Eidgenössische Technische Hochschule (ETH) Zürich

Leray-Serre Spectral Sequences

A Morse Homological Approach

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Abstract and Acknowledgments

In this work we present a proof of the Leray-Serre theorem entirely based on Morse homology.

After a short informal introduction, chapters two and three introduce the reader to the notation and the results of homological algebra and Morse theory that will be used in the proof. In these two chapters for most of the statements a proof is presented (but for conciseness not for all). The last chapter focuses on the proof of the Leray-Serre theorem giving at the end some examples on where and how this theorem is used.

I would like to thank Dr. S. Sivek for all the precious advices he has given me during the realisation of this thesis and for presenting me the subject of Morse homology in relation to the Leray-Serre spectral sequences in the first place.

I would also like to thank Prof. Dr. A. Sisto for insightful discussions.

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Chapter 1

Introduction

Morse theory is a rather simple but powerful tool that allows to recover the structure of a smooth manifold M through the study of a single smooth function $f : M \to \mathbb{R}$. A well-known example of such a set-up is the height function f on the torus $\mathbb{T}^2$ shown in Figure 1.1.

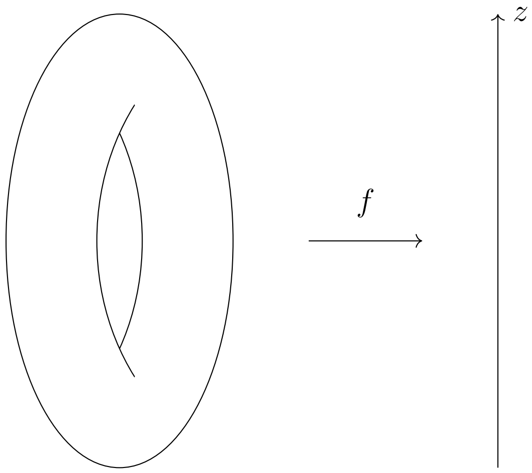

Figure 1.1: The height function on $\mathbb{T}^2$.

A first key result shows that it is possible to construct a CW-complex structure on M, as shown in Figure 1.2, analysing the sub-level sets $S_c = f^{-1}(\{x \le c\})$ of f.

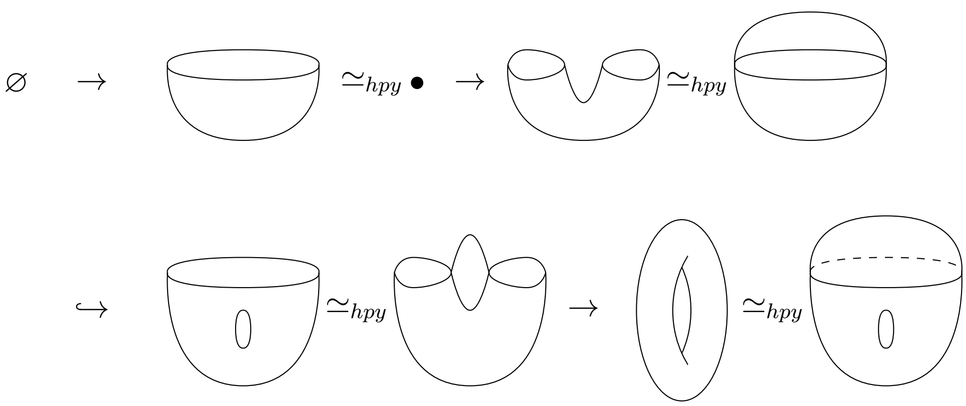

Figure 1.2: Sub-level sets of f and homotopic equivalent cells forming a CW decomposition of $\mathbb{T}^2$.

However, the tool that we will use in this work is the one of Morse homology. Under suitable conditions, it is possible to associate a chain complex to M, and thus a homology, looking at the critical points of f and how they relate to each other.

The second key theory that will be used in this work is spectral sequences. Spectral sequences are a broad argument (generalised to category theory), but for the purpose of this work they are an algebraic construct associated to chain complexes that allow to recover the homology of the original chain based on the homology on subspaces of the chain elements.

The homology of a direct sum is the direct sum of the homologies – or, more formally, given a chain complex

$$\ldots \to \bigoplus_{i=1}^{d} A_{n+1,i} \xrightarrow{\alpha_{n+1}} \bigoplus_{i=1}^{d} A_{n,i} \xrightarrow{\alpha_n} \bigoplus_{i=1}^{d} A_{n-1,i} \to \ldots$$

such that $\alpha_k(A_{k,i}) \subseteq A_{k-1,i}$, then

$$H_*\left(\bigoplus_{i=1}^{d} A_{*,i}\right) = \bigoplus_{i=1}^{d} H_*(A_{*,i}). \tag{$\star$}$$

But this is not the case in the situation where we just have a filtration

$$\ldots \to \bigcup_{i=1}^{d} A_{n+1,i} \xrightarrow{\alpha_{n+1}} \bigcup_{i=1}^{d} A_{n,i} \xrightarrow{\alpha_n} \bigcup_{i=1}^{d} A_{n-1,i} \to \ldots \tag{$\star\star$}$$

satisfying the same condition $\alpha_k(A_{k,i}) \subseteq A_{k-1,i}$. Spectral sequences are the "correct" generalisation of property ($\star$) to ($\star\star$).

Chapter 2

Homological algebra[a]

The aim of this chapter is to introduce the few results and definitions that will be used further on in the text.

2.1 Chains

Definition 2.1 (Chain complexes and homology groups)**.** A chain complex (C_*, ∂) is a sequence $C_* = (C_n)_n$ of abelian groups with associated homomorphisms

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

such that $\partial_{n-1}\partial_n = 0$ for all n. We define the n^{th} *homology group* of the chain complex to be the quotient

$$H_n(C_*) = \ker(\partial_n)/\mathrm{im}(\partial_{n+1}). \tag{2.1}$$

The elements of $\ker(\partial_n)$ are called *cycles* and the elements of $\mathrm{im}(\partial_{n+1})$ *boundaries.* Two cycles representing the same homology class $\mathrm{im}(\partial_{n+1})$ are said to be *homologous.*

Chain complex of R-modules[b] are the objects of a category $\mathfrak{Ch}_R$. The relative morphisms are chain maps.

Definition 2.2 (Chain maps)**.** Let (A_*, ∂) and (B_*, d) be two chain complexes. A *chain map* is a sequence f_* of homomorphisms $f_n : A_n \to B_n$ such that f_* commutes with the boundary operator, i.e. $d_n \circ f_n = f_{n-1} \circ \partial_n$ for all n. On a diagram this means that

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "d_{n+1}"] & B_n \arrow[r, "d_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd} \tag{2.2}$$

every cell commutes.

Lemma 2.3. *Chain maps send cycles to cycles and boundaries to boundaries. They thus induce a map on the homology.*

[a] For this chapter, where not stated differently, the information are taken form [6, Section 2.1].

[b] R can be any commutative ring, for $R \neq \mathbb{Z}$ one just have to slightly modify Definition 2.1.

PROOF. If $a \in A_n$ such that $\partial_n(a) = 0$ then $d_n(f_n(a)) = f_{n-1}(\partial_n(a)) = 0$ so the image of a cycle is again a cycle.

Say instead $a = \partial_{n+1}(b)$ then $f_n(a) = f_n(\partial_{n+1}(b)) = d_{n+1}(f_{n+1}(b))$ so the image of boundary is still a boundary.

Therefore the maps

$$\begin{aligned} \tilde{f}_* : H_*(A_*) &\to H_*(B_*) \\ [a] &\mapsto [f_*(a)] \end{aligned}$$

are well-defined. ■

2.1.1 Exact sequences

Definition 2.4 (Exact sequence)**.** Let $(\alpha_i)_i$ be a sequence of homomorphisms of the from

$$\ldots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \ldots .$$

$(\alpha_i)_i$ is said to be *exact* if $\mathrm{im}(\alpha_{i+1}) = \ker(\alpha_i)$ for all i.

Remark 2.5.

1. Since the condition $\mathrm{im}(\alpha_{i+1}) = \ker(\alpha_i)$ implies $\alpha_i\alpha_{i+1} = 0$, an exact sequence defines a chain complex.

2. $0 \to A \xrightarrow{\alpha} B$ is exact $\Leftrightarrow$ α is injective.

3. $A \xrightarrow{\alpha} B \to 0$ is exact $\Leftrightarrow$ α is surjective.

4. $0 \to A \xrightarrow{\alpha} B \to 0$ is exact $\Leftrightarrow$ α is bijective.

5. $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact $\Leftrightarrow$ α is injective, β surjective and $\mathrm{im}(\alpha) = \ker(\beta)$.

A sequence of the form of point 5 is called *short exact sequence*.

Definition 2.6 (Splitting sequence)**.** [c]We say that a short exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

splits if there exists a right inverse $\xi : C \to B$ to β, i.e. $\beta\xi = \mathrm{id}_C$. We call ξ splitting map.

Proposition 2.7. *If a short exact sequence* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *splits, then* $B \cong A \oplus C$*.*

[c]This last part of Section 2.1.1 is due to [8, Lecture 12].

PROOF. Let ξ be a splitting map. Let $b \in B$ then $\beta(b) \in C$. Since

$$\beta(b - \xi\beta(b)) = \beta(b) - \underbrace{\beta\xi}_{\text{id}}(\beta(b)) = 0,$$

$b - \xi\beta(b) \in \ker\beta$. Using exactness we get an $a \in A$ such that $\alpha(a) = b - \xi\beta(b)$, i.e. $b = \xi\beta b + \alpha a$. But this means that $B = \text{im}\alpha + \text{im}\xi$. For an element $x \in \text{im}\alpha \cap \text{im}\xi$ for which $x = \alpha(a) = \xi(c)$. $\beta(x) = \beta\alpha(a) = 0$ using exactness and $\beta(x) = \beta\xi(c) = c$ imply that $x = \xi(c) = \xi(0) = 0$, i.e. $\text{im}\alpha \cap \text{im}\xi = \emptyset$. We have shown $B = \text{im}\alpha \oplus \text{im}\xi$. The first isomorphism theorem leads to conclusion. ■

A splitting sequence, is therefore a sequence which is composed in the "simplest possible way". We will use these concepts for stating Theorem 2.32 (The Universal Coefficients Theorem) in Section 2.4.

2.2 Simplicial and singular homology

Definition 2.8 (n-simplex)**.** The *n-simplex* in $\mathbb{R}^m$ is the smallest convex set containing $n+1$ points $v_0, \ldots, v_n$ that do not lay in a hyperplane of dimension less than n. We denote this simplex by $[v_0, \ldots, v_n]$ and the points $v_0, \ldots, v_n$ are called the *vertices* of the simplex.

Deleting one of the $n+1$ vertices of a n-simplex, say v_i, generates a $(n-1)$-simplex denoted by $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$ called a *face* of $[v_0, \ldots, v_n]$.

The order of the vertices does play a role in homology and we adopt the convention that *the vertices of any sub-simplex spanned by a subset of vertices (e.g. a face), will always be ordered according to their order in the larger simplex.*

Example 2.9. The *standard n-simplex* is

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | \sum_i t_i = 1,\ 0 \leq t_i\ \forall i \right\}$$

where the edges are simply the standard basis e_i. The union of the faces of Δ^n is the *boundary* of Δ^n written $\partial\Delta^n$. We call *open simplex* $\mathring{\Delta}^n = \Delta^n - \partial\Delta^n$ the interior of Δ^n.

Definition 2.10 (Δ-complex)**.** A Δ-complex structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \to X$, with n dependent on α, such that

1. The restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective.

2. Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. Here we identify the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.

3. A set $A \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

Definition 2.11. We denote by $\Delta^n(X)$ the free abelian group with basis the open n-simplices e^n_α of X. Elements of $\Delta^n(X)$ have the from of finite formal sums $\sum_\alpha n_\alpha e^n_\alpha$ with $n_\alpha \in \mathbb{Z}$ and are called *n-chains*. We may write $\sum_\alpha n_\alpha e^n_\alpha = \sum_\alpha n_\alpha \sigma_\alpha$ with $\sigma_\alpha : \Delta^n \to X$ the characteristic map of e^n_α.

The boundary of the n-simplex $[v_0, \ldots, v_n]$ consists in many $(n-1)$-simplices that we might wish just sum up to form the boundary. However inserting alternating signs and leting

$$\partial_n[v_0, \ldots, v_n] = \sum_i (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

be the boundary of $[v_0, \ldots, v_n]$ leads to better results.

Definition 2.12 (Boundary maps)**.** We define for a general Δ-complex on X the differential ∂_n as the *boundary homomorphism* defined by its action on the basis elements

$$\begin{aligned} \partial_n : \Delta_n(X) &\longrightarrow \Delta_{n-1}(X) \\ \sigma_\alpha &\longmapsto \partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]} \end{aligned}$$

Lemma 2.13. *In the sequence*

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$$

we have that $\partial_{n-1}\partial_n = 0$

PROOF. We need to show that $\mathrm{im}(\partial_n) \subseteq \ker(\partial_{n-1})$. To that end we have

$$\begin{aligned} \partial_{n-1}\partial_n(\sigma) &= \partial_{n-1} \sum_i (-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]} \\ &= \sum_{j<i} (-1)^i (-1)^j \sigma|_{[v_0,\ldots,\hat{v}_j,\ldots,\hat{v}_i,\ldots,v_n]} + \sum_{i\le j} (-1)^i (-1)^j \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_n]} \\ &\overset{\diamondsuit}{=} \sum_{j<i} (-1)^i (-1)^j \sigma|_{[v_0,\ldots,\hat{v}_j,\ldots,\hat{v}_i,\ldots,v_n]} + \sum_{i<j} (-1)^i (-1)^{j+1} \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_n]} \\ &= \sum_{j<i} (-1)^i (-1)^j \sigma|_{[v_0,\ldots,\hat{v}_j,\ldots,\hat{v}_i,\ldots,v_n]} - \sum_{i<j} (-1)^i (-1)^j \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_n]} = 0. \end{aligned}$$

Where in $\diamondsuit$ we used that delete index $j \ge i$ when index i was already deleted is as delete index $j+1$ in the original setting. ■

Remark 2.14. Lemma 2.13 states that $\Delta_n(X)$ defines a chain complex. We will denote this n^{th} *simplicial homology group of* X as $H^\Delta_n(X)$.

2.2.1 Singular homology

In Definition 2.10 we have introduced the notion of Δ-complex. This involved maps σ_α which had to fulfil three different conditions that made the embedding of the standard n-simplex (very) "nice". We can loosen up this additional condition by just requiring continuity and we get the singular complex.

Definition 2.15 (Singular-complex)**.** A singular n-simplex on a space X is a continuous map $\sigma_\alpha : \Delta^n \to X$. A singular n-complex is a collection of such maps.

We call n-chains the elements of the free abelian group generated by the singular n-simplices, i.e. elements of the form $\sum_{i\in\mathbb{Z}} n_i\sigma_i$ with $n_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$. We denote this group by $C_n(X)$.

The boundary maps for the singular complexes is defined exactly as in Definition 2.12.

Definition 2.16 (Singular-homology groups)**.** The n-singular homology groups of X is the quotient group

$$H_n = \ker(\partial_n)/\mathrm{im}(\partial_{n+1})$$

where $\partial_n : C_n(X) \to C_{n-1}(X)$ is the boundary map.

The same proof as presented for Lemma 2.13 applied instead to $C_n(X)$ shows that Definition 2.16 is indeed well-defined.

Remark 2.17. One can easily see from Definition 2.15 and Definition 2.16 that homeomorphic spaces have isomorphic singular homology groups[d]. This is not the case for simplicial homology. The embedding conditions happen to be too strict. Singular homology behaves even better, since it's invariant under homotopy equivalence [3, Proposition 6.3].

Singular homology is therefore the tool we want to use for analysing the structure of the spaces of our interest, nevertheless one can show (see [6, Section "*Singular Homology*"]) singular homology is just a special case of simplicial homology.

We point to any book on Algebraic Topology (e.g. [3] or [6]) for a vast outlook on properties and application of singular homology.

2.3 Spectral Sequences

We will mainly follow the path delineated in [11] and [7][e].

Definition 2.18 (Spectral sequence)**.** A *spectral sequence* (E, d) is a sequence of bigraded groups, i.e. a sequence $E = (E^r)_r$ whose elements are a family of doubly indexed groups $E^r = \{E^r_{p,q} : p, q \in \mathbb{Z}\}$ together with a sequence of maps $d = (d^r)_r$ such that:

- $d^r : E^r \to E^r$ and more precisely $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$
- $d^r \circ d^r \equiv 0$
- $E^{r+1} = \ker(d^r)/\mathrm{im}(d^r)$

[d] The proof is basically composing the homeomorphism with any σ and show that everything is fine.

[e] Since [4] was also one of the reading during the writing of this part, some ideas might also come from this source.

Definition 2.19 (Limit of a spectral sequence)**.** A spectral sequence $(E^r)_r$ is *convergent* if it is eventually stationary, i.e. it exists a $N \in \mathbb{N}$ such that

$$\forall r \geq N, \forall p, q \in \mathbb{Z}: \; E^r_{p,q} = E^N_{p,q}.$$

We denote $E^N_{p,q} = E^\infty_{p,q}$ and write $E^r \to E^\infty$. E^∞ is called the *limit* of the sequence.

2.3.1 Exact couples

Definition 2.20 (Exact couple)**.** An *exact couple* is an exact triangle of groups A and E with maps i, j, k of the form

$$\begin{tikzcd} A \arrow[r, "i"] & A \arrow[d, "j"] \\ & E \arrow[ul, "k"] \end{tikzcd} \tag{2.3}$$

In this case exactness means that the image of each map is the kernel of the following.

Given an exact couple as in (2.3) we define

$$d = j \circ k : E \longrightarrow E$$

where $d^2 = jkjk = 0$ due to $jk = 0$ because of the exactness assumption of (2.3).

Definition 2.21 (Derived couple)**.** Considering an exact couple as in Definition 2.20 we obtain a *derived couple*

$$\begin{tikzcd} i(A) \arrow[r, "i'"] & i(A) \arrow[d, "j'"] \\ & H(E;d) \arrow[ul, "k'"] \end{tikzcd} \tag{2.4}$$

Where

- $i' = i|_{i(A)}$
- $H(E;d) = \ker(d)/\mathrm{im}(d)$ is well-defined as homology group as in (2.1) because of $d^2 = 0$.
- $j'(i(a)) = [j(a)]$ for all $a \in A$ is well-defined since
  - $d(j(a)) = jkj(a) = 0 \; \forall a \in A$ due to exactness of (2.3)
  - if $a_1, a_2 \in A$ are such that $i(a_1) = i(a_2)$ then $i(a_1 - a_2) = 0$ and so $a_1 - a_2 \in \ker(i) = \mathrm{im}(j)$. Therefore $j(a_1) - j(a_2) \in \mathrm{im}(jk) = \mathrm{im}(d)$ which means $[j(a_1)] = [j(a_2)]$.
- $k'[e] = k(e)$ which is well-defined since
  - $[e] \in H(E;d)$ implies $e \in \ker(d)$ so $jk(e) = d(e) = 0$ so $e \in \ker(j) = \mathrm{im}(i)$.
  - $[e_1] = [e_2]$ implies $e_1 - e_2 \in \mathrm{im}(d) \subseteq \mathrm{im}(j) = \ker(k)$ so $k(e_1) = k(e_2)$.

Lemma 2.22. *The derived couple of an exact couple is also exact.*

Proof. We have to prove 6 things

$\ker(j') \supseteq \mathrm{im}(i')$: for $ia \in i(A)$ we have $j'i'ia = [j(i(a))] = 0$ since $ia \in \mathrm{im}(i) = \ker(j)$

$\ker(j') \subseteq \mathrm{im}(i')$: if $ia \in \ker(j')$ then $0 = j'ia = [ja]$ so $ja \in \mathrm{im}(d)$ which implies that $\exists e \in E$ such that $ja = de = jke$. But this means $a - ke \in \ker(j) = \mathrm{im}(i)$ so $\exists b \in A$ such that $ib = a - ke$. Applying i on both sides we get $i^2 b = ia - ike = ia$ so $ia \in \mathrm{im}(i^2) = \mathrm{im}(i')$

$\ker(k') \supseteq \mathrm{im}(j')$: for $ia \in i(A)$ we have $k'j'ia = k'[ja] = kja = 0$ since $\mathrm{im}(j) = \ker(k)$.

$\ker(k') \subseteq \mathrm{im}(j')$: if $0 = k[e] = ke$ then $e \in \ker(k) = \mathrm{im}(j)$ so $\exists a \in A$ such that $e = ja$ and thus $[e] = [ja] = j'ia \in \mathrm{im}(j')$

$\ker(i') \supseteq \mathrm{im}(k')$: $i'k'[e] = ike = 0$ since $\mathrm{im}(k) = \ker(i)$

$\ker(i') \subseteq \mathrm{im}(k')$: if $0 = i'(i(a)) = i^2 a$ $ia \in \ker(i) = \mathrm{im}(k)$ so $\exists e \in E$ such that $ia = ke = k[e] \in \mathrm{im}(k')$.

■

Remark 2.23. Let

$$\begin{tikzcd} A \arrow[r, "i"] & A \arrow[d, "j"] \\ & E \arrow[ul, "k"] \end{tikzcd}$$

be an exact couple. Then if i is an inclusion map we have that $0 = \ker(i) = \mathrm{im}(k)$ so $k \equiv 0$ and therefore $d \equiv 0$. The derived couple of this exact couple is the exact couple itself.

Definition 2.24 (Filtration)**.** Let (C_*, d) be a chain complex. A *filtration* for (C_*, d) is a finite sequence of abelian groups for each C_k:

$$0 = F_{-1,k} \subseteq F_{0,k} \subseteq F_{1,k} \subseteq \cdots \subseteq F_{n,k} = C_k \tag{2.5}$$

such that the boundary map respects the filtration: $d(F_{p,k}) \subseteq F_{p,k-1}$. If (C_*, d) has a filtration we call it a *filtered chain complex.*

Lemma 2.25. *Let (C_*, d) be a filtered chain complex as in (2.5). Then*

$$0 \to F_{p-1,k} \xrightarrow{\iota} F_{p,k} \xrightarrow{q} F_{p,k}/F_{p-1,k} \to 0$$

is a short exact sequence for every p and k.

Proof. The fact that ι is injective and q is surjective follows directly from the definitions. Further $\ker(q) = F_{p-1,k} = \mathrm{im}(\iota)$. ■

2.3.2 Construction of a spectral sequence for filtered chain complexes

We consider a chain complex (C_*, d) which is filtered as in (2.5). Define further

$$\begin{aligned} F_{p,k} &= 0 \quad \forall\, p < 0 \\ F_{p,k} &= C_k \quad \forall\, p > n. \end{aligned} \tag{2.6}$$

and

$$E^0_{p,q} = E^0_{p,k-p} = \bigoplus_s F_{s,k}/F_{s-1,k}. \tag{2.7}$$

Because the boundary map respects the filtration, the homology groups

$$H_k(F_{p,k}) = \frac{\ker(d : F_{p,k} \to F_{p,k-1})}{\operatorname{im}(d : F_{p,k+1} \to F_{p,k})}$$

are well-defined and we have a situation as in Figure 2.1 where d^0 is the induced map by d on the quotients $E^0_{p,q}$.

$$\begin{array}{c|cccc}
q & \vdots & \vdots & \vdots & \iddots \\
2 & E^0_{0,2} & \xleftarrow{d^0} E^0_{1,2} & \xleftarrow{d^0} E^0_{2,2} & \cdots \\
1 & E^0_{0,1} & \xleftarrow{d^0} E^0_{1,1} & \xleftarrow{d^0} E^0_{2,1} & \cdots \\
0 & E^0_{0,0} & \xleftarrow{d^0} E^0_{1,0} & \xleftarrow{d^0} E^0_{2,0} & \cdots \\
\hline
 & 0 & 1 & 2 & p
\end{array}$$

Figure 2.1: The $E^0_{p,q}$ groups on the $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ plane.

One might notice that defining $\tilde{i}^1 = \iota : H_k(F_{p-1,k}) \to H_k(F_{p,k})$, $\tilde{j}^1 = q : H_k(F_{p,k}) \to H_k(E^0_{p,k-p})$ and $\tilde{\kappa}^1 = \iota : H_k(E^0_{p,k-p}) \to H_{k-1}(F_{p-1,k-1})$ these maps extend to the following exact couple

$$\begin{tikzcd}
A^1 = \bigoplus_p H_k(F_{p-1,k}) \arrow[rr, "i^1"] & & \bigoplus_p H_k(F_{p,k}) = A^1 \arrow[d, "j^1"] \\
 & & H_k(\bigoplus_p E^0_{p,k-p}) = \bigoplus_p E^1_{p,k-p} \arrow[ull, "\kappa^1"]
\end{tikzcd} \tag{2.8}$$

where we might define $E^1 = \bigoplus_p E^1_{p,k-p} = H_k(\bigoplus_p E^0_{p,k-p})$ since the homology of the direct sum is the direct sum of the homologies: in particular we define

$$E^1_{p,q} = H_k(E^0_{p,q}) = \frac{\ker(d^0 : E^0_{p,q} \to E^0_{p,q-1})}{\operatorname{im}(d^0 : E^0_{p,q+1} \to E^0_{p,q})}. \tag{2.9}$$

To compute the derived exact couple of (4.27) we need the map

$$d^1 = j^1 \circ k^1 : E^1_{p,q} = E^1_{p,k-p} \to E^1_{p-1,(k-1)-(p-1)} = E^1_{p-1,q}$$

remembering that the k^1 map reduces the coefficient k by one. Finally the derived exact couple is

$$\begin{tikzcd} A^2 \arrow[rr, "i^2"] & & A^2 \arrow[d, "j^2"] \\ & H_k(E^1, d^1) = E^2 \arrow[ul, "\kappa^2"] & \end{tikzcd} \tag{2.10}$$

where the groups A^2 and E^2 are defined as in Definition 2.21.

One might now define successively E^r groups using the maps $d^r : j^r \circ k^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ as done above with the case $r = 2$:

$$E^{r+1}_{p,q} = H_k(E^r_{p,q}) = \frac{\ker(d^r : E^r_{p,q} \to E^r_{p-r,q+r-1})}{\operatorname{im}(d^r : E^r_{p+r,q-r+1} \to E^r_{p,q})} \tag{2.11}$$

In Figure 2.2 we see the $E^r_{p,q}$ groups and d^r.

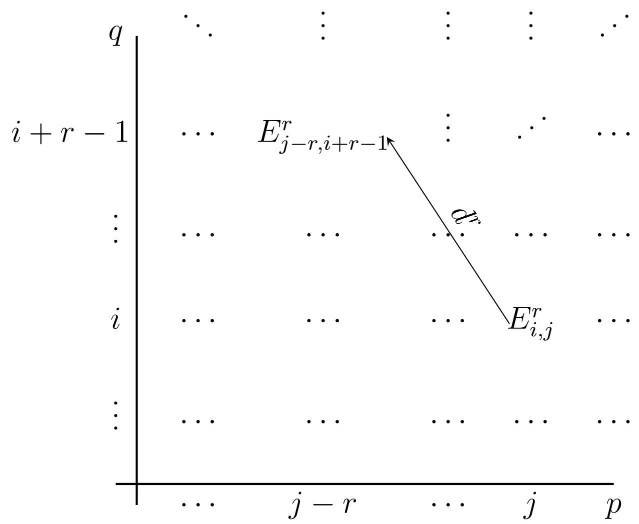

Figure 2.2: Two $E^r_{p,q}$ groups and the d^r maps between them on the $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ plane.

Proposition 2.26 (Convergence). *A filtered chain complex (C_*, d) has an associated spectral sequence which is eventually stationary.*

Proof. Because with (2.6) we assumed $F_{p,k} = 0$ for all $p < 0$ and $F_{p,k} = C_k$ for all $p \geq n$, we have that

$$\begin{aligned} E^0_{p,q} = E^0_{p,k-p} = F_{p,k}/F_{p-1,k} = C_k/C_k &= 0 \quad \forall p \geq n+1 \\ E^0_{p,q} = E^0_{p,k-p} = 0/0 &= 0 \quad \forall p < 0. \end{aligned} \tag{2.12}$$

and this must also hold for all $E^r_{p,q}, r \geq 0$, because of the recurrent definition of the E^r as in (2.11).

Since the boundary maps $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ "go left r cells and up $r-1$ cells", for r big enough (in particular $r \geq n+1$) all d^r maps are of the type $d^r : E^r_{p,q} \to 0$ or $d^r : 0 \to E^r_{p,q}$ or just trivial 0 maps. This implies that $E^r_{p,q}$ is independent of r for r big enough, since by the argument above in (2.11) we would either have a $0/0$ quotient or $E^{r+1}_{p,q} = E^r_{p,q}/0$. ■

Recall now that

$$A^1_k = \bigoplus_{p\in\mathbb{Z}} H_k(F_{p-1,k}) = \bigoplus_{p\geq 0}^{n-1} H_k(F_{p-1,k}) \oplus \bigoplus_{p\geq n} H(C_k).$$

So the image of A^1_k under i^n is

$$i^n A^1_k = \bigoplus_{p\geq 0}^{n-1} i^n(H_k(F_{p-1,k})) \oplus \bigoplus_{p\geq n} i^n(H_k(C_*)).$$

Lemma 2.27. *For a given k define $G_s = \operatorname{im}(i^n : H(F_{s,k}) \to H_k(C_*))$. Then the subsequent quotients $\bigoplus_s(G_s/G_{s-1})$ are isomorphic to $H_k(C_*)$:*

$$\bigoplus_s (G_s/G_{s-1}) \cong H_k(C_*)$$

where for a given k we define $G_s = \operatorname{im}(i^n : H(F_{s,k}) \to H_k(C_*))$.

PROOF. By definition of i^1 in (4.27) we have $i^k = \iota : H(F_{p-1,k}) \to H_k(F_{p,k})$, but (2.6) means $F_{p-1,k} = F_{p,k}$ for $p > n+1$, so $i^k = id$ for those p. Further $F_{p,k} = 0$ for $p < 0$. Fix now a k, then the given equation becomes

$$\begin{aligned}\bigoplus_s (G_s/G_{s-1}) = & \bigoplus_{s<0} \frac{\operatorname{im}(i^n : H_k(F_{s,k}) \to H_k(C_*))}{\operatorname{im}(i^n : H_k(F_{s-1,k}) \to H_k(C_*))} \oplus \bigoplus_{0\leq s\leq n+2} \frac{\operatorname{im}(i^n : H_k(F_{s,k}) \to H_k(C_*))}{\operatorname{im}(i^n : H_k(F_{s-1,k}) \to H_k(C_*))} \oplus \\ & \bigoplus_{n+2<s} \frac{\operatorname{im}(i^n : H_k(F_{s,k}) \to H_k(C_*))}{\operatorname{im}(i^n : H_k(F_{s-1,k}) \to H_k(C_*))} \\ = & \bigoplus_{0\leq s\leq n+2} \frac{\operatorname{im}(i^n : H_k(F_{s,k}) \to H_k(C_*))}{\operatorname{im}(i^n : H_k(F_{s-1,k}) \to H_k(C_*))} \end{aligned} \tag{2.13}$$

since both the first direct sum sums $0/0$ groups while the third sums $H(F_{s,k})/H(F_{s,k}) = 0$ groups. Finally (2.13) evolves into

$$\bigoplus_{0\leq s\leq n+1} \frac{\operatorname{im}(i^n : H_k(F_{s,k}) \to H_k(C_*))}{\operatorname{im}(i^n : H_k(F_{s-1,k}) \to H_k(C_*))} = \frac{H_k(F_{0,k})}{0} \oplus \frac{H_k(F_{1,k})}{H_k(F_{0,k})} \oplus \cdots \frac{H_k(F_{n+1,k})}{H_k(F_{n,k})} \overset{(2.6)}{\cong} H_k(C_*)$$

■

Lemma 2.28. *The E^r groups are subsequent quotients of the form*

$$E^r_{p,r-p} = \bigoplus_p (G_s/G_{s-1}) \tag{2.14}$$

where for a given k we define $G_s = \operatorname{im}(i^n : H_k(F_{s,k}) \to H_k(C_*))$.

Proof (reference). This is shown at the end of [7, Section 5.1] □

The two lemmas we just proved immediately lead to the following theorem.

Theorem 2.29 (Spectral Sequence limit formula). *For filtered chain complex (C_*, d) with associated spectral sequence $E^r \to E^\infty$ the homology groups $H_k(C_*)$ are given by*

$$H_k(C_*) \cong \bigoplus_p E^\infty_{p,k-p}. \tag{2.15}$$

We will write $E^r \rightrightarrows H_k(C_*)$.

PROOF. Proposition 2.26 implies the convergence of the associated spectral sequence. Lemma 2.27 and Lemma 2.28 lead to the desired equivalence. ■

2.4 Universal coefficient theorem

We introduce in this section the concept of homology with coefficient in any abelian group. Consequence of this new concept is Theorem 2.32 (The Universal Coefficients Theorem) which we will use to simplify interpretation and computations further on (see Section 4.3).

Definition 2.30 (Homology with coefficients in a general abelian group). Let X be a topological space and A an abelian group. We define $C_*(X; A) \overset{\text{(def)}}{=} C_*(X) \otimes A$ where $C_*(X)$ the singular chain complexes.

The *n-th singular homology of X with coefficients in A* is then

$$H_n(X; A) \overset{\text{(def)}}{=} H_n(C_*(X; A))$$

Remark 2.31. Since for A abelian

$$\begin{aligned} \mathbb{Z} \otimes A &\to A \\ m \otimes a &\mapsto ma \end{aligned}$$

is an isomorphism. So for $A = \mathbb{Z}$ we recover the normal singular homology: $H_n(X; \mathbb{Z}) = H_n(X)$.

Theorem 2.32 (The Universal Coefficients Theorem). *Let X be a topological space and let A be an abelian group. Then for every $n \in \mathbb{N}_0$ there is an exact sequence*

$$0 \longrightarrow H_n(X) \otimes A \xrightarrow{\omega} H_n(X; A) \longrightarrow \mathrm{Tor}(H_{n-1}(X), A) \longrightarrow 0$$

where $\omega([c] \otimes a) = [c \otimes a]$. Moreover this sequence splits and therefore

$$H_{n-1}(X; A) \cong H_n(X) \otimes A \oplus \mathrm{Tor}(H_{n-1}(X), A) \tag{2.16}$$

Proof (reference). This is Theorem 25.10 in [8, Lecture 25] □

Corollary 2.33. We will mostly be interested in $A = \mathbb{Z}, \mathbb{F}_2$ which are luckily torsion free. Therefore (2.16) becomes

$$\begin{aligned} H_{n-1}(X; \mathbb{Z}) &\cong H_n(X) \otimes \mathbb{Z} \cong H_n(X) \\ H_{n-1}(X; \mathbb{F}_2) &\cong H_n(X) \otimes \mathbb{F}_2. \end{aligned}$$

Chapter 3

Morse homology[a]

We will implicitly assume that every mentioned manifold or function is smooth.

3.1 Morse functions

Definition 3.1 (Critical point)**.** Let M be a manifold and $f : M \to \mathbb{R}$ a function. A *critical point* of f is a point $c \in M$ such that $(Df)_c = 0$.
A critical point c is said to be *non degenerate* if the Hessian of f is non singular at that point:

$$\det H_c(f) = \det \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(c) \right)_{i,j} \neq 0$$

for some local coordinate $(x_\ell)_\ell$. A critical point c which is not non-degenerate is said to be *degenerate*.

We are now ready to define Morse functions.

Definition 3.2 (Morse function)**.** Let M be a manifold and $f : M \to \mathbb{R}$ a function. We call f a *Morse function* if all its critical point are nondegenerate.

Example 3.3 (Height function on $\mathbb{S}^2$)**.** Consider $\mathbb{S}^2 \subseteq \mathbb{R}^3$ and the function

$$\begin{aligned} h : \mathbb{S}^2 &\to \mathbb{R} \\ (x, y, z) &\longmapsto z \end{aligned}$$

then the critical point of h are the point where the (df) has zero rank. Considering $\chi(x, y, z) = (x, y)$ to be the coordinate chart on $\{z > 0\} \cap S^2$ we have:

$$D\tilde{h}|_{(x,y)} = h \circ \chi^{-1}|_{(x,y)} = \frac{1}{\sqrt{1 - x^2 - y^2}}(x, y)$$

has rank 0 only if $x = y = 0$ so the only critical point on $S^2 \cap \{z > 0\}$ is $(0, 0, 1)$. Analogous we find $(0, 0, -1)$ to be the only other critical point. The matrix associated to $(d^2\tilde{h})$ at $(0, 0)$ is

$$\begin{pmatrix} \frac{1-y^2}{(1-x^2-y^2)^{3/2}} & \frac{xy}{(1-x^2-y^2)^{3/2}} \\ \frac{xy}{(1-x^2-y^2)^{3/2}} & \frac{1-x^2}{(1-x^2-y^2)^{3/2}} \end{pmatrix}_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is non degenerate so h is a Morse function.

[a]For this chapter, where not stated differently, the information are taken form [1, Chapters 1-4].

The following lemma will play a crucial role in defining Morse complexes.

Lemma 3.4 (Morse lemma)**.** *Let c be a nondegenerate critical point of the function $f : M \to \mathbb{R}$. Then there exist a neighbourhood $\Omega(c)$ of c and a diffeomorphism $\varphi : (\Omega(c), c) \to (U_{h(c)}, 0)$ such that*

$$f \circ \varphi^{-1}(c, \dots, x_n) = f(c) - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2$$

Such charts are called Morse charts. *We define Ind$(c) \overset{(\mathrm{def})}{=} i$ to be the* index *of the critical point c.*

PROOF. The key idea of the proof is use Taylor expansion and the implicit function theorem.

Since it is a statement about a local property we can without loss of generality assume $M = \mathbb{R}^n$ and $c = 0$. Further, by change of basis, we assume $(Df)_0$ to be diagonal. For $n = 1$, Taylor expansion of f at 0 gives

$$f(x) = f(0) + 0 + \tfrac{1}{2} f''(0)x^2 + \varepsilon(x)x^2 = f(0) \pm ax^2(1 + \varepsilon(x))$$

with ε smooth and $a \in \mathbb{R}_{>0}$. Since $\varphi(x) = x\sqrt{a(1 + \varepsilon(x)}$ has $\varphi'(0) = \sqrt{a} > 0$ it's locally a diffeomorphism (by the implicit function theorem) and the statement holds by $x_1 = \varphi(x)$.
We proceed by induction on n. Let $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ with elements (x, y). Letting $f(x, y) = f_y(x)$ we can express f as Taylor expansion of f_y by keeping y fixed:

$$f(0, 0) = f_y(0) + f_y'(0)x + \tfrac{1}{2} f_y''(0)x^2 + \varepsilon(x, y)x^2$$

Since f is Morse and thus $(Df)_0$ non degenerate $f_y''(0)$ is non zero since if zero the first column would be a zero column and so $(D^2 f)_0$ degenerate. Assume now $f_y'(0) = 0$ then $(x_1, y_1)\varphi(x, y) = (x\sqrt{a(1 + \varepsilon(x, y)}, 1)$ as in the case $n = 1$ leads to $f \circ \varphi^{-1}(x_1, y_1) = \pm x_1^2 + f(0, y)$ and we can conclude by induction. If $f_y'(0) \neq 0$ we can compose the chosen coordinate chart with a diffeomorphism making $f_y'(0) = 0$ but keeping $(d^2 f)_0$ fix. This is done by applying the inverse function theorem[b] to the solutions (x, y) to $\frac{\partial f}{\partial x} = 0$ obtaining $x = \varphi(y)$ with $(D\varphi)_0 = 0$ and then shift $(x, y) \to (x + \varphi(y), y)$. ∎

Therefore for $c \in \mathrm{Crit}(f)$ we define $\Omega(c) \subseteq M$ to be the image under the diffeomorphism h given by the Morse lemma of an open $U_{h(c)} \subseteq \mathbb{R}^n$.

Corollary 3.5. The critical points of a function on M are isolated. In particular, a Morse function on a compact manifold has only finitely many critical points.

Remark 3.6. Using the definition in Lemma 3.4 we notice that:

1. A local maximum of f is a critical point of index 0.

[b] $\frac{\partial^2 f}{\partial x^2}(0, 0) \neq 0$ since f Morse.

2. A local minimum of f is a critical point of index n, the dimension of the manifold.

3. An index 1 critical point of a function on a two dimensional submanifold of $\mathbb{R}^3$ is a saddle point.

We have now the tools to test whether a function is Morse or not. But do Morse functions exist at all on any manifold? The following theorem answers very satisfactorily this question.

Theorem 3.7 (Existence and Abundance of Morse Functions)**.** *Let M be a compact manifold. The set of Morse function on M is a dense open subset of $C^\infty(M)$.*

Proof (reference). This is Theorem 1.2.5 in [1, Page 12]. □

3.2 Pseduo-Gradients

Morse functions are one of the two very important tools we need in Morse theory. We will introduce here the second one.

Definition 3.8 (Pseudo-gradient)**.** Let $f : M \to \mathbb{R}$ a Morse function. A *pseudo-gradient* adapted to f is a vector field $X : M \to TM$ on M such that the following holds[c]

1. We have $(Df)_x(X_x) \leq 0$, with equality if and only if x is a critical point.

2. In a Morse chart in the neighbourhood of a critical point, X coincides with the negative gradient for the canonical metric on $\mathbb{R}^n$.

Remark 3.9 (Metrics)**.** If (M, m) is a Riemann manifold then we can define the gradient of f as the unique function $\nabla_x f$ such that for every $Y \in T_xM$

$$m(\nabla f, Y) = (df)_x Y.$$

Since every manifold admits a riemannian metric we will often interchange ∇f and df where one notation is more intuitive.

On the other hand a metric m immediately defines a gradient of f and so a vector field, thus providing f and X or f and m is equivalent.

Definition 3.10 (Stable and unstable manifolds)**.** Let $f : M \to \mathbb{R}$ a function and c a critical point. Let X be a pseudo gradient and denote by φ^t its flow[d]. Then we define its *stable manifold* to be

$$W^s(c) = \left\{ x \in M | \lim_{t\to\infty} \varphi^t(x) = c \right\}$$

and conversely its *unstable manifold* to be

$$W^u(c) = \left\{ x \in M | \lim_{t\to-\infty} \varphi^t(x) = c \right\}.$$

[c] We use the following notation: for $x \in M$ we denote $X(x) = (x, X_x)$ with $X_x \in T_xM$.

[d] Recall that given a smooth $F : (-\varepsilon, \varepsilon) \times M \to M$ such that $F_s(x) = F(s, x)$ is a diffeomorphism and $F_0 = id_M$, then $\frac{\partial F_x}{\partial s}|_{s=0}$ defines a vector field and vice versa. We denote φ^s the $F(s, \cdot)$ such that $\frac{\partial F_x}{\partial s}|_{s=0} = X(x)$ and call it flow of X.

On submanifolds of $\mathbb{R}^n$ this two manifolds $W^s(c)$ and $W^u(c)$ have a very geometric interpretation. If the pseudo gradient is the vector field of a force, we can imagine $W^s(c)$ to be the set of points that "fall down to c" under this force. On the other hand $W^u(c)$ can be thought to be the set of point that "pushed away from c" under this force.

Theorem 3.11 (Dimensions). *The stable and unstable manifolds of the critical point c are submanifolds of M that are diffeomorphic to open disks. Moreover, we have*

$$\dim W^u(c) = \operatorname{codim} W^s(c) = \operatorname{Ind}(c).$$

Proof. Away from a critical point c the stable manifold is the image of the embedding $(x, s) \mapsto \varphi^s(x)$ and in the neighbourhood $\Omega(c)$ it is the image of the locus where the quadratic form associated to f in c is positive. Thus the stable manifold is diffeomorphic to a disk of dimension $n - \operatorname{Ind}(c)$ and likewise the unstable manifold is diffeomorphic to a disk of dimension $\operatorname{Ind}(c)$.
For a more detailed proof we point to [1, Section 2.1.d]. ∎

We can again ask ourself if a pseudo-gradient exist for any Morse function. The answer is yes as well.

Theorem 3.12 (Existence of Pseudo-Gradient). *Let M be a manifold and f be a Morse function on it. Then there exist a pseudo-gradient adapted to f on M.*

Proof (reference). This is Section 2.1.c in [1, Page 26]. □

The following is a technical lemma that we will need to prove a key theorem further on.

Lemma 3.13. *If $f : M \to \mathbb{R}$ is a Morse function on a finite dimensional compact smooth M, and $p \in \operatorname{Crit}(f)$, then there exist*

$$E^s : T_p^s M \to W_p^s(\varphi)$$
$$E^u : T_p^u M \to W_p^u(\varphi)$$

homeomorphisms onto their images.

Proof (reference). This is Lemma 4.20 in [2, Page 115] □

3.2.1 The Smale condition and its consequences

Definition 3.14 (Trasversality). Let M be a manifold and $Q_1, Q_2 \subset M$ be two submanifolds. We say that Q_1, and Q_2 are *transverse* submanifolds, noted $Q_1 \pitchfork Q_2$, if

$$T_q Q_1 + T_q Q_2 = T_q M \quad \forall q \in Q_1 \cap Q_2.$$

A function $f : M \to N$ is transversal to $Z \subseteq N$, $f \pitchfork Z$, if $(Df)_x(T_x M) + (D\iota)_z(T_z Z) = T_y N$ where $\iota : Z \to N$ is the usual inclusion map.

The visual interpretation of transversality in $\mathbb{R}^n$ is that the two submanifolds should not be tangent at any point in the intersection.

Definition 3.15 (Smale condition)**.** We say that a pseudo-gradient adapted to f satisfies the *smale condition* if all stable and unstable manifolds of the critical points of f meet transversally, i.e.

$$W^u(a) \pitchfork W^s(b) \quad \forall a, b \in \mathrm{Crit}(f) \tag{3.1}$$

It is usual to call (f, X) a Morse-Smale pair if X is a pseudo-gradient, adapted to the Morse function f, satisfying the Smale condition.

The following proposition states that if you perturb any pseudo-gradient that satisfies the Smale condition then the resulting gradient will still satisfy the condition as well.

Proposition 3.16. *Let $f : M \to \mathbb{R}$ be a Morse function and let X be a pseudo-gradient field adapted to f that has the Smale property. Every vector field $\tilde{X}$ sufficiently close to X (in the C^1 sense) is an adapted pseudo-gradient field that still has the Smale property. Moreover, we have*

$$(C_*(f), \partial_X) = (C_*(f), \partial_{\tilde{X}})$$

Proof (reference). This is Proposition 3.4.3 in [1, Page 71] □

Theorem 3.17 (Inverse Image)**.** *Let M, N be manifolds and $Z \subseteq N$ a submanifold. Then if $f \pitchfork Z$, then $f^{-1}(Z)$ is a submanifold of M such that*

$$\dim M - \dim f^{-1}(Z) = \dim N - \dim Z \tag{3.2}$$

Proof (reference). This is Theorem 5.11 in [2, Page 131] □

Applying the above theorem to the inclusion $\iota : N \hookrightarrow M$ we get the next useful corollary.

Corollary 3.18. If M and Z are submanifolds of N of dimension m, z and n respectively and $M \pitchfork Z$, then $M \cap Z$ is an immersed submanifold of N of dimension $m + z - n$.

We are finally ready to prove the following proposition (and its corollary), which will play a central role in all our next computations.

Proposition 3.19. *Let X be a pseudo-gradient adapted to f on M that satisfies the Smale condition. Then we have that for any two critical points a and b*

$$\mathrm{codim}(W^u(a) \cap W^s(b)) = \mathrm{codim}(W^u(a)) + \mathrm{codim}(W^s(b)) \tag{3.3}$$

Proof. Now that we have Corollary 3.18 and Lemma 3.13 we can just write down the codimentions:

$$\begin{aligned}
\mathrm{codim}(W^u(a) \cap W^s(b)) &= \dim M - \dim(W^u(a) \cap W^s(b)) \\
&\overset{\text{Cor3.18}}{=} \dim M - (\dim W^u(a) + \dim W^s(b) - \dim M) \\
&= \dim M - (\dim M - \mathrm{codim} W^u(a) + \dim M - \mathrm{codim} W^s(b) - \dim M) \\
&= \mathrm{codim}(W^u(a)) + \mathrm{codim}(W^s(b))
\end{aligned}$$

■

Definition 3.20 (Trajectories)**.** Given a vector field X defined on M, a *trajectory* on M along X is a curve $\gamma(t)$ such that

- $\gamma : (-\varepsilon, \varepsilon) \to M$
- $\forall t \in (-\varepsilon, \varepsilon): \ \gamma'(t) = X(\gamma(t))$.

We usually denote γ by γ_x where $x = \gamma(0)$ saying that γ_x is the trajectory of x.
A trajectory along X from p to q is a trajectory $\gamma_{p,q}$ such that

$$\lim_{t \to -\varepsilon} \gamma_{p,q}(t) = p \quad \text{and} \quad \lim_{t \to \varepsilon} \gamma_{p,q}(t) = q$$

Moreover $W_f(p, q)$ denotes the set of trajectories from p to q along $-\nabla f$.

Under the assumption that the smale condition holds, we have a very useful identification for $W^u(a) \cap W^s(b)$. We denote in the following by $\mathcal{M}(p, q)$ the set of points on some trajectory from p to q, i.e.

$$\mathcal{M}(p, q) = \{x \in M | \exists \gamma \in W_f(p, q) : \exists t \in (-\varepsilon, \varepsilon) \text{ for which } \gamma(t) = x\}.$$

Lemma 3.21. *Let M be a manifold with a Morse function f and a pseudo-gradient which satisfies the smale condition. Then*

$$W^u(p) \cap W^s(q) = \mathcal{M}(p, q). \tag{3.4}$$

Further, this is a manifold of dimension

$$\dim \mathcal{M}(p, q) = \operatorname{Ind}(p) - \operatorname{Ind}(q). \tag{3.5}$$

PROOF. The dimension equation (3.5) is a direct consequence of (3.4) using Proposition 3.19 to get $\operatorname{codim}(W^u(a) \cap W^s(b)) = \operatorname{codim}(W^u(a)) + \operatorname{codim}(W^s(b))$ and Theorem 3.11 to get $\dim W^u(x_0) = \ \operatorname{codim} W^s(x_0) = \operatorname{Ind}(x_0)$.
Equation (3.4) is a consequence of the more general Proposition 3.22 ■

Proposition 3.22 (Characterisation of trajectories)**.** *Let M be a compact Manifold and f a Morse function on it. If $\gamma : (-\varepsilon, \varepsilon) \to M$ is a trajectory along a pseudo-gradient field X adapted to f on M, then there exist $p, q \in \operatorname{Crit}(f)$ such that*

$$\lim_{t \to -\varepsilon} \gamma(t) = p \quad \textit{and} \quad \lim_{t \to \varepsilon} \gamma(t) = q. \tag{3.6}$$

PROOF. We show that $\gamma(t)$ has a limit for $t \to \varepsilon$ and that this limit is a critical point. The same proof will apply for $t \to -\varepsilon$. We must show that $\gamma(t)$ reaches $\Omega(c)$ for some critical point c. Assume not, then every time γ enters one of the finitely many $\Omega(c_i)$ it must also leave it, and since f is decreasing along γ[e] once γ left a neighbourhood it can not return into it. Let $s_0 \in (-\varepsilon, \varepsilon)$ be the last time that γ leaves $\bigcup_{i=1}^{N} \Omega(c_i)$. We first need an intermediate result.

[e] $\frac{d}{dt} f(\gamma(t)) = -\|\nabla_{\gamma(t)} f\|^2 < 0$.

Claim. There exists an $\varepsilon_0 > 0$ such that

$$\forall x \in M \setminus \bigcup_{i=1}^{N} \Omega(c_i) : \ (df)_x(X_x) \leq -\varepsilon_0$$

Proof of Claim. Since $(df)_x$ is a continuous function and $M \setminus \bigcup_{i=1}^{N} \Omega(c_i)$ is (a compact set without an open and thus) compact this function has a maximum. The maximum must be ≤ 0 because of Point 1 in Definition 3.8 and can not be zero since $(df)_x$ is zero only at critical points and we precisely exclude them considering $M \setminus \bigcup_{i=1}^{N} \Omega(c_i)$. $\square$

So for any $s \in (-\varepsilon, \varepsilon)$ such that $s \geq s_0$:

$$f(\gamma(s)) - f(\gamma(s_0)) = \int_{s_0}^{s} \frac{\mathrm{d}(f(\gamma(u)))}{\mathrm{d}u} \mathrm{d}u = \int_{s_0}^{s} (df)_{\gamma(u)} X_{\gamma(u)} \mathrm{d}u \leq -\varepsilon_0 (s - s_0)$$

but since we can diffeomorphicaly map $(-\varepsilon, \varepsilon)$ to $\mathbb{R}$ this implies that $f \circ \gamma$ has no lower bound (by letting s be arbitrary big). But since it factorises trough M which is compact this is a contradiction. $\blacksquare$

We will not be much interested in $\mathcal{M}(p,q)$ spaces, but more on trajectories and in particular trajectories up to re-parametrisation (we are interested in the trajectory itself and do not want to differentiate based on the speed with which we walk along it). But this up to re-parametrisation translates in a quotient of the previous manifold.

Corollary 3.23. Quotienting $\mathcal{M}(p,q)$ by the action of time translation we obtain a new manifold

$$\mathcal{L}(p,q) = \{\text{trajectories along } -\nabla f\} / (\text{reparametrisation}).$$

of dimension

$$\dim \mathcal{L}(p,q) = \mathrm{Ind}(p) - \mathrm{Ind}(q) - 1.$$

3.3 Morse Complexes

Let for this section f be a Morse function on a compact manifold M and X a pseudo-gradient field satisfying the Smale condition.

We will consider the $\mathbb{Z}/2\mathbb{Z}$-vector space

$$MC_k(f) = \left\{ \sum_{c \in \mathrm{Crit}_k(f)} \lambda_c c \mid \lambda_c \in \mathbb{Z}/2\mathbb{Z} \right\} = \mathbb{Z}/2\mathbb{Z}[\mathrm{Crit}_k(f)] \tag{3.7}$$

where $\mathrm{Crit}_k(f)$ denotes the set of critical points of index k of f. We further define ∂_X on $MC_k(f)$ by extending the definition for $a \in \mathrm{Crit}_k(f)$:

$$\partial_X(a) = \sum_{b \in \mathrm{Crit}_{k-1}(f)} n_X(a,b) b \tag{3.8}$$

where $n_X(a,b) \in \mathbb{Z}/2\mathbb{Z}$ is the number (modulo 2) of trajectories (up to re-parametrisation) of X going from a to b, i.e. $|\mathcal{L}(a,b)|$.

Theorem 3.24. *Given a Morse function f and an adapted pseudo-gradient X on a compact manifold M, then equations (3.7) and (3.8) define a chain complex structure $MC_*(f)$.*

This is *the* building block for Morse theory, but it's proof requires some non trivial ideas.

3.3.1 The compactification theorem

The proof of Theorem 3.24 goes as follows. We need to show that $\partial_X \circ \partial_X(a) = 0$ for all $a \in \mathrm{Crit}(f)$. This is

$$\partial_X \circ \partial_X(a) = \sum_{b \in \mathrm{Crit}_{k-2}(f)} \left(\sum_{c \in \mathrm{Crit}_{k-1}(f)} n_X(a,c) n_X(c,b) \right) b$$

and so if we can show $\sum_{c \in \mathrm{Crit}_{k-1}(f)} n_X(a,c) n_X(c,b) = 0$ for any $a \in \mathrm{Crit}_k(f)$ and $b \in \mathrm{Crit}_{k-2}(f)$ we are done. So by definition of $n_X(\cdot,\cdot)$ we get

$$\sum_{c \in \mathrm{Crit}_{k-1}(f)} n_X(a,c) n_X(c,b) = \left| \bigsqcup_{c \in \mathrm{Crit}_{k-1}(f)} \mathcal{L}(a,c) \times \mathcal{L}(c,b) \right|.$$

It is a know fact form topology that the boundary of a one dimensional compact manifold is given by an even number of point, we will therefore show that this set is such a boundary. Let's define the set *of broken trajectories*

$$\overline{\mathcal{L}}(a,b) = \bigcup_{c_i \in \mathrm{Crit}(f)} \prod_{\substack{j=i \\ \mathrm{Ind}(c_{i_j}) > \mathrm{Ind} c_{i_{j+1}} \\ c_{i_1} = a, c_{i_{n_i}} = b}}^{n_i} \mathcal{L}(c_{i_j}, c_{i_{j+1}}).$$

Since the $\mathcal{L}(c_i, c_j)$ have a manifold structure we will be able to construct a topology. This topology makes $\overline{\mathcal{L}}(a,b)$ compact (as the notation suggests). Furthermore if $\mathrm{Ind}(a) = \mathrm{Ind}(b) + 2$, $\overline{\mathcal{L}}(a,b)$ turns out to be a one dimensional compact manifold with boundaries. This will prove the theorem. The missing proofs for the following statements can be found in [1, Section 3.2]

Let $\lambda = (\lambda_i)_{i=1}^q \in \overline{\mathcal{L}}(a,b)$ be a broken trajectory. Then λ connects q critical points c_i each of which admits a Morse neighbourhood $\Omega(c_i)$ as by Lemma 3.4 (Morse lemma). Therefore λ_i exits $\Omega(c_{i-1})$ to enter $\Omega(c_i)$.

Let now $\mathbb{U}^+$ (and $\mathbb{U}^-$) be the collection of U_i^+ (U_i^-) open neighbourhood of the exit point (entry point) of λ_i from $\Omega(c_{i-1})$ ($\Omega(c_i)$).

Lemma 3.25 (Topology on $\overline{\mathcal{L}}(a,b)$)**.** *The set*

$$\mathcal{W}(\lambda, \mathbb{U}^+, \mathbb{U}^-) = \left\{ \begin{array}{c} \mu = (\mu_j)_{j=1}^k : \mu_j \in \mathcal{L}(c_i, c_j), k \leq q, \mu_j \text{ enters and exists} \\ \Omega(c_{i_j}) \text{ throught the interiors of } U_i^+, U_i^- \end{array} \right\}$$

form a basis for a topology on $\overline{\mathcal{L}}(a,b)$.

From now on we will refer to $\overline{\mathcal{L}}(a,b)$ as a topological space with the topology from Lemma 3.25.

Theorem 3.26 (Compactness). *The topological space $\overline{\mathcal{L}}(a,b)$ is compact.*

Corollary 3.27. For $\operatorname{Ind}(a) = \operatorname{Ind}(b) + 1$ the space $\mathcal{L}(a,b)$ consist in a finite set of points.

PROOF. $\mathcal{L}(a,b)$ is a 0-dimensional submanifold of the compact space $\overline{\mathcal{L}}(a,b)$. ∎

Proposition 3.28. *Let f be a Morse function on a compact manifold M and with pseudo-gradient X satisfying the Smale condition. Let $a, c, b \in \operatorname{Crit}(f)$ with index $k-1, k, k+1$ respectively and $\lambda_1 \in \mathcal{L}(a,c)$ and $\lambda_2 \in \mathcal{L}(c,b)$. Then there is a open $O_{(\lambda_1,\lambda_2)} \subseteq \overline{\mathcal{L}}(a,b)$ and a continuous injection $\psi : [0,\delta) \to O_{(\lambda_1,\lambda_2)}$ differentiable on the interior and such that*

$$\begin{cases} \psi(0) = (\lambda_1, \lambda_2) \in \overline{\mathcal{L}}(a,b) \\ \psi(s) \in \mathcal{L}(a,b) \ \forall s > 0 \end{cases}.$$

Theorem 3.29. *If $\operatorname{Ind}(a) = \operatorname{Ind}(b) + 2$, then $\overline{\mathcal{L}}(a,b)$ is a compact 1-manifold with boundaries.*

Definition 3.30 (Morse homology). The homology of the chain complex structure $MC_*(f)$

$$\ldots \to MC_{n+1} \xrightarrow{\partial_X} MC_n \xrightarrow{\partial_X} MC_{n-1} \to \ldots .$$

as in Theorem 3.24 with boundary map

$$\partial_X(a) = \sum_{b \in \operatorname{Crit}_{k-1}(f)} n_X(a,b) b$$

where $n_X(a,b) \in \mathbb{Z}/2\mathbb{Z} = |\mathcal{L}(a,b)|$ is called the *Morse homology* $MH_*(f)$ of M based on f.

Theorem 3.24 shows that this is indeed well-defined.

Example 3.31 ($\mathbb{S}^n$). We have shown in Example 3.3 that the height function on $\mathbb{S}^2$ has two critical points: N & S.

This holds in general for the height function $h : \mathbb{S}^n \to \mathbb{R}$ with $\operatorname{Ind}(N) = n$ and $\operatorname{Ind}(S) = 0$. Therefore we have

$$MC_k(h) = \begin{cases} \mathbb{Z}/2\mathbb{Z}N, & k = n \\ \mathbb{Z}/2\mathbb{Z}S, & k = 0 \\ 0, & k \neq 0, n \end{cases}.$$

The Morse homology groups are then

$$MH_k(h) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & k = n \\ \mathbb{Z}/2\mathbb{Z}, & k = 0 \\ 0, & k \neq 0, n \end{cases}.$$

Generalisation and outlook

It was easy to introduce the concept of Morse complexes as a $\mathbb{Z}/2\mathbb{Z}$-vector space. We can extend this by defining $MC_k(f)$ as the $\mathbb{Z}$-module generated by by $\mathrm{Crit}_k(f)$. The ∂ maps will work in the same way but we will not have to reduce modulo 2 each time. ∂ will count the elements in $\mathcal{L}(a, c)$ with orientation signs[f].

Morse theory can also be used to give a CW structure to M and thus do topology with that. Morse homology is also considered to be the finite dimensional version of the very powerful Floer homology where one studies symplectic manifolds and orbits of hamiltonians.

3.4 Continuation maps

We will introduce in this section the concept of continuation maps. These maps will play a key role in two important proofs that we present in this work.

3.4.1 Construction

Considering (f_0, X_0) and (f_1, X_1) two Morse-Smale pairs, let

$$\begin{aligned} F : M \times [0,1] &\to \mathbb{R} \\ (x,s) &\mapsto F(x,s) = F_s(x) \end{aligned}$$

such that

$$\begin{cases} F_s = f_0 & \text{for } s \in [0, \frac{1}{3}] \\ F_s = f_1 & \text{for } s \in [\frac{2}{3}, 1] \end{cases}.$$

We can extend F to $M \times [-\frac{1}{3}, \frac{4}{3}]$ by setting

$$\begin{cases} F_s = f_0 & \text{for } s \in [-\frac{1}{3}, 0] \\ F_s = f_1 & \text{for } s \in [1, \frac{4}{3}] \end{cases}.$$

Consider now a Morse function $g : \mathbb{R} \to \mathbb{R}$ with critical points in 0 (index 0) and 1 (index 1) increasing on $(-\infty, 0) \cup (1, \infty)$ and such that

$$\frac{\partial F}{\partial s}(x,s) + g'(s) < 0 \quad \forall x \in M, \forall s \in (0,1).$$

A sketch of such a function is given in Figure 3.1.

Then the function $\tilde{F} = F + 0 \oplus g : M \times [-\frac{1}{3}, \frac{4}{3}] \to \mathbb{R}$ is a Morse function with critical points:

$$\mathrm{Crit}(\tilde{F}) = \mathrm{Crit}(f_0) \times \{0\} \cup \mathrm{Crit}(f_1) \times \{1\}$$

[f]Orientations of moduli spaces are quite technical and we avoid this discussion in this work.

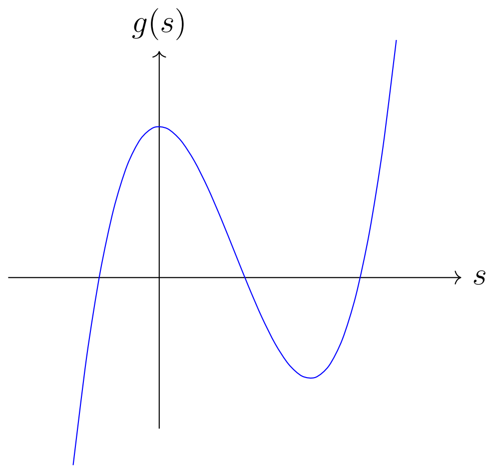

Figure 3.1: Sketch of a possible g function.

Lemma 3.32. *For any $a \in \mathrm{Crit}(f_0)$ or $b \in \mathrm{Crit}(f_1)$ we have*

$$\mathrm{Ind}_{\tilde{F}}((a,0)) = \mathrm{Ind}_{f_0}(a) + 1 \qquad \mathrm{Ind}_{\tilde{F}}((b,1))) = \mathrm{Ind}_{f_1}(b). \tag{3.9}$$

where we add a subscript to the index for avoiding ambiguity.

Proof. By considering now the product $M \times [-\frac{1}{3}, \frac{4}{3}]$ the dimension of the manifold increased by one. It's a direct consequence of Lemma 3.4 (Morse lemma) that in local coordinates around $(\cdot, 0)$ we will have one more dimension with negative sign while at $(\cdot, 1)$ we will have one more dimension with positive sign but in this last case – since the dimension of the manifold increased by one – the number of negative dimensions will remain the same and so, by its definition, will be the index. ■

Two pseudo-gradient fields adapted to $\tilde{F}$ come in consideration at this point: $X_0 - \nabla g$ and $X_1 - \nabla g$. Using a partition of unity we can fill all this into one single pseudo-gradient X such that

$$\begin{cases} X = X_0 - \nabla g, & \text{on } M \times [-\frac{1}{3}, \frac{1}{3}] \\ X = X_1 - \nabla g, & \text{on } M \times [\frac{2}{3}, \frac{4}{3}] \end{cases}.$$

By perturbing X we can achieve that it is transversal to $M \times \{s\}$ for all $s \in \{-\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}\}$ and that is satisfies the Smale condition (applying Proposition 3.16).

All this means that we can choose X such that

$$\left(C_*(\tilde{F}|_{M\times[-\frac{1}{3},\frac{1}{3}]}), \partial_X\right) = \left(C_*(f_0 + g|_{[-\frac{1}{3},\frac{1}{3}]}), \partial_{X_0+\nabla g}\right) \overset{(3.9)}{=} (C_{*+1}(f_0), \partial_{X_0})$$

and similarly

$$\left(C_*(\tilde{F}|_{M\times[\frac{2}{3},\frac{4}{3}]}, \partial_X\right) = (C_*(f_1), \partial_{X_1}).$$

Considering now the whole manifold $M \times [-\frac{1}{4}, \frac{4}{3}]$ we are interested in the trajectories along X. As Proposition 3.22 shows, these connect critical points of $\tilde{F}$ and can be divided into two categories:

1. γ starting and finishing in the same sections $[-\frac{1}{3}, \frac{1}{3}]$ or $[\frac{2}{3}, \frac{4}{3}]$, i.e. trajectories of either X_0 or X_1.

2. γ that start in the section $[-\frac{1}{3}, \frac{1}{3}]$ but finish up in section $[\frac{2}{3}, \frac{4}{3}]$, connecting critical points of f_0 to f_1[g].

Using Lemma 3.32 this leads to

$$C_{k+1}(\tilde{F}) = C_k(f_0) \oplus C_{k+1}(f_1). \tag{3.10}$$

Acting on this complex ∂_X has the form

$$\begin{aligned} \partial_X : C_k(f_0) \oplus C_{k+1}(f_1) &\to C_{k-1}(f_0) \oplus C_k(f_1) \\ \begin{pmatrix} a \\ b \end{pmatrix} &\mapsto \begin{pmatrix} \partial_{X_0} & 0 \\ \Phi^F & \partial_{X_1} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \end{aligned}$$

where Φ^F counts trajectories along X connecting $a \in M \times \{0\} \cap \mathrm{Crit}(\tilde{F})$ to $b \in M \times \{1\} \cap \mathrm{Crit}(\tilde{F})$

$$\begin{aligned} \Phi^F : C_k(f_0) &\to C_k(f_1) \\ a &\mapsto \sum_{b \in \mathrm{Crit}_k(f_1)} n_X(a,b) b. \end{aligned} \tag{3.11}$$

Since $\partial_X^2 = 0$ we get

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \partial_{X_0} & 0 \\ \Phi^F & \partial_{X_1} \end{pmatrix}^2 = \begin{pmatrix} \star & \star \\ \Phi^F \circ \partial_{X_0} + \partial_{X_1} \circ \Phi^F & \star \end{pmatrix}$$

So that Φ^F is a morphism of complexes.

3.4.2 Definition

Definition 3.33 (Continuation map)**.** Let (f_0, X_0) and (f_1, X_1) be two Morse-Smale pairs. Let further $\Gamma = \{(f_s)\}_{s \in [0,1]}$ be a path of functions[h] from f_0 to f_1. A map

$$\Phi : (C_*(f_0), \partial_{X_0}) \to (C_*(f_1), \partial_{X_1})$$

is a *continuation map* if for a critical point $c \in \mathrm{Crit}_k(f_0)$ it respects

$$\Phi(c) = \sum_{b \in \mathrm{Crit}_k(f_1)} n_X(c,b) c \tag{3.12}$$

using the construct of Section 3.4.1 and (3.11).

The next two corollaries follow directly from the discussion of Section 3.4.1 above.

Corollary 3.34 (Existence)**.** Given (f_0, X_0) and (f_1, X_1) two Morse-Smale pairs. With a path $\Gamma = \{(f_s)\}_{s \in [0,1]}$ of functions and gradients connecting them. Then a continuation map form (f_0, X_0) to (f_1, X_1) exists.

Corollary 3.35 (Morphism)**.** Continuation maps are morphism of chain complexes, i.e. chain maps.

[g] Note that since $\tilde{F}$ is decreasing along the trajectories start in the section $[\frac{2}{3}, \frac{4}{3}]$ and finish in the section $[-\frac{1}{3}, \frac{1}{3}]$ is not an option.

[h] Note that we do not require f_s to be Morse or the associated pseudo-gradients X_s to satisfy the Smale condition.

3.5 Reassuring equivalences up to isomorphism

We have now constructed a new type of homology a priori very different from the well know singular homology introduced in Section 2.2 and that at a first sight very much depends on the choices of the Morse function and the pseudo gradient. The next results are thus important in order to give general validity to this field of study.

3.5.1 Independence of choices

We know from Theorem 3.7 that there is an abundance of Morse functions. The just introduced concept of Morse homology groups would be quite unusable if for any two distinct such function we would obtain different homology groups. The next theorem states that this does not happen and, on the contrary, there is just one interesting group.

Theorem 3.36 (Uniqueness of Morse homology). *Let M be a compact manifold and $f_0, f_1 : M \to \mathbb{R}$ be two Morse functions on M respectively with adapted pseudo-grandients X_0, X_1 which respect the Smale condition. Then there is a morphism of complexes*

$$\Phi_* : (C_*(f_0), \partial_{X_0}) \to (C_*(f_1), \partial_{X_1})$$

which induces and isomorphism on the homology.

Idea of the proof

The proof is based on three steps. Considering

$$\begin{aligned} F : M \times [0,1] &\to \mathbb{R} \\ (x,s) &\mapsto F(x,s) = F_s(x) \end{aligned}$$

such that

$$\begin{cases} F_s = f_0 & \text{for } s \in [0, \frac{1}{3}] \\ F_s = f_1 & \text{for } s \in [\frac{2}{3}, 1] \end{cases}$$

First step: From F one deduces a chain complex morphism

$$\Phi^F : (C_*(f_0), \partial_{X_0}) \to (C_*(f_1), \partial_{X_1})$$

Second step: One shows that if $(f_0, X_0) = (f_1, X_1)$ and $F_s = f_0$ for every s then

$$\Phi^F = \mathrm{id}_{(C_*(f_0), \partial_{X_0})}$$

Third step: Considering another Morse-Smale pair (f_2, X_2), one finally shows that if G satisfies the same conditions as F for (f_1, X_1) and (f_2, X_2) and H does for (f_0, X_0) and (f_2, X_2) then the induced morphisms on the homologies

$$\Phi^G \circ \Phi^F,\ \Phi^H : MH_*(M, f_0, \partial_{X_0}) \to MH_*(M, f_2, \partial_{X_2})$$

coincide.

The statement follows form the last step since Φ^F must induce an isomorphism on the homology. Indeed one can use $H = f_0 = f_2$ as in Step two. Then Φ^F and Φ^G must be each others inverse.

PROOF.
FIRST STEP
We can just take the continuation map from (f_0, X_0) to (f_1, X_1) on the path $\Gamma = \{(F_s)\}_{s\in[0,1]}$ as introduced in Definition 3.33.
SECOND STEP
Let's set $f_1 = f_0$, $X_1 = X_0$ and

$$\begin{aligned} F : M \times [0,1] &\to \mathbb{R} \\ F(x,s) &\mapsto f_0(x). \end{aligned}$$

The function g will have the same form as in the construction of continuation maps in Section 3.4.1.

Claim. The vector field $X = X_0 + \nabla g$ is an adapted pseudo-gradient field for F and satisfies the Smale condition.

Considering $a \in \mathrm{Crit}(f_0)$ then the trajectory from $(a, 0)$ to $(a, 1)$ constant if projected to M is the unique trajectory along X that connects $(a, 0)$ to $(c, 1)$ for any c such that $\mathrm{Ind}_{f_0}(c) = \mathrm{Ind}_{f_0}(a)$. Therefore $\Phi^F = \mathrm{id}$ by the definition of Φ in (3.12).
THIRD STEP
Assume that we have interpolations H between f_0, f_2; F between f_0, f_1 and G between f_1, f_2. We do an interpolation of interpolations by considering a map:

$$\begin{aligned} K : M \times [-\frac{1}{3}, \frac{4}{3}] \times [\frac{1}{3}, \frac{4}{3}] &\to \mathbb{R} \\ (x,s,t) &\mapsto K(x,s,t) = K_{s,t}(x) \end{aligned}$$

satisfying

$$\begin{cases} K_{s,t} = H_t, & s \in [-\frac{1}{3}, \frac{1}{3}] \\ K_{s,t} = G_t, & s \in [\frac{2}{3}, \frac{4}{3}] \\ K_{s,t} = F_s, & t \in [-\frac{1}{3}, \frac{1}{3}] \\ K_{s,t} = f_2, & t \in [\frac{2}{3}, \frac{4}{3}] \end{cases}.$$

Considering again a Morse function g as in Section 3.4.1 but now increasing outside $[0, 1]$ and satisfying

$$\begin{aligned} \frac{\partial K}{\partial s}(x,s,t) + g'(s) < 0 \quad \forall (x,s,t) \in M \times (0,1) \times \left[\tfrac{1}{3}, \tfrac{4}{3}\right] \\ \frac{\partial K}{\partial t}(x,s,t) + g'(s) < 0 \quad \forall (x,s,t) \in M \times \left[\tfrac{1}{3}, \tfrac{4}{3}\right] \times (0,1). \end{aligned}$$

We finally define

$$\tilde{K}(x,s,t) = K_{s,t}(x) + g(s) + g(t)$$

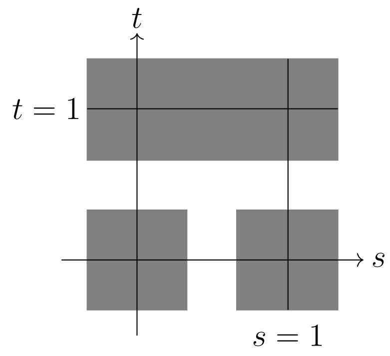

Figure 3.2: The various definition areas of K.

that is a Morse function and in the shaded area of Figure 3.2, is given by $f_i(x)+g(s)+g(t)$. A similar procedure as in Section 3.4.1 shows that we can define a pseudo gradient $\tilde{X}$ adapted to $\tilde{K}$ that satisfies the Smale property such that the differential $\partial_{\tilde{X}}$ is given by

$$\partial_{\tilde{X}} = \begin{pmatrix} \partial_{X_0} & 0 & 0 & 0 \\ \Phi^F & \partial_{X_1} & 0 & 0 \\ \Phi^H & 0 & \partial_{X_2} & 0 \\ S & \Phi^G & \mathrm{id} & \partial_{X_2} \end{pmatrix}$$

and so we are able to conclude. For more detail on this part see [1, Section 3.4] ■

Theorem 3.36 thus shows that the Morse homology of M is independent of f. We will therefore just write $MH_*(M)$ meaning the Morse homology of M based on any function if we do not want to specify a Morse function.

3.5.2 Morse homology is the singular homology

Now that we have shown with Theorem 3.36 that Morse homology is precisely one object for any manifold, we could be ask ourself if it is a new invariant that could help us distinguish two spaces.

Theorem 3.37 (Equivalence to singular homology)**.** *The Morse homology $MH_*(M)$ is isomorphic to the singular homology $H_*(M)$.*

Actually, a stronger statement is true. A Morse function allows us to define a cellular decomposition of the manifold; this cellular complex and the Morse complex are isomorphic. A standard result in algebraic topology states that singular and cellular homology agree.

Strongly inspired by Floer's approach to similar problems Conley [5] and Salamon [13] build up this proof.

Proof (reference). This is Theorem 7.4 in [2, Page 198] □

Morse homology is thus not a new invariant but a very useful tool to define and compute (often easily) the already well-known singular homology from which we can use the known results.

3.6 Miscellaneous

3.6.1 Bundles

Definition 3.38 (Fibre bundle)**.** Let F, E and B be topological spaces and $\pi : E \to B$ a surjective map. (F, E, π, B) is a *fibre bundle*, also denoted

$$F \to E \xrightarrow{\pi} B,$$

if π respects the following *local triviality condition*: for every $x \in B$ there exists an open neighbourhood $U \subseteq B$ of x such that there exists a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ which makes the following diagram commute

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & U \times F \arrow[d, "\mathrm{pr}_1"] \\ & U \end{tikzcd} \tag{3.13}$$

We call B *base space*, E *total space*, F *fibre*, π *projection map* and the collection $\{(U_i, \varphi_i)\}_i$ *local trivialisation* of the fibre bundle.

The intuition behind this definition is that E locally looks like a product of F and B.

We often denote $E_x = \pi^{-1}(x)$ the *fibre* of x. Trivially $E_x \cong F$.

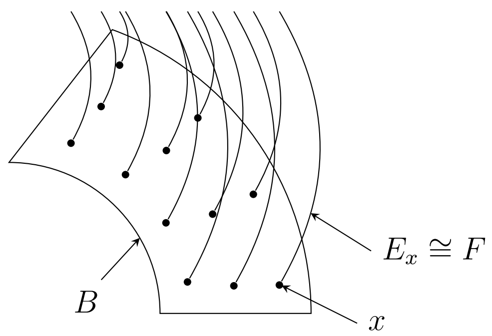

Figure 3.3: Visual representation of a Fibre bundle.[i]

Definition 3.39 (Vector Bundle)**.** A fibre bundle (F, E, π, B) is a n-dimensional *vector bundle* (over $\mathbb{R}$) if the fibre F is a n-dimensional vector field $F \cong \mathbb{R}^n$ such that the homeomorphism φ given from the local triviality condition induces a k-linear transformation on each fibre.

3.6.2 Parallel Transport[j]

Definition 3.40 (Section)**.** Let (F, E, π, B) be a fibre bundle and $\varphi : M \to B$ a smooth map. A *section of E along φ* is a smooth map $s : M \to E$ such that $s(x) \in E_{\varphi(x)}$ for every $x \in B$. We denote by $\Gamma_\varphi(E)$ all the sections of E along φ.

[i]The code for this image was published on `https://tex.stackexchange.com/questions/289165/drawing-a-fiber-bundle?rq=1`.

[j]For this section the information are taken form [9, Lecture 29].

Definition 3.41 (Parallel Transport)**.** Let (F, E, π, B) be a vector bundle. A *parallel transport system* $\mathbb{P}$ on E assigns to every point $p \in E$ and every curve $\gamma : [0,1] \to B$ on the base space with $\gamma(0) = \pi(p)$ a unique section $\mathbb{P}_\gamma(p) \in \Gamma_\gamma(E)$ with initial condition p, i.e. $\mathbb{P}_\gamma(p)(0) = p$. We call $\mathbb{P}_\gamma(p)$ parallel lift of γ starting at p. This association should satisfy the following four axioms:

1. For every smooth curve $\gamma : [0,1] \to B$

   $$\begin{aligned} \hat{\mathbb{P}}_\gamma : E_{\gamma(0)} &\to E_{\gamma(1)} \\ p &\mapsto \mathbb{P}_\gamma(p)(1) \end{aligned}$$

   is a linear isomorphism. Moreover $\hat{\mathbb{P}}_\gamma^{-1} = \hat{\mathbb{P}}_{\gamma^-}$.

2. Let $\gamma : [0,1] \to B$ be a smooth curve and $h : [0,1] \to [0,1]$ a diffeomorphism $h(0) = 0$ and $h(1) = 1$. Then for every $p \in E_{\gamma(0)}$ and every $t \in [0,1]$ we have

   $$\mathbb{P}_{\gamma \circ h}(p)(t) = \mathbb{P}_\gamma(p)(h(t))$$

3. The section $\mathbb{P}_\gamma(p)$ depends smoothly on both p and γ.

4. Let $\gamma, \delta : [0,1] \to B$ be two smooth curves with same start point and $\gamma'(0) = \delta'(0)$. Then for each $p \in E_{\gamma(0)}$

   $$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbb{P}_\gamma(p)(t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbb{P}_\delta(p)(t).$$

Chapter 4

Leray-Serre spectral sequence

The goal of this chapter is to prove Theorem 4.1. We will mainly follow the path delineated in [10, Chapter 3] and [11].

4.1 The theorem

The following Theorem 4.1 (Leray-Serre) is a key tool in understanding the homology (and thus the structure) of fibre bundles. First presented by Serre in [14], this theorem expresses the homology of the total space in terms of the one of the base space and fibre.

Theorem 4.1 (Leray-Serre). *Let*

$$F \to E \xrightarrow{\pi} B$$

be a fibre bundle with a simply connected base B and with a fibre F, where all E, B and F are closed manifold. Then there exists a spectral sequence given by

$$E^2_{p,q} = MH_p(B) \otimes MH_q(F). \tag{4.1}$$

Further we have convergence

$$E^2_{p,*-p} \rightrightarrows MH_*(E). \tag{4.2}$$

4.2 Construction of the proof

The most known proof of this theorem uses just tools from algebraic topology, see for this [7, Theorem 5.3]. We show here that one can also prove this theorem using differential geometry tools.

The proof is divided in the following subsections dedicated to each part of it. In Section 4.2.4 we summarise our findings and conclude the proof.

4.2.1 Morse-Smale data

Consider

$$b : B \to \mathbb{R} \quad \text{and} \quad g_{B,p} : T_pB \times T_pB \to \mathbb{R} \tag{4.3}$$
$$f : F \to \mathbb{R} \quad \text{and} \quad g_{F,p} : T_pF \times T_pF \to \mathbb{R} \tag{4.4}$$

to be Morse functions and riemannian metrics of B and F respectively. Let further $\mathrm{Crit}(b) = \{b_1, \ldots, b_n\}$ and $\mathrm{Crit}(f) = \{y_1, \ldots, y_m\}$ which we know to be finite because of Corollary 3.5.

Let $B_i \subseteq B$ be disjoint open sets contained in a local trivialisation of the fibre bundle with $b_i \in B_i$. We define the functions $\mathfrak{f}_i : E \to \mathbb{R}$ to be the ones that make the following commute

$$\begin{tikzcd} \pi^{-1}(B_i) \arrow[r, "\varphi"] \arrow[dr, "\mathfrak{f}_i"'] & B_i \times F \arrow[d, "f \circ \mathrm{proj}_2"] \\ & \mathbb{R} \end{tikzcd} . \tag{4.5}$$

The behaviour away from the $\pi^{-1}(B_i)$ is not going to matter as one can see from their use in (4.6). Let $\rho_i : B \to [0, 1]$ be bump functions such that

$$\rho_i(x) = \begin{cases} 1, & \text{if } x \in B_i' \subsetneq B_i \\ 0, & \text{if } x \notin B_i \end{cases} \qquad \text{for some fix } B_i' \subsetneq B_i'.$$

Morse function on E

We want first to construct a Morse function on E using f and b which we both (suppose to) know well. The newly defined $\mathfrak{f}_i$ look like a pre-composition of f with the trivialisation map, so we might combine these and $b \circ \pi$ to get a function that encodes all the useful information we need. The following Lemma shows that perturbing $b \circ \pi : E \to \mathbb{R}$ around its critical values gives a good result.

For simplify the notation we will write $\mathfrak{b} = b \circ \pi$ from now on.

Lemma 4.2. *Let*

$$F \to E \xrightarrow{\pi} B$$

where all E, B and F are closed manifold.

$$\begin{aligned} h : E &\longrightarrow \mathbb{R} \\ x &\longmapsto h(x) = \mathfrak{b}(x) + \varepsilon \sum_i (\rho_i \circ \pi)(x) \cdot \mathfrak{f}_i(x) \end{aligned} \tag{4.6}$$

Then h is a Morse function for $\varepsilon > 0$ small enough.

PROOF. On $\{\rho_i = 1\}$ we have that $h = \mathfrak{b} + \varepsilon \mathfrak{f}_i$ so $h = \mathfrak{b} \oplus \varepsilon f$ on $\{\rho_i = 1\} \times F$. Since (3.13) implies that the π map is locally bijective and by our choice of B_i in a trivialisation we get that $\tilde{h} = b \oplus \varepsilon f : \{\rho_i = 1\} \times F \to \mathbb{R}$ is locally a sum of Morse functions and thus also a Morse function, with $\tilde{h} = h \circ \varphi|_{\varphi^{-1}(\{\rho_i = 1\} \times F)}$.

We consider now $S = \bigcap_i \{\rho_i < 1\}$. We notice that $(db)_\beta = 0$ holds only for $\beta \in \mathrm{Crit}(b)$ but since we can cover $\mathrm{Crit}(b)$ with opens such that this cover does not intersect $\overline{S}$ (by our definition of the ρ_i) we have $db(T\overline{S}) \neq 0$ so there must exist a δ such that $0 < \delta < |(db)_s|$ for all $s \in S$. Because F compact by assumption, all ρ_i and f are bounded. Therefore by taking ε small enough we can achieve $0 < \frac{1}{2}\delta < |(dh)_s|$ for all $s \in S$ which implies that h has not critical points on S and so is Morse also there. ■

Corollary 4.3. From the first part of the proof of Lemma 4.2 we conclude that in the trivialisations

$$\mathrm{Crit}(h) = \mathrm{Crit}(\mathfrak{b}) \times \mathrm{Crit}(\mathfrak{f}). \tag{4.7}$$

Pseudo-gradient on E adapted to f

We want to construct a pseudo vector field v for h such that the following two conditions hold:

$$d\pi \circ v = \nabla \mathfrak{b} \tag{4.8}$$
$$v = \nabla h = \nabla \mathfrak{b} \oplus \varepsilon \nabla \mathfrak{f}_i \quad \text{on } \rho_i = 1 \tag{4.9}$$

Lemma 4.4 (Modified g_E pseudo-gradient)**.** *Define*

$$\begin{aligned} &V = \ker(d\pi) \subseteq TE \\ &H = V^{\perp} \quad \textit{perpendicular accoring to } g_E \end{aligned}$$

and let further

$$\tilde{g}_E = \begin{cases} g_E = g_B \oplus g_F, & \textit{on } V^{a} \\ \pi^* g_B, & \textit{on } H \end{cases} \quad : T_pE \times T_pE \to \mathbb{R}. \tag{4.10}$$

Then denoting $\tilde{\nabla}$ the gradient with respect to $\tilde{g}_E$[b], the pseudo-gradient vector field

$$v = \tilde{\nabla}\mathfrak{b} + \varepsilon \sum_i \rho_i \tilde{\nabla} \mathfrak{f}_i \tag{4.11}$$

respects the conditions (4.8) and (4.9).

Notice that using the trivialisation $V \cong TF$ and above $E|_{B_i}$ we have $H \cong TB_i$.

PROOF. We look at first at (4.8). We may remark that by definition of a gradient

$$d\mathfrak{b}Y = \tilde{g}_E(\tilde{\nabla}\mathfrak{b}, Y) = \tilde{g}_E(\tilde{\nabla}\mathfrak{b}, \mathrm{pr}_H Y)$$

because on V by definition $d\mathfrak{b} \equiv 0$. But $H = V^{\perp}$ so

$$\tilde{g}_E(\tilde{\nabla}\mathfrak{b}, \mathrm{pr}_H Y) = \tilde{g}_E(\mathrm{pr}_H \tilde{\nabla}\mathfrak{b}, \mathrm{pr}_H Y) = \pi^* g_B(\mathrm{pr}_H \tilde{\nabla}\mathfrak{b}, \mathrm{pr}_H Y)$$

where the last equality holds because we are now on H in (4.10). Writing out what $\pi^* g_B$ is, we get

$$\pi^* g_B(\mathrm{pr}_H \tilde{\nabla}\mathfrak{b}, \mathrm{pr}_H Y) = g_B(d\pi \mathrm{pr}_H \tilde{\nabla}\mathfrak{b}, d\pi \mathrm{pr}_H Y) = g_B(d\pi \tilde{\nabla}\mathfrak{b}, d\pi Y)$$

[a] Actually we should write $g_{E,p} = \pi^* g_{B,\pi(p)} \oplus (\mathrm{pr}_2 \circ \varphi)^* g_{F,\mathrm{pr}_2\varphi(p)}$, which further reduces to $(\mathrm{pr}_2 \circ \varphi)^* g_{F,\mathrm{pr}_2\varphi(p)}$ since we are on the kernel of $d\pi$.

[b] See Remark 3.9, i.e. $X = \tilde{\nabla}\psi \Leftrightarrow \tilde{g}_E(X, Y) = d\psi Y$.

with the last equality holding because $H = \ker(d\pi)^\perp$ and so we can just forget the projection $d\pi \mathrm{pr}_H Z = d\pi Z$. Since the defining property of ∇b is $dbY = g_B(\nabla b, Y)$, this leads to

$$d\pi \cdot \tilde{\nabla}\mathfrak{b} = \nabla b.$$

One now notices that for $\kappa \in H$ we have $\tilde{g}_E(\tilde{\nabla} f_i, \kappa) = df_i\kappa$ and over B_i $H \cong TB_i$ which implies $df_i\kappa = 0$. Outside B_i, $\rho_i = 0$ and thus $d\pi \sum_i \rho_i \tilde{\nabla}\mathfrak{f}_i = 0$. This directly implies $d\pi \cdot v = \nabla b$, i.e (4.8).

For (4.9) we have that on $\rho_i = 1$ we must be on V since $d\pi$ acts trivially. Therefore there $g_E = \tilde{g}_E$ holds and v simplifies to (4.9). ■

4.2.2 Morse complex

Let v the vector field of Lemma 4.4 that respects (4.8) and (4.9) and $e \neq e' \in \mathrm{Crit}(h)$, then we define

$$\mathcal{L}(e, e') = \{-v \text{ trajectory from } e \text{ to } e'\}/(-\varepsilon, \varepsilon) \tag{4.12}$$

where the quotient represents re-parametrisation as already done in Corollary 3.23. Equation (4.7) shows that

$$C_* = MC_*(h) = MC_*(b) \otimes MC_*(f) \tag{4.13}$$

and the boundary map[c] is naturally given by

$$\begin{aligned} \partial : MC_*(h) &\to MC_{*-1}(h) \\ e \mapsto \partial e &= \sum_{e' \neq e:\ \dim(\mathcal{L}(e,e'))=0} |\mathcal{L}(e, e')| e' \\ &= \sum_{p=0}^{\infty} d_p e \end{aligned} \tag{4.14}$$

where

$$d_p e = \sum_{\substack{e' \neq e:\ \mathrm{Ind}(\pi e) - \mathrm{Ind}(\pi e') = p \\ \dim(\mathcal{L}(e,e'))=0}} |\mathcal{L}(e, e')| e' \tag{4.15}$$

as we defined it for any Morse function. Finally we define

$$F_p = \bigoplus_{e \in \mathrm{Crit}(h):\ \mathrm{Ind}(\pi e) \leq p} \mathbb{Z}/2\mathbb{Z} \cdot e. \tag{4.16}$$

Lemma 4.5. *The F_p groups as in (4.16) form a filtration for $MC_*(h)$ in (4.13).*

[c]We prove that d as defined is well-defined as boundary map in the proof of Lemma 4.5.

PROOF. Let $C_k = MC_k(h)$, then defining

$$F_{p,k} = \bigoplus_{e \in \mathrm{Crit}_k(h)\colon\ \mathrm{Ind}(\pi e) \leq p} \mathbb{Z}/2\mathbb{Z} \cdot e \tag{4.17}$$

we clearly get that the defining (2.5) of a filtration

$$0 = F_{-1,k} \subseteq F_{0,k} \subseteq F_{1,k} \subseteq \cdots \subseteq F_{n,k} = C_k$$

holds. Remark that the index for the critical point must be finite and we have finitely many critical points, therefore we can just put $n \in \mathbb{N}$ to be the maximal $\mathrm{Ind}(\pi e)$ of any critical point .

We still need to check that the boundary map respects the filtration. Let so $e \in F_{p,k}$ then de is a (weighted) sum over all the other critical points e' such that $\dim(\mathcal{L}(e,e')) = 0$. But Corollary 3.23 shows that this implies that $\mathrm{Ind}(e') = \mathrm{Ind}(e) - 1$ and so $e' \in \mathrm{Crit}_{k-1}(h)$. Remark that if γ is a trajectory along $-v$, then $\pi \circ \gamma$ is a trajectory along $-\nabla b$ since

$$(\pi \circ \gamma(t))' = (d\pi)_{\gamma(t)} \circ (-v)_{\gamma(t)} = -(d\pi \circ v)_{\gamma(t)} \overset{(4.8)}{=} -(\nabla b)_{\gamma(t)}.$$

Therefore we conclude that $0 = \dim \mathcal{L}(e,e') \leq \dim V(\pi(e), \pi(e')) = \mathrm{Ind}(\pi(e)) - \mathrm{Ind}(\pi(e')) - 1$ which leads to

$$\mathrm{Ind}(\pi(e')) \leq \mathrm{Ind}(\pi(e)) - 1 \leq p - 1 \leq p.$$

This immediately implies $d(F_{p,k}) \subseteq F_{p,k-1}$ ∎

Lemma 4.6. *Define the $E^0_{p,q}$ groups as in (2.7) by $E^0_{p,k-p} = F_{p,k}/F_{p-1,k}$, then the following equality holds*

$$E^0_{p,k-p} = MC_p(b) \otimes MC_{k-p}(f). \tag{4.18}$$

The d^0 map is the d_0 map defined in (4.15).

PROOF. This follows directly form (4.13) using that for $e = e_b \otimes e_f \in MC_p(b) \otimes MC_q(f)$ the condition $\mathrm{Ind}(e) = k$ is $\mathrm{Ind}(e_b) + \mathrm{Ind}(e_f) = k$[d] and the condition $\mathrm{Ind}(\pi e) \leq p$ is $\mathrm{Ind}(e_b) \leq p$. Remark that taking the quotient makes so that the index $\mathrm{Ind}(\pi e)$ is exactly p.

We recall that d^0 for a (to be) spectral sequence for filtered chain complex (see Section 2.3.2) is defined as $d^0[\omega] = [\partial \omega]$ for $[\omega] \in E^0_{p,q}$. Therefore in this case for a basis element $e = e_b \otimes e_f \in MC_p(b) \otimes MC_q(f)$

$$d^0 e = \partial(e_b \otimes e_f) = \sum_{p=0}^{\infty} d_p e = d_0 e + \underbrace{\sum_{p=1}^{\infty} d_p e}_{\diamondsuit} = d_0 e$$

where $\diamondsuit = 0$ since by taking the quotient we allow only change of $\mathrm{Ind}(\pi e)$ of order 0 (any other change goes in the 0 coset). ∎

[d]This follows directly from the definition of index arisen form Lemma 3.4 (Morse lemma).

Lemma 4.7. *The d^0 boundary map counts the trajectories along $-\nabla f$. More specifically this means that $|\mathcal{L}(e,e')| = |W(\pi e, \pi e')|$ for any two $e, e' \in MC_*(h)$. We therefore can suggestively denote $d^0 = \partial_{fibre}$.*

PROOF. Recall the definition of d_0 in (4.15)

$$d_0 e = \sum_{\substack{e' \neq e:\ \mathrm{Ind}(\pi e) - \mathrm{Ind}(\pi e') = 0 \\ \dim(\mathcal{L}(e,e')) = 0}} |\mathcal{L}(e,e')| e'.$$

Let so e' be such that $\mathrm{Ind}(\pi e) = \mathrm{Ind}(\pi e')$. Then since trajectories go to critical points with bigger index we must conclude that either $W(\pi e, \pi e')$ is empty which means that we do not have any trajectory at all (and so $|\mathcal{L}(e,e')| = 0$) or $\pi e = \pi e'$ which gives us only one trajectory in B, i.e. the constant trajectory. Since the trajectories from e to e' are the product of trajectories from e_b to e'_b and e_f to e'_f we get the result that $|\mathcal{L}(e,e')|$ counts actually only the trajectories in F. ∎

Theorem 4.8. *The derived groups $E^1_{p,q}$ using (2.9) on $E^0_{p,q}$ are*

$$E^1_{p,q} \cong MC_p(b) \otimes MH_q(f). \tag{4.19}$$

PROOF.

Claim. We first claim that $E^1_{p,q} = \bigoplus_{b_i \in \mathrm{Crit}(b)} R_q(b_i)$ where

$$R_q(b_i) = MH_q(E_{b_i}, h|_{E_{b_i}}) = MH_q(E_{b_i}, \mathfrak{f}_i|_{E_{b_i}}). \tag{4.20}$$

Proof of Claim. Recall that by definition $E^1_{p,q}$ is

$$E^1_{p,q} = H_k(MC_p(b) \otimes MC_q(f)) = \frac{\ker(d^0 : E^0_{p,q} \to E^0_{p,q-1})}{\mathrm{im}(d^0 : E^0_{p,q+1} \to E^0_{p,q})}.$$

Let now $e_b \otimes e_f$ be a basis element of $MC_p(b) \otimes MC_q(f)$. Then Lemma 4.7 states that $d^0(e_b \otimes e_f)$ only depends on e_f. Thus by fixing e_b if we can now show (4.20) we are done. The second equality is clear since restricting to E_{b_i} makes h collapse to $\mathfrak{f}_i$. But restricting the action of d^0 on $MC_q(f)$ and taking the homology we get $MH_q(f)$ by definition. □

Claim. The second claim is that $R_q(b_i) \cong MH_q(f)$

Proof of Claim. By picking a trivialisation we have that once we fixed b_i, E_{B_i} is homeomorphic to F and composing this homeomorphism with $\mathfrak{f}_i$ we recover exactly f. □

Composing these two claims we get the isomorphism (4.19) by seeing that every $MH_q(F,f)$ depends on the $b_i \in \mathrm{Crit}(b)$ and fixing q we fix the index of b_i. ∎

4.2.3 The base space

Let C^p be the space generated by $\{e \in \mathrm{Crit}(h) | \mathrm{Ind}(\pi e) = p\}$. Then we get that (4.16) is actually

$$F_p = C^p \oplus F_{p-1}.$$

Since the F_* groups can be viewed as subsets of C_*, the restriction of the ∂ map defined in (4.14) to F_p is well-defined. We will denote

$$\partial' : F_p = C^p \oplus F_{p-1} \to F_{p-1} \tag{4.21}$$
$$\partial'' : F_p = C^p \oplus F_{p-1} \to F_{p-1} \tag{4.22}$$

the maps $d \circ \mathrm{proj}_{C^p}$ and $d \circ \mathrm{proj}_{F_{p-1}}$ respectively. We may form a larger operator d on the whole space F_p by defining

$$\begin{aligned} d : F_p = C^p \oplus F_{p-1} &\longrightarrow C^p \oplus F_{p-1} \\ \begin{pmatrix} a \\ b \end{pmatrix} &\longmapsto \begin{pmatrix} d_0 & 0 \\ \partial' & \partial'' \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \end{aligned} \tag{4.23}$$

which is well-defined because of (4.21)-(4.22) and since (4.15) shows that $d_0(C^p) \subseteq C^p$.

The above data comes naturally with some maps which form an exact couple exactly as in Section 2.3.2. But we are interested in a slightly different i map here:

$$\begin{aligned} i : F_{p-1} = C^{p-1} \oplus F_{p-2} &\to F_p = C^p \oplus F_{p-1} \\ (a, b) &\mapsto (0, a + b) \end{aligned}$$

Following the same procedure as there in deriving the exact couple we get

$$\begin{tikzcd} H_*(F_{p-1}) \arrow[r, "i^1"] & H_*(F_p) \arrow[d, "j^1"] \\ & H_*(F_p/F_{p-1}) = E^1_{p,*-p} \arrow[ul, "k^1"] \end{tikzcd} \tag{4.24}$$

where $j^1 : H_*(F_p) \to H_*(F_p/F_{p-1})$ sending $[(a,b)] \mapsto [a]$ is well-defined since $F_p/F_{p-1} = C^p$.

We need a better understanding of the k^1 map in order to understand the d^1 map. To do that we look at the short exact sequences[e]

$$\begin{tikzcd} 0 \arrow[r] & (F_{p-1})_* \arrow[r] \arrow[d, "d"] & (F_p)_* \arrow[r] \arrow[d, "d"] & (F_p/F_{p-1})_* \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & (F_{p-1})_{*-1} \arrow[r] & (F_p)_{*-1} \arrow[r] & (F_p/F_{p-1})_{*-1} \arrow[r] & 0 \end{tikzcd} \tag{4.25}$$

and recall that $d : (F_p/F_{p-1})_* \to (F_p/F_{p-1})_{*-1}$ is just $j \circ k$. Inserting j and k in the diagram we get

$$\begin{tikzcd} & & (F_p/F_{p-1})_* \arrow[r] \arrow[d, "d"] \arrow[dll, "k"'] & 0 \\ (F_{p-1})_{*-1} \arrow[r, "\iota"] & (F_p)_{*-1} \arrow[r, "j"] & (F_p/F_{p-1})_{*-1} \arrow[r] & 0 \end{tikzcd} \tag{4.26}$$

[e] We use the notation $(F_p)_* = F_{p,*}$ of (4.17).

plugging in what these maps concretely do on the elements we get

$$\begin{tikzcd} & & (a,b) \arrow[r] \arrow[d, "d"] & a \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & (\partial' a, 0) \arrow[r] & d(a,b) = (d_0 a, \partial' a + \partial'' b) \arrow[r] & d_0 a \arrow[r] & 0 \end{tikzcd} \quad . \tag{4.27}$$

We diagram chase what $k(a)$ should be. Since $d_0 a = 0$[f] and because the bottom of (4.27) is exact $d(a,b) = (d_0 a, \partial' a + \partial'' b) \in \mathrm{im}(i : (F_{p-1})_{*-1} \to (F_p)_{*-1})$, but the domain of i implies that this is of the form $d(a,b) = i(\partial' a, 0)$. Applying this to the homology (and using the $\cdot^1$ maps) we get that $k^1[a] = [(\partial' a, 0)]$.

Then just using the definition $d^1 = j^1 \circ k^1$ and noticing that $j^1 k^1 [a] = j^1[(\partial' a, 0)] = [\partial' a]$ and since we quotient by F_{p-2} this is $[d_1 a]$ we finally get

$$d^1 : H_*(F_p/F_{p-1}) \to H_{*-1}(F_{p-1}/F_{p-2}) \tag{4.28}$$
$$[a] \mapsto d^1[a] = [d_1 a]. \tag{4.29}$$

So we actually have to understand better d_1 in order to understand d^1.

By its definition in (4.14), d_1 acts on an element $e \in MC_*(h)$ of the from $e = e_b \otimes e_f$ by

$$d_1(e_b \otimes e_f) = \sum_{\substack{e' \in MC_*, \mathrm{Ind}(e_b) - \mathrm{Ind}(\pi e') = 1 \\ \dim \mathcal{L}(e,e') = 0}} |\mathcal{L}(e,e')| \cdot e'. \tag{4.30}$$

But letting $e' = e'_b \otimes e'_f$ we have that $\dim \mathcal{L}(e,e') = 0$ says that $\mathrm{Ind}(e) - \mathrm{Ind}(e') = 1$ which together with $\mathrm{Ind}(e_b) - \mathrm{Ind}(e'_b) = 1$ implies that e_f and e'_f have same index. Exactly how we have shown in the proof of Lemma 4.7 the trajectories $u \in \mathcal{L}(e,e')$ can therefore be simply visualised as the $\pi \circ u$ trajectories along $-\nabla b$.

Lemma 4.9. *Along a trivialization over any path $\pi \circ \gamma : \mathbb{R} \to B$ joining $b_i, b_j \in \mathrm{Crit}(b)$ we have a unique identification between $R(b_i)$ and $R(b_j)$.*

PROOF. We want to choose an appropriate trivialisation of E along the path γ of the form $\mathbb{R} \times F$[g] such that it agrees with the already chosen trivialisations when we defined B_i. This is done by parallel transport (see Definition 3.41) the one over B_i over γ to define (setting it equal to the first one) a trivialisation for $B_j \times F$ around b_j. Since $\pi_1(B)$ is trivial by assumption this definition is well-defined.

Consider two $b_i, b_j \in \mathrm{Crit}(b)$ and let's consider an homotopy

$$F : E \times [0,1] \to E$$
$$(x,s) \mapsto F(x,s) = F_s(x)$$

with $F_0 = F_1 = \mathfrak{f}$, with $\mathfrak{f} = f \circ \mathrm{pr}_2 \circ \varphi$, induced by $\pi_1(B) = 0$. If γ is a path joining b_i, b_j we consider the path of functions $\{h_s\}_{s \in [0,1]}$ with $h_s(x) = b(\pi(x)) + \varepsilon \sum_i (\rho_i \circ \pi)(x) \cdot F_s(x)$.

[f] This holds because with d_0 is d^0 and if we follow the path of a on the exact couple, we notice that it must be mapped in the 0 coset.

[g] $\mathbb{R}$ represents the image of γ, with F the usual fibre.

We assume without loss of generality that the whole path of γ is covered by the B_i. Indeed we can just complete these B_i by adding more (finitely many) opens. Then the first part of the summands becomes useless, so we get $h_s(x) = b(\pi(x)) + \varepsilon F_s(x)$. We can rewrite $h_s(x) = b(\pi(x)) + \varepsilon\mathfrak{f}(x) + c(s)$. Using the above trivialisation we obtain that h above b_i"$\approx -\infty$" or b_j"$\approx \infty$" looks like

$$h = \begin{cases} \varepsilon f + c_1, & \text{at } (-\infty, \cdot) \\ \varepsilon f + c_2, & \text{at } (+\infty, \cdot) \end{cases} \tag{4.31}$$

where $c_i = c(i) + b(b_i)$ for $i \in \{0, 1\}$. We denote now the continuation map defined by $\{h_s\}_{s\in[0,1]}$ as

$$\Phi : MC_*\left(h|_{E_{b_i}}\right) \to MC_*\left(h|_{E_{b_j}}\right)$$

In the SECOND STEP of the proof of Theorem 3.36 we showed that the continuation map between the same complex is the identity. Following the same proof one shows that the continuation map between two homotopic maps is an isomorphism. We conclude that the count of $-v$ trajectories on the trivialisation is chain homotopic to an isomorphism and hence is an isomorphism on the homology. Now we are done recalling the definition $MH_q(E_{b_i}, h|_{E_{b_i}}) = R_q(b_i)$. ∎

Lemma 4.10. *The d_1 map on $E^1 = \bigoplus_{b_i} R(b_i) = MC_p(b) \otimes MH_q(f)$ counts the trajectories on the base space. We can therefore suggestively denote $d_1 = \partial_{base}$ on $MC_*(b) \otimes MH_*(f)$.*

PROOF. This is a direct consequence of Lemma 4.9. Indeed, a trajectory along $-v$ projects to a trajectory along g_B on B. Lemma 4.9 shows that d_1 keeps the $MH_*(f)$ component unaltered. So by d_1 having the form (4.30) we conclude that it must count the $-\nabla b$ trajectories. ∎

Theorem 4.11. *The second derived group is given by*

$$E^2_{p,q} = MH_p(b) \otimes MH_q(f) \tag{4.32}$$

PROOF. Follows directly from the construction of E as spectral sequence arising from the exact couple (4.27), the characterisation of d^1 in (4.29) and Lemma 4.10. ∎

4.2.4 Conclusion

We have now all the necessary ingredients to conclude the proof of Theorem 4.1 (Leray-Serre).

PROOF OF THEOREM 4.1. Consider Morse functions and pseudo-gradients as in (4.3) and (4.4). From Theorem 3.7 (Existence and Abundance of Morse Functions) and Theorem 3.12 (Existence of Pseudo-Gradient) we know that these always exist. Then the arguments in Sections 4.2.1 to 4.2.3 leading to Theorem 4.11 show that a spectral sequence given by $E^2_{p,q} = MH_p(b) \otimes MH_q(f)$ exist. Proposition 2.26 states that this must be eventually stationary, thus converge, say $E^r_{p,q} \to E^\infty_{p,q}$. We only need to check (4.2), i.e. $\bigoplus_p E^\infty_{p,k-p} \cong MH_k(E)$.

In the over mentioned construction we defined $C_* = MC_*(h,v) = MC_*(b) \otimes MC_*(f)$ in (4.13). Remark now that in Lemma 4.4 we showed that v is an adapted pseudo-gradient to h thus by Theorem 3.36 (Uniqueness of Morse homology) we get that $MC_*(h,v) \cong MC_*(h)$. Furthermore Theorem 2.29 (Spectral Sequence limit formula) states $H_k(C_*) \overset{(\text{def})}{=} MH_k(h) \cong \bigoplus_p E^\infty_{p,k-p}$ which concludes the proof. ■

4.3 Applications

4.3.1 The Künneth formula

We present now the Künneth Theorem, which allows us to compute the homology of products.

Theorem 4.12 (Künneth Theorem)**.** *Let k be a field, M and N be closed manifolds with at least one of them simply connected. Then considering homology with coeficents in k*

$$MH_j(M \times N) \cong \bigoplus_s MH_s(M) \otimes MH_{j-s}(N). \tag{4.33}$$

PROOF. We show the $k = \mathbb{F}_2$ case. We assume without loss of generality that N is simply connected. Consider the trivial fibre bundle

$$M \to N \times M \xrightarrow{\pi} N$$

with $\pi = \mathrm{pr}_1$. By assumption all the requirements of Theorem 4.1 are satisfied. Therefore we get that

$$E^2_{p,q} = MH_p(N) \otimes MH_q(M)$$

and

$$E^2_{p,*-p} \rightrightarrows MH_*(N \times M)$$

so we just need to show that $E^\infty_{p,q} \cong E^2_{p,q}$ and we are done. To show this it suffices to show that the d^2 maps are trivial, but this is true since the d^2 map is induced by the boundary map on $MH_*(N)$ and $MH_*(M)$. ■

4.3.2 Computation of $MH_*(\mathbb{CP}^n)$

In order to compute $MH_*(\mathbb{CP}^n)$ we consider the *Hopf fibration*

$$\mathbb{S}^1 \to \mathbb{S}^{2n+1} \xrightarrow{\pi} \mathbb{CP}^n.$$

$\mathbb{CP}^n$ is well known to be ${}^{\mathbb{S}^{2n+1}}\!\big/_{(|z|=1)}$ and this clarifies why we have such a fibration.

We can thus apply Theorem 4.1 (Leray-Serre) and we get

$$E^2_{p,q} = MH_p(\mathbb{CP}^n) \otimes MH_q(\mathbb{S}^1)$$

and we know from Example 3.31 that $MH_q(\mathbb{S}^k) = \mathbb{Z}$ for $q = 0, k$ and $MH_q(\mathbb{S}^k) = 0$ for $q \neq 0, k$. By Theorem 2.32 (The Universal Coefficients Theorem) we get that $E^2_{p,q} \cong MH_p(\mathbb{CP}^n)$ for $q = 0, 1$ and $E^2_{p,q} = 0$ for $q \geq 2$ which we can display on $\mathbb{Z}^2$ as in Figure 4.1.

$$\begin{tikzcd}
q & \ddots & \vdots & \vdots & \vdots & \iddots \\
\vdots & 0 & 0 & 0 & 0 & \cdots \\
1 & MH_0(\mathbb{CP}^n) & MH_1(\mathbb{CP}^n) & MH_*(\mathbb{CP}^n) \arrow[ull, "d^2"'] & MH_n(\mathbb{CP}^n) & \cdots \\
0 & MH_0(\mathbb{CP}^n) & MH_1(\mathbb{CP}^n) & MH_*(\mathbb{CP}^n) \arrow[ull, "d^2"'] & MH_n(\mathbb{CP}^n) & \cdots \\
 & 0 & 1 & \cdots & n & p
\end{tikzcd}$$

Figure 4.1: The $E^2_{p,q}$ groups for the Hopf fibration.

Considering then the $E^3_{p,q}$ groups one notices that the all the d^3 arrows either start or finish in a 0 group and are thus trivial, so $E^3 = E^\infty$. Recall that $MH_k(S^{2n+1}) = \bigoplus_p E^\infty_{p,k-p}$ which is $\mathbb{Z}$ for $k = 0, 2n+1$ and 0 else. So $E^3 = E^\infty$ looks like presented in Figure 4.2.

$$\begin{array}{c|ccccc}
q & \ddots & \vdots & \vdots & \vdots & \iddots \\
\vdots & 0 & \star & 0 & 0 & \cdots \\
\vdots & 0 & 0 & \star & 0 & \cdots \\
0 & \mathbb{Z} & 0 & 0 & \star & \cdots \\
\hline
 & 0 & 1 & \cdots & 2n+1 & p
\end{array}$$

Figure 4.2: The $E^3_{p,q}$ groups for the Hopf fibration. The $\star$ can not all be 0.

We can already conclude

$$\mathbb{Z} = E^3_{0,0} = \frac{\ker(d^2 : E^2_{0,0} \to E^2_{-2,1})}{\operatorname{im}(d^2 : E^2_{2,-1} \to E^r_{0,0})} = MH_0(\mathbb{CP}^n) \big/ 0 = MH_0(\mathbb{CP}^n).$$

Further for $1 < k < 2n$ we have $E^3_{k-1,1} = E^3_{k+1,0} = 0$ and so $d^2 : E^2_{k+1,0} \to E^2_{k-1,1}$ must have been injective since $0 = E^3_{k+1,0} = \ker d^2/0$ and on the other hand also surjective since $0 = E^3_{k+1,0} = E^2_{k-1,1}/\mathrm{im} d^2$. So these above mentioned d^2 are all isomorphism, i.e. all the odd homology groups are isomorphic and all the even are isomorphic: $MH_k(\mathbb{CP}^n) \cong MH_{k+2}(\mathbb{CP}^n)$.

Therefore it remains to compute $MH_1(\mathbb{CP}^n)$. For this one we consider $d^2 : E^2_{1,0} \to E^2_{-1,1}$ which must be an isomorphism by the same reason given above. It's furthermore clear that $MH_k(\mathbb{CP}^n) = 0$ for $k > 2n+1$ since the index of the critical points are bounded by the dimension of the manifold.

Concluding

$$MH_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z}, & \text{for } k \text{ even, } 0 \leq k \leq 2n+1 \\ 0, & \text{else} \end{cases}.$$

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