Classical Systems, Chapter 3

3.1 Basics of Ergodic Theory

Definition 0.1 *A dynamical system (X, φ_t, μ) is **mixing** if for any $\mu-$absolutely continuous measure ρ and all observable $f \in L^\infty(X, d\mu)$ it holds:*

$$\lim_{t\to\infty} \rho_t(f) = \mu(f).$$

Proposition 0.2 *Mixing implies ergodicity.*

Proof Let A be an invariant set modulo μ, such that $\mu(A) > 0$. Then $\rho(f) = \mu(f_{\chi_A})/\mu(A)$ defines a $\mu-$absolutely continuous invariant measure with $\rho(A) = 1$. If μ is mixing, then

$$1 = \rho(A) = \rho_t(A) = \lim_{t\to\infty} \rho_t(A) = \mu(A).$$

With Theorem 3.6 from the book, we can conclude that μ is ergodic. □

Note that the reverse statement (ergodicity implies mixing) is not true.

3.2 Classical Koopmanism

Definition 0.3 *The **Koopman space** of the dynamical system (X, φ_t, μ) is the Hilbert space $\mathcal{H} = L^2(X, d\mu)$. On this space, the **Koopman operators** U^t are defined by*

$$U^t f \equiv f \circ \varphi_t.$$

In the following we assume that the Koopman space is separable.

Lemma 0.4 (Koopman Lemma) *If $\mathcal{H}$ is separable, then U^t is a strongly continuous group of unitary operators on $\mathcal{H}$.*

Proof It has already been shown in the book that U^t is a group of isometries on $\mathcal{H}$. Since $U^t U^{-t} = I$, we have $ran(U^t) = \mathcal{H}$ and therefore U^t is unitary. Since the map $t \mapsto (f, U^t g)$ is measurable and $\mathcal{H}$ is separable, it follows from theorem 0.6 that $t \mapsto U^t$ is strongly continuous. □

Definition 0.5 *There exists a self-adjoint operator L on $\mathcal{H}$, such that $U^t = e^{-itL}$. We call L the **Liouvillean** of the system.*

Theorem 0.6 (Mean ergodic Theorem) *Let $U^t = e^{-itA}$ be a strongly continuous group of unitaries on a Hilbert space $\mathcal{H}$, P the orthogonal projection on $Ker(A)$. Then we have for all $f \in \mathcal{H}$*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T U^t f \, dt = Pf.$$

Proof Since U^t is continuous, $\langle f\rangle_T = \frac{1}{T}\int_0^T U^t f\,dt$ is well defined by the Riemann integral. For all $f \in ran(A)$ and some $g \in D(A)$, we have:

$$U^t f = U^t A g = i\delta_t U^t g.$$

Putting this into the integral, we get

$$\lim_{T\to\infty}\langle f\rangle_T = \lim_{T\to\infty}\frac{1}{T}\int_0^T i\delta_t U^t g\,dt = \lim_{T\to\infty}\frac{i}{T}(U^T - I)g = 0.$$

With $\|\langle u\rangle_T - \langle v\rangle_T\| \leq \|u - v\|$ the above result holds for all $f \in \overline{ran(A)} = Ker(A)^{\perp}$. Note that $f \in Ker(A)$ implies $\langle f\rangle_T = f$. Therefore

$$\lim_{T\to\infty}\langle f\rangle_T = \lim_{T\to\infty}\langle Pf\rangle_T + \lim_{T\to\infty}\langle (I-P)f\rangle_T = Pf. \qquad \square$$

Theorem 0.7 (Koopman Ergodicity Criterion) *A dynamical system is ergodic if and only if* 0 *is a simple eigenvalue of its Liouvillean L.*

Proof $\underline{\Rightarrow:}$ Let $f \in Ker(L)$. Then f is an invariant function in $L^1(X, d\mu)$ and by Theorem 3.6, we know that if μ is ergodic, then $f = \mu(f)$. And therefore the dimension of $Ker(L)$ is 1.
$\underline{\Leftarrow:}$ Assume now that $Ker(L)$ is one-dimensional and let A be an invariant set modulo μ. So $\chi_A \in \mathcal{H}$ is invariant and therefore $\chi_A \in Ker(L)$. Since $\chi_A = \mu(A)$, it follows that $\mu(A) \in \{0,1\}$ and with Theorem 3.6 we can conclude that μ is ergodic. $\square$

Theorem 0.8 (Koopman Mixing Criterion) *A dynamical system is mixing if and only if*

$$w - \lim_{t\to\infty} U^t = (1,\cdot)1. \qquad (0.1)$$

Proof $\underline{\Rightarrow:}$ Set $\mathcal{H}_{1+} := \{g \in \mathcal{H} | g \geq 0, \mu(g) = 1\}$. Any $g \in \mathcal{H}_{1+}$ is the Randon-Nikodym derivative of some $\mu-$absolutely continuous probability ρ, so we get for all $f \in L^\infty(X, d\mu)$

$$(g, U^t f) = \rho_t(f) \xrightarrow{t\to\infty} \mu(f) = (g,1)(1,f). \qquad (0.2)$$

Because every $g \in \mathcal{H}$ is a finite linear combination of elements in $\mathcal{H}_{1+}$, this holds for all $g \in \mathcal{H}$. Since the left and the right hand side of equation 0.2 are $\mathcal{H}$-continuous in f and uniformly in t and with the fact that L^∞ is dense in $\mathcal{H}$, equation 0.1 follows.
$\underline{\Leftarrow:}$ Suppose that ρ is a $\mu-$absolutely continuous probability and g its Randon-Nikodym derivative. Assume that $g \in \mathcal{H}$. With equation 0.1 we get

$$\lim_{t\to\infty}\rho_t(f) = \mu(f)$$

for all $f \in L^\infty$. $\rho_t(f)$ is L^1-continuous in g and uniformly in t. So with the fact that $\mathcal{H}$ is dense in L^1, it follows that the system is mixing. $\square$