# 4.2 The modular group

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**Abstract**

This summary focuses on section 4.2 of the chapter about modular theory in the published lecture notes on Open Quantum Systems by S. Attal , C.-A. Pillet, and A. Joye [AJ06].
Central to the chapter is the concept of the modular group, defined by $\sigma_t(A) = \Delta^{it} A \Delta^{-it}$ for $A \in$ B(H).
We give a proof of Tomita-Takesaki's theorem in the case when $\Delta$ is a bounded operator. Then one shows that

$$w(A\sigma_t(B)) = w(\sigma_{t+i}(B)A)$$

for all $A, B \in$ M and remarks that the automorphism group satisfying that equation is unique for a given state $w$. At the end, one summarizes a proof of an equivalence concerning the group $\sigma_t(A) = e^{itH} A e^{-itH}$ of automorphisms on B$(K)$, when H is a self-adjoint operator on $K$.

## 1 Repetition of modular theory

First let us reintroduce the operators that will be important for the understanding of the modular group. Let $(\mathrm{M}, w)$ be a pair of a von Neumann algebra acting on some Hilbertspace and $w$ a normal, faithful state on the Hilbertspace.
We can consider its Gelfand-Naimark-Segal (G.N.S.) representation which consists of the triple $(H, \pi, \Omega)$, where $H$ is the Hilbertspace, $\pi$ the representation of M in B$(H)$ and $\Omega$ a unit vector in $H$ such that

(i) $\pi$ is a morphism from M to B$(H)$

(ii) $w(A) =< \Omega, \pi(A)\Omega > \forall A \in \mathrm{M}$

(iii) $\{\pi(A)\Omega\,,\, A \in \mathrm{M}\}$ is dense in $H$.

One can then identify each element $A \in$ M with its representation $\pi(A) \in$ B$(H)$. This is important in the definition of the following operators:

$$S_0 := \mathrm{M}\,\Omega \to \mathrm{M}\,\Omega$$
$$A\Omega \to A^*\Omega$$

and

$$F_0 := \mathrm{M'}\,\Omega \to \mathrm{M'}\,\Omega$$
$$A\Omega \to A^*\Omega$$

where M' is the commutant of M . The operators $S$ and $F$ are the closed extensions of the operators defined beforehand.
As showed in a previous chapter of the book [AJ06], it holds that $S = S^{-1}$ and $F = S^*$.

We now define the modular operator

$$\Delta := FS = S^*S$$

that is invertible with inverse $\Delta^{-1} := SF = SS^*$.

Besides that we define the modular conjugation $J$ such that

$$S = J(S^*S)^{1/2}$$

is an anti-isometry from $H \to H$, which means that $< Sv, Sw > = < w, v >$ for all $v, w \in H$.
With the help of the modular conjugation, we get

$$S = J\Delta^{1/2} = \Delta^{-1/2}J \quad (1)$$

$$F = \Delta^{1/2}J = J\Delta^{-1/2}. \quad (2)$$

The following properties were also shown in the chapter beforehand:

$$J = J^{-1} \quad (3)$$

$$\Delta^{it}J = J\Delta^{it}. \quad (4)$$

## 2 Tomita-Takesaki's theorem

For the well-definedness of the modular group that will be introduced later, we show the following lemma for bounded operators which is then generalized by Tomita-Takesaki's theorem that holds for unbounded operators as well. The proof of the theorem can be found here [Tak03] and uses left and right Hilbertspace algebras.

**Lemma 1** *Let us assume that $\Delta$ is bounded.*

*(i)* SMS ⊂ M'

*(ii)* FM'F ⊂ M

*(iii)* $\Delta^n \, \mathrm{M} \, \Delta^{-n} \subset \mathrm{M} \; \forall n \in \mathbb{N}_0$

*(iv)* $\Delta^{\mathrm{z}} \, \mathrm{M} \, \Delta^{\text{-z}} = \mathrm{M} \; \forall z \in \mathbb{C}$

*(v)* JMJ = M'

**Proof 1** *(i) We want to show that $SAS$ and $B$ commute for $A, B$ arbitrary in $\mathrm{B}(H)$, then $SAS$ lies in $\mathrm{M}' = \{B \in \mathrm{B}(H) | B\,\mathrm{M} = \mathrm{M}\,B\}$.*
*Let $C \in \mathrm{M}$, then by the definition of $S$:*

$$SAS(BC\Omega) = S(AC^*B^*\Omega) = BCA^*\Omega = BS(AC^*\Omega) = BSAS(C\Omega)$$

*So by the the density of $\{A\Omega\,,\, A \in \mathrm{M}\}$ in $H$, the first inclusion follows.*

*(ii) follows by a similar argument as i)*

*(iii) We show $\Delta\,\mathrm{M}\,\Delta^{-1} \subset \mathrm{M}$ and the statement follows by induction over $\mathbb{N}$.*

$$\Delta\,\mathrm{M}\,\Delta^{-1} = (FS)\,\mathrm{M}(SF) \overset{i)}{\subset} F\,\mathrm{M'}\,F \overset{ii)}{\subset} \mathrm{M}$$

(iv) *In order to extend the statement from N to the complex plane C, one uses complex analysis argumentation. Carlson's theorem states that if*

  (a) *$f(z)$ is an entire function of exponential type (i.e. such that $|f(z)| < ce^{\tau|z|}$ for $c, \tau \in \mathbb{R}$).*
  (b) *$\exists c < \pi$ such that $|f(iy)| < ce^{c|y|}$ for $y \in \mathbb{R}$*
  (c) *$f(n) = 0 \, \forall n \in \mathbb{N}$ implies $f = 0$,*

*then $f(z) = 0$ for all $z \in \mathbb{C}$.*
*We can apply this theorem to the function*

$$f(z) = ||\Delta||^{-2z} < \phi, [\Delta^z A \Delta^{-z}, A']\psi > \tag{5}$$

*for any $A \in \mathrm{M}, A' \in \mathrm{M'}$ and $\phi, \psi \in H$. (Some details left out)*

(v)

$$J \, \mathrm{M} \, J \overset{iv)}{=} J\Delta^{1/2} \, \mathrm{M} \, \Delta^{-1/2} J \qquad = S \, \mathrm{M} \, S \overset{i)}{\subset} \mathrm{M'}$$
$$J \, \mathrm{M'} \, J \overset{iv)}{=} J\Delta^{-1/2} \, \mathrm{M'} \, \Delta^{1/2} J \qquad = F \, \mathrm{M'} \, F \overset{ii)}{\subset} \mathrm{M}$$

*As $J = J^{-1}$, the second equation can be rewritten as $\mathrm{M'} \subset J \, \mathrm{M} \, J$ and $\mathrm{M'} = J \, \mathrm{M} \, J$ follows.*

**Theorem 1 (Tomita-Takesaki's theorem)** *For $\Delta$ arbitrary, it holds that*

$$\mathrm{JMJ} = \mathrm{M'} \tag{6}$$
$$\Delta^{it} \, \mathrm{M} \, \Delta^{-it} = \mathrm{M} \tag{7}$$

# 3 The modular group

Let us define the following automorphism group of M :

$\sigma_t(A) := \Delta^{it} A \Delta^{-it}$ , $A \in \mathrm{B}(H)$(8)
We will first prove a property of this automorphism group that we will later see is unique to this automorphism group:

**Theorem 2** *For all $A, B \in \mathrm{M}$*

$$w(A\sigma_t(B)) = w(\sigma_{t+i}(B)A) \tag{9}$$

**Proof 2**

$$\begin{aligned}
w(A\sigma_t(B)) &= < \Omega, A\Delta^{it} B \Delta^{it} \Omega > \\
&= < \Delta^{-it} A^* \Omega, B\Omega > \\
&= < \Delta^{-it-1} \Delta^{1/2} A^* \Omega, \Delta^{1/2} B\Omega > \\
&= < J\Delta^{1/2} B\Omega, J^2 \Delta^{-it+1} J \Delta^{1/2} A^* \Omega > \\
&= < SB\Omega, \Delta^{-it+1} S A^* \Omega > \\
&= < B^* \Omega, \Delta^{-i(t+i)} A\Omega > \\
&= < \Delta^{-i(t+i)} \Omega, B\Delta^{-i(t+i)} A\Omega > \\
&= w(\sigma_{t+i}(B)A)
\end{aligned}$$

*where we used that $J$ is an anti-isometry, that $J^2 = Id$ and that $\Delta\Omega = \Omega$.*
*We also used that $\Delta^{-it}\Delta^1 = \Delta^{it} SF = \Delta^{-it} J \Delta^{1/2} \Delta^{1/2} J = J\Delta^{it+1} J$, where we exploited the fact that $J$ and $\Delta^{it}$ commute.*

Next we will just state a uniqueness statement concerning $\sigma_t$. A proof is found in the book[AJ06].

**Theorem 3** *$\sigma_t$ is the only automorphism group to satisfy 9 on* M *for a given $w$.*

So the property $w(A\sigma_t(B)) = w(\sigma_{t+i}(B)A)$ for all $A, B \in \mathrm{M}$ uniquely defines the modular group.

Finally we will define another automorphism group by

$$\sigma_t(A) := e^{itH} A e^{-itH} \tag{10}$$

for H self-adjoint on $K$.

**Theorem 4** *Let $w$ be a state such that $w(A) = tr(\rho A)$ on* $\mathrm{B}(K)$ *with $\rho$ being a positive trace-class operator with $tr(\rho) = 1$, for example a normal state.*
*Then it holds that for all $A, B \in \mathrm{B}(K)$ and $t, \beta \in \mathbb{R}$:*

$$w(A\sigma_t(B)) = w(\sigma_{t-\beta_i}(B)A) \tag{11}$$

*if and only if*

$$\rho = \frac{1}{Z} e^{-\beta H} \tag{12}$$

*where $Z = tr(e^{-\beta H})$.*

**Proof 3** *Lets assume that $\rho = \frac{1}{Z}e^{-\beta H}$, then by straightforward calculation using the cyclic permutation property of the trace, it follows*

$$\begin{aligned} w(A\sigma_t(B)) &= tr(\rho A e^{itH} B e^{itH}) \\ &= \frac{1}{Z} tr(e^{-\beta H} A e^{itH} B e^{itH}) \\ &= \frac{1}{Z} tr(A e^{-\beta H} e^{(it+\beta)H} B e^{-(it+\beta)H}) \\ &= \frac{1}{Z} tr(e^{-\beta H} \sigma_{t-\beta_i}(B) A) \\ &= w(\sigma_{t-\beta_i}(B)A). \end{aligned}$$

*For showing the other implication, one sees that by setting $t$ to $0$ that*

$$w(AB) = w(A\sigma_0(B)) = w(\sigma_{-\beta i}(B)A) = tr(\rho e^{\beta H} B e^{-\beta H} A) = tr(A\rho e^{\beta H} B e^{-\beta H})$$

*holds for all $A \in \mathrm{B}(K)$ as well as*

$$w(AB) = tr(\rho AB) = tr(AB\rho).$$

*As $A$ is chosen arbitrarily, we get that*

$$\begin{aligned} B\rho &= \rho e^{\beta H} B e^{-\beta H} \\ B(\rho e^{\beta H}) &= (\rho e^{\beta H})B \end{aligned}$$

*which again holds for all $B \in \mathrm{B}(K)$.*
*So it follows that for some $\alpha \in \mathbb{R}$*

$$\rho e^{\beta H} = \alpha Id$$

*As $tr(\rho) = 1$, we get that $\alpha = \frac{1}{tr(e^{-\beta H})}$ and the statement follows.*

# References

[AJ06] C.-A. Pillet Attal, S. and Alain Joye. Open quantum systems- the hamiltonian approach. *Springer*, page 86 to 99, 2006.

[Tak03] Masamichi Takesaki. Theory of operator algebras ii. *Encyclopaedia of Mathematical Sciences*, 125, 2003.