Completely Positive Maps

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We follow ROLANDO REBOLLEDOS' section *Complete Positivity and the Markov structor of Open Quantum Systems* in *Open Quantum Systems II.* by ATTAL et al. [2, Chapter 2, p.157 et seq.]

1 Definitions

Definition 1.1 (Completely Positive)**.** *Let $\mathcal{A}$, $\mathcal{B}$ be two $*$-algebras and $S \subset \mathcal{A}$ an operators system. A linear map $\Phi : S \to \mathcal{B}$ is called completely positive if*

$$\sum_{i,j=1}^{n} b_i^* \Phi(a_i^* a_j) b_j \in \mathcal{B} \tag{1}$$

is positive for all $n \in \mathbb{N}$ and for all $a_i \in S, b_i \in \mathcal{B}$. The space of all such maps is denoted by $\mathrm{CP}(S, \mathcal{B})$

From now on we will mostly consider the case of C^*-algebras and $\mathcal{B} \subset \mathcal{B}(H)$ for some Hilbert space H.

Remark 1.1. *Notice that the expression 1 is positive if and only if:*

$$\sum_{i,j=1}^{n} \langle u, b_i^* \Phi(a_i^* a_j) b_j u \rangle \leq 0 \quad \forall u \in H$$

which is equivalent to

$$\sum_{i,j=1}^{n} \langle u_i, \Phi(a_i^* a_j) u_j \rangle \leq 0 \quad \forall u_i \in H$$

2 Commutative Case

Theorem 2.1 (Stinespring, Arveson)**.** *Let $\Phi : \mathcal{A} \to \mathcal{B}$ be positive for some C^*-algebras $\mathcal{A}, \mathcal{B}$. Then if (a): $\mathcal{A}$ or (b): $\mathcal{B}$ is commutative we have that Φ is completely positive.*

For the proof of this we will first introduce two results

Theorem 2.2 (Gelfands Representation)**.** *Let $\mathcal{A}$ be a commutative unital C^*-algebra then $X = \sigma(\mathcal{A})$ is compact T_2. And we have*

$$\mathcal{A} \cong C(X) \cong C_0(X)$$

See: Open Quatum Systems I *by* ATTAL *et al.* *[1, p. 83].*

Theorem 2.3 (Riesz-Markov-Kakutani)**.** *Let X be compact T_2 then we have*

$$C(X)^* \cong \{\mu : \mu \text{ complex baire measure on } X\}.$$

Where the isomorphism is given by:

$$\mu \mapsto \int_X f \mathrm{d}\mu$$

We can now proof (a) and (b) from the Stinespring, Arveson theorem.

Proof (a). We assume that $\mathcal{A}$ is unital and $\mathcal{B} \subset \mathcal{B}(H)$. Thus we get

$$\mathcal{A} \cong C(X)$$

for X compact and T_2. We identify $a \in \mathcal{A}$ with $a(x) \in C(X)$ Thus for $u, v \in H$ we can write the linear functional:

$$\langle v, \Phi(a)u \rangle = \int_X a(x) \mathrm{d}\mu_{u,v}$$

as an integral with complex bair measure $\mu_{u,v}$. Let now $u_1, \dots, u_n \in H$ we define:

$$\mathrm{d}\mu := \sum_{i,j=1}^{n} |\mathrm{d}\mu_{u_i,u_j}|, \quad u := \sum_{i=1}^{n} \lambda_i u_i$$

for some $\lambda_i \in \mathbb{C}$. Using Radon-Nykodim we can write:

$$\mathrm{d}\mu_{u,u} = \left(\sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j h_{u_i,u_j} \right) \mathrm{d}\mu$$

for h_{u_i,u_j} the derivatives of $\mathrm{d}\mu_{u_i,u_j}$ with respect to $\mathrm{d}\mu$. Since Φ is positive we get that $\mathrm{d}\mu_{u,u} \geq 0$. But since $\mathrm{d}\mu \geq 0$ by construction we get that

$$\sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j h_{u_i,u_j} \geq 0 \quad \mu\text{-a.e}$$

Let now $a_1, \dots, a_n \in \mathcal{A}$ then we get

$$\sum_{i,j=1}^{n} \langle u_i, \Phi(a_i^* a_j) u_j \rangle = \int_X \left(\sum_{i,j=1}^{n} \overline{a_i(x)} a_j(x) h_{u_i,u_j} \right) \mathrm{d}\mu \geq 0$$

□

Proof (b). As above we identify $B \cong C(X)$ thus by linearity of Φ we get:

$$\begin{aligned}\sum_{i,j\,=1}^{n} b_i^* \Phi(a_i^* a_j) b_j &= \sum_{i,j=1}^{n} \overline{b_i(x)} \Phi(a_i^* a_j) b_j(x) \\ &= \Phi\left(\sum_{i,j=1}^{n} \overline{b_i(x)} a_i^* a_j b_j(x) \right) \\ &= \Phi\left(\left[\sum_{i=1}^{n} b_i(x) a_i \right]^* \left[\sum_{i=1}^{n} b_i(x) a_i \right] \right)\end{aligned}$$

which is positive by positivity of Φ. □

3 General Case

Definition 3.1. *Let $\Phi : \mathcal{A} \to \mathcal{B}$ be any map. We define:*

$$\begin{aligned}\Phi_n : \mathcal{M}_n(\mathcal{A}) &\to \mathcal{M}_n(\mathcal{B}) \\ (a_{ij})_{ij} &\mapsto (\Phi(a_{ij}))_{ij}\end{aligned}$$

Definition 3.2. *A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ between $*$-algebras is called n-positive if Φ_n is positive.*

Proposition 3.1. *A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ between $*$-algebras is completely positive if and only if it is n-positive for all $n \in \mathbb{N}$*

Remark 3.1. *Let $\mathcal{A} = \mathcal{B}(H)$ with $P, Q \in \mathcal{A}^+$ and $A \in \mathcal{A}$ then the following two statements are equivalent.*

1. $\begin{pmatrix} P & A \\ A^* & Q \end{pmatrix}$ *is positive*

2. $|\langle u, Av \rangle|^2 \leq \langle u, P, u \rangle \langle v, Q, v \rangle \quad \forall u, v \in H$

Proposition 3.2. *Let $\mathcal{A}, \mathcal{B}$ be two unital C^*-algebras then*

1. *for $a \in \mathcal{A} : ||a|| \leq 1$ if and only if the matrix* $\begin{pmatrix} \mathbb{1} & a \\ a^* & \mathbb{1} \end{pmatrix}$ *is positive*

2. *for $b \in \mathcal{A}^+$ and $a \in \mathcal{A}$ we have $a^* a \leq b$ if and only if* $\begin{pmatrix} \mathbb{1} & a \\ a^* & b \end{pmatrix}$ *is positive*

3. *let $\Phi : \mathcal{A} \to \mathcal{B}$ be 2-positive then Φ is contractive.*

4. *let Φ as above then for $a \in \mathcal{A}$ we have $\Phi(a)^* \Phi(a) \leq \Phi(a^* a)$*

Proof (1) & (2). Notice that $\mathbb{1}$ is positive. Thus we can apply the result from the previous remark. This yields inequalities from functional analysis that give the statements. □

Proof (3): Take $a \in \mathcal{A}$ such that $||a||= 1$ apply Φ_2 to the matrix $\begin{pmatrix} \mathbb{1} & a \\ a^* & \mathbb{1} \end{pmatrix}$, which is positive by (1) so its image is also positive (since Φ is 2-positive). Thus again using (1) we get that $||\phi(x)||\leq 1$ by linearity we get the statement. □

Proof (4): Let $a \in \mathcal{A}$ consider the calculation

$$\begin{pmatrix} \mathbb{1} & a \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} \mathbb{1} & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & a \\ a^* & a^*a \end{pmatrix}$$

which is positive by construction so its image under Φ_2

$$\begin{pmatrix} \mathbb{1} & \Phi(a) \\ \Phi(a)^* & \Phi(a^*a) \end{pmatrix}$$

is also positive and so by (2) we have that $\Phi(a)^*\Phi(a) \leq \Phi(a^*a)$. □

Theorem 3.1 (Arveson)**.** *Let $\mathcal{A}$ be a C^*-algebra, $S \subset \mathcal{A}$ an operator system and H a Hilbert space. For a given $\varphi \in \mathrm{CP}(S, \mathcal{B}(H))$ there always exists a $\Phi \in \mathrm{CP}(\mathcal{A}, \mathcal{B}(H))$ such that:*

$$\Phi|_S = \varphi$$

References

[1] Stéphane Attal, Alain Joye, and Claude-Alain Pillet, editors. *Open Quantum Systems I.* Springer Berlin Heidelberg, 2006.

[2] Stéphane Attal, Alain Joye, and Claude-Alain Pillet, editors. *Open Quantum Systems II.* Springer Berlin Heidelberg, 2006.