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C^* - Algebras

C^* - Algebras (and Von-Neumann algebras) have historically been studied in the context of linear operators on Hilbert-Spaces. In particular we will see further on in the course that they can be represented as sub-algebras of the algebra of bounded operators acting on a Hilbert-space $\mathcal{B}(H)$.

However, due to the work of Gelfand and Naimark (1943), there exists an abstract definition of C^*-Algebras that does not refer to any specific representation.

(Def. Algebra)

<u>Definition</u>: C^*-Algebras ($*$-Algebras, Banach*-Algebras) (Note that the algebra is over $\mathbb{C}$)

A <u>C^*-Algebra</u> is an <u>algebra</u> $\mathcal{A}$ equipped with an involution $* : \mathcal{A} \to \mathcal{A}$, $A \mapsto A^*$ and a norm $\|\cdot\|$ such that for all $A, B \in \mathcal{A}$, $\lambda, \mu \in \mathbb{C}$

$*$-algebra:
- i.) $A^{**} = A$ ($*$ is an involution)
- ii.) $(\lambda A + \mu B)^* = \bar{\lambda} A + \bar{\mu} B$ ($*$ is anti-linear)
- iii.) $(AB)^* = B^* A^*$ ($*$ is „anti-distributive")

$\|\cdot\|$ is a norm:
- i'.) $\|A\| \geq 0$ and $\|A\| = 0$ iff $A = 0$ (positive definiteness of $\|\cdot\|$)
- ii'.) $\|\lambda A\| = |\lambda| \, \|A\|$ (absolute homogenity of $\|\cdot\|$)
- iii'.) $\|A + B\| \leq \|A\| + \|B\|$ (triangle inequality for $\|\cdot\|$)
- iv'.) $\|AB\| \leq \|A\| \, \|B\|$ (product inequality) (guarantees mult. is cont.)
- i''.) $\mathcal{A}$ is complete for $\|\cdot\|$ ($\mathcal{A}$ is a Banach-space)
- ii''.) $\|AA^*\| = \|A\|^2$ (C^*-Norm-identity)

If i.), ii.) and iii.) are fulfilled, we speak of a <u>$*$-algebra</u>.
If we replace ii''.) by

ii'''.) $\|A^*\| = \|A\|$ ($*$ is an isometry),

then we speak of a <u>Banach*-algebra</u>.

The most well-known example of such a $*$-map is the adjoint of an operator.

In fact, a C^*-Algebra is always a Banach*-algebra:

<u>Proposition 2.1.</u>
On a C^*-algebra $\mathcal{A}$ we have for all $A \in \mathcal{A}$
$$\|A^*\| = \|A\|$$

Proof:

- $\|A\|^2 = \|A^*A\| \leq \|A^*\| \, \|A\| \Rightarrow \|A\| \leq \|A^*\|$
- $\|A^*\|^2 = \|(A^*)^* A^*\| = \|A A^*\| \leq \|A\| \, \|A^*\| \Rightarrow \|A^*\| \leq \|A\|$

$$\Rightarrow \|A^*\| = \|A\|$$

$\square$

<u>Examples</u>

1.) The set of bounded operators on a Hilbert-space $\mathcal{H}$: $\mathcal{B}(\mathcal{H})$
(The involution is the adjoint map and the norm is the usual operator norm: $\|A\| = \sup\limits_{\substack{x \in \mathcal{H} \\ \|x\| = 1}} \|Ax\|$)
(Exercise: prove C^*-norm identity, submultiplicative

2.) The set of compact operators on a Hilbert-space $\mathcal{H}$: $\mathcal{K}(\mathcal{H})$
(This is a sub-C^*-algebra of $\mathcal{B}(\mathcal{H})$ since $\mathcal{K}(\mathcal{H})$ is norm-closed in $\mathcal{B}(\mathcal{H})$ and hence complete.)

3.) The set of continuous functions vanishing at infinity on a locally compact set X : $C_0(X)$
(Recall that a function f on X vanishes at infinity if for all $\varepsilon > 0$ there exists $K \subseteq X$ kompakt s.t. $|f| < \varepsilon$ outside of K.) (pointwise add., mult. $\Rightarrow$ commutative)

As we will see further on in the course, this examples will more than basic.
In particular every C^*-algebra will be a sub-algebra of a type 1.) example, and every commutative C^*-algebra of the form 3.).

Definition

An element I of a C^*-Algebra $\mathcal{A}$ is called the unit if

$$\forall A \in \mathcal{A} : \quad IA = AI = A$$

$\circledast$ **Lemma:**

If a unit exists, then it is unique and has norm 1 (unless $\mathcal{A} = 0$, trivial case).

Proof:

i.) Assume there exists two units I, I'. Then

$$I I' = I, \quad I I' = I' \quad \Rightarrow \quad I = I'$$

ii.) $I = I^* \Rightarrow \|I\| = \|I I^*\| = \|I\|^2 \Rightarrow \|I\| = 1$

$\square$

$\circledast$ **Lemma:**

Let $I \in \mathcal{A}$ be the unit of a C^*-algebra (or $*$-algebra / Banach*-algebra) $\mathcal{A}$. Then

$$I^* = I$$

Proof:

- $I^* A = (A^* I)^* = (A^*)^* = A$ (Prop. i.), iii.))
- $A I^* = (I A^*)^* = (A^*)^* = A$

$\Rightarrow I^*$ is a unit

$\Rightarrow I^* = I$ by uniqueness of unity

Remark: The unit must not always exist.

For example, the unit in $\mathcal{B}(\mathcal{H})$ is the identity operator I. However, I is compact if and only if $\mathcal{H}$ is finite-dimensional (to show this one can use sequences that do not contain convergent subsequences.). Hence, for $\mathcal{H}$ infinite-dimensional, $I \notin \mathcal{K}(\mathcal{H})$.

In the case of $C_0(X)$, the unit should clearly be the function $\mathbb{1}$ that maps all of X to 1. Then $\mathbb{1} \in C_0(X)$ iff X is kompakt

$\Leftarrow$: vacous truth

$\Rightarrow$: Kontra-position

Let us now show how we can add a unit to a non-unital C^*-algebra.

<u>Proposition</u>

Let $\mathcal{A}$ be a C^*-algebra without a unit.
Then $\mathcal{A}' := \mathcal{A} \oplus \mathbb{C}$ with elementwise addition and scalar multiplication and product $(A,\lambda)(B,\mu) = (AB + \lambda B + \mu A, \lambda\mu)$,
involution

$$(A,\lambda)^* := (A^*, \bar{\lambda})$$

and norm

$$\|(A,\lambda)\| := \sup_{\substack{\|B\|=1 \\ B\in\mathcal{A}}} \|AB + \lambda B\|$$

is a C^*-algebra with unit $(0,1)$. We identify the original C^*-algebra $\mathcal{A}$ with the set $\{(A,0) \mid A \in \mathcal{A}\} \subseteq \mathcal{A}'$

<u>Proof</u>:

- It is clear from the definition of the multiplication in $\mathcal{A}'$ that $(0,1)$ is the unit.
- We check that $\mathcal{A}'$ is indeed a C^*-algebra.

(Trivial) i.) $((A,\lambda)^*)^* = (A^*, \bar{\lambda})^* = (A, \lambda)$

(Trivial) ii.)
$$\begin{aligned}(x(A,\lambda) + y(B,\mu))^* &= ((xA, x\lambda) + (yB, y\mu))^* \\ &= (xA + yB, x\lambda + y\mu)^* \\ &= (\bar{x}A^* + \bar{y}B^*, \bar{x}\bar{\lambda} + \bar{y}\bar{\mu}) \\ &= (\bar{x}A^*, \bar{x}\bar{\lambda}) + (\bar{y}B^*, \bar{y}\bar{\mu}) \\ &= \bar{x}(A,\lambda)^* + \bar{y}(B,\mu)^*\end{aligned}$$

(Trivial) iii.)
$$\begin{aligned}((A,\lambda)(B,\mu))^* &= (AB + \lambda B + \mu A, \lambda\mu)^* \\ &= (B^*A^* + \bar{\mu}A^* + \bar{\lambda}B^*, \bar{\mu}\bar{\lambda}) \\ &= (B^*, \bar{\mu})(A^*, \bar{\lambda}) \\ &= (B,\mu)^*(A,\lambda)^*\end{aligned}$$

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(non-trivial!) ii.) Reminder: Z: $\| (A,\lambda) \| \geq 0$ and $\| (A,\lambda) \| = 0 \Leftrightarrow A = 0, \lambda = 0$

- $\| (A,\lambda) \| = \sup\limits_{\|B\|=1} \overbrace{\| AB + \lambda B \|}^{\geq 0} \geq 0$
- Lemma: $\| A \| = \sup\limits_{\|B\|=1} \| AB \|$

Proof: $- \ \sup\limits_{\|B\|=1} \| AB \| \leq \sup\limits_{\|B\|=1} \| A \| \|B\| = \| A \|$

$$- \ \sup_{\|B\|=1} \| AB \| \geq \| A \frac{A^*}{\|A\|} \| \qquad \left(\| \frac{A^*}{\|A\|} \| = \frac{\|A^*\|}{\|A\|} \overset{\text{Prop.}}{=} 1 \right)$$

$$= \frac{1}{\|A\|} \overbrace{\| A A^* \|}^{= \|A\|^2}$$

$$= \| A \|$$

- Assume $\lambda = 0$. Then

$$\| (A, 0) \| = \sup_{\|B\|=1} \| AB \| \overset{\text{Lemma}}{=} \| A \| = 0,$$

so $\| (A,0) \| = 0 \Leftrightarrow \|A\| = 0$

- Hence we can assume $\lambda \neq 0$ and w.l.o.g. $\lambda = 1$.

We compute

$$\| B - AB \| = \| B \| \, \| \tfrac{B}{\|B\|} - A \tfrac{B}{\|B\|} \|$$

$$\leq \| B \| \sup_{\|C\|=1} \| C - AC \|$$

$$= \| B \| \, \| (-A, 1) \|$$

Thus, if $\| (-A, 1) \| = 0$, then

$$B = AB \quad \forall B \in \mathcal{A}$$

and applying the involution $*$ on $B^* = AB^*$

$$B = BA^* \qquad \forall B \in \mathcal{A}$$

Now choosing $B = A$ and $B = A^*$ we find

$$A = AA^* = A^*$$

and we conclude that

$$B = AB = BA \quad \forall B \in \mathcal{A}$$

and hence A is a unit, in contradiction to our assumption that $\mathcal{A}$ is non-unital.

(trivial) ii'.) $\| x(A,\lambda)\| = \|(xA, x\lambda)\|$

$$= \sup_{\|B\|=1} \| xAB + x\lambda B\|$$

$$= |x| \sup_{\|B\|=1} \|AB + \lambda B\|$$

$$= |x| \, \|(A,\lambda)\|$$

(trivial) iii'.) $\|(A,\lambda) + (B,\mu)\| = \|(A+B, \lambda+\mu)\|$

$$= \sup_{\|C\|=1} \|(A+B)C + (\lambda+\mu)C\|$$

$$= \sup_{\|C\|=1} \| AC + \lambda C + BC + \mu C\|$$

$$\leq \sup_{\|C\|=1} \left(\|AC + \lambda C\| + \|BC + \mu C\|\right)$$

$$= \|(A,\lambda)\| + \|(B,\mu)\|$$

v'.) (Exercise)

i''.) Completeness : follows from completeness of A and $\mathbb{C}$

ii.''.) (Exercise)