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SPOOKY ACTIONS AT A DISTANCE

On the problem of quantum correlations

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Sinistre azioni a distanza

Sulla natura delle correlazioni nella teoria quantistica

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Preface

> In the case of all things which have several parts and in which the totality is not, as it were, a mere heap, but the whole is something beside the parts, there is a cause.
>
> Aristotle – Metaphysics VIII

ONCE that it was given me the opportunity to carry out a research work in the field of the fascinating and upsetting quantum physics, the decision for the subject was easy. The knowledge in the field at the beginning of this research work was almost inexistent; it was exactly this opportunity to build a theory from its foundation, on a free basis, that has fascinated me and has stimulated the curiosity to undertake this way.

The theme of deepening was a choice affected by the recent experimental development in the field. This brought to square the circle giving a logical coherence, a begin and a end to the work.

Acknowledgements

I would like to thank Prof. Christian FERRARI for his help and availability that have allowed to realize this research project.

I thank Prof. Gianni BOFFA for the interesting discussions on the philosophical sides of the subject, Prof. Frederick JOPLING for the help in the realisation of the English production and Prof. Gian Michele GRAF for all his interesting hints.

I thank my family for their willingness to read the text and the huge amount of helpful advices.

Abstract

Subject

The description of quantum pure states in terms of unit vectors of certain complex spaces (so called Hilbert spaces, in this research simply $\mathbb{C}^2$) creates the possibility for superposition states due to linear combinations of pure states. If superposition occurs in a joint system, the state is called "entangled". In an entangled state the proprieties of the subsystems are not determined.

An important phenomenon related to entangled states is the two-particle quantum correlation. The measurement of the property of one subsystem yields the unambiguous determination of the other subsystem proprieties.

This research project attempts to determine, if possible, the nature of the two-particle quantum correlation phenomenon.

Methods

The research carried out is theoretical, and it includes an analysis of experimental results found in the literature. To begin the foundations of quantum physics were studied and modelled. The procedure was carried out with mathematical rigour in order to make it possible to create a structurally and logically sound theory and give validity to the conclusions; subsequently, quantum correlations were examined using the model so obtained. This research followed the historical development with references to the literature describing models and theories which have attempted to explain the nature of the two-particle quantum correlation phenomenon. Finally, the results obtained were compared to the results of the most recent experimentation in the field.

Results

The main result obtained in this research is a definite demonstration that quantum theory and the microscopic world violate Bell's inequality. A consequence of the violation of Bell's inequality is the impossibility of completing the quantum theory with a local hidden variable theory – or, in any case, it is shown that there would be a conflict between this theory and the quantum theory. This result had first been obtained only theoretically – many difficulties delayed an experimental demonstration. A demonstration which can be considered definitive was presented only recently.

The hypothesis of completing the quantum theory with additional variables – as proposed by the Einstein, Podolsky and Rosen group (known as EPR argument) – has thus been proved fallacious.

Discussion

The obtained main result does not lend itself to interpretation: the experimental demonstration of the theoretical result concerning the violation of Bell's inequality must be considered as definitive. Quantum theory can therefore not be completed with any local hidden variable theory.

However, the nonlocality of the microscopic world has yet to be proved beyond doubt. The discussion is therefore open as to whether Nature includes a nonlocal aspect, or if a classical and local model, not yet present in the literature, can explain the quantum correlation phenomenon.

Some scientists retain the quantum theory itself to be the local theory that explains the two-particle quantum correlation phenomenon.

Conclusion

In the end it was not possible to determine in a definitive way the nature of the two-particle quantum correlation phenomenon. The most important conclusion of this research project is that completing the quantum theory with any local hidden theory is definitely impossible. The only existing explanation for this phenomenon (which is in agreement with the experimental results) is a nonlocal action. Nevertheless, a local model cannot be excluded. The discussion is still open.

Part I

The quantum theory

Chapter 1

Foundations of the quantum theory

THIS his chapter is an introduction to the basics of the quantum theory. Its purpose is to create a solid base to allow the understanding of the subjects of study in the following chapters.

The fundamental concepts of the quantum physics will be introduced starting from the postulate of quantum logic and deducing from this the general rules of the quantum theory. All passages are explained step by step with the help of examples with photons. In appendix A you will find the basic mathematical concepts needed to follow the argumentation.

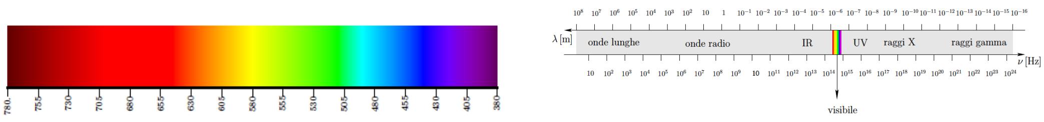
The theoretical aspects are taken from [21, 35, 41], while the exemplifications are from the author.

1.1 Light and Electromagnetic radiation

The debate on the nature of light has dominated scientific discussions for centuries, in particular the 18th and 19th century. In this period, more or less significant proof was presented both for the corpuscular theory, defended by NEWTON, and the wave theory, that saw in its most important exponents Huygens. Since Newton was more influential than the less acknowledged HUYGENS, the corpuscular theory was the more accredited one. However in the 20th century, mainly thanks to YOUNG and FRESNEL's works, the wave theory had taken the lead and became the accepted theory by the scientific community [48].

Crucial years for the light-study were these between 1861 and 1873. During this period MAXWELL formulated 4 equations known nowdays as Maxwell's laws, which sanction the union of the electrical and magnetic theory giving shape to the electromagnetism. The unification of this two theories established (in particular with the second $\nabla \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}$ and the fourth $\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ Maxwell's equations) that a magnetic field, respectively an electric field, variable in time generates an electric field, respectively a magnetic field. This unification includes also the theory of light; it is indeed possible to demonstrate that, in vacuum, the Maxwell's equations give origin to a wave equation both for the electrical and the magnetic field conferring them wave properties, in this case know as electromagnetic waves (EMR) of which the light is a special case [48].

Here below we offer a brief demonstration of the wave properties of an electrical field.



(a) Visible spectrum (in nm). (b) Electromagnetic spectrum.

Figure 1.1: Rapresentation of the light's and the EMR's spectrum.

From Maxwell's equations follows

$$\begin{aligned}\nabla \times \left(\nabla \times \vec{E}\right) &= \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\nabla \times \vec{B} \\ &= -\frac{\partial}{\partial t}\mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t} = -\mu_0\varepsilon_0\frac{\partial^2 \vec{E}}{\partial t^2}\end{aligned} \tag{1.1}$$

An important theorem postulates that for a vector field $\vec{F}$

$$\nabla \times \left(\nabla \times \vec{F}\right) = \nabla\left(\nabla \cdot \vec{F}\right) - \triangle \vec{F}$$

holds and so for the electrical field since in vacuum $\nabla \cdot \vec{E} = 0$

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \triangle \vec{E} = - \triangle \vec{E} \tag{1.2}$$

matching equations (1.1) and (1.2) it results

$$- \triangle \vec{E} = -\mu_0\varepsilon_0\frac{\partial^2 \vec{E}}{\partial t^2} \Longleftrightarrow \triangle \vec{E} - \mu_0\varepsilon_0\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

that is a wave equation for the electrical field[1].

Classical physics interprets so light as a transversal EMR that propagates in space at a speed, in vacuum[2], of $c = 299\ 792\ 458\ \frac{\mathrm{m}}{\mathrm{s}}$, and as such it has a characteristic frequency ν that depends of the color. These frequencies and their relative wavelength λ for the propagation in vacuum, calculated with the equation $\lambda\nu = c$, are illustrated in figure 1.1(b).

Polarisation[3]

Figure 1.2 shows an additional property of the EMRs (besides speed and frequency): the polarization. This property is defined by the direction of propagation of the electric field.

[1] We used here the notation

- ∇ for the gradient
- $\nabla\cdot$ for the divergence
- $\nabla\times$ for the curl
- $\triangle$ for the Laplace operator

[2] Every speed c' of light propagation in a material respects the physical law $c' = \frac{c}{n} \leq c$, where $n \geq 1$ is the refractive index of the material.

[3] Information taken, and partially modified, form [20].

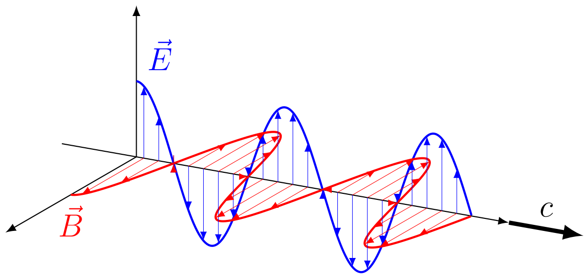
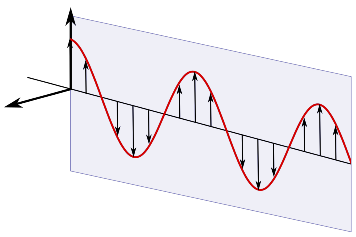


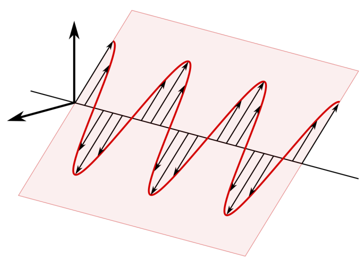
Figure 1.2: Scheme of the electronic $\vec{E}$ and magnetic $\vec{B}$ field in the EMRs.

It is possible to identify three categories of EMRs on the base of it's polarisation.

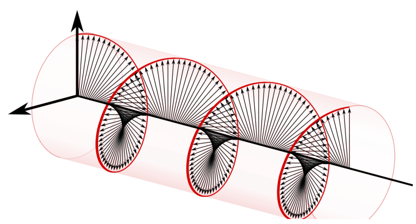
- When the electric field swings long a well defined direction, the wave has a *linear polarisation* as shown in figure 1.3(a) and 1.3(b).
- When the electric field rotates in a periodic way on an axis perpendicular to the wave's propagation, the wave has a *circular polarisation* as shown in figure 1.3(c).
- When the electric field does not have a privileged swing direction, the wave is called *non-polarised*.



(a) Linear (vertical) polarisation.



(b) Linear (oblique) polarisation.



(c) Circular polarisation.

Figure 1.3: Some images about the wave's polarisation.

The property "linear polarization" is associated to an axis defined by a vector $\vec{e} \in \mathbb{R}^2$, whose direction has no value ($\vec{e}$ and $-\vec{e}$ identify the same polarization) that indicates the oscillation's axis of the field during his propagation. Two particular linear polarizations are:

- The **linear polarisation** H/V (horizontal/vertical) represented by the vectors

$$\vec{e}_H = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{e}_V = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad .$$

This pair makes up the canonical basis of $\mathbb{R}^2$. It is so particular convenient to express a general polarisation's state with axis α trough a linear combination of this two vectors as follows:

$$\vec{e_\alpha} = \cos\alpha\ \vec{e}_H + \sin\alpha\ \vec{e}_V = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \quad . \tag{1.3}$$

- The **linear polarisation** $+/-$ $(+45°/-45°)$, represented by the vectors

$$\vec{e}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{e} \quad \vec{e}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad .$$

A linear combination similar to equation (1.3), but with complex coefficients leads to the definition of the circular polarisation.

- The **circular polarisation** R/L (right/left) represented by the vectors

$$\vec{e_R} = \frac{1}{\sqrt{2}}(\vec{e_H} + i\vec{e_V}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{e} \quad \vec{e_L} = \frac{1}{\sqrt{2}}(\vec{e_H} - i\vec{e_V}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad .$$

Measurement tools

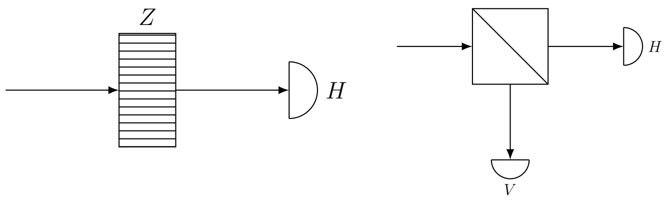
The determination of every EMRs' polarization is possible thanks to two measurement tools and two accessories:

The polarizer is a filter that, thanks to its crystal structure, has a preferential axis. EMRs oscillating parallel to the axis of the polarizer can pass it, whereas EMRs oscillating perpendicular to its axis will be absorbed or reflected, as shown in figure 1.4(a). This filter permits to determine the polarization along an axis.

The polarizing beam splitter (PBS) is a material that permits to separate a light beam according to its polarization. In the case of figure 1.4(b) the horizontal polarization H is transmitted, while the vertical V is reflected. This tool is ideal to determinate the polarization in the two directions H and V.

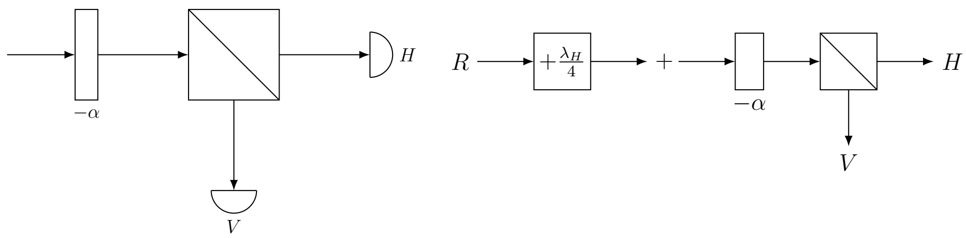
The polarization rotator is a material that permits to rotate the polarization of a light beam by an angle $-\alpha$. The measurement of a light beam with polarization axis α and $\alpha^\perp$ (for example the polarization $+/-$) is achieved as shown in figure 1.5(a). The transmitted light had, before the polarization rotor, polarization α while the reflected had polarization $\alpha^\perp$.

Quarter-wave plate is a material that permits to convert circular polarization into linear polarization, or vice versa, unequivocally. This material adjusts the phase between the component $\vec{e_H}$ and $\vec{e_V}$. Used in combination with a polarization rotator this device permits to determine circular polarization as shown in figure 1.5(b).



(a) Schem of a polariser. (b) Schema of a PBS.

Figure 1.4: Measurement tools.



(a) Scheme of the use of a polarization rotator whit a PBS. (b) Scheme of the use of a quarter-wave plate with a PBS.

Figure 1.5: Accessories for the measurement of polarization.

1.2 Introduction to the quantum nature of light

Planck introduced in 1900, during his study of the black body, for the first time the concept of "quantum" [42]. He proposed a model in which the radiation, when interacting with matter, is composed of quanta that can be emitted or absorbed by the atoms. This quanta were considered discrete amounts of energy, whose value depend on the frequency of the radiation. Einstein introduced in 1905 the idea that not only the atoms emitted and absorbed energy in discrete amounts (as claimed by Planck), but also that the EMR itself is constituted out of quanta [18], i.e. discrete amount of energy, later called *photons* by Lewis in 1926 [37].

Therefore, when speaking about light in quantum physics it is necessary to study the photons, the quanta associated to the EMR. The photon is a fundamental particle that has infinite lifetime: it can be created and destroyed by the interaction with other particles, but it can not decay spontaneously. Even though a photon has no mass, it is influenced by gravity, has energy and momentum; in vacuum it moves at light speed.

Photon's polarization

For quantum physics light is so composed by particles called photons; it is therefore natural to define a property of the photon that can be, in case of huge amounts of particles, put in relation with the polarization of light, whose quantum physical interpretation is exactly the photon.

After having introduced the formalism of quantum physics, in section 1.4 the model for the polarization will be built.

First experimental evidences

The experiment taken here in account consists in sending a photon to a polariser, paying attention to always have only one photon in the tool at the same time. The experience can give two complementary results, independently by the measure and by the direction of the polariser:

- the photon is transmitted by the polariser and is successively detected;
- the photon is absorbed by the polariser and is therefore not detected.

The transmission of a photon by the polariser is never certain, except for the case in which the photon has exactly the same polarization axis as the polariser axis that it is going to meet.

Furthermore there are no compatible axis when taking in account more polarisers. Indeed if three tools of this kind, but with different axis, are positioned in a row, i.e. $Z \mapsto X \mapsto Z$ as shown in figure 1.6, results of incompatibility are obtained. This means that if a photon goes through the first two polarisers, *it is not certain* that the photon will also go through the third polariser, despite it has already passed a polariser with the same axis.

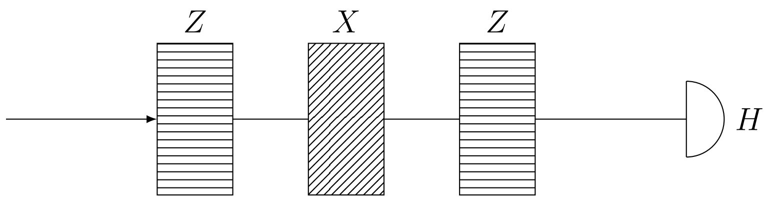


Figure 1.6: Series of three polarisers: $Z \mapsto X \mapsto Z$.

1.3 Quantum logic

Considering a *system* Σ, it is possible to define *properties* $\mathcal{P}_k$ that Σ posses at that istant or that it can aquire. For each of these properies $\mathcal{P}_k$ exist a relative test[4] $\mathcal{T}_k$, whose result can only be "yes" or "no". The property $\mathcal{P}_k$ is a *potential property* if the system Σ can aquire it, i.e. the result "yes" is possible. In the case that the result "yes" is certain, then $\mathcal{P}_k$ is an *actual property.*

On the basis of the property $\mathcal{P}_k$ we define the *negation*, noted $\neg\mathcal{P}_k$. $\neg\mathcal{P}_k$ is defined as having opposite results to the test $\mathcal{T}_k$. To $\neg\mathcal{P}_k$ corresponds the test $\neg\mathcal{T}_k$ whose results are the same as of $\mathcal{T}_k$ for the double negation of terms.

Another operation on the properties is the *conjunction* noted $\mathcal{P}_k \wedge \mathcal{P}_l$. To this property corresponds the test $\mathcal{T}_{k\wedge l}$ that consists in performing randomly one of the two tests $\mathcal{T}_k$ or $\mathcal{T}_l$. The property $\mathcal{P}_k \wedge \mathcal{P}_l$ is *actual* in the case that the result of the test $\mathcal{T}_{k\wedge l}$ is "yes" for certain.

[4] A *test* is different from a "normal experience" in the way that it admits only a determinate number of possible result default at the startup [35].

Finally we introduce the *trivial property* $\mathcal{V}$ that is always actual and the *absurd property* $\mathcal{F}$ that is never actual. We note that $\neg\mathcal{V} = \mathcal{F}$ and vice versa.

Overall the set of all the property of a system Σ is:

$$\mathcal{P} = \{\mathcal{P}_k, \mathcal{V}, \mathcal{F}\}$$

where the operation conjunction$\wedge$ and the negation $\neg$ are defined.

Postulate of quantum logic

The fundamental concept for the mathematical formalization of quantum physics is the **Hilbert space** $\mathcal{H}$. To every subspaces E of $\mathcal{H}$ is associated a property $\mathcal{P}$ and vice versa. The following correspondences apply:

- Property $\mathcal{P} \longleftrightarrow$ subspace E;
- Trivial property $\mathcal{V} \longleftrightarrow \mathcal{H}$;
- Absurd property $\mathcal{F} \longleftrightarrow \{0_{\mathcal{H}}\}$;
- Property $\mathcal{P}_1 \wedge \mathcal{P}_2 \longleftrightarrow$ Intersection $E_1 \cap E_2$;
- Property $\neg\mathcal{P} \longleftrightarrow E^{\perp}$.

In this research work the Hilbert space $\mathcal{H}$ will be of finite dimension, and mainly simply of dimension 2 as result of section 1.4.

1.3.1 State

In quantum physics there are two types of states: the pure states and the mixed states. In this work we will concentrate on the pure states, this means the states that specify the maximal information that Nature permits us to have. These states do not contemplate any kind of probability due to lack of knowledge by the side of the observer (unlike the mixed states).

Rule 1

The *states* of a quantum system are represented by normalized vectors $|\psi\rangle$ of the Hilbert space $\mathcal{H}$.
We notice that linear dependent vectors that differ by a modulus 1 factor represent the same state.

A property $\mathcal{P}$ is actual if and only if $|\psi\rangle \in E$, with E the subspace associate to $\mathcal{P}$. As a quantum state is represented by a vector, we can apply the principle linear combination. The linear combination of two vectors $|\psi_1\rangle$, $|\psi_2\rangle$, that represent two different states, results in a vector that represents a third state given by

$$|\psi\rangle = \frac{\lambda|\psi_1\rangle + \mu|\psi_2\rangle}{\|\lambda|\psi_1\rangle + \mu|\psi_2\rangle\|} \qquad \lambda, \mu \in \mathbb{C} \quad . \tag{1.4}$$

This third state is called *superposition state*. The superposition states, that in classical physics do not exist, are very important in quantum physics; every property of the superposition is indeed potential and the system can so be in every state of it.

1.3.2 Observable

To every test $\mathcal{T}_k$ are associate the two results "yes" and "no". These two results represent the properties of the system and can therefore be associate to a subspace: "yes" $\rightleftarrows \mathcal{P}_1 \rightleftarrows E_1$ and "no" $\rightleftarrows \mathcal{P}_2 \rightleftarrows E_2$. If the result is "yes" with certain the state will be $|\psi\rangle \in E_1$, while if the result is "no" with certain the state will be $|\psi\rangle \in E_2$.

The subspaces E_1 and E_2 are orthogonal to each other, this because the answer "yes" with certain to one of the properties implies the impossibility of the same answer to the other property: the properties of the system associated to every test $\mathcal{T}_k$ are *mutually exclusive*. Mathematically:

$$E_2 = E_1^{\perp} \ . \tag{1.5}$$

We observe that $\mathcal{H} = E_1 \oplus E_2$ and therefore, if we introduce the projectors P_{E_1} and P_{E_2} on the two subspaces, we obtain $P_{E_1} + P_{E_2} = I$. Thanks to this, and on the basis of the spectral theorem, to every test $\mathcal{T}_k$ it is possible to associate a physical quantity, the *observable*, that can be written as:

$$A = \lambda_1 P_{E_1} + \lambda_2 P_{E_2} \tag{1.6}$$

where P_{E_1}, P_{E_2} are the projectors associated to the properties and λ_1, λ_2 two real values associated to the results "yes" and "no" of the test. For its construction (the spectral theorem) the matrix A of equation (1.6) satisfies the condition $A = A^*$ and is so self-adjoint.

Rule 2

The *observables* of a quantum system are represented by self-adjoint matrices defined on a Hilbert space $\mathcal{H}$, i.e. $A : \mathcal{H} \to \mathcal{H}$.

1.4 Construction of the model for the polarization

In our context we will consider the photon not to be characterized by both spatial and polarization's variables; we will consider only its polarization. This choice permits to model a system with a Hilbert space of dimension 2, that means $\mathcal{H} = \mathbb{C}^2$ as deducted here below.

We indicate with $\mathcal{P}_\alpha$ the polarization's properties, associated to the tests $\mathcal{T}_\alpha$ made with polarisers (may with rotator and/or quarter-wave plate). The experience shows that

1. if a photon goes through the polariser $\mathcal{T}_\alpha$, and therefore has a determinated polarisation by the latter, then for any polariser $\mathcal{T}_{\alpha'}$ different from $\mathcal{T}_\alpha$ the result "yes"[5] will never be certain. That means that there is just one property $\mathcal{P}_\alpha$ that can be actual at the same time.

2. if for a photon the property $\mathcal{P}_\alpha$ is actual, there exists only one polarisers $\mathcal{T}_\beta$ for which the result "no" is ceratain.

From 1. follows that for every $\alpha \neq \alpha'$

$$\mathcal{P}_\alpha \wedge \mathcal{P}_{\alpha'} = \mathcal{F} \tag{1.7}$$

indeed the property $\mathcal{P}_\alpha \wedge \mathcal{P}_{\alpha'}$ is potential if and only if there is a possibility that both tests $\mathcal{T}_\alpha$ and $\mathcal{T}_{\alpha'}$ would give the result "yes" with certain, what is experimentally impossible as exposed in 1.

From 2. follows that $\mathcal{T}_\beta \neq \mathcal{T}_\alpha$ and that $\mathcal{P}_\beta$ is the negation of $\mathcal{P}_\alpha$, i.e.

$$\neg\mathcal{P}_\alpha = \mathcal{P}_\beta \quad . \tag{1.8}$$

If E_α are the subspaces of $\mathcal{H}$ associated to the properties $\mathcal{P}_\alpha$ then equation (1.7) implies

$$E_\alpha \cap E_{\alpha'} = \{0_\mathcal{H}\} \tag{1.9}$$

and equation (1.8) implies

$$E_\alpha^\perp = E_\beta \quad . \tag{1.10}$$

Now, because only one property $\mathcal{P}_\alpha$ can be actual, the property of having a given polarization characterizes completely the state of the system, and therefore

$$\dim(E_\alpha) = 1 \quad . \tag{1.11}$$

Concluding, equations (1.9-1.11) lead to

$$\dim(\mathcal{H}) = \dim\left(E_\alpha \oplus E_\alpha^\perp\right) = \dim\left(E_\alpha \oplus E_\beta\right) = \dim\left(E_\alpha\right) + \dim\left(E_\beta\right) = 2 \tag{1.12}$$

and so

$$\mathcal{H} = \mathbb{C}^2 \quad .$$

Examples on states and observables for the polarization

To better understand the last paragraphs, we propose an application of these concepts to a photon.

[5] We consider here that "yes" is associate to the transmittion of the photon trough the polariser and the result "no" with the absorption/reflection.

States of polarised photons

The state of a linear polarised photon is generally represented by:

$$|\alpha\rangle = \cos(\alpha)|H\rangle + \sin(\alpha)|V\rangle \tag{1.13}$$

where $|H\rangle$ is the state associated to the linear horizontal polarization $|H\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|V\rangle$ the state associated to the linear vertical polarization $|V\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ and α the angle between the polarization axis of the photon and the axis of $|H\rangle$. We notice the affinity of equation (1.13) with the equation (1.3).

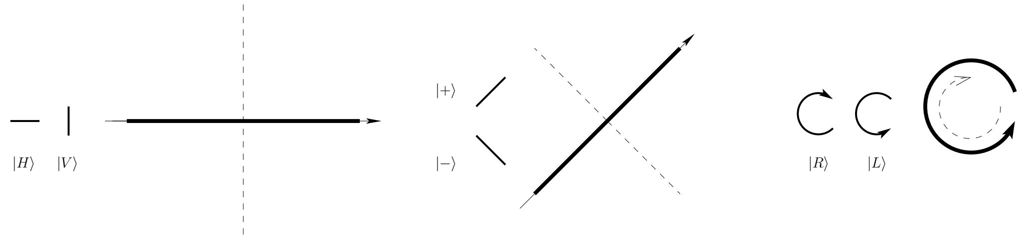


Figure 1.7: The three most usual polarization: H/V, $+/-$ and R/L.

We take in account, for example, a system in which a source emits photons to an horizontal polariser ($\alpha = 0°$). The photons that are transmitted by the polariser will have a horizontal polarization, their state will so be described by the vector $|H\rangle = \begin{pmatrix}1\\0\end{pmatrix}$.

If instead the polariser would have been vertical, the photons would have been polarised vertically ($\alpha = 90°$), i.e. $|V\rangle = \begin{pmatrix}0\\1\end{pmatrix}$.

A similar process can be executed with photons of circular polarisation after being tramuted by a quarter-wave plate into a linear polarization. Once linearized, the circular polarization can be analyzed with a polarization rotator and a polarizer or PBS.

Observables associated to polarisers and PBS

Applying the spectral theorem associated with **Rule 2**, we calculate the observable associated with the polarization measured by a particular PBS. In some exemplar cases:

- The PBS without polarization rotator separates the states $|H\rangle$ and $|V\rangle$. Associating $+1$ to the property "horizontal polarization" and -1 to "vertical polarization" we obtain

$$A = 1 \cdot P_{|H\rangle} - 1 \cdot P_{|V\rangle} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \sigma_z \quad ;$$

- The PBS with a polarization rotator of angle $\alpha = -45°$ separates the states

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{e} \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$

  Associating the value $+1$ to the property "polarization $+45°$" and -1 to the property "polarization $-45°$" we obtain

$$A = 1 \cdot P_{|+\rangle} - 1 \cdot P_{|-\rangle} = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \sigma_x;$$

- The PBS with polarization rotator of angle $\alpha = -45°$ on which are sent the linear transposition of

$$|R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \quad \text{e} \quad |L\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

  separates these two states. Associating the value $+1$ to the propriety "right polarization" and -1 to "left polarization" we obtain

$$A = 1 \cdot P_{|R\rangle} - 1 \cdot P_{|L\rangle} = \frac{1}{2}\begin{pmatrix}1 & -i\\i & 1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1 & i\\-i & 1\end{pmatrix} = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix} = \sigma_y$$

  where σ_x, σ_y e σ_z are called *Pauli matrices.*

In general the observable of a polariser with angle α is give by

$$A(\alpha) = \begin{pmatrix}\cos 2\alpha & \sin 2\alpha\\ \sin 2\alpha & -\cos 2\alpha\end{pmatrix} \tag{1.14}$$

The choice of the values λ_i is indeterminate provided that they are elements of $\mathbb{R}$. However, some logical factors must be considered: To one of the propererties (usually the property of being transmitted) we associate, for convenience, the value $+1$. In the case of a normal polariser, in which the photons that are not transmitted by the tool are destructed, it makes sense to associate the value 0 to this second component with orthogonal axis of polarization. Whereas using a PBS (in which the photons non transmitted are deviated) it makes sense to assign the value of -1 to the component with a value different of $+1$.

1.5 Ideal measurement and probability

The measurement is an experience bound to an observable. In fact it depends from the measurement's tool (which determinates the observable). The probability to observe a certain value is influenced by the state of the system before the measurement.

This part of quantum physics is the first aspect that contains a fundamental difference with classical physics. If, indeed, in classical theories the measurement shows something pre-existent to the measurement, in quantum physics the measurement influences indeterministically and irreversibly the state of the system.

We take into account the property associated to the subspace $E \subset \mathcal{H}$ on which is based the test represented by the self-adjoint matrix A defined by the projector $P_{|\varphi\rangle}$, where we associate 1 to the result "yes" and 0 to "no", then $A = P_{|\varphi\rangle}$. We want to measure this property. In the case of an ideal measurement, given the state before the measurement $|\psi\rangle$ the result of the test is "yes" with certain if and only if the property E is actual (i.e. $|\psi\rangle \in E$). If the result is "yes" then, after the measurement, the property E is actual and $|\psi'\rangle \in E$, where $|\psi'\rangle$ is the state after the measurement.

We consider now the transition from the state $|\psi\rangle$ to the state $|\varphi\rangle$:

Rule 3a

The *probability* to find the state $|\varphi\rangle$ in the measurement of the observable $P_{|\varphi\rangle}$ given the system's state $|\psi\rangle$ is

$$\text{Prob}\{|\psi\rangle \to |\varphi\rangle\} = |\langle\varphi|\psi\rangle|^2 \quad . \tag{1.15}$$

$\text{Prob}\{|\psi\rangle \to |\varphi\rangle\}$ is the probability to observe the value 1 in the measurement of the observable $P_{|\varphi\rangle}$ given the state $|\psi\rangle$, this probability is also noted $\text{Prob}_{|\psi\rangle}\{P_{|\varphi\rangle} = 1\}$.

We can enlarge the result of equation (1.15) for the tests to every other experience. To do so we consider an observable $A = \sum_i \lambda_i P_{E_i}$ in which every projector P_{E_i} is associated to a property $\mathcal{P}_i$. We consider so each of this property in a single way and interpret the experience as a test for each of this property. Doing so the consideration we made for the ideal measurements are enlarged to every measurement, in the case we notice that immediately after the measurement of the observable A, in which was observed the value λ_i, the state of the system is a state of "absolute knowledge" (see section 1.6) for A, that means a state for which

$$\begin{aligned}\text{Prob}_{|\psi\rangle}\{A = \lambda_i\} &= \text{Prob}\{|\psi\rangle \to |\varphi\rangle\} \\ &= |\langle\varphi,\psi\rangle|^2 = \|P_{E_i}|\psi\rangle\|^2 = 1 \quad \Leftrightarrow \quad \psi \in E_i\end{aligned}$$ [6].

The **Rule 3a** can be generalized and is then known as Born*'s rule*:

Rule 3b

The *probability* to observe the value λ_i in the measurement of the observable A, given the state of the system $|\psi\rangle \in \mathcal{H}$, is

$$\text{Prob}_{|\psi\rangle}\{A = \lambda_i\} = \|P_{E_i}|\psi\rangle\|^2 = \langle\psi|P_{E_i}\psi\rangle \tag{1.16}$$

where P_{E_i} is the projector associated to the eigenvalue λ_i.

[6] Here we used the property $|\langle\varphi,\psi\rangle|^2 = \|P_\varphi|\psi\rangle\|^2$ where $P_{E_i} = P_\varphi$. This is only true for projector associated to subspaces of dimension 1.

This rule leads to a very interesting observation.

The equation (1.16) is the fundamental rule that translates the randomness of quantum physics into mathematics. It shows that the *objective probability* to observe the value λ_i in a measurement of the observable A, given the state $|\psi\rangle$, depends only on the observable A and on the state $|\psi\rangle$. The searched probability depends only on the observable but not on the specific measurement experience.

The objective probability has nothing to do with a lack of information on the side of the observer in the description of the state, it is intrinsic to the physical situation itself and independent of the observer. The specification "objective" serves to differentiate it from the probability in statistical physics, where probability is used to describe states that are not of maximal information (mixed states):*Nature at a microscopical level is objectively random*, independently by the observer. At microscopic levels the result of a measurement is not deterministically predictable, but randomness intervenes only in the measurement and not in the description of the state.

The description of the state after the measurement, considering to have observed the value λ_i, is possible with a complement of the **Rules 3**.

Rule 4

If the state of a system before the measurement is $|\psi\rangle$, then immediately after the measurement of the observable A, in which was observed the value λ_i associated to the subspace E_i, the state of the system is given by

$$|\psi'\rangle = \frac{P_{E_i}|\psi\rangle}{\|P_{E_i}|\psi\rangle\|} \quad . \tag{1.17}$$

This result is known as *postulate of state reduction.*

Measurement of the polarization of a photon

The tools to measure the polarization are the polarizer or the polarizing beam splitter (described in section 1.1). We want to measure the observable of polarization +45°, that is described by the matrix σ_x, of a photon with initial state $|H\rangle$. We use equation (1.16) to calculate the probability of measuring the eigenvalues associated to the eigenvectors described above. For example $\lambda = +1$:

$$\mathrm{Prob}_{|H\rangle}\{\sigma_x = +1\} = \langle H|P_{|+\rangle}H\rangle = \left\langle \begin{pmatrix}1\\0\end{pmatrix} \middle| \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}1\\0\end{pmatrix} \middle| \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\right\rangle = \frac{1}{2}.$$

We notice that this result coincides with the one that we would obtain considering a big quantity of photons (a light beam) and using the Malus' law. Indeed

$$I = I_0\cos^2\theta \Leftrightarrow \frac{I}{I_0} = \cos^2\theta, \tag{1.18}$$

so, considering the intensity as the product of the energy of a single particle multiplied by the number of particles, the ratio of the intensities $\frac{I}{I_0}$ is equal to the ration of the number of photons transmitted and absorbed by the polariser. The Malus' law is the macroscopic equivalent to the measurement of a photon's polarization.

Considering the situation described above with the observable +45° we calculate the factor $\frac{I}{I_0}$ with equation (1.18) obtaining the same result:

$$\frac{I}{I_0} = \cos^2(45°) = \frac{1}{2} \quad .$$

1.6 Mean value and standard deviation

We do some statistics with the results of the experiences. Given an initial state $|\psi\rangle$ and an observable A, repeating the experiment for a at least representative amount of tries, it is possible to identify the mean value of the observable A, noted $\langle A\rangle$. Using the probabilistic definition of mean value we obtain the mathematical definition:

$$\begin{aligned}\langle A\rangle &= \sum_i \lambda_i \mathrm{Prob}_{|\psi\rangle}\{A = \lambda_i\} = \sum_i \lambda_i \langle\psi|P_{E_i}\psi\rangle \\ &= \left\langle \psi \middle| \sum_i \lambda_i P_{E_i}\psi \right\rangle = \langle\psi|A\psi\rangle \quad . \end{aligned} \tag{1.19}$$

If there are some fluctuations around the mean value we define the standard deviation ΔA based on the theory of probability, as

$$\Delta A = \sqrt{\langle (A - \langle A\rangle)^2\rangle} = \sqrt{\langle A^2\rangle - \langle A\rangle^2} \quad . \tag{1.20}$$

Considering the special case when $\Delta A = 0$ we can derive important conclusions. In fact, considering a situation like this we have

$$0 = \Delta A = (\Delta A)^2 = \langle (A - \langle A\rangle)^2\rangle = \langle\psi|(A - \langle A\rangle)^2\psi\rangle \tag{1.21}$$

$$\overset{7}{=} \langle (A - \langle A\rangle)\psi|(A - \langle A\rangle)\psi\rangle = \|(A - \langle A\rangle)|\psi\rangle\|^2 \Leftrightarrow (A - \langle A\rangle)|\psi\rangle = 0. \tag{1.22}$$

The equation (1.22) leads to the conclusion that in the cases where $|\psi\rangle$ is a eigenvector of A (whose eigenvalue is necessarly $\langle A\rangle$), the standard deviation ΔA is 0, because

$$\Delta A = 0 \Leftrightarrow (A - \langle A\rangle)|\psi\rangle = 0 \Leftrightarrow A|\psi\rangle = \langle A\rangle|\psi\rangle.$$

Systems in these particular states are called of *absolute knowledge* for this observable.

Mean value and standard deviation for polarized photons

We can examine a practical example to enlighten the equations described above. Considering a system where a source emits photons with polarization +45° that

[7] The matrix A are self adjoint, so $(A - \langle A\rangle)^* = (A - \langle A\rangle)$.

go to a horizontal polariser. On the initial state of the system described by the vector $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, we want to calculate the mean value for the observable σ_z. Applying equation (1.19) we obtain the mean value:

$$\langle\sigma_z\rangle = \langle +|\sigma_z +\rangle = \frac{1}{2}\left\langle \begin{pmatrix}1\\1\end{pmatrix}\middle| \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right\rangle = 0$$

or:

$$\langle\sigma_z\rangle = 1\ \mathrm{Prob}\{\sigma_z = +1\} - 1\ \mathrm{Prob}\{\sigma_z = -1\} = \frac{1}{2} - \frac{1}{2} = 0 \quad .$$

We can now calculate the standard deviation

$$\Delta\sigma_z = \sqrt{\langle\sigma_z^2\rangle - \langle\sigma_z\rangle^2} = \sqrt{\frac{1}{2} - 0} = \frac{\sqrt{2}}{2} \quad . \tag{1.23}$$

The equation (1.23) shows that the analysed system is not of absolute knowledge.

1.7 Incompatibility of observables

We call two observables A and B mutually incompatible if measuring B we perturb the precedent measurement of A. Mathematically we define the compatibility as commutativity of the operators, i.e.:

- if $[A, B] \neq 0$ then A and B are incompatible, it exist indeed a $|\psi\rangle \in \mathcal{H}$ for which the result of B perturbs a precedent measurement of A;
- if $[A, B] = 0$ then A and B are compatible, it does not exist indeed a $|\psi\rangle \in \mathcal{H}$ for which the result of B perturbs a precedent measurement of A;

Incompatibility of observables for polarized photons

Given a system with a source that emits photons to two PBS in a row one after the other; the first with axis $\alpha = +45°$ and the second with axis $\alpha = 0°$. We calculate $[\sigma_x, \sigma_z]$, because, as shown in 1.3.2, these Pauli matrices are associated to the observables of the two PBS:

$$\begin{aligned}[\sigma_x, \sigma_z] &= \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} - \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\\ &= \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} = \begin{pmatrix}0 & -2\\2 & 0\end{pmatrix} \neq 0 \quad .\end{aligned}$$

σ_x and σ_z are therefore incompatible.

1.8 Time Evolution

In the previous section we considered a physical system in a well defined moment or in a measurement's process. We take now in account the time evolution of the state $|\psi\rangle \in \mathcal{H}$, noting with the index t the time dependence of the state $|\psi_t\rangle$.

We consider a test (associate with the property $\mathcal{P}_t$) defined at the final moment t. We suppose that the result of the test $\mathcal{T}_t$ will be "yes" with certain, then $|\psi_t\rangle \in E_t$ and the vector $|\varphi_t\rangle \in E_t^{\perp}$ is orthogonal to $|\psi_t\rangle$. In the case where the *time evolution is deterministic* we can associate to $\mathcal{T}_t$ a test $\mathcal{T}_0$ (associated to the property E_0) defined at the initial moment $t_0 = 0$ so that the result of $\mathcal{T}_0$ is "yes" with certain. With this assumption $|\psi_0\rangle \in E_0$ and $|\varphi_0\rangle \in E_0^{\perp}$, and their deterministic time evolution is given by $|\psi_0\rangle \to |\psi_t\rangle$ and $|\varphi_0\rangle \to |\varphi_t\rangle$. We can also reason in terms of negation, indeed if the test $\neg\mathcal{T}_t$ gives result "yes" with certain then the state of the system is given by $|\varphi_t\rangle \in E_t^{\perp}$; because $\neg\mathcal{T}_0$ will give result "yes" with certain, this is the test associated to $\neg\mathcal{T}_t$ and so $|\varphi_0\rangle \in E_0^{\perp}$. The schema of figure 1.8 can help to understand.

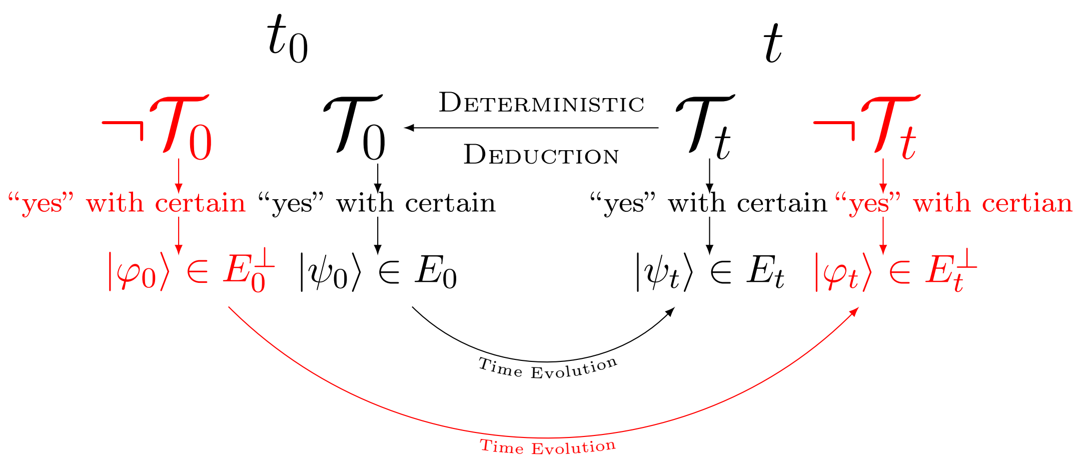


Figure 1.8: Scheme of the deterministic time evolution of states.

Because

$$\langle\varphi_t|\psi_t\rangle = 0 \Leftrightarrow \langle\varphi_0|\psi_0\rangle = 0 \tag{1.24}$$

we can conclude that two initial state that are orthogonal to each other have to keep their orthogonality during the deterministic time evolution. Mathematically the time evolution is given by a matrices U_t such that the previous conclusion is verified. Analysing equation (1.24) we conclude that

$$\langle\psi_t|\varphi_t\rangle = \langle U_t\psi_0|U_t\varphi_0\rangle = \langle U_t^* U_t\psi_0|\varphi_0\rangle = \langle\psi_0|\varphi_0\rangle \tag{1.25}$$

where for we deduce that the matrices U_t have to verify the condition $U_t^* U_t^{-1} = I \Leftrightarrow U_t^* = U_t^{-1}$, characteristic of the unitary matrices. This characterization of the matrices is fundamental, only so the scalar product of the state does not change during the evolution.

Rule 6

The **time evolution** of state is give by

$$|\psi_t\rangle = U_t|\psi_0\rangle$$

where $|\psi_t\rangle$ is the state at time t, $|\psi_0\rangle$ the initial state ($t_0 = 0$) and $\{U_t\}_{t\in\mathbb{R}_+}$ a set of *unitary operators*.

Observations

- The time evolution is represented by a unitary matrix, but time is not always shown; this means that given the state $|\psi\rangle$ before the experimental tool, the state after the tool will be given by $|\psi'\rangle = U|\psi\rangle$ where U is a unitary matrix.

- Knowing the state at an instant t_0 it is possible to determinate uniquely which will be the state at a moment $t > t_0$[8], therefore the evolution of a state in a quantum system is a *deterministic process*.

[8]Assuming that between time t_0 and time t no measurement occurred.

Chapter 2

Single particle quantum interference

HAVING established the necessary basis to move in the field of quantum physics, we start the analysis of the more amazing and counterintuitive phenomena of the quantum theory.

This chapter handles the quantum interference phenomenon of a single particle and serves as introduction to the following chapters. The importance of underlining the study of just one particle during the experiment will be more clear in the following chapter 3.

2.1 Experimental motivation

Already in the first years of quantum physics, a strange behavior of subatomic particles has been observed. 1927 DAVISSON and GERMER did some experiments [15] sending electrons towards a slit. The expectation from classical physics – i.e. that the electron, being a particle, would pass through the split without any modification of its linear trajectory from the source to the detector – was not confirmed. They observed a wave-diffraction-like behavior. Later in 1961 some experiments with two or more slits were done with analog results [33].

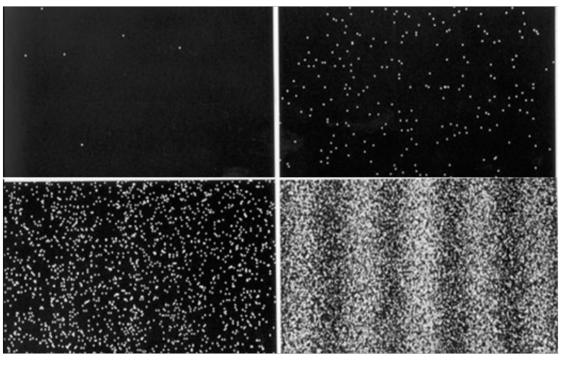


Figure 2.1: Result of an interference double-slit experiment with 8, 270, 2′000 and 160′000 electrons [31].

2.1.1 Wave-particle duality

The phenomenon described above, apparently counterintuitive, could be explained with the theory of *wave-particle duality*.

In 1905 EINSTEIN, with the article on the photoelectric effect [18][1], associates to every EMR of frequency ν and wave vector $\vec{k}$ a particle of energy $E = h\nu$ and momentum $\vec{p} = \hbar\vec{k}$.

In 1924 DE BROGLIE elaborated the argument in his doctorate thesis [16] postulating that every free particle of energy E and momentum $\vec{p}$ is possibly associable to a wave of frequency $\nu = \frac{E}{h}$ and of wave length $\lambda = \frac{h}{\|\vec{p}\|}$.

We could deduce that the observed interference in the experiments with electrons is interpretable on the basis of the wave-like characteristic associated to these particles by the theory of DE BROGLIE[2] called in fact "wave-particle duality".

However, it is important to notice that the wave-particle duality is not the cause of the interferences but just a model that can be applied after the experiment, and that it does not clarify its causes.

In classical physics, the interference is observed in every single instant of the experiment, because the wave splits into two parts in the slit and recombines itself on the screen creating the interference's figures.

However, at a microscopical level, a particle can not divide itself into two parts and, in principle, in the experiment there is always just one particle at any given moment; the interference's figure is reconstructed only at the end of the experiment. Therefore, the origin of this phenomenon can not be lead back to this cause. The wave-particle duality is now a classical way to interpret Nature, about that we propose the following interesting text of LÉVY-LEBLOND.

> «[...] Vous comprenez aussi pourquoi on a pendant longtemps caractérisé les choses suivant une terminologie qui se révèle aujourd'hui inadaptée, mais que j'explicite pour la critiquer, qui a été une façon de parler au début du XX^{e} siècle (mais ça a persisté et ça persiste encore dans pas mal de livres de vulgarisation, voire d'enseignement) de parler de la dualité ondes-corpuscules dans le cadre de la théorie quantique. Ça s'explique historiquement, puisque historiquement on découvre effectivement que ce qu'on croyait être un corpuscule: l'électron, ah! Présente des aspects ondulatoires, que ce que l'on croyait être une onde: les ondes électromagnétiques, ah! Présente un aspect corpusculaire. On s'est dit:" Tiens, c'est tantôt l'un, tantôt l'autre, c'est bizarre quand même, il y a une dualité tout de même, tantôt l'un, tantôt l'autre." Ceci n'est pas une bonne façon de penser, d'abord c'est contradictoire, un objet ne peut pas être tantôt un type d'objet, tantôt un autre type d'objet. Et il faut bien se rendre à l'évidence, après quelques décennies, la réalité c'est que les objets quantiques, les quantons, ce ne sont ni des ondes ni des corpuscules, mais que, dans certaines conditions, ils peuvent ressembler à des ondes et dans certaines autres conditions, ils peuvent ressembler à des corpuscules.
>
> Quand on était au début de la théorie quantique les conditions de types classiques

[1]Research that valued him the Nobel prize for phyisics in 1921 [40].

[2]In principle, the diffraction of light could also be reconducted to this theory, however the wave nature of light was already demonstrated [32], it is therefore less curious.

prévalaient, c'est-à-dire que la plupart des objets apparaissaient soit comme corpuscules soit comme ondes, mais depuis que notre connaissance du monde quantique s'est approfondie et que la sophistication des expérimentateurs s'est donnée libre cours, la plupart des quantons que nous manipulons se présentent d'une façon qui n'est ni ondulatoire ni corpusculaire et ils révèlent leur nature propre en plein. [...] Nous avons exactement la même chose ici, si j'ose dire, les quantons sont les ornithorynques, en ce sens que ce ne sont que des aspects très particuliers qui peuvent nous les faire prendre soit pour des particules, soit pour des ondes et que leur nature propre est d'un autre genre.» [36]

2.1.2 Interferences and "which-way" information

Another interesting phenomenon is related to the particle's quantum interferences. In every experiment, in which it was tried to find out in which of the two (or more) slits the particles had passed (i.e. the *"which-way" information*), the interference's figure disappeared. In 1991 was thought up a *Gedankenexperiment*[3] which would permit to observe the interferences and to know the "which-way" information. If this experiment would have worked, this would have eliminated the possibility that the disappearance of the interferences was not intrinsical but bound to technology problems [45].

The experiment was thought as follows: a source emits particles that interact with a double-slit interferometer giving place to interferences shown on a screen. To determine the which-way information, we proceed as follows: before the double slit we position two cavities preceded by a laser beam. When the particles are sent, they are excited by the laser and when they then pass through the two cavities they de-excite themselves by emitting a photon. To determine the which-way information it is sufficient to look into which of the two cavities the photon was emitted. The original scheme is proposed in figure 2.2.

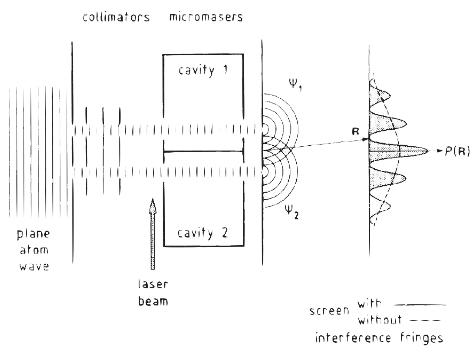


Figure 2.2: Scheme of the *Gedankenexperiment* [45].

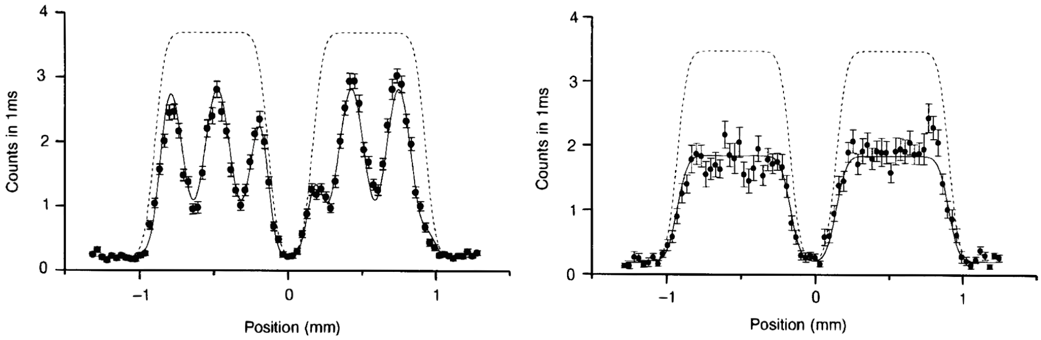
It is important to notice that with this experimental set-up to determine the trajectory of a particle we do not influence the particle's trajectory.

In 1998 an experiment [17] tried to determine the which-way information as proposed in [45]. The conclusion of this experiment was that, the interference

[3]Form German, a "thought experiment": «Thought experiments are devices of the imagination used to investigate the nature of things. » [13].

disappears if the which-way information is collected. The disappearance of the interference is not caused by an interaction with the experimental tools.

The experimental results (summarized in figure 2.3) show that there is no interference if it is possible to determine the which-way information, even if the state of the particle is not influenced.



(a) Interference's figure without "wich-way" information.

(b) Interference's figure destructed by the "wich-way" information.

Figure 2.3: Result of experiments with and without "wich-way" information [17].

2.2 Mach–Zehnder interferometer[4]

The experimental results discussed in the previous paragraph are surprising and are not explained by any classical theory. Quantum physics provides an exhaustive explanation of these phenomena, but needs a mathematical modelling of them. We will build a simple model called *Mach-Zehnder interferometer*, that will permit to analyse the one particle quantum correlation and that will show the same phenomena of the previous paragraph.

2.2.1 Set-up

As shown in figure 2.4, the set-up consists in a source that sends particles, e.g. photons, into an interferometer composed by two beam separators (beam splitter BS_1 e BS_2) and two mirrors (M_A e M_B). At the end of the interferometer two detectors (D_X e D_Y) that count the number of particles are positioned. It is possible to add a factor (a path variation) ϕ that allows to differentiate the two ways (A and B) that the particles can take.

2.2.2 Model, states and observables

The *Mach-Zehnder interferometer* allows to put aside all the characteristic of the particles except for their propagation's direction, that can be – according to the

[4]The informations, partially modified, are from [21].

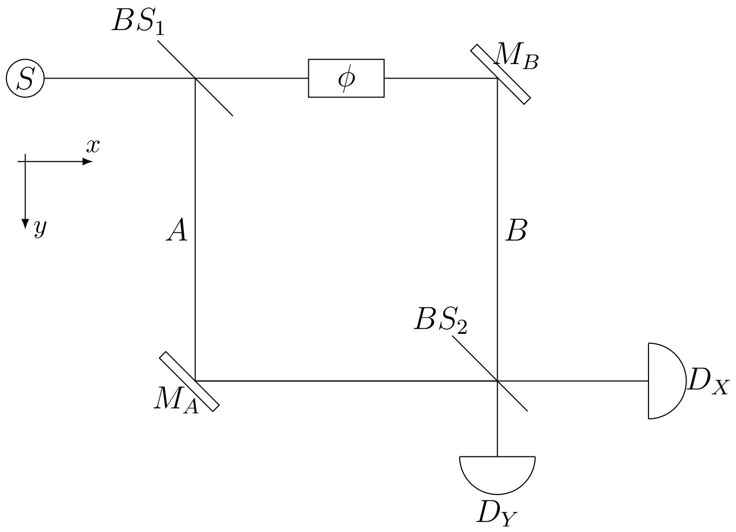


Figure 2.4: The Mach-Zehnder intereferometer.

coordinate system of figure 2.4 – in x or in y. The described system is therefore a two-level system whose Hilbert space is

$$\mathcal{H} = \mathbb{C}^2 \quad . \tag{2.1}$$

The pure states characterized by the two possible directions of propagation are identified with the two orthogonal vectors

$$|\psi_x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{e} \quad |\psi_y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad .$$

The projectors of the correspondent subspaces generated by $|\psi_x\rangle$ e $|\psi_y\rangle$

$$X = P_{|\psi_x\rangle} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{e} \quad Y = P_{|\psi_y\rangle} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

are associated respectively with properties "propagation in direction x" and "propagation in direction y".

The observables associated to the detectors D_X and D_Y measure the propagation's direction of the particles. A particle can only be detected or not detected, we assign the values 0 and 1 to these results.

We build the two matrices that represent the observables in D_X and D_Y

$$1X + 0Y = X \quad \text{e} \quad 0X + 1Y = Y$$

that are the same X and Y.

Recapitulating the state are given by the vectors

$$|\psi_x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{e} \quad |\psi_y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.2}$$

and the observable by the matrices

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{e} \quad Y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad . \tag{2.3}$$

2.2.3 Time evolution of the states

To complete the model, we need to describe the effect of the beam splitters, the mirrors and eventually the "factor ϕ" on the state of the particles. These elements modify the initial state $|\psi_{in}\rangle$ into the final $|\psi_{out}\rangle$ that results after the interferometer. The modifications occur, as seen in section 1.8, through an unitary matrix U.

Beam splitter

The matrix, because unitary and therefore which respects the property

$$U^*_{BS} = U^{-1}_{BS} \Leftrightarrow \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \frac{1}{\det U_{BS}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

is of the type

$$U_{BS} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad .$$

considering that $\det U_{BS} = 1$. This condition does not lead to a loss of generality since the matrix is applied to a vector that would be linearly dependent through a factor $(\det U_{BS})^{-1}$ with the state considering $\det U_{BS} = 1$.

A particle with the initial state $|\psi_{in}\rangle = |\psi_x\rangle$[5] after the beam splitter will be characterized by a superposition state of the type

$$|\psi_{out}\rangle = \alpha|\psi_x\rangle + \beta|\psi_y\rangle \quad \text{with} \quad |\alpha|^2 + |\beta|^2 = 1$$

where the probability that the particle will continue horizontally is:

$$t = |\alpha|^2 = |\langle\psi_x|\psi_{out}\rangle|^2 = \text{Prob}\{X = 1\}$$

and the probability that the particle will continue vertically is:

$$r = |\beta|^2 = |\langle\psi_y|\psi_{out}\rangle|^2 = \text{Prob}\{Y = 1\}$$

Since the particles can only be reflected or transmitted $t + r = 1$ holds.

We can write the factors α and β of the superposition state in function of the values t and r as follows[6]

$$\alpha = |\alpha|e^{i\varphi_1} = \sqrt{t}\ e^{i\varphi_1} \quad \text{and} \quad \beta = |\beta|e^{i\varphi_2} = \sqrt{r}\ e^{i\varphi_2}$$

with φ_1, $\varphi_2 \in [0, 2\pi[$. Factors that inserted in the equation of the state give:

$$|\psi_{out}\rangle = \sqrt{t}\ e^{i\varphi_1}|\psi_x\rangle + \sqrt{r}\ e^{i\varphi_2}|\psi_y\rangle \quad .$$

[5] The reasoning is the same – and finishes with the same matrix – for a particle with $|\psi_{in}\rangle = |\psi_y\rangle$.

[6] The adding of $e^{i\varphi}$ brings α and β in the usual polar form of the complex numbers, where $|\alpha|$ is the modulus φ the argument.

Since $|\psi\rangle$ and $e^{-i\varphi_1}|\psi\rangle$ describe the same state (being linearly dependent and both normed), multiplying each of them by $e^{-i\varphi_1}$ gives

$$e^{-i\varphi_1}|\psi_{out}\rangle = \sqrt{t}\ |\psi_x\rangle + \sqrt{r}\ e^{i\varphi_2}e^{-i\varphi_1}|\psi_y\rangle$$

and setting $e^{i\varphi} = e^{i(\varphi_2-\varphi_1)}$ we obtain

$$|\psi_{out}\rangle = \sqrt{t}\ |\psi_x\rangle + \sqrt{r}\ e^{i\varphi}|\psi_y\rangle$$

In this equation the value φ identifies the phase relative to the "reflected" and the "transmitted" state.

So, the initial state $|\psi_{in}\rangle = |\psi_x\rangle$ evolves into $|\psi_{out}\rangle = \sqrt{t}\ |\psi_x\rangle + \sqrt{r}\ e^{i\varphi}|\psi_y\rangle$ after the beam splitter. We obtain

$$U_{BS}\psi_{in} = \psi_{out} \Leftrightarrow \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sqrt{t}\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{r}\ e^{i\varphi}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} a \\ -\bar{b} \end{pmatrix} = \begin{pmatrix} \sqrt{t} \\ \sqrt{r}\ e^{i\varphi} \end{pmatrix}$$

following

$$\begin{cases} a = \sqrt{t} \\ b = -\sqrt{r}\ e^{-i\varphi} \end{cases} .$$

The unitary matrix that identifies the time evolution of a beam splitter is therefore:

$$U_{BS}(r,t) = \begin{pmatrix} \sqrt{t} & -\sqrt{r}\ e^{-i\varphi} \\ \sqrt{r}\ e^{i\varphi} & \sqrt{t} \end{pmatrix} .$$

It is important that the set up is symmetrical (e.g. that the phase relative between the "reflected" and the "transmitted" state is equal independent from $|\psi_{in}\rangle$). We set $\varphi = \frac{\pi}{2}$so that $e^{i\varphi} = e^{i\frac{\pi}{2}} = i$ and $e^{-i\frac{\pi}{2}} = -i$. We can write the matrix in function of the only value r obtaining:

$$U_{BS}(r) = \begin{pmatrix} \sqrt{1-r} & i\sqrt{r} \\ i\sqrt{r} & \sqrt{1-r} \end{pmatrix} .$$

We are looking for a set-up for which the probability Prob$\{X = 1\}$ and Prob$\{Y = 1\}$ are equal, that means that the beam splitter is balanced. To $t + r = 1$ we add the parameter $t = r$, it follows so $t = r = \frac{1}{2}$. The matrix that represents the time evolution of the beam splitters used for this set-up is therefore:

$$U_{BS}\left(r = \frac{1}{2}\right) = U_{BS} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} . \tag{2.4}$$

Mirrors

The mirrors M_A and M_B can be thought as non-balanced beam splitters for which $r = 1$ and $t = 0$. The matrices associated to the mirrors are so:

$$U_S = U_{BS}(r = 1) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} . \tag{2.5}$$

Factor ϕ

The modify of the path does not have any influence on the direction of propagation of the particles. Therefore, the state remains unchanged. The modification that this element puts to the state has to make $|\psi_{out}\rangle$ linearly dependent from $|\psi_{in}\rangle$:

$$|\psi_{out}\rangle = e^{i\phi}|\psi_{in}\rangle \quad .$$

Basing on these considerations we build the matrix $U(\phi)$: hypothesising an initial state $|\psi_{in}\rangle = |\psi_x\rangle$, $U(\phi)$ has to be such that

$$|\psi_{out}\rangle = U|\psi_{in}\rangle \Leftrightarrow e^{i\phi}|\psi_x\rangle = U|\psi_x\rangle \Leftrightarrow \begin{pmatrix} e^{i\phi} \\ 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a = e^{i\phi} \\ c = 0 \end{cases} \quad .$$

On a particle of initial state $|\psi_{in}\rangle = |\psi_y\rangle$ it has to have no effect, so

$$|\psi_y\rangle = U|\psi_y\rangle \Leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} b = 0 \\ d = 1 \end{cases} \quad .$$

The unitary matrix that describes a modification on the way during the propagation in direction x direction is therefore

$$U(\phi) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix} \quad . \tag{2.6}$$

The matrix of equation (2.6) modifies the state of the the system leaving it linearly dependent to the state before of it. But before the modification put by ϕ, the state is of superposition and ϕ acts only on the element that depends from $|\psi_x\rangle$:

$$U(\phi)|\psi_{in}\rangle = U(\phi)\left[\alpha|\psi_x\rangle + \beta|\psi_y\rangle\right] = U(\phi)\alpha|\psi_x\rangle + \beta|\psi_y\rangle$$

Therefore the final state after ϕ is different form the initial state.

2.2.4 Study of the interferences

From the multiplication of the matrices (2.4), (2.5) and (2.6) we obtain the unitary matrix U_{MZ} that describes the time evolution of the whole interferometer:

$$\begin{aligned} U_{MZ} = U_{BS}U_M U(\phi) U_{BS} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\ &= -\frac{1}{2} \begin{pmatrix} 1+e^{i\phi} & -i(1-e^{i\phi}) \\ i(1-e^{i\phi}) & 1+e^{i\phi} \end{pmatrix} \quad . \end{aligned} \tag{2.7}$$

The matrix (2.7) permits us to study the operation's principle of this interferometer and to draw important conclusions. Here follow the more significant cases.

$\phi = 0$

The initial state of the system is $|\psi_x\rangle$ and the factor $\phi = 0$. The final state is

$$|\psi_{out}\rangle = U_{MZ}|\psi_{in}\rangle = -\frac{1}{2}\begin{pmatrix} 1+e^0 & -i(1-e^0) \\ i(1-e^0) & 1+e^0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\stackrel{7}{\Longleftrightarrow} |\psi_{out}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.8}$$

and as a consequence, the probabilities to observe the particle in D_X or D_Y are

$$\begin{aligned} \text{Prob}\{X=1\} &= |\langle\psi_x|\psi_{out}\rangle|^2 = 1 \\ \text{Prob}\{Y=1\} &= |\langle\psi_y|\psi_{out}\rangle|^2 = 0 \end{aligned} \tag{2.9}$$

The 100% of the particles arrive to the detector D_X. This result is quite surprising and against intuition. We would have expected that the distribution would have been equally divided between the two detectors.

$\phi \neq 0$

In the situation where the factor ϕ is not zero and the initial state is $|\psi_{in}\rangle = |\psi_x\rangle$, the final state is

$$|\psi_{out}\rangle = U_{MZ}|\psi_{in}\rangle = -\frac{1}{2}\begin{pmatrix} 1+e^{i\phi} & -i(1-e^{i\phi}) \\ i(1-e^{i\phi}) & 1+e^{i\phi} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \stackrel{8}{=} \begin{pmatrix} -e^{i\frac{\phi}{2}}\cos(\frac{\phi}{2}) \\ -e^{i\frac{\phi}{2}}\sin(\frac{\phi}{2}) \end{pmatrix}$$
$$= -e^{i\frac{\phi}{2}}\cos(\frac{\phi}{2})|\psi_x\rangle - e^{i\frac{\phi}{2}}\sin(\frac{\phi}{2})|\psi_y\rangle \tag{2.10}$$

and consequently the probabilities to observe the particle in D_X or D_Y are

$$\begin{aligned} \text{Prob}\{X=1\} &= |\langle\psi_x|\psi_{out}\rangle|^2 = \cos^2\left(\frac{\phi}{2}\right) \\ \text{Prob}\{Y=1\} &= |\langle\psi_y|\psi_{out}\rangle|^2 = \sin^2\left(\frac{\phi}{2}\right) \end{aligned} \tag{2.11}$$

We notice that introducing this modification to the ways, the result depends from ϕ. In particular we have a periodic probability to observe the particle in D_X or D_Y. The situation is opposite in respect to an unchanged path in case of $\phi = 0$ for $\phi = \pi$ and again equal for $\phi = 2\pi$.

2.2.5 "Which-way" information

We introduce now a detector $\tilde{D}_X$ in one of the two ways, for example in way B, between the first beam splitter and the mirror. We take $\phi = 0$. The detector detects if a particles passing through it, but does not stop the particles. The particles go on to the detectors D_X and D_Y. The situation is illustrated in figure 2.5.

[7] We eliminate for simplicity the factor -1, in any case the result is linearly dependent, so we are allowed to do that.

[8] We use the relation $\cos\alpha = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha})$ and $\sin\alpha = \frac{1}{2i}(e^{i\alpha} - e^{-i\alpha})$.

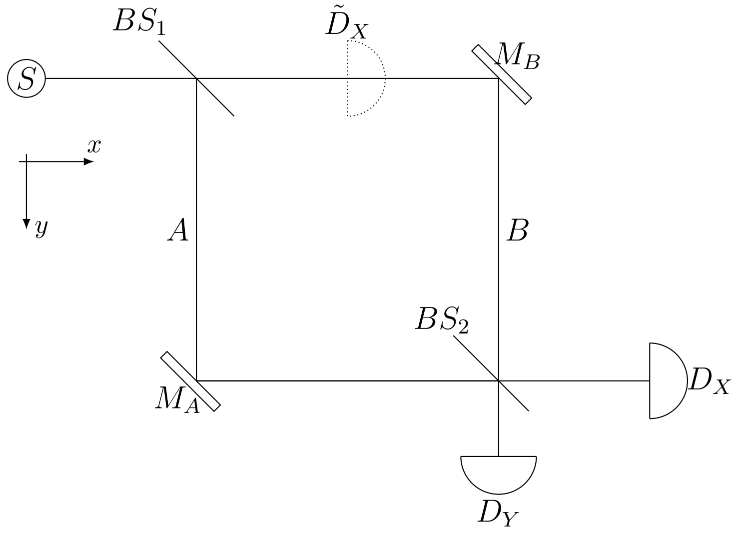


Figure 2.5: The Mach-Zehnder interferometer with detecotr $\tilde{D}_X$ on B.

The measurement with $\tilde{D}_X$ generates a modification of the state. Immediately after BS_1, the state is represented by a superposition as shown in the application of the BS matrix U_{BS} in equation (2.4) to the initial state $|\psi_{in}\rangle = |\psi_x\rangle$:

$$|\psi_{out}\rangle = U_{BS}|\psi_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|\psi_x\rangle + i|\psi_y\rangle\right) \quad .$$

But after the measurement the state is of absolute knowledge for X (and Y). In fact the state is $|\psi\rangle = |\psi_x\rangle$ is if the detector did detect the particle, and $|\psi\rangle = |\psi_y\rangle$ in the opposite case.

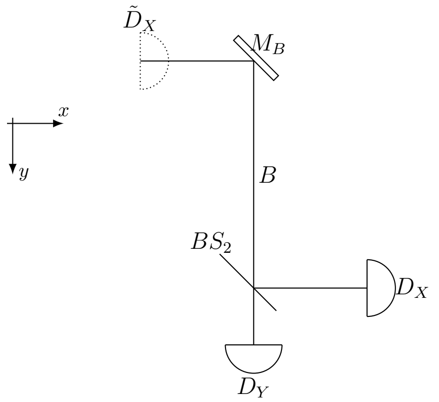
We can create a model that describes this new situation. Since the state is of absolute knowledge – if we introduce the detector $\tilde{D}_X$ – we can redesign the interferometer cancelling everything before the detector, creating a situation as in 2.6, where the situation 2.6(a) is realized if the particle is detected while 2.6(b) when this does not occur.

To calculate the probability of detecting a particle in D_X or D_Y is now, with the new model, simple. The time evolution after the detection or not in $\tilde{D}_X$ is so

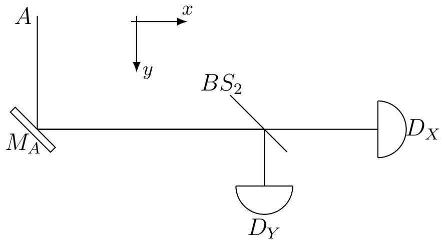
$$|\psi_{x,out}\rangle = U_{BS}U_S|\psi_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\i\end{pmatrix} \tag{2.12}$$

$$|\psi_{y,out}\rangle = U_{BS}U_S|\psi_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}i\\-1\end{pmatrix}. \tag{2.13}$$

From the states (2.12), that identifies the situation if the particle is detected in $\tilde{D}_X$, and (2.13), that identifies the situation if the particle is not detected in $\tilde{D}_X$,



(a)



(b)

Figure 2.6: The two possible situation of the Mach-Zehnder interferometer with detector $\tilde{D}_X$.

we calculate the probability of detection in X or Y for the two cases:

$$\begin{aligned}
\text{Prob}\{X=1\}_{|\psi_{x,out}\rangle} &= |\langle\psi_x|\psi_{x,out}\rangle|^2 = \left|\left\langle\begin{pmatrix}1\\0\end{pmatrix}\middle|\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\i\end{pmatrix}\right\rangle\right|^2 = \frac{1}{2}\\
\text{Prob}\{Y=1\}_{|\psi_{x,out}\rangle} &= |\langle\psi_y|\psi_{x,out}\rangle|^2 = \left|\left\langle\begin{pmatrix}0\\1\end{pmatrix}\middle|\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\i\end{pmatrix}\right\rangle\right|^2 = \frac{1}{2}\\
\text{Prob}\{X=1\}_{|\psi_{y,out}\rangle} &= |\langle\psi_x|\psi_{y,out}\rangle|^2 = \left|\left\langle\begin{pmatrix}1\\0\end{pmatrix}\middle|\frac{1}{\sqrt{2}}\begin{pmatrix}i\\-1\end{pmatrix}\right\rangle\right|^2 = \frac{1}{2}\\
\text{Prob}\{Y=1\}_{|\psi_{y,out}\rangle} &= |\langle\psi_y|\psi_{y,out}\rangle|^2 = \left|\left\langle\begin{pmatrix}0\\1\end{pmatrix}\middle|\frac{1}{\sqrt{2}}\begin{pmatrix}i\\-1\end{pmatrix}\right\rangle\right|^2 = \frac{1}{2} \quad .
\end{aligned} \tag{2.14}$$

The results of (2.14) – as shown in figure 2.5 – are equal to the situation where just one BS is present and therefore the probability is equally distributed.

However, this result is completely different from the case without the which-way information in equation (2.9). We can therefore conclude that if we want to know which way the particle will take, we will lose the interference's effects.

2.3 Interpretation and final considerations[9]

The analysis through the Mach-Zehnder interferometer model allows to draw three important conclusions about the single particle quantum interference phenomenon.

Delocalization

The differentiation of the two paths A e B through the factor ϕ determinates the proportion of the particle detected in D_X and D_Y. If this factor is positioned

[9]The inforamtions, partially modified, are from [21].

in the path B and has a value of $\phi = \pi$ all the particles are detected in D_Y. This means that the modification of just one of the two path influences the result of all the particles, even though the modification is only on one of the two possible paths. Considering that in the interferometer there is always just one particle at the same time, we can exclude effects of possible collisions and we have to acknowledge that the modification of one of the paths influences also the other. We therefore talk about delocalization of the particle in the two paths: the particles potentially explore every possible path (but that does not mean that the particle splits).

Complementarity

The knowledge of which of the path the particle took and the effects of interference are two complementary aspects, they cannot manifest them self at the same time. We speak of complementarity in the sense of the principle of complementarity presented by Bohr [11], which states that entities may have mutually exclusive properties, such as being a wave or a particle, depending from the kind of observation we perform.

Indiscernibles

We can predict if in an experiment there will be some quantum interferences thanks to the principle of indiscernibles, which states that

> «In an experiment there will be interferences if a particle go through many paths to arrive at the same detector and this paths are not distinguishable.»[10]

[10]Translation of [44, p.13].

Chapter 3

Entanglement

In this chapter we begin to analyse system composted of two particles. We will be faced with the phenomena of entanglement and the quantum correlation, two phenomena at the heart of quantum physics, which will be object of deeper analysis in the following chapters.

The theoretical information[1] for this chapter are taken (if not differently noted) from [21].

3.1 Formalism: The tensor product space

The result obtained in the sections form 1.3 to 1.8 are valid also in this context despite the Hilbert space is a little different.

The Hilbert space $\mathcal{H}$, which is the model of the pure state of a system$\Sigma = \Sigma_1 \cup \Sigma_2$, is the tensor product of the two states that describe the two single systems[2] noted

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

An important mathematical result affirms that: given two basis $\mathcal{B}_{\mathcal{H}_1} = \{|\psi_1\rangle, \ldots, |\psi_m\rangle\}$ and $\mathcal{B}_{\mathcal{H}_2} = \{|\varphi_1\rangle, \ldots, |\varphi_n\rangle\}$ of $\mathcal{H}_1$ and $\mathcal{H}_2$, then a base of $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is given by

$$\mathcal{B} = \{|\psi_1\rangle \otimes |\varphi_1\rangle, |\psi_1\rangle \otimes |\varphi_2\rangle, \ldots, |\psi_m\rangle \otimes |\varphi_n\rangle\} \Rightarrow \dim\mathcal{H} = m \cdot n$$

States

The most general vector element of $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is

$$|\Psi\rangle = \sum_{i,j} \gamma_{i,j} |\psi_i\rangle \otimes |\varphi_j\rangle \ . \tag{3.1}$$

[1]We operated a transposition from spin to polarization.

[2]The mathematical nature of the tensor product goes beyond the goal of this text, it will only be used formally to describe two particles systems (it is just importatn to know that it benefits of the usual property of multiplication, but is not commutative [22]). However, for sake of completeness, in appendix A there is the elementar mathematical formalism for the calculus of quantum physics.

The form

$$|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \tag{3.2}$$

is indeed not exhaustive, because we can always represent a pure state $|\Psi\rangle$ with the linear composition of two states of the kind $|\psi_1\rangle \otimes |\psi_2\rangle$ as seen in section 1.3.1. So not every state can be expressed in this factorised form. For example, given the states $|\psi_i\rangle, |\varphi_i\rangle \in \mathcal{H}_i$ $(i = 1, 2)$, assuming that they are orthogonal, the state vector

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|\psi_1\rangle \otimes |\psi_2\rangle + |\varphi_1\rangle \otimes |\varphi_2\rangle\right) \tag{3.3}$$

does not admit a factorisation in $|\chi_1\rangle \otimes |\chi_2\rangle$ with $|\chi_i\rangle \in \mathcal{H}_i$. Every[3] state of $\mathcal{H}$ that can not be expressed in the factorisation of equation (3.2) are called *entangled states*, these are the not factorisable states of $\mathcal{H}$.

Application to the polarisation

The Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ gives a description of the properties of a system composed by two photons.

Given the basis

$$\mathcal{B} = \left\{|V^a\rangle \otimes |V^b\rangle, |V^a\rangle \otimes |H^b\rangle, |H^a\rangle \otimes |V^b\rangle, |H^a\rangle \otimes |H^b\rangle\right\}$$

and $|\Psi\rangle \in \mathcal{H}$ the superposition state

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|V^a\rangle \otimes |H^b\rangle - |H^a\rangle \otimes |V^b\rangle\right) \tag{3.4}$$

then equation (3.4)) is an entangled state that can not be factorised in the form of equation (3.2).

Indeed supposing we can express (3.4) as $|\psi^a\rangle \otimes |\psi^b\rangle$ and with $(\alpha = a, b)$ these are

$$|\psi^\alpha\rangle = \sum_{i=H,V} \lambda_i^\alpha |i^\alpha\rangle \quad \lambda_i^\alpha \in \mathbb{C},$$

then we can re-write $|\psi^a\rangle \otimes |\psi^b\rangle$ as

$$\begin{aligned}
|\psi^a\rangle \otimes |\psi^b\rangle &= \sum_{i=H,V} \lambda_i^a |i^a\rangle \otimes \sum_{j=H,V} \lambda_j^b |j^b\rangle \\
&= \sum_{i=H,V} \sum_{j=H,V} \lambda_i^a \lambda_j^b |i^a\rangle \otimes |j^b\rangle \\
&= \lambda_V^a \lambda_V^b |V^a\rangle \otimes |V^b\rangle + \lambda_V^a \lambda_H^b |V^a\rangle \otimes |H^b\rangle \\
&+ \lambda_H^a \lambda_V^b |H^a\rangle \otimes |V^b\rangle + \lambda_H^a \lambda_H^b |H^a\rangle \otimes |H^b\rangle \quad .
\end{aligned} \tag{3.5}$$

[3]The orthogonal condition is here not necessary any more.

Comparing equations(3.4) and (3.5) we obtain the following equation system for the coefficients λ^α:

$$\begin{cases} \lambda_V^a \lambda_V^b = 0 \\ \lambda_V^a \lambda_H^b = \dfrac{1}{\sqrt{2}} \\ \lambda_H^a \lambda_V^b = -\dfrac{1}{\sqrt{2}} \\ \lambda_H^a \lambda_H^b = 0 \end{cases} . \tag{3.6}$$

It is clear that (3.6) has no solution $\lightning$. Since we concluded with a contradiction,(3.4) has no factorised form $|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ i.e.:

$$\frac{1}{\sqrt{2}} \left(|V^a\rangle \otimes |H^b\rangle - |H^a\rangle \otimes |V^b\rangle\right) \neq |\psi^a\rangle \otimes |\psi^b\rangle. \tag{3.7}$$

3.2 Entangled polarization and correlations

As well as superposition states in a one particle system, the entangled states in a system of two particles, that are superposition's states, shows interesting properties. This section is dedicate to the study of this properties in a two particles system.

The Hilbert space associate the the studied system is

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$$

while the studied state is

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|V^a\rangle \otimes |H^b\rangle - |H^a\rangle \otimes |V^b\rangle\right) \quad . \tag{3.8}$$

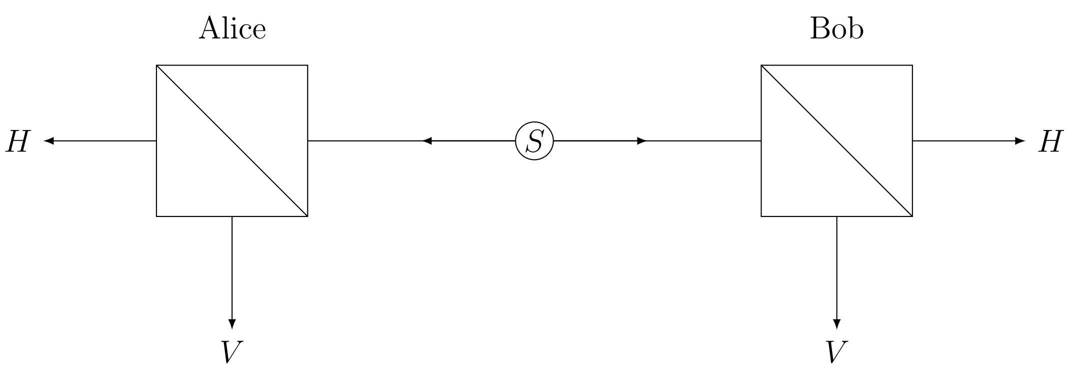


Figure 3.1: Schema of experiment with two entangled photons.

The state (3.8) does not represent a state of absolute knowledge neither for the observable σ_z associated to the measurement tool of photon a (noted $\sigma_z \otimes I$) nor for observable σ_z associated to the measurement tool of photon b (noted $I \otimes \sigma_z$). If (3.8) would be such a state, it should be an eigenvector of this observable, i.e.:

$$\sigma_z \otimes I\,|\Psi\rangle = \lambda|\Psi\rangle$$

but for $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|V^a\rangle \otimes |H^b\rangle - |H^a\rangle \otimes |V^b\rangle\right)$ this equation is not possible, because for every $\lambda \in \mathbb{R}$

$$\begin{aligned}&\sigma_z \otimes I\left[\frac{1}{\sqrt{2}}\left(|V^a\rangle \otimes |H^b\rangle - |H^a\rangle \otimes |V^b\rangle\right)\right] = \frac{1}{\sqrt{2}}\left(\sigma_z|V^a\rangle \otimes I|H^b\rangle - \sigma_z|H^a\rangle \otimes I|V^b\rangle\right)\\ &= -\frac{1}{\sqrt{2}}\left(|V^a\rangle \otimes |H^b\rangle + |H^a\rangle \otimes |V^b\rangle\right) \neq \lambda\left[\frac{1}{\sqrt{2}}\left(|V^a\rangle \otimes |H^b\rangle - |H^a\rangle \otimes |V^b\rangle\right)\right]\end{aligned} \tag{3.9}$$

and analogue of $I \otimes \sigma_z$.

However, the state (3.8) is a sate of absolute knowledge for the observable $\sigma_z \otimes \sigma_z$ because

$$\begin{aligned}&\sigma_z \otimes \sigma_z\left[\frac{1}{\sqrt{2}}\left(|V^a\rangle \otimes |H^b\rangle - |H^a\rangle \otimes |V^b\rangle\right)\right]\\ &= -\frac{1}{\sqrt{2}}(|V^a\rangle \otimes |H^b\rangle - |H^a\rangle \otimes |V^b\rangle) = \lambda\frac{1}{\sqrt{2}}\left(|V^a\rangle \otimes |H^b\rangle - |H^a\rangle \otimes |V^b\rangle\right)\end{aligned} \tag{3.10}$$

for $\lambda = -1$.

We notice therefore that there is no absolute knowledge for the polarization of the single photons. In the entangled state states the properties of the single subsystem is not defined, but just the property of the whole system.

The values $+1$ and -1 of the observables σ_z are therefore both associated to potential properties for the system in the state $|\Psi\rangle$.

We suppose now that Alice (who measures photon a) and Bob (who measures photon b) execute a series of N measurement using always particles in the state (3.8). In these experiment both analyse the observable σ_z.

The obtained result are compared the table 3.1.

We notice that for every couple of results there is a perfect (anti)correlation: if Alice observes the value $+1$ for the polarization of photon a, then Bob observes value -1 for photon b. The properties are so potentially correlated.

However, the couple of values are random (it is impossible to predict their value a priori) and present them-self with the following probability

$$\mathrm{Prob}_{|\Psi\rangle}\{\sigma_z \otimes I = \pm 1\,; I \otimes \sigma_z = \mp 1\} = \frac{1}{2} \quad .$$

The correlation is also a clear result of the calculus of the objective probabilities

Measurement	Alice's result	Bob's result
I	-1	1
II	-1	1
III	1	-1
VI	-1	1
V	1	-1
...	...	...
$N-1$	1	-1
N	-1	1

Table 3.1: Compare of Alice and Bob's result in an experiment with entangled photons.

bound with the measurements of the polarizations:

$$\begin{aligned}
&\mathrm{Prob}_{|\Psi\rangle}\left\{\sigma_z \otimes I = +1\, ;\, I \otimes \sigma_z = -1\right\} = \left\| P_{|H^a\rangle} \otimes P_{|V^b\rangle} \left|\Psi\right\rangle \right\|^2 \\
&= \left\| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \left[\frac{1}{\sqrt{2}} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right] \right\|^2 \\
&= \left\| \frac{1}{\sqrt{2}} \left[\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \otimes \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \right] \right\|^2 = \frac{1}{2} \left\| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|^2 = \frac{1}{2} \qquad (3.11)
\end{aligned}$$

$$\mathrm{Prob}_{|\Psi\rangle}\left\{\sigma_z \otimes I = -1\, ;\, I \otimes \sigma_z = +1\right\} = \left\| P_{|V^a\rangle} \otimes P_{|H^b\rangle} \left|\Psi\right\rangle \right\|^2 = \frac{1}{2} \qquad (3.12)$$

$$\mathrm{Prob}_{|\Psi\rangle}\left\{\sigma_z \otimes I = +1\, ;\, I \otimes \sigma_z = +1\right\} = \left\| P_{|H^a\rangle} \otimes P_{|H^b\rangle} \left|\Psi\right\rangle \right\|^2 = 0 \qquad (3.13)$$

$$\mathrm{Prob}_{|\Psi\rangle}\left\{\sigma_z \otimes I = -1\, ;\, I \otimes \sigma_z = -1\right\} = \left\| P_{|V^a\rangle} \otimes P_{|V^b\rangle} \left|\Psi\right\rangle \right\|^2 = 0 \qquad (3.14)$$

Form equations (3.13) and (3.14) we conclude clearly that it is impossible that Alice and Bob would observe the same value in their measurement. The quations (3.11) and (3.12) show that the two possible result $(+1, -1)$ and $(-1, +1)$ are equiprobable.

State collapse

The shown correlations can be understood on the basis of the *state collapse* interpretation. The initial state (3.8) "collapse" into one of the two states that build the superposition, i.e. (3.15) or (3.16) depending on the result of the measurement performed by Alice or Bob. The measurement $+1$ for the observable $\sigma_z \otimes I$ leads to the following collapse

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|V^a\rangle \otimes |H^b\rangle - |H^a\rangle \otimes |V^b\rangle \right) \xrightarrow{\sigma_z \otimes I = +1} |\Psi'\rangle = |H^a\rangle \otimes |V^b\rangle \qquad (3.15)$$

keeping only the part of state (3.8) (in this case the "right part") in accord with the measurement.

Analogously for and observed value +1 for $I \otimes \sigma_z$ the following collapse occurs

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|V^a\rangle \otimes |H^b\rangle - |H^a\rangle \otimes |V^b\rangle\right) \xrightarrow{I\otimes\sigma_z=+1} |\Psi'\rangle = |V^a\rangle \otimes |H^b\rangle \ . \tag{3.16}$$

We notice that, in the moment that a measurement occurs, the collapsed state $|\Psi'\rangle$ is of absolute knowledge for both particles.

3.3 Modified Franson interferometer

Similarly to the section 2.2 with the Mach-Zehnder interferometer, in this section we will study a simplified model that allows us to come up with important experimental conclusions about the two-particle systems.

The model – called *modified Franson interferometer* – shown in Figure 3.2, is constructed in such away that each of a pair of particles emitted propagates in a Mach-Zehnder interferometer (in fact we note that this assembly is the union of two of these interferometers). The source of coupled particles, named EPR, emits two particles that start in opposite directions, but without it being possible to know which direction the particles will take. Only one pair of particles is emitted at a time, so there is always only one pair inside the unit.

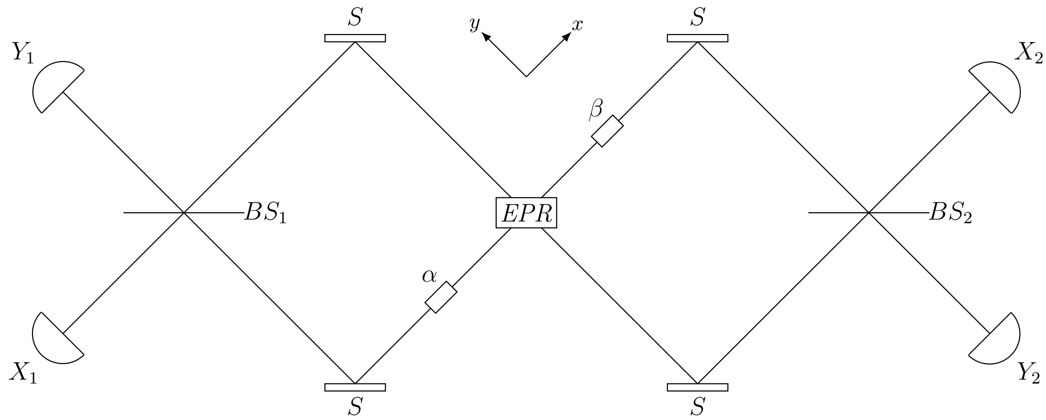


Figure 3.2: The modified Franson interferometer.

3.3.1 Model, states and observables

Each of the particles is described by a Hilbert's space $\mathcal{H} = \mathbb{C}^2$ where the states vectors $|\psi_x\rangle, |\psi_y\rangle$ e $|\varphi_x\rangle, |\varphi_y\rangle$ describe the direction of propagation. The model for the two particles is therefore the Hilbert's space

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \quad .$$

The initial space $|\Psi\rangle_{in}$ is the state for which the propagation's direction of the pair is defined but not that of each individual particle:

$$|\Psi\rangle_{in} = \frac{1}{\sqrt{2}}(|\psi_x\rangle \otimes |\varphi_x\rangle + |\psi_y\rangle \otimes |\varphi_y\rangle) \quad . \tag{3.17}$$

Analysing (3.17) we note that $|\psi_x\rangle \otimes |\varphi_x\rangle$ represents the state of the system in which the particles propagate in x direction, while $|\psi_y\rangle \otimes |\varphi_y\rangle$ represents the one in which they propagate in direction y. We can therefore rewrite equation (3.17) as follows

$$|\Psi\rangle_{in} = \frac{1}{\sqrt{2}}(|\Psi\rangle_x + |\Psi\rangle_y) \tag{3.18}$$

where $|\Psi\rangle_x = |\psi_x\rangle \otimes |\varphi_x\rangle$ and $|\Psi\rangle_y = |\psi_y\rangle \otimes |\varphi_y\rangle$. In this formalism it is clear that this is a superposition state.

The observables associated with the detectors are similar to those obtained in section 2.2.2.

3.3.2 Time evolution of the states

The matrices associated with the mirrors, beam splitters and path differentials are similar to those obtained in section 2.2.3.

The time evolution of the initial state $|\Psi\rangle_{in}$ is, due to linearity, the sum of the single evolutions of the states $|\Psi\rangle_x$ and $|\Psi\rangle_y$:

$$U\left(\frac{1}{\sqrt{2}}(|\Psi\rangle_x + |\Psi\rangle_y)\right) = \frac{1}{\sqrt{2}}(U|\Psi\rangle_x + U|\Psi\rangle_y) \quad .$$

Time evolution also acts on the pair of particles: the unitary operator U that acts on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is composed by the tensor $U = U_1 \otimes U_2$, where U_1 acts on the space $\mathcal{H}_1$ while U_2 acts on the space $\mathcal{H}_2$.

The unitary matrix that identifies the evolution of the modified Franson interferometer is therefore:

$$U = [U_{BS}U_SU_\alpha] \otimes [U_{BS}U_SU_\beta] = [U_{BS} \otimes U_{BS}]\,[U_S \otimes U_S]\,[U_\alpha \otimes U_\beta] \ . \tag{3.19}$$

The state evolution of the state $|\Psi\rangle_x$ is therefore:

$$\begin{aligned}
U|\Psi\rangle_{x,in} &= U(|\psi_x\rangle \otimes |\varphi_x\rangle) = [U_{BS}U_SU_\alpha] \otimes [U_{BS}U_SU_\beta]\,(|\psi_x\rangle \otimes |\varphi_x\rangle) \\
&= [U_{BS}U_S] \otimes [U_{BS}U_S]\,(e^{i\alpha}|\psi_x\rangle \otimes e^{i\beta}|\varphi_x\rangle) = [U_{BS}] \otimes [U_{BS}]\,(ie^{i\alpha}|\psi_y\rangle \otimes ie^{i\beta}|\varphi_y\rangle) \\
&= \left[ie^{i\alpha}\frac{1}{\sqrt{2}}(|\psi_y\rangle + i|\psi_x\rangle)\right] \otimes \left[ie^{i\beta}\frac{1}{\sqrt{2}}(|\varphi_y\rangle + i|\varphi_x\rangle)\right] = |\Psi\rangle_{x,out}
\end{aligned} \tag{3.20}$$

while for $|\Psi\rangle_y$ it is:

$$\begin{aligned}
U|\Psi\rangle_{y,in} &= U(|\psi_y\rangle \otimes |\varphi_y\rangle) = [U_{BS}U_SU_\alpha] \otimes [U_{BS}U_SU_\beta]\,(|\psi_y\rangle \otimes |\varphi_y\rangle) \\
&= [U_{BS}U_S] \otimes [U_{BS}U_S]\,(|\psi_y\rangle \otimes |\varphi_y\rangle) = [U_{BS}] \otimes [U_{BS}]\,(i|\psi_x\rangle \otimes i|\varphi_x\rangle) \\
&= \left[i\frac{1}{\sqrt{2}}(|\psi_x\rangle + i|\psi_y\rangle)\right] \otimes \left[i\frac{1}{\sqrt{2}}(|\varphi_x\rangle + i|\varphi_y\rangle)\right] = |\Psi\rangle_{y,out}
\end{aligned} \tag{3.21}$$

As written in (3.18) the equations of the final states (3.20) and (3.21) are added, to obtain $|\Psi\rangle_{out}$:

$$\begin{aligned}|\Psi\rangle_{out} &= \frac{1}{\sqrt{2}}\left[ie^{i\alpha}\frac{1}{\sqrt{2}}(|\psi_y\rangle + i|\psi_x\rangle)\right] \otimes \left[ie^{i\beta}\frac{1}{\sqrt{2}}(|\varphi_y\rangle + i|\varphi_x\rangle)\right] \\ &+ \frac{1}{\sqrt{2}}\left[i\frac{1}{\sqrt{2}}(|\psi_x\rangle + i|\psi_y\rangle)\right] \otimes \left[i\frac{1}{\sqrt{2}}(|\varphi_x\rangle + i|\varphi_y\rangle)\right] \\ &= \frac{ie^{i\theta}}{\sqrt{2}}\left[\sin\theta(|\psi_x\rangle \otimes |\varphi_x\rangle - |\psi_y\rangle \otimes |\varphi_y\rangle) + \cos\theta(|\psi_x\rangle \otimes |\varphi_y\rangle + |\psi_y\rangle \otimes |\varphi_x\rangle)\right]\end{aligned} \tag{3.22}$$

where $\theta = \dfrac{\alpha+\beta}{2}$.

3.3.3 Study of correlations

From state in equation (3.22) we can calculate the probabilities of observing the particles in the detectors X and Y and consequently analyse some special cases.

$$\text{Prob}_{|\Psi\rangle_{out}}\{X_1 = 1; X_2 = 1\} = \left\|P_{|\psi_x\rangle} \otimes P_{|\varphi_x\rangle}|\Psi\rangle_{out}\right\|^2 = \frac{1}{2}\sin^2\left(\frac{\alpha+\beta}{2}\right) \tag{3.23}$$

$$\text{Prob}_{|\Psi\rangle_{out}}\{Y_1 = 1; Y_2 = 1\} = \left\|P_{|\psi_y\rangle} \otimes P_{|\varphi_y\rangle}|\Psi\rangle_{out}\right\|^2 = \frac{1}{2}\sin^2\left(\frac{\alpha+\beta}{2}\right) \tag{3.24}$$

$$\text{Prob}_{|\Psi\rangle_{out}}\{X_1 = 1; Y_2 = 1\} = \left\|P_{|\psi_x\rangle} \otimes P_{|\varphi_y\rangle}|\Psi\rangle_{out}\right\|^2 = \frac{1}{2}\cos^2\left(\frac{\alpha+\beta}{2}\right) \tag{3.25}$$

$$\text{Prob}_{|\Psi\rangle_{out}}\{Y_1 = 1; X_2 = 1\} = \left\|P_{|\psi_y\rangle} \otimes P_{|\varphi_x\rangle}|\Psi\rangle_{out}\right\|^2 = \frac{1}{2}\cos^2\left(\frac{\alpha+\beta}{2}\right) \tag{3.26}$$

Case $\alpha = \beta = 0$

In this situation the equations (3.23), (3.24), (3.25) and (3.26) show that the particles are always observed in two opposite detectors (X_1 and Y_2 or Y_1 and X_2) with probability distribution 50%. This is a perfect (anti)correlation.

Case $\alpha + \beta = \frac{\pi}{2}$

In this situation instead the equations (3.25), (3.26), (3.23) and (3.24) show that the particles are always observed in similar detectors (X_1 and X_2 or Y_1 and Y_2) with probability distribution 50%. It is therefore a perfect correlation. $\alpha+\beta = \frac{\pi}{2}$ allows the choice of $\alpha = \frac{\pi}{2}$ and $\beta = 0$. This condition allows us to leave one of the two paths unaltered.

This phenomenon is called interference at two particles, similar to that of a single particle. In this case, however, the phenomenon can be observed only when considering the two particles as a single system. In fact, observing them separately, a normal random distribution is obtained:

$$\text{Prob}_{|\Psi\rangle_{out}}\{X_1 = 1\} = \left\|P_{|\psi_x\rangle} \otimes I\,|\Psi\rangle_{out}\right\|^2 = \frac{1}{2}\left(\sin^2\theta + \cos^2\theta\right) = \frac{1}{2}$$
$$\text{Prob}_{|\Psi\rangle_{out}}\{Y_2 = 1\} = \left\|I \otimes P_{|\varphi_y\rangle}\,|\Psi\rangle_{out}\right\|^2 = \frac{1}{2}\left(\sin^2\theta + \cos^2\theta\right) = \frac{1}{2} \qquad (3.27)$$

independent of the values of θ[4] and therefore free from interference phenomenon.

3.4 Interpretation and final considerations

Entangled states are quantum phenomena without precedent in classical physics [44]. Their existence has a couple of important consequences.

Entangled

It emerges both from the purely mathematical point – equation (3.3) – and from the physical point – with the modified Franson interferometer in the equations (3.23-3.26) and (3.27) – that the two particles must be considered as a single entities, since, as shown in section 3.3.3, acting only on one path of the Franson interferometer changed the overall outcome of the experience.

Communication tool

As mentioned in section 3.2, the values observed during experiments with interlaced states are random, so there is no way of predicting them.

However, it may be thought that correlations, as they produce a correlated result, could be used to communicate at very high speeds.

That is not the case, though.

As described in [44, p.73],], the correlation phenomenon cannot be used to communicate at a speed faster than light speed since it is necessary to use a "traditional" communication system in addition.

> «Cependant, ce phénomène ne peut pas etre employé pour communiquer, c'est-à-dire pour envoyer un message. La raison en est la suivante: qu'on soit en situation de corrélation parfaite, d'anti-corrélation parfaite, ou en n'importe quelle situation intermédiaire en ce qui concerne les corrélations à deux particules, rien ne change aux résultats que on observe pour chaque particule séparément. En particulier, pour l'interféromètre de Franson que nous avons considéré, nous avons dit que de chaque côté, la moitié des particules est détectée dans un détecteur, et l'autre moitié dans l'autre. Alice, qui observe seulement les particules qui sont parties vers la gauche, voit des détections aléatoires; à droite, Bob a beau modifier son interféromètre, rien ne va changer chez Alice.
> C'est seulement lorsque Alice et Bob se parlent (par téléphone par exemple) et qu'ils comparent leurs résultats, qu'ils remarquent l'existence de corrélations entre les particules. Un moyen de communication ordinaire (téléphone, Internet, se rencontrer dans un bistrot) est donc absolument nécessaire pour connaître les

[4] In the same way the result for the other 2 cases can be obtained.

> corrélations quantiques; ces corrélations à elles seules ne permettent pas de communiquer.»
>
> [44, p.73]

Since the measure is random, Bob, once observed his result (if he knows the values of α and β), can deduct Alice's results. But if you want to be able to communicate with this system you have to change the values of α and or β. In this case, however, to communicate with Alice, one should be able to modify the path quicker than the particle propagation, which is not possible.

Part II

Spooky action at a distance

Chapter 4

EPR argument

THIS chapter of the research will deal with the important criticism expressed by EINSTEIN, PODOLSKY, and ROSEN (EPR) to the quantum theory in order to highlight the possible gaps in it.

4.1 Introduction

The overview offered so far in this text on quantum physics is part of what is called the "*Copenhagen Interpretation*", i.e. the standard and axiomatized theory. As seen in the preceding chapters, this interpretation includes in its base the ineliminable random probabilities associated with the measurement act, a principle of complementarity, and a formalism that does not allow to visualize reality, which representation becomes useless [39].

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

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(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

1.

ANY serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concepts with which the theory operates. These concepts are intended to correspond with the objective reality, and by means of these concepts we picture this reality to ourselves.

In attempting to judge the success of a physical theory, we may ask ourselves two questions: (1) "Is the theory correct?" and (2) "Is the description given by the theory complete?" It is only in the case in which positive answers may be given to both of these questions, that the concepts of the theory may be said to be satisfactory. The correctness of the theory is judged by the degree of agreement between the conclusions of the theory and human experience. This experience, which alone enables us to make inferences about reality, in physics takes the form of experiment and measurement. It is the second question that we wish to consider here, as applied to quantum mechanics.

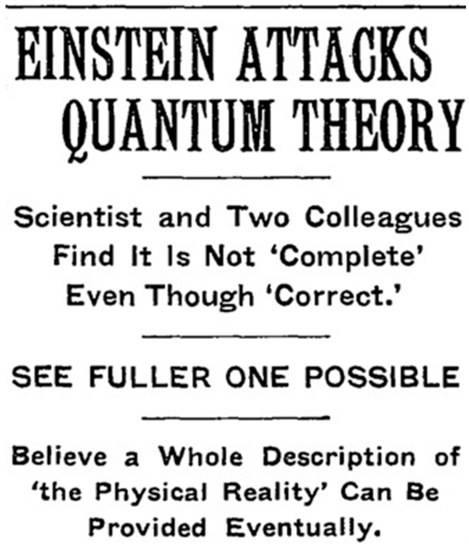
Whatever the meaning assigned to the term *complete*, the following requirement for a complete theory seems to be a necessary one: *every element of the physical reality must have a counterpart in the physical theory*. We shall call this the condition of completeness. The second question is thus easily answered, as soon as we are able to decide what are the elements of the physical reality.

The elements of the physical reality cannot be determined by *a priori* philosophical considerations, but must be found by an appeal to results of experiments and measurements. A comprehensive definition of reality is, however, unnecessary for our purpose. We shall be satisfied with the following criterion, which we regard as reasonable. *If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.* It seems to us that this criterion, while far from exhausting all possible ways of recognizing a physical reality, at least provides us with one

Figure 4.1: Original version of the article appeared in *Physical review.*

It is on this basis that the criticism of the title goes on, which challenges the completeness of quantum formalism as assessed by the *Copenhagen Interpretation.*

As you can see from the title of the first page of the renowned newspaper The New York Times reported in Figure 4.2 [34], at that time this criticism of quantum theory made a stir as to be reported even outside the usual specialized scientific journals.


EINSTEIN ATTACKS QUANTUM THEORY

Scientist and Two Colleagues Find It Is Not 'Complete' Even Though 'Correct.'

SEE FULLER ONE POSSIBLE

Believe a Whole Description of 'the Physical Reality' Can Be Provided Eventually.

Figure 4.2: Headlines of the article published on the NYT on May 4, 1935 [34].

4.2 The EPR article

4.2.1 Preconditions

In the first lines of the article in question[1], from now just EPR, we can read that «*in attempting to judge the success of a physical theory, we may ask ourselves two questions: (1) "Is the theory correct?" and (2) "Is the description given by the theory complete?"*»[2].

In the field of quantum physics, the authors' interest, the answer to the first question is clearly positive. «*The correctness of the theory is judged by the degree of agreement between the conclusions of the theory and human experience.*», it is therefore to determine whether the experimental results confirm the theoretical predictions, which happens in this area with great precision.

«*It is the second question that we wish to consider here, as applied to quantum mechanics.*». EPR therefore does not discuss the correctness of quantum theory but its completeness.

[1] EINSTEIN Albert et al., "Can quantum-mechanical description of physical reality be considered complete?", in: *Physical review* 47.10 (1935), p. 777.

[2] The notation «*text*» inicates from now a citation from the article [19].

Condition of completeness in EPR

According to EPR «*the following requirement for a complete theory seems to be a necessary one: every element of the physical reality must have a counter part in the physical theory We shall call this the condition of completeness.*»

EPR's completeness condition is closely related to the concept of physical reality, which - as said in the article - is far from easy to define. The authors decided that «*A comprehensive definition of reality is, however, unnecessary for our purpose*»», thus giving a simplified definition, but «*in agreement with classical as well as quantum-mechanical ideas*», of physical reality: «*If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.*».

4.2.2 The EPR argument

The argument of EPR starts with the analysis of a particle having only one degree of freedom[3] and whose momentum has a determined value p_0 and concludes, that the amount of motion and position cannot be simultaneously real as their operators do not commute. EPR deducts from this mathematical evidence a fundamental logical disjunction. On the basis of this logical disjunction EPR develops an argument that analyzing «*two systems, which we permit to interact from the time $t = 0$ to $t = T$, after which time we suppose that there is no longer any interaction between the two parts*» comes to the conclusion that the quantum theory is not complete, since a state whose position and momentum can be simultaneously real is obtainable. Fundamental to this argument is the interaction that the two systems have had: the creation of an entangled state.

4.3 The argument

Here below we propose the argument with the analysis of polarized photons[4].

4.3.1 Logical disjunction

Taking a polarized photon with initial state

$$|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.1}$$

of which we want to know the values (always ± 1) associated to the observables

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ .$$

[3] I.e. that the particle may move only in one direction [20].

[4] The proceding is similar to the one proposed by Bohm and Aharonov in [9].

We start with the measurement of σ_z:

$$\mathrm{Prob}_{|H\rangle}\{\sigma_z = +1\} = \left\| P_{E_{|H\rangle}} |H\rangle \right\|^2 = \left\| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|^2 = 1 \quad . \tag{4.2}$$

Equation 4.2 allows to conclude that for a status particle $|H\rangle$, the value of the observable σ_z can be predicted with certainty. Therefore, for the definition of physical reality, «*there exists an element of physical reality corresponding to this physical quantity*».

On the particle, whose state is unaltered by the measurement of σ_z[5], we want now to know the value associated with the observable σ_x. This value cannot be known a priori since a measurement is required in which both values appear with 50% of probability[6]. The problem associated with this measurement – necessary to know the value of the observable σ_x with certainty, and thus associating an element of physical reality to this physical quantity – is that the state after the measurement , $|+\rangle$ or $|-\rangle$, no longer fulfils the criterion needed to associate an element of reality for observable σ_z, i.e. the value of σ_z cannot be predicted with certainty:

$$\mathrm{Prob}_{|+\rangle}\{\sigma_z = +1\} = \left\| P_{E_{|H\rangle}} |+\rangle \right\|^2 = \left\| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|^2 = \frac{1}{2} \quad .$$

$$\mathrm{Prob}_{|-\rangle}\{\sigma_z = +1\} = \left\| P_{E_{|H\rangle}} |-\rangle \right\|^2 = \left\| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\|^2 = \frac{1}{2} \quad .$$

From this we deduce that, for a particle in the state (4.1), when the value of the observable σ_z is known the value of the observable σ_x does not have a physical reality and vice versa.

EPR extends the concept explained above by saying «*More generally, it is shown in quantum mechanics that, if the operators corresponding to two physical quantities, say A and B, do not commute, that is, if $AB \neq BA$, then the precise knowledge of one of them precludes such a knowledge of the other. Furthermore, any attempt to determine the latter experimentally will alter the state of the system in such a way as to destroy the knowledge of the first*».

This leads to the central logic disjunction of the article: «*from this follows that either*

(a) *the quantum mechanical description of reality given by the wave function* [state vector $|\psi\rangle$] *is not complete*

or

(b) *when the corresponding operators of two physical quantities do not commute the two quantities cannot have simultaneous reality*».

[5] $|\psi\rangle' = \dfrac{P_{E_{|H\rangle}}|H\rangle}{\|P_{E_{|H\rangle}}|H\rangle\|} = |H\rangle$

[6] $\mathrm{Prob}_{|H\rangle}\{\sigma_x = +1\} = \|P_{E_{|H\rangle}}|H\rangle\|^2 = \left\| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|^2 = \left\| \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|^2 = \frac{1}{2}$

$\mathrm{Prob}_{|H\rangle}\{\sigma_x = -1\} = \|P_{E_{|V\rangle}}|H\rangle\|^2 = \left\| \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|^2 = \left\| \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\|^2 = \frac{1}{2}$

4.3.2 Simultaneous reality for two observables that do not commute

EPR performs a general demonstration of how two non-commuting physical quantities can simultaneously be associated to an element of reality. In this section we present a similar argument, made with two entangled state photons, which leads to the same conclusion.

We consider two photons a and b in the entangled state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|V^a \otimes |H^b\rangle - |H^a\rangle \otimes |V^b\rangle\rangle\right) \quad . \tag{4.3}$$

We assume that the description of the state given by this vector is complete (this is equivalent to denying the alternative a of the logic disjunction).

The state (4.3) is expressed with the eigenvectors $|V\rangle$ and $|H\rangle$ of the observable σ_z, which is, however, expressible also with eigenvectors

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

of the observable $\sigma_x$using the following equalities

$$|V\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right) \quad \text{and} \quad |H\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) \quad . \tag{4.4}$$

So we rewrite the state as

$$\begin{aligned}
|\Psi\rangle &= \frac{1}{\sqrt{2}}\left(|V^a \otimes |H^b\rangle - |H^a\rangle \otimes |V^b\rangle\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(|+^a\rangle - |-^a\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|+^b\rangle + |-^b\rangle\right)\right] \\
&- \left[\frac{1}{\sqrt{2}}\left(|+^a\rangle + |-^a\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|+^b\rangle - |-^b\rangle\right)\right] \\
&= \frac{1}{\sqrt{2}}\left[-\frac{1}{2}|-^a\rangle \otimes |+^b\rangle + \frac{1}{2}|+^a\rangle \otimes |-^b\rangle\right. \\
&\left.- \frac{1}{2}|-^a\rangle \otimes |+^b\rangle + \frac{1}{2}|+^a\rangle \otimes |-^b\rangle\right] \\
&= -\frac{1}{\sqrt{2}}\left(|-^a\rangle \otimes |+^b\rangle - |+^a\rangle \otimes |-^b\rangle\right) \quad .
\end{aligned} \tag{4.5}$$

The equality (4.5) is particularly interesting analysing the problem from the point of view of the state reduction as shown in section 3.2. It is evident from this point of view that there exist an (anti)correlation also for observable $\sigma_x \otimes \sigma_x$. Measuring a value $+1$ for observable $\sigma_x \otimes I$ the system is reduced into the state

$$|\Psi'\rangle = \frac{1}{\sqrt{2}}\left(|+^a\rangle \otimes |-^b\rangle\right)$$

and vice versa with the photon b.

From the measurement it results that the frequency of the correlations is the same for both pairs of observables[7]:

$$\mathrm{Prob}_{|\Psi\rangle}\left\{\sigma_z \otimes I = +1\,; I \otimes \sigma_z = -1\right\} = \left\| P_{|H^a\rangle} \otimes P_{|V^b\rangle} |\Psi\rangle \right\|^2 = \frac{1}{2}$$
$$\mathrm{Prob}_{|\Psi\rangle}\left\{\sigma_x \otimes I = -1\,; I \otimes \sigma_x = +1\right\} = \left\| P_{|-^a\rangle} \otimes P_{|+^b\rangle} |\Psi\rangle \right\|^2 = \frac{1}{2}$$
$$\mathrm{Prob}_{|\Psi\rangle}\left\{\sigma_z \otimes I = +1\,; I \otimes \sigma_z = +1\right\} = \left\| P_{|H^a\rangle} \otimes P_{|H^b\rangle} |\Psi\rangle \right\|^2 = 0$$
$$\mathrm{Prob}_{|\Psi\rangle}\left\{\sigma_x \otimes I = -1\,; I \otimes \sigma_x = -1\right\} = \left\| P_{|-^a\rangle} \otimes P_{|-^b\rangle} |\Psi\rangle \right\|^2 = 0$$

Local causes

At this point it is crucial to clarify a principle given for granted in EPR. In the text we read that considering systems that do not interact «*no real change can take place in the second system in consequence of anything that may be done to the first system*». The statement here is known as the principle of local causes, which formally states that: events occurring in a given time-space region cannot be affected by a change of parameters located in a space-time region separated by a range of type space L within a time period $\tau < \frac{L}{c}$ [20].

In this perspective, two entangled photons can be considered as two separate systems[8].

Following the principle of local causes, given the entangled state (4.1) Alice's measurement on photon a does not disturb the polarization of the photon b measured by Bob, but because of the correlations, it is possible to know with certainty the values of the observer associated with the latter. Indeed, as seen in section 3.2, considering a measurement +1 by Alice for polarization σ_z we always observe, and therefore with probability 1, a measurement −1 by Bob. The same reasoning applies to the observable σ_x.

By performing two separate measurements for two different observables, as shown in figure 4.3, you can safely know the value associated to both. Considering the measurement +1 for observable σ_z and −1 for σ_x by Alice, the state collapses as follows

$$|\Psi\rangle \overset{\sigma_z\otimes I=+1}{\longrightarrow} |\Psi'_{\sigma_z}\rangle = \frac{1}{\sqrt{2}}\left(|H^a\rangle \otimes |V^b\rangle\right)$$
$$|\Psi\rangle \overset{\sigma_x\otimes I=-1}{\longrightarrow} |\Psi'_{\sigma_x}\rangle = \frac{1}{\sqrt{2}}\left(|-^a\rangle \otimes |+^b\rangle\right) \tag{4.6}$$

allowing us to know with certainty that Bob will observe opposite values, i.e. −1 for observable σ_z and +1 for σ_x.

These two observables therefore possess both an element of physical reality – respecting the criterion set out in the article to be able to predict with certainty the value associated with them – although the two operators do not commute as

$$\left[\sigma_x^{Bob}, \sigma_z^{Bob}\right] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \neq 0\ .$$

[7] The proceeding is analog to equations 3.11-3.14 in section 3.2.

[8] We notice that the correlations are observed even thought if this condition is satisfied

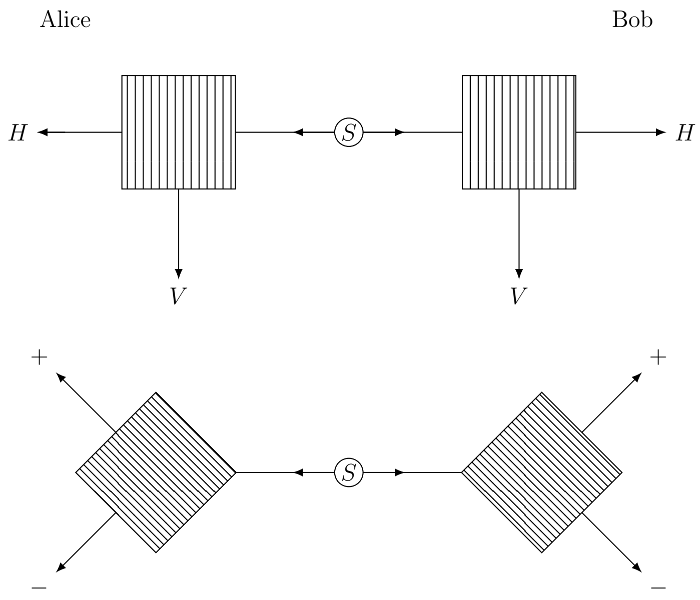


Figure 4.3: Scheme of the experiment proposed by EPR.

We conclude therefore that the part *(b)* of the logical disjunction is wrong.

4.3.3 Conclusion of reasoning

The argument just presented shows that *Starting then with the assumption that the wave function* [i.e. state vector] *does give a complete description of the physical reality, we arrived at the conclusion that two physical quantities, with noncommuting operators, can have simultaneous reality. Thus the negation of (a) leads to the negation of the only other alternative (b). We are thus forced to conclude that the quantum-mechanical description of physical reality given by wave functions is not complete.*».

4.4 Criticism

EPR is logically structured in a rigorous and correct manner. Despite this, it is not perfect. It has been criticized on the foundations on which the article is based. However, since it is a subject that goes beyond the experimental results, the scientific community's response was not really wide. In his article [10] published shortly after EPR and having the same title, Bohr raises an attack on EPR's thesis. His criticism is mainly based on the inadequacy of the criterion of reality, indeed we read in his article that «From our point of view we now see that the wording of the above-mentioned criterion of physical reality proposed by Einstein, Podolsky and Rosen contains an ambiguity in regard of the meaning of the expression "without in any way disturbing a system".» The criticism is expressed in many aspects as can be read below

«Bohr agrees that the indirect measurement of Bob' system achieved when one makes a measurement of Alice's system does not involve any "mechanical disturbance" of Bob' system. (Thus Bohr takes for granted that one may raise the question of a disturbance between the two systems, and hence he takes separability, that there are distinct systems, for granted.) Still, Bohr claims that a measurement on Alice's system does involve "an influence on the very conditions which define the possible types of predictions regarding the future behavior of [Bob's] system"»[24].

4.5 Comments and conclusions

The conclusion of what is stated in this chapter is clear.

Assuming the two fundamental principles – the principle of reality and the principle of locality – as correct, it is necessary to conclude that the quantum theory is incomplete. The reasoning that leads to this conclusion is unassailable because it is logically correct. The only problems can therefore arise in the evaluation of the starting hypotheses, that is, the two above principles, which, however, are said to be unassailable by the authors of EPR, as they are extremely intrinsic to the reality we perceive. The discussion on these two principles as such[9], would risk spreading on a philosophical, rather than physical, level losing touch with real data obtained in the experiments.

It is thus open « *the question of whether or not such a* [complete] *description exists*».

In the EPR group, EINSTEIN is certainly the best known and respected figure in the scientific community. So much attention has been paid to his personal opinion on the entanglement phenomenon. In particular, we read in the correspondence between BORN and EINSTEIN EINSTEIN's following consideration:

> «I cannot make a case for my attitude in physics which you would consider at all reasonable. I admit, of course, that there is a considerable amount of validity in the statistical approach which you were the first to recognise clearly as necessary given the framework of the existing formalism. *I cannot seriously believe in it because the theory cannot be reconciled with the idea that physics should represent a reality in time and space, free from spooky actions at a distance.* I am, however, not yet firmly convinced that it can really be achieved with a continuous field theory, although I have discovered a possible way of doing this which so far seems quite reasonable. The calculation difficulties are so great that I will be biting the dust long» [12].

This represents the situation in 1935. Later, new and breathtaking discoveries will give fire to the discussion of whether the description by quantum physics of reality is complete.

[9]A principle is an statement that constitutes the generalization of a vast experimental evidence and that is assumed as true for every possible further experience[49], it is therefore particularly difficult to question its terms.

Chapter 5

Bell's Theorem

THIS chapter speaks of what has been called « one of the most important works in the history of physics »[1]. This is the definitive and unambiguous answer to the question posed by EINSTEIN, PODOLSKY, and ROSEN in the previous chapter.

5.1 Introduction to a classical nature of correlations

The quantum correlations observed in laboratory was still an open question for the scientific community of the first half of the 20th century. The analysis of this correlation leads to the formulation of two classical options for its nature.

5.1.1 Exchange of information

The first option consists in the method according to which the first particle, after being measured, sends to the second particle the information of the result of its measurement so that the second one can react accordingly to it and show the correlations.

According to the theory of relativity, the maximum speed for information propagation has a limit given by the speed of light c. This principle – since correlations can also be observed in events separated by a spatial range, as demonstrated with distant photons 10.9 km [47]– contradicts the possibility of the exchange of information between the particles.

This first option is therefore denied by experiments in accordance with the theory of relativity.

5.1.2 Correlations established at the source

This second possibility consists in the assumption that every particle of the couple, when it is emitted from the source, already "knows" how it has to react when it meets a particular type of measuring device and this regardless of the possible measurements made on the other particle.

[1] Quote by Alain ASPECT, talking about the article: "On the Einstein Podolsky Rosen paradox" [7].

This possibility corresponds to what is called a local theory, namely that the two pair particles are independent of each other and are considered as separate entities. In particular, this type of theory is that supported by EPR.

5.2 Hidden variables theory

The EPR thesis on the incompleteness of quantum theory gave rise to what is known as "theory of local hidden variables". According to this theory, the element of randomness in quantum theory is due to incomplete knowledge of the state of the system since the Hilbert space vector, as in statistical mechanics, would be an average of better defined states for which the individual result would be determined. These hypothetical "dispersion free" states would be defined not only by the quantum state's vector but also from further "hidden variables" -"hidden" because if states with assigned values of these variables would be prepared, the results of quantum physics would be inadequate [5, 8], since the random character of the theory would disappear.

Model

Considering the schematic assembly of figure 5.1 in which Alice and Bob analyze a pair of entangled state particles on which both can measure two observables.

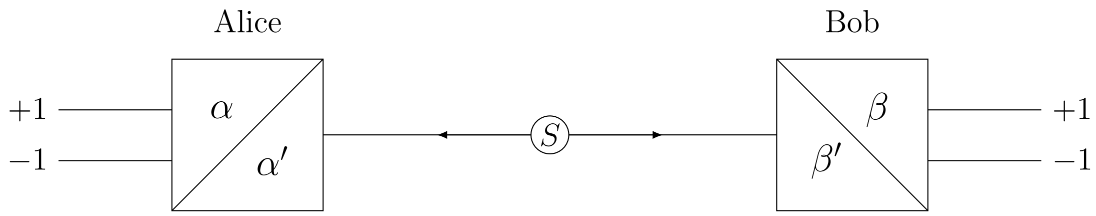


Figure 5.1: Scheme of measurement on two particles and two observables for each observer.

For both observables the only two results are ± 1. In the case of two polarized photons, $+1$ corresponds to the crossing of the polarizer while -1 to its deviation.

Alice's measurements results are noticed a, a' (a corresponds to the result ± 1 after having chosen α as observable, and likewise with a') while those of Bob are b, b'.

According to theory, the principle of locality[2] is valid. The idea proposed is that the results of each measurement are already set at the source, which means that the particle leaves the source with a list

$$\lambda_A = \{a(\lambda), a'(\lambda)\} \quad \lambda_B = \{b(\lambda), b'(\lambda)\}$$

where the results of the possible measurements are indicated. All values depend on λ which varies in all pairs of particles.

[2] In this case it means that the results a, a' are not influenced by Bob's measurement and vice versa.

It is therefore important to perform two different measurements, as otherwise you might create a template where starting lists do not depend on the measurements (in the case of polarization from the polarizer's angles).

For example, in the case of two entangled state photons, if the particle directed to Alice encounters a type of measurement $\alpha = H/V$, then at the origin it is established that $a = +1$ and, regardless of Alice's measurement, it is always established at the origin that for the particle directed to Bob, meeting a type of measurement $\beta = H/V$ the result will be $b = -1$. Inverted values can be established as well and so do congruent results. In this way we see that the correlations observed on the entangled state $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|V^a\rangle \otimes |H^b\rangle - |H^a\rangle \otimes |V^b\rangle\right)$ are established at the origin.

The union of these two lists

$$\Lambda = \{\lambda_A, \lambda_B\}$$

is the hidden variable (local, for how it was constructed) for this system, which completes the state, creating a "dispersion free state". We notice that for each pair of particles Λ is different.

5.3 Bell's theorem

5.3.1 Bell's inequality

The results outlined in this section were obtained by John Stewart Bell in the two fundamental articles "On the problem of hidden variables in quantum mechanics" and "On the Einstein Podolsky Rosen paradox" [7, 8]. Regarding the formalism, the one exposed is the *CHSH* variant (by the authors Clauser, Horne, Shimony, and Holt) of the inequalities, exposed in [14], more simple since directly addressed to the experimental verification [3] and by dealing only with one pair of entangled particles.

We consider the following magnitude

$$S(\Lambda) = (a + a')\, b + (a - a')\, b' = ab + a'b + ab' - a'b' \quad . \tag{5.1}$$

Since only the following cases are possible

$$\begin{cases} (a + a') = 0 \\ (a - a') = \pm 2 \end{cases} \quad \text{o} \quad \begin{cases} (a + a') = \pm 2 \\ (a - a') = 0 \end{cases} \quad \Rightarrow S = \pm 2 \quad . \tag{5.2}$$

For each pair of particles, Alice can measure only α or α' and the same goes for Bob with β and β', the value of $S(\Lambda)$ can therefore not be obtained from measurements. Doing many measurements, however, we can calculate the average value $\langle S(\Lambda)\rangle$:

$$|\langle S(\Lambda)\rangle| = |\langle ab + a'b + ab' - a'b'\rangle| = |\langle ab\rangle + \langle a'b\rangle + \langle ab'\rangle - \langle a'b'\rangle| \quad . \tag{5.3}$$

[3] As we can notice already for the title: "Proposed experiment to test local hidden-variable theories".

From equations (5.2) and (5.3) we obtain *the **CHSH** variant of Bell's inequality*:

$$|\langle S(\Lambda)\rangle| \leq 2 \quad . \tag{5.4}$$

It is important to notice that in making the equation (5.4) no quantum assumption has been made, it has so a general value: every local theory[4] must comply with this inequality.

In conclusion, for each local theory, and in particular if the correlations are established at the source, $|\langle S(\Lambda)\rangle| \leq 2$ must hold. This result is comparable to the prediction of quantum physics and the experimental results as we will show in the next section 5.3.2.

Considering instead a non-local theory – where a, a' can depend on the measurement of Bob's b, b' and vice versa – then the magnitude

$$S = a_{(\beta)}b_{(\alpha)} + a'_{(\beta)}b_{(\alpha')} + a_{(\beta')}b'_{(\alpha)} - a'_{(\beta')}b'_{(\alpha')} \tag{5.5}$$

can assume the values $S = 0, \pm 2, \pm 4$. Therefore we clearly obtain to the inequality

$$|\langle S(\Lambda)\rangle| \leq 4 \tag{5.6}$$

which must be respected by any non-local theory.

5.3.2 Bell's inequality in quantum physics[5]

The inequalities (5.4) and (5.6) for how they have been obtained, can be applied to any physical theory. Also, within quantum physics there are many applications, in this section we will apply them to a system of two entangled photons.

As described in section 1.6, the average value associated with an observable A given the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|V^a\rangle \otimes |H^b\rangle - |H^a\rangle \otimes |V^b\rangle\right) \overset{6}{\equiv} \frac{1}{\sqrt{2}}(|VH\rangle - |HV\rangle) \tag{5.7}$$

is

$$\langle A\rangle_{|\Psi\rangle} = \langle \Psi|A\Psi\rangle \quad . \tag{5.8}$$

The observable associated with two correlated polarizations, where θ and λ indicate the orientation of the axis of the measuring instrument, is obtained through the spectral theorem:

$$\begin{aligned} A_{\theta\otimes\lambda} &= \left(P_{|\psi_\theta\rangle} - P_{|\psi_{\theta^\perp}\rangle}\right) \otimes \left(P_{|\psi_\lambda\rangle} - P_{|\psi_{\lambda^\perp}\rangle}\right) = A_\theta \otimes A_\lambda \\ &= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \otimes \begin{pmatrix} \cos 2\lambda & \sin 2\lambda \\ \sin 2\lambda & -\cos 2\lambda \end{pmatrix} \quad . \end{aligned} \tag{5.9}$$

[4]Since we considered a hidden variable Λ that respects the principle of local causes and therefore λ_A and λ_B are independent from the measurement.

[5]This section in partially based on[21, 23, 35] and in particular for the part with random variables on [29, p.418-420].

[6]For typographical reasons we will no more write explicitly the tensor product $\otimes$ in the states, so $|V\rangle \otimes |H\rangle \equiv |VH\rangle$.

From the equation (5.8) and (5.9) follows that the mean value for the state (5.7) is:

$$
\begin{aligned}
\langle A_{\theta\otimes\lambda}\rangle &= \langle\Psi|A_{\theta\otimes\lambda}\Psi\rangle \\
&= \left\langle \frac{1}{\sqrt{2}}(VH-HV)\middle| A_\theta\otimes A_\lambda\left[\frac{1}{\sqrt{2}}(VH-HV)\right]\right\rangle \\
&= \left\langle \frac{1}{\sqrt{2}}VH-\frac{1}{\sqrt{2}}HV\middle|\frac{1}{\sqrt{2}}A_\theta VA_\lambda H-\frac{1}{\sqrt{2}}A_\theta HA_\lambda V\right\rangle \\
&= \left\langle \frac{1}{\sqrt{2}}VH\middle|\frac{1}{\sqrt{2}}A_\theta VA_\lambda H\right\rangle - \left\langle \frac{1}{\sqrt{2}}VH\middle|\frac{1}{\sqrt{2}}A_\theta HA_\lambda V\right\rangle \\
&- \left\langle \frac{1}{\sqrt{2}}HV\middle|\frac{1}{\sqrt{2}}A_\theta VA_\lambda H\right\rangle + \left\langle \frac{1}{\sqrt{2}}HV\middle|\frac{1}{\sqrt{2}}A_\theta HA_\lambda V\right\rangle \\
&= \frac{1}{2}\left(\langle V|A_\theta V\rangle\langle H|A_\lambda H\rangle - \langle V|A_\theta H\rangle\langle H|A_\lambda V\rangle\right. \\
&- \left.\langle H|A_\theta V\rangle\langle V|A_\lambda H\rangle + \langle H|A_\theta H\rangle\langle V|A_\lambda V\rangle\right) \\
&= -\frac{1}{2}\cos 2\theta\cos 2\lambda - \frac{1}{2}\sin 2\theta\sin 2\lambda - \frac{1}{2}\sin 2\theta\sin 2\lambda - \frac{1}{2}\cos 2\theta\cos 2\lambda \\
&= -\cos 2\theta\cos 2\lambda - \sin 2\theta\sin 2\lambda = -\cos(2\theta-2\lambda) = -\cos 2(\theta-\lambda)\ .
\end{aligned}
\tag{5.10}
$$

The value $-\cos 2(\theta-\lambda)$ obtained in equation (5.10) is confirmed by the calculation considering Alice's and Bob's measurements as random variables. The path is analogue to the one proposed above with the calculation of the mean values according to the quantum rule obtained in section (1.19). In fact, two measurements with angles θ and λ are taken in account and the same state as equation (5.7). Given the notation.:

$$
\begin{gathered}
|\Psi_{out,++}\rangle = |\psi_\theta\rangle\otimes|\psi_\lambda\rangle \equiv |\psi_\theta\psi_\lambda\rangle \quad |\Psi_{out,+-}\rangle = |\psi_\theta\rangle\otimes|\psi_{\lambda\perp}\rangle \equiv |\psi_\theta\psi_{\lambda\perp}\rangle \\
|\psi_\theta\rangle = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \quad |\psi_\lambda\rangle = \begin{pmatrix}\cos\lambda\\ \sin\lambda\end{pmatrix} \quad |\psi_{\lambda\perp}\rangle = \begin{pmatrix}-\sin\lambda\\ \cos\lambda\end{pmatrix}
\end{gathered}
\tag{5.11}
$$

and with the probability of observing the measurement $(+1,+1)$ or $(-1,-1)$

$$
\begin{aligned}
&\mathrm{Prob}\{A_\theta\otimes I = +1, I\otimes A_\lambda = +1\} = |\langle\Psi_{out,++}|\Psi\rangle|^2 \\
&= \left|\langle\psi_\theta\psi_\lambda|\frac{1}{\sqrt{2}}(VH-HV)\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\langle\psi_\theta|H\rangle\langle\psi_\lambda|V\rangle - \frac{1}{\sqrt{2}}\langle\psi_\theta|V\rangle\langle\psi_\lambda|H\rangle\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}(\cos\theta\sin\lambda - \sin\theta\cos\lambda)\right|^2 = \frac{1}{2}\sin^2(\theta-\lambda)
\end{aligned}
$$

and the probability of observing the measurement $(+1,-1)$ or $(-1,+1)$

$$
\begin{aligned}
&\mathrm{Prob}\{A_\theta\otimes I = +1, I\otimes A_\lambda = -1\} = |\langle\Psi_{out,+-}|\Psi\rangle|^2 \\
&= \left|\langle\psi_\theta\psi_{\lambda\perp}|\frac{1}{\sqrt{2}}(VH-HV)\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\langle\psi_\theta|H\rangle\langle\psi_{\lambda\perp}|V\rangle - \frac{1}{\sqrt{2}}\langle\psi_\theta|V\rangle\langle\psi_{\lambda\perp}|H\rangle\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}(\cos\theta\cos\lambda + \sin\theta\sin\lambda)\right|^2 = \frac{1}{2}\cos^2(\theta-\lambda)
\end{aligned}
$$

we can calculate the probability of observing a measurement +1 for photon a as the sum of all possible results with this outcome:

$$\text{Prob}\{A_\theta = +1\} = \frac{1}{2}\cos^2(\theta - \lambda) + \frac{1}{2}\sin^2(\theta - \lambda) = \frac{1}{2} \quad . \tag{5.12}$$

The same reasoning leads to an identical result also for the measurement of photon b.

From equation (5.12) follows therefore that the measurement for a single polarization is random, it can therefore be considered as a random variable. We notice therefore $\mathcal{A}(\theta)$ and $\mathcal{B}(\lambda)$ the random variables associated with the measurements for which the probabilities are:

$$\begin{aligned}\text{Prob}[\mathcal{A}(\theta) = +1] = \text{Prob}[\mathcal{A}(\theta) = -1] &= \frac{1}{2} \\ \text{Prob}[\mathcal{B}(\lambda) = +1] = \text{Prob}[\mathcal{B}(\lambda) = -1] &= \frac{1}{2} \; .\end{aligned}$$

Given two random variables $\mathcal{A}(\theta)$ and $\mathcal{A}(\theta)$, it is possible to study the *correlation* between them defined as

$$\text{Corr}(\mathcal{A}(\theta), \mathcal{B}(\lambda)) = \frac{E\left[\left(\mathcal{A}(\theta) - E\left[\mathcal{A}(\theta)\right]\right)\left(\mathcal{B}(\lambda) - E\left[\mathcal{B}(\lambda)\right]\right)\right]}{\sqrt{E\left[\left(\mathcal{A}(\theta) - E\left[\mathcal{A}(\theta)\right]\right)^2\right]}\sqrt{E\left[\left(\mathcal{B}(\lambda) - E\left[\mathcal{B}(\lambda)\right]\right)^2\right]}} \tag{5.13}$$

where the notation $E\left[\mathcal{A}(\theta)\right]$indicates the expected value for the random variable $\mathcal{A}(\theta)$ defined as

$$E\left[\mathcal{A}(\theta)\right] = \sum_i x_i \text{Prob}\{\mathcal{A}(\theta) = x_i\} \tag{5.14}$$

where x_i is a possible result of $\mathcal{A}(\theta)$.

Through equation (5.14) we calculate the numeric result of the expected values for the interested random variable:

$$E\left[\mathcal{A}(\theta)\right] = E\left[\mathcal{B}(\lambda)\right] = 1\frac{1}{2} + (-1)\frac{1}{2} = 0 \quad . \tag{5.15}$$

From equation (5.13), (5.14), (5.15) and with the property

$$E\left[\left(\mathcal{A}(\theta) - E\left[\mathcal{A}(\theta)\right]\right)\left(\mathcal{B}(\lambda) - E\left[\mathcal{B}(\lambda)\right]\right)\right] = E\left[\mathcal{A}(\theta)\mathcal{B}(\lambda)\right] - E\left[\mathcal{A}(\theta)\right]E\left[\mathcal{B}(\lambda)\right]$$

we obtain the numeric value of the correlation between $\mathcal{A}(\theta)$ and $\mathcal{B}(\lambda)$:

$$\begin{aligned}\text{Corr}(\mathcal{A}(\theta), \mathcal{B}(\lambda)) = & \frac{E\left[\mathcal{A}(\theta)\mathcal{B}(\lambda)\right] - E\left[\mathcal{A}(\theta)\right]E\left[\mathcal{B}(\lambda)\right]}{\sqrt{E\left[\left(\mathcal{A}(\theta) - E\left[\mathcal{A}(\theta)\right]\right)^2\right]}\sqrt{E\left[\left(\mathcal{B}(\lambda) - E\left[\mathcal{B}(\lambda)\right]\right)^2\right]}} \\ & = E\left[\mathcal{A}(\theta)\mathcal{B}(\lambda)\right] = \text{Prob}\{(+1,+1)\} + \text{Prob}\{(-1,-1)\} \\ & - \text{Prob}\{(+1,-1)\} - \text{Prob}\{(-1,+1)\} = \sin^2(\theta - \lambda)\cos^2(\theta - \lambda) \\ & = -\cos 2(\theta - \lambda) \, .\end{aligned} \tag{5.16}$$

The result is identical to the mean value calculated in the equation (5.10) and therefore it confirms it.

With the result just obtained, which is to be interpreted as one of the four elements of equation (5.1), we can calculate the mean value of the magnitude $S(\Lambda)$ as

$$|\langle S(\Lambda)\rangle| = |-\cos 2\,(\theta-\lambda) - \cos 2\,(\theta'-\lambda) - \cos 2\,(\theta-\lambda') + \cos 2\,(\theta'-\lambda')\,|\ . \tag{5.17}$$

Given the magnitude S we can calculate its maximum as follows.

We consider the function

$$f(\theta,\lambda,\theta',\lambda') = -\cos 2\,(\theta-\lambda) - \cos 2\,(\theta'-\lambda) - \cos 2\,(\theta-\lambda') + \cos 2\,(\theta'-\lambda')\,. \tag{5.18}$$

It is a function of C^1 class, therefore its stationary points are $a \in \mathbb{R}^4$ for which

$$\nabla f(a) = 0 \quad .$$

These points are the solution of the following equation system

$$2\sin 2(\theta-\lambda) + 2\sin 2(\theta-\lambda') = 0 \tag{5.19a}$$
$$-2\sin 2(\theta-\lambda) - 2\sin 2(\theta'-\lambda) = 0 \tag{5.19b}$$
$$2\sin 2(\theta'-\lambda) - 2\sin 2(\theta'-\lambda') = 0 \tag{5.19c}$$
$$-2\sin 2(\theta-\lambda') - 2\sin 2(\theta'-\lambda') = 0 \tag{5.19d}$$

We assume without loss of generality that $\theta = 0$.

From equation (5.19a) we obtain

$$2\sin 2(\theta-\lambda) + 2\sin 2(\theta-\lambda') = 0 \Leftrightarrow \sin 2\lambda = -\sin 2\lambda' \Leftrightarrow \lambda = -\lambda' \tag{5.20}$$

Inserting this result into equation (5.19b) we get

$$\begin{aligned} 0 &= \sin(2\theta'-2\lambda) - \sin(2\theta'-2\lambda') = \sin(2\theta'+2\lambda') - \sin(2\theta'-2\lambda') \\ &= \cos(2\theta')\sin(2\lambda') + \cos(2\theta')\sin(2\lambda') \Leftrightarrow 0 = \cos(2\theta')\sin(2\lambda') \end{aligned} \tag{5.21}$$

considering in the result of equation (5.21) the factor $\cos(2\theta')$ as "determinant" we get

$$\cos(2\theta') = 0 \Leftrightarrow \theta' = \frac{\pi}{4}. \tag{5.22}$$

Inserting the result of equation (5.22) into (5.19d) we obtain the last value

$$-\sin 2(\theta-\lambda') - \sin 2(\theta'-\lambda') \tag{5.23}$$
$$= \sin 2\lambda' - \sin\left(\frac{\pi}{2} - 2\lambda'\right) = \sin 2\lambda' - \cos(2\lambda') \Leftrightarrow \lambda' = -\frac{\pi}{8} \tag{5.24}$$

We notice that equation (5.24) gives as result $\pm\frac{\pi}{8}$. The choice of $-\frac{\pi}{8}$ is motivated by the sbsequent steps. So the point

$$\alpha = (0, \frac{\pi}{8}, \frac{\pi}{4}, -\frac{\pi}{8})$$

is a stationary point for the function f.

To identify the category of this stationary point it is necessary to analyze the Hessian matrix of f, i.e. the matrix with components $h_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$.

The function $f(x)$ is periodical, i.e. it exist a $\lambda \in \mathbb{R}^4$ such that

$$f(x+\lambda) = f(x) \quad \forall x \in \mathbb{R}^4 \quad .$$

Because of this there exist no maxima. However, we can obtain the highest assumed value by removing form the Hessian matrix a row and its corresponding column. This leads to the the elimination of the dimension in which the function is constant.

The resulting matrix is

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial \theta^2} & \frac{\partial^2 f}{\partial \theta \partial \lambda} & \frac{\partial^2 f}{\partial \theta \partial \theta'} \\ \frac{\partial^2 f}{\partial \lambda \partial \theta} & \frac{\partial^2 f}{\partial \lambda^2} & \frac{\partial^2 f}{\partial \lambda \partial \theta'} \\ \frac{\partial^2 f}{\partial \theta' \partial \theta} & \frac{\partial^2 f}{\partial \theta' \partial \lambda} & \frac{\partial^2 f}{\partial \theta'^2} \end{pmatrix}$$

$$= \begin{pmatrix} 4\cos(2(\theta-\lambda)) + 4\cos(2(\theta-\lambda')) & -4\cos(2(\theta-\lambda)) & 0 \\ -4\cos(2(\theta-\lambda)) & 4\cos(2(\theta'-\lambda)) + 4\cos(2(\theta-\lambda)) & -4\cos(2(\theta'-\lambda)) \\ 0 & -4\cos(2(\theta'-\lambda)) & 4\cos(2(\theta'-\lambda)) - 4\cos(2(\theta'-\lambda')) \end{pmatrix}$$

$$\Leftrightarrow H(f)(\alpha) = \begin{pmatrix} 4\sqrt{2} & -2\sqrt{2} & 0 \\ -2\sqrt{2} & 4\sqrt{2} & -2\sqrt{2} \\ 0 & -2\sqrt{2} & 4\sqrt{2} \end{pmatrix}$$

This matrix is positive definite, since all its eigenvalues μ

$$\mu \in \left\{ 2\sqrt{2}\left(\sqrt{2}+2\right), 4\sqrt{2}, 2\sqrt{2}\left(2-\sqrt{2}\right) \right\}$$

are positive. The point

$$\alpha = (0, \frac{\pi}{8}, \frac{\pi}{4}, -\frac{\pi}{8})$$

is therefore an argument whose function values is a maximum. The magnitude $|\langle S(\Lambda)\rangle|$ for the angle configuration α assumes the value

$$|\langle S(\Lambda)\rangle| = 2\sqrt{2} \quad . \tag{5.25}$$

A comparison with equations (5.4) and (5.6) leads to

$$2 < |\langle S(\Lambda)\rangle| \leq 4 \quad .$$

The result obtained with equation (5.25) shoes how a suitable choice of the directions of measurement leads to the violation of equation (5.4) determining that the quantum theory is not a theory that can be completed with hidden variables of local type. To confirm this conclusion, the equation does not violate the equation (5.6) respecting the values for a non-local theory.

5.3.3 Statement of Bell's Theorem

Bell himself formulates his theorem as follows:

Bell's theorem

«*But if [a hidden variable theory] is local it will not agree with quantum mechanics, and if it agrees with quantum mechanics it will not be local. This is what the theorem says.*» [6, p.9]

5.4 Bell's inequalities in experiments

The construction of Bell's inequality follows a logical-deductive path; to ensure its validity it needs to be confirmed with experimental evidence.

5.4.1 First experiment: Freedman et al. and Aspect et al.

The first experiment [25] related to Bell's theorem was done in 1972 and attributed to Freedman et al. The experiment, whose experimental apparatus is shown in figure 5.2(a), is performed with pairs of entangled state photons emitted by a source of calcium atoms and is based on a slightly different inequality from the CHSH one: :

$$-1 \leq \langle S \rangle \leq 0 \quad .$$

The average values obtained (which according to the authors « do not show evidence of deviation from the predictions of quantum physics ») associated to angles for which the magnitude S is maximum are

$$\langle S_1 \rangle = 0.104 \pm 0.026$$
$$\langle S_2 \rangle = -1.097 \pm 0.018$$

These values are in clear violation with the proposed inequality. With this result we have the first experimental confirmation of the violation of Bell's inequalities.

In 1982 Aspect et al. propose a new experiment, which is also performed with a pair of entangled photons emitted by a source of calcium, and which is based on the same inequality (5.4) obtained in section 5.3:

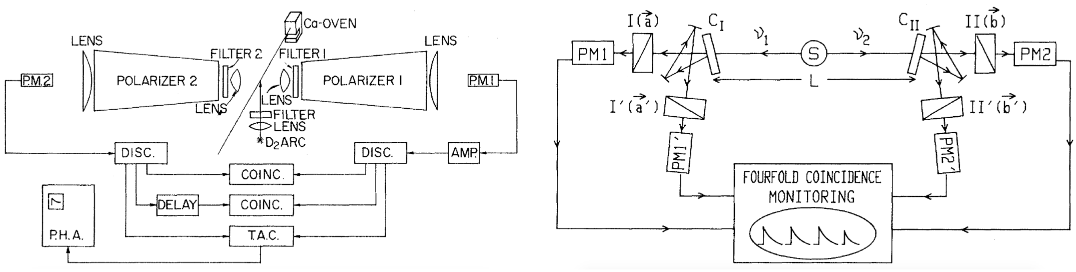
$$|\langle S \rangle| \leq 2 \quad .$$

The average value expected for the experiment was

$$\langle S_{QM} \rangle = 2.70 \pm 0.05 \quad .$$

The experimental result obtained from 5 measurements made with the configuration $\beta - \alpha = \alpha' - \beta = \beta' - \alpha' = 22.5°$ and $\beta' - \alpha = 67.5°$ is

$$\langle S_{Exp} \rangle = 2.697$$



(a) Experiment of Freedman et al. [25]

(b) Experiment of Aspect et al. [4]

Figure 5.2: Scheme of the first experiment on Bell's theorem.

with a margin of experimental error of ± 0.015.

Aspect et al. proposes in 1982 a second experiment, shown schematically in Figure 5.2(b) where we see particularly well the possibility to perform two measurements for each photon (noted ν_1 and ν_2). Unlike the previous experiment, in this experiment the types of measurement to which the particles (i.e. the angle of the polarizers) are subjected are not static but are subject to an (almost[7]) random and independent change between Alice's and Bob's apparatus. The "switch" time between a direction of measurement and the other is 10 ns that, related to the value $c/L = 40$ ns allows to affirm the impossibility of signal exchange between the particles[8].

The inequality to respect is:

$$-1 \leq \langle S \rangle \leq 0 \quad .$$

For the configuration of angles $\beta - \alpha = \alpha' - \beta = \beta' - \alpha' = 22.5°$ and $\beta' - \alpha = 67.5°$ leads to a theoretical prediction for the average value of:

$$\langle S_{QM} \rangle = 0.101 \pm 0.020$$

which it is in perfect agreement with the value obtained from the experiment

$$\langle S_{Exp} \rangle = 0.112 \quad .$$

5.4.2 "Loopholes"

Despite Bell's theorem has been confirmed by three different experiments, its validity is not yet proven absolutely. Especially the experiments Aspect show a clear and obvious violation of the inequality, but the experimental set-ups are not optimal. In particular, some critical voices about the following issues were raised:

[7] «The ideal scheme has not been completed since the change is not truly random, but rather quasiperiodic. Nevertheless, the two switches on the two sides are driven by different generators at different frequencies. It is then very natural to assume that they function in an uncorrelated way.» [4]

[8] «In this experiment, switching between the two channels occurs about each 10 ns. Since this delay, as well as the lifetime of the intermediate level of the cascade (5 ns), is small compared to L/c (40 ns), a detection event on one side and the corresponding change of orientation on the other side are separated by a spacelike interval.»[4]

Locality loophole

The choices of which measurements are done must be separated by a space type interval. That is, the choice of measuring α and α' and β and β' of figure 5.1 must be separated by space and time from each other and must be random, so that it cannot affect the behaviour of the particles. To respect this restriction the use of fast random number generators (RNGs) is needed.

Detection loophole

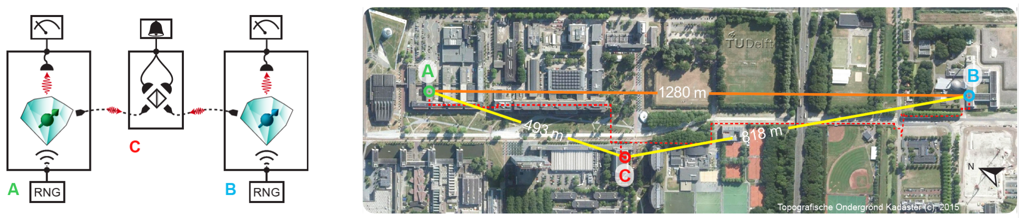
The detectors must be efficient enough to measure a representative portion of the particles[9]

The experiments set forth in section 5.4.1 do not respect exhaustively both loopholes.

5.4.3 Definitive demonstration: Hensen et al.

In 1998 the locality loophole was filled by an experimt with detectors positioned 400 m apart [50]. In 2001 a second experiment closed the detection loophole with high-efficency detectors [43].

The above mentioned experiments, however, fill only one of the loopholes at a time. Only in 2015 the final proof of the violation of Bell's inequalities came for the first time with the experiment by Hensen et al. [30]. This result was confirmed by two other experiments [28, 46], both in 2015, in which with slightly different experimental apparatuses they came to the same results.



(a) Experimental setup

(b) Aerial photograph of the experiment

Figure 5.3: Illustrations of the first definitive demonstration of Bell's theorem [30].

The expeiment [30] is based on the CHSH version of Bell's inequalities.

The experimental setup uses high efficiency spin detectors based on detection by microwaves. This experimental setup eliminates the detection loophole. This type of measurement is carried out detecting a change in the state of the electron which is brought, in the case of certain spin, to emit a red or yellow pulse on the basis of which it is possible to analyse the initial state of the electron.

[9]In particular, as exposed in [27], «for the conventional experiment with particles in the singlet state (or its photon analogue), the data predicted by the quantum theory do not violate this condition unless the quantum efficiency of the detectors exceeds 83%».

The path selectors are based on RNGs, constructed using laser devices, thanks to which the creation of random numbers is possible, with negligible margin of predictability in the order of 10^{-5}, in an acceptable time for the experiment, i.e. 36 ns [1].

The detectors placed 1.3 km apart open a time interval of $\frac{L}{c} = 4.27\ \mu$s in wich the mesurements must be performed. The comple process leaves a window of 90 ns of margin to satisfy the condition of the locality loophole.

The result of 245 measurements lead to an average value S of

$$\langle S_{Expe} \rangle = 2.42 \pm 0.20$$

in agreement with the theoretical calculation of

$$\langle S_{QM} \rangle = 2.30$$

with a margin of experimental error of ± 0.07.

The experiments [28, 46] are instead based on slightly different inequalities from that CHSH but using the experimental apparatuses more similar to those presented in Figure 5.1. The experimental setup that allowed to perform these experiments are concisely described in a similar way by the two articles: for Shalm et al. «Using a well-optimized source of entangled photons, rapid setting generation, and highly efficient superconducting detectors, we observe a violation of a Bell inequality with high statistical significance» and for Giustina et al. «A high-quality polarization entangled source of photons, combined with high-efficiency, low-noise, single-photon detectors, allows us to make measurements without requiring any fair-sampling assumptions».
An important consideration on experiments without loopholes is made by Hensen et al. at the end of the article: «*Our experiment realizes the first Bell test that simultaneously addresses both the detection loophole and the locality loophole. Being free of the experimental loopholes, the setup can test local realist theories of nature without introducing extra assumptions* [...]. *This result places the strongest restrictions on local realistic theories of nature to date.*»

This result is to be interpreted as the definitive experimental demonstration of Bell's Theorem.

5.5 Final consideration and conclusions

The conclusions expressed in Chapter 4 opened the need to prove the correctness of the classical view of Nature proposed by EPR or to constate its fallacy With Bell's inequality presented in article "On the Einstein Podolsky Rosen paradox" [7] we answer to the question proposed by EPR stating that quantum physics is not compatible with any completation with hidden variables as it does not meet the criteria imposed by the Bell inequality.

Recent research, particularly those presented in section 5.4.3, also bring the final experimental confirmation of the just mentioned question. Note the interesting consideration made by Hensen et al. about the importance of the development of a loophole-free Bell test: «*Because of unclosed loopholes, Bell's inequality could not be tested in previous experiments without introducing additional assumptions. Therefore, a Bell test that closes all experimental loopholes at the same time – commonly referred to as a loophole-free Bell test – is of foundational importance to the understanding of nature. In addition, a loophole-free Bell test is a critical component for device-independent quantum security protocols and randomness certification. In such adversarial scenarios all loopholes must be closed, since they allow for security breaches in the system*».

Nevertheless, the discussion has in fact not yet entirely ended, as Aspect argues in [2]: «*can we say that the debate over local realism is resolved? There is no doubt that these are the most ideal experimental tests of Bell's inequalities to date. Yet no experiment, as ideal as it is, can be said to be totally loophole-free. In the experiments with entangled photons, for example, one could imagine that the photons' properties are determined in the crystal before their emission, in contradiction with [a] reasonable hypothesis. [...] The random number generators could then be influenced by the properties of the photons, without violating relativistic causality. Farfetched as it is, this residual loophole cannot be ignored. [...] Should experimentalists decide they want to close this far-fetched loophole, they could base the polarizers' orientations on cosmologic radiation received from opposite parts of the Universe [26]*».

There are therefore still some far fetched loopholes to fulfill.

However, from Bell's theorem one cannot deduce that quantum theory is a non-local theory; there may be a theory, not considered in section 5.1 and still absent in all the literature, explaining locally correlations. Such a theory is, however, (in my opinion, and probably for many others, since nobody ever presented such a theory) difficult to imagine, the option of non-local action seems more likely.

The scientific community is divided on this point. Contrary to the proposed considerations made so far in the text, a part of the scientific community considers the same quantum theory the local theory that explains the quantum correlations. The quantum theory would therefore be a non-classical theory, therefore not of the type proposed in section 5.1, which explains the correlations in local mode.

Chapter 6

Conclusion

The EPR argument brings in the landscape of quantum physics the statement that – assuming for correct two fundamental principles, the principle of reality and the principle of locality – the quantum theory is not complete, supposing so the possibility to complete it with some kind of local hidden variables.

The inequalities proposed by BELL allow us to analyse from a mathematical point of view the idea of EPR leading to the conclusion that complete the quantum theory with local hidden variables is in fact impossible.

This result was now experimentally proven in a to be considered definitive[1] way by the recent experiment. It is maybe the first time in the contemporary world that a philosophical problem can be solved mathematically and experimentally.

However, an explanation of the two particles quantum correlation with a local model can not be excluded, despite such a model is not present in all the literature. To date, the only existing and not experimentally confuted explanation is a non local action between the particles. The discussion on the nature of this phenomenon is thus still open. Open is also the problem whether or not the quantum theory itself can be considerate the local but not classical theory that clarifies how this correlations are possible. The scientific community is indeed split on this argument.

Starting so form what exposed by EPR we end up in an important conclusion: the quantum theory is valid and it is not possible to complete it with any local hidden variable theory, the indeterminism associated to the measurement's objective probabilities are thus conserved.

The quantum theory is so *complete* and *incompatible with hidden variables of local type* and the Nature at a microscopical level is *objectively indeterministic* at measurement level.

[1]The "far fetched" proposed by ASPECT in [2] are to be considered important in a context of quantum technology as the quantum cryptography, probably for the non locality of the theory these are irrelevant.

The incompatibility of the quantum theory with any local model present in literature leads to an *incompatibility between the two fundamental pillars of contemporary theory*[5].

Presumably it will be necessary a new revolution in the world of contemporary science, on which reflect Bell himself:

> «*The consequences of events at one place propagate to other places faster than light. This happens in a way that we cannot use for signalling. Nevertheless it is a gross violation of relativistic causality* [...] *For me then this is the real problem with quantum theory: the apparently essential conflict between any sharp formulation and fundamental relativity. That is to say, we have an apparent incompatibility, at the deepest level, between the two fundamental pillars of contemporary theory.* [...] *It may be that a real synthesis of quantum and relativity theories requires not just technical developments but radical conceptual renewal* [5, p.228]».

List of Figures

Appendix A

Mathematical Basis

The build the fundations of the quantum theory we need some precise mathematical concepts, in particular the one of a Hilbert space. Here we present the most important.

The vector space $\mathbb{C}^n$

The set $\mathbb{C}^n$, composed by n-tuple of elements $x \in \mathbb{C}$ noted $(x_1, \ldots, x_n)$, with the operations *addition*

$$\begin{gathered} \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n \\ ((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \mapsto (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \end{gathered}$$

and *multiplication with scalar*

$$\begin{gathered} \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n \\ (\alpha, (x_1, \ldots, x_n)) \mapsto \alpha(x_1, \ldots, x_n) = (\alpha x_1, \ldots, \alpha x_n) \end{gathered}$$

is a $\mathbb{C}$-*vector space*.

A basis $\mathcal{B}$ of $\mathbb{C}^n$ is a sub set of $\mathbb{C}^n$ whose elements are linearly independent and through a linear combination of whose we can write every vector element of $\mathbb{C}^n$. Every basis has the same cardinality and this number is called dimension of $\mathbb{C}^n$, noted $\dim\mathbb{C}^n$.

The set of $x_i \in \mathbb{C}^n, \forall i \in \{1, \ldots, n\}$

$$\{x_1 = (1, 0, \ldots, 0), x_2 = (0, 1, \ldots, 0), \ldots, x_n = (0, 0, \ldots, 1)\}$$

is called *canonical basis of* $\mathbb{C}^n$. The dimension of $\mathbb{C}^n$ is thus $\dim\mathbb{C}^n = n$.

Subspaces of $\mathbb{C}^n$

A subset $\mathbb{C}^n$, $W \subset \mathbb{C}^n$, with

- $x + y \in W, \ \forall x, y \in W;$
- $\alpha x \in W, \ \forall x \in W \text{ e } \alpha \in \mathbb{C};$
- $0_{\mathbb{C}^n} \in W$

is called *subspace of* $\mathbb{C}^n$.

Scalar product in $\mathbb{C}^n$

Lert $\psi, \varphi \in \mathbb{C}^n$, the the sesquilinear map

$$\begin{aligned} \mathbb{C}^n \times \mathbb{C}^n &\longrightarrow \mathbb{C} \\ (\psi, \varphi) &\longmapsto \langle \psi, \varphi \rangle = \sum_{i=1}^{n} \overline{\psi_i} \varphi_i \end{aligned}$$

is called *standard scalar product.*

Norm

Let $\psi \in \mathbb{C}^n$ then

$$\begin{aligned} \mathbb{C}^n &\longrightarrow \mathbb{R}_+ \\ \psi &\longmapsto \|\psi\| = \sqrt{\langle \psi, \psi \rangle} \end{aligned}$$

Normalized vector

Let $\psi \in \mathbb{C}^n$ so that

$$\|\psi\| = 1$$

then ψ is said *normalized.*

Orthogonality

Let $\psi, \varphi \in \mathbb{C}^n$ such that

$$\langle \psi, \varphi \rangle = 0$$

then ψ, φ are said *orthogonal.*

Orthogonal complement

Let W a subspace of $\mathbb{C}^n$. With $W^\perp$ we indicate the set of all vectors of $\mathbb{C}^n$ orthogonal to every vector of W, i.e.:

$$W^\perp = \{\psi \in \mathbb{C}^n : \langle \psi, \varphi \rangle = 0, \quad \forall \varphi \in W\}.$$

$W^\perp$ is called *orthogonal complement* of W and is a subspace of $\mathbb{C}^n$.

Every vector $\psi \in \mathbb{C}^n$ may be written as

$$\psi = \psi_W + \psi_{W^\perp}$$

where $\psi_W \in W$ and $\psi_{W^\perp} \in W^\perp$. It follows that $\mathbb{C}^n$ may be written as the direct sum of W and $W^\perp$, i.e.:

$$\mathbb{C}^n = W \oplus W^\perp.$$

Hilbert space

$(\mathbb{C}^n, \langle\cdot,\cdot\rangle)$ is an Hilbert space, with $\mathbb{C}^n$ intended as $\mathbb{C}$-vector space.

Matrix product

Let $A \in \mathbb{M}_{pm}(\mathbb{C}), B \in \mathbb{M}_{mq}(\mathbb{C})$, then $C = AB$ is defined as:

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

Where a_{ik}, b_{kj}, c_{ij} are the entries of A, B, C.

Adjoint and self-adjoint matrices

Let A , $B \in \mathbb{M}_m(\mathbb{C})$ so that

$$\langle B\psi, \varphi\rangle = \langle \psi, A\varphi\rangle \quad \forall\ \psi, \varphi \in \mathbb{C}^n$$

then B is said the*adjoint* of A and is noted $B = A^*$.

If $A^* = A$ then A is said *self-adjoint*.

Projectors

A matrix $P \in \mathbb{M}_m(\mathbb{C})$ such that

$$P^2 = P \quad \text{e} \quad P = P^*$$

is a *projector*.

P_ψ intended as orthogonal projector on the direction defined by the vector ψ is computed as follows:

$$P_\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix} \begin{pmatrix} \overline{\psi_1} & \overline{\psi_2} & \cdots & \overline{\psi_n} \end{pmatrix} = \begin{pmatrix} |\psi_1|^2 & \psi_1\overline{\psi_2} & \cdots & \psi_2\overline{\psi_n} \\ \psi_2\overline{\psi_1} & |\psi_2|^2 & \cdots & \psi_1\overline{\psi_n} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_n\overline{\psi_1} & \psi_n\overline{\psi_2} & \cdots & |\psi_n|^2 \end{pmatrix}$$

Unitary matrices

A matrix $U \in \mathbb{M}_m(\mathbb{C})$ such that

$$\langle U\psi, U\varphi\rangle = \langle \psi, \varphi\rangle$$

is said to be *unitary*, we note that U preserves the scalar product. The equalities $U^{-1} = U^*$ hold for unitary matrices.

Eigenvalues and eigenvectors

Let $\psi \in \mathbb{C}^n$ not zero and $A \in \mathbb{M}_n(\mathbb{C})$ so that A and $A\psi$ linearly dependent

$$\exists \lambda \in \mathbb{C} : A\psi = \lambda\psi$$

then ψ is said an *eigenvector* of A. The scalar λ is the *eigenvalue* associated to the eigenvector ψ.

For self-adjoint matrices eigenvector associated to different eigenvalues ($\lambda_1 \neq \lambda_2$) are orthogonal:

$$\begin{cases} A\psi_1 = \lambda_1\psi_1 \\ A\psi_2 = \lambda_2\psi_2 \end{cases} \Rightarrow \langle \psi_1, \psi_2 \rangle = 0$$

Spectral theorem

Let $A \in \mathbb{M}_n(\mathbb{C})$ self-adjoint, then

$$A = \sum_{i=1}^{m} \lambda_i P_{\lambda_i}$$

where
λ_i: m eigenvalues of A
P_{λ_i}: orthogonal projector associated to the eigenvalue λ_i.

Tensor product space [35]

Let $\mathcal{H}_1$ an Hilbert space of dimension n on $\mathbb{C}$ and let $\mathcal{H}_2$ an Hibert space of dimension m on $\mathbb{C}$. The tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$ is the Hilbert space noted $\mathcal{H}_1 \otimes \mathcal{H}_2$ and an map

$$\otimes : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$$

such that $\otimes$ is bilinear.
A basis of $\mathcal{H}$ is expressed with the elements of the basis $\mathcal{B}_{\mathcal{H}_1} = \{\psi_1, \ldots, \psi_n\}$ and $\mathcal{B}_{\mathcal{H}_2} = \{\varphi_1, \ldots, \varphi_m\}$ as

$$\mathcal{B}_{\mathcal{H}} = \{\psi_i \otimes \varphi_j | i \leq n, j \leq m\} \quad .$$

A vector $\Phi \in \mathcal{H}$ based on the vectors $\psi \in \mathcal{H}_1$ and $\varphi \in \mathcal{H}_2$ as follows

$$\psi = \sum_{i=1}^{n} c_i \psi_i \quad \varphi = \sum_{j=1}^{m} d_j \varphi_i$$

is noted $\Phi = \psi \otimes \varphi$ and is expressed in therms of the basis $\mathcal{B}_{\mathcal{H}}$ as

$$\Phi = \psi \otimes \varphi = \sum_{i,j} c_i d_j \ \psi_i \otimes \varphi_j \quad .$$

More generally every vector $\Psi \in \mathcal{H}$ may be written as a linear combination of the basis $\mathcal{B}_{\mathcal{H}}$ as

$$\Psi = \sum_{i,j} \alpha_{i,j} \psi_i \otimes \varphi_j$$

where $\alpha_{i,j}$ is not necessary factorisable in $c_i d_j$.

The scalar product of two elements $\psi_1 \otimes \psi_2, \varphi_1 \otimes \varphi_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is defined as

$$\langle \psi_1 \otimes \psi_2, \varphi_1 \otimes \varphi_2 \rangle_{\mathcal{H}} = \langle \psi_1, \varphi_1 \rangle_{\mathcal{H}_1} \langle \psi_2, \varphi_2 \rangle_{\mathcal{H}_2} \quad .$$

So the norm of $\psi_1 \otimes \psi_2$ is

$$\|\psi_1 \otimes \psi_2\|_{\mathcal{H}_1 \otimes \mathcal{H}_2} = \|\psi_1\|_{\mathcal{H}_1} \|\psi_2\|_{\mathcal{H}_2} \quad .$$

If A is a linear operator on $\mathcal{H}_1$ and B a linear operator on $\mathcal{H}_2$, then in the space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ the operator $A \otimes B$ acts on $\psi_1 \otimes \psi_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$ as follows:

$$(A \otimes B)(\psi_1 \otimes \psi_2) = A\psi_1 \otimes B\psi_2 \quad .$$

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