \sqrt{H * H}$.

Theorem 3.2. *$\mathcal{B}(\mathcal{H})$ is the topological dual of $\mathcal{T}(\mathcal{H})$ thanks to the duality* $(A, T) \mapsto \mathrm{tr}(AT) \quad A \in \mathcal{B}(\mathcal{H}), T \in \mathcal{T}(\mathcal{H})$. *Moreover, the ∗-weak topology on $\mathcal{B}(\mathcal{H})$ associated to this duality is the σ-weak topology.*

Proof. I'm giving just the idea here, for further details see the references.
For the first part we need to show both inclusions, the more challenging part hereby is to show that any element $\omega \in \mathcal{T}(\mathcal{H})$ coincides with the linear form $\mathrm{tr}(A\cdot)$. For this we first check the restriction of ω to rank one operators and find with Riesz theorem that they coincide, we conclude by the density of rank one operators in $\mathcal{T}(\mathcal{H})$.

For the second part one notices that every trace class operator T can be written as

$$\mathrm{T} = \sum_n \lambda_n \; |\xi_n\rangle\langle\nu_n|$$

where $(\xi_n)_n$ and $(\mu_n)_n$ are orthonormal systems and the sequence of complex numbers $(\lambda_n)_n$ is absolutely summable. Therefore

$$\mathrm{tr}(AT) = \sum_n \lambda_n \langle \nu_n, \; A\xi_n$$

and thus the semi norms $\mathrm{P}_\mathrm{T} = |\,\mathrm{tr}(AT)|$ are as in the definition of the σ-weak topology.

□

Corollary 3.3. *Every σ-weakly continuous linear form on $\mathcal{B}(\mathcal{H})$ is of the form $A \mapsto \mathrm{tr}(AT)$.*

3.2 Commutant

We start this section with another example of a von Neumann algebra. Let (X, μ) be a locally compact measure space with σ-finite measure μ. Let $\mathcal{H} = L^2(X, \mu)$, and consider $L^\infty(X, \mu)$ acting on $\mathcal{H}$ by multiplication, i.e. we associate $f \in L^\infty(X, \mu)$ with $m_f : u \in \mathcal{H} \mapsto f \cdot u \in \mathcal{H}$. Then, $L^\infty(X, \mu)$ is a von Neumann algebra. This can be shown with the help of the Bicommutant Theorem, which we are going to prove in the following section.

Definition. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a subset. We define the* ***commutant*** *of $\mathcal{M}$ by*

$$\mathcal{M}' = \{B \in \mathcal{B}(\mathcal{H}) \mid BM = MB, \; \forall M \in \mathcal{M}\}.$$

Inductively we also define $\mathcal{M}^{(n)} = (\mathcal{M}^{(n-1)})'$ for $n \geq 1$, where $\mathcal{M}^{(0)} = \mathcal{M}$.

Proposition 3.4. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a subset. Then,*

1. *$\mathcal{M}'$ is weakly closed, and*
2. *$\mathcal{M}' = \mathcal{M}^{(2k+1)}$ and $\mathcal{M} \subseteq \mathcal{M}'' = \mathcal{M}^{(2+2k)}$ for all $k \geq 1$.*

Proof. 1. Let $(A_i)_{n \in I} \subseteq \mathcal{M}'$ be a net with $A_i \to A \in \mathcal{B}(\mathcal{H})$ with respect to the weak operator topology. Let $B \in \mathcal{M}$ and $x, y \in \mathcal{H}$. Then,

$$\begin{aligned}
\langle (AB - BA)x, y\rangle &= \langle ABx, y\rangle - \langle BAx, y\rangle \\
&= \lim_{i \in I} (\langle A_i Bx, y\rangle - \langle A_i x, B^* y\rangle) \\
&= \lim_{i \in I} (\langle BA_i x, y\rangle - \langle BA_i x, y\rangle) = 0.
\end{aligned}$$

Since x, y were arbitrary, $AB = BA$, and the first statement follows.

2. Let $A \in \mathcal{M}$. Then, for all $B \in \mathcal{M}'$, we have $AB = BA$. Thus, $A \in \mathcal{M}''$, and $\mathcal{M} \subseteq \mathcal{M}''$. The same argument for $\mathcal{M}'$ gives $\mathcal{M}' \subseteq \mathcal{M}'''$. Note that $\mathcal{M}_1 \subseteq \mathcal{M}_2$ implies that $\mathcal{M}_2' \subseteq \mathcal{M}_1'$. So $\mathcal{M}''' \subseteq \mathcal{M}'$ again by the first argument. Now the statement follows by induction.

□

Proposition 3.5. *Let $\mathcal{M}$ be a self-adjoint subset of $\mathcal{B}(\mathcal{H})$, i.e. $M^* \in \mathcal{M}$, $\forall M \in \mathcal{M}$. Let $\mathcal{E} \subseteq \mathcal{H}$ be a closed subspace and P be the orthogonal projection onto $\mathcal{E}$. Then,*

$$M(\mathcal{E}) \subseteq \mathcal{E},\ \forall M \in \mathcal{M} \iff P \in \mathcal{M}'.$$

Proof. First, assume that $P \in \mathcal{M}'$. Then,

$$PM(\mathcal{E}) = MP(\mathcal{E}) = M(\mathcal{E}),\ \forall M \in \mathcal{M},$$

which implies that $M(\mathcal{E}) \subseteq \mathcal{E}$ for all $M \in \mathcal{M}$.
Conversely, assume that $\mathcal{E}$ is invariant under $\mathcal{M}$. This is equivalent to $MP = PMP$ for all $M \in \mathcal{M}$. Taking the adjoint gives

$$PM^* = PM^*P,\ \forall M \in \mathcal{M},$$

where we used that P is self-adjoint. Since $\mathcal{M}$ is self-adjoint, we get that $MP = PM$ for all $M \in \mathcal{M}$, and $P \in \mathcal{M}'$. □

Theorem 3.6 (Von Neumann Density Theorem)**.** *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a sub-*-algebra which contains the identity I. Then $\mathcal{M}$ is dense in $\mathcal{M}''$ with respect to the weak (strong) topology.*

Proof. By proposition 3.1, the weak and strong closures of $\mathcal{M}$ agree. We show density with respect to the weak operator topology. Let $B \in \mathcal{M}''$. Let $\varepsilon > 0$, $x_1, \ldots, x_n \in \mathcal{H}$, and

$$V = V(B; x_1, \ldots, x_n; \varepsilon) = \{A \in \mathcal{B}(\mathcal{H}) \mid \|(B - A)x_j\| < \varepsilon,\ 1 \le j \le n\}$$

a neighborhood around B.
Define the Hilbert space $\widetilde{\mathcal{H}} = \bigoplus_{j=1}^n \mathcal{H}$, and $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\widetilde{\mathcal{H}})$ by $\pi(A) = \oplus_{j=1}^n A$. Set $x = (x_1, \ldots, x_n) \in \widetilde{\mathcal{H}}$ and $\mathcal{E} = \overline{\pi(\mathcal{M})x} \subseteq \widetilde{\mathcal{H}}$, where $\pi(\mathcal{M})x = \{\pi(A)x \mid A \in \mathcal{M}\}$. Let P be the orthogonal projection onto $\mathcal{E}$. Let $z \in \mathcal{E}$, i.e. $z = \lim_{k\to\infty} \pi(C_k)x$, $C_k \in \mathcal{M}$. Then,

$$\pi(A)z = \pi(A)\left(\lim_{k\to\infty} \pi(C_k)x\right) = \lim_{k\to\infty} \pi(AC_k)x \in \mathcal{E}$$

for all $A \in \mathcal{M}$. Hence, $P \in \pi(\mathcal{M})'$ by proposition 3.5.
Identify $\mathcal{B}(\widetilde{\mathcal{H}})$ with $M_n(\mathcal{B}(\mathcal{H}))$, the set of operator valued $(n \times n)$-matrices. We want to show that $\pi(\mathcal{M})' = M_n(\mathcal{M}')$. So, let $A \in \pi(\mathcal{M})'$ and identify it with $\underline{A} = (A_{ij})_{i,j=1,\ldots,n} \in M_n(\mathcal{B}(\mathcal{H}))$. Then, for $C \in \mathcal{M}$, $\underline{A}\pi(C) = \pi(C)\underline{A}$, i.e.

$$\sum_{j=1}^n A_{ij} C(z_j) = C\left(\sum_{j=1}^n A_{ij}(z_j)\right), \forall 1 \le i \le n,\ \forall z = (z_1, \ldots, z_n) \in \widetilde{\mathcal{H}}.$$

Thus, $A_{ij}C = CA_{ij}$, $\forall 1 \le i, j \le n$, and since C was arbitrary, $\underline{A} \in M_n(\mathcal{M}')$. For $\underline{A} \in M_n(\mathcal{M}')$, we have $\underline{A}\pi(C) = \pi(C)\underline{A}$ for all $C \in \mathcal{M}$ such that $\underline{A} \in \pi(\mathcal{M})'$. A similar argument shows that $\pi(\mathcal{M}'') \subseteq M_n(\mathcal{M}')'$.
It follows that $\pi(B) \in \pi(\mathcal{M})''$, and in particular, $P\pi(B) = \pi(B)P$ such that $\mathcal{E}$ is invariant under $\pi(B)$. Since $I \in \mathcal{M}$, we have $\pi(I)x \in \mathcal{E}$, and

$$\pi(B)(\pi(I)x) = \begin{pmatrix} Bx_1 \\ \vdots \\ Bx_n \end{pmatrix} \in \mathcal{E}.$$

Remember that $\mathcal{E} = \overline{\mathcal{M}x}$. Hence, there exists $A \in \mathcal{M}$ such that $\|Bx_j - Ax_j\| < \varepsilon$ for $1 \le j \le n$, and $A \in V$. $\square$

Corollary 3.7 (Bicommutant Theorem)**.** *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a sub-*-algebra with $I \in \mathcal{M}$. Then,*

$$\mathcal{M} \textit{ is weakly (strongly) closed} \iff \mathcal{M} = \mathcal{M}''.$$

Note that the identity I is always an element of $\mathcal{M}''$ such that we have the following characterisation for von Neumann algebras: A C*-algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if $\mathcal{M} = \mathcal{M}''$.