Seminar on Operator Algebras and Quantum Information Theory

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Contents

1 Self-dual cone and standard form

First of all, we define

$$\mathcal{P} = \{\overline{AJAJ\Omega \; A \in \mathcal{M}}\}$$

Proposition 1.1.

1. *$\mathcal{P} = \overline{\Delta^{1/4}\mathcal{M}_{+}\Omega} = \overline{\Delta^{1/4}\mathcal{M}'_{+}\Omega}$ and thus $\mathcal{P}$ is a convex cone.*
2. *$\Delta^{it}\mathcal{P} = \mathcal{P}\ \forall t$.*
3. *If f is of positive type then $f(\log\Delta)\mathcal{P} \subset \mathcal{P}$.*
4. *If $\xi \in \mathcal{P}$, then $J\xi = \xi$.*
5. *If $A \in M$ then $AJAJ\mathcal{P} \subset \mathcal{P}$*

Proof.

1. Let $\mathcal{M}_0$ be the $*$-algebra of elements of $\mathcal{M}$ which are entire for the modular group σ. (that is, $t \to \sigma_t(A)$ admits an analytic extension). We shall admit here that $\mathcal{M}_0$ is σ-weakly dense in $\mathcal{M}$. For every $A \in \mathcal{M}_0$ we have

$$\begin{aligned}\Delta^{1/4}AA^*\Omega &= \sigma_{-i/4}(A)\sigma_{i/4}(A)^*\Omega \\ &= \sigma_{-i/4}(A)J\Delta^{1/2}\sigma_{1/4}(A)\Omega \\ &= \sigma_{i/4}(A)J\sigma_{-i/4}(A)J\Omega \\ &= BJBJ\Omega\end{aligned}$$

where $B = \sigma_{-i/4}(A)$. Then, since $\sigma_{-i/4}(\mathcal{M}_0) = \mathcal{M}_0$ and the fact that $\mathcal{M}_0$ is dense in $\mathcal{M}$, we have

$$BJBJ\Omega \in \overline{\Delta^{1/4}\mathcal{M}_{+}\Omega} \subset \overline{\Delta^{1/4}\overline{\mathcal{M}_{+}\Omega}}$$

for every $B \in \mathcal{M}$. Hence,

$$\mathcal{P} \subset \overline{\Delta^{1/4}\mathcal{M}_{+}\Omega} \subset \overline{\Delta^{1/4}\overline{\mathcal{M}_{+}\Omega}}.$$

On the contrary, $\mathcal{M}_0^{+}$ is dense in $\overline{\mathcal{M}_{+}\Omega}$. Let $\psi \in \overline{\mathcal{M}_{+}\Omega}$. Thern there exists a sequence $(A_n) \subset \mathcal{M}_0^{+}$ s.t. $A_n\Omega \to \psi$. We know thanks to the above relation that $\Delta^{1/4}A_n\Omega \in \mathcal{P}$. However,

$$J\Delta^{1/2}A_n\Omega = A_n\Omega \to \psi = J\Delta^{1/2}\psi$$

and thus

$$a$$

Therefore, $\Delta^{1/4}\psi$ belongs to $\mathcal{P}$ and $\overline{\Delta^{1/4}\overline{M_1\Omega}} \subset \mathcal{P}$. This proves the first equality. Analogously, one can prove the second one.

2. Immediate since we have

$$\Delta^{it}\Delta^{1/4}\mathcal{M}_+\Omega = \Delta^{1/4}\Delta^{it}\mathcal{M}_+\Omega = \Delta^{1/4}\sigma_t(\mathcal{M}_+)\Omega = \Delta^{1/4}\mathcal{M}_+\Omega.$$

3. If f is of positive type, then f is the Fourier transform of some positive, finite, Borel measure μ on $\mathbb{R}$. In particular

$$f(\log\Delta) = \int \Delta^{it} d\mu(t).$$

By 2, one concludes.

4. $JAJAJ\Omega = JAJA\Omega = AJAJ\Omega$

5. $AJAJBJBJ\Omega = ABJAJJBJ\Omega = ABJABJ\Omega.$

□

Theorem 1.2.

1. *$\mathcal{P}$ is self-dual, i.e. $\mathcal{P} = \mathcal{P}^\vee$, where*

$$\mathcal{P}^\vee = \{x \in \mathcal{H}; \langle y, x\rangle \geq 0, \forall y \in \mathcal{P}\}$$

2. *$\mathcal{P}$ is pointed, that is*

$$\mathcal{P} \cap (-\mathcal{P}) = \{0\}$$

3. *If $J\xi = \xi$, then ξ admits a unique decomposition as $\xi = \xi_1 - \xi_2$, with $\xi_i \in \mathcal{P}$ and $\xi_1 \perp \xi_2$.*

4. *The span of $\mathcal{P}$ is the whole of $\mathcal{H}$*

Proof.

1. If $A \in \mathcal{M}$ and $A' \in \mathcal{M}'_+$, then

$$\langle \Delta^{1/4}A\Omega,\ \Delta^{-1/4}A'\Omega\rangle = \langle A\Omega,\ A'\Omega\rangle = \langle \Omega,\ A^{1/2}A'A^{1/2}\Omega\rangle \geq 0.$$

Thus, $\mathcal{P} \subset \mathcal{P}^\vee$.

Conversely, if $\xi \in \mathcal{P}^\vee$, that is $\langle \xi, \nu\rangle \geq 0\ \forall\ \nu \in \mathcal{P}$, we set

$$\xi_n = f_n(log\Delta)\xi$$

where $f_n(x) = \exp(-x^2/2n^2)$. Then, ξ_n belongs to $\cap_{\alpha\in\mathbb{C}}\mathrm{Dom}\Delta^\alpha$ and ξ_n converges to ξ. We know that $f_n(log\Delta)\nu$ belongs to $\mathcal{P}$ and thus

$$\langle \Delta^{1/4}\xi_n, A\Omega\rangle = \langle \xi_n, \Delta^{1/4}A\Omega\rangle \geq 0.$$

Therefore, $\Delta^{1/4}\xi_n$ belongs to $\overline{\mathcal{M}_+\Omega}^\vee$, which coincides with $\overline{\mathcal{M}'_+\Omega}$ (admitted). This finally gives that ξ_n belongs to $\Delta^{1/4}\overline{\mathcal{M}'_+\Omega} \subset \mathcal{P}$.

2. If $\xi \in \mathcal{P} \cap (-\mathcal{P}) = \mathcal{P} \cap (-\mathcal{P}^\vee)$, then $\langle \xi, -\xi \rangle \geq 0$ and $\xi = 0$.

3. If $J\xi = \xi$ then, as $\mathcal{P}$ is convex and closed, there exists a unique $\xi_1 \in \mathcal{P}$ such that

$$\|\xi - \xi_1\| = \inf\{\|\xi - \nu\|; \nu \in \mathcal{P}\}.$$

   We set $\xi_2 = \xi_1 - \xi$. Let then $\nu \in \mathcal{P}$ and $\lambda > 0$. Then $\xi_1 + \lambda\nu$ belongs to $\mathcal{P}$ and

$$\|\xi - \xi_1\|^2 \leq \|\xi_1 - +\lambda\nu\xi\|^2.$$

   That is $\|\xi_2\|^2 \leq \|\xi_2 + \lambda\nu\|^2$, or else $\lambda^2\|\nu\|^2 + 2\lambda\mathcal{R}\langle \xi_2, \nu\rangle \geq 0$. This implies that $\mathcal{R}\langle \xi_2, \nu\rangle > 0$. But as $J\xi_2 = \xi_2$ and $J\nu = \nu$, then

$$\langle \xi_2, \nu\rangle = \langle J\xi_2, J\nu\rangle = \overline{\langle \xi_2, \nu\rangle}.$$

   That is $\langle \xi_2, \nu\rangle > 0$ and $\xi_2 \in \mathcal{P}^\vee = \mathcal{P}$.

4.) If ξ is orthogonal to the linear span of $\mathcal{P}$ then ξ belongs to $\mathcal{P}^\vee = P$. Therefore $\langle \xi, \xi\rangle = 0$ and $\xi = 0$.

□

Theorem 1.3 (**Universality**).

1. *If $\xi \in \mathcal{P}$ then ξ is cyclic for $\mathcal{M}$ if and only if it is separating for $\mathcal{M}$.*

2. *If $\xi \in \mathcal{P}$ then ξ is cyclic for $\mathcal{M}$ then $J_\xi, \mathcal{P}_\xi$ associated to $(\mathcal{M}, \xi)$ satisfy*

$$J_\xi = J \quad \text{and} \quad \mathcal{P}_\xi = \mathcal{P}.$$

Proof.

1. If ξ is cyclic for $\mathcal{M}$ then $J\xi$ is cyclic for $\mathcal{M}' = J\mathcal{M}J$ and thus $\xi = J\xi$ is separating for $\mathcal{M}$. And conversely.

2. Define as before (the closed version of)

$$\begin{aligned} S_\xi &: A\xi \longmapsto A^*\xi \\ F_\xi &: A'\xi \longmapsto A'^*\xi. \end{aligned}$$

   Then, we have

$$\begin{aligned} JF_\xi JA\xi &= JF_\xi JAJ\xi \\ &= J(JAJ)^*\xi \\ &= A^*\xi \\ &= S_\xi A\xi. \end{aligned}$$

This proves that $S\xi \subset JF_\xi J$. By a symmetric argument $F_\xi \subset JS_\xi J$ and thus

$$JS_\xi = F_\xi J.$$

Note that

$$(JS_\xi)^* = S_\xi^* J = F_\xi J = JS_\xi.$$

This means that JS_ξ is self-adjoint. Let us prove that it is positive.
We have

$$S_\xi = J_\xi \Delta_\xi^{1/2} = J(JS_\xi).$$

By uniqueness of the polar decomposition we must have $J = J_\xi$. Finally, we have that $\mathcal{P}_\xi$ is generated by the $AJ_\xi AJ_\xi \xi = AJAJ_\xi$. But as ξ belongs to $\mathcal{P}$ we have that $AJAJ\xi$ belongs to $\mathcal{P}$ and thus $\mathcal{P}\xi \subset \mathcal{P}$. Finally,

$$\mathcal{P} = \mathcal{P}^\vee \subset \mathcal{P}_\xi^\vee = \mathcal{P}_\xi \quad \text{and} \quad \mathcal{P} = \mathcal{P}_\xi.$$

□

Theorem 1.4.

1. *For every $\omega \in \mathcal{M}_{*+}$, there exists a unique $\xi \in \mathcal{P}$ such that*

$$\omega = \omega_\xi$$

2. *The mapping $\xi \longmapsto \omega_\xi$ is an homeomorphism and*

$$\|\xi - \nu\|^2 \leq \|\omega_\xi - \omega_\nu\|^2 \leq \|\xi - \nu\| \|\xi + \nu\|.$$