Operator Algebras and Quantum Information Theory, Sec. 3.3 and 4.1

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October 2023

1 3.3 Preduals, normal states

Definition 1. Let $\mathcal{M}$ be a von Neumann algebra. Define $\mathcal{M}_1 = \{M \in \mathcal{M} : ||M|| \leq 1\}$

$\mathcal{M}_1$ is a weakly compact subset of $\mathcal{B}(\mathcal{H})$ which is weakly compact by Banach-Alaoglu. Hence on $\mathcal{M}_1$ the weak and σ-weak topology coincide. A proof can be found here `https://almostsuremath.com/2020/01/04/operator-topologies/`.

Definition 2. Define M_* as the space of all weakly continuous linear forms on $\mathcal{M}$ which are continuous on $\mathcal{M}_1$.

One can show that for all elements $\Psi \in \mathcal{M}_*$, the image of $\mathcal{M}_1$ is a compact subset in $\mathbb{C}$ which implies the norm continuity of Ψ. Thus $\mathcal{M}_* \subset \mathcal{M}^*$, the topological dual of $\mathcal{M}$.

Proposition 1.

1. *$\mathcal{M}_*$ is a closed subset of $\mathcal{M}^*$*
2. *$\mathcal{M}$ is the dual of $\mathcal{M}_*$*

Proof. Idea: For the first part we show that for any converging sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{M}_*$, for which a limit $f \in \mathcal{M}^*$ exists, $f \in \mathcal{M}_*$. To show $f \in \mathcal{M}_*$ it is sufficient to prove, that f is weakly continuous on $\mathcal{M}_1$. Choose a weakly convergent sequence $(A_n)_{n\in\mathbb{N}}$ and show by using the triangle inequality that $|f(A_n) - f(A)| = 0 \quad (n \to \infty)$
For the second statement remember that any surjective linear isometry on a linear Banach space is an isomorphism.
First we show that the inclusion map

$$\iota : \mathcal{M} \to (\mathcal{M}_*)^* \quad A \mapsto A = (\omega \mapsto \omega(A))$$

is a linear isometry. Define the norm of A in the dual space as

$$||A||_{du} = \sup_{\substack{||w||=1 \\ \omega\in\mathcal{M}_*}} |\omega(A)|$$

Note: This is just the natural operator norm.
One shows now that those norms are equal. Hence ι is a linear isometry.
For the surjectivity one choose $\phi \in (M_*)^*$ and $\phi' : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, $(x, y) \mapsto \phi(\omega_{x,y}|_{\mathcal{M}})$. Where $\omega_{x,y} : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$, $A \mapsto \langle y, Ax, \rangle$ By the Riesz representation theorem $\exists A \in \mathcal{B}(\mathcal{H})$ s.t. $\phi'(x, y) = \langle y, Ax \rangle \; \forall x, y \in \mathcal{H}$
Next one shows that $A \in \mathcal{M}'' = \mathcal{M}$, which implies that $\iota(A) = \phi'$ for the A given by Riesz, and that $\iota(A)$ coincides with ϕ on all $\omega \in \mathcal{M}_*$ in of the form $\omega = \omega_{x,y}$ for some $x, y \in \mathcal{H}$.
$\forall \omega \in \mathcal{M}_*$ we can write $\omega = tr(\rho \, \cdot)$. Using this we show

$$\omega = \sum_{n \in \mathbb{N}} \lambda_n \omega_{x_n, y_n}$$

Since $\iota(A)$ coincides with ϕ on all ω_{x_n,y_n} they are the same. □

Two examples where given :

1. $\mathcal{M} = \mathcal{B}\mathcal{H} \Rightarrow \mathcal{M}_* = \mathcal{T}(\mathcal{H})$
2. $\mathcal{M} = L^\infty(X, \mu) \Rightarrow \mathcal{M}_* = L^1(X, \mu)$

Theorem 1. ***Sakai Theorem:*** *A C^*- Algebra is a von Neumann algebra if and only if it is the dual of some Banach space.*

Definition 3. A state on a von Neumann algebra is called normal if it is σ-weakly continuous.

Theorem 2. *On a von Neumann algebra $\mathcal{M}$ and a state ω the following are equivalent:*

1. *ω is normal*
2. *$\exists \rho > 0$, $\rho \in \mathcal{T}(\mathcal{H})$ s.t. $tr(\rho) = 1$ and $\omega(A) = tr(\rho A) \; \forall A \in \mathcal{M}$*

2 4.1 The modular operators

We have a pair $(\mathcal{H}, \omega)$, where M is a von Neumann algebra acting on some Hilbert space and ω a normal faithful state on $\mathcal{H}$.

Definition 4. ω is faithful on $\mathcal{H}$ if $\forall x \in M$, $\omega(x^*x) = 0 \implies x = 0$.

We know consider the GNS (Gelfand-Naimard-Segal) representation of $(\mathcal{H}, \omega)$.

Definition 5. The GNS (Gelfand-Naimard-Segal) representation of $(\mathcal{M}, \omega)$ is the triple $(\mathcal{H}, \Pi, \Omega)$ with:

1. Π is a morphism from $\mathcal{H}$ to $\mathcal{B}(\mathcal{H})$
2. $\omega(A) =< \Omega, \Pi(A)\Omega >$
3. $\Pi(\mathcal{M})\Omega$ is dense in $\mathcal{H}$.

Notation: We identify $\mathcal{M}$ and $\mathcal{M}'$ with $\Pi(\mathcal{M})$ and $\Pi(\mathcal{M}')$. This implies that $\omega(A) =< \Omega, A\Omega >$.

Proposition 2. *The vector Ω is cyclic and separating for $\mathcal{M}$ and $\mathcal{M}'$*

A quick reminder,

- Ω is cyclic for $\mathcal{M}$ if Ω, $\mathcal{M}\Omega$, $\mathcal{M}^2\Omega$, ... span $\mathcal{H}$. Or equivalently, that

$$\mathcal{M}\Omega = \{A\mathcal{M} : A \in \mathcal{M}\} \text{ is norm dense in } \mathcal{H}$$

- Ω is separating for $\mathcal{M}$ if $\forall A \in \mathcal{M}$ such that $A\Omega = 0$ then $A = 0$

Proof. Let us first prove it for $\mathcal{M}$:

- Cyclic: As by definition, we have $\mathcal{M}\Omega$ is dense in $\mathcal{H}$ so Ω is cyclic for $\mathcal{M}$.
- Separating: If $A \in \mathcal{M}$ is such that $A\Omega = 0$ then $\omega(A^*A) =< \Omega, A^*A\Omega >=$ 0 but as ω is faithful, this implies that $A = 0$.

Now we prove that it also holds on $\mathcal{M}'$:

- Separating: If $A' \in \mathcal{M}'$ and $A'\Omega = 0$ then, using that A' is in the commutant:

$$A'B\Omega = BA'\Omega = 0 \forall B \in \mathcal{M}$$

  Thus A' vanishes on a dense subspace of $\mathcal{H}$ which implies that $A' = 0$. Thu Ω is separating for $\mathcal{M}'$.

- Cyclic: Let P be the projection on $\mathcal{M}\Omega$. Then $P \in \mathcal{M}'$ and $(I - P)\Omega = 0$ as $PI\Omega = I\Omega = \Omega$ with $I \in \mathcal{M}$ the identity. Hence $I - P$ as Ω is separating in $\mathcal{M}$ and thus Ω is cyclic for $\mathcal{M}'$ because $P = I$ implies that $\mathcal{M}'\Omega$ is dense.

□

Definition 6. We define the operators (which are anti-linear):

$$\begin{aligned} S_0 : \mathcal{M}\Omega &\to \mathcal{M}\Omega \\ A\Omega &\to A^*\Omega \end{aligned}$$

$$\begin{aligned} F_0 : \mathcal{M}'\Omega &\to \mathcal{M}'\Omega \\ B\Omega &\to B^*\Omega \end{aligned}$$

Proposition 3. *The operator S_0 and F_0 are closable and $\overline{S_0} = F_0^*$, $\overline{F_0} = S_0^*$.*

We know put $S = \overline{S_0} = F_0^*$ and $F = \overline{F_0} = S_0^*$.

Theorem 3. *We have $S = S^{-1}$.*

Proof. Let $z \in DomS*$. We have:

$$\begin{aligned} < S_0 A\Omega, S^* z > &=< A^* z, S_0^* z > \text{ because } S^* = (F_0^*)^* = F_0 \text{ and } \overline{F_0} = S_0^*, \\ &=< z, S_0 A^* \Omega > \text{ as } S_0 \text{ anti-linear,} \\ &=< z, A\Omega > \text{ by definition of } S_0. \end{aligned}$$

This means that $S^* z$ belongs to $DomS_0^* \in DomS^*$ because as we have

$$< S_0 A\Omega, S^* z >=< z, A\Omega >$$

so we can do $< S_0^* S^* z, A\Omega >$) and $(S^*)^2 z = z$.
Let $y \in DomS$ and $z \in DomS*$, we have $S * z \in DomS*$ and

$$\begin{aligned} < S^* z, Sy > &=< y, (S^*)^2 z > \text{ by anti-linearity} \\ &=< y, z > \text{ as } (S^*)^2 z = z. \end{aligned}$$

Thus $Sy \in DomS^{**} = DomS$ and

$$S^2 y = S * *Sy = y \text{as} < y, (S^*)^2 z >=< y, z >$$

Thus we have that $S^2 = I$ on $DomS$ which implies that $S = S^{-1}$. □

Let us define Δ as $\Delta = FS = S^* S$.

Theorem 4. *There exists an anti-unitary operator J from $\mathcal{H}$ to $\mathcal{H}$ and an (unbouded) invertible, positive operator Δ such that:*

$$\begin{aligned} &\Delta = FS, \Delta^{-1} = SF, J^2 = I \\ &S = J\Delta^{1/2} = \Delta^{-1/2} J \\ &F = J\Delta^{-1/2} = \Delta^{1/2} J \\ &J\Delta^{it} = \Delta^{it} J \\ &J\Omega = \Delta\Omega = \Omega \end{aligned}$$

J is actually the polar decomposition of S:

$$S = J(S^* S)^{1/2}$$

Proof. We will prove only some of the equalities.

$$\begin{aligned} \Delta^{-1} = (FS)^{-1} = S^{-1} F^{-1} = SF. \\ S = J\Delta^{1/2} = (SS^*)^{1/2} J = \Delta^{-1/2} J. \end{aligned}$$

Let $x \in DomS$. Then

$$x = S^2 x = J\Delta^{1/2} \Delta^{-1/2} J = J^2 x.$$

and thus $J^2 = I$.
Finally, note that $S\Omega = F\Omega = \Omega$ by taking $A = I \in \mathcal{M}$ and thus $\Delta\Omega = FS\Omega = \Omega$ and similarly for J, $J\Omega = \Omega$. □

Example:
If the state ω was tracial, that is $\omega(AB) = \omega(BA), \forall A, B$, we would have

$$\begin{aligned} ||S_0 A\Omega||^2 = ||A * \Omega||^2 &= < A^*\Omega, A^*\Omega > \\ &= \omega(AA^*) \\ &= \omega(A * A) \\ &= ||A\Omega||^2 \end{aligned}$$

Thus S_0 would be an isometry and

$$\begin{aligned} S = J = F \\ \Delta = I. \end{aligned}$$