Eidgenössische Technische Hochschule Zürich

SYMPLECTIC INVARIANTS OF MAGNETIC COTANGENT BUNDLES

Silvio Barandun

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Advisor: Prof. Dr Will Merry

Contents

Abstract

We first introduce various homological theories — and the relations between them — classically used to study symplectic manifolds.

Then we compute various symplectic invariants of magnetic cotangent bundles. In particular, we will be able to compute the symplectic cohomology of the magnetic cotangent bundle of the two-sphere. This will be shown to vanish using the methods developed by RITTER [Rit14].

CHAPTER 1

Introduction

The first occurrence of symplectic geometry is dated to 1808 when LAGRANGE [Lag08] studied the motion of planets under the interaction of gravitational force [Wei81]. This way of studying mechanical systems without pictures was new with respect to the much more geometrically oriented approaches of his predecessors like Huygens and Newton. The Lagrangian approach was then further developed by Hamilton. No wonder that two central objects in the theory of symplectic geometry carry the names of these two mathematicians.

Since Lagrange, symplectic geometry developed into a rich field of modern mathematics with many connection to other branches such as algebraic geometry, dynamical systems and theoretical physics. In particular the work of FLOER [Flo88a; Flo88b; Flo89] signalled a striking development of the theory with the introduction of the homonymous homology groups. Originally introduced by Floer is the nowadays called *Lagrangian Floer homology*, but in this text we will focus on *Hamiltonian Floer homology*.

Hamiltonian Floer homology is known as an infinite dimensional Morse theory. On a manifold M, Morse theory would consider a function $f : M \to \mathbb{R}$. Floer theory does consider a — typically time-dependent Hamiltonian — function $H_t : M \to \mathbb{R}$, but then looks at the base manifold $C^\infty_{\mathrm{contr}}(S^1, M)$ of contractible smooth loops on M. The role of the actual Morse function is played by the action functional, which maps a loop γ to

$$\mathcal{A}(\gamma) = \int_D \widetilde{\gamma}^*\omega + \int_0^1 H_t(\gamma(t))dt$$

where $\widetilde{\gamma}$ is an extension of γ to the disk. We remark that $\mathcal{A}$ is not some God-given entity, but is the action integral whose critical points are in one-to-one correspondence with solutions to Hamilton's equations in Hamiltonian mechanics. Floer homology is thus a Morse theory on the loop space, which indeed is infinite dimensional.

Using this new homology theory, Floer was able to prove the Arnold conjecture stating that every Hamiltonian symplectomorphism has at least as many fixed points as the number of critical points of any generic smooth function $M \to \mathbb{R}$.

This text focuses on a specific class of symplectic manifolds, namely magnetic cotangent bundles (these are sometimes also called *twisted cotangent bundles* or *twisted tangent bundles* after having identified T^*M and TM via a Riemannian metric). For a manifold M, the cotangent bundle T^*M carries a natural symplectic form $\omega = d\theta$, where $\theta_{p,v}\xi = v(D\pi(p)\xi)$ and $\pi : T^*M \to M$ the natural projection. Magnetic cotangent bundles generalise this construction by considering $\omega_\sigma = d\theta + \pi^*\sigma$ for some closed $\sigma \in \Omega^2(M)$. This generalisation allows for a controlled introduction of non exactness. Magnetic cotangent bundles owe their name to the fact that for M a surface and σ and area form on it, the flow of the kinetic energy Hamiltonian $H(p,v) = \frac{1}{2}\|v\|^2$ are magnetic geodesics, that is trajectories of charged particles on M under the action of the magnetic field symbolised by σ. Magnetic cotangent bundles are open manifolds and thus do not satisfy the requirements needed to apply the tools of standard Floer homology. We will thus need to introduce *symplectic cohomology* — a variation of Floer homology for certain open manifolds. Symplectic cohomology is generally very difficult to compute as it is defined as a direct limit of Floer homology groups. When the symplectic form is exact, common results in the literature limit themselves to showing that the rank of symplectic cohomology is zero or infinite. Nonetheless the importance of its understanding has already been clear since its very introduction during VITERBO's ICM lecture [Vit95], where he sketched a proof that the symplectic cohomology of cotangent bundles of oriented manifolds is the homology of their free loop space.

In this work we first analyse various symplectic structures of magnetic cotangent bundles. Then we focus on T^*S^2 and show that RITTER's methods [Rit14] for negative line bundles may be applied to this well studied manifold.

Notation

Mostly for the author's own whim, we try to keep a consistent notation throughout the text. Manifolds will mostly be denoted by M or N and p or q will be used for points on them, that is $p \in M$. Typically $\xi, \zeta \in T_pM$ will be tangent vectors and $v, w \in T_p^*M$ cotangent vectors. When possible we will denote $(p,\xi) \in TM$ for $\xi \in T_pM$ and similarly for general vector bundles. Vector bundles $\pi : E \to M$ often carry an index π_E in the footprint map, in particular in Chapter 4 when we will be working with many vector bundles and π will also be needed as the area of the unit circle. For $\varphi : M \to N$ a smooth map between manifolds, the derivative at p is $D\varphi(p) : T_pM \to T_{\varphi(p)}N$, while for $f : M \to \mathbb{R}$ we reserve the notation $df \in \Omega^1(M)$ for the one-form induced by f. Derivates along coordinates $x : M \to \mathbb{R}$ will be ∂_x. For smooth paths $\gamma : \mathbb{R} \to M$ the derivate $\dot\gamma(t) \in T_{\gamma(t)}M$ satisfies $df\dot\gamma(t) = \frac{\partial}{\partial s}|_t f \circ \gamma(s)$.

Outline

The text is structured as follows

Chapter 2: exposes various homological theories in symplectic geometry.

Chapter 3: analyses the famous paper by Piunikhin, Salamon and Schwarz [PSS96] developing a link between quantum and Floer homology.

Chapter 4: is the heart of the text computing first the first Chern class and quantum cohomology of magnetic cotangent bundles and then showing the vanishing of symplectic cohomology of the magnetic T^*S^2.

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As the conclusion of this thesis marks the conclusion of my (at least undergraduate) studies, I would also like to take the chance to thank

- my parents for their unconditioned support, whatever I have been doing;
- the ones — you know who you are — who made my studies the wonderful moment they have been;
- the mathematical world itself.

CHAPTER 2

Preliminaries

This chapter is dedicated to the introduction of different concepts related to the homological study of symplectic manifolds. In particular, we introduce in Section 2.2 Hamiltonian Floer homology and later in Section 2.5 symplectic homology. The latter is a generalisation of the former to non-compact manifolds, typically of cotangent bundles.

2.1. Symplectic manifolds

With the goal of fixing the notation, we present here a very brief introduction to symplectic manifolds. The main reference we follow is the book by McDuff and Salamon [MS12].

Definition 2.1 (Symplectic manifold). A *symplectic structure* on an m-dimensional smooth manifold M is a closed 2-form $\omega \in \Omega^2(M)$, which is non-degenerate in the sense that for every $p \in M$ if $\xi \in T_pM$ is such that

$$\omega(\xi, \zeta) = 0 \quad \forall \zeta \in T_pM \tag{2.1}$$

then $\xi = 0$. We call the pair (M, ω) a symplectic manifold.

Standard results from linear algebra show that a symplectic manifold must have even dimension $m = 2n$. The natural structure preserving maps between symplectic manifolds are a special type of diffeomorphisms.

Definition 2.2 (Symplectomorphism). Let (M, ω) and (N, η) be symplectic manifolds. A diffeomorphism $\varphi \in \mathrm{Diff}(M, N)$ is called a *symplectomorphism* if it preserves the symplectic structure, that is

$$\varphi^*\eta = \omega.$$

In particular when considering endomorphisms, we denote the subgroup of symplectomorphisms $\mathrm{Sym}(M, \omega) \subseteq \mathrm{Diff}(M)$ or simply $\mathrm{Sym}(M)$ if the symplectic structure is clear form the context.

Definition 2.3 (Almost complex structure). An *almost complex structure* is a section of the automorphism bundle $J \in \Gamma(\mathrm{End}(TM))$ satisfying the identity $J^2 = -\,\mathrm{id}$ at every point.

An almost complex structure J is said to be *ω-tame* if $\omega(\xi, J\xi) > 0$ for every non-zero tangent vector ξ. We will denote by $\mathcal{J}_\tau(M, \omega)$ the set of almost complex structure tamed by ω. On the other hand, one calls J *ω-compatible* if it is ω-tame and $\omega(J\xi, J\zeta) = \omega(\xi, \zeta)$. We will denote by $\mathcal{J}(M, \omega)$ the set of ω-compatible almost complex structures.

An almost complex structure can be thought of as an element that plays the role of i making TM into a complex vector bundle of dimension n over $\mathbb{C}$.

It is well-known that the set $\mathcal{J}_\tau(M,\omega)$ is always non-empty, path-connected, and contractible[a]. Being path-connected, any choice of $J \in \mathcal{J}_\tau(M,\omega)$ gives rise the same complex vector bundle (TM,J) up to isomorphism.

For a $\pi : E \to B$ be a complex vector bundle of complex dimension n, the *Chern classes* $c_i(E) \in H^{2i}(B,\mathbb{Z})$ of E are characteristic classes. We point to the standard reference by Milnor and Stasheff [MS74] for a precise definition. In this text, we are, however, only interested in some properties of the first Chern class, which we summarise in the following remark.

Remark 2.4 (Chern classes).

- The top Chern class equals the Euler class of the underling real vector bundle.
- For a continuous $\varphi : B \to B'$, the Chern class of the pullback bundle is the pullback of the Chern class: $c_i(\varphi^* E) = \varphi^* c_i(E)$.
- Whitney sum formula: $c_1(E \oplus F) = c_1(E) + c_1(F)$.
- Denoting by E^* the dual vector bundle, $c_1(E^*) = -c_1(E)$.

Associated to the first Chern class is the concept of minimal Chern number which is the integer

$$N = \inf\left\{ k > 0 \;\middle|\; \exists \varphi : S^2 \to M : \int_{S^2} \varphi^* c_1 = k \right\}.$$

Definition 2.5 (J-holomorphic). Given two almost complex manifolds[b] (M,J) and (M',J'), a map $\varphi : M \to M'$ is said to be (J,J')*-holomorphic* if its derivative is complex linear, that is

$$D\varphi(p) \circ J(p) = J'(p) \circ D\varphi(p) \in \mathrm{Hom}(T_p M, T_{\varphi(p)} M')$$

for all $p \in M$.

Of course this game can also be played on manifolds whose complex structure on the tangent bundle arises from the complex nature of the manifold. The one dimensional case — that is, of Riemannian surfaces — is of particular interest.

Definition 2.6 (J-holomorphic curve). A J-holomorphic curve $u : \Sigma \to M$ is a (j,J)-holomorphic map from a Riemann surface (Σ, j) to an almost complex manifold (M,J).

[a] By definition such a J together with ω defines a Riemannian metric on M, so this property follows from the analogue for Riemannian metrics.

[b] That is, a pair of a manifold and an almost complex structure on it.

Like many concepts in geometry, the theory of symplectic forms starts at the level of linear algebra. Considering vector bundles with a symplectic structure is thus a natural step.

Definition 2.7 (Symplectic vector bundle). A symplectic vector bundle over a manifold M is a pair (E, ω) consisting of a real vector bundle $\pi : E \to M$ and a family of symplectic bilinear forms $\omega \in \Gamma(M, E^* \wedge E^*)$ that are non-degenerate in a similar fashion as in (2.1).

Example 2.8. Let $E \to M$ be any vector bundle. Then $E \oplus E^*$ is a symplectic vector bundle with symplectic form given by

$$\omega_{can}((v,a),(w,b)) := b(v) - a(w).$$

for $(v,a),(w,b) \in E \oplus E^*$.

2.1.1. Hamiltonian dynamics

Symplectic geometry arose traditionally as a tool to study celestial mechanics or more generally Hamiltonian systems. It is thus no surprise that Hamiltonian functions play an important role in the theory.

Definition 2.9 (Hamiltonians and co.). A (time-dependent) Hamiltonian, is a smooth function

$$H : S^1 \times M \to \mathbb{R},$$

which we also denote by $H_t(\cdot) = H(t, \cdot)$. The Hamiltonian vector field $X_{H_t} \in \Gamma(TM)$ or also simply X_H or X_t associated to H is the unique (time-dependent) vector field such that

$$dH_t = i_{X_t}\omega \in \Omega^1(M).$$

We denote by φ_H^t the *Hamiltonian isotopy*

$$\partial_t \varphi_H^t = X_t(\varphi_H^t) \quad \varphi_H^0 = \mathrm{id}$$

generated by H and by $\varphi_H^1 =: \varphi_H$ time-1 maps. These are called *Hamiltonian diffeomorphism* and form the group

$$\mathrm{Ham}(\omega, M) := \{\varphi_H : H \text{ is a Hamiltonian on } M\}.$$

Depending on the situation this set will be denoted as $\mathrm{Ham}(M) = \mathrm{Ham}(M, \omega) = \mathrm{Ham}(\omega)$.

Let now H be a Hamiltonian. For a Hamiltonian diffeomorphism $\varphi_H \in \mathrm{Ham}(M)$ and a fix point $p \in \mathrm{fix}(\varphi_H)$, the curve $\gamma_p(t) = \varphi_H^t(p)$ satisfies

$$\dot{\gamma_p}(t) = X_H(\gamma_p(t)).$$

For p being a fix point, γ_p must be a 1-periodic orbit of X_H. Indeed,

$$\gamma_p(0) = \varphi_H^0(p) = p = \varphi_H(p) = \varphi_H^1(p) = \gamma_p(1).$$

Thus we call γ_p a 1-periodic orbit of X_H and write $\gamma_p \in \mathcal{P}(H)$. We denote by $\mathcal{P}_0(H) \subset \mathcal{P}(H)$ the subset of contractible periodic orbits.

One notices that there is a bijection $\mathcal{P}(H) \leftrightarrow \mathrm{fix}(\varphi_H)$ given by $\gamma(t) \mapsto \gamma(0)$. A periodic orbit $\gamma \in \mathcal{P}(H)$ corresponding to p is said to be *non-degenerate* if the linear map

$$D\varphi_H(p) : T_pM \to T_pM \tag{2.2}$$

does not have 1 as eigenvalue. That is, $\det(\mathrm{id} - D\varphi_H(p)) \neq 0$.

2.2. Hamiltonian Floer homology

As mentioned in the introduction, Floer homology was first introduced by FLOER in 1988. In this section we introduce the Hamiltonian version of this homology theory, mainly following [Aud13] and [Mer13], where the interested reader can find all the proofs that we decided to omit because of conciseness. For this first introduction we limit ourselves to a set of symplectic manifolds that behave particularly nicely (see Assumption 2.10). Further development is presented in Section 2.4.

A symplectic manifold (M, ω) is said to be *symplectically aspherical* if for every smooth map $\varphi : S^2 \to M$, we have

$$\int_{S^2} \varphi^*\omega = 0.$$

For the rest of Section 2.2 we assume the following.

ASSUMPTION 2.10. We consider a symplectic manifold (M, ω) for which

(1) M is compact.
(2) (M, ω) is symplectically aspherical.
(3) The first Chern class vanishes on the image of $\pi_2(M)$ in $H_2(M)$ under the Hurewicz homomorphism: $c_1(\pi_2(M)) = 0$.

Furthermore, we fix an almost complex structure J compatible with ω.

2.2.1. Morse theory

Hamiltonian Floer homology is a Morse theory on the loop space of a symplectic manifold. The loop space we are particularly interested in is the one of contractible smooth loops

$$\mathcal{L}M := C^\infty_{\mathrm{contr}}(S^1, M)$$

with elements mostly denoted by $\gamma\colon S^1 \to M$. This space has only the structure of a Frechet manifold; it will thus sometimes be convenient to work with the completed loop space $\overline{\mathcal{L}}M := W^{1,2}_{\mathrm{contr}}(M)$ with respect to the $W^{1,2}$-Sobolev norm. This completion gives $\overline{\mathcal{L}}M$ the structure of a Banach manifold, which is nicer than the one of a Frechet manifold. Nevertheless, we won't consider this analytical finesse. The interested reader may find the details in any of [Mer13; Aud13; MS17].

In Floer homology the role of the Morse function is played by the action functional.

DEFINITION 2.11 (Action functional). Let $H : S^1 \times M \to \mathbb{R}$ be a 1-periodic Hamiltonian. We denote by

$$\begin{aligned} \mathcal{A}_H : \mathcal{L}M &\to \mathbb{R} \\ \gamma &\mapsto \mathcal{A}_H(\gamma) := \int_D \widetilde{\gamma}^* \omega + \int_0^1 H_t(\gamma(t))dt \end{aligned}$$

the action functional of H. Here $\widetilde{\gamma}$ is any extension of γ to the disk, that is $\widetilde{\gamma} : \{z \in \mathbb{C} : |z| \leq 1\} \to \mathbb{R}$ such that $\widetilde{\gamma}|_{S^1} = \gamma$ seeing $S^1 = \partial\{z \in \mathbb{C} : |z| \leq 1\}$.

The functional $\mathcal{A}_H$ depends a priori on the extension $\widetilde{\gamma}$. However, Assumption 2.10, 2 allows us to show that it actually does not and thus its well-definedness.

Morse theory is constructed using the critical points of the Morse function. The following lemma shows that these are well-known for the action functional.

LEMMA 2.12. *A loop $\gamma \in \mathcal{L}M$ is a critical point of $\mathcal{A}_H$ if and only if γ is a contractible periodic solution of the Hamiltonian system, that is*

$$\operatorname{crit}(\mathcal{A}_H) = \mathcal{P}_0(H).$$

We have seen in Definition 2.3 that a ω-tame almost complex structure J defines a Riemannian metric on M via $g(\xi, \zeta) := \omega(\xi, J\zeta)$. We can even lift g to define a non-complete metric on the loop space. We firstly identify the tangent space $T_\gamma \mathcal{L}M$ with $\Gamma_\gamma(TM)$. The metric is then given for two vector fields X and Y along γ via

$$g_\gamma^{\mathcal{L}M}(X, Y) := \int_0^1 g_{\gamma(t)}(X(t), Y(t))dt.$$

Having a metric we can talk about the gradient of the action functional $\operatorname{grad}_\gamma \mathcal{A}_H \in T_\gamma \mathcal{L}M$, which is uniquely determined by the equation

$$g_\gamma^{\mathcal{L}M}(\operatorname{grad}_\gamma \mathcal{A}_H, X) = d\mathcal{A}_H(\gamma)(X).$$

From these equalities, one may show that the gradient of $\mathcal{A}_H$ is given by

$$(\operatorname{grad}_\gamma \mathcal{A}_H)(t) = J_{\gamma(t)}(\dot{\gamma}(t)) + \operatorname{grad}_{\gamma(t)} H_t$$

where $\operatorname{grad}_{\gamma(t)} H_t$ is the gradient with respect to the metric g. So the integral curves of the negative gradient of the action functional are maps

$$\begin{aligned} \mathbb{R} &\to \mathcal{L}M \\ s &\mapsto u(s) \end{aligned}$$

satisfying the Floer equation

$$\partial_s u = -J_{u(s,t)}(\partial_t u) - \operatorname{grad}_{u(s,t)} H_t(u(s,t)). \tag{2.3}$$

The notation used in (2.3) suggests that we will usually consider $u : \mathbb{R} \to \mathcal{L}M$ as maps $u : \mathbb{R} \times S^1 \to M$ with variables $(s, t) \in \mathbb{R} \times S^1$.

Not every function is a Morse function; there is a regularity condition that must be satisfied. In order to show this for the action functional, we need the concept of vertical derivative.

Let $E \to M$ be a vector (or Banach) bundle over M. Remark that for any base point $p \in M$ the tangent space of the zero section splits as $T_{0_p}E \cong T_pM \oplus E_p$. We may thus consider the projection $\mathrm{pr}_p \colon T_{0_p}E \to E_p$. Let $\sigma \in \Gamma(E)$ be a section and consider a base point $p \in M$ with $\sigma(p) = 0_p$. The *vertical derivative* of σ at p, denoted by $D^v\sigma(p)$, is the map

$$\begin{aligned} D^v\sigma(p) : T_pM &\to E_p \\ \xi &\mapsto \mathrm{pr}_p(D\sigma(p)\xi). \end{aligned}$$

We say that a zero p of σ is *regular* if the vertical derivative at p is surjective.

Of our interest is the Banach bundle $E \to \overline{\mathcal{L}}M$ with fibre over γ given by $L^2(S^1, \gamma^*TM)$. On this bundle, the gradient $\operatorname{grad}\mathcal{A}_H$ defines a section and the following lemma shows that regular zeros of $\operatorname{grad}\mathcal{A}_H$ correspond to non-degenerate periodic orbits.

LEMMA 2.13. *A periodic orbit $\gamma \in \mathcal{P}_0(H)$ is a regular zero of* $\operatorname{grad}\mathcal{A}_H$ *if and only if γ is a non-degenerate element of $\mathcal{P}_0(H)$ in the sense of* (2.2).

Exactly as in Morse theory, there is an abundance of Hamiltonians sufficiently nice to work with.

THEOREM 2.14. *There exists a subset $\mathcal{H}_{reg} \subset C^\infty(S^1 \times M)$ of second category with the property that each element $H \in \mathcal{H}_{reg}$ has only non-degenerate periodic orbits.*

ASSUMPTION 2.15 (Hamiltonians). For the remaining of Section 2.2 we fix an Hamiltonian $H \in \mathcal{H}_{\mathrm{reg}}$ and denote by $\varphi = \varphi_H$ the Hamiltonian diffeomorphism associated to it.

2.2.2. Gradient flow lines of the action functional

We define the Floer operator $\bar\partial_{J,H}$ to be

$$\bar\partial_{J,H}(u) = \partial_s u + J(u)\partial_T u - J(u)X_{H_t}(u)$$

so that u is a solution of the Floer equation (2.3) if and only if $\bar\partial_{J,H}(u) = 0$.

In this subsection, we show that the set of solutions of the Floer equation has nice properties.

The most important theorem is know as elliptic regularity of the Floer operator. This says that solutions of the Floer equation are always smooth.

THEOREM 2.16 (Elliptic regularity). *Let $u \in C^1(\mathbb{R} \times S^1, M)$ be a zero of the Floer operator $\bar\partial_{J,H}(u) = 0$. Then $u \in C^\infty(\mathbb{R} \times S^1, M)$.*

Corollary 2.17. Every C^1 solution of the Floer equation is of class C^∞. Moreover the topologies C^0_{loc}, C^1_{loc} and C^∞_{loc} all agree on the space of solutions.

We define the energy function of a solution u of (2.3) by

$$E(u) := -\int_{\mathbb{R}} \frac{\partial}{\partial s}\mathcal{A}_H(u(s))ds = \int_{\mathbb{R}\times S^1} \left|\frac{\partial u}{\partial s}\right|^2 dsdt. \tag{2.4}$$

The energy of any solution u is non negative. It is zero if and only if u is independent of s. This is equivalent to u being equal to a critical point of $\mathcal{A}_H$. If a solution u connects two critical points $\gamma^\pm$ of $\mathcal{A}_H$, meaning that

$$\lim_{s\to\pm\infty} u(s,t) = \gamma^\pm,$$

then

$$E(u) = \mathcal{A}_H(\gamma^-) - \mathcal{A}_H(\gamma^+) < \infty. \tag{2.5}$$

We will now see that the converse of (2.5) holds as well, that is, solutions with finite energy connect critical points. This is the same behaviour of trajectories of Morse gradient flows (see Figure 2.1). We present an approach in three steps ending with Proposition 2.21 and motivating Definition 2.22.

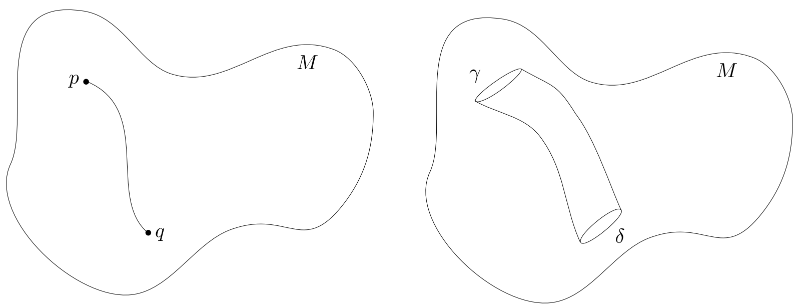

Figure 2.1. Floer theory is a Morse theory on the loop space.

The first step says that if the energy is finite, then there exists critical points taking the value of the limit of the action functional.

Lemma 2.18. *Let u a solution of* (2.3) *with $E(u) < \infty$. Then there exist two critical points $\gamma^\pm$ of $\mathcal{A}_H$ such that*

$$\lim_{s\to-\infty} \mathcal{A}_H(u(s)) = \mathcal{A}_H(\gamma^-) \quad \text{and} \quad \lim_{s\to\infty} \mathcal{A}_H(u(s)) = \mathcal{A}_H(\gamma^+).$$

The second step is showing that there are only finitely many critical points of the action functional if the Hamiltonian is regular.

Lemma 2.19. *If all periodic trajectories of X_t are non-degenerate, then there are finitely many critical points of $\mathcal{A}_H$ (i.e. periodic trajectories of X_t).*

The last preparatory step is showing that u has a subsequence tending to a critical point.

Lemma 2.20. *Let u a solution of the Floer equation with finite energy. Let also (s_k) be a sequence in $\mathbb{R}$ with $\lim_{k\to\infty} s_k = \infty$. Then there exists a subsequence $(s_{s'})$ of (s_k) and a critical point δ of $\mathcal{A}_H$ such that*

$$\lim_{k'\to\infty} u(s_{k'}) = \delta.$$

We can finally put everything together and state that solutions of finite energy tend to critical points.

Proposition 2.21. *Suppose that integral curves of the Hamiltonian vector fields $X_t = X_{H_t}$ are non-degenerate. Then for every solution of the Floer equation with finite energy, there exist two critical points $\gamma_\pm$*

$$\lim_{s\to-\infty} u(s) = \gamma_- \quad \textit{and} \quad \lim_{s\to-\infty} u(s) = \gamma_+$$

in $C^\infty(S^1, M)$. Moreover,

$$\lim_{s\to\pm\infty} \partial_s u(s,t) = 0$$

uniformly in t seeing u as $u : \mathbb{R}\times S^1 \to M$.

Proposition 2.21 motivates the following definition, making it the space of solutions that connect critical points.

Definition 2.22 (Moduli space of solutions). We define the moduli space of contractible solutions of the Floer equation with finite energy:

$$\mathcal{M}(H,J) := \left\{ u \in C^\infty(\mathbb{R}\times S^1, M) \;\middle|\; \begin{array}{c} u \text{ is contractible,} \\ \bar\partial_{J,H}(u) = 0 \text{ and } E(u) < \infty \end{array} \right\}.$$

Moreover, we will denote the subspace having fix ends by

$$\mathcal{M}(\gamma,\delta,H,J) := \{u \in \mathcal{M}(H,J) : \lim_{s\to-\infty} u = \gamma \text{ and } \lim_{s\to\infty} u = \delta\}$$

where γ, δ are critical points of the action functional.

If the Hamiltonian and the almost complex structure are clear from the context we will just write $\mathcal{M}$ or $\mathcal{M}(\gamma,\delta)$.

The space $\mathcal{M}$ plays a major role in defining Floer homology. There is a natural right action that comes with it: $\mathcal{M} \curvearrowleft \mathbb{R}$ given by

$$\begin{aligned} &\mathcal{M}\times\mathbb{R} \to \mathcal{M} \\ &(u(\cdot),\sigma) \mapsto u(\cdot+\sigma). \end{aligned} \tag{2.6}$$

2.2.3. Compactness of the moduli space of solutions

In this subsection, we will derive a compactness result due to Gromov. This is one of the few proofs we present in this section. We chose this one because of its manageable size and for the exposition of the concept of bubbling. We follow [Aud13, Section 6.6]. We recall that we work under Assumption 2.10, so in particular we have $\omega(A) = 0$ for all $A \in \pi_2(M)$.

The first lemma belongs to a course on metric spaces.

LEMMA 2.23 (Half maximum). *Let $g\colon X \to \mathbb{R}_{>0}$ be a continuous function on a complete metric space. Let furthermore x_0 be a point and $\varepsilon_0 > 0$. Then there exists $y \in X$ and $0 < \varepsilon \leq \varepsilon_0$ such that*

$$\begin{cases} d(y, x_0) \leq 2\varepsilon \\ \varepsilon g(y) \geq \varepsilon_0 g(x_0) \\ g(x) \leq 2g(y) \qquad \forall x \in B_\varepsilon(y) \end{cases}.$$

Lemma 2.23 is an important ingredient of the following proposition, which will make the proof of the Gromov compactness theorem a simple corollary of elliptic regularity.

PROPOSITION 2.24 (Bounded gradient). *There exists a universal $A > 0$ such that for all $u \in \mathcal{M}$ and for all $(s,t) \in \mathbb{R} \times S^1$*

$$\| \operatorname{grad}_{(s,t)} u\| \leq A.$$

PROOF. Let $u\colon \mathbb{R} \times S^1 \to M$ be a Floer solution. We consider its continuation as a periodic map $\mathbb{R}^2 \to M$ that we still denote by u. Assume for the sake of contradiction that there exists a sequence of elements $u_k \in \mathcal{M}$ and a sequence $(r_k) = (s_k, t_k) \subset \mathbb{R}^2$ such that

$$\| \operatorname{grad}_{r_k} u_k\| \longrightarrow +\infty.$$

It is possible to make this sequence a little smaller keeping its unbounded asymptotic. That is, there exists a sequence $(\varepsilon_k) \in \mathbb{R}_{>0}$ such that $\varepsilon_k \to 0$ and

$$\varepsilon_k \| \operatorname{grad}_{r_k} u_k\| \longrightarrow +\infty.$$

Lemma 2.23 applied to the functions $g_k\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $g_k(r) = \| \operatorname{grad}_r u_k\|$ provides new sequences (which we substitute for the old ones, keeping the same notation) ε_k and r_k with the two properties

$$\varepsilon_k \| \operatorname{grad}_{r_k} u_k\| \longrightarrow +\infty \tag{2.7}$$
$$2\| \operatorname{grad}_{r_k} u_k\| \geq \| \operatorname{grad}_r u_k\| \quad \forall r \in B_{\varepsilon_k}(r_k). \tag{2.8}$$

We lighten a bit the notation defining $R_k = \| \operatorname{grad}_{r_k} u_k\|$ and we introduce a new sequence of functions

$$\begin{aligned} v_k\colon \mathbb{R}^2 &\to M \\ r &\mapsto u_k\left(\frac{r}{R_k} + r_k\right). \end{aligned}$$

A direct computation shows that

$$\operatorname{grad}_r v_k = \frac{1}{R_k} \operatorname{grad}_{\frac{r}{R_k}+r_k} u_k \tag{2.9}$$

so that for $r = (0,0)$ this reduces to

$$\operatorname{grad}_{(0,0)} v_k = \frac{1}{R_k} \operatorname{grad}_{r_k} u_k.$$

By definition of R_k, $\|\operatorname{grad}_{(0,0)} v_k\| = 1$, while the gradient remains uniformly bounded by 2 in a neighbourhood of the origin $B_{\varepsilon_k R_k}((0,0))$:

$$\operatorname{grad}_r v_k \overset{(2.9)}{=} \frac{1}{R_k} \operatorname{grad}_{\frac{r}{R_k}+r_k} u_k \overset{(2.8)}{\leq} \frac{2}{R_k} \operatorname{grad}_{r_k} u_k \leq 2.$$

Again by (2.9), and as the u_k are solutions of the Floer equation, the v_k satisfy

$$\partial_s v_k + J(v_k)\partial_t v_k + \frac{1}{R_k} \operatorname{grad}_{\frac{r}{R_k}+r_k} H = 0$$

so that we may apply elliptic regularity. This means that, after possibly reducing to a subsequence, $(v_k)_k$ tends to a limit $v \in C^\infty_{loc}(\mathbb{R}^2, M)$ that is J-holomorphic and satisfies

$$\begin{cases} \|\operatorname{grad}_{(0,0)} v\| = 1 \\ \|\operatorname{grad}_r v\| \leq 2 \qquad \forall r \in \mathbb{R}^2 \end{cases}. \tag{2.10}$$

<u>v has finite energy</u>. We show now that v has finite energy. Denote by $B_k = B_{\varepsilon_k}(r_k)$; then

$$\int_{B_{\varepsilon_k R_k}((0,0))} \|\operatorname{grad} v_k\|^2 dr = \int_{B_k} \|\operatorname{grad} u_k\|^2 dr \overset{\diamond}{\leq} 3C + 2\int_{B_k} \|X_t\|^2 dtds$$

where C is a universal constant bounding the action functional $-C \leq \mathcal{A}_H(u) \leq C$ and the energy $0 \leq E(u) \leq C$ for every solution of the Floer equation u. Obviously we have skipped a few steps in $\diamond$. As the radius of B_k tends to zero as k grows, the last integral tends to zero as well, so that for k large enough

$$\int_{B_{\varepsilon_k R_k}((0,0))} \|\operatorname{grad} v_k\|^2 dr \leq 4C.$$

But $\bigcup B_{\varepsilon_k R_k}((0,0)) = \mathbb{R}^2$, so by Fatou's lemma

$$E(v) = \int_{\mathbb{R}\times S^1} |\partial_s u|^2 dsdt < \infty.$$

<u>v has finite and nonzero area</u>. This follows from a direct computation. In the first place we notice that by (2.10), v is not constant and thus has non-vanishing area

$$\begin{aligned} 0 \neq \int_{\mathbb{R}^2} v^*\omega &= \int_{\mathbb{R}^2} \omega(\partial_s v, \partial_t v) ds dt \stackrel{\star}{=} \int_{\mathbb{R}^2} \omega(-J(v)\partial_t v, \partial_t v) ds dt \\ &= \int_{\mathbb{R}^2} \omega(\partial_t v, J(v)\partial_t v) ds dt = \int_{\mathbb{R}^2} \|\partial_t v\|^2 ds dt < \infty \end{aligned} \tag{2.11}$$

where the last inequality follows form the previous step and in $\star$ we used that v is J-holomorphic.

On the other hand, the next step will show that in some region, v has a sort of a spike as shown in Figure 2.2.

<u>v has a spike</u>. For v being J-holomorphic, the form $v^*\omega$ is symplectic. Thus using polar coordinates we can write $v^*\omega_{\rho,\theta} = f(\rho,\theta)\rho d\theta \wedge d\rho$ for $f > 0$. This induces a Riemannian metric $f(\rho,\theta)(d\rho^2 + \rho^2 d\theta^2)$ that allows to compute $\ell(r) := r\int_0^{2\pi} \sqrt{f(r,\theta)} d\theta$, the length of the boundary $v(\partial B_r)$. The same can be done with the area $A(r) = \int_{B_r} v^*\omega$ showing that $A'(r) = \frac{dA(r)}{dr} = \int_0^{2\pi} f(r,\theta) r d\theta$, so that applying the Cauchy–Schwarz inequality, we get

$$\ell(r) \leq r\sqrt{2\pi \frac{A'(r)}{r}}.$$

Since A is bounded, we have that

$$0 = \lim_{k\to\infty} \frac{A(k^2) - A(k)}{\ln k} = \lim_{k\to\infty} r_k A'(r_k)$$

for $k \leq r_k \leq k^2$. This shows that $\ell(r)$ tends to zero as r tends to infinity.

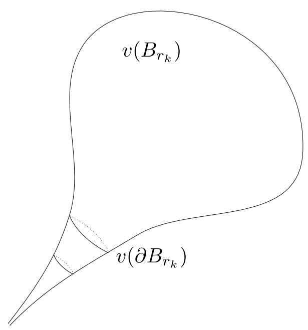

Figure 2.2. A bubble

<u>Contradiction: ω does not vanish</u>. By the previous step, the image $v(\partial B_{\rho_k}((0,0))$ tends to a point in M. So for k big enough, the image is contained in a Darboux chart $U \subset M$ that we may assume to be a closed ball. On that chart ω is exact $\omega = d\theta$. Furthermore $v(\partial B_r)$ is the

boundary of a small disc $D_\rho \subset U$ so that $D_\rho \cup v(B_\rho) = S^2_\rho$. Now, on one hand Assumption 2.10 says

$$\omega(S^2_\rho) = 0. \tag{2.12}$$

But on the other,

$$\begin{aligned}\omega(S^2_\rho) &= \int_{D_\rho} \omega + \int_{v(B_\rho)} \omega = \int_{D_\rho} d\lambda + \int_{v(B_\rho)} \omega = \int_{v(\partial B_\rho)} \lambda + \int_{v(B_\rho)} \omega \\ &\longrightarrow 0 + \int_{\mathbb{R}^2} v^*\omega. \end{aligned} \tag{2.13}$$

Inserting (2.11) into (2.13) and comparing the latter with (2.12) leads to contradiction and thus finishes the proof. ∎

The rather lengthy proof of Proposition 2.24 is the main ingredient for the compactness of the moduli space.

Theorem 2.25 (Gromov compactness). *For a symplectically aspherical compact manifold (M, ω) the space $\mathcal{M}$ is compact in $C^\infty_{loc}(\mathbb{R} \times S^1, M)$.*

Proof. Let u_n be a sequence in $\mathcal{M}$. Then it is enough to show that u_n admits a subsequence converging in $C^0_{loc}(\mathbb{R} \times S^1, M)$ such that the limit is smooth and a solution of the Floer equation. This is enough as the topologies agree by Corollary 2.17.

By Proposition 2.24 we have equicontinuity of elements in $\mathcal{M}$, so by the Arzelà–Ascoli theorem we do have a limit in the continuous case. Elliptic regularity shows the other required properties of this limit. ∎

2.2.4. Index computation

Morse theory relies on the fact that one can assign an index to critical points. In that framework, the index is given by the number of negative eigenvalues of the Hessian. In this section, we develop a similar index for critical points of the action functional.

We denote by

$$\mathrm{Sp}(2n)^\star := \{A \in \mathrm{Sp}(2n) : \det(A - \mathrm{id}) \neq 0\}.$$

the complement of the hypersurface $\Sigma = \{A \in \mathrm{Sp}(2n) : \det(A - \mathrm{id}) = 0\}$ in $\mathrm{Sp}(2n)$ Denote the two connected components of $\mathrm{Sp}(2n)$ by

$$\begin{aligned}\mathrm{Sp}(2n)^+ &= \{A \in \mathrm{Sp}(2n;\mathbb{R}) : \det(A - \mathrm{id}) > 0\} \\ \mathrm{Sp}(2n)^- &= \{A \in \mathrm{Sp}(2n;\mathbb{R}) : \det(A - \mathrm{id}) < 0\}\end{aligned}$$

and fix two matrices in these sets

$$W^+ := -\mathrm{id} \quad \text{and} \quad W^- := \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -\mathrm{id}_{2n-2}\,. \end{pmatrix}.$$

Let $\gamma(t) = \varphi^t(p)$ be the periodic orbit corresponding to a fixed point p of the Hamiltonian diffeomorphism φ. Choose a symplectic basis

$$Z(0) = (Z_1(0), \ldots, Z_{2n}(0))$$

for T_pM. As the Hamiltonian flow preserves the symplectic form, the matrix $A(1)$ of $D\varphi^1(p)$ in the above basis is symplectic, i.e. $A(1) \in \mathrm{Sp}(2n)$. As γ is not degenerate, $A(1)$ does not have 1 as eigenvalue. It is possible to find a family of symplectic bases $Z(t)$ of $T_{\gamma(t)}M$ smooth in t [Aud13, Theorem 7.1.1]. We can thus consider the map

$$D\varphi^t(p) : T_pM \to T_{\gamma(t)}M$$

and its matrix $A(t)$ in the bases $Z(0)$ and $Z(t)$. This defines a path

$$\begin{aligned} A_\gamma : [0,1] &\to \mathrm{Sp}(2n)^* \\ t &\mapsto A(t) \end{aligned} \tag{2.14}$$

starting at $A(0) = \mathrm{id}$. As a consequence of the same theorem cited above for the existence of a symplectic trivialisation along γ, we have that the path A_γ from (2.14) is unique (i.e. independent of the choice of frame Z) up to homotopy.

The following theorem is a key technical instrument towards the definition of the index.

Theorem 2.26 (The ρ map). *For every $n \in \mathbb{N}$, there exists a continuous map*

$$\rho : \mathrm{Sp}(2n) \to S^1$$

with the following properties. Let $A, T \in \mathrm{Sp}(2n)$ and $B \in \mathrm{Sp}(2m)$.

Naturality: $\rho(TAT^{-1}) = \rho(A)$.

Product:

$$\rho \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \rho(A)\rho(B).$$

Determinant: *If $A \in U(n)$ is of the form*

$$A = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$

then $\rho(A) = \det(X + iY)$.

Normalisation: *Let m_0 be the total multiplicity of the negative real eignevalues of A, then $\rho(A) = (-1)^{m_0/2}$.*

Transposition and conjugation: $\rho(A^T) = \rho(A^{-1}) = \overline{\rho(A)}$.

Using the ρ map we are now able to define the index of a path of matrices. Let $\delta : [0,1] \to \mathrm{Sp}(2n)$ be any path and $\alpha : [0,1] \to \mathbb{R}$ a lift of $\rho \circ \delta$. We set

$$\Delta(\delta) \coloneqq \frac{\alpha(0) - \alpha(1)}{\pi} \in \mathbb{R}.$$

For $A \in \mathrm{Sp}(2n)^*$ if we choose a path δ_A in $\mathrm{Sp}(2n)^*$ from A to $W^\pm$ depending on the connected component of A, the homotopy class of δ_A is independent of choices. Thus, again by [Aud13, Theorem 7.1.1], $\Delta(\delta_A)$ depends only on A. We may so define

$$\begin{aligned} r : \mathrm{Sp}(2n)^* &\to \mathbb{R} \\ A &\mapsto \Delta(\delta_A). \end{aligned}$$

We are now set to define the index of a path in $\mathrm{Sp}(2n)^*$, the last step before defining the index of a periodic solution.

Definition 2.27 (Maslov index of a matrix path). Let $\psi : [0,1] \to \mathrm{Sp}(2n)^\star$ be a path in starting at id and ending in $\mathrm{Sp}(2n)^\star$. We define the Maslov index $\mu(\psi)$ of ψ by

$$\mu(\psi) \coloneqq \Delta(\psi) + r(\psi(1)).$$

The Maslov index enjoys some nice properties that we summarise here below.

Proposition 2.28. *The Maslov index is an integer. Two paths ψ_0 and ψ_1 are homotopic with endpoints in* $\mathrm{Sp}(2n)^\star$ *if and only if they have the same index. Moreover:*

- $|\det(\psi(1) - \mathrm{id})| = (-1)^{\mu(\psi)-n}$
- *If S is invertible and symmetric with norm $\|S\| < 2\pi$ and if $\psi(t) = \exp(tJ_0S)$, then $\mu(\psi) = \mathrm{ind}(S) - n$ with $\mathrm{ind}(S)$ the number of negative eigenvalues of S.*

Definition 2.29 (Maslov index). The Maslov index of a non-degenerate periodic solution γ of the Hamiltonian system is

$$\mu(\gamma) \coloneqq \mu(A_\gamma),$$

the Maslov index of the path associated to γ in $\mathrm{Sp}(2n)^\star$ in (2.14).

2.2.5. Dimension theory — Functional analysis

We will reduce this section to the bare minimum necessary to state the most important result and give an idea of where its proof comes from. In reality, there is a considerable amount of analysis behind it that one can catch up reading [Mer13, Chapter 5] or [Aud13, Chapter 8].

Theorem 2.30 (Dimension of moduli space). *Let H be a non degenerate Hamiltonian and J an almost complex structure. Then there exists a perturbation h such that $\mathcal{P}_0(H+h) = \mathcal{P}_0(H)$ and $\mathcal{M}(\gamma, \delta, H+h, J)$ is a manifold of dimension $\mu(\gamma) - \mu(\delta)$.*

Floer theory uses solutions of the Floer equation, which is an elliptical partial differential equation, as a building stone. Despite few explicit computations (notably by Gromov [Gro85]) showing the existence of exactly one solution, in modern symplectic geometry we never really compute solutions explicitly. Theorem 2.30 is the tool that allows us to work with solutions of the Floer equation without ever seeing one. It states that solutions exist and how many they are.

We will call (H, J) a *regular pair* if the statement of Theorem 2.30 holds.

2.2.6. The Floer chain complex

Definition 2.31 (Space of trajectories). Let γ and δ be two distinct critical points of $\mathcal{A}_H$. Then we define the space of trajectories connecting γ to δ as

$$\mathcal{L}(\gamma, \delta) = \mathcal{M}(\gamma, \delta)/\mathbb{R},$$

where $\mathbb{R}$ acts on $\mathcal{M}$ as in (2.6).

A solution of the Floer equation connecting γ to δ induces a trajectory between the two. However, in general, there are many solutions associated with the same trajectory.

The topology on $\mathcal{L}(\gamma, \delta)$ is the quotient topology. Thus $(\widetilde{u}_n)$ in $\mathcal{L}(\gamma, \delta)$ converges to $\widetilde{u} \in \mathcal{L}(\gamma, \delta)$ if and only if, seeing $\widetilde{u}_n$ and $\widetilde{u}$ as elements of $\mathcal{M}$, there is a sequence (s_n) in $\mathbb{R}$ so that $\widetilde{u}_n(s + s_n) \to \widetilde{u}(s)$ in $\mathcal{M}(\gamma, \delta)$.

Lemma 2.32. *The topology on $\mathcal{L}(\gamma, \delta)$ is Hausdorff.*

Solutions of the Floer equation behave nicely when taking limits. Although this is not quite sequential compactness, it induces compactness on the space of trajectories.

Proposition 2.33. *Let (u_n) be a sequence of elements of $\mathcal{M}(\gamma, \delta)$. Then, after possibly reducing to a subsequence also denoted by (u_n), there exist:*

- *critical points $\gamma_0 = \gamma, \gamma_1, \dots, \gamma_{\ell+1} = \delta$,*
- *sequences (s_n^k) for $0 \leq k \leq \ell$,*
- *solutions $u^k \in \mathcal{M}(\gamma_k, \gamma_{k+1})$,*

such that for ever $0 \leq k \leq \ell$

$$\lim_{n \to \infty} u_n \cdot s_n^k = u^k.$$

Corollary 2.34. Let $\mu(\gamma) - \mu(\delta) = 1$. Then the space $\mathcal{L}(\gamma, \delta)$ is compact.

We are now ready to define the underlying chain complex of Floer homology.

Definition 2.35 (Floer complex and differential). The Floer chain complex is constituted by

$$CF_k(H) = \left\{ \sum_{\substack{i \in I \\ |I| < \infty}} n_i \gamma_i : n_i \in \mathbb{Z}/2\mathbb{Z} \text{ and } \mu(\gamma_i) = k \right\}.$$

The Floer differential is then defined by

$$\begin{aligned} \partial : CF_k(H) &\to CF_{k-1}(H) \\ \gamma &\mapsto \sum_{\mu(\delta)=k-1} n(\gamma,\delta)\delta \qquad \text{extended linearly,} \end{aligned} \tag{2.15}$$

with $n(\gamma,\delta) = \#\mathcal{L}(\gamma,\delta)$.

As with Morse theory and many other aspects in geometry, signs and orientations are a plague that many (including the author) do not wish to encounter. One can work over $\mathbb{Z}/2\mathbb{Z}$ and just forget about them or work over $\mathbb{Z}$ and acknowledge that care is needed. We will work over $\mathbb{Z}/2\mathbb{Z}$ but never encounter a place where orientations actually matter, so one could just substitute it everywhere with $\mathbb{Z}$.

The well-definedness of (2.15) follows from Theorem 2.30 and Corollary 2.34 . These two results show that, under the given index assumption, $\mathcal{L}(\gamma,\delta)$ is a compact manifold of dimension 0.

Of course what one wants to show now is that $CF_\bullet(H)$ is a chain complex, that is $\partial^2 = 0$. For this, let γ and δ be two critical points of $\mathcal{A}_H$ with $\mu(\gamma) = \mu(\delta) + 2$. Then we define

$$\overline{\mathcal{L}}(\gamma,\delta) \coloneqq \mathcal{L}(\gamma,\delta) \cup \bigcup_{\substack{\eta: \\ \mu(\eta)=\mu(\gamma)+1}} \mathcal{L}(\gamma,\eta) \times \mathcal{L}(\eta,\delta).$$

Theorem 2.36. *Let (H,J) be a regular pair and let γ and δ be two periodic solutions of the Hamiltonian system with*

$$\mu(\gamma) = \mu(\delta) + 2.$$

Then $\overline{\mathcal{L}}(\gamma,\delta)$ is a compact manifold of dimension 1 with boundary

$$\partial\overline{\mathcal{L}}(\gamma,\delta) = \bigcup_{\substack{\eta: \\ \mu(\gamma)<\mu(\eta)<\mu(\delta)}} \mathcal{L}(\gamma,\eta) \times \mathcal{L}(\eta,\delta).$$

Corollary 2.37. The Floer differential ∂ of (2.15) respects $\partial \circ \partial = 0$ and makes thus $(CF_\bullet(H), \partial)$ into a chain complex. Its homology $HF_\bullet(H,J)$ is called Floer homology.

Proof. The only thing one has to notice is that

$$\sum_{\substack{\eta \\ \mu(\eta)=\mu(\gamma)+1}} n(\gamma,\eta)n(\eta,\delta) = \#\left(\bigcup_{\substack{\eta: \\ \mu(\gamma)<\mu(\eta)<\mu(\delta)}} \mathcal{L}(\gamma,\eta) \times \mathcal{L}(\eta,\delta) \right).$$

Then the result follows from Theorem 2.36 as the boundary of a compact manifold of dimension 1 is an even number of points. ∎

2.2.7. Invariance

In Morse theory, a Morse function is needed in order to define the Morse complex. However, one of the most important results in the theory is that the resulting homology theory does not depend on the Morse function one started with. This subsection is dedicated to the same statement for Floer homology concluding with Proposition 2.38.

In this subsection we will consider two *regular pairs* (H^a, J^a) and (H^b, J^b). We will furthermore assume that there is a smooth homotopy between the pairs, that is

$$H : \mathbb{R} \times S^1 \times M \to \mathbb{R} \qquad J : \mathbb{R} \to \operatorname{End}(TM)$$

$$\begin{cases} H(s,\cdot,\cdot) = H^a & s \leq -R \\ H(s,\cdot,\cdot) = H^b & s \geq R \end{cases} \qquad \begin{cases} J(s) = J^a & s \leq -R \\ J(s) = J^b & s \geq R \end{cases}$$

for some positive constant R. We will denote $H(s,t,p) = H^s_t(p)$ and $J(s) = J^s$ and the whole homotopy by Γ.

The new and revised version of the Floer equation is

$$\partial_s u + J^s(u)\partial_t u + \operatorname{grad}^s_u H^s_t = 0 \tag{2.16}$$

inducing the new moduli space

$$\mathcal{M}^\Gamma := \left\{ u \in C^\infty(\mathbb{R} \times S^1, M) \;\middle|\; \begin{array}{c} u \text{ contractible solution of} \\ (2.16) \text{ and } E(u) < \infty \end{array} \right\}.$$

The same game played in Section 2.2.5 may be done with $\mathcal{M}^\Gamma$ showing that, after some small perturbation not affecting the critical points of the action functional, the subset of this moduli space of solutions to (2.16) connecting a critical point γ of $\mathcal{A}_{H^a}$ to δ of $\mathcal{A}_{H^b}$ is a manifold of dimension $\mu(\gamma) - \mu(\delta)$. One can thus define

$$\begin{aligned} \Phi^\Gamma : CF_\bullet(H^a, J^a) &\to CF_\bullet(H^b, J^b) \\ \gamma &\mapsto \sum_{\substack{\delta: \\ \mu(\gamma)=\mu(\delta)}} n^\Gamma(\gamma,\delta)\delta \end{aligned} \tag{2.17}$$

where $n^\Gamma(\gamma,\delta)$ denotes the number of elements in $\mathcal{M}^\Gamma(\gamma,\delta)$.

The independence of the chosen regular pair is then an easy consequence of the following result.

Proposition 2.38. *Let* Φ^Γ *be the map from* (2.17). *Then:*

- Φ^Γ *is a morphism of complexes.*
- *If* $(H^a, J^a) = (H^b, J^b)$ *and* $\Gamma = \mathrm{id}$, *then* Φ^Γ *is the identity.*
- *Given three regular pairs* $(H^a, J^a), (H^b, J^b)$ *and* (H^c, J^c) *and two homotopies* Γ' *and* Γ'' *connecting* (H^a, J^a) *to* (H^b, J^b) *and* (H^b, J^b) *to* (H^c, J^c) *respectively. Then there exists a homotopy* Γ *connecting* (H^a, J^a) *to* (H^c, J^c) *such that* $\Phi^{\Gamma''} \circ \Phi^{\Gamma'}$ *and* Φ^Γ *induce the same homomorphism at the homology level.*

2.3. Gromov–Witten theory and quantum cohomology

In this section, we introduce the main ideas of the Gromov–Witten invariants from a symplectic point of view. This theory is important for the construction of a ring structure on quantum cohomology, the second object introduced in this section, although it is also of great interest on its own. The main reference is [MS12, Chapters 7 and 11] with some intuition and pictures form [PSS96].

2.3.1. Gromov–Witten Invariants

Heuristically, Gromov–Witten invariants are the number of specific J-holomorphic spheres in a symplectic manifold (M, ω). See for example the defining (2.19). Recall, however, that we usually work with perturbed data, so spheres are not the usual spheres.

Let (M, ω) be a symplectic manifold and $J \in \mathcal{J}_\tau(M, \omega)$. Fix also a homology class $A \in H_2(M)$ and $k \in \mathbb{N}_0$. We define the *moduli space of simple k-pointed J-holomorphic spheres in the class A*

$$\mathcal{M}^*_{0,k}(A; J)$$

to be the moduli space of equivalence classes of tuples $(u, z_1, \dots, z_k)$ where $u : S^2 \to M$ is a simple[c] J-holomorphic curve representing the class A and z_i are pairwise distinct points on S^2. The equivalence relation is given by the action of reparametrization of $\mathrm{PSL}(2, \mathbb{C})$. The definition for the case $A = 0$ needs particular care. If $k \geq 3$ we set $\mathcal{M}^*_{0,k}(0; J)$ to be the constant maps $u : S^2 \to M$ while for $k < 3$ we just define it to be the empty set as the costraints are too broad and too many curves would satisfy them.

The moduli space $\mathcal{M}^*_{0,k}(A; J)$ comes with a natural evaluation map

$$\begin{aligned} \mathrm{ev}\colon\ \mathcal{M}^*_{0,k}(A; J) &\to M^k \\ [u, z_1, \dots, z_k] &\mapsto (u(z_1), \dots, u(z_k)). \end{aligned} \tag{2.18}$$

In order to define the Gromov–Witten invariants, we need a suitable notion of a cycle.

Definition 2.39 (Pseudocycle). A d-dimensional *pseudocycle* in a manifold M is a smooth map

$$f : N \to M$$

[c]A *simple* J-holomorphic curve is a curve $u : \Sigma \to M$ that is not multiply covered, i.e. for all compact Riemann surfaces (Σ', j'), holomorphic curve $u' : \Sigma' \to M$ and holomorphic branched covering $\varphi : \Sigma \to \Sigma'$ such that $u = u' \circ \varphi$, we have $\deg(\varphi) \leq 1$.

defined on an oriented d-dimensional manifold N such that $f(N)$ has compact closure and

$$\dim \Omega_f \leq \dim N - 2 \quad \text{with} \quad \Omega_f := \bigcap_{\substack{K \subset N, \\ K \text{compact}}} \overline{f(V \setminus K)}.$$

Strictly speaking the dimension of Ω_f is not well-defined as it does not have to be a manifold. For an arbitrary set $L \subset M$ we say it is of *dimension at most d* if it is contained in the image of a smooth map defined on a manifold whose components have dimension less or equal to d.

Poincaré duality provides a bridge between homology and cohomology so that we can talk about the Poincaré dual of a homology class. In a similar flavour, a singular cohomology class $a \in H^\bullet(M)$ is said to be Poincaré dual to $(2n-k)$-dimensional pseudocycle $f : N \to M$ if

$$\int_X a = f \cdot X$$

holdsfor every closed oriented k-dimensional submanifold $X \subset M$. Here $f \cdot X$ is the intersection number.

Proposition 2.40. *Let (M, ω) be a closed semipositive (see Definition 2.48) symplectic $2n$-dimensional manifold and let $J \in \mathcal{J}_{reg}(M, \omega)$. Let $A \in H_2(M; \mathbb{Z})$ such that*

$$A = mB, \quad c_1(B) = 0 \Rightarrow m = 1$$

for every positive integer m and every $B \in \widetilde{\Gamma}$. Then the evaluation map

$$\mathrm{ev} : \mathcal{M}^*_{0,k}(A : J) \to M^k$$

is a pseudocycle of dimension $\mu(A, k) = 2n + 2c_1(A) + 2k - 6$.

Definition 2.41 (Gromov–Witten invariants). Let $a_i \in H^\bullet(M)$ be cohomology classes dual to submanifolds $X_i \subset M$ in general position for $1 \geq i \geq k$. Then the *Gromov–Witten* invariant $\mathrm{GW}^M_{A,k}(a_1, \ldots, a_k)$ is the number

$$\mathrm{GW}^M_{A,k}(a_1, \ldots, a_k) = |\{[u, z_1, \ldots, z_k] \in \mathcal{M}^*_{0,k}(A : J) : u(z_i) \in X_i\}|. \tag{2.19}$$

By Proposition 2.40 this is a moduli space of dimension $2n + 2c_1(A) - 2k - 6$.

This formula works modulo 2 whenever the degrees of the classes a_i sum up to the dimension of the moduli space, that is

$$\sum_i \deg(a_i) = 2n + 2c_1(A) + 2k - 6. \tag{2.20}$$

If condition (2.20) is not satisfied the invariant is defined to be zero.

LEMMA 2.42. *Let (M,ω) be a compact semipositive symplectic manifold and $A \neq 0$. Then*

$$\mathrm{GW}_{A,k}(a_1,\dots,a_{k-1},\mathrm{PD}([M])) = 0$$

for any choice of a_i.

2.3.2. Quantum (co-)homology

The goal of this section is to introduce a new homology theory, which is a deformation of the standard singular homology. This new homology theory is strongly dependent on the symplectic structure of the manifold and will allow us to compute Floer homology more easily (see Chapter 3).

We denote by $\widetilde{\Gamma}$ the image of the Hurewicz homomorphism $\pi_2(M) \to H_2(M)$ and the quotient

$$\Gamma = \frac{\widetilde{\Gamma}}{\ker\omega \cap \ker c_1}. \tag{2.21}$$

DEFINITION 2.43 (Novikov ring). Let Γ as in (2.21) and denote for an integer $k \in \mathbb{Z}$

$$\Gamma_k \coloneqq \{A \in \Gamma : 2c_1(A) = k\}.$$

For a commutative ring R we define the degree $k \in \mathbb{Z}$ Novikov ring as

$$\Lambda_k \coloneqq \left\{\sum n_j q^{A_j} : n_j \in R, A_j \in \Gamma_k\right\},$$

where the terms satisfy the finiteness condition

$$|\{A_j \colon n_j \neq 0, \omega(A_j) \leq c\}| < \infty \quad \forall c > 0$$

and q is a formal variable . The total *Novikov ring* is then $\Lambda = \bigoplus_k \Lambda_k$.

The commutative ring R is usually taken to be $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Q}$. We will mainly opt for the former when not specified otherwise. The above definition focuses on cohomology, meaning that for homology the sign of the grading is inverted. Both versions are clearly equivalent. Being a ring of formal sums the ring structure is the one of polynomial rings.

The Novikov ring is the coefficient ring of the new homology we will define now.

DEFINITION 2.44 (Quantum cohomology and homology). The *quantum cohomology* ring $QH^\bullet(M)$ is a Λ-module with underlying chain complex given by

$$QC^k(M) \coloneqq \bigoplus_{i+j=k} C^i(M) \otimes \Lambda_j$$

whereas the *quantum homology* is the Λ-module

$$QH_\bullet(M) \coloneqq H_\bullet(M;\Lambda).$$

If the manifold of interest is not compact, we implicitly take locally finite chains for the definition of quantum homology.

The last remark considering non compact manifolds is needed in order to make the duality statement Proposition 2.45 hold in all generality. The elements of $QH^k(M)$ are thus formal sums

$$a = \sum_{A\in\Gamma} a_A q^A \quad \alpha_A \in H^{k+2c_1(A)}(M, \mathbb{Z}/2\mathbb{Z}) \tag{2.22}$$

while elements of $QH_k(M)$ are formal sums

$$\alpha = \sum_{A\in\Gamma} \alpha_A q^A \quad \alpha_A \in H_{k-2c_1(A)}(M, \mathbb{Z}/2\mathbb{Z}) \tag{2.23}$$

both satisfying the finiteness condition of Definition 2.43.

Quantum cohomology enjoys many of the nice properties of singular homology. The two most important ones are the following.

PROPOSITION 2.45 (Pairing and duality). *There is a natural pairing induced by the pairing $\langle\cdot,\cdot\rangle$ between the homology and cohomology of M given by*

$$\begin{aligned}\langle\cdot,\cdot\rangle : QH_k(M) \times QH^k(M) &\to \Lambda_0\\ (\alpha, a) &\mapsto \langle\alpha, a\rangle = \sum_{\substack{A\in\Gamma:\\ c_1(A)=0}} \sum_{B\in\Gamma} \langle a_{A-B}, \alpha_B\rangle q^A\end{aligned}$$

for α as in (2.23) *and a as in* (2.22). *Furthermore, the Poincaré duality isomorphism* $\mathrm{PD} : H^k(M) \to H_{2n-k}(M)$ *between homology and cohomology induces an isomorphism*

$$\begin{aligned}\mathrm{PD}_Q : QH^k(M) &\to QH_{2n-k}(M)\\ \sum_{A\in\Gamma} a_A q^A &\mapsto \sum_{A\in\Gamma} \mathrm{PD}(a_A) q^A.\end{aligned}$$

Quantum cohomology and the Gromov–Witten invariants are closely related as the latter is used to define the ring structure on the former. In order to define the ring structure, it is enough to define the product of two elements $H^\bullet(M)$ as these generate $QH^\bullet(M)$. The product on $QH^\bullet(M)$ is called the *quantum cup product*, while its dual on $QH_\bullet(M)$ is called the *quantum intersection product*.

DEFINITION 2.46 (Quantum cup product). For two cohomology elements $a, b \in QH^\bullet(M)$ the quantum cup product $a * b \in QH^\bullet(M)$ is defined as

$$a * b := \sum_{A\in\Gamma} (a*b)_A q^A$$

where $(a*b)_A$ is the unique element such that for all c,

$$\int_M (a*b)_A \smile c := \mathrm{GW}^M_{A,3}(a, b, c).$$

PROPOSITION 2.47 (Properties of the quantum cup product).

(1) *The quantum cup product is distributive over addition and skew-commutative*

$$a * b = (-1)^{\deg(a)\deg(b)} b * a$$

for $a, b \in H^\bullet(M)$ *of pure degree. It is associative and commutes with the action of* Λ.

(2) *The leading term* $(a * b)_0$ *is the standard cup product*

$$(a * b)_0 = a \smile b$$

for all $a, b \in H^\bullet(M)$. *Moreover higher terms vanish whenever one of the classes is of degree zero or one*

$$a * b = a \smile b \quad \forall a \in H^i(M), i = 0, 1.$$

Thus $1 \in H^0(M)$ *is the unit in quantum cohomology.*

Proposition 2.47 might seem harmless, but its proof is not trivial. The interested reader may consult [MS12, Proposition 11.1.11].

When the quantum cup product agrees with the standard cup product, we say that the quantum cohomology is *undeformed*.

2.4. Non aspherical manifolds

In this section, we will lose two of the points in Assumption 2.10. Many results remain unchanged, but this generalisation allows us on the one hand to consider more symplectic manifolds and on the other hand, have a very useful result on the nature of Floer homology. The main reference for the section is [PSS96] while some details and background come from [Sal99].

DEFINITION 2.48 (Semipositive). A symplectic manifold (M, ω) is said to be *semipositive* if at least one of the following three conditions hold:

(1) There exists a $\lambda \geq 0$ such that for all $A \in \pi_2(M)$

$$c_1(A) = \lambda\omega(A).$$

(2) $c_1(A) = 0$ for all $A \in \pi_2(M)$.

(3) The minimal Chern number of (M, ω) is grater than or equal to $\dim(M) - 2$.

One notices that all symplectic manifolds of dimension less than or equal to six are semipositve.

ASSUMPTION 2.49. For this section we consider a symplectic manifold (M, ω) which is semipositive.

2.4.1. Floer homology revisited

As in Section 2.2 we will assume throughout the rest of this section that H is a time-dependent regular Hamiltonian on H and J an ω-compatible almost complex structure.

In Section 2.2.1 we introduced the contractible loops $\mathcal{L}M$ on M on which the action functional was successively defined. The proof that that action functional was well-defined does not extend to the non aspherical case, but it directly points to the correct generalisation of the loop space on which it will. The cover of $\mathcal{L}M$ introduced by [HS95] is given by

$$\widetilde{\mathcal{L}M} := \{(\gamma, v) | \gamma \in \mathcal{L}M, v : \mathbb{D} \to M, \gamma(t) = v(e^{2\pi i t})\}/\sim$$

with the equivalence relation

$$(\gamma, u) \sim (\delta, v) \Leftrightarrow \left(\gamma = \delta, \quad \int_{\mathbb{D}} u^* c_1 = \int_{\mathbb{D}} v^* c_1 \quad \text{and} \quad \int_{\mathbb{D}} u^* \omega = \int_{\mathbb{D}} v^* \omega\right),$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. We often denote an element $[\gamma, v] \in \widetilde{\mathcal{L}M}$ by $\widetilde{\gamma}$.

There is an action $\Gamma \curvearrowright \widetilde{\mathcal{L}M}$ of gluing spheres that we denote by $A\#\widetilde{\gamma} = A\#[\gamma, u] = [\gamma, A\#u]$. Explicitly, for $A \in \Gamma$ we choose $v \in A$ with $u(0) = v(\infty) = p \in M$ for some fixed point $p \in M$. Then the action is given by the connected sum

$$A\#u := v\#u(z) = \begin{cases} v\left(\frac{z}{1-2|z|}\right), & |z| \leq \frac{1}{2} \\ u\left(z\left(2 - |z|^{-1}\right)\right), & |z| \geq \frac{1}{2} \end{cases}$$

We use similar notation as before and consider the action functional

$$\begin{aligned} \mathcal{A}_H : \widetilde{\mathcal{L}M} &\to \mathbb{R} \\ [\gamma, v] &\mapsto \int_{\mathbb{D}} v^* \omega + \int_0^1 H_t(\gamma(t)) dt. \end{aligned}$$

With this new action functional (which is just the same as the old one for the aspherical case), for $A \in \widetilde{\Gamma}$ and $\widetilde{\gamma} \in \widetilde{\mathcal{L}M}$ we have

$$\mathcal{A}_H(A\#\widetilde{\gamma}) = \mathcal{A}_H(\widetilde{\gamma}) - \omega(A).$$

The set of contractible 1-periodic orbits will again be denoted by $\mathcal{P}(H)$. We denote by $\widetilde{\mathcal{P}}(H)$ the pairs $[\gamma, v]$ for which $\gamma \in \mathcal{P}(H)$. Integral curves along the negative gradient of the action functional are again solutions of the Floer equation

$$\partial_s u + J(u)\partial_t u + \operatorname{grad} H_t(u) = 0. \tag{2.24}$$

As before for two critical points $\widetilde{\gamma}^{\pm}$ we define the moduli space of solutions

$$\mathcal{M}(\widetilde{\gamma}^-, \widetilde{\gamma}^+) := \left\{ u : \mathbb{R} \times S^1 \to M \;\middle|\; \begin{array}{c} u \text{ is a contractible smooth solution} \\ \text{of (2.24) with } \lim_{s \to \pm\infty} u(s,t) = \widetilde{\gamma}^{\pm} \\ \text{for which } \widetilde{\gamma}^+ \# u = \widetilde{\gamma}^+ \end{array} \right\}.$$

The energy of a solution is defined as in (2.4) and as before solution of (2.24) have finite energy if and only if they connect critical points.

For a generic Hamiltonian the moduli space $\mathcal{M}(\widetilde{\gamma},\widetilde{\delta})$ is a manifold of dimension

$$\dim \mathcal{M}(\widetilde{\gamma},\widetilde{\delta}) = \mu_{CZ}(\widetilde{\gamma}) - \mu_{CZ}(\widetilde{\delta})$$

where the function $\mu_{CZ} : \widetilde{\mathcal{P}}(H) \to \mathbb{Z}$ is the *Conley–Zehnder* index, a variation of the Maslov index. A precise definition of μ_{CZ} would be superfluous for this text, so we limit ourself to notice that it has the nice property that

$$\mu_{CZ}(A\#\widetilde{\gamma}) = \mu_{CZ}(\widetilde{\gamma}) - 2c_1(A)$$

for $A \in \widetilde{\Gamma}$. When working with semipositive manifolds we will abuse notation and use μ for μ_{CZ}.

Definition 2.50. The Floer chain complex $CF_k(H)$ is defined to be the set of formal sums

$$\xi = \sum_{\substack{\widetilde{\gamma}\in\widetilde{\mathcal{P}}(H)\\ \mu(\widetilde{\gamma})=k}} \xi_{\widetilde{\gamma}}\widetilde{\gamma} \qquad \xi_{\widetilde{\gamma}} \in \mathbb{Z}/\mathbb{Z}.$$

Again, we impose the finiteness condition

$$|\{\widetilde{\gamma} \in \widetilde{\mathcal{P}} | \xi_{\widetilde{\gamma}} \neq 0, \mathcal{A}_H(\widetilde{\gamma}) \geq c\}| < \infty$$

for all $c \in \mathbb{R}$.

Floer homology $HF_k(H)$ is a module over the Novikov ring Λ via

$$\lambda \cdot \xi = \sum_{\substack{\widetilde{\gamma}\in\widetilde{\mathcal{P}}(H)\\ \mu(\widetilde{\gamma})=k}} \sum_{A\in\Gamma} \lambda_A \xi_{(-A)\#\widetilde{\gamma}}\widetilde{\gamma}$$

for λ and ξ as above.

The boundary map is given as in the aspherical case. We may now say that all the proofs given in Section 2.2 hold also for the non-aspherical case. The curious reader may find further details in [MS12].

Nothing new happens in cohomology either.

Definition 2.51 (Floer cohomology). The Floer cochain complex is given by the groups $CF^k(H)$ of formal sums

$$\eta = \sum_{\mu(\widetilde{\gamma})=k} \eta_{\widetilde{\gamma}}\widetilde{\gamma}$$

satisfying the opposite finiteness condition

$$|\{\widetilde{\gamma} \in \widetilde{\mathcal{P}} | \xi_{\widetilde{\gamma}} \neq 0, \mathcal{A}_H(\widetilde{\gamma}) \leq c\}| < \infty$$

for all $c \in \mathbb{R}$.

Also, the Floer cochain complexes carry a Novikov-module structure via

$$\lambda \cdot \eta = \sum_{\substack{\widetilde{\gamma} \in \widetilde{\mathcal{P}}(H) \\ \mu(\widetilde{\gamma})=k}} \sum_{A \in \Gamma} \lambda_A \eta_{(A)\#\widetilde{\gamma}} \widetilde{\gamma}$$

for λ and η as above. This allows us to construct a pairing

$$\begin{aligned} CF^k(H) \times CF_k(H) &\to \Lambda_0 \\ (\eta, \xi) &\mapsto \sum_{\substack{A \in \Gamma: \\ c_1(A)=0}} \sum_{\widetilde{\gamma}} \eta_{\widetilde{\gamma}} \xi_{A\#\widetilde{\gamma}} q^A. \end{aligned}$$

Remark furthermore that there is a one-to-one correspondence between solutions $\gamma \in \widetilde{\mathcal{P}}(H)$ and $\bar{\gamma} \in \widetilde{\mathcal{P}}(\bar{H})$, where $\bar{H}_t = -H_{-t}$, via $\bar{\gamma}(t) = \gamma(-t)$. This induces a correspondence in the cover $[\gamma, v] \leftrightarrow [\bar{x}, \bar{v}]$ with $\bar{v}(z) = v(\bar{z})$ so that

$$\mathcal{A}_{\bar{H}}([\bar{x}, \bar{v}]) = -\mathcal{A}_H([\gamma, v]) \quad \mu([\bar{x}, \bar{v}]) = 2n - \mu([\gamma, v]).$$

Furthermore, u is a solution for the Floer equation for H if and only if $\bar{u} := u(-s, -t)$ is a solution using the Hamiltonian $\bar{H}$.

Formally we thus have the following result.

Lemma 2.52. *There are isomorphisms of groups*

$$CF^k(H) \cong CF_{2n-k}(\bar{H})$$

which extend to natural isomorphism

$$HF^k(H) \cong HF_{2n-k}(\bar{H})$$

In order to show that the identification descends to homology, one has to show that boundaries and cycles get mapped to boundaries and cycles.

The composition of the above result with the Floer continuation map from $HF_\bullet(\bar{H})$ to $HF_\bullet(H)$ leads to the following corollary.

Corollary 2.53 (Poincaré duality). There exists a Poincaré duality isomorphism

$$\mathrm{PD}_F : HF^k(H) \to HF_{2n-k}(H)$$

and Poincaré duality pairings

$$HF^k(H,J) \times HF^{2n-k}(H,J) \to \Lambda_0 \qquad HF_k(H,J) \times HF_{2n-k}(H,J) \to \Lambda_0.$$

2.5. Symplectic cohomology

As mentioned in the introduction, symplectic geometry arose as a tool to study Hamiltonian dynamics, where the natural symplectic manifold one considers is the cotangent bundle of a smooth manifold with the Poincaré 2-form. This large and interesting set of symplectic manifolds are semipositive but not compact. So a further generalisation of the theory is needed in order to apply the powerful tools of Floer homology to cotangent bundles. For this section we mainly follow the respective sections in [Sei08; Rit09; Abo15].

Open symplectic manifolds are far too general to develop a reasonable Floer theory on them. So we will have to restrict our attention to a subclass having nice properties, in particular, behaving nicely outside of compact sets. Viterbo [Vit99] introduced the class most used in the literature: Liouville domains. These manifolds parametrise the manifold outside of a compact set (at infinity) via the Reeb flow of a bounded contact manifold.

2.5.1. Liouville domains

Definition 2.54 (Liouville domain). A compact manifold with boundary M together with a one-form $\theta \in \Omega^1(M)$ such that $\omega = d\theta$ is symplectic and such that the vector field $Z \in \mathcal{X}(M)$ given by $i_Z\omega = \theta$ points outwards along ∂M is called a *Liouville domain*.

Let us now fix a Liouville domain $(M, \omega = d\theta)$. For M being compact, Z has a flow defined for all negative times. This flow gives rise to a canonical collar

$$(-\infty, 0] \times \partial M$$

of ∂M inside M.

Let $\alpha = \theta|_{\partial M} \in \Omega^1(\partial M)$. Then, using the flow of Z, we may glue a symplectic cone

$$[0, \infty) \times \partial M \quad \text{with} \quad \omega_c := d(e^r\alpha)$$

where r is the coordinate on $[0, \infty)$. This gluing defines the *completion*

$$\widehat{M} := M \cup_{\partial M} [0, \infty) \times \partial M.$$

We call $\kappa := \mathbb{R} \times \partial M$ the *collar* of $\widehat{M}$. The completion comes with a natural exact symplectic form given by $\omega = d\widetilde{\theta}$ where

$$\widetilde{\theta} = \begin{cases} \theta & \text{on } M \\ e^r\alpha & \text{on } [0, \infty) \times \partial M \end{cases}. \tag{2.25}$$

We will abuse notation and call $\widetilde{\theta}$ just θ.

As promised, this section handles cotangent bundles, as the next lemma shows.

Lemma 2.55 (Cotangent bundle). *Let (M, g) a Riemannian manifold. Then the unit disk bundle*

$$D^*M := \{(q, p) \in T^*M : g^*(p, p) \leq 1\}$$

is a Liouville domain with the usual one form

$$\theta = qdp.$$

The completion $\widehat{D^*M}$ is symplectomorphic to the cotangent bundle T^*M.

Proof. The first part of the statement is clear. So one only has to show that $\widehat{D^*M}$ and T^*M are symplectomorphic. We denote by S^*M the unit cotangent bundle of M, that is ∂D^*M. By definition of attaching space, a map

$$\varphi\colon \widehat{D^*M} \to T^*M$$

is the same thing as two maps

$$\varphi_1\colon D^*M \to T^*M$$
$$\varphi_2\colon [0,\infty) \times S^*M \to T^*M$$

smoothly agreeing on S^*M. We take the most natural choices: $\varphi_1 \coloneqq \iota\colon D^*M \to T^*M$ and $\varphi_2(r,(q,p)) \coloneqq (q, e^r \cdot p)$. They clearly agree on ∂M, so it remains to show that they build a symplectomorphism. Bijectivity and

$$\varphi_2^*(dp \wedge dq) = d(e^r \cdot q) \wedge dp = d(e^r \cdot \alpha).$$

are both easy to check. ■

The exponential in (2.25) might seem arbitrary at first sight, but this is really the form we wish to have as after time t one typically has flown for distance $\log t$.

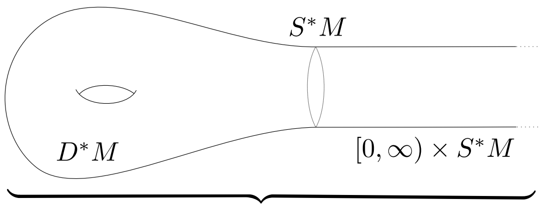

Figure 2.3. The cotangent bundle as completion of a Liouville domain.

The main problem of non-compact manifolds is the behaviour outside of compact sets. We tackle this by considering Hamiltonians on $\widehat{M}$ that have a special structure.

Definition 2.56. Suppose that an Hamiltonian $H \in C^\infty(\widehat{M})$ only depends on e^r on the collar, that is $H|_\kappa = h(e^r)$ for some $h \in C^\infty(\mathbb{R})$. Then we call H a Hamiltonian which is *conical at infinity*.

For Hamiltonians H conical at infinity the corresponding vector field has a nice form. Indeed, on the collar X_H is given by

$$X_H|_\kappa = h'(e^r)\mathcal{R}, \tag{2.26}$$

where $\mathcal{R}$ denotes the extension of the Reeb vector field associated to α on κ. We recall that the Reeb vector field determined by α is the unique vector field $\mathcal{R}$ such that $i_{\mathcal{R}} d\alpha = 0$ and $i_{\mathcal{R}}\alpha = 1$. In (2.26) we slightly cheated as $\mathcal{R}$ is a priori only defined on ∂M. Nonetheless, it may be easily extended to the whole collar.

Let γ be a non constant 1-periodic Hamiltonian orbit. By (2.26), if γ intersects the collar, then all of γ must live in $\{e^\rho\} \times \partial M$ for some $\rho \in \mathbb{R}$. Assume that δ is a T-periodic orbit of $\mathcal{R}$ and that $r_T \in \mathbb{R}$ is such that $h'(e^{r_T}) = T$. Then $\gamma(t) := (r_T, \delta(Tt))$ is a 1-periodic orbit of $X_H|_\kappa$.

Lemma 2.57. *Let (N, ξ) be a contact manifold with contact form α. We denote by* $\operatorname{Spec}(\alpha)$ *the* spectrum *of Reeb periods*

$$\operatorname{Spec}(\alpha) := \{T \in \mathbb{R} \colon \mathcal{R} \textit{ has an orbit of period } T\}.$$

Then $\operatorname{Spec}(\alpha)$ *is countable and closed.*

Lemma 2.57 together with the above findings on the relationship between Reeb orbits and Hamiltonian orbits show that taking h with constant slope

$$h'(e^r) = m \in \mathbb{R} \setminus \operatorname{Spec}(\alpha) \tag{2.27}$$

for $r > 1$ prevents the existence of 1-periodic orbits of X_H outside of a compact set of $\widehat{M}$. For an H as above and γ a 1-periodic orbit of X_H in $\{e^\rho\} \times \partial M$, the value of the action functional at γ is given by

$$\mathcal{A}_H(\gamma) = h(e^\rho) - e^\rho h'(e^\rho).$$

Similar properties are required for the almost complex structure in order to control the behaviour at infinity of solutions of the Floer equation.

Definition 2.58 (Contact type almost complex structures). We call an almost complex structure $J \in \Gamma(\operatorname{End}(T\widehat{M}))$ on $\widehat{M}$ of *contact type* if

$$d(e^r) \circ J = -\theta.$$

on the collar.

We have now introduced the particularities of the structures that we want to consider. We will tacitly assume them for the rest of the section.

Assumption 2.59. We consider Hamiltonians respecting (2.33) and almost complex structures of contact type.

2.5.2. Maximum principle

The only danger in this non-compact set up — with respect to the theory developed in Section 2.2 and Section 2.4 for closed manifolds — is that there may be sequences of Floer trajectories $u \in \mathcal{M}(\gamma, \delta)$ leaving all compact sets in $\widehat{M}$. The solution to this problem is the following proposition.

PROPOSITION 2.60 (Maximum principle). *Let u be a solution to the Floer equation. Then for any bounded open subset $\Omega \subset \mathbb{R} \times S^1$, the maximum of u on the closure Ω is taken on the boundary, that means*

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

PROOF. We follow [Sei08, Section 3c]. Consider the function $v = e^r \circ u : \mathbb{R} \times S^1 \to \mathbb{R}$. The conditions of Assumption 2.59 guarantee that

$$\begin{aligned} \partial_s v &= \theta(\partial_t u) - v h'(v), \\ \partial_t v &= -\theta(\partial_s u), \end{aligned}$$

which are equivalent to (identifying the tangent space of the collar with $\mathbb{C} \times \xi$, where ξ is the contact distribution)

$$dv \circ i = -u^*\theta + v \cdot h'(v) dt.$$

Remark that

$$\begin{aligned} |\partial_s u|^2 = \omega(\partial_s u, \partial_t u - X) &= \omega(\partial_s u, \partial_t u) - dH(\partial_s u) \\ &= \omega(\partial_s u, \partial_t u) - h'(v)\partial_s v. \end{aligned} \tag{2.28}$$

So we may compute

$$\begin{aligned} \Delta v = \partial_s^2 v + \partial_t^2 v &= \partial_s(\theta(\partial_t u)) - \partial_t(\theta(\partial_s u)) - (h'(v)\partial_s v + v h''(v)\partial_s v) \\ &= \partial_s(\theta(\partial_t u)) - \partial_t(\theta(\partial_s u)) + |\partial_s u|^2 - \omega(\partial_s u, \partial_t u) - v h'' \partial_s v \\ &= |\partial_s u|^2 - v h''(v) \partial_s v. \end{aligned} \tag{2.29}$$

Consider the differential operator $L(\varphi) \coloneqq \Delta\varphi + \varphi \cdot h'' \cdot \partial_s \varphi$ on $\mathbb{R} \times S^1 \to \mathbb{R}$. We may rephrase (2.29) as

$$L(v) = |\partial_s u|^2 \geq 0. \tag{2.30}$$

The statement follows now from Hopf's Weak Maximal Principle [Eva10; Hop52].

■

The immediate corollary of Proposition 2.60 is the following.

COROLLARY 2.61. Let u be a solution of the Floer equation connecting $\gamma^- = (r^-, y^-)$ to $\gamma^+ = (r^+, y^+)$. Then the entire image of u is contained in the subset where $r \leq \max(r^-, r^+)$, that is,

$$u(\mathbb{R} \times S^1) \subset [0, \max(r^-, r^+)] \times \partial M. \tag{2.31}$$

Analogously, we can look at solutions of (2.16) considering a Hamiltonian $H_s(r, p) = h_s(e^r)$ and almost complex structures J_s of contact type at infinity. The same proof as for Proposition 2.60 shows that if

$$\partial_s h_s' \leq 0 \tag{2.32}$$

then (2.31) still holds.

2.5.3. Definition of symplectic cohomology

Let $H = H^{c,C,m}$ be a time dependent Hamiltonian such that on the collar κ for $r > 1$

$$H|_\kappa(r,p) = h^{c,C,m}(e^r) = m(e^r - c) + C \tag{2.33}$$

for $c, C, m \in \mathbb{R}$. Then H is clearly of the type of (2.27) for a generic choice of m and we call it *linear* at infinity. The generality in the choice of m follows from the discussion in the previous paragraph on the period of Reeb orbits. Consider furthermore an homotopy H^s between two such Hamiltonians $H^0 = H^{c_0,C_0,m_0}$ and $H^1 = H^{c_1,C_1,m_1}$, that is $H^s = H^{c_s,C_s,m_s}$ with $m_s = m_0$ for $s < -1$ and $m_s = m_1$ for $s > 1$. Then Proposition 2.60 states that if

$$\partial_s m_s \leq 0,$$

we have a continuation map

$$\begin{aligned} \varphi : CF^\bullet(H^1) &\to CF^\bullet(H^0) \\ \gamma &\mapsto \sum_{u \in \mathcal{M}(\gamma,\delta)} \varepsilon(u)\delta \end{aligned}$$

when γ and δ have index difference 1. The map ε is determined by orientation signs, see [Abo15] and [Rit13, Section 17], so working on $\mathbb{Z}/2\mathbb{Z}$ we can ignore them. These maps behave nicely under composition at the level of cohomology (on the chain, they are chain homotopic), that is

$$\begin{tikzcd} HF^\bullet(H^0) & HF^\bullet(H^2) \arrow[l] \arrow[dl] \\ HF^\bullet(H^1) \arrow[u] & \end{tikzcd}$$

commutes.

Lemma 2.62. *For a Hamiltonian H^s of the form* (2.33) *the Floer homology $FH^\bullet(H^s)$ is independent of c_s and C_s. Furthermore if the slopes $m_1 = m_2$ are the same, then the continuation map is an isomorphism.*

We will thus simply denote H^m the Hamiltonian $H^{c,C,m}$. The relation

$$H^{m^-} \preceq H^{m^+} \Leftrightarrow m^- \leq m^+$$

defines a preorder on the set of linear Hamiltonians [Abo15] and thus it defines an index category. Actually, it indexes a diagram on the category whose objects are $HF^\bullet(H^m)$ for $m \in \mathbb{R} \setminus \mathrm{Spec}(\alpha)$ and morphisms are given by continuation maps. We may thus consider the induced colimit.

Definition 2.63 (Symplectic cohomology). Let (M, ω) be a Liouville domain and $\widehat{M}$ its completion. Then we define the symplectic cohomology of M to be

$$SH^\bullet(M) \coloneqq \varinjlim_{m_i} FH^\bullet(H^{m_i})$$

for a sequence of Hamiltonian H^{m_i} with $m_i \to \infty$ as $i \to \infty$.

The above definition lends itself to a couple of remarks.

- By definition of colimit we have a commutative diagram

$$\begin{tikzcd}
HF^\bullet(H^{m^-}) \arrow[rr, "\varphi(m^-;m^+)"] \arrow[dr, "\psi_{m^-}"] \arrow[ddr, "\widetilde{\psi}_{m^-}"'] & & HF^\bullet(H^{m^+}) \arrow[dl, "\psi_{m^+}"'] \arrow[ddl, "\widetilde{\psi}_{m^+}"] \\
& SH^\bullet(M) \arrow[d, "\wr"'] & \\
& \varinjlim_{m_i} FH^\bullet(H^{m_i}) &
\end{tikzcd} \quad . \tag{2.34}$$

- The concrete choice of m_i does not affect the resulting colimit up to an isomorphism induced by the continuation maps.
- The symplectic cohomology is an invariant of the Liouville domain (using an appropriate notion of isomorphism in this category, see [Sei08]). However, we will mostly abuse notation and look at the symplectic cohomology of the completion, which we will also denote by $SH^\bullet(M) = SH^\bullet(\widehat{M})$.

CHAPTER 3

The PSS map

This chapter is dedicated to the introduction of a homomorphism $\mathrm{PSS} : QH^\bullet(M) \to HF^\bullet(H)$ first defined by PIUNIKHIN, SALAMON and SCHWARZ [PSS96]. The main results will be Theorem 3.2 and Theorem 3.3.

3.1. Relative Donaldson type invariants

In this section we consider a Riemann surface Σ of genus g and with ℓ cylindrical ends, that is, with $1 \leq i \leq \ell$ embeddings

$$\phi_i : (0, \infty) \times S^1 \to \Sigma,$$

whose images (the cylinders) we denote by $Z_i \subset \Sigma$. We assume without loss of generality that the pullback of the almost complex structure j on Σ and the standard structures on the cylinders agree. Fix also ℓ time-dependent Hamiltonians

$$\begin{aligned} H_i : (0, \infty) \times S^1 \times M &\to \mathbb{R} \\ (s, t, p) &\mapsto H_i(s, t, p) \end{aligned}$$

that vanish near $s = 0$ and are independent of s for $s \geq 1$. Assume that the periodic solutions to the ℓ Hamiltonian systems are non-degenerate and denote by $\widetilde{\mathcal{P}}(H_i)$ the set of contractible solutions lifted to $\widetilde{\mathcal{L}M}$ as in Section 2.4. Remark that we are actually not interested in ℓ Hamiltonians on all of M. We want to control the behaviour of the Hamiltonian on each of the cylindrical ends and this is done by considering the sum $H = H_1 + \cdots + H_\ell$ and then the compositions of H and ϕ_i.

For $\widetilde{\gamma}_i \in \widetilde{\mathcal{P}}(H_i)$ we consider the space

$$\mathcal{M}_\Sigma(\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_\ell) = \mathcal{M}_\Sigma(\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_\ell, H_1, \ldots, H_\ell, J)$$

of smooth maps $u : \Sigma \to M$ that satisfy the three following conditions:

(1) u is J-holomorphic on the complement

$$\Sigma_0 = \Sigma \setminus \bigcup_i Z_i.$$

(2) The maps $u_i \coloneqq u \circ \phi_i$ are solutions of the respective Floer equation

$$\partial_s u_i + J(u_i)\partial_t u_i + \operatorname{grad} H_i(s, t, u_i) = 0$$

and tend to the loops γ_i:

$$\gamma_i(t) = \lim_{s\to\infty} u_i(s,t).$$

The original paper contains a third condition, which we removed as we are considering a slightly different Novikov ring.

Under these conditions, the space $\mathcal{M}_\Sigma(\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_\ell)$ has dimension

$$\dim \mathcal{M}_\Sigma(\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_\ell) = 2n(1-g) - \sum_{i=1}^{\ell} \mu(\widetilde{\gamma}_i).$$

We fix a non-negative integer $d \in \mathbb{N}_0$, distinct points $z_1, \dots, z_d \in \Sigma$ and homology classes $\alpha_1, \dots, \alpha_d \in H_\bullet(M)$ such that

$$\sum_{i=1}^{\ell} \mu(\widetilde{\gamma}_i) = 2n(1-g) - \sum_{\nu=1}^{d} (2n - \deg(\alpha_\nu)).$$

The space

$$\mathcal{M}_\Sigma((\alpha_1, \dots, \alpha_d), (\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_\ell))$$

is then defined to be the set of curves $u \in \mathcal{M}_\Sigma(\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_\ell)$ that map z_ν into α_ν. This set is finite and we denote its cardinality by

$$n_\Sigma((\alpha_1, \dots, \alpha_d), (\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_\ell)) := |\mathcal{M}_\Sigma((\alpha_1, \dots, \alpha_d), (\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_\ell))|.$$

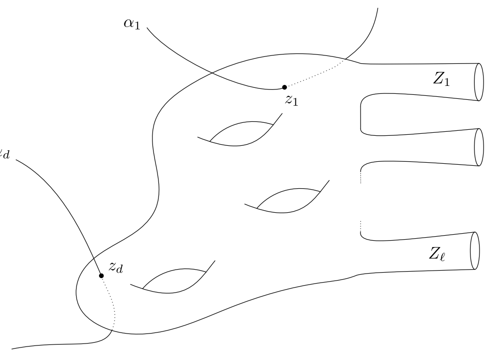

FIGURE 3.1. The space $\mathcal{M}_\Sigma$.

Counting the elements in this moduli spaces defines a map

$$\begin{aligned}
\psi_\Sigma : H_\bullet(M)^{\otimes d} &\to CF_\bullet(H_1, \dots, H_\ell) = \bigotimes_{i=1}^{\ell} CF_\bullet(H_i) \\
(\alpha_1, \dots, \alpha_d) &\mapsto \sum_{\widetilde{\gamma}_i \in \widetilde{\mathcal{P}}(H_i)} n_\Sigma((\alpha_1, \dots, \alpha_d), (\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_\ell)) \langle \widetilde{\gamma}_1 \rangle \otimes \dots \otimes \langle \widetilde{\gamma}_\ell \rangle,
\end{aligned}$$

which descends to a map in homology as if one of the cycles α_i is a boundary then so is ψ_Σ. This last map is denoted by

$$\Psi_\Sigma : H_\bullet(M)^{\otimes d} \to HF_\bullet(H_1) \otimes \cdots \otimes HF_\bullet(H_\ell) \tag{3.1}$$

and is called a *relative Donaldson invariant*. In this last two equations we used $(H_1, \ldots, H_\ell)$, however, as noted before this is only notationally different from considering the Floer homology and complex of the Hamiltonian resulting as the sum of the different H_i.

Theorem 3.1 (Properties of Ψ_Σ). *Let Ψ_Σ be as in* (3.1), *then*

(1) *The class $\Psi_\Sigma(\alpha_1, \ldots, \alpha_d)$ has degree $2n(1-g) - \sum_{\nu=1}^{d}(2n - \deg(\alpha_\nu))$.*
(2) *The Floer homology class $\Psi_\Sigma(\alpha_1, \ldots, \alpha_d)$ does not depend on the choice of points z_ν.*
(3) *The Floer homology class $\Psi_\Sigma(\alpha_1, \ldots, \alpha_d)$ is natural under the variation of the Hamiltonians and the almost complex structure. That is, for two choices $(H_1^\alpha, \ldots, H_\ell^\alpha, J^\alpha)$ and $(H_1^\beta, \ldots, H_\ell^\beta, J^\beta)$ the isomorphism Φ^Γ from Section 2.2.7 relates the two classes*

$$\Psi_\Sigma^\alpha(\alpha_1, \ldots, \alpha_d) = \Phi^\Gamma \circ \Psi_\Sigma^\beta(\alpha_1, \ldots, \alpha_d).$$

The map Ψ_Σ extends naturally to the quantum homology as a linear map over the Novikov ring. Explicitly this means that for

$$\alpha = \sum_A \alpha_A q^A, \quad \alpha_A \in H_\bullet(M, \mathbb{Z}/2\mathbb{Z})^{\otimes d}, \quad \deg(a_A) = k - 2c_1(A),$$

we define for $\widetilde{\gamma} = [\widetilde{\gamma}_1, \ldots \widetilde{\gamma}_\ell]$

$$\psi_\Sigma(\alpha) = \sum_{\substack{A \in \Gamma \\ \widetilde{\gamma} \in \widetilde{\mathcal{P}}(H_1, \ldots, H_\ell)}} n_\Sigma(\alpha_A, (-A)\#\widetilde{\gamma})\widetilde{\gamma},$$

and the induced map on cohomology again denoted by

$$\Psi_\Sigma : QH_\bullet(M)^{\otimes d} \to HF_\bullet(H_1) \otimes \cdots \otimes HF_\bullet(H_\ell).$$

The construction of Ψ is more natural in homology and we thus defined it in this framework. Nevertheless, Poincaré duality allows us to switch between homology and cohomology effortless, defining so

$$\Psi_\Sigma \colon QH^\bullet(M)^{\otimes d} \to HF^\bullet(H_1) \otimes \cdots \otimes HF^\bullet(H_\ell)$$

by abuse of notation. We will mainly use this latter map, but the behaviour is just the same.

3.2. Examples

The Ψ_Σ maps from (3.1) provide various bridges from the quantum to the Floer theory, one for each Riemannian surface Σ. We consider in this section three of these bridges.

3.2.1. Complex plane — The PSS map

We first consider the case $\Sigma = \mathbb{C}$, thinking of the complex plane as a half infinite cylinder closed on the finite side (that is $\ell = 1$) as in Figure 3.2. In this case, the space

$$\mathcal{M}(\widetilde{\gamma}) = \mathcal{M}(\widetilde{\gamma}, H, J) \tag{3.2}$$

consists in perturbed J-holomorphic curves such that the map

$$\mathbb{R} \times S^1 \ni (s,t) \mapsto u(e^{2\pi(s+it)})$$

satisfies the Floer equation for $s \geq 0$ and the limit for $s \to \infty$ is $\widetilde{\gamma} \in \widetilde{\mathcal{P}}(M)$. This moduli space is of dimension $2n - \mu(\widetilde{\gamma})$.

In this setup we consider one cycle (so $d = 1$) representing a homology class of degree $\deg(\alpha) = \mu(\widetilde{\gamma})$. We denote by $\mathcal{M}(\alpha, \widetilde{\gamma})$ the moduli spaces of curves $u \in \mathcal{M}(\widetilde{\gamma})$ with $u(0) \in \alpha$. This moduli space has dimension $\dim \mathcal{M}(\widetilde{\gamma}) - (2n - \deg(\alpha))$ so that for the above choice of α it is of null dimension and we may denote its cardinality by $n(\alpha, \widetilde{\gamma})$. These numbers determine a homomorphism

$$\begin{aligned} \psi_{\mathbb{C}} \colon QH_k(M) &\to CF_k(H) \\ \alpha &\mapsto \sum_{A, \widetilde{\gamma}} n(\alpha_A, (-A)\#\widetilde{\gamma})\widetilde{\gamma} \end{aligned}$$

for $\alpha = \sum_{A \in \Gamma} \alpha_A q^A$ as usual. We will denote by

$$\Psi_{\mathbb{C}} : QH_k(M) \to HF_k(H) \tag{3.3}$$

the induced map in homology and call it the PSS map, usually denoting it by PSS referring to the authors PIUNIKHIN, SALAMON and SCHWARZ who firstly introduced it.

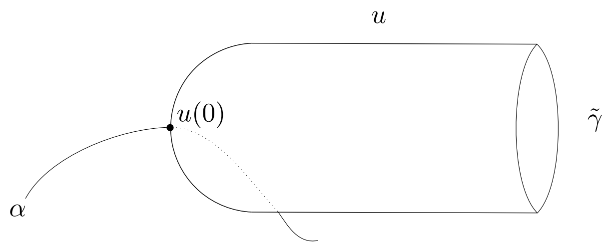

FIGURE 3.2. The PSS map counts half-closed J-holomorphic cylinders intersecting α and converging to $\widetilde{\gamma}$.

3.2.2. A cylinder — The cap product

Similarly one defines the cup product. The Riemann surface one considers is now an infinite cylinder $Z = \mathbb{R} \times S^1$ (thus $\ell = 2$) and one homology class (again $d = 1$). For a class $\alpha \in H_\bullet(M)$ and $\widetilde{\gamma}^+ \in \widetilde{\mathcal{P}}(H_1), \widetilde{\gamma}^- \in \widetilde{\mathcal{P}}(H_2)$ we consider the space

$$\mathcal{M}(\widetilde{\gamma}^+, \widetilde{\gamma}^-, \alpha) := \{u : \Sigma \to M : \lim_{s\to\pm\infty} u(s,t) = \widetilde{\gamma}^\pm, u(\Sigma) \cap \alpha \neq \varnothing\}$$

whose dimension is

$$\dim \mathcal{M}(\widetilde{\gamma}^+, \widetilde{\gamma}^-, \alpha) = \mu(\widetilde{\gamma}^+) - \mu(\widetilde{\gamma}^+) - (2n - \deg(\alpha))$$

so that defining the cardinality of this set by $n(\widetilde{\gamma}^+, \widetilde{\gamma}^-, \alpha)$ when the dimension of $\mathcal{M}(\widetilde{\gamma}^+, \widetilde{\gamma}^-, \alpha)$ is zero induces a homomorphism

$$\Psi_Z \colon H_\bullet(M) \to HF_\bullet(H_1) \otimes HF_\bullet(H_2).$$

Remark that elements $\beta = \beta_1 \otimes \beta_2 \in HF_k(H_1) \otimes HF_j(H_2)$ and $\eta \in HF^s(H_1)$ may be combined to obtain $(\beta_1 \frown \eta) \otimes \beta_2 \in HF_{k-s}(H_1) \otimes HF_j(H_2)$. Linearly extending this procedure using $\Phi_Z(\alpha)$ for $\deg(\alpha) = j$ yields a map $HF_k(H_1) \to HF_{k-j}(H_2)$. With continuation maps we finally get a map

$$\frown_F \colon H^j(M) \times HF_k(H, J) \to HF_{k-j}(H, J) \tag{3.4}$$

called the cap-product in Floer homology.

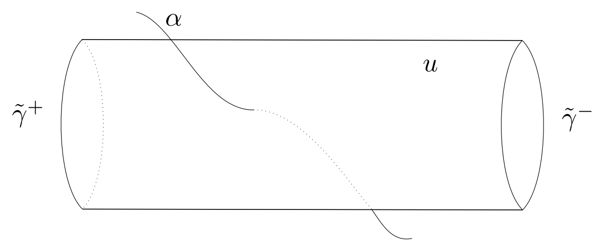

Figure 3.3. The cap product counts J-holomorphic cylinders intersecting a homology class.

3.2.3. Pair-of-pants product

The last surface we consider is a pair-of-pants, that is, a surface with genus zero and three cylindrical ends ($\ell = 3$) as in Figure 3.4 with Hamiltoninas $H_1 = H_2 = \bar{H}_3 = H$. As usual (3.1) provides a map

$$\Psi_{PoP} \colon H_\bullet(M) \to HF_\bullet(H_1) \otimes HF_\bullet(H_2) \otimes HF_\bullet(\bar{H}_3).$$

In this case we are not interested in the map per se, but on the image of the fundamental class $\Phi_{PoP}([M])$. As $[M]$ is of top degree, Poincaré duality provides the map

$$\succ \colon HF^j(H) \otimes HF^k(H) \to HF^{j+k}(H), \tag{3.5}$$

which is known as the *pair-of-pants product* on Floer homology.

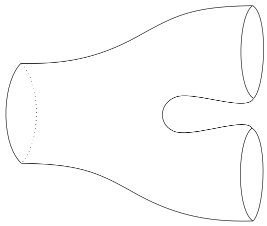

Figure 3.4. A pair of pants surface.

3.2.4. General remarks

We end this section with two related remarks.

It is useful to think about these relative Donaldson invariants as operators. Considering cylindrical ends with orientations — left and right — and making the respective Hamiltonian flow mapping the first orientation to the second (flow from left to right), we can consider Ψ_Σ as an operator from the Floer homology of the Hamiltonian on the left to the Floer homology of the Hamiltonian on the right.

The above interpretation of Ψ_Σ allows also to define a composition of these maps. For well suited Σ and Σ' — Σ' has the same number of cylindrical ends on the left as Σ has on the right — we define

$$\Psi_{\Sigma\#\Sigma'} = \Psi_{\Sigma'} \circ \Psi_\Sigma. \tag{3.6}$$

It is intuitively clear that $\Psi_{\Sigma\#\Sigma'}$ and the actually relative Donaldson invariant associated to $\Sigma\#\Sigma'$ must agree. So $\Psi_{\Sigma\#\Sigma'}$ is well-defined. It is now an almost combinatorial exercise to show that all relative Donaldson invariants my be computed using (3.6) and the three invariants (3.3), (3.4) and (3.5).

3.3. Module isomorphism $QH_\bullet(M) \cong HF_\bullet(H)$

The goal of this section is to show the following theorem.

Theorem 3.2. *Let (M,ω) be a closed semipositive symplectic manifold. Then*

$$\mathrm{PSS} : QH_\bullet(M) \to HF_\bullet(H)$$

from (3.3) *defines an isomorphism of modules.*

Proof. We prove the theorem in 5 steps. For this proof, we will denote $\Phi = \mathrm{PSS}$ as we are going to define another map Φ similar to Φ that will allow us to show the statement.

Homomorphism. This follows directly from the definition of Ψ_Σ so what remains to show is bijectivity.

Morse theory. We start by choosing a Morse function $H_0 : M \to \mathbb{R}$ and considering the Morse complex $CM_\bullet(H_0, \omega)$ consisting in the Λ-module generated by the critical points of H_0. We think of $CM_k(H_0, \omega)$ as the set of formal sums

$$\xi = \sum_{\substack{p \in \mathrm{crit}(H_0),\ A \in \Gamma \\ \mathrm{ind}_{H_0}(p) - 2c_1(A) = k}} \xi_{p,A} \langle p, A \rangle \quad \xi_{p,A} \in \mathbb{Z}/2\mathbb{Z}$$

satisfying the finiteness condition

$$|\{\langle p, A \rangle : \xi_{p,A} \neq 0, \omega(A) \leq c\}| < \infty$$

for all $c \in \mathbb{R}$. The boundary operator is defined as usual. As an additive group, the homology of this complex is naturally isomorphic to the quantum homology of M

$$HM_\bullet(M, \omega) \cong QH_\bullet(M) = H_\bullet(M) \otimes \Lambda. \tag{3.7}$$

$\Phi = \Phi$. Given a critical point $p \in \mathrm{crit}(H_0)$ and a periodic orbit $\widetilde{\gamma} \in \widetilde{\mathcal{L}M}$ we consider the space

$$\mathcal{M}(p, \widetilde{\gamma}) \coloneqq \{u \in \mathcal{M}(\widetilde{\gamma}) : u(0) \in W^u(p)\},$$

with $\mathcal{M}(\widetilde{\gamma})$ as in (3.2). This space has dimension $\dim \mathcal{M}(p, \widetilde{\gamma}) = \mathrm{ind}_{H_0}(p) - \mu(\widetilde{\gamma})$. In the case of dimension zero we denote the cardinality by $n(p, \widetilde{\gamma})$.

We define now a map

$$\begin{aligned} \phi : CM_\bullet(H_0, \omega) &\to CF_\bullet(H) \\ \langle p, A \rangle &\mapsto \sum_{\substack{\widetilde{\gamma}: \\ \mu(\widetilde{\gamma}) = \mathrm{ind}_{H_0}(p) - 2c_1(A)}} n(p, (-A)\#\widetilde{\gamma}) \langle \widetilde{\gamma} \rangle \end{aligned}$$

As this map is linear over the Novikov ring and a chain map[a], it induces a homomorphism on the level of homology:

$$\Phi : HM_\bullet(H_0, \omega) \to HF_\bullet(H).$$

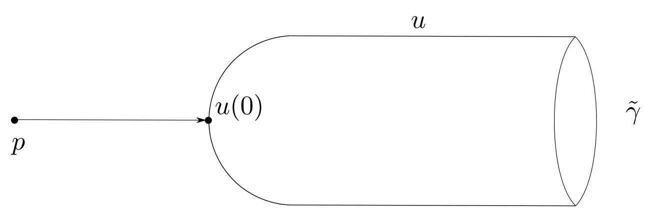

FIGURE 3.5. The Φ map counts spiked cylinders.

[a]This last fact follows from the fact that the boundary operator counts in both homology theories the number of connecting orbits. Roughly speaking, ϕ sends the orbits in one theory to the orbits in the other as Figure 3.5 shows. So it must send boundaries to boundaries.

The goal of this part is to show that $\Phi = \Phi$ under the identification of (3.7). In order to show this, take $\alpha \in H_\bullet(M)$ and represent it as a chain $\xi = \sum_{p\in \mathrm{crit}_{H_0}} \xi(\alpha,p)p$ in the Morse complex. By the construction of Morse homology, this means that α is represented by a cycle that is arbitrarily close to $W_\alpha \coloneqq \sum_{p\in \mathrm{crit}_{H_0}} \xi(\alpha,p)W^u(p)$. For this cycle the Floer homology class $\phi(\xi)$ is given by the intersection number of $\mathcal{M}(\widetilde{\gamma})$ and W_α. This was the definition of $\psi_{\mathbb{C}}$.

$\underline{\Psi \circ \Phi = \mathrm{id}}$. To show bijectivity we will construct an explicit inverse. We denote by $\mathcal{M}^-(\widetilde{\gamma}) = \mathcal{M}^-([\bar{\gamma},\bar{v}],\bar{H},J)$ with the notation introduced after Definition 2.51. We remark that $\mathcal{M}^-(\widetilde{\gamma})$ has dimension $\mu(\widetilde{\gamma})$. Analogously, we define

$$\mathcal{M}^-(\widetilde{\gamma},p) \coloneqq \{u \in \mathcal{M}^-(\widetilde{\gamma}) : u(0) \in W^s(p)\},$$

which has instead dimension $\mu(\widetilde{\delta}) - \mathrm{ind}_{H_0}(p)$. As usual we denote by $n^-(\widetilde{\gamma},p)$ the cardinality of $\mathcal{M}^-(\widetilde{\gamma},p)$ when the space is zero dimensional. These numbers define a chain map

$$\begin{aligned} \psi : CF_\bullet(H,J) &\to CM_\bullet(H_0,\omega) \\ \widetilde{\gamma} &\mapsto \sum_{p,A} n^-((-A)\#\widetilde{\gamma},p)\langle p,A\rangle. \end{aligned}$$

and a corresponding induced map in homology

$$\Psi : HF_\bullet(H,J) \to HM_\bullet(H_0,\omega).$$

Consider now the composition

$$\Psi \circ \Phi$$

which at the chain level takes the form

$$\psi \circ \phi(\langle p,A\rangle) = \sum_{\substack{\widetilde{\gamma}\in\widetilde{\Lambda}M \\ \mu(\widetilde{\gamma})=\mathrm{ind}_{H_0}(p)}} \sum_{\substack{q,B \\ \mathrm{ind}(p)-\mathrm{ind}(q)+2c_1(A)=0}} n(p,\widetilde{\gamma})n^-((-B)\#\widetilde{\gamma},q)\langle q,B\rangle. \tag{3.8}$$

The $\langle q,B\rangle$ coefficient in (3.10) is

$$\sum_{\widetilde{\gamma}} n(p,\widetilde{\gamma})n^-((-B)\#\widetilde{\gamma},q),$$

which by gluing is the number of perturbed J-holomorphic spheres $u : S^2 \to M$ such that

$$u(0) \in W^u(p), \quad u(\infty) \in W^s(q).$$

We choose a homotopy of perturbations from the given one to $H = 0$. For this last one, by dimension reasons (that is because $\mathrm{ind}(p) - \mathrm{ind}(q) + 2c_1(A) = 0$), we can only have such a solution if $p = q$ and $B = 0$. By the arguments above this count is independent of B so we may choose $B = 0$. This implies that $p = q$, so that there can be only one holomorphic sphere satisfying the condition above. This means that we have exactly the solutions

$$\sum_{\widetilde{\gamma}} n(p,\widetilde{\gamma})n^-((-B)\#\widetilde{\gamma},q) = \begin{cases} 1, & p=q, B=0, \\ 0, & \text{else,} \end{cases}$$

so that

$$\psi \circ \phi(\langle p, A\rangle) = \langle p, A\rangle. \tag{3.9}$$

As the induced map in Floer homology is independent of the choice of perturbations and (3.9) means that for $H=0$ at the chain level $\psi\circ\phi$ is chain homotopy equivalent to the identity, this shows that $\Psi\circ\Phi = \mathrm{id}$.

$\Phi\circ\Psi = \mathrm{id}$. It remains to show that Ψ is a right inverse to Φ. We work again on the level of chain maps:

$$\phi\circ\psi(\widetilde{\gamma}) = \sum_{\substack{\widetilde{\delta}\in\widetilde{\Lambda}M \\ \mu(\widetilde{\gamma})=\mu(\widetilde{\delta})}} \sum_{\substack{p,A \\ \mathrm{ind}(p)=\mu(\widetilde{\gamma})+2c_1(A)}} n((-A)\#\widetilde{\delta},p)n^-(p,(-A)\#\widetilde{\delta})\langle\widetilde{\delta}\rangle. \tag{3.10}$$

so that the coefficient of $\langle\widetilde{\delta}\rangle$ is given by

$$n(\widetilde{\gamma},\widetilde{\delta}) = \sum_{\substack{p,A \\ \mathrm{ind}(p)=\mu(\widetilde{\gamma})+2c_1(A)}} n((-A)\#\widetilde{\delta},p)n^-(p,(-A)\#\widetilde{\delta}),$$

which is the number of triples (u^-,σ,u^+) such that $u^-\in\mathcal{M}^-(\widetilde{\gamma})$, $u^+\in\mathcal{M}^-(\widetilde{\delta})$ and $\dot{\sigma} = -\operatorname{grad} H_0(\sigma)$ with $\sigma(\pm T) = u^\pm(0)$. We can, however, set $T=0$ without changing the induced map in homology and get rid of σ so that

$$n(\widetilde{\gamma},\widetilde{\delta}) = |\{(u^-,u^+) : u^-\in\mathcal{M}^-(\widetilde{\gamma}), u^+\in\mathcal{M}^-(\widetilde{\delta}), u^+(0)=u^-(0)\}|.$$

By gluing we can show that $n(\widetilde{\gamma},\widetilde{\delta})$ is the number of J-holomorphic cylinders running from $\widetilde{\gamma}$ to $\widetilde{\delta}$. A modification of the homotopy argument in [SZ92, Theorem 6.1] or [Aud13, Chapter 11] shows that $\phi\circ\psi$ is chain homotopy equivalent to the identity. Thus $\Phi\circ\Psi = \mathrm{id}$, which completes the proof. ■

3.4. Ring isomorphism

We introduced with Definition 2.46 (Quantum cup product) the ring structure on $QH^\bullet(M)$ and in Section 3.2.3 the one on $HF^\bullet(H)$. In this section, we show that the PSS map is not only a group isomorphism but also a ring isomorphism mapping the quantum cup product to the pair-of-pants product.

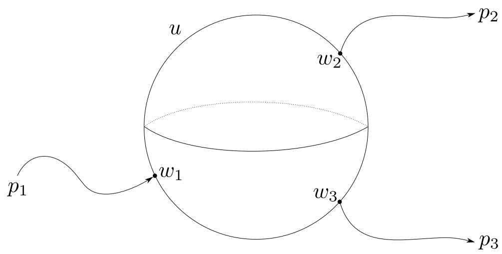

FIGURE 3.6. Morse version of the quantum cup product.

THEOREM 3.3. *The PSS isomorphism from Theorem 3.2 intertwines the quantum cup product with the pair-of-pants product, that is*

$$\mathrm{PSS}(a * b) = \mathrm{PSS}(a) \curlyeqsucc \mathrm{PSS}(b)$$

for all $a, b \in QH^\bullet(M)$. *Furthermore, it intertwines the quantum cup product with the cap product via*

$$\mathrm{PD}_F^{-1}(a \frown_F \mathrm{PD}_F(\mathrm{PSS}(b))) = \mathrm{PSS}(a * b)$$

for all $a, b \in QH^\bullet(M)$.

PROOF. We provide a proof only for the first statement, which is the only one we will use in the rest of this text and also the most used in the literature.

The proof argues with a Morse argument similar to the one of Theorem 3.2. To that end, consider three (generic) Morse functions $f_1, f_2, f_3 : M \to \mathbb{R}$, three critical points $p_1, p_2, p_3 \in M$ and three distinct points $w_1, w_2, w_3 \in \mathbb{C}P^1$. We will omit the details about the regular family of almost complex structures that one has to choose for each point in S^2. For a spherical representative $A \in H_2(M)$ we consider the space

$$\mathcal{M}_A(p_1; p_2, p_3)$$

of J-holomorphic A-spheres $u \colon \mathbb{C}P^1 \to M$ respecting

$$u(w_1) \in W^u(p_1, f_1), \quad u(w_2) \in W^s(p_2, f_2), \quad u(w_3) \in W^s(p_3, f_3),$$

where we use the standard Morse theoretic notation of W^s and W^u for the stable and unstable manifold of a critical point. See Figure 3.6 for a picture of this set-up.

From standard Morse theoretic results, one derives

$$\dim W^s(p, f) = 2n - \mathrm{ind}_f(p), \quad \dim W^u(p, f) = \mathrm{ind}_f(p),$$

which imply

$$\dim \mathcal{M}_A(p_1; p_2, p_3) = 2c_1(A) + \mathrm{ind}_{f_1}(p_1) - \mathrm{ind}_{f_2}(p_2) - \mathrm{ind}_{f_3}(p_3).$$

As usual we denote by $n_A(p_1; p_2, p_3)$ the cardinality of $\mathcal{M}_A(p_1; p_2, p_3)$ whenever the dimension of the moduli space is zero.

These numbers define a new chain map

$$MC^\bullet(f_2) \otimes MC^\bullet(f_3) \to MC^\bullet(f_1) \colon (q_2, q_3) \to q_2 * q_1$$

defined by

$$q_2 * q_3 \coloneqq \sum_{\substack{A \\ q_1 \in \mathrm{crit}(f_1)}} n_A(q_1, q_2, q_3) q_1. \tag{3.11}$$

As this induces a product on Morse cohomology, we have a product on quantum cohomology which agrees with the quantum cup product as the stable and unstable manifolds of Morse critical points represent cohomology classes generating $H^\bullet(M)$.

Consider now the gluing of three J-holomorphic spiked discs to the boundary of a pair-of-pants surface as shown in Figure 3.7. By definition, the pair-of-pants product takes two loops on the left and returns the loop on the right of this surface counting J-holomorphic spheres. By the description above, under identification via PSS as in Theorem 3.2, this is the same as (3.11).

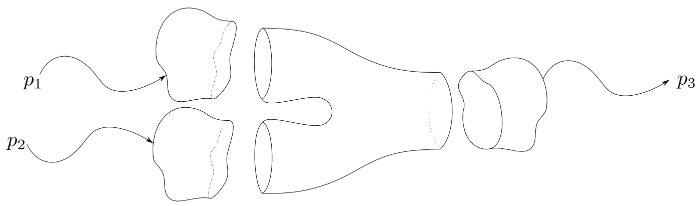

FIGURE 3.7. From a pair-of-pants to the quantum cup product. After gluing this picture is the same as Figure 3.6.

More concretely, if PSS(a) and PSS(b) are the two loops on the left of the pair-of-pants surface, then — by the last paragraph — PSS(a) $\succ$ PSS(b) is the loop on right of the surface, say PSS(c) $\coloneqq$ PSS(a) $\succ$ PSS(b). Consider now the three spiked cylinders that the PSS map gives us (as in Figure 3.5) and glue them to the pair-of-pants structure. Then we get a sphere with three spikes identified by $a, b, a * b$ as in Figure 3.6. By construction we must have $c = a * b$, which concludes the proof. ■

CHAPTER 4

Magnetic cotangent bundles

This is the last but central chapter of the text. We analyse various symplectic aspects of magnetic cotangent bundles. We show the vanishing of the first Chern class for general magnetic cotangent bundles in Section 4.1 and that the quantum cohomology of a large class of these is undeformed in Section 4.2. In the end, we follow RITTER's approach to show that the symplectic cohomology vanishes in the presence of an Hamiltonian circle action linear at infinity [Rit14]. Whilst we do not present deep new results the treatment of $SH^\bullet(T^*S^2, \omega_\sigma) = 0$ uses an Hamiltonian circle action developed in Section 4.4 not present in the literature.

DEFINITION 4.1 (Magnetic symplectic form). Let (M, g) be a closed Riemannian manifold and denote by $\pi : T^*M \to M$ its cotangent bundle. Denote by $\omega = d\theta$ the canonical symplectic form and consider a closed 2-form $\sigma \in \Omega^2(M)$. The symplectic form

$$\omega_\sigma \coloneqq \omega + \pi^*\sigma \tag{4.1}$$

is called *magnetic symplectic form*. We refer to (T^*M, ω_σ) as a *magnetic cotangent bundle*.

In particular when considering symplectic cohomology we will look at cotangent bundles of surfaces, actually exclusively T^*S^2, where there is a natural closed 2-form to consider, that is $s\sigma$ for σ the area form of M and some $s \in \mathbb{R}$.

4.1. Vanishing first Chern class

We consider a closed connected orientable Riemannian manifold (M, g). In this section we will pedantically denote by

$$\pi_E : E \to M$$

the foot point map of a vector bundle $E \to M$. We reserve a special notation for the bundle

$$\pi_E : E \coloneqq TM \oplus T^*M \to M \quad \text{with} \quad \pi_E = \pi_{TM} \oplus \pi_{T^*M}.$$

The musical isomorphism between tangent and cotangent bundle induced by g will be denoted by

$$\widetilde{g} : TM \to T^*M \quad (p, \xi) \mapsto g_p(\cdot, \xi)$$

and the induced metric on the cotangent bundle by

$$g^* \in \mathcal{T}^{2,0}(M) \quad g^*_p(a,b) := g_p(\widetilde{g}^{-1}(a), \widetilde{g}^{-1}(b)) \quad \text{for} \quad a, b \in T^*_pM.$$

Let us recall first some standard results form gauge theory [Mer21, Lecture 31]. For any vector bundle $\pi : F \to N$ with connection Δ the connection map $K : TF \to F$ is defined as follows

$$\begin{tikzcd}
TF \arrow[r, "\sim"] \arrow[d] \arrow[rrrr, bend left, "K"] & \Delta \oplus VF \arrow[r, "\mathrm{pr}_2"] \arrow[d] & VF \arrow[r, "\mathcal{J}^{-1}"] \arrow[d] & \pi^*F \arrow[r, "\mathrm{pr}_1"] \arrow[d] & F \arrow[d] \\
F \arrow[r, "\mathrm{id}"] & F \arrow[r, "\mathrm{id}"] & F \arrow[r, "\mathrm{id}"] & F \arrow[r, "\pi"] & N
\end{tikzcd} . \tag{4.2}$$

By making (4.2) commute, K is a vector bundle morphism along π. With this definition, it is well known that

$$\begin{tikzcd}
TF \arrow[r, "(D\pi, K)", "\sim"'] \arrow[d] & TN \oplus F \arrow[d] \\
F \arrow[r, "\pi"] & N
\end{tikzcd}$$

is a vector bundle isomorphism along π. If we now consider the specific vector bundle $\pi : T^*M \to M$ the diagram above becomes

$$\begin{tikzcd}
TT^*M \arrow[r, "(D\pi_{T^*M}, K)", "\sim"'] \arrow[d] & TM \oplus T^*M \arrow[d] \\
T^*M \arrow[r, "\pi_{T^*M}"] & M
\end{tikzcd} . \tag{4.3}$$

We will use (4.3) to define some extra structure on E.

Riemannian vector bundle: The metrics on the tangent and cotangent bundle make E into a Riemannian vector bundle via

$$g_E := g \oplus g^* \in \Gamma(E^* \otimes E^*) \quad g_{E_p}((\xi, a), (\zeta, b)) = g(\xi, \zeta) + g^*(a, b)$$

for $\xi, \zeta \in T_pM$ and $a, b \in T^*_pM$.

Symplectic vector bundle: The form

$$\widehat{\omega} \in \Gamma(E^* \otimes E^*) \quad \widehat{\omega}_p((\xi, a), (\zeta, b)) = b(\xi) - a(\zeta),$$

where $\xi, \zeta \in T_pM$ and $a, b \in T^*_pM$, defines a symplectic form on each fiber. Recall Example 2.8.

Complex vector bundle: For

$$J \in \Gamma(\mathrm{End}(E)) \quad J_p(\xi, a) = (-\widetilde{g}^{-1}(a), \widetilde{g}(\xi)),$$

where $\xi \in T_pM$ and $a \in T^*_pM$, defines a complex structure on E making it into a complex vector bundle.

LEMMA 4.2. *The triple* $(\widehat{\omega}, J, g_E)$ *is compatible, that is*

$$\widehat{\omega}((\xi, a), J(\zeta, b)) = g_E((\xi, a), (\zeta, b))$$

for $\xi, \zeta \in T_pM$ *and* $a, b \in T_p^*M$.

PROOF. Straightforward computation. ■

Let (M, g) be a Riemannian manifold and $\sigma \in \Omega^2(M)$ closed so that (T^*M, ω_σ) is a magnetic cotangent bundle as in Definition 4.1. We follow [Mer10, Section 2.2, v2] modifying the isomorphism (4.3) adapting it to this situation.

Related to the magnetic symplectic form is the Lorentz force $Y \in \Gamma(\mathrm{End}(TM))$ defined by

$$Y_p(\xi) := \widetilde{g}_p^{-1}(\sigma_p(\xi, \cdot)) \quad \xi \in T_pM,$$

with the property that

$$\sigma(\xi, \zeta) = g(Y(\xi), \zeta). \tag{4.4}$$

DEFINITION 4.3. We define

$$\begin{aligned} F_\sigma : TT^*M &\to E \\ ((p, a), \xi) &\mapsto (\xi^h, \xi^\sigma) \end{aligned}$$

with

$$\xi^h := D\pi_{T^*M}(p, a)\xi \quad \xi^\sigma := K(\xi) - \frac{1}{2}Y(\xi^h).$$

PROPOSITION 4.4. *The map*

$$F_\sigma : (TT^*M, \omega_\sigma) \to (E, \widehat{\omega})$$

is a symplectic vector bundle isomorphism along π_{T^*M}, *that is*

$$F_\sigma^*(\widehat{\omega}) = \omega_\sigma.$$

PROOF. We have to show that F_σ is a vector bundle isomorphism and that it respects the symplectic structure on the fibers. For the first point, we slightly adapt the proof that shows that (4.3) is a vector bundle isomorphism. Indeed, F_σ is a bundle morphism along π_{T^*M} by definition. So, as the fibers have the same dimension, one only has to show that the map is injective on the fibers. But if $F_\sigma(\xi) = 0$ then $\xi^h = 0$ and $\xi^\sigma = 0$. The composition of these two implies that $K(\xi) = 0$ so that $\xi \in \Delta \cap V(T^*M) = \{0\}$ as (4.3) is an isomorphism on the fibers.

It remains to show that F_σ respects the symplectic structure. We do this in two steps handling first cotangent bundles with the canonical symplectic form, that is $\sigma = 0$, and then the general magnetic case.

$\underline{\sigma = 0}$. Let $(p, a) \in T^*M$ and $\xi \in T_{(p,a)}T^*M$ so that $\xi = \dot\gamma(0)$ for $\gamma(t) = (\gamma_b(t), \gamma_f(t))$ passing though (p, a). Similarly let $\zeta \in T_{(p,a)}T^*M$ be given as tangent to the curve σ. Then the connection map K has the following form

$$K(\xi) = \nabla^*_{\dot\gamma_b(0)}\gamma_f$$

thus

$$\begin{aligned}
F_0^*(\widehat{\omega})(\xi, \zeta) &= K(\xi)(D\pi_{T^*M}(p, a)\zeta) - K(\zeta)(D\pi_{T^*M}(p, a)\xi) \\
&= \left(\nabla^*_{\dot\gamma_b(0)}\gamma_f\right)\zeta^h - \left(\nabla^*_{\dot\sigma_b(0)}\sigma_f\right)\xi^h \\
&= \nabla_{\dot\gamma_b(0)}(a\zeta^h) - \nabla_{\dot\sigma_b(0)}(a\xi^h) \\
&= da(\xi^h, \zeta^h) = d(a \circ D\pi_{T^*M})(\xi, \zeta) \\
&= \omega_0(\xi, \zeta).
\end{aligned}$$

<u>General case</u>. For a general closed $\sigma \in \Omega^2(M)$ the result follows quickly. Indeed, using the $\sigma = 0$ case, the only thing one still has to show is

$$Y(\xi^h)\zeta^h - Y(\zeta^h)\xi^h = \pi^*_{T^*M}\sigma(\xi, \zeta).$$

But this follows directly form (4.4). ■

F_σ induces also a preferred choice of almost complex structure J_σ on T^*M given by

$$J_\sigma(\xi) = F_\sigma^* J(\xi) = J(\xi^h, \xi^\sigma.)$$

and of metric

$$h_\sigma(\xi, \zeta) := F^* g_E(\xi, \zeta).$$

Because of Lemma 4.2, also $(\omega_\sigma, J_\sigma, h_\sigma)$ forms a compatible triple.

LEMMA 4.5. *Let*

$$H^\sigma = F_\sigma^{-1}(TM \oplus \{0\}) \quad V = F_\sigma^{-1}(\{0\} \oplus T^*M).$$

Then

$$TT^*M = H^\sigma \oplus V$$

and both H^σ and V are Lagrangian subbundles. We call such a splitting a Lagrangian splitting.

PROOF. The fact that H^σ and V are Lagrangian subbundles of TT^*M follows directly from Proposition 4.4 and the fact that $TM \oplus \{0\}$ and $\{0\} \oplus T^*M$ are clearly Lagrangian in E. The splitting follows from the isomorphism property of F_σ. ■

PROPOSITION 4.6. *For an orientable manifold M, the first Chern class of the symplectic manifold (T^*M, ω_σ) vanishes:*

$$c_1(T^*M, \omega_\sigma) = 0. \tag{4.5}$$

Proof. Being a Lagrangian splitting implies that $V^* = H^\sigma$. Indeed, we have a map

$$\alpha : H^\sigma \to V^* \quad \xi \mapsto \omega_\sigma(\xi, \cdot),$$

that is injective, as if $\alpha(\xi) = 0$ then $\xi \in V^{\omega_\sigma}$, the symplectic complement of V. But V is Lagrangian so $V^{\omega_\sigma} = V$ and V intersects H^σ trivially so that $\xi = 0$. By a dimension argument one then sees that α must be an isomorphism. Remark that this is a complex linear map (considering V^* with the dual complex structure $J^*(f)(\xi) = f(J(\xi)))$ as $(J^*\alpha)(\xi) = \alpha(\xi) \circ J$. See [Ben18, Proposition 2.6] for a similar argument. The result now follows from the additivity and duality of the first Chern class, i.e.

$$c_1(E \oplus F) = c_1(E) + c_1(F) \qquad c_1(E^*) = -c_1(E).$$

■

4.2. Quantum cohomology

Quantum (co-)homology was initially introduced in Section 2.3.2 for compact manifolds. As this is not the case for cotangent bundles, we need to consider an appropriate modification with locally finite homology in order for Poincaré duality to work. Here we closely follow [Rit14, Section 2.12].

For a non-compact M, quantum cohomology is define exactly as in Section 2.3.2 as a Λ-module is $QH^\bullet(M, \omega) = H^\bullet(M; \Lambda)$, that is,

$$QH^k(M, \omega) = H^k(M, \Lambda) = \bigoplus_{i+j=k} H^i(M) \otimes \Lambda_j.$$

On the other hand, in order to make the Poincaré duality work, we consider *locally finite quantum homology*

$$QH_\bullet^{\mathrm{LF}}(M) \coloneqq H_\bullet^{\mathrm{LF}}(M; \Lambda).$$

As said, it is useful to remark that Poincaré duality (which does not hold in the non-compact case) may be extended to the non-compact case with this generalisation:

$$PD_c : H^k(M) \xrightarrow{\sim} H_{n-k}^{\mathrm{LF}}(M)$$

for a manifold of dimension n.

Considering that $c_1(T^*S^2) = 0$ by (4.5), the only non zero subring is Λ_0 so that

$$QH^k(M, \omega) = H^k(M) \otimes \Lambda_0.$$

Proposition 4.7 (Quantum cup product). *Let N be a symplectic manifold with $H^k(N) = 0$ for $k \geq 4$. Then the quantum cup product on $QH^\bullet(N)$ is undeformed, that is for $a \in H^\bullet(M)$ and $b \in H^\bullet(M)$*

$$a * b = a \smile b.$$

For $N = (T^*M, \omega_\sigma)$ a magnetic cotangent bundle of a surface, this follows form the vanishing of the first Chern class $c_1(T^*M) = 0$.

Proof. The second part follows directly from Proposition 2.47 and degree considerations.

For magnetic cotangent bundles with vanishing first cohomology group, we can argue without the need of Proposition 2.47 (whose proof is technical and was omitted) using the vanishing of $c_1(T^*M)$ from Section 4.1. Let t be the generator of $H^0(M)$, $f \in H^1(M)$ and m a generator of $H^2(M)$. Because of $c_1(T^*M) = 0$, we are only interested in three invariants, namely $\mathrm{GW}_{A,3}(t,t,0)$, $\mathrm{GW}_{A,3}(t,m,m)$ and $\mathrm{GW}_{A,3}(f,f,m)$. The latter vanishes by sign considerations in the computation of the invariant. By Lemma 2.42 any invariant in a non zero cohomology class A involving the generator of $H^0(M)$ is zero. So for $0 \neq A \in H_2(M)$, we have

$$\mathrm{GW}_{A,3}(t,t,0) = 0 \quad \mathrm{GW}_{A,3}(t,m,m) = 0. \tag{4.6}$$

By simple degree considerations $m * m = 0 = m \smile m$. Recalling the formula defining the quantum cup product

$$\langle (a*b)_A, c\rangle = \mathrm{GW}_{A,3}(a,b,c),$$

one concludes from (4.6) that $t*t = t \smile t = t$, $t*m = (t*m)_0 = t \smile m$ and $f*f = f \smile f$. ■

4.3. Symplectic cohomology of magnetic cotangent bundles

First, in Section 4.3.1, we follow the work of Benedetti and Ritter [BR20, Section 6] and show that despite ω_σ not being exact, we can still see magnetic cotangent bundles as Liouville domains. Then, we follow [Rit14, Section 2-] and provide a tool to express the symplectic cohomology of certain manifolds as a quotient of quantum homology.

4.3.1. Well-definiteness

We introduced the symplectic cohomology for Liouville domains in Section 2.5. It is a priori not clear if magnetic cotangent bundles fit into this category, as the restriction of the symplectic form ω_σ to the unit tangent bundle does not seem to be exact. There are two things that have to be checked. First, one should show that on the boundary the symplectic form is exact; furthermore the primitive should be positive (i.e. the Liouville vector field should be outwards pointing). We work on T^*S^2, but similar results are true for other surfaces, see [BR20, Section 6].

Consider the set of pairs

$$\mathcal{N} := \left\{ (g,\eta) : g \text{ is any metric and } \int_{S^2} \eta = 4\pi \right\}.$$

The condition on η is not restrictive, as we may substitute η with $c\eta$ and use a different scaling in (4.1). By the Gauss–Bonnet theorem, the form $\eta' \coloneqq \eta - K\sigma$ is exact, where σ is the natural volume form on S^2 with respect to g and K the curvature. We thus get a primitive $\beta \in \Omega^2(S^2)$ of η', which we pack into the new form

$$\theta_{s,\beta} \coloneqq \theta - s\pi^*\beta + s\tau \tag{4.7}$$

where $\tau_{(p,v)}(w) \coloneqq \frac{1}{|v|^2} g_p(w^\sigma, \jmath v)$, with $\jmath : T^*M \to T^*M$ the fibrewise rotation by $\frac{\pi}{2}$, is the S^1-connection form. We denote $\alpha_{s,\beta} \coloneqq \theta_{s,\beta}|_{S^*S^2}$.

PROPOSITION 4.8. *Let $D^*S^2 \coloneqq \{(q,p) \in T^*S^2 : |p| \leq 1\}$ the unit co-disk bundle over S^2 and assume that σ is non-exact and nowhere vanishing. Then $(D^*S^2, \omega_{s\sigma})$ is a Liouville domain for s large enough.*

PROOF. By construction, $\theta_{s,\beta}$ from (4.7) is a primitive of ω_σ away from the zero section (which is all that we need). So it remains to show that the induced vector field on the boundary is outwards pointing. In order to show this last point let $f_\eta^g \colon S^2 \to \mathbb{R}$ be the density of η with respect to μ, that is, $\eta = f_\eta^g \mu$. Then for a pair $(g,\eta) \in \mathcal{N}$ we set

$$s_-(g,\eta) = \sup_{d\beta = \eta'} \left\{ s_* \geq 0 : \forall 0 \leq s \leq s_*,\ 1 - \|\beta\| s + \min(f_\eta^g) s^2 > 0 \right\},$$

and

$$A \coloneqq \{(g,\eta,s) \in \mathcal{N} \times \mathbb{R} : s < s_-(g,\eta)\}.$$

We claim that this set is nonempty and connected. Furthermore, for each triple $(g,\eta,s) \in A$, $(D, \omega_\sigma, \alpha_{s,\beta})$ is a Liouville domain, where β is a primitive of η' with $s < s_-(g,\eta,\beta)$.

A is not empty as the standard choices for g and σ clearly live in A for any choice of s. Indeed, $(g,\sigma) \in \mathcal{N}$ as we defined $\mathcal{N}$ exactly so and $s_-(g,\sigma) = +\infty$ as for $\beta = 0$ the polynomial in the definition of s_- has no positive real root. Connectedness follows form the fact that $\mathcal{N}$ is path connected, which is inherited by A. The evaluation on the Liouville vector field Z gives

$$\alpha_{\beta,s}(Z) \geq 1 - \|\beta\| s + s^2 \min_{q \in S^2} s^2 > 0$$

by the choice of s. This same procedure works for s large enough instead of s small enough. ∎

4.3.2. Hamiltonian S^1-action on Floer cohomology

This subsection follows the exposition by SEIDEL in [Sei97].

We denote by G the group of loops of Hamiltonian diffeomorphisms based at the identity

$$G \coloneqq \{g : S^1 \to \mathrm{Ham}(M) : g(0) = \mathrm{id}\}.$$

For the rest of this section, we will work with a $g \in G$, so we assume that such a g exists.

Assumption 4.9. We assume that there exists a non-trivial $g \in G$. Furthermore from Section 4.3.3 we assume that the Hamiltonian generating g is linear at infinity.

We say that a Hamiltonian $K^g \in C^\infty(S^1 \times M, \mathbb{R})$ generates g if

$$\partial_t g_t(p) = X_{K^g_t}(g_t(p))$$

for all $p \in M$. The group G acts on the loop space $\mathcal{L}M \subset C^\infty(S^1, M)$ via

$$(g \cdot \gamma)(t) = g_t(\gamma(t)).$$

Lemma 4.10 (Seidel). *The action of G on $\mathcal{L}M$ lifts to an action on the universal cover $\widetilde{\mathcal{L}M}$.*

Proof. As $\widetilde{\mathcal{L}M}$ is a connected cover, it is sufficient to show that the action preserves smooth maps $S^1 \to \mathcal{L}M$ that can be lifted to $\widetilde{\mathcal{L}M}$. Such maps are given by $B \in C^\infty(S^1 \times S^1, M)$ having $\omega(B) = 0$ (and $c_1(B) = 0$ which is always the case for cotangent bundles). Denoting by $B'(s,t) := g_t(B(s,t))$ one sees that $B'^*\omega = B^*\omega + d\eta$ where $\eta(s,t) = K_g(t, g_t(B(s,t)))dt$. So $\omega(B') = 0 = \omega(B)$. ∎

We put

$$\widetilde{G} \subset G \times \mathrm{Homeo}(\widetilde{\mathcal{L}_0 M})$$

the subgroup of pairs $(g, \widetilde{g})$ such that $\widetilde{g}$ is a lift of the action of g.

The following lemma is just an application of parallel transport. See for example [Moo01, Chapter 1, Proposition 2] for a neat proof.

Lemma 4.11. *Let $F_1, F_2 \colon N \to M$ be two smoothly homotopic maps and $E \to M$ a vector bundle over M. Then F_1^*E and F_2^*E are isomorphic vector bundle over N. In particular, vector bundles over contractible manifolds are trivial.*

Thus a point $c = [v, \gamma]$ defines a symplectic trivialisation of the pullback bundle along γ in the way that v^*TM is trivial, and we may restrict it to γ^*TM obtaining

$$\tau_c : \gamma^*TM \to S^1 \times \mathbb{R}^{2n}.$$

Via this trivialisation, an element $(g, \widetilde{g})$ of $\widetilde{G}$ induces a map

$$\begin{aligned} \ell : S^1 &\to \mathrm{Sp}(2n, \mathbb{R}) \\ t &\mapsto \tau_{\widetilde{g}(c)}(t) \circ Dg_t \circ \tau_c(t)^{-1}. \end{aligned} \tag{4.8}$$

This defines the Maslov index of $\widetilde{g}$ via

$$I(\widetilde{g}) = \deg(\ell_{\widetilde{g}}). \tag{4.9}$$

Here deg: $H_1(\mathrm{Sp}(2n,\mathbb{R})) \to \mathbb{Z}$ is the isomorphism induced by the determinant on $U(n) \subset \mathrm{Sp}(2n,\mathbb{R})$.

Recall the definition of Γ in (2.21) and its action on $\widetilde{\mathcal{L}M}$ as in Section 2.4.1. Then choosing $g_0 = \mathrm{id}$ and $\widetilde{g}_0$ the multiplication by $A \in \Gamma$ leads to

$$2I(\widetilde{g}_0) = 2c_1(A). \tag{4.10}$$

The pullback of a pair (H, J) of Hamiltonian and almost complex form is defined by

$$g^*H_t(p) := H_t(g_t(p)) - K_t^g(g_t(p)) \quad J_t^g := Dg_t^{-1} \circ J_t \circ Dg_t.$$

Defining the pullback like this has its motivation in the following lemma.

Lemma 4.12. *Let K^g be the Hamiltonian generating $g \in G$. Consider a Hamiltonian H and denote by $\alpha_H := d\mathcal{A}_H$ the action one-form. Then*

$$g^*\alpha_H = \alpha_{g^*H}.$$

Proof. Once we have noticed that the action one-form is given by

$$\alpha_H(\gamma)\xi = \int_{S^1} \omega(\dot{\gamma}(t) - X_{H_t}(\gamma(t)), \xi(t))dt,$$

we see that

$$\begin{aligned}
\alpha_{g^*H}(\gamma)\xi &= \int_{S^1} \omega(\dot{\gamma}(t), \xi(t))dt - \int_{S^1} d(g^*H)(\xi(t))dt \\
&= \int_{S^1} \omega(\dot{\gamma}(t), \xi(t))dt - \int_{S^1} dH(Dg_t\xi(t))dt + \int_{S^1} dK_t^g(Dg_t\xi(t))dt \\
&= g^*\alpha_H(\gamma)\xi.
\end{aligned}$$

■

As a direct consequence of Lemma 4.12 the action functionals relate via

$$\mathcal{A}_{g^*H} = \widetilde{g}^*\mathcal{A}_H + C$$

for some constant $C \in \mathbb{R}$, so that also critical points coincide: $\mathrm{crit}(\mathcal{A}_H) = \widetilde{g}(\mathrm{crit}(\mathcal{A}_{g^*H})$. On the same wavelength, one may show that the moduli spaces are in bijection.

Lemma 4.13. *Let γ, δ be critical points of $\mathcal{A}_{g^*H}$. Then there is a bijection*

$$\mathcal{M}(\gamma, \delta; g^*H, g^*J)/\mathbb{R} \longleftrightarrow \mathcal{M}(\widetilde{g}(\gamma), \widetilde{g}(\delta); H, J)/\mathbb{R}$$

Furthermore, if (H, J) is a regular pair then so is (g^*H, g^*J).

Lemma 4.13 means that we can do Floer homology on the pulled-back pair. See [Sei97, Section 4] for a formal proof. The Maslov index has an interesting property that we will exploit later on.

Proposition 4.14 (Maslov index of pullback). *Let* $\widetilde{\gamma} \in \operatorname{crit}(\mathcal{A}_{g^*H})$ *be a non degenerate critical point. Then*

$$\mu_H(\widetilde{g}(\widetilde{\gamma})) = \mu_{g^*H}(\widetilde{\gamma}) - 2I(g, \widetilde{g})$$

where I *is defined as in* (4.9).

Proof. Let (v', δ) be some representative of $\widetilde{g}(\widetilde{\gamma})$ and $\tau_{\widetilde{g}(c)}$ be a trivialisation of δ^*TM as discussed when introducing I in (4.9). Then the loop used in the definition of $\mu_H(\widetilde{g}(\widetilde{\gamma}))$ reads

$$\Psi_{g^*H,\widetilde{g}(c)}(t) = \tau_{\widetilde{g}(c)}(t) D\varphi^t_{g^*H}(\delta'(0))\tau_{\widetilde{g}(c)}(0)^{-1}$$

with $\varphi^t_{g^*H}$ the Hamiltonian flow of g^*H. The definition of g^*H says that $\varphi^t_H = g_t \varphi^t_{g^*H} g_0^{-1}$. So the above loop may be rewritten as $\ell(t)\Psi_{g^*H,c}(t)\ell(0)^{-1}$ where ℓ is as in (4.8) and $\Psi_{g^*H,c}(t)$ is the loop used to define the index at c. If two paths $\Psi_1, \Psi_2 : [0,1] \to \mathrm{Sp}(2n, \mathbb{R})$ are related by a loop $\rho : S^1 \to \mathrm{Sp}(2n, \mathbb{R})$ via $\Psi_2 = \rho(t)\Psi_1(t)\rho(0)^{-1}$, then

$$\mu(\Psi_2) = \mu(\Psi_1) - 2\deg(\rho)$$

by Definition 2.27, and thus the proof is complete. ■

Theorem 4.15. *For any* $g \in G$ *with lift* $(g, \widetilde{g}) \in \widetilde{G}$, *there is an isomorphism*

$$\begin{aligned} s_{\widetilde{g}} : CF^\bullet(H) &\to CF^{\bullet + 2I(\widetilde{g})}(g^*H) \\ \langle \gamma \rangle &\mapsto \langle \widetilde{g}^{-1} \cdot \gamma \rangle \end{aligned}$$

and an induced isomorphism on cohomology

$$\begin{aligned} \mathcal{S}_{\widetilde{g}} : FH^\bullet(H) &\to FH^{\bullet + 2I(\widetilde{g})}(g^*H) \\ \langle \gamma \rangle &\mapsto \langle \widetilde{g}^{-1} \cdot \gamma \rangle. \end{aligned}$$

Proof. The proof is easy having the above results. Lemma 4.13 give us an identification of the moduli spaces, which according to Proposition 4.14 maps from the left-hand side to the right-hand side. ■

4.3.3. Isomorphism on symplectic cohomology

In the previous subsection, we constructed an isomorphism at the level of Floer cohomology starting from an Hamiltonian circle action. Now we show that if the circle action is generated by a Hamiltonian linear at infinity, then the isomorphism on Floer cohomology descends to an isomorphism on symplectic cohomology. We closely follow [Rit14, Section 4].

Denote by $\mathrm{Ham}_\ell(M, \omega)$ the set of Hamiltonian diffeomorphisms linear at infinity. Similarly, G_ℓ denotes the subgroup of loops $g : S^1 \to \mathrm{Ham}_\ell \subset \mathrm{Ham}$ so that $g \in G_\ell$ is generated by $K^g_t : M \to \mathbb{R}$ of the form $K^g_t(p) = \kappa_t R(y)$ for some $\kappa_t \in \mathbb{R}$. We use the notation $\ell \geq 0$ for linear Hamiltonians with non negative slope. Let us assume for this subsection that the Hamiltonian circle action g is generated by a Hamiltonian linear at infinity of slope κ_t, that is, $K_t^g(e^r) = \kappa_t e^r$. One may homotope g_t within G_ℓ to ensure that $\kappa_t = \kappa$ is time independent, see [Rit14, Remark 19].

LEMMA 4.16. *For a g as above, H_m a Hamiltonian linear at infinity with slope m and J an almost complex form of contact type, g^*J is of contact type and g^*H_m is linear at infinity with slope $m - \kappa$.*

PROOF. We only have to consider what happens on the collar. K^g is linear there, so g_t must preserve r and consequently Dg_t preserves the Liouville vector field $e^r\partial_r$. On the other hand

$$g^*H = H \circ g_t - K^g \circ g_t = (m - \kappa)e^r,$$

which completes the proof. ■

The following result is a rephrasing of Theorem 4.15.

PROPOSITION 4.17. *The direct limit of the maps $\mathcal{S}_{\widetilde{g}}$ induce a Λ-module automorphism*

$$\mathcal{S}_{\widetilde{g}} : SH^*(M) \to SH^{*+2I(\widetilde{g})}(M).$$

PROOF. This follows directly by taking colimits of the maps in Theorem 4.15. By Lemma 4.16 this is well-defined, as after taking the pullback we end up again with a linear Hamiltonian. ■

4.4. Hamiltonian circle action

In this section we derive a Hamiltonian circle action on the magnetic cotangent bundle $(T^*S^2, \omega_{s\sigma})$ where σ is the standard volume form on S^2. We consider $S^2 \subset \mathbb{R}^3$ so that $\times$ denotes the cross product and $\langle \cdot, \cdot \rangle$ the usual inner product. Identifying $T_pS^2 = p^\perp \subset \mathbb{R}^3$, the usual volume form on S^2 is given by $\sigma \in \Omega^2(S^2)$

$$\sigma_p(v, w) = \langle p, v \times w \rangle.$$

THEOREM 4.18 (BIMMERMANN [Bim20]). *For any $s > 0$ there exists a symplectomorphism*

$$F : (D_{\sqrt{s+1}/2}S^2, \omega_{s\sigma}) \to (S^2 \times S^2 \setminus \Delta, \pi_1^*\sigma + (s+1)\pi_2^*\sigma),$$

where $\Delta \coloneqq \{(p,p) \colon p \in S^2\}$. The symplectomorphism is $SO(3)$ invariant, so it satisfies

$$\pi_1(F(p, v)) + (s+1)\pi_2(F(p, v)) = p \times v - s \cdot p \tag{4.11}$$

As BIMMERMANN notices, $S^2 \times S^2 \setminus \Delta$ is the configuration space of a pair of particles moving on S^2 (we remove the diagonal as they can not be on exactly the same position at the same time). From (4.11) we can derive a useful relation between the kinetic energy Hamiltonian $\mathsf{E}^{\mathsf{kin}}(p,v) = \frac{1}{2}\|v\|^2$ and the Hamiltonian

$$\begin{aligned}\mathsf{E}^{\mathsf{pot}} : S^2 \times S^2 &\to \mathbb{R}\\ (p,q) &\mapsto (s+1)(1-\langle p,q\rangle),\end{aligned}$$

which describes the potential energy on $S^2 \times S^2 \setminus \Delta$ considering massless particles connected by a spring of constant $s+1$ as shown in Figure 4.1. One may see this by applying the cosine law $d^2 = 2 - 2\langle p,q\rangle$ and the formula for the elastic potential energy $\frac{1}{2}(s+1)d^2$.

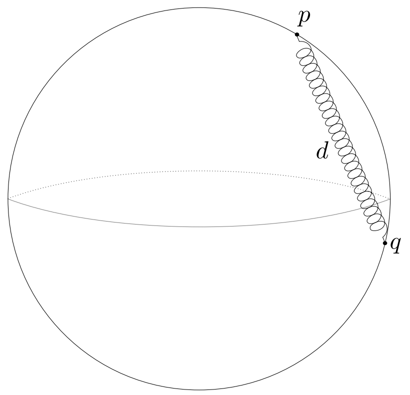

Figure 4.1. Particles on S^2 coupled by a spring.

Back to the relation between $\mathsf{E}^{\mathsf{kin}}$ and $\mathsf{E}^{\mathsf{pot}}$ we notice that (4.11) implies

$$\|N + RS\|^2 = \|p \times v - sp\|^2 \Leftrightarrow \langle N, S\rangle = \frac{1}{s+1}\mathsf{E}^{\mathsf{kin}}(p,v) - 1,$$

where $(N,S) = F(p,v)$. So

$$\mathsf{E}^{\mathsf{pot}} \circ F = \frac{s+1}{8} - \mathsf{E}^{\mathsf{kin}}.$$

Thus, it is natural to investigate $\mathsf{E}^{\mathsf{pot}}$ on $S^2 \times S^2$ where we have a nice expression for the symplectic form. We will shorthand $R = s+1$ and $\omega_s := \pi_1^*\sigma + (s+1)\pi_2^*\sigma$.

Considering γ, δ two curves going through $p, q \in S^2$ we have that[a]

$$d\mathsf{E}^{\mathsf{pot}}|_{p,q}(\dot\gamma, \dot\delta) = -R(\langle\dot\gamma, q\rangle + \langle p, \dot\delta\rangle)$$

while taking other two curves η, χ trough the same points

$$\omega_s|_{p,q}((\dot\eta, \dot\chi), (\dot\gamma, \dot\delta)) = \langle p, \dot\eta \times \dot\gamma\rangle + R\langle q, \dot\chi \times \dot\delta\rangle.$$

Imposing the condition for Hamiltonian vector fields $\iota_{X_{\mathsf{E}^{\mathsf{pot}}}}\omega = d\mathsf{E}^{\mathsf{pot}}$ yields

$$X_{\mathsf{E}^{\mathsf{pot}}}(p,q) = (Rp \times q, -p \times q).$$

[a]We suppress the evaluation at 0 and leave it implicit in the notation.

We investigate if the flow of $X_{\mathsf{E}^{\mathsf{pot}}}$ may provide a circle action. To do so let (p_t, q_t) denote the flow at time t of the point (p_0, q_0). This satisfies

$$\frac{d}{dt}(p_t, q_t) = X_{\mathsf{E}^{\mathsf{pot}}}(p_t, q_t) \Leftrightarrow \begin{cases} \dot{p}_t = Rp_t \times q_t \\ \dot{q}_t = -p_t \times q_t \end{cases} \tag{4.12}$$

One notices that $\dot{p}_t = -R\dot{q}_t$ so that

$$p_t = -Rq_t + d \tag{4.13}$$

for some constant $d \in \mathbb{R}^3$ given by $p_0 = -Rq_0 + d$. Thus (4.12) reduces to

$$\dot{q}_t = -(-Rq_t + d) \times q_t = -d \times q_t. \tag{4.14}$$

The Rodriguez formula gives an explicit solution to (4.14) in the form of

$$q_t = d\frac{d \cdot q_0}{\|d\|^2} + \left(q_0 - d\frac{d \cdot q_0}{\|d\|^2}\right)\cos(\|d\|t) + \frac{d}{\|d\|} \times q_0 \sin(\|d\|t), \tag{4.15}$$

and p_t is then determined via (4.13).

The problem with (4.15) is that it is periodic, but of period $\frac{2\pi}{\|c\|} = \frac{2\pi}{\|p_0 + Rq_0\|}$ dependent on the starting point of the flow. This is incompatible with a circle action.

In order to obtain a period independent of the initial condition we modify the starting equation (4.12) and then try to recover a Hamiltonian that matches the new ODE.

We consider for $a \in \mathbb{R}$

$$\begin{cases} \dot{p}_t = \frac{-aR}{\|p_t + Rq_t\|} p_t \times q_t \\ \dot{q}_t = \frac{a}{\|p_t + Rq_t\|} p_t \times q_t \end{cases}. \tag{4.16}$$

The same considerations as above lead to

$$p_t = -Rq_t + d$$

with $d = p_0 + Rq_0$. However, now (4.14) reads

$$\dot{q}_t = \frac{ad}{\|d\|} \times q_t, \tag{4.17}$$

so that (4.15) translates into

$$q_t = c\frac{c \cdot q_0}{-a} + \left(q_0 - c\frac{c \cdot q_0}{a}\right)\cos(-at) + \frac{c}{-a} \times q_0 \sin(-at) \tag{4.18}$$

for $c = \frac{da}{\|d\|}$ of period $\frac{2\pi}{|a|}$.

So we want to find a Hamiltonian H that has as corresponding vector field

$$X_{\mathsf{H}}(p, q) = \left(\frac{-aR}{\|p + Rq\|} p \times q, \frac{a}{\|p + Rq\|} p \times q\right)$$

The imposed conditions means

$$\begin{aligned} d\mathsf{H}_{p,q}(\dot\gamma,\dot\eta) &= \iota_{X_{\mathsf{H}}}\omega(\dot\gamma,\dot\eta) \\ &= \left\langle p, \frac{-aR}{\|p+Rq\|}(p\times q)\times\dot\gamma\right\rangle + R\left\langle q, \frac{a}{\|p+Rq\|}(p\times q)\times\dot\eta\right\rangle \end{aligned} \tag{4.19}$$

Remark that $\langle p, (p\times q)\times\dot\gamma\rangle = \langle -q,\dot\gamma\rangle$ and similarly for p, q and $\dot\eta$ so that (4.19) reads

$$dH_{p,q}(\dot\gamma,\dot\eta) = \frac{-aR}{\|p+Rq\|}(\langle q,\dot\gamma\rangle + \langle p,\dot\eta\rangle). \tag{4.20}$$

The Hamiltonian

$$\begin{aligned} \mathsf{H} : S^2\times S^2 &\to \mathbb{R} \\ (p,q) &\mapsto a\|p+Rq\| \end{aligned}$$

satisfies (4.20). So the flow of H generates a Hamiltonian S^1-action. We choose $a\in 1+2\pi\mathbb{N}$ so that the period is not a multiple of the minimal Reeb flow (see discussion in Section 2.5). Furthermore from (4.16) it is clear that the action restricts to the identity on the diagonal Δ, inducing so a circle action on $S^2\times S^2\setminus\Delta$.

By (4.11)

$$\langle \pi_1 F(p,v), \pi_2 F(p,v)\rangle = \frac{1}{R}E(p,v) - 1,$$

so as

$$\|p+Rq\| = \sqrt{1+2R\langle p,q\rangle + R^2},$$

H on the disk bundle is given via

$$\mathsf{H}_a \coloneqq \mathsf{H}\circ F = a\sqrt{\|v\|^2+s^2}. \tag{4.21}$$

4.4.1. Linearity

A natural question about H_a in this non-compact setting is whether or not it is linear at infinity. In particular, the methods that we developed in Section 4.3 only work for a circle action generated by a Hamiltonian of that type. H_a does not look linear in the common radial coordinate $\rho(p,v) = \|v\|$. Recall, however, that the Liouville structure we are working with is not the standard one because of the non-exactness of $\omega_{s\sigma}$.

Lemma 4.19. *Consider the primitive $\theta_{s,0}$ for $\omega_{s\sigma}$. Then the flow of the Liouville vector field induced by $\omega_{s\sigma}$ is given by*

$$\rho_s(p,v) = \sqrt{\frac{\|v\|^2+s^2}{1+s^2}}.$$

PROOF. Denoting by Z_s the Liouville vector field of $\theta_{s,0}$, i.e. $i_{Z_s}\omega_{s\sigma} = \theta + s\tau$, and by r_s the coordinate on the collar induced by the flow of Z_s we have to show that $\rho_s = e^{r_s}$. To do so, we first differentiate

$$\frac{d\rho}{dr_s} = d\rho(Z_s) = \frac{1}{\rho}(\theta + s\tau)(X_\rho) = \frac{1}{\rho}(\rho^2 + s^2),$$

and then integrate the result multiplied by $\frac{\rho}{\rho^2+s^2}$ obtaining

$$\int \frac{\rho}{\rho^2 + s^2} d\rho = \int dr_s \Leftrightarrow r_s = \frac{1}{2}\ln(\rho^2 + s^2),$$

thus proving the statement. ■

So H_a is linear at infinity being of the form

$$\mathsf{H}_a = a\sqrt{s^2+1} \cdot \rho_s.$$

From (4.18) it is easy to see that we obtain circle actions of periods $\frac{2\pi}{|a|}$.

4.4.2. Index

Recall from Section 2.5 that linear Hamiltonians with slope not a multiple of the period of a Reeb orbit have 1-periodic orbits lying on some hypersurface of constant radial coordinate. In particular, there are no 1-periodic solutions outside a compact set. In this specific case, the only 1-periodic orbits are constants $(p, 0) \in T^*S^2$. Remark that the isomorphism F maps $(p, 0)$ to $(N, -N) \in S^2 \times S^2$.

We will compute the index using a different approach than RITTER. The Morse index theorem by DUISTERMAAT [Dui76] states that for cotangent bundles the Lagrangian version of the Conley–Zehnder of a critical point of the Lagrangian action functional corresponds to its Morse index. The Lagrangian version of the Conley–Zehnder index differs from the one used in this work by a factor of 2. DUISTERMAAT's result is stated for the standard set-up and not the magnetic one. However, the same result holds for ω_σ as shown by MERRY [Mer10, Appendix A, v2]. Another trick may be used to avoid computing the Morse index explicitly. The following is an early result by Morse [Mil63, Theorem 15.1].

PROPOSITION 4.20 (Morse). *The index of a geodesic γ is equal to the number of points $\gamma(t)$, with $0 < t < 1$, such that $\gamma(t)$ is conjugate to $\gamma(0)$ along γ (counting with multiplicity).*

Recall that two points p and q are said to be conjugate along a geodesic δ that connects them if there exists a non-zero Jacobi field along δ that vanishes at both p and q.

Having the above result it is now easy to see that the constant orbit $(N, -N) \in S^2 \times S^2$ has Morse index 2 and thus Conley–Zehnder index 1. Meaning $I(\widetilde{g}) = 1 \neq 0$. Indeed, any point has only itself as conjugate and its order must be two as there are two linearly independent Jacobi fields along a constant geodesic on this $S^2 \times S^2$.

4.5. Vanishing of $SH^\bullet(T^*S^2, \omega_\sigma)$

This last section aims to use the circle action from Section 4.4 and apply the tools from Section 4.3 to the magnetic cotangent bundle (T^*S^2, ω_σ).

We adopt the notation from Section 4.3. The circle action g is the one from Section 4.4. The careful reader will notice that the only tools one needs is Section 4.3 and a Hamiltonian circle action linear at infinity with non-vanishing index, so the results of this section generalise to the class of manifolds satisfying these conditions. Thus we will present the intermediate statements for a general symplectic manifold (M, ω) satisfying the following assumption.

ASSUMPTION 4.21. We consider a Liouville domain (M, ω) with undeformed quantum cohomology and a Hamiltonian S^1-action g linear at infinity with non-vanishing index $I(\widetilde{g}) \neq 0$.

Consider a time-independent Hamiltonian H_0, which is a C^2-small perturbation of 0 and has a positive slope δ at infinity smaller than the minimal Reeb period.

Because of the special form of this Hamiltonian, the PSS map

$$\mathrm{PSS} : QH^\bullet(M) \to HF^\bullet(H_0) \tag{4.22}$$

is still a ring isomorphism, despite the non-compact nature of the manifold. In fact despite non-compactness, for small Hamiltonians, the Floer cohomology reduces to Morse cohomology ([Aud13, Chapter 10]) which is the quantum cohomology by the same procedure as in Chapter 3.

We then denote by

$$H_k := H_0 + k\mathsf{H}_{1+2\pi}$$

a generic Hamiltonian with slope $\delta + k\kappa$. By Lemma 4.12 we know

$$g^* H_k = H_{k-1}.$$

DEFINITION 4.22. Use the notation from above and consider the continuation map $\varphi_0 : CF^\bullet(H_{-1}) \to CF^\bullet(H_0)$. Then we define the maps $\mathcal{R}_{\widetilde{g}}$ and $r_{\widetilde{g}}$ by making the following diagram commute

$$\begin{tikzcd}
 & & QC^{\bullet+2I(\widetilde{g})}(M) \\
 & & CF^{\bullet+2I(\widetilde{g})}(H_0) \arrow[u, "\mathrm{PSS}^{-1}"] \\
QC^\bullet(M) \arrow[r, "\mathrm{PSS}"] \arrow[uurr, "r_{\widetilde{g}}"] & CF^\bullet(H_0) \arrow[r, "\mathcal{S}_{\widetilde{g}}"] \arrow[ur, "\mathcal{R}_{\widetilde{g}}"] & CF^{\bullet+2I(\widetilde{g})}(H_{-1}) \arrow[u, "\varphi_0"]
\end{tikzcd}.$$

As we work with a fixed g, we will drop it from the notation. Furthermore the index will be denoted by $I(g, \widetilde{g}) =: j$.

We have a product structure on the Floer cohomology, so one may ask if the maps $\mathcal{S}$ and $\mathcal{R}$ are homomorphisms. This is not true. However, a weaker property holds.

Proposition 4.23. *The maps $\mathcal{R}$ and $\mathcal{S}$ are both compatible with products, meaning*

$$\mathcal{R}(a \succ b) = \mathcal{R}(a) \succ b$$

And similarly for $\mathcal{S}$.

Proof. We only give a sketch of the proof. The first thing one has to notice is that the definition of the pair-of-pants product given in Section 3.2.3 has a compactness issue when working on open manifolds. This can be fixed by slightly changing it as in [Rit14, Seciton 4.3]. At the homological level, $\mathcal{S}$ only depends on $(\widetilde{g}, g)$ up to homotopy. We can ensure that g is the identity not only at zero but also on a greater closed interval containing zero leaving $\mathcal{S}$ invariant. This can be done maintaining the positive slope. By doing so we ensure that near one of the cylindrical ends $K_t^g = 0$, so that the data (g^*H_α, g^*J) is the same as (H_α, J) (where H_α is the Hamiltonian on this specific cylinder). So $\mathcal{S}$ leaves this cylinder invariant showing the statement. ■

Lemma 4.24. *The sequence of modules*

$$HF^\bullet(H_0) \xrightarrow{\varphi_1} HF^\bullet(H_1) \xrightarrow{\varphi_2} \cdots \tag{4.23}$$

defining the symplectic cohomology $SH^\bullet(M)$ is equivalent to a sequence

$$V \xrightarrow{\varphi} V \xrightarrow{\varphi} \cdots \tag{4.24}$$

meaning that there exist isomorphisms ψ_k so that the following commutes

$$\begin{tikzcd} HF^\bullet(H_0) \arrow[r, "\varphi_1"] \arrow[d, "\psi_0"] & HF^\bullet(H_1) \arrow[r, "\varphi_2"] \arrow[d, "\psi_1"] & \cdots \\ V \arrow[r, "\varphi"] & V \arrow[r, "\varphi"] & \cdots \end{tikzcd} . \tag{4.25}$$

Furthermore, φ stabilised. That is, there exists a $N \in \mathbb{N}$ for which

$$\varphi^N(V) = \varphi^{N+1}(V).$$

Proof. Fundamental are the diagrams in Definition 4.22 and (2.34). By Theorem 4.15 we have a commutative diagram

$$\begin{tikzcd} HF^\bullet(H_k) \arrow[r, "\mathcal{S}"] & HF^{\bullet+2j}(H_k) \\ HF^\bullet(H_{k+1}) \arrow[r, "\mathcal{S}"] \arrow[u, "g^*\varphi_k"] & HF^{\bullet+2j}(H_{k+1}) \arrow[u, "\varphi_k"'] \end{tikzcd} . \tag{4.26}$$

So we get that

$$\varphi_{k+1} = g^*\varphi_k = \mathcal{S}^{-1} \circ \varphi_k \circ \mathcal{S} \tag{4.27}$$

by Lemma 4.12 and by induction $\varphi_k = \mathcal{S}^{-k} \circ \varphi_0 \circ \mathcal{S}^k$. On the other hand, by the properties of continuation maps

$$\varphi_{k+1} = \varphi_k \circ \cdots \circ \varphi_1 = \mathcal{S}^{-k}\mathcal{R}^k.$$

We set $V = HF^\bullet(H_0)$. Then conjugation by $\mathcal{S}^k$ as in (4.27) gives the required maps ψ_k. By definition V has finite rank so that an induction argument shows that $\varphi^k(V)$ must stabilise. ■

PROPOSITION 4.25 ([Rit14]). *Let M be a symplectic manifold satisfying Assumption 4.21. Then there exits a submodule $S \subset QH^\bullet(M)$ such that*

$$SH^\bullet(M) \cong QH^\bullet(M)/S.$$

Furthermore S is identifiable with $\ker(\varphi^N \circ \mathrm{PSS})$, *where N is as in Lemma 4.24.*

PROOF. It is now trivial to show using the result of Lemma 4.24 and the definition of colimit that

$$SH^\bullet(M) \cong HF^\bullet(H_0)/\ker(\varphi^N).$$

We then pass to the quantum cohomology using the PSS isomorphism. ■

THEOREM 4.26. *The symplectic cohomology of (T^*S^2, ω_σ) vanishes, that is,*

$$SH^\bullet(T^*S^2, \omega_\sigma) = 0.$$

FIRST PROOF. Since the map $\mathcal{R}$ is compatible with products, it is uniquely determined by $\mathcal{R}(1)$ as

$$\mathcal{R}(a) = \mathcal{R}(1 \succ a) = \mathcal{R}(1) \succ a.$$

Consequently also $\mathcal{R}^N$ is uniquely determined by $\mathcal{R}(1)$.

Using the PSS isomorphism from (4.22) and the fact that the quantum cohomology is undeformed as shown in Proposition 4.7, we conclude as follows: Denoting by $a = \mathcal{R}(1)$ and N the integer from Lemma 4.24

$$\Phi^{-1}(\mathcal{R}^N(1)) = \Phi^{-1}(a) * \ldots * \Phi^{-1}(a) = \Phi^{-1}(a) \smile \cdots \smile \Phi^{-1}(a) = 0.$$

The last equality holds for our choice of N big enough and the fact that $\Phi^{-1}(a)$ has non-zero degree. As φ^N factors via $\mathcal{R}^N$ this concludes the proof. ■

SECOND PROOF. This second proof is more direct. Proposition 4.17 shows that the rank of $SH^\bullet(T^*S^2, \omega_{s\sigma})$ must be zero or infinity as we have an isomorphism of degree two. But the rank of $QH^\bullet(M)$ is always finite, so Proposition 4.25 shows that it is zero. ■

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