Seminar Introduction to Nonlinear Analysis

# THE SCHRÖDINGER SEMIGROUP

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Francesco Mussin, David Ziener
ETH Zürich

# 1 Preliminaries

## 1.1 Distributions

The notion of distribution first came about in physics, where, for example, it was needed to compute the derivative on $\mathbb{R}$ of the Heaviside function $H : \mathbb{R} \longrightarrow \mathbb{R}$ (defined by $H(x) = 1$ for $x \geq 0$ and 0 otherwise): Clearly such a derivative, which takes the name $\delta$ (Dirac delta), should be 0 everywhere, and for any $\varepsilon > 0$, it should be the case that:

$$\int_{[-\varepsilon,\varepsilon]} \delta(x)dx = H(\varepsilon) - H(-\varepsilon) = 1.$$

Obviously such a measurable function does not exist, nonetheless this precise property was crucial to physicists.
So, there was the necessity to have a more reliable notion of function, one that could allow for the behaviour of the Dirac delta, and that could allow for discussing about derivatives without any worries regarding regularity (the "derivatives" of the Dirac delta were also needed in physics).
The idea for such a generalization comes from the observation that in functional analysis, functions and function classes of any kind are often described by how they behave when one integrates them against smooth and compactly supported functions, so it was somehow natural to think of this new kind of function (which takes the name of *distribution*) as a linear functional on the space of smooth and compactly supported functions (often called *test functions* for short), with the hope in mind to generalize the assignment:

$$\varphi \longmapsto \int f(x)\varphi(x)dx.$$

Evidently, hoping to generalize integration against test functions only through linear functionals on the space of test functions is not enough, so one needs some continuity properties. But what topology should we put on the space of test functions? Here come the difficulties regarding distributions: the topology that mathematicians ended up choosing is called the canonical LF topology, and although it makes the space of test functions a locally convex topological vector space, it's easier to describe it through the local base at 0 of open, balanced, and convex sets, instead of describing it through seminorms.
For our purposes, it will be enough to state what the continuity of linear functionals means and what the convergence of test functions means. Before moving on, it is common in distribution theory to use the symbol $\mathcal{D}(\Omega)$ to denote the vector space of test functions over an open set $\Omega \subseteq \mathbb{R}^d$.
Regarding the continuity of the linear functional, we can have a rough idea of what the definition should be by taking a look at the map

$$T_f : \mathcal{D}(\Omega) \ni \varphi \longmapsto \int_\Omega f(x)\varphi(x)dx \in \mathbb{C},$$

for a given $f \in L^1_{\text{loc}}(\Omega)$. Then, for any compact set $K \subseteq \Omega$ we have that, given $\varphi \in \mathcal{D}(\Omega)$ supported on $K$:

$$|T\varphi| \leq \int_\Omega |f(x)||\varphi(x)|dx \leq \|\varphi\|_\infty \int_K |f(x)|dx,$$

In short, for each compact subset $K \subseteq \Omega$, we have a linear bound on the image of $\varphi$ through $T$ provided that $\text{supp}(\varphi) \subseteq K$. The space of test functions on $\Omega$ that is supported on the compact set $K$ is often denoted with $\mathcal{D}_K(\Omega)$.

**Proposition 1.** *Let $\Omega \subseteq \mathbb{R}^d$ be open, and let $\{\varphi_n\}_n \subseteq \mathcal{D}(\Omega)$ be a sequence. Then $\varphi_n$ converges to $\varphi \in \mathcal{D}(\Omega)$ if and only if there is a compact set $K \subseteq \Omega$ such that:*

- $\text{supp}(\varphi_n) \subseteq K$ *for $n$ large enough.*
- *For all $\alpha \in \mathbb{N}^d$, $\lim_{n\to\infty} \sup_{x\in K} |\partial^\alpha \varphi_n(x) - \partial^\alpha \varphi(x)| = 0$.*

**Proposition 2.** *Let $\Omega \subseteq \mathbb{R}^d$ be open. Then the linear functional $T : \mathcal{D}(\Omega) \longrightarrow \mathbb{C}$ is continuous (i.e. a distribution) if and only if for any compact set $K \subseteq \Omega$, there is $C_K > 0$ and $p_K \in \mathbb{N}$ such that given $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}(\varphi) \subseteq K$, it holds that:*

$$|T\varphi| \leq C_K \max_{|\alpha| \leq p_K} \|\partial^\alpha \varphi\|_\infty.$$

If the integer $p_K$ can be taken independently of $K$, then that integer is called the order of $T$. We denote with $\mathcal{D}'(\Omega)$ the space of distributions on $\Omega$.
As an example, consider $\Omega \subseteq \mathbb{R}^d$ open and $f \in L^1_{\text{loc}}(\Omega)$. We defined the linear functional:

$$T_f : \mathcal{D}(\Omega) \ni \varphi \longmapsto \int_\Omega f(x)\varphi(x)dx \in \mathbb{C},$$

and we have seen how it is a distribution of order 0. Important to note is the property that for any two $f, g \in L^1_{\text{loc}}(\Omega)$, $T_f = T_g$ if and only if $f = g$ (almost everywhere).
Let us now delve into the topic of how we generalize derivatives to distributions. We start by observing that, for $f \in \mathcal{C}^k(\Omega)$, and any multiindex $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$, it holds for any $\varphi \in \mathcal{D}(\Omega)$ that:

$$\int_\Omega f(x)\partial^\alpha \varphi(x)dx = (-1)^{|\alpha|} \int_\Omega \partial^\alpha f(x)\varphi(x)dx,$$

(the above is just an application of integration by parts, helped by the fact that $\varphi$ is compactly supported, making the boundary terms disappear). As distributions generalize integration against test functions, it is natural to come up with the following definition of partial derivative of a distribution.

**Definition 1.** *Given $\Omega \subseteq \mathbb{R}^d$ open, $T \in \mathcal{D}'(\Omega)$, and a multiindex $\alpha \in \mathbb{N}^d$, we define the $\alpha$-th partial derivative of $T$ as the (linear) map:*

$$\partial^\alpha T : \mathcal{D}(\Omega) \ni \varphi \longmapsto (-1)^{|\alpha|} T\partial^\alpha \varphi \in \mathbb{C}.$$

Derivatives of distributions are indeed distributions: This property is immediate once one takes a look at our definition of distributions. A natural question that one might ask is whether or not for $f \in \mathcal{C}^k(\Omega)$, and $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$ it holds that $\partial^\alpha T_f = T_{\partial^\alpha f}$. That indeed is the case, as integration by parts dictates.
Another construction that distributions allow is multiplication against smooth functions. Indeed, thinking again at what distribution generalize, if we take $f \in L^1_{\text{loc}}(\Omega)$ and $a \in \mathcal{C}^\infty(\Omega)$, then given $\varphi \in \mathcal{D}(\Omega)$ it holds that:

$$T_{af}\varphi = \int_\Omega a(x)f(x)\varphi(x)dx = \int_\Omega f(x)a(x)\varphi(x)dx = T_f(a\varphi),$$

so we write down the following definition.

**Definition 2.** *Given $\Omega \subseteq \mathbb{R}^d$ open, $T \in \mathcal{D}'(\Omega)$, and $a \in \mathcal{C}^\infty(\Omega)$, we define $aT : \mathcal{D}(\Omega) \longrightarrow \mathbb{C}$ as follows:*

$$aT : \mathcal{D}(\Omega) \ni \varphi \longmapsto T(a\varphi) \in \mathbb{C}$$

Surely enough, the above defines a distribution on $\Omega$, as the Leibniz product rule shows.
Before moving on, it is important that we set some notation, namely: Duality brackets. From now on, when dealing with distributions or tempered distributions, we'll use the notation $\langle T, \varphi \rangle$ to mean $T\varphi$.

### 1.2 The Schwartz space and the Fourier transform

The Schwartz function space can be defined as the following subset of $\mathcal{C}^\infty(\mathbb{R}^d)$:

$$\mathcal{S}(\mathbb{R}^d) = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^n) : \ \forall \alpha, \beta \in \mathbb{N}^d, \ \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta f(x)| < +\infty \right\}.$$

In other words, $\mathcal{S}(\mathbb{R}^d)$ is the set of all smooth functions whose derivatives fall to zero at $\infty$ faster than any polynomial, indeed it can be easily showed that $f \in \mathcal{S}(\mathbb{R}^d)$ means:

$$\forall \alpha, \beta \in \mathbb{N}^d, \lim_{|x|\to+\infty} |x^\alpha \partial^\beta f(x)| = 0.$$

$\mathcal{S}(\mathbb{R}^d)$ is clearly a subspace of $\mathcal{C}^\infty(\mathbb{R}^d)$, however topologically speaking, it is far more interesting to consider it with its own family of seminorms. There are many candidates, and they're all related to the quantity $\sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)|$ as $\alpha$ and $\beta$ range in $\mathbb{N}^d$, for example, one such family could be $\{N_p : p \in \mathbb{N}\}$, where for each $p \in \mathbb{N}$ $N_p$ is defined in the following fashion:

$$N_p : \mathcal{S}(\mathbb{R}^d) \ni f \longmapsto \sum_{|\alpha|,|\beta|\le p} \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)| \in [0, +\infty)$$

this is easily checked to be a seminorm on $\mathcal{S}(\mathbb{R}^d)$. Thus, we have a locally convex linear topology on $\mathcal{S}(\mathbb{R}^d)$. This gets us a couple of things for free:

**Proposition 3.** *Any sequence $\{\varphi_n\}_n \subseteq \mathcal{S}(\mathbb{R}^d)$ converges to $\varphi \in \mathcal{S}(\mathbb{R}^d)$ if and only if for every $p \in \mathbb{N}$, $N_p(\varphi_n - \varphi) \to 0$ as $n \to \infty$.*

Due to the nature of the family of seminorms $\{N_p : p \in \mathbb{N}\}$, we have the following characterization of a continuous linear functional:

**Proposition 4.** *Let $T : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathbb{C}$ be a linear functional, then $T$ is continuous if and only if there is a $p \in \mathbb{N}$ and a constant $C > 0$ such that for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $|T\varphi| \le CN_p(\varphi)$.*

To better explore some of the most basic properties of $\mathcal{S}(\mathbb{R}^d)$, we set the following definition.

**Definition 3.** *A function $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ is said to have polynomial growth if there is a positive integer $n$ such that:*

$$\sup_{x\in\mathbb{R}^d} \left| \frac{f(x)}{\langle x\rangle^n} \right| < +\infty,$$

*where we use the shorthand notation $\langle x\rangle = \sqrt{1 + |x|^2}$.*

With this, we have the following:

**Proposition 5.** *Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then:*

- *(i) For any $\alpha \in \mathbb{N}^d$, $\partial^\alpha \varphi \in \mathcal{S}(\mathbb{R}^d)$.*
- *(ii) For any $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ of polynomial growth, $f\varphi \in \mathcal{S}(\mathbb{R}^d)$.*

The first point is obvious while for the second one is just a verification.
Let us now jump right into the Fourier transform.

**Definition 4.** *Given $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we define the Fourier transform of $f$ as the map $\widehat{\varphi} = \mathcal{F}\varphi : \mathbb{R}^d \longrightarrow \mathbb{C}$ given by:*

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}^d} \varphi(x) e^{-ix\cdot\xi} dx,$$

*for every $\xi \in \mathbb{R}^d$.*

Notice how for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\widehat{\varphi}$ is always a bounded function, more precisely $\|\widehat{\varphi}\|_\infty \le \|\varphi\|_{L^1(\mathbb{R}^d)}$. This simple fact can help us understand better the meaning behind the basic properties of the Fourier transform.

**Proposition 6.** *Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then*

- *(i) For any $\alpha \in \mathbb{N}^d$, $\partial^\alpha \mathcal{F}\varphi = \mathcal{F}((-i)^{|\alpha|} x^\alpha \varphi)$.*
- *(ii) For any $\alpha \in \mathbb{N}^d$, $\mathcal{F}(\partial^\alpha \varphi) = (-i)^{|\alpha|} \xi^\alpha \mathcal{F}(\varphi)$.*
- *(iii) The Fourier transform is continuous. More precisely for any $p \in \mathbb{N}$, $N_p(\mathcal{F}\varphi) \le C_p N_{p+d+1}(\varphi)$ for some constant $C_p$ independent of $\varphi$.*

(*iv*) *Given* $\psi \in \mathcal{S}(\mathbb{R}^d)$*, it holds that* $\mathcal{F}(\varphi * \psi) = \mathcal{F}(\varphi)\mathcal{F}(\psi)$*.*

*Proof.* $(i)$. This is just a matter of recursion and applying Lebesgue's dominated convergence theorem.
$(ii)$. Like the abobve, use recursion and integration by parts.
$(iii)$. This involves using $(i)$ and $(ii)$ to bound the seminorms. We spare the details.
$(iv)$. This is just a verification using Fubini's theorem. □

As it turns out, the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$ is not just continuous, but invertible with continuous inverse.

**Proposition 7.** *The Fourier transform* $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^d)$ *is an automorphism of* $\mathcal{S}(\mathbb{R}^d)$*, with inverse given by:*

$$\mathcal{F}^{-1}\varphi(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\varphi(\xi)d\xi$$

*for all* $\varphi \in \mathcal{S}(\mathbb{R}^d)$ *and* $x \in \mathbb{R}^d$*. Moreover, for all* $\varphi \in \mathcal{S}(\mathbb{R}^d)$ *it holds that* $\mathcal{F}\mathcal{F}\varphi = \varphi^\vee$*, where* $\varphi^\vee : \mathbb{R}^d \ni x \longmapsto \varphi(-x) \in \mathbb{C}$*.*

### 1.3 Tempered distributions and their Fourier Transform

Unfortunately we can not define the Fourier Transform for distributions, but we can use the same ideas behind distributions to generalize the Fourier transform to a greater class of functions than $L^2$. The objects we are referring to are called tempered distributions.
Tempered distributions are just the elements of the dual of $\mathcal{S}(\mathbb{R}^d)$, or in other words, linear functionals $T : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathbb{C}$ satisfying $|T\varphi| \leq CN_p(\varphi)$ for some $p \in \mathbb{N}$ and some constant $C >$ (and for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$).
As with distributions, we can define derivatives of tempered distributions precisely like we did for elements of $\mathcal{D}'(\Omega)$, and moreover we can multiply tempered distributions by smooth functions of polynomial growth (this too is defined the same way as for distributions).
The main inspiration for the notion of Fourier transform of a tempered distribution is the following property of the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$ (which is a simple application of Fubini's theorem):

**Proposition 8.** *Let* $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$*. Then:*

$$\int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\psi(\xi)d\xi = \int_{\mathbb{R}^d} \varphi(\xi)\mathcal{F}\psi(\xi)d\xi.$$

Thus we make the following definition.

**Definition 5.** *Given* $T \in \mathcal{S}(\mathbb{R}^d)'$*, we define its Fourier Transform,* $\mathcal{F}T$*, or* $\widehat{T}$*, by the following:*

$$\forall\varphi \in \mathcal{S}(\mathbb{R}^d),\ \langle \widehat{T}, \varphi\rangle = \langle T, \widehat{\varphi}\rangle$$

This inded defines a tempered distribution because if, say, $|T\varphi| \leq CN_p(\varphi)$ for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, then $|\widehat{T}\varphi| = |T\widehat{\varphi}| \leq CN_p(\widehat{\varphi}) \leq CN_{p+d+1}(\varphi)$.
This definition allows us to recover the usual properties of the Fourier transform.

**Proposition 9.** *Let* $T \in \mathcal{S}(\mathbb{R}^d)'$ *be a tempered distribution. Then:*

(*i*) *For all* $\alpha \in \mathbb{N}^d$*,* $\mathcal{F}\partial^\alpha T = (i)^{|\alpha|}\xi^\alpha \mathcal{F}T$*.*

(*ii*) *For all* $\alpha \in \mathbb{N}^d$*,* $\mathcal{F}(x^\alpha T) = (i)^{|\alpha|}\partial^\alpha \mathcal{F}T$*.*

*Moreover, if one sets* $T^\vee : \varphi \longmapsto T\varphi^\vee$*, then* $\mathcal{F} : \mathcal{S}(\mathbb{R}^d)' \longrightarrow \mathcal{S}(\mathbb{R}^d)'$ *is an automorphism with:*

$$\mathcal{F}^{-1}T = \frac{1}{(2\pi)^d}(\mathcal{F}T)^\vee.$$

### 1.4 The Sobolev space $H^s(\mathbb{R}^d)$

In this section we will define the Sobolev spaces $H^s(\mathbb{R}^d)$, which are a generalization to (tempered) distributions of the usual Sobolev spaces $W^{2,k}(\mathbb{R}^d)$. Their motivation comes from the the following fact about the latter family of spaces.

**Proposition 10.** *The following norms are equivalent on $W^{2,k}(\mathbb{R}^d)$:*

$$\|f\|_{W^{2,k}(\mathbb{R}^d)} = \sum_{|\alpha|\leq k} \|\partial_w^\alpha f\|_{L^2(\mathbb{R}^d)} \qquad \text{and} \qquad \|f\| = \left(\int_{\mathbb{R}^d} (1+|\xi|^2)^k \widehat{f}(\xi) d\xi\right)^{1/2},$$

*where $\partial_w^\alpha f$ denotes the weak derivative of order $\alpha$ of $f$.*

Recall that we adopt the notation of the japanese bracket, namely for $\xi \in \mathbb{R}^d$:

$$\langle \xi \rangle := \sqrt{1+|\xi|^2}.$$

**Definition 6.** *Given $s \in \mathbb{R}$, we define the Sobolev space $H^s(\mathbb{R}^d)$ as the space of all tempered distributions $u \in \mathcal{S}(\mathbb{R}^d)'$ for which there is $f \in L^2(\mathbb{R}^d; \langle\xi\rangle^{2s} d\xi)$ representing $\widehat{u} : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathbb{C}$. Committing abuse of notation, we'll denote this (unique) function class with $\widehat{u}$.*

We can define an inner product on this space, namely:

$$\langle u, v \rangle_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi,$$

and sure enough, this makes $H^s(\mathbb{R}^d)$ a Hilbert space.

**Proposition 11.** $(H^s(\mathbb{R}^d), \langle \cdot, \cdot \rangle_{H^s(\mathbb{R}^d)})$ *is a Hilbert space.*

As we will later see, this provides a good setting when it comes to weak solutions of our problem of interest, the free Schrödinger wave equation.

## 2 Classical Solutions

We now look into the free Schrödinger wave equation in $\mathbb{R}^d$, i.e. the following Cauchy problem:

$$\begin{cases} i\partial_t u(t,x) = -\Delta u(t,x) \\ u(0,x) = u_0(x) \end{cases} \qquad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^d \tag{1}$$

for a given initial condition $u_0$.

**Proposition 12** (Solution by means of Fourier transform)**.** *Let $u_0 \in \mathcal{S}(\mathbb{R}^d)$, then the unique solution $u \in \mathcal{C}^1(\mathbb{R}, \mathcal{S}(\mathbb{R}^d))$ to (1) is given by*

$$u(t,\cdot) = \mathcal{F}^{-1}(e^{-it|\cdot|^2} \widehat{u}_0). \tag{2}$$

*Proof.* We start by taking (1), and applying the Fourier transform in $\mathbb{R}^d$ (viewing the time variable as a parameter) on both sides. We get the following:

$$\begin{cases} i\partial_t \widehat{u}(t,\xi) = |\xi|^2 \widehat{u}(t,\xi) \\ \widehat{u}(0,\xi) = \widehat{u}_0(\xi) \end{cases} \qquad \forall (t,\xi) \in \mathbb{R} \times \mathbb{R}^d \tag{3}$$

Viewing $\xi$ as a parameter, the above is actually a Cauchy problem for an ordinary differential equation, with solution given by $\widehat{u}(t,\xi) = e^{-it|\xi|^2} \widehat{u}_0(\xi)$, so the statement can be obtained by taking the inverse Fourier transform. ☐

The expression in (2) is the central object of study in the next few sections, and we'll set for every $t \in \mathbb{R}$ and $u_0 \in \mathcal{S}(\mathbb{R}^d)$, $S(t)u_0 = \mathcal{F}^{-1}(e^{-it|\cdot|^2}\widehat{u}_0)$. We can give a more concrete expression to $S(t)u_0$ with the following lemma.

**Lemma 1.** *Let $z \in \mathbb{C}$ have positive real part. Then for all $\xi \in \mathbb{R}^d$:*

$$\mathcal{F}(e^{-z|\cdot|^2})(\xi) = \left(\frac{\pi}{z}\right)^{d/2} e^{-\frac{|\xi|^2}{4z}},$$

*where we take $z^{d/2}$ to mean $|z|^{d/2}e^{id\theta/2}$ whenever $z = |z|e^{i\theta}$ for $\theta \in [-\pi/2, \pi/2]$.*

*Proof.* This comes from the identity principle of holomorphic functions. Let $\Omega = \{z \in \mathbb{C} : \Re z > 0\}$, and for a given $\xi \in \mathbb{R}^d$ consider the two following functions on $\Omega$:

$$\Omega \ni z \longmapsto \int_{\mathbb{R}^d} e^{-z|x|^2}e^{-ix\cdot\xi} \in \mathbb{C} \quad \text{and} \quad \Omega \ni z \longmapsto \left(\frac{\pi}{z}\right)^{d/2} e^{-\frac{|\xi|^2}{4z}} \in \mathbb{C}.$$

From the expression for the Fourier transform of the Gaussian $\mathbb{R}^d \ni x \mapsto e^{-|x|^2} \in \mathbb{C}$ we can easily see that the two above functions agree on $\{x \in \mathbb{C} : x \in \mathbb{R},\ x > 0\}$, so being those two holomorphic on $\Omega$, they must coincide on the whole set $\Omega$. $\square$

Even though this might not seem that useful (our $z$ is purely imaginary), we can work around this issue and still end up with the same expression.

**Corollary 1** (Solution by means of convolution)**.** *Let $u_0 \in \mathcal{S}(\mathbb{R}^d)$. Then trivially $S(0)u_0 = u_0$, and for $t \in \mathbb{R} \setminus \{0\}$:*

$$S(t)u_0 = S_t * u_0,$$

*where $S_t : \mathbb{R}^d \longrightarrow \mathbb{C}$ is defined by:*

$$S_t(x) = \frac{1}{(4\pi it)^{d/2}} e^{i\frac{|x|^2}{4t}}.$$

*Proof.* Let $\Omega$ be the denoted earlier half plane. Given $t \in \mathbb{R} \setminus \{0\}$, take a sequence $\{z_n\}_n \subseteq \Omega$ converging to $it$. The Dominated Convergence Theorem ensures for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$:

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} e^{-z_n|x|^2}\varphi(x)\ dx = \int_{\mathbb{R}^d} e^{-it|x|^2}\varphi(x)\ dx,$$

together with:

$$\lim_{n\to\infty} \left(\frac{\pi}{z_n}\right)^{d/2} \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{4z_n}}\varphi(\xi)\ d\xi = \left(\frac{\pi}{it}\right)^{d/2} \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{4it}}\varphi(\xi)\ d\xi,$$

hence seeing $e^{-it|\cdot|^2}$ and $e^{-z_n|\cdot|^2}$ as tempered distributions yields for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$:

$$\begin{aligned}\langle \widehat{e^{-it|\cdot|^2}}, \varphi\rangle &= \langle e^{-it|\cdot|^2}, \widehat{\varphi}\rangle = \lim_{n\to\infty} \langle e^{-z_nt|\cdot|^2}, \widehat{\varphi}\rangle = \lim_{n\to\infty} \langle \widehat{e^{-z_n|\cdot|^2}}, \varphi\rangle = \lim_{n\to\infty} \left(\frac{\pi}{z_n}\right)^{d/2} \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{4z_n}}\varphi(\xi)\ d\xi \\ &= \left(\frac{\pi}{it}\right)^{d/2} \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{4it}}\varphi(\xi)\ d\xi.\end{aligned}$$

Now that we have established whose the transform of $e^{-it|\cdot|^2}$, we can write $S(t)u_0 = \mathcal{F}(e^{-it|\cdot|^2}\widehat{u}_0)$ as:

$$S(t)u_0 = \mathcal{F}\left(\mathcal{F}^{-1}\left(\left(\frac{\pi}{it}\right)^{d/2} e^{-\frac{|\cdot|^2}{4it}}\right)\widehat{u}_0\right) = S_t * u_0. \qquad \square$$

**Lemma 2** (Duhamel formula)**.** *Let $u_0$ be a function in $\mathcal{S}(\mathbb{R}^d)$, and $f \in \mathcal{C}(\mathbb{R}, \mathcal{S}(\mathbb{R}^d))$. Then the unique solution $u \in \mathcal{C}^1(\mathbb{R}, \mathcal{S}(\mathbb{R}^d))$ to the nonhomogeneous problem*

$$\begin{cases} i\partial_t u(t,x) + \Delta u(t,x) = f(t,x) \\ u(t,x) = u_0(x) \end{cases} \qquad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^d \tag{4}$$

*is given by:*

$$u(t,\cdot) = S(t)u_0 - i\int_0^t S(t-\tau)f(\tau,\cdot)\ d\tau, \tag{5}$$

*for all $t \in \mathbb{R}$.*

*Proof.* As in Lemma 12, we take the Fourier Transform on both sides, ending up with:

$$\begin{cases} i\partial_t \widehat{u}(t,\xi) - |\xi|^2 \widehat{u}(t,\xi) = \widehat{f}(t,\xi) \\ \widehat{u}(0,\xi) = \widehat{u}_0(\xi), \end{cases}$$

and again, treating $\xi$ as a parameter, this is a Cauchy problem of the given ODE, with unique global solution given by:

$$\widehat{u}(t,\xi) = e^{-it|\xi|^2}\widehat{u}_0(\xi) - i\int_0^t e^{-i|\xi|^2(t-\tau)}\widehat{f}(\tau,\xi)\ d\tau,$$

so we end up with (5) by taking the inverse Fourier Transform on both sides. □

# 3 The Schrödinger semi group

The representation formula (2) for the Schrödinger equation allows for a different approach of studying the equation. Solutions to the equation are fully described by the family $(S(t))_{t\in\mathbb{R}}$. Thus, it makes sense to study this family and its structural properties, which should in turn help us to derive further properties of solutions. We have the following observation.

**Theorem 1.** *Let $s \in \mathbb{R}$. Define $S(t) : H^s(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$ by*

$$S(t)u_0 = \mathcal{F}^{-1}(e^{-it|\cdot|^2}\widehat{u}_0).$$

*Then $(S(t))_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group on $H^s$ called the Schrödinger semi group. More precisely:*

*(i) Continuity: $t \mapsto S(t)u_0 \in \mathcal{C}(\mathbb{R}; H^s)$.*

*(ii) Unitarity: $||S(t)u_0||_{H^s} = ||u_0||_{H^s}$.*

*(iii) Group Property: For $t, r \in \mathbb{R}$, $S(t)S(r) = S(t+r)$ and $S(0) = Id$.*

*(iv) Adjoint: $S(t)^* = S(-t)$.*

*Proof.* The proof of the properties $(ii) - (iv)$ are direct consequences of the definition. $(i)$ is an application of the Lebesgue dominated convergence Theorem. □

Strongly continuous one-parameter unitary groups $(U(t))_{t\in\mathbb{R}}$ are characterized by Stone's Theorem. This states that these objects are generated by a unique self-adjoint operator $A : D(A) \subset H \to H$ in the sense that

$$U(t) = e^{itA},$$

where this equality is understood in the sense of an unbounded Borel functional calculus.

A further property of the Schrödinger semi group is the pointwise decay.

**Proposition 13** (Pointwise decay)**.** *Let $t \in \mathbb{R}\setminus\{0\}$ and $p \in [2,\infty]$. Then $S(t) \in \mathcal{L}(L^q; L^p)$ and*

$$||S(t)u_0||_{L^q} \le \frac{1}{|4\pi t|^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}}||u_0||_{L^p},$$

*where $q$ is the Hölder conjugate of $p$.*

*Proof.* Note that

$$S(t) \in \mathcal{L}(L^1; L^\infty)$$

from Young's inequality for convolutions. Moreover,

$$S(t) \in \mathcal{L}(L^2; L^2),$$

by $L^2$ isometry. Therefore, we can apply the Riesz-Thorin interpolation Theorem to derive the boundedness in $\mathcal{L}(L^q; L^p)$. The explicit estimate follows from the explicit form of the convolution. □

# 4 Weak Solutions

For now we are able to evaluate the semi group for very general functions in $H^s$ but it is not clear in which sense the corresponding function

$$u(t) = S(t)u_0$$

solves the Schrödinger equation. For this function to be a solution, we have to broaden our notion of solutions. The key is that we can only take derivatives in the sense of distributions.

**Definition 7.** *A distribution $u \in \mathcal{C}(\mathbb{R}; \mathcal{S}')$ is a weak solution of the nonhomogeneous problem* (4) *if for all $\varphi \in \mathcal{C}^1(\mathbb{R}; \mathcal{S}(\mathbb{R}^d))$, it holds*

$$\int_0^t \langle u(r), \Delta\varphi(r) - i\partial_r\varphi(r)\rangle dr = -i\langle u_0, \varphi(0)\rangle + i\langle u(t), \varphi(t)\rangle + \int_0^t \langle f(r), \varphi(r)\rangle dr,$$

*where we understand $\langle \cdot, \cdot \rangle$ as dual pairing.*

The next proposition states that the semi group actually produces weak solutions to the Schrödinger equation, just in the way we saw it for smooth data.

**Proposition 14.** *Let $u_0 \in \mathcal{S}'$. Then the distribution*

$$u(t) = S(t)u_0$$

*is a weak solution to* (1)*.*

The proof is a direct calculation using the definition of the Fourier transform for distributions, so we do not present it here.

Instead, we ask ourselves if it is possible to also extend the Duhamel formula to the weak setting. Clearly, we have to assume some regularity of the data for the formula to make sense, but it turns out that the assumptions are more general compared to the smooth setting.

**Proposition 15.** *Let $u_0 \in L^2$ and $f \in L^1_{loc}(\mathbb{R}; L^2)$. Then the nonhomogeneous Schrödinger equation* (4) *has a unique weak solution $u \in \mathcal{C}(\mathbb{R}; L^2)$ given by*

$$u(t) = S(t)u_0 - i\int_0^t S(t-t')f(t')dt'. \tag{6}$$

*Moreover,*

$$||u(t)||_{L^2}^2 = ||u_0||_{L^2}^2 + 2\Im \int_0^t \int_{\mathbb{R}^d} f(\tau, x)\overline{u}(\tau, x)dxd\tau. \tag{7}$$

*Proof.* We only give a sketch.

**Uniqueness:** If we assume $u \in \mathcal{C}(\mathbb{R}; L^2)$ solves the equation with $f \equiv 0$ and $u_0 \equiv 0$, we obtain by convolution a sequence of smooth solutions with the same data. Then using uniqueness for the equation in the strong form and the convergence of this sequence the claim follows.

**Existence:** This follows from the previous proposition and a calculation involving again standard properties of the Fourier transform.

**Mass Equality:** First, we assume that $u_0 \in \mathcal{S}(\mathbb{R}^d)$ and $f \in \mathcal{C}(\mathbb{R}; \mathcal{S}(\mathbb{R}^d))$. Then equality (7) follows by testing the weak formulation against $\overline{u}$, where $u \in \mathcal{C}^1(\mathbb{R}; \mathcal{S}(\mathbb{R}^d))$ is given by (6). For less regular data as stated above, we construct approximating sequences for the data to get a sequence of solutions. This sequence turns out to be a Cauchy sequence from Duhamel's formula. Taking the limit proves the claim. $\square$