Seminar Operator Algebra

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Let $\mathfrak{A}$ and $\mathfrak{B}$ denote C*-algebras. The first 5 lemmas are used for the proof of some properties of completely bounded maps. If not mentioned otherwise, the statements are from the chapter by Rolando Rebolledo, Section 3 or 4, from the Book 2 in this seminar. Otherwise they are from the book by Takesaki (https://link.springer.com/book/10.1007/978-1-4612-6188-9) or from the script of Paul Skoufranis (https://pskoufra.info.yorku.ca/files/2016/07/Completely-Positive-Maps.pdf).

Lemma 0.1 *(Lemma 2.16 in Skoufranis)*
Let $\Phi : \mathfrak{A} \to \mathfrak{B}$ be a positive map, then $\Phi(x^*) = \Phi(x)^* \forall x \in \mathfrak{A}$

Lemma 0.2 *$M_n(\mathfrak{A})$ denotes the set of all $n \times n$ matrices with entries in $\mathfrak{A}$ with the involution $[a_{i,j}]^* = [(a_{j,i})^*]$. Then there is a norm $||(\cdot)||$ such that $M_n(\mathfrak{A})$ is a C*-algebra.*

Lemma 0.3 *(Lemma 3.1 and 3.2 from Chapter 4 Takesaki) The following are equivalent: Let $[a_{i,j}] \in M_n(\mathfrak{A})$*

1) $[a_{i,j}] \in M_n(\mathfrak{A})$ is positive

2) $[a_{i,j}] = \sum c_k$ where $(c_k)_{i,j} = [(b_i^{(k)})^* b_j^{(k)}]$ for $b_i^{(k)} \in \mathfrak{A}$ ($\sum$ is a finite sum)

3) $\forall b_1, ..., b_n \in \mathfrak{A}$ the sum $\sum_{i,j} b_i^* a_{i,j} b_j$ is positive in $\mathfrak{A}$

Lemma 0.4 *There is a bijective *-morphism from $M_2(M_n(\mathfrak{A}))$ to $M_{2n}(\mathfrak{A})$, namly* $\Phi(\begin{pmatrix} [A] & [B] \\ [C] & [D] \end{pmatrix}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

Lemma 0.5 *(Proposition 2.6 from Chapter by Rolando Rebolledo, Book 2 of this Seminar and Lemma 3.17 in Skoufranis)*

1) $\begin{pmatrix} p & a \\ a^* & p \end{pmatrix} \in M_2(\mathfrak{A})$ positive, then $||a|| \leq ||p||$

2) Let $\mathfrak{A}$ be a C-algebra with unit. Then $\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} \in M_2(\mathfrak{A})$ positive if and only if $||a|| \leq 1$*

Definition 0.6 *$\Phi : \mathfrak{U} \rightarrow \mathfrak{B}$ linear. Then Φ is called completely bounded if $||\Phi||_{cb} := sup\{||\Phi_n||_{op} : n \in \mathbb{N}\} < \infty$. (Recall the definition of Φ_n from the previous talk).*

Proposition 0.7 *Let $\mathfrak{U}$ be a C*-algebra with unit. $\Phi : \mathfrak{U} \rightarrow \mathfrak{B}$ a completely positive map. Then Φ is completely bounded and $||\Phi||_{op} = ||\Phi(1)||$.*

Proof $||\Phi(1)|| \leq ||\Phi||_{op} \leq ||\Phi||_{cb}$ holds because $||1|| = 1$ and Φ_n for $n = 1$ is Φ. We only need to prove $||\Phi||_{cb} \leq ||\Phi(1)||$. Let n be fix and consider $A \in M_n(\mathfrak{U})$ with $||A|| = 1$. $\begin{pmatrix} 1_n & A \\ A^* & 1_n \end{pmatrix}$ is positive in $M_2(M_n(\mathfrak{U}))$ by one of the previous lemmas because $||A||$ is 1. Hence the same matrix, but now considered to be in $M_{2n}(\mathfrak{U})$, is positive. Φ_{2n} is a positive map because Φ is completely positive and hence $\Phi_{2n}(\begin{pmatrix} 1_n & A \\ A^* & 1_n \end{pmatrix}) = \begin{pmatrix} \Phi_n(1_n) & \Phi_n(A) \\ \Phi_n(A^*) & \Phi_n(1_n) \end{pmatrix} = \begin{pmatrix} \Phi_n(1_n) & \Phi_n(A) \\ \Phi_n(A)^* & \Phi_n(1_n) \end{pmatrix}$ is positive in $M_{2n}(\mathfrak{U})$, and hence it is positive if considered to be in $M_2(M_n(\mathfrak{U}))$. This implies that $||\Phi_n(A)|| \leq ||\Phi_n(1_n)|| \leq ||\Phi(1)||$ Where the last inequality is due to a relation of the norm on $\mathfrak{B}$ and the norm on $M_n(\mathfrak{B})$ which was not proved in this talk. Hence the operator norm of Φ_n is less or equal than $||\Phi(1)||$ for all n, which proves the proposition. □

Theorem 0.8 *Let Φ be a bounded linear map from $\mathfrak{U}$ to $\mathfrak{B}$ where the latter is commutative. Then Φ is completely bounded and $||\Phi||_{op} = ||\Phi||_{cb}$.*

Theorem 0.9 *Let $\mathfrak{U}$ and $\mathfrak{B}$ be C*-algebra with unit such that $\mathfrak{B} \subset B(\mathfrak{h})$ for $\mathfrak{h}$ a separable Hilbertspace. Φ linear map from $\mathfrak{U}$ to $\mathfrak{B}$. The the following are equivalent*

1) Φ is completely positive

*2) There exists a representation of $\mathfrak{U}$, denoted by $(\Pi, \mathfrak{k})$, where Π is a *-morphism from $\mathfrak{U}$ to $\mathfrak{k}$, $\mathfrak{k}$ Hilbertspace. Moreover there exists a bounded linear map $V : \mathfrak{h} \rightarrow \mathfrak{k}$ such that for all x in $\mathfrak{U}$ we have that*

$$\Phi(x) = V^* \circ \Pi(x) \circ V$$

Where $V^* : \mathfrak{k} \rightarrow \mathfrak{h}$ satisfies $< x, Vy >_{\mathfrak{k}} = < V^*x, y >_{\mathfrak{h}}$