Seminar on Operator Algebras HS23

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03.11.2023

1 An algebraic view on probability

At first, we will recap some of the main definitions. Then we will introduce algebraic notions that are motivated by objects from standard probability theory.

Definition 1.1. Let $\mathfrak{U}$ be a $\mathbb{C}$-vector space with a bilinear multiplication operation $\mathfrak{U} \times \mathfrak{U} \ni (u, v) \mapsto uv \in \mathfrak{U}$ that is associative, i.e. $\forall u, v, w \in \mathfrak{U} : (uv)w = u(vw)$. Then we call $\mathfrak{U}$ an algebra. If it contains a unity $\mathbb{1}$, we call $\mathfrak{U}$ unital.

Definition 1.2. Let $\mathfrak{U}$ be an algebra endowed with a map $\mathfrak{U} \ni u \mapsto u^* \in \mathfrak{U}$ that satisfies:

(i) for all $u \in \mathfrak{U} : (u^*)^* = u$ (involution).

(ii) for all $\alpha, \beta \in \mathbb{C}$ and $u, v \in \mathfrak{U} : (\alpha u + \beta v)^* = \bar{\alpha} u + \bar{\beta} v$ (antilinearity).

(iii) for all $u, v \in \mathfrak{U} : (uv)^* = v^* u^*$.

Them we call $\mathfrak{U}$ an ***-algebra**.
We further define the cone of positive elements $\mathfrak{U}^+ = \{u^* u \in \mathfrak{U} \mid u \in \mathfrak{U}\}$, which induces a partial order[1] on $\mathfrak{U} : u \leq v \iff v - u \in \mathfrak{U}^+$.

Definition 1.3. Let $\mathfrak{U}$ be an unital *-algebra, that satisfies:

(i) for all $u \in \mathfrak{U}^+$ there is a $\lambda \in \mathbb{R}^+ : u \leq \lambda \mathbb{1}$.

(ii) if an increasing net $(x_j)_{j \in J}$ has an upper bound in $\mathfrak{U}^+$ it has a least upper bound $\sup_j u_j \in \mathfrak{U}^+$.

Then we call $\mathfrak{U}$ a **D^*-algebra**.

Until now, we did not consider any topological notions. However, we will now come back to a central notion of this seminar with the next definition:

Definition 1.4. A normed *-algebra $\mathfrak{U}$ is called a **Banach algebra** if it is complete w.r.t. the induced metric and the norm satisfies $\|u\| = \|u^*\|$ for all $u \in \mathfrak{U}$. The induced topology is called the uniform topology.
We call a Banach algebra $\mathfrak{U}$ a **C*-algebra** if $\|u^* u\| = \|u\|^2$ for all $u \in \mathfrak{U}$. A subspace $\mathcal{S}$ of an unital C*-algebra is called operator system if it contains the unity and $\mathcal{S}^* \subset \mathcal{S}$.
If $\mathfrak{U} \subset \mathfrak{B}(\mathcal{H})$ is a sub-C*-algebra of the bounded operators on some Hilbert space that satisfies $\mathfrak{U}'' = \mathfrak{U}$ where $\mathfrak{U}''$ denotes the double commutant, we call $\mathfrak{U}$ a **von Neumann algebra**.

[1] To see that this not a total order, take for example matrices A, B s.t. $A - B$ is indefinite.

Lemma 1.1 (characterization of von Neumann algebras)**.** Let $\mathfrak{U} \subseteq \mathfrak{B}(\mathcal{H})$ be a sub-C*-algebra of the bounded operators on a Hilbert space, then TFAE:

- $\mathfrak{U}'' = \mathfrak{U}$.
- $\mathfrak{U}$ is weakly closed.
- there exist a predual $\mathfrak{U}_*$, which is a Banach space.

Proof. was already given on 13.10., see handout.

Definition 1.5. Let $\mathfrak{U}$ be an unital *-algebra and $\mathbb{E} : \mathfrak{U} \to \mathbb{C}$ a linear functional on $\mathfrak{U}$ that is positive: $E(u^*u) \geq 0$ for all $u \in \mathfrak{U}$ and is normalized: $E(\mathbb{1}) = 1$. We call such linear forms a state and a tuple $(\mathfrak{U}, \mathbb{E})$ an **algebraic probability space**. The set of all states is denoted $\mathfrak{S}(\mathfrak{U})$.

Definition 1.6. If $\mathfrak{B}$ is another *-algebra and $\mathfrak{U}$ is as above, then we call a map: $\phi : \mathfrak{B} \to \mathfrak{U}$ an **algebraic random variable**[2], if it is a *-morphism (linearity and $\forall a, b \in \mathfrak{B} : \phi(a^*b) = \phi(a)^*\phi(b)$). We call the state $\mathbb{E} \circ \phi$ the image state or the law/distribution of ϕ.

The following example shows, how the standard description of probability theory fits into this framework.

Example 1.1 (standard probability theory)**.** Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\mathfrak{U} = L^\infty(\Omega, \mathcal{A})$ be the set of real valued bounded measurable functions. Then $\mathfrak{U}$ is a D^*-algebra and the expectation value $\mathbb{E} : \mathfrak{U} \ni X \mapsto \mathbb{E}(X) := \int_\Omega X dP$ a linear form that makes $(\mathfrak{U}, \mathbb{E})$ into an algebraic probability space. If now $X \in \mathfrak{U}$ is a standard random variable, we can define $\phi_X : L^\infty(\mathbb{R}) \ni f \mapsto f \circ X \in \mathfrak{U}$, an algebraic random variable. Notice that $\mathbb{E}_{\phi_X}(f) = \mathbb{E}(f \circ X)$ induces a measure on $\mathbb{R}$ given by the distribution function of X, which is the image state of ϕ_X. ⊣

Definition 1.7. Let $(\mathfrak{U}, \mathbb{E})$ be an algebraic probability space.
The state $\mathbb{E}$ is called **normal**, if for all increasing nets $(x_j)_{j \in J}$ in $\mathfrak{U}^+$ with a least upper bound $x = \sup_j x_j$ it holds that $\mathbb{E}(x) = \sup_j \mathbb{E}(x_j)$. We denote the set of normal states on $\mathfrak{U}$ as $\mathfrak{S}_n(\mathfrak{U})$.[3]
A **pure** state is a normal state $\mathbb{E}$ s.t. if φ is a positive linear functional on $\mathfrak{U}$ that is majorized by $\mathbb{E}$, it follows that $\varphi = \lambda\mathbb{E}$ for some $\lambda \in [0, 1]$.
We call non pure states **mixed**.
Idempotent elements $p \in \mathfrak{U}$ (i.e. $p = p^2$) will be called **algebraic events**.

Example 1.2. Let us see how to interpret those definitions in a standard probability framework. Therefore we continue example 1.1. By the dominated convergence theorem, we have that the expectation value is a normal state on $L^\infty(\Omega, \mathcal{A})$. If P is e.g. a Dirac measure we have that $\mathbb{E}$ is pure. The events are exactly those functions with values in $\{0, 1\}$, i.e. (measurable) characteristic functions. Notice that we have a natural isomorphism between $\mathcal{A}$ and $\{\mathbb{1}_A \mid A \in \mathcal{A}\}$, thus we have that we have a natural isomorphism between algebraic events and events as a standard notion in probability theory. ⊣

Proposition 1.1. Let $\mathfrak{U}$ be an unital C*-algebra and $\mathbb{E}$ a state. Then $\mathbb{E}$ is pure if and only if it is an extremal point of the convex set $\mathfrak{S}_n(\mathfrak{U})$.

Proof. First let $\mathbb{E}$ be a pure state and assume it is a convex combination $\mathbb{E} = \lambda\mathbb{E}_1 + (1 - \lambda)\mathbb{E}_2$ with $\mathbb{E}_1, \mathbb{E}_2 \in \mathfrak{S}_n(\mathfrak{U})$, but then $\mathbb{E}_1, \mathbb{E}_2 \leq \mathbb{E}$, so by purity the combination has to be trivial.

[2] "algebraic" is left out in the reference, however, for the sake of clarity it makes sense to distinguish between standard RVs and algebraic RVs, it is also consistent with the term algebraic probability space.
[3] This definition is probably motivated by the dominated convergence theorem.

On the other hand if $\mathbb{E}$ is extremal, suppose there was a non trivial linear functional $\varphi < \mathbb{E}$ that was not of the form $\lambda\mathbb{E}$ for some $\lambda \in [0,1]$. Then we can write

$$\mathbb{E} = \varphi(\mathbb{1})\underbrace{(\frac{1}{\varphi(\mathbb{1})}\varphi)}_{=:\mathbb{E}_1} + (1-\varphi(\mathbb{1}))\underbrace{(\frac{1}{1-\varphi(\mathbb{1})})(\mathbb{E}-\varphi)}_{=:\mathbb{E}_2}.$$

showing that $\mathbb{E}$ is not an extremal point in $\mathfrak{S}_n\mathfrak{U}$. ⊣

Example 1.3 (a non commutative algebraic probability space)**.** Let $\mathfrak{U} = \mathbb{C}^{n\times n}$ be the complex square matrices and $\rho \in \mathbb{C}^{n\times n}$ be of unit trace. Define $\mathbb{E}(A) = \mathrm{tr}(\rho A)$ Then $(\mathfrak{U}, \mathbb{E})$ is a non commutative algebraic probability space.
Now let $U \in \mathfrak{U}$ be an unitary matrix, then the map $\phi_U : \mathfrak{U} \ni A \to U^*AU \in \mathfrak{U}$ defines an algebraic random variable.
The events P are exactly the matrices, that represent projections s.t. $P^2 = P$ ⊣

Example 1.4. Let $(\Omega, \mathcal{B}(\Omega), P)$ be a connected compact probability space. Set $\mathfrak{U} = C(\Omega, \mathbb{C})$, then $(\mathfrak{U}, \mathbb{E})$ is an algebraic probability space. Notice that all events are trivial characteristic functions on Ω are constant by connectedness and continuity. ⊣

2 Completely positive maps

2.1 transition kernels

Definition 2.1. Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be measurable spaces. We call a map $\kappa : \Omega_2 \times \mathcal{A}_1 \to [0,\infty]$ a **transition kernel** from Ω_2 to Ω_1, if

(i) $\omega_2 \mapsto \kappa(\omega_2, A_1)$ is $\mathcal{A}_2$ measurable $\forall A_1 \in \mathcal{A}_1$.

(ii) $P_{\omega_2} : A_1 \mapsto \kappa(\omega_2, A_1)$ is a probability measure on $(\Omega_1, \mathcal{A}_1)$.[4]

Example 2.1 (product measures)**.** [5] Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are two σ-finite measure spaces and let $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ be the product measurable space. Then

$$\kappa : X \times (\mathcal{A} \otimes \mathcal{B}) \to [0,\infty], (x, Q) \mapsto \nu(\{y \in Y \mid (x,y) \in Q\}).$$

defines a (not necessarily Markov) from X to $X \times Y$. ⊣

Example 2.2 (conditional probability)**.** [6] If X_1 is a random variable with values in $(\Omega_1, \mathcal{A}_1, P)$ and $\mathcal{F} \subseteq \mathcal{A}_1$ be a sub-σ-algebra. Then the conditional distribution of X_1 given $\mathcal{F}$ is a transition kernel:

$$\kappa_{X_1,\mathcal{F}}(\omega_2, A_1) = P[\{X_1 \in A_1\} \mid \mathcal{F}] := \mathbb{E}[\mathbb{1}_{A_1}(X_2) \mid \mathcal{F}].$$

A special case of this is when $\mathcal{F} = \sigma(X_2)$, where X_2 is some random variable, then

$$\kappa_{X_2,\sigma(X_1)}(\omega_2, A_1) = P[\{X_1 \in A_1\} \mid X_2 = \omega_2].$$

Both of these are transition kernels from Ω_2 to Ω_1.
If we now consider (X_1, X_2) as tuple of random variables that takes values in $\Omega_1 \times \Omega_2$ and Q is the probability distribution of X_2 then P_{ω_2} is the conditional probability of X_1 given $X_2 = \omega_2$. ⊣

[4](ii) is usually less strict, where we only demand $A_2 \mapsto \kappa(\omega_1, A_2)$ to be a measure, if it is also a probability measure $\forall \omega_1 \in \Omega_1$ one usually calls κ Markov kernel.
[5]for details see chapter 7 in Salamon [3]
[6]for details see chapter 8 in Klenke [4]

Example 2.3 (first encounter with complete positivity)**.** Now consider the two C*-algebras: $\mathfrak{U} = L^\infty(\Omega_1, \mathcal{A}_1)$ and $\mathcal{B} = L^\infty(\Omega_2, \mathcal{A}_2)$ with the usual operations. Then a transition kernel κ induces a linear map

$$\Phi_\kappa : \mathfrak{U} \to \mathfrak{B}, (\Phi_\kappa(u))(\omega_2) = \int_{\Omega_1} u(\omega_1) dP_{\omega_2}.$$

This map is positive, as it maps positive functions/elements to positive functions/elements. It is also **completely positive** (CP): for all $u_i \in \mathfrak{U}, b_i \in \mathfrak{B}, i = 1, ..., n$ we have that

$$\sum_{i,j=1}^{n} b_i^* \Phi_\kappa(u_i^* u_j) b_j \in \mathfrak{B}.$$

is positive as for any $\omega_2 \in \Omega_2$.[7].

Now suppose that $(\Omega_2, \mathcal{A}_2, Q)$ is a probability space and define $\Omega = \Omega_2 \times \Omega_1$. Let $\mathcal{A} = \mathcal{A}_2 \otimes \mathcal{A}_1$ be the product σ-algebra. Define the function $\mathbb{E}$ by:

$$\mathbb{E}[b \otimes u] = \int_{\Omega_2} (\Phi_\kappa(u))(\omega_2) b(\omega_2) dQ = \int_{\Omega_2} \int_{\Omega_1} u(\omega_1) dP_{\omega_2} b(\omega_2) dQ.$$

Notice that $\mathbb{E}[\mathbb{1}_{\Omega_2} \otimes \mathbb{1}_{\Omega_1}] = 1$ therefore this map induces[8] a probability measure $\mathbb{P}$ on $\mathcal{A}$ and a state by extending linearly.

At last consider $\mathfrak{U} = L^\infty(\Omega_1, \mathcal{A}_1)$ and $\mathfrak{B} = L^2(\Omega_2, \mathcal{A}_2, Q)$ with the usual inner product. Take their algebraic tensor product $\mathfrak{B} \otimes \mathfrak{U}$ and define for $b \otimes u, c \otimes v \in \mathfrak{B} \otimes \mathfrak{U}$:

$$\langle b \otimes u, c \otimes v \rangle_{\mathfrak{h}} = \langle b, \Phi_\kappa(u^* v) c \rangle_{L^2(\Omega_2, \mathcal{A}_2)} = \int_{\Omega_2} b^*(\omega_2)(\Phi_\kappa(u^* v))(\omega_2) c(\omega_2) dQ.$$

By extending this linearly it becomes an inner product[9]. By modding out the kernel and taking the completion, one gets the Hilbert space $\mathfrak{h}$ and the following equation

$$\mathbb{E}[b \otimes u] = \langle \mathbb{1}_{\Omega_2} \otimes \mathbb{1}_{\Omega_1}, b \otimes u \rangle_{\mathfrak{h}} = \langle \mathbb{1}_{\Omega_2} \otimes \mathbb{1}_{\Omega_1}, (b \otimes u)(\mathbb{1}_{\Omega_2} \otimes \mathbb{1}_{\Omega_1}) \rangle_{\mathfrak{h}}.$$

which is reminiscent of the Gelfand-Naimark-Segal representation [10].

$\dashv$

[7] Use the embedding $L^\infty(\Omega_1, \mathcal{A}_1, P_{\omega_2}) \hookrightarrow L^2(\Omega_1, \mathcal{A}_1, P_{\omega_2})$ for finite measure spaces, we get $(\sum_{i,j=1}^n b_i^* \Phi_\kappa(u_i^* u_j) b_j)(\omega_2) = \|\sum_{i=1}^n b_i(\omega_2) u_i\|^2_{L^2(\Omega_1, \mathcal{A}_1, P_{\omega_2})}$.

[8] First use characteristic functions to get a premeasure on boxes, then use Caratheodory's extension theorem.

[9] Note that the commutativity of the underlying algebras guarantees symmetry, positive definiteness follows from CP.

[10] see [1] chapter by Stephane Attal

3 Table

The following table gives an overview of how to include standard notions from probability into algebraic notions in operator theory. The notation should be clear from the previous discussion.

notion	standard	algebraic	inclusion
probability space	$(\Omega, \mathcal{A}, P)$	$(\mathfrak{U}, \mathbb{E})$	$\mathfrak{U} = L^\infty(\Omega, \mathcal{A})$, $\mathbb{E}(X) = \int_\Omega X dP$
event	$A \in \mathcal{A}$	$p = p^2 \in \mathfrak{U}$	$p = \mathbb{1}_A$
random variable	$X : \Omega \to \mathbb{C}$	$\phi : \mathfrak{B} \to \mathfrak{U}$	$\mathfrak{B} = L^\infty(\mathbb{C})$, $\phi(f) = X^* f$
transition kernel	$\kappa : \Omega_2 \times A_1 \to [0, 1]$	$\Phi : \mathfrak{U} \to \mathfrak{B}$ CP	Φ_κ see example 2.3

Table 1: From standard notions to algebraic notions

References

[1] Stéphane Attal, Alain Joye, Claude-Alain Pillet, *Open Quantum Systems I - The Hamiltonian Approach*, Springer-Verlag, 2006

[2] Stéphane Attal, Alain Joye, Claude-Alain Pillet, *Open Quantum Systems II - The Markovian Approach*, Springer-Verlag, 2006

[3] Dietmar Salamon, *Measure and Integration*, European Mathematical Society, 2016

[4] Achim Klenke, *Probability Theory*, Springer-Verlag, 3rd Ed., 2020