ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Continuity of Magnitude

Bachelor's Thesis

Maria Morariu

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Advisor: Dr. Sara Kališnik Hintz
Department of Mathematics, ETH Zürich

Abstract

The magnitude of a finite metric space is a numerical invariant, introduced by Leinster in 2011 as a measure for the 'effective number of species'. In this thesis, we follow Leinster's approach in [1] to define the magnitude, as well as the magnitude function of finite metric spaces and show some examples and ways to compute it. Based on Meckes' work in [2], we consider possible generalizations to compact metric spaces and discuss their equivalence for positive definite spaces. We introduce a further numerical invariant, similar to the magnitude, the maximum diversity of a compact metric space. We prove a continuity result for the maximum diversity and see how this translates to continuity of magnitude. In the end, we develop a new perspective on magnitude relying on Hilbert spaces and Fourier theory. We then prove that magnitude is continuous on the class of compact subsets of $\mathbb{R}^n$ with non-empty interior with respect to the Hausdorff distance.

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Chapter 1

Introduction

Magnitude was first introduced by Solow and Polasky under the name 'effective number of species'. They were interested in large-scale reduction in biological diversity through the extinction of plant and animal species. In order to implement a decision-making framework for species conservation, they needed a measure that would quantify species diversity. They characterized diversity in terms of distances arising from biology[1], but arbitrary distances also yield interesting examples for us.

Solow and Polasky wanted to capture the idea that a set consisting of three ant species is less diverse than a set consisting of an ant, a fish and an elephant. They determined three natural requirements that their measure should respect:

(1) *Monotonicity in species*: If a set of species contains an other, the diversity of the first should be larger than that of the second.

(2) *Twinning*: Diversity should not be increased by the addition of a species that is already in the set.

(3) *Monotonicity in distance*: Larger distances between the species should result in larger diversity.

The measure that Solow and Polasky introduced as 'effective number of species' corresponds to what is now called magnitude. The notion of magnitude was first introduced in mathematics in 2011 by Tom Leinster in [1]. His approach was to generalize the Euler characteristic of a category to a similar invariant for metric spaces. With this motivation, he introduced magnitude in the general context of enriched categories, an example of which are precisely metric spaces. It has been proved that under suitable assumptions, magnitude captures some of the most important invariants of a space –

[1]Such distances can be based on morphological, behavioral differences, or molecular biology methods.

dimension, perimeter, area, volume etc. (see Theorem 4.6 in [3]) and it has been conjectured that this is true even more generally (see Conjecture 4.5 in [3]).

A first step towards defining magnitude is considering a special class of finite metric spaces for which magnitude is particularly easy to define. For an arbitrary finite metric space (A, d), the distances between any two points are encoded by the similarity matrix ζ_A with entries $\zeta_A(a, b) = e^{-d(a,b)}$. If the similarity matrix is invertible, we can define the magnitude $|A|$ of A as the sum of the elements of ζ_A^{-1}.

Let us illustrate this on the example of the two-point space $A = \{a, b\}$. Let $d := d(a, b)$. The similarity matrix of A is $\begin{pmatrix} 1 & e^{-d} \\ e^{-d} & 1 \end{pmatrix}$, which has inverse

$$\frac{1}{1 - e^{-2d}} \begin{pmatrix} 1 & -e^{-d} \\ -e^{-d} & 1 \end{pmatrix}$$

and therefore the magnitude is

$$|A| = \frac{2 - 2e^{-d}}{1 - e^{-2d}} = 1 + \frac{1 - 2e^{-d} + e^{-2d}}{1 - e^{-2d}} =$$
$$1 + \frac{(1 - e^{-d})^2}{(1 - e^{-d})^2(1 + e^{-d})^2} = 1 + \tanh\frac{d}{2}.$$

Figure 1.1 shows the magnitude of the space A for different values of the distance. We can observe that for $d = 0$, the magnitude is 1. The two points are effectively just one point and as we had hoped, magnitude captures this. As d grows larger, magnitude grows towards 2, the number of points in our space. As shown in Figure 1.1 for $d = 3$, we already have $2 - |A| < 0.1$.

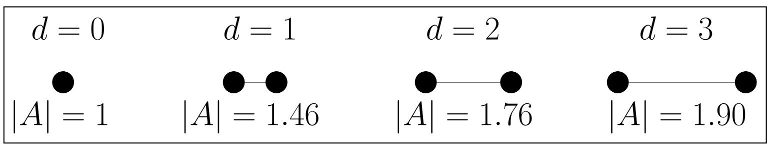


Figure 1.1: Magnitude of the 2-point space A for different distances d.

It is also interesting to note that viewing magnitude at different scales gives additional information about the space. This means that for $t > 0$, the magnitude of the rescaled space[2] tA cannot generally be deduced from the magnitude of the space A. This observation motivates the definition of the magnitude function $t \mapsto |tA|$.

[2]The metric space tA has the same point set as A but distances rescaled by a factor t, meaning that for $a, b \in tA$, the distance in tA is $d_{tA}(a, b) = td(a, b)$.

We can now rewrite the requirements given by Solow and Polasky in modern terms as follows:

(1) Magnitude is monotone with respect to inclusion: $B \subset A \implies |B| \leq |A|$.

(2) Magnitude is continuous with respect to inclusion.

(3) The magnitude function is monotone.

We will see that, while not true in general, magnitude does satisfy these properties under suitable assumptions.

The main goal of this thesis is to review some important results regarding the magnitude of metric spaces, with a focus on continuity results. The category theory approach to magnitude is beyond the scope of this thesis, so we directly define the magnitude of metric spaces.

By relying on Leinster's [1], we introduce the magnitude of finite metric spaces. We consider some possible constructions: expansion, union, tensor product, fibration and constant-distance gluing. By relying on these, we can simplify computations for the magnitude of a finite metric space, by deducing it from the magnitudes of simpler spaces. An interesting feature of magnitude is that it is not scale invariant. In fact, the magnitude function, contains much more information than just the magnitude. Among other aspects, we show that the number of points in the space can be deduced from knowing the magnitude function. However, we also discuss the limitations of the magnitude function. In general, it is not well-defined everywhere and it is not necessarily increasing. To get around this problem, we restrict to the special class of positive definite finite metric spaces, where magnitude does (or is conjectured to) behave better.

A natural question that arises after having discussed magnitude for finite metric spaces is, 'Can it be generalized to infinite ones?' We review some possible generalizations of magnitude to compact metric spaces and discuss strengths and weaknesses of each definition, based on the work of Meckes [2]. There is no generally accepted convention for how to extend the definition from finite metric spaces to infinite ones. Therefore, we restrict ourselves again to positive-definite metric spaces, where all proposed generalizations are equivalent. In this context, our first continuity result for magnitude arises, in the form of lower semicontinuity. We explicitly compute some magnitudes, such as that of a line segment and a circle. As a useful tool, we introduce a further invariant, the maximum diversity, for which a continuity result exists and we show what implications this has for the continuity of magnitude.

In our final chapter, we develop a new perspective on the magnitude of metric spaces based on the theory of Hilbert spaces and Fourier analysis. Equipped with this new tool, we can prove the main known continuity result for magnitude, as presented by Leinster and Meckes in [3]:

Theorem *Let $\mathcal{K}_n$ be the class of compact subsets of $\mathbb{R}^n$ with non-empty interior and suppose that $A \in \mathcal{K}_n$ is star-shaped. Then magnitude, as a map $\mathcal{K}_n \to \mathbb{R}$ is continuous*[3] *at A.*

[3] Here continuity means continuity with respect to the Euclidean distance on $\mathbb{R}$ and the Hausdorff distance on $\mathcal{K}_n$, which we define in the first chapter.

Chapter 2

Magnitude of Finite Metric Spaces

In this chapter, we introduce the magnitude and magnitude function of finite metric spaces, based on the work of Leinster in [1]. We analyze some properties of magnitude and ways to compute it using the magnitude of smaller spaces. We explicitly compute the magnitude for some examples, which illustrate that it does not always behave as nicely as one might expect.

2.1 Metric Spaces – Basic Definitions

Let us start by reviewing some basic definitions about metric spaces.

Definition 2.1 *A **metric space** (A, d) is a set A, together with a map $d\colon A \times A \to [0, \infty]$, such that for all $a, b, c \in A$:*

- $d(a, b) = 0$ *if and only if* $a = b$*;*
- $d(a, b) = d(b, a)$*;*
- $d(a, c) \leq d(a, b) + d(b, c)$*.*

Furthermore, by the metric space tA, for $t > 0$, we mean the metric space with the same points as A, but distances $d_{tA}(a, b) = t \cdot d(a, b)$, for any $a, b \in A$.

In a metric space (A, d), we denote by $B(x, r)$ (or $B^d(x, r)$ if there is some ambiguity about the distance) the open ball of radius r around x, i.e. the set $\{y \in A : d(x, y) < r\}$.

Example 2.2 (Graphs as Metric Spaces) *Let G be a (non-oriented) graph, $t > 0$. We can define the metric space tG, with points the vertices of G and distances minimal path-lengths, where the length of one edge is t.*

Figure 2.1 shows a concrete example of a graph with four points and four edges, viewed as a metric space.

We introduce notions of distance on the class of compact metric spaces.

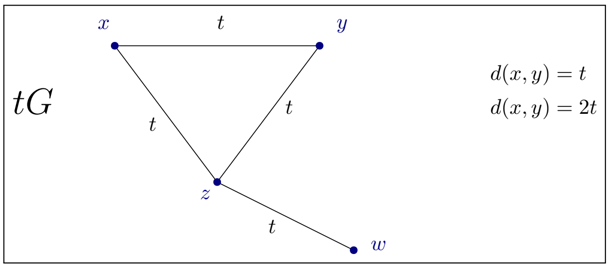


Figure 2.1: Example of graph as metric space.

Definition 2.3 *Given a metric space (X, d) and two compact subspaces $A, B \subset X$, the **Hausdorff distance** between A and B is*

$$d_H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\}.$$

On the class of compact subsets of X, the Hausdorff distance defines a metric.

Let us explicitly compute the Hausdorff distance for some subspaces of the Euclidean space.

Example 2.4 *Consider the subsets $A = \{1, 2, 3\}$ and $B = \{5, 6\}$ of $\mathbb{R}$ (equipped with the Euclidean distance). We have $d(A, 6) = 3$, $d(A, 5) = 2$, so $\sup_{b \in B} d(A, b) = 3$. Also, $d(3, B) = 2$, $d(2, B) = 3$ and $d(1, B) = 4$, so $\sup_{a \in A} d(a, B) = 4$ and thus $d_H(A, B) = 4$.*

Example 2.5 *Let $X = \mathbb{R}^2$, $A = B((0, 0), 1)$ and $B = \{4\} \times [-3, 3]$. (see Figure 2.2) Then $\sup_{a \in A} d(a, B) = 5$ and it is attained for $a = (-1, 0)$. Also, $\sup_{b \in B} d(A, b)$ is attained at $(4, -3)$ and $(4, 3)$. The origin, $(4, 0)$ and $(4, 3)$ form a right triangle with cathetes of lengths $3, 4$, so the hypothenuse has length 5. The radius of the circle is 1, so $\sup_{b \in B} d(A, b) = 4$. Thus, $d_H(A, B) = 5$.*

The following distance notion for metric spaces is slightly more general, since it does not require the two spaces to be embedded into the same space.

Definition 2.6 *Given two metric spaces A, B, the **Gromov-Hausdorff distance** between A and B is*

$$d_{GH}(A, B) = \inf d_H(\varphi(A), \psi(B)),$$

where the infimum is taken over all metric spaces X and all isometric embeddings $\varphi\colon A \to X$, $\psi\colon B \to X$.

This defines a metric on the class of compact metric spaces.

2.2 Definition of Magnitude for Finite Metric Spaces and First Examples

Before introducing magnitude, let us fix some notation. For a finite set A, we denote by $\#A$ the number of points in A. Given finite sets X, Y, we say that ζ is a $X \times Y$-matrix if it has dimensions $\#X \times \#Y$. We view the matrix as indexed by the two sets and refer to elements as $\zeta(x, y)$ for $x \in X$ and $y \in Y$. By δ we denote the identity matrix (the size can be read from the context). We write ζ^T for the transpose of the matrix ζ.

Let us start by defining the magnitude of a matrix.

Definition 2.7 *Given an $A \times A$-matrix ζ, a* ***weighting*** *of ζ is an A-vector w such that $\zeta \cdot w = 1$, where by* $\mathbf{1}$ *we denote the A-vector with all entries 1. The* ***magnitude of the matrix*** *of ζ is $|\zeta| = \sum_{a \in A} w(a)$.*

Relying on the last definition, we can define the magnitude of a metric space:

Definition 2.8 *Given a finite metric space (A, d), we define the* ***similarity matrix*** *of A as the $A \times A$-matrix ζ_A with elements $\zeta_A(a, b) = e^{-d(a,b)}$ for $a, b \in A$. The* ***magnitude of the metric space*** *A is the magnitude of the similarity matrix ζ_A.*

Note that if a weighting exists, then the magnitude of a metric space is well-defined, i.e. it does not depend on the weighting: Given two weightings w, v on A and using the symmetry of the similarity matrix, we have

$$\sum_{a \in A} w(a) = \mathbf{1}^T \cdot w = (\zeta_A v)^T w = (v^T \zeta_A^T) w = v^T (\zeta_A w) = v^T \cdot \mathbf{1} = \sum_{a \in A} v(a).$$

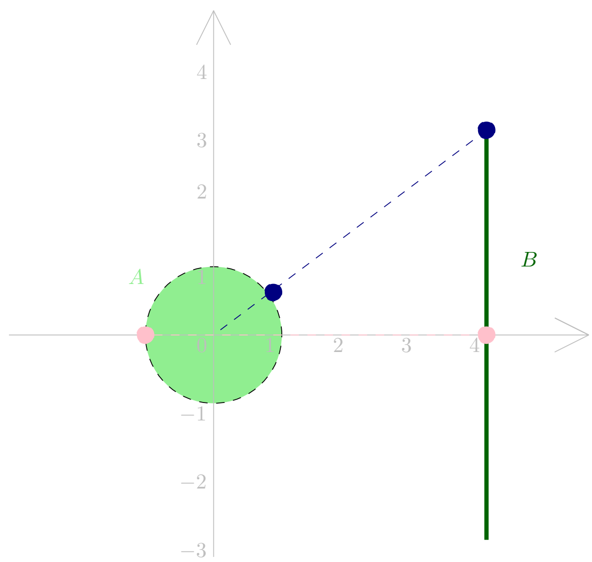


Figure 2.2: The Hausdorff distance between the two spaces is 5.

Let us see how this plays out in a simple example.

Example 2.9 *Let us compute the magnitude of the two-point space* $A = \{a, b\}$. *Let* $d := d(a, b)$. *The similarity matrix of A is* $\zeta_A = \begin{pmatrix} 1 & e^{-d} \\ e^{-d} & 1 \end{pmatrix}$, *which has inverse*

$$\zeta_A^{-1} = \frac{1}{1 - e^{-2d}} \begin{pmatrix} 1 & -e^{-d} \\ -e^{-d} & 1 \end{pmatrix}.$$

We want the weighting to satisfy $\zeta_A w = \mathbf{1}$, *so* $w = \zeta_A^{-1}\mathbf{1}$ *and we can compute the weighting*

$$w = \frac{1}{1 - e^{-2d}} \begin{pmatrix} 1 & -e^{-d} \\ -e^{-d} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{1 - e^{-2d}} \begin{pmatrix} 1 - e^{-d} \\ 1 - e^{-d} \end{pmatrix}.$$

Hence, the magnitude is

$$|A| = \frac{2 - 2e^{-d}}{1 - e^{-2d}} = 1 + \frac{1 - 2e^{-td} + e^{-2td}}{1 - e^{-2td}} =$$
$$1 + \frac{(1 - e^{-td})^2}{(1 - e^{-td})^2(1 + e^{-td})^2} = 1 + \tanh \frac{d}{2}.$$

As seen in the last example, computing magnitude by relying on the inverse of the similarity matrix (if it exists) is very practical. This motivates the following definition:

Definition 2.10 *If* ζ_A *is invertible, we define the* ***Möbius inversion*** *of A as* $\mu_A = \zeta_A^{-1}$.

In the case where the Möbius inversion exists, the weighting is unique, $w_A = \mu_A \cdot \mathbf{1}$ and the magnitude can be directly computed as the sum of the elements of μ_A.

Our definition does not guarantee the existence of magnitude. In fact, we will see that magnitude is not always defined. However, given some 'good' assumptions on a finite metric space, it follows that it has magnitude. A good example is that of a homogeneous metric space.

Definition 2.11 *A metric space* (A, d) *is called* ***homogeneous*** *if its isometry group acts transitively on points.*

Example 2.12 (Complete Graphs are Homogeneous) *Let* K_n *be the complete graph with n vertices,* $t > 0$. *Take two vertices* $v, w \in tK_n$. *Then the map* $\varphi \colon tK_n \to tK_n$ *that swaps v and w and keeps everything else fixed is an isometry, since the distance between any two points is t. Therefore, the metric space* tK_n *is homogeneous.*

We can give an explicit formula for the magnitude of a finite homogeneous metric space:

Example 2.13 (Magnitude of a Finite Homogeneous Metric Space) *We show that a homogeneous metric space A with n elements has magnitude*

$$|A| = \frac{n}{\sum\limits_{a \in A} e^{-d(x,a)}}.$$

To see this, fix $x \in A$ and set $S := \sum\limits_{a \in A} \zeta_A(x,a) = \sum\limits_{a \in A} e^{-d(x,a)}$. By the definition of homogeneity, this sum is independent of x. Therefore, we can define a weighting $w(a) = \frac{1}{S}$ on A. Indeed, this is a weighting, since for any $a' \in A$, $\sum\limits_{a \in A} \zeta_A(a',a)w(a) = \frac{S}{S} = 1$. Therefore, the magnitude of A is $\sum\limits_{a \in A} \frac{1}{S} = n \cdot \frac{1}{S}$.

This allows us to compute the magnitude of a complete graph.

Example 2.14 (Magnitude of Complete Graph) *Consider the complete graph with n vertices K_n. Then, K_n with distance given by the shortest path-length, such that the length of one edge is 1 is a homogeneous metric space with n points such that the distance between any two distinct points is 1 and the distance from a point to itself is 0. Therefore, we can apply Example 2.13: Fix $x \in K_n$. Then $S = e^0 + (n-1) \cdot e^{-1} = 1 + (n-1)e^{-1}$, so $|K_n| = \frac{n}{1+(n-1)e^{-1}}$.*

2.3 Constructions

As with other invariants, we would like to determine the behavior of the magnitude when a space is obtained from others through certain operations. This allows us to reduce the problem of computing the magnitude of a metric space to a (hopefully) simpler one.

2.3.1 Expansion

Intuitively, one might expect to have some kind of monotonicity statement for magnitude with respect to inclusion and dilation. We will see examples that show that a general monotonicity result does not exist. However, if a non-negative weighting exists, we do have a monotonicity statement.

Definition 2.15 *Given two metric spaces A and B, we say that A is an* ***expansion*** *of B if there exists a distance-decreasing surjection $A \to B$.*

Lemma 2.16 (Monotonicity with Respect to Expansion) *Given two metric spaces (A, d_A) and (B, d_B), such that each of them admits a non-negative weighting and A is an expansion of B, it follows that $|A| \geq |B|$.*

Proof Let $f\colon A \to B$ be a distance-decreasing surjection. As for any surjection, we can choose $g\colon B \to A$ such that $f \circ g = id_B$. Then for any $a \in A, b \in B$ we have $d_A(a, g(b)) \geq d_B(f(a), f(g(b)) = d_B(f(a), b)$. This implies that $\zeta_B(f(a), b) = e^{-d_B(f(a),b)} \geq e^{-d_A(a,g(b))} = \zeta_A(a, g(b))$.

Let w_A and w_B be non-negative weightings on A, respectively B. Then for every $a \in A$, we have $\sum_{b \in B} \zeta_B(f(a), b) w_B(b) = 1$. Thus

$$\begin{aligned}|A| &= \sum_{a \in A} w_A(a) = \sum_{a \in A} \left(\sum_{b \in B} \zeta_B(f(a), b) w_B(b) \right) w_A(a) = \\ &= \sum_{a \in A, b \in B} \zeta_B(f(a), b) w_B(b) w_A(a) \geq \sum_{a \in A, b \in B} \zeta_B(a, g(b)) w_B(b) w_A(a) = \\ &= \sum_{b \in B} w_B(b) \left(\sum_{a \in A} \zeta_B(a, g(b)) w_A(a) \right) = \sum_{b \in B} w_B(b) = |B|.\end{aligned}$$

And we obtain the desired inequality. □

Note that in particular, if $B \subset A$ and both admit a non-negative weighting, the lemma immediately implies that $|B| \leq |A|$.

2.3.2 Unions

We want to adapt the inclusion-exclusion principle for the cardinality of the union of to finite space to the magnitude. Again, the statement does not hold in general, however we can add the assumption that the sets *project* onto each other in order to obtain the desired principle.

Definition 2.17 *Given a metric space X and subspaces $A, B \subset X$, we say that A* ***projects*** *onto B if $\forall a \in A, \exists \pi(a) \in A \cap B$ such that*

$$\forall b \in B, d(a, b) = d(a, \pi(a)) + d(\pi(a), b).$$

Then $d(a, \pi(a)) = \inf_{b \in B} d(a, b)$.

Proposition 2.18 (Inclusion Exclusion Principle for Magnitude) *Let X be a metric space and subspaces $A, B \subset X$, such that A projects onto B and B projects onto A. If A, B and $A \cap B$ have magnitude, so does $A \cup B$ and*

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Proof Let w_A, w_B, $w_{A \cap B}$ be weightings on A, B, respectively $A \cap B$. We define the following weighting on $A \cup B$:
For $x \in A \cup B$ set

$$w_{A \cup B}(x) = \begin{cases} w_A(x), \text{ if } x \in A \setminus B \\ w_B(x), \text{ if } x \in B \setminus A \\ w_A(x) + w_B(x) - w_{A \cap B}(x), \text{ if } x \in A \cap B. \end{cases}$$

We show that this is indeed a weighting, so we check that $\zeta_{A\cup B} w_{A\cup B} = \mathbf{1}$.

We treat the following cases:

1. $x \in A \setminus B$. We compute

$$\begin{aligned}
&\sum_{a\in A\cup B} \zeta_{A\cup B}(x,a) w_{A\cup B}(a) = \\
&= \sum_{a\in A\setminus B} \zeta_{A\cup B}(x,a) w_{A\cup B}(a) + \sum_{b\in B\setminus A} \zeta_{A\cup B}(x,b) w_{A\cup B}(b) + \sum_{a\in A\cap B} \zeta_{A\cup B}(x,a) w_{A\cup B}(a) = \\
&= \sum_{a\in A\setminus B} \zeta_{A\cup B}(x,a) w_A(a) + \sum_{b\in B\setminus A} \zeta_{A\cup B}(x,b) w_B(b) + \\
&+ \sum_{a\in A\cap B} \zeta_{A\cup B}(x,a)(w_A(a) + w_B(a) - w_{A\cap B}(a)) = \\
&= \sum_{a\in A} \zeta_{A\cup B}(x,a) w_A(a) + \sum_{b\in B} \zeta_{A\cup B}(x,b) w_B(b) - \sum_{a\in A\cap B} \zeta_{A\cup B}(x,a) w_{A\cap B}(a)
\end{aligned}$$

Observe that since $x \in A$, we have $\zeta_{A\cup B}(x,a) = \zeta_A(x,a)$ for any $a \in A$. So since w_A is a weighting, $\sum_{a\in A} \zeta_{A\cup B}(x,a) w_A(a) = 1$.

Also, A projects onto B, so $\exists \pi(x) \in A \cap B$ such that

$$\forall b \in B, d(x,b) = d(x,\pi(x)) + d(\pi(x),b).$$

This means that for any $b \in B$,

$$\begin{aligned}
&\zeta_{A\cup B}(x,b) = e^{-d(x,b)} = e^{-(d(x,\pi(x))+d(\pi(x),b))} = e^{-d(x,\pi(x))} \cdot e^{-d(\pi(x),b)} = \\
&= \zeta_{A\cup B}(x,\pi(x)) \cdot \zeta_{A\cup B}(\pi(x),b).
\end{aligned}$$

Thus, we can continue the chain of equalities above:

$$\begin{aligned}
&\sum_{a\in A\cup B} \zeta_{A\cup B}(x,a) w_{A\cup B}(a) = \\
&= 1 + \sum_{b\in B} \zeta_{A\cup B}(x,\pi(x)) \zeta_{A\cup B}(\pi(x),b) w_B(b) - \\
&- \sum_{a\in A\cap B} \zeta_{A\cup B}(x,\pi(x)) \zeta_{A\cup B}(\pi(x),a) w_{A\cap B}(a) = \\
&= 1 + \zeta_{A\cup B}(x,\pi(x)) \cdot \left(\sum_{b\in B} \zeta_{A\cup B}(\pi(x),b) w_B(b) - \sum_{a\in A\cap B} \zeta_{A\cup B}(\pi(x),a) w_{A\cap B}(a) \right) = \\
&= 1 + \zeta_{A\cup B}(x,\pi(x)) \cdot (1-1) = 1
\end{aligned}$$

where we used that $\pi(x) \in A \cap B$, so in particular $\pi(x) \in B$ and $w_{A\cap B}, w_B$ are weightings, so

$$\sum_{b\in B} \zeta_{A\cup B}(\pi(x), b) w_B(b) = \sum_{a\in A\cap B} \zeta_{A\cup B}(\pi(x), a) w_{A\cap B}(a) = 1.$$

2. $x \in B \setminus A$. This case is completely analogous to the one above.
3. $x \in A \cap B$. As above, we compute

$$\begin{aligned}
&\sum_{a\in A\cup B} \zeta_{A\cup B}(x,a) w_{A\cup B}(a) = \\
&= \sum_{a\in A} \zeta_{A\cup B}(x,a) w_A(a) + \sum_{b\in B} \zeta_{A\cup B}(x,b) w_B(b) - \sum_{a\in A\cap B} \zeta_{A\cup B}(x,a) w_{A\cap B}(a) = \\
&= 1 + 1 - 1 = 1,
\end{aligned}$$

where the last equality follows since $x \in A \cap B$ implies $x \in A$ and $x \in B$ and from the definition of a weighting.

Thus, $w_{A\cup B}$ defines a weighting. Therefore, we can compute the magnitude as

$$\begin{aligned}
|A \cup B| &= \sum_{x\in A\cup B} w_{A\cup B}(x) = \sum_{a\in A\setminus B} w_A(a) + \sum_{b\in B\setminus A} w_B(b) + \\
&+ \sum_{a\in A\cap B} (w_A(a) + w_B(a) - w_{A\cap B}(a)) = \\
&= \sum_{a\in A} w_A(a) + \sum_{b\in B} w_B(b) - \sum_{a\in A\cap B} w_{A\cap B}(a) = |A| + |B| - |A \cap B| \qquad \square
\end{aligned}$$

A quite trivial example of why the condition that the two sets project onto each other is necessary is the 2-point space:

Example 2.19 *Given the* 2*-point space* $\{a, b\}$ *with* $a \neq b$ *we can set* $A = \{a\}$ *and* $B = \{b\}$*. Then* $|A| = |B| = 1$*, however, as we have seen in Example 2.9, the magnitude of* $A \cup B$ *is not* 2.

Our inclusion-exclusion principle immediately implies the following:

Corollary 2.20 *Let* X *be a finite metric space,* $A, B \subset X$*, such that* $A \cap B = \{c\}$ *and* $\forall a \in A, \forall b \in B$*,* $d(a, b) = d(a, c) + d(c, b)$*. If* A *and* B *have magnitude, then* $A \cup B$ *has magnitude* $|A| + |B| - 1$*.*

Moreover, if A *and* B *have Möbius inversions* μ_A, μ_B*, then so does* $A \cup B$ *and*

$$\mu_{A\cup B}(x,y) = \begin{cases} \mu_A(x,y), & \textit{if } x,y \in A, (x,y) \neq (c,c) \\ \mu_B(x,y), & \textit{if } x,y \in B, (x,y) \neq (c,c) \\ \mu_A(c,c) + \mu_B(c,c) - 1, & \textit{if } (x,y) = (c,c) \\ 0, & \textit{otherwise.} \end{cases}$$

Proof This is a direct application of the above proposition. We can see that A and B project onto each other by setting $\pi(a) = \pi(b) = c$ for any $a \in A, b \in B$. Now, the proposition above gives $|A \cup B| = |A| + |B| - |A \cap B|$ and since the magnitude of the one-point space is 1, we obtain the first part.

Let us now assume that A and B have Möbius inversions μ_A, μ_B. We show that the given formula gives an inverse of the matrix $\zeta_{A\cup B}$, i.e. that $\mu_{A\cup B} \cdot \zeta_{A\cup B} = \delta$. We treat the following cases:

(1) $a, b \in A$, $(a, b) \neq (c, c)$. Then

$$(\mu_{A\cup B} \cdot \zeta_{A\cup B})(a, b) = \sum_{x \in A\cup B} \mu_{A\cup B}(a, x)\zeta_{A\cup B}(x, b) =$$
$$= \sum_{x\in A} \mu_A(a, x)\zeta_A(x, b) = \delta_{a,b}.$$

(2) $a, b \in B$, $(a, b) \neq (c, c)$. This case follows similarly to the previous one.

(3) $(a, b) = (c, c)$

$$(\mu_{A\cup B} \cdot \zeta_{A\cup B})(c, c) = \sum_{x\in A\cup B} \mu_{A\cup B}(c, x)\zeta_{A\cup B}(x, c) =$$
$$= \sum_{x\in A\setminus\{c\}} \mu_A(c, x)\zeta_A(x, c) + (\mu_A(c, c) + \mu_B(c, c) - 1)\zeta_{A\cup B}(c, c) +$$
$$+ \sum_{x\in B\setminus\{c\}} \mu_B(c, x)\zeta_B(x, c) =$$
$$= \sum_{x\in A} \mu_A(c, x)\zeta_A(x, c) + \sum_{x\in B} \mu_B(c, x)\zeta_B(x, c) - 1 = 1 + 1 - 1 = 1.$$

Therefore, $\mu_{A\cup B} \cdot \zeta_{A\cup B} = \delta$ and we are done. □

We can use this to determine the magnitude of any finite subset of $\mathbb{R}$.

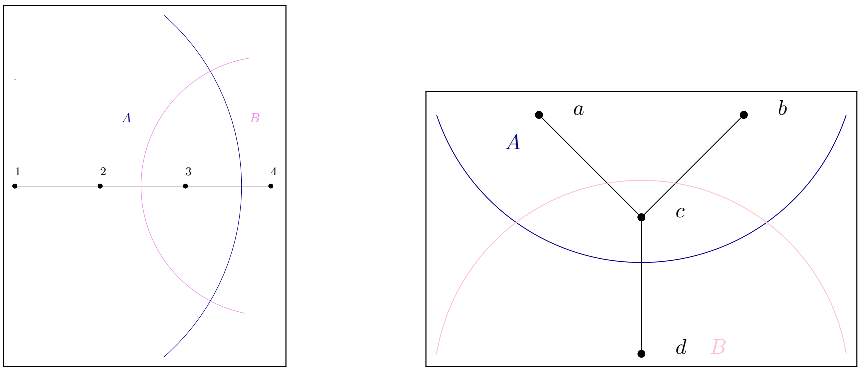
Corollary 2.21 *Let $A = \{a_0, ..., a_n\}$ be a subset of $\mathbb{R}$ with $a_0 < ... < a_n$ and set $d_i = a_i - a_{i-1}$. Then A has magnitude $|A| = 1 + \sum_{i=1}^{n} \tanh \frac{d_i}{2}$. (We make the convention that $d_0 = d_{n+1} = \infty$ and $\tanh \infty = 1$)*

Proof We show this by induction on n. If $n = 0$, there is nothing to prove. For $n = 1$, the claim follows directly from Example 2.9.

Let $n > 1$ and assume that the statement holds for any $k < n$. We can define the sets $B = \{a_0, ..., a_{n-1}\}$ and $C = \{a_{n-1}, a_n\}$. By the previous corollary, we deduce $|A| = |B| + |C| - 1$ and by induction hypothesis we get the desired conclusion. □

The magnitude is not a complete invariant of finite metric spaces. In fact, there exist non-isometric spaces with the same magnitude function, as shown by the following example.

Example 2.22 *Consider the spaces $X = \{0, 1, 2, 3\} \subset \mathbb{R}$ and the Y-shaped set Y with the metric from Example 2.2 (see Figure 2.3), where all edges have equal length. Clearly, the two spaces are not isometric, but for both we can take a set A containing 3 points ($\{1, 2, 3\}$ respectively $\{a, b, c\}$) and a set B containing 2 points ($\{3, 4\}$, respectively $\{c, d\}$) as depicted in Figure 2.3. We can now apply Corollary 2.20 to see that the magnitudes are equal.*



(a) X and the sets A, B

(b) Y and the sets A, B

Figure 2.3: Magnitude is not a complete invariant of metric spaces.

2.3.3 Tensor Product

One can determine the cardinality of the cartesian product of two sets as the product of their cardinalities. An analogous relation holds for the magnitude of the tensor product of two metric spaces.

We can define the tensor product of two metric spaces (A, d_A) and (B, d_B) as follows:

Definition 2.23 *Let $A \otimes B$ be the metric space whose point-set is $A \times B$ and with distance given by*

$$d((a, b), (a', b')) = d_A(a, a') + d_B(b, b'), \forall a, a' \in A, \forall b, b' \in B.$$

With this definition, we get the following result:

Proposition 2.24 (Magnitude of Tensor Product) *If two finite metric spaces A and B have magnitude, so does their tensor and*

$$|A \otimes B| = |A| \cdot |B|.$$

Proof Let w_A, w_B be weightings on A, respectively B. We define

$$w(a, b) = w_A(a) \cdot w_B(b)$$

and show that this is a weighting on $A \otimes B$.

Fix $x, y \in A \otimes B$. Then

$$\sum_{(a,b)\in A\otimes B} \zeta_{A\otimes B}((x,y),(a,b))w(a,b) = \sum_{a\in A, b\in B} e^{-d((x,y),(a,b))}w(a,b) =$$
$$= \sum_{a\in A, b\in B} e^{-d_A(x,a)}e^{-d_B(y,b)}w_A(a)w_B(b) =$$
$$= \sum_{a\in A} e^{-d_A(x,a)}w_A(a)\left(\sum_{b\in B} e^{-d_B(y,b)}w_B(b)\right) = \sum_{a\in A} e^{-d_A(x,a)}w_A(a)\cdot 1 = 1.$$

So, w gives a weighting and therefore $A \otimes B$ has magnitude. We can compute it as

$$|A\otimes B| = \sum_{a\in A, b\in B} w_A(a)w_B(b) = |A|\cdot|B|,$$

which is exactly the desired equality. □

We can now determine the magnitude of the space $\mathbb{F}_q^N$ endowed with the Hamming metric.

Example 2.25 *For a prime power q, let $\mathbb{F}_q$ be the field with q elements. We can endow it with a metric defined as $d(a,b) = 1$ for any $a \neq b$. Let $N \in \mathbb{N}$ and denote $\mathbb{F}_q^N := \bigotimes_{i=1}^{N} \mathbb{F}_q$, endowed with the Hamming metric, i.e. the distance between two points is the number of positions where the two N-tuples are different. By Proposition 2.24, $|\mathbb{F}_q^N| = |\mathbb{F}_q|^N$. We can view $\mathbb{F}_q$ as a complete graph with q vertices, so by Example 2.36, the magnitude is given by $|\mathbb{F}_q^N| = \left(\frac{q}{1+(q-1)e^{-1}}\right)^N$.*

2.3.4 Fibrations

It is well-known that the Euler characteristic is multiplicative with respect to fibration. We develop a similar result in the context of the magnitude of metric fibrations.

Definition 2.26 *For two metric spaces A and B, a **metric fibration** from A to B is a distance-decreasing map $p\colon A \to B$ such that $\forall a \in A$, $\forall b' \in B$ with $d(p(a), b') < \infty$, $\exists a_{b'} \in p^{-1}(b')$ satisfying*

$$\forall a' \in p^{-1}(b') : d(a,a') = d(p(a),b') + d(a_{b'},a'). \tag{2.1}$$

We want to define a fibre of B as $p^{-1}(b)$ for a fibration p and some $b \in B$. It is important to note that $p^{-1}(b)$ is independent of b (up to isometry), as shown by the following lemma.

Lemma 2.27 *Let $p\colon A \to B$ be a fibration of metric spaces and $b, b' \in B$ with $d(b,b') < \infty$. Then $p^{-1}(b)$ and $p^{-1}(b')$ are isometric.*

Proof Fix $a \in p^{-1}(b)$. We show that $a_{b'}$ is unique. Assume there exist $a^1_{b'}, a^2_{b'}$, both satisfying (2.1).
Take $a' = a^1_{b'}, a_{b'} = a^1_{b'}$ in (2.1). Then

$$d(a, a^1_{b'}) = d(b, b') + d(a^1_{b'}, a^1_{b'}) = d(b, b'). \tag{2.2}$$

Now, take $a' = a^1_{b'}, a_{b'} = a^2_{b'}$ in (2.1). Then $d(a, a^1_{b'}) = d(b, b') + d(a^2_{b'}, a^1_{b'})$. The two equations, together with finiteness of $d(a, c)$, imply $d(a^2_{b'}, a^1_{b'}) = 0$, so by the definition of a metric, $a^2_{b'} = a^1_{b'}$.

Therefore, we can define a map $\gamma_{b,b'} \colon p^{-1}(b) \to p^{-1}(b')$, by $a \mapsto a_{b'}$. We show that it is distance-decreasing: for $a, c \in p^{-1}(b)$

$$\begin{aligned} d(b, b') + d(\gamma_{b,b'}(a), \gamma_{b,b'}(c)) &= d(b, b') + d(a_{b'}, \gamma_{b,b'}(c)) = \\ &= d(a, \gamma_{b,b'}(c)) \leq d(a, c) + d(c, \gamma_{b,b'}(c)) = d(a, c) + d(b, b') \end{aligned}$$

where the last equality follows from (2.2). Since $d(b, b')$ is finite, we obtain

$$d(\gamma_{b,b'}(a), \gamma_{b,b'}(c)) \leq d(a, c).$$

By symmetry, there exists a distance-decreasing map $\gamma_{b',b} \colon p^{-1}(b') \to p^{-1}(b)$ defined in the same way. We check that these are mutually inverse. Fix $a' \in p^{-1}(b')$. Then $\gamma_{b,b'} \circ \gamma_{b',b}(a') = \gamma_{b,b'}(a'_b)$, which we denote by x. Now Equation (2.1) for $a = a'_b$ gives $d(a'_b, a') = d(b, b') + d(x, a')$. But, by Equation (2.2), $d(a'_b, a') = d(b, b')$, so $d(x, a') = 0$, and therefore $x = a'$. Thus, $\gamma_{b,b'} \circ \gamma_{b',b} = id_{p^{-1}(b')}$. Completely analogously, it follows $\gamma_{b',b} \circ \gamma_{b,b'} = id_{p^{-1}(b)}$. Hence, the two maps are mutually inverse and distance-decreasing, which implies that they are isometries and we are done. □

The lemma now allows us to introduce the following definition:

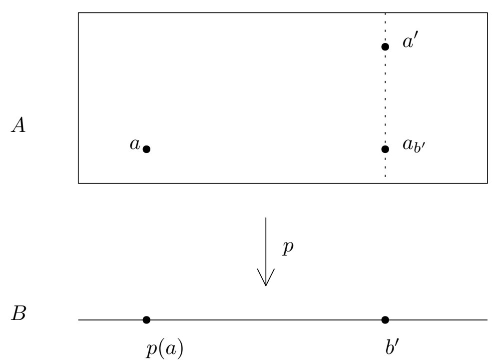


Figure 2.4: Vizualization of fibration

Definition 2.28 *Given a nonempty metric space, such that all distances are finite and a fibration $p\colon A \to B$, we say that the space $p^{-1}(b)$ is the* ***fibre*** *of B.*

By Lemma 2.27, the fibre of B is well-defined up to isometry. The following theorem gives the desired result about fibres.

Theorem 2.29 *Let $p\colon A \to B$ be a fibration of finite metric spaces, such that B is nonempty with finite distances and F the fibre of B. If F and B have magnitude, then so does A and $|A| = |B| \cdot |F|$.*

Proof Let w_B be a weighting on B and for each $b \in B$ let w_b be a weighting on $p^{-1}(b)$. For $a \in A$, define $w_A(a) = w_{p(a)}(a) \cdot w_B(p(a))$. We show that this is indeed a weighting. Fix $a' \in A$ and for $b \in B$ let $a'_b \in p^{-1}(b)$ be as in (2.1). Then

$$
\begin{aligned}
&\sum_{a\in A} \zeta_A(a',a) w_A(a) = \sum_{b\in B} \sum_{a\in p^{-1}(b)} e^{-d(a',a)} w_A(a) = \\
&= \sum_{b\in B} \sum_{a\in p^{-1}(b)} e^{-d(p(a'),b)-d(a'_b,a)} \cdot w_b(a) w_B(b) = \\
&= \sum_{b\in B} e^{-d(p(a'),b)} w_B(b) \left(\sum_{a\in p^{-1}(b)} e^{-d(a'_b,a)} \cdot w_b(a) \right) = \\
&= \sum_{b\in B} e^{-d(p(a'),b)} w_B(b) \cdot 1 = 1.
\end{aligned}
$$

Where in the last two equalities, we have used that w_B and w_b are weightings. Therefore, w_A is a weighting and

$$
\begin{aligned}
|A| &= \sum_{a\in A} w_A(a) = \sum_{b\in B} \sum_{a\in p^{-1}(b)} w_b(a) w_B(b) = \\
&= \sum_{b\in B} w_B(b) \cdot \left(\sum_{a\in p^{-1}(b)} w_b(a) \right) = \sum_{b\in B} w_B(b) \cdot |F| = |B| \cdot |F|. \qquad \square
\end{aligned}
$$

Let us start with a somewhat trivial example of a fibration.

Example 2.30 *Consider the product projection $p\colon B \otimes F \to B$. This is a fibration and by Theorem 2.29, $|B \otimes F| = |B| \cdot |F|$, which is consistent with Proposition 2.24.*

Let us also look at a more interesting example, taken from [4].

Example 2.31 *Consider the example represented in Figure 2.5, where B is a complete graph with three vertices and A is a graph consisting of 6 vertices, as in the figure. We view them as metric spaces with distance given by the shortest path-length, where the length of one edge is equal to 1 (see Example 2.2). We consider the map $p\colon A \to B$ that takes the three inner vertices of A to corresponding vertices in B and* *the three outer vertices of A to the corresponding ones in B. Then this is a metric fibration, as can be seen by directly checking that Equation (2.1) is satisfied for any pair* $(a, b') \in A \times B$. *A fibre is given by* $F = K_2$, *the complete graph with two points. Therefore, by Theorem 2.29,* $|A| = |F| \cdot |B|$. *Now by Example 2.14, we have* $|B| = \frac{3}{1+2e^{-1}}$ *and* $|F| = \frac{2}{1+e^{-1}}$. *Together, these results imply that* $|A| = \frac{6}{1+3e^{-1}+2e^{-2}}$.

2.3.5 Constant-Distance Gluing

We want to see what happens to the magnitude if we 'glue' together two spaces.

Definition 2.32 *Given metric spaces* (A, d_A) *and* (B, d_B) *and* $D \geq \frac{\max(diam(A),\ diam(B))}{2}$, *we define* $\mathbf{A} +_{\mathbf{D}} \mathbf{B}$ *as the metric space with points* $A \sqcup B$ *and metric*

$$d(x, y) = \begin{cases} d_A(x, y), \textit{if } x, y \in A \\ d_B(x, y), \textit{if } x, y \in B \\ D, \textit{otherwise.} \end{cases}$$

Proposition 2.33 *Let* A, B *be finite metric spaces,* $D \geq \frac{\max(diam(A),\ diam(B))}{2}$. *If* A *and* B *have magnitude and* $|A| \cdot |B| \neq e^{2D}$, *then* $A +_D B$ *has magnitude*

$$|A +_D B| = \frac{|A| + |B| - 2e^{-D}|A||B|}{1 - e^{-2D}|A||B|}.$$

Proof As above, let us denote d_A, d_B, d the distances in A, B and $A +_D B$. Let w_A and w_B be weightings on A, respectively B. We define

$$w(a) = \frac{1 - e^{-D}|B|}{1 - e^{-2D}|A||B|} w_A(a) \text{ for } a \in A,$$

$$w(b) = \frac{1 - e^{-D}|A|}{1 - e^{-2D}|A||B|} w_B(b) \text{ for } b \in B.$$

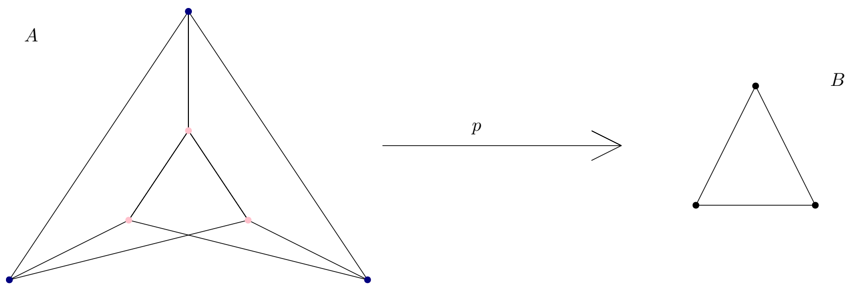


Figure 2.5: Example of a nontrivial fibration

We show that this is a weighting. Fix $a' \in A$. Then

$$\sum_{x \in A+_D B} \zeta_{A+_D B}(a', x) w(x) = \sum_{a \in A} e^{-d(a',a)} w(a) + \sum_{b \in B} e^{-d(a',b)} w(b) =$$
$$= \sum_{a \in A} e^{-d_A(a',a)} \frac{1 - e^{-D}|B|}{1 - e^{-2D}|A||B|} w_A(a) + \sum_{b \in B} e^{-D} \frac{1 - e^{-D}|A|}{1 - e^{-2D}|A||B|} w_B(b) =$$
$$= \frac{1 - e^{-D}|B|}{1 - e^{-2D}|A||B|} \sum_{a \in A} e^{-d_A(a',a)} w_A(a) + e^{-D} \frac{1 - e^{-D}|A|}{1 - e^{-2D}|A||B|} \sum_{b \in B} w_B(b) =$$
$$= \frac{1 - e^{-D}|B|}{1 - e^{-2D}|A||B|} \cdot 1 + e^{-D} \frac{1 - e^{-D}|A|}{1 - e^{-2D}|A||B|} \cdot |B| =$$
$$= \frac{1 - e^{-D}|B| + e^{-D}|B| - e^{-2D}|A||B|}{1 - e^{-2D}|A||B|} = 1.$$

We can do the same computation for fixed $b' \in B$, and conclude that w defines a weighting. Therefore, we can compute the magnitude as

$$|A +_D B| = \sum_{x \in A+_D B} (a', x) w(x) = \sum_{a \in A} w(a) + \sum_{b \in B} w(b) =$$
$$= \sum_{a \in A} \frac{1 - e^{-D}|B|}{1 - e^{-2D}|A||B|} w_A(a) + \sum_{b \in B} \frac{1 - e^{-D}|A|}{1 - e^{-2D}|A||B|} w_B(b) =$$
$$= \frac{1 - e^{-D}|B|}{1 - e^{-2D}|A||B|} \cdot |A| + \frac{1 - e^{-D}|A|}{1 - e^{-2D}|A||B|} \cdot |B| = \frac{|A| + |B| - 2e^{-D}|A||B|}{1 - e^{-2D}|A||B|}. \square$$

This theorem is useful for computing the magnitude of certain graphs, as we will see in the next section.

2.4 The Magnitude Function

Magnitude does not behave predictably with respect to scaling. In fact, viewing the magnitude of all possible rescalings of a space, rather than just the magnitude gives significantly more information. This motivates the following definition:

Definition 2.34 *The **magnitude function** is a partially defined function $(0, \infty) \to \mathbb{R}$, taking $t \mapsto |tA|$.*

Let us first return to the example of the 2-point space.

Example 2.35 *Let $A = \{a, b\}$ and $d = d(a, b)$. We can directly apply Example 2.9 on the space tA, the 2-point space where the distance between the two points is td. Thus, $|tA| = 1 + \tanh \frac{td}{2}$.*

We can now extend Example 2.14 to determine the magnitude function of a complete graph.

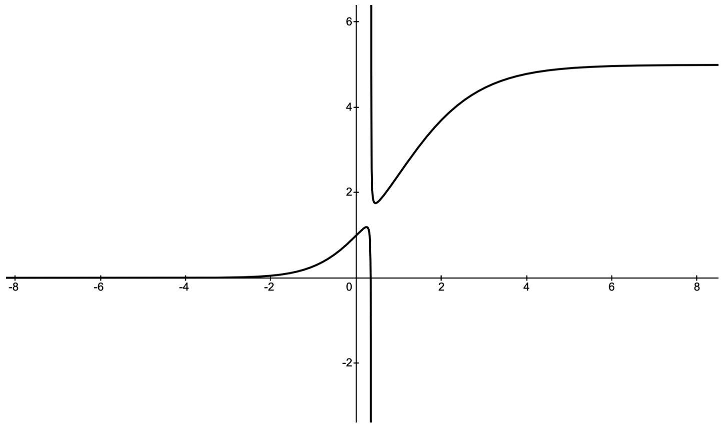


Figure 2.6: Magnitude function for $K_{3,2}$

Example 2.36 (Magnitude Function of a Complete Graph) *Consider the complete graph with n vertices K_n. Then, tK_n with the metric from Definition 2.2 is a homogeneous space with n points such that the distance between any two distinct points is t and the distance from a point to itself is* 0. *Therefore, we can apply Example 2.13: Fix $x \in K_n$. Then $S = e^0 + (n-1) \cdot e^{-t} = 1 + (n-1)e^{-t}$, so $|tK_n| = \frac{n}{1+(n-1)e^{-t}}$.*

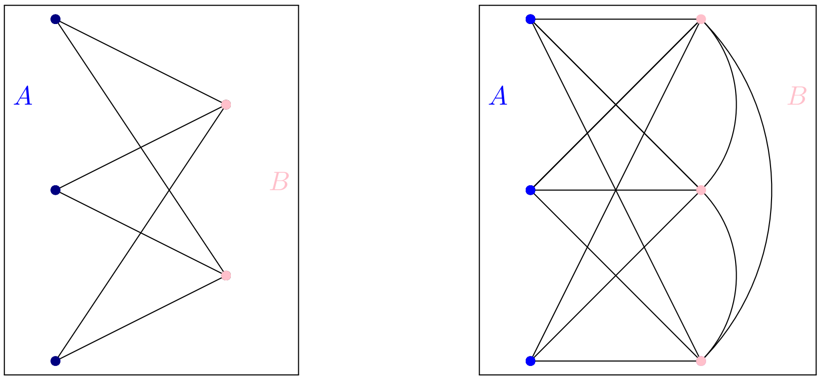
Let us consider some examples that show that the magnitude does not always behave as one might hope.

The following is a fruitful counterexample. It shows that the magnitude function is not necessarily defined everywhere, it is not always increasing and it can take negative values.

Example 2.37 (Pathological Behaviour of Magnitude Function) *Consider the bipartite graph $K_{3,2}$ (see Figure 2.7a). We can pick A as the three elements on one part, B as the two elements in the other. Then both tA and tB are complete graphs with distance* 2*t between their points. Thus by Example 2.36, we have $|tA| = \frac{3}{1+2e^{-2t}}$ and $|tB| = \frac{2}{1+e^{-2t}}$ We can pick $D = t$ in Proposition 2.33. Then $tK_{3,2} = tA +_D tB$. So with Proposition 2.33, we can compute the magnitude function:*

$$|tK_{3,2}| = \frac{\frac{3}{1+2e^{-2t}} + \frac{2}{1+e^{-2t}} - 2e^{-t}\frac{3}{1+2e^{-2t}}\frac{2}{1+e^{-2t}}}{1 - e^{-2t}\frac{3}{1+2e^{-2t}}\frac{2}{1+e^{-2t}}} = \frac{5 - 12e^{-t} + 7e^{-2t}}{1 - 3e^{-2t} + 2e^{-4t}} =$$
$$= \frac{(5-7e^{-t})(1-e^{-t})}{(1-2e^{-2t})(1+e^{-t})(1-e^{-t})} = \frac{5-7e^{-t}}{(1-2e^{-2t})(1+e^{-t})}.$$

This shows that the magnitude function of $K_{3,2}$ is undefined for $t = \log\sqrt{2}$. This magnitude function also has negative values, as well as larger values than the number of points. Also, the magnitude function is decreasing on certain intervals as can be seen clearly from the plot (see Figure 2.6).



(a) $K_{3,2}$ and the sets A, B **(b)** G and the sets A, B

Figure 2.7

The following is an example of a metric space, whose magnitude function does not tend to 1 as t shrinks.

Example 2.38 (Magnitude does not always go to 1 for $t \to 0$) *Consider the graph G, obtained from the bipartite graph $K_{3,3}$ with two pairs of 3 vertices, where we add 3 additional edges, uniting all vertices in the second group (see Figure 2.7b). Then, in the above proposition, we can take A as the first group of three points and B as the second one. Fix $t > 0$. Then for each t, the distance between any two points of tA is $2t$ and the distance between any two points of tB is t. Thus, we can take $D = t$ in Proposition 2.33 and we see $tG = tA +_D tB$. Also, we can view tA and tB as complete graphs with edge lengths $2t$, respectively t, so by Example 2.36, their magnitudes are $|tA| = \frac{3}{1+2e^{-2t}}$ and $|tB| = \frac{3}{1+2e^{-t}}$. Thus, Proposition 2.33 gives*

$$|tG| = \frac{\frac{3}{1+2e^{-t}} + \frac{3}{1+2e^{-t}} - 2e^{-t}\frac{3}{1+2e^{-t}}\frac{3}{1+2e^{-t}}}{1 - e^{-2t}\frac{3}{1+2e^{-t}}\frac{3}{1+2e^{-t}}} = \frac{3 + 6e^{-t} + 3 + 6e^{-2t} - 18e^{-t}}{1 + 2e^{-t} + 2e^{-2t} + 4e^{-3t} - 9e^{-2t}} =$$

$$= \frac{6(1 - 2e^{-t} + e^{-2t})}{(1 + 4e^{-t})(1 - 2e^{-t} + e^{-2t})} = \frac{6}{1 + 4e^{-t}}.$$

Now note that $\lim_{t \to 0} |tG| = \frac{6}{5}$, which is clearly different from 1.

Despite the examples we have just seen, the magnitude function does behave well for t sufficiently large.

Proposition 2.39 *Given a finite metric space A, the following hold:*

(1) tA has Möbius inversion and thus magnitude for sufficiently large $t > 0$.

(2) For t sufficiently large, there is a unique positive weighting on A.

(3) For t sufficiently large, the magnitude function of A is increasing.

(4) $|tA| \to \#A$ as $t \to \infty$.

Proof We start by noting that any $\zeta \in GL(A)$ has weighting $w_\zeta(a) = \sum_{b \in A} \zeta^{-1}(a,b)$. Let us write $\tilde{\zeta}$ for the adjoint matrix of ζ. We then have an explicit formula for the inverse and deduce $w_\zeta(a) = \frac{\tilde{\zeta}(a,b)}{det(\zeta)}$.

(1) Note that as t becomes larger, $\zeta_{tA} \to \delta$. In particular, ζ_{tA} is invertible for t sufficiently large. This means precisely that tA has Möbius inversion and therefore magnitude.

(2) The explicit formula we gave in the beginning of the proof, along with continuity of the determinant and the adjoint show that for each $a \in A$, the map $\zeta \mapsto w_\zeta(a)$ is continuous on $GL(A)$.

    Now $w_\delta(a) = 1, \forall a \in A$, so by continuity, there exists a neighborhood $U \subset GL(A)$ of δ such that $w_\delta(a) > 0, \forall \zeta \in U, \forall a \in A$. But $\zeta_{tA} \to \delta$ for $t \to \infty$, so for t large enough, $\zeta_{tA} \in U$, which is exactly what we claimed.

(3) From the last point, it follows that for $t < u$ sufficiently large, the weightings w_{tA} and w_{uA} are positive and $tA = \frac{t}{u} \cdot uA$, meaning that uA is an expansion of tA. By the lemma above, we can deduce $|tA| \leq |uA|$.

(4) By continuity (for t sufficiently large), it follows that

$$\lim_{t\to\infty} |tA| = |\lim_{t\to\infty} \zeta_{tA}| = |\delta| = \#A.$$

2.5 Magnitude of Finite Positive Definite Spaces

The examples in the previous chapters have shown that the behaviour of the magnitude function can sometimes be pathological. Therefore, we would like to restrict ourselves to a special class of metric spaces, such that magnitude has 'nice' properties (it is defined, non-decreasing, etc.). A category of spaces that satisfies this is that of positive definite metric spaces.

Definition 2.40 *A finite metric space A is called* ***positive definite*** *if its similarity matrix ζ_A is positive definite. A is called* ***positive semidefinite****, if its similarity matrix is positive semidefinite.*

We start by noting that a positive definite matrix is invertible, so a positive definite space has Möbius inversion and therefore it has magnitude and a unique weighting.

The following way of viewing the magnitude of a positive-definite metric space will turn out particularly useful for generalizing to compact metric spaces.

Proposition 2.41 *A positive definite metric space A has magnitude*

$$|A| = \sup_{v \in \mathbb{R}^A \setminus \{0\}} \frac{\left(\sum_{a \in A} v(a)\right)^2}{v^T \zeta_A v},$$

where the supremum is attained at v if and only if v is a nonzero scalar multiple of the (unique) weighting w_A on A.

Proof ζ_A is positive definite, so we can apply the Cauchy-Schwarz inequality to obtain $(v^T \zeta_A v)(w^T \zeta_A w) \geq (v^T \zeta_A w)^2$, for any $v, w \in \mathbb{R}^A$, with equality if and only if $v = \lambda w$ for some $\lambda \in \mathbb{R}$. Now take $w = w_A$. Then $\zeta_A w_A = \mathbf{1}$ and $w_A^T \zeta_A w_A = |A|$, so the inequality above becomes

$$(v^T \zeta_A v) \cdot |A| \geq \left(\sum_{a \in A} v(a)\right)^2, \text{ i.e. } |A| \geq \frac{\left(\sum_{a \in A} v(a)\right)^2}{v^T \zeta_A v},$$

with equality if and only if v is a scalar multiple of w_A, which gives the conclusion. □

The last proposition quickly implies that in the case of finite positive definite metric spaces, magnitude is monotone with respect to inclusion.

Corollary 2.42 *If A is a positive definite metric space and $B \subset A$, then $|B| \leq |A|$.*

Proof Let w be the weighting on B. Define $v \in \mathbb{R}^A$ as $v(b) = w(b)$ for any $b \in B$ and 0 otherwise. Then the last proposition implies

$$|B| = \frac{\left(\sum_{a \in A} w(a)\right)^2}{w^T \zeta_B w} = \frac{\left(\sum_{a \in A} v(a)\right)^2}{v^T \zeta_A v} \leq |A|,$$

which is what we wanted to show. □

A further property of magnitude that is easily obtained for finite positive definite metric spaces is the following:

Corollary 2.43 *A nonempty positive definite finite metric space A has magnitude at least* 1.

Proof Take v as the vector with all entries 1 in $\mathbb{R}^A$. Then $\sum_{a \in A} v(a) = \#A$ and $\forall a, b \in A, d(a,b) \geq 0$, so $\zeta_A(a,b) = e^{-d(a,b)} \leq 1$. Therefore,

$$v^T \zeta_A v = \sum_{a,b \in A} \zeta_A(a,b) \leq \sum_{a,b \in A} 1 = (\#A)^2$$

and hence, $|A| \geq \frac{\left(\sum_{a \in A} v(a)\right)^2}{v^T \zeta_A v} = \frac{(\#A)^2}{(\#A)^2} = 1.$ □

The following proposition gives us an important class of examples for positive definite metric spaces.

Proposition 2.44 *Every finite subspace X of $\mathbb{R}$ (with the euclidean metric) is positive-definite with positive weighting.*

Proof We say that a metric space is *good* if it has Möbius inversion μ_A and $\forall v \in \mathbb{R}^A : v^T \mu_A v \geq \max_{a \in A} v(a)^2$. Let $A, B \subset X$ such that $A \cap B = \{c\}$ and $\forall a \in A, b \in B : d(a,b) = d(a,c) + d(b,c)$.

We show that if A and B are good, then so is their union. Let $v \in \mathbb{R}^{A \cup B}$. Then by Corollary 2.20, $A \cup B$ has Möbius inversion $\mu_{A \cup B}$ and

$$v^T \mu_{A \cup B} v = v|_A^T \mu_A v|_A + v|_B^T \mu_B v|_B - v|_{A \cap B}^T v|_{A \cap B} = v|_A^T \mu_A v|_A + v|_B^T \mu_B v|_B - v(c)^2.$$

Let $x \in A \cup B$. We can assume without loss of generality that $x \in A$. A is good, so $v|_A^T \mu_A v|_A \geq v(x)^2$. B is good, so $v|_B^T \mu_B v|_B \geq v(c)^2$. Therefore, we have $v^T \mu_{A \cup B} v \geq v(x)^2$, so it follows that $A \cup B$ is good.

Let us now conclude by induction. Clearly, every metric space with $0, 1, 2$ points is good. Let X be a subset of $\mathbb{R}$ with n points. We assume that $n \geq 3$ and and that any subset of $\mathbb{R}$ with $n - 1$ points is good. We can order the points in X in an increasing order, say $X = \{x_1, ..., x_n\}$ with $x_1 < x_2 < ... < x_n$. Let $A = \{x_1, ..., x_{n-1}\}$ and $B = \{x_{n-1}, x_n\}$. Then $X = A \cup B$ with $A \cap B = \{x_{n-1}\}$ and for all $a \in A$ and $b \in B$, we have $d(a,b) = d(a, x_{n-1}) + d(b, x_{n-1})$ (note that this is where we use the assumption that X is a subset of $\mathbb{R}$). By induction hypothesis, the set A is good. B has two elements and is therefore good, so the considerations above show that X must also be good. Thus every subset of $\mathbb{R}$ is good and in particular positive-definite with positive weighting. □

Chapter 3

Magnitude of Compact Metric Spaces

After having discussed magnitude of finite metric spaces, we analyse how this notion can be generalized to infinite ones. This chapter is based on the approach of Meckes in [2]. We offer some possibilities for defining the magnitude of a compact metric space, and discuss advantages and disadvantages of each definition. We focus on the class of positive definite metric spaces, where we show that the four definitions for the magnitude of compact metric spaces are equivalent. In the process, we show our first continuity result: lower semicontinuity of magnitude. In the end, we introduce an invariant similar to magnitude called the maximum diversity and formulate a continuity result for the maximum diversity.

Throughout this section, we restrict the notion of metric to finite-valued maps, satisfying the properties in Definition 2.1.

3.1 Measure Theoretic Preliminaries

Since two of the definitions for magnitude we suggest rely on measures, we start this section with some basic measure theoretic notions. We introduce the definition of a signed measure and total variation, based on [5] and the support of a measure based on [6].

Throughout this section, let X be a space and $\mathcal{M}$ a σ-algebra of subsets of X.

Definition 3.1 *A* ***(signed) measure*** *on $\mathcal{M}$ is a map $\nu\colon \mathcal{M} \to (-\infty, \infty]$, such that for any countable set $\{E_j\}_{j=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{M}$, $\nu(\bigcup\limits_{j=1}^{\infty} E_j) = \sum\limits_{j=1}^{\infty} \nu(E_j)$. By a* ***positive measure*** *we mean a signed measure with non-negative values.*

Example 3.2 *Let μ be a positive measure and $f\colon E \to \mathbb{R}$ a μ-measurable, μ-integrable function f, where by μ- integrable, we mean $\int_E f^- d\mu < \infty$ and $\int_E f^+ d\mu$*

exists. Then $\nu(A) := \int_A f d\mu$ for subsets $A \subset E$ is a signed measure, which can take negative values depending on the function f.

Definition 3.3 *Given a set X, the* ***Dirac delta measure*** *δ_x is* $\delta_x(A) = \begin{cases} 1, \text{ if } x \in A \\ 0, \text{ otherwise.} \end{cases}$

Definition 3.4 *The* ***total variation*** *of a signed measure ν is*

$$|\nu|(E) = \sup \sum_{j=1}^{\infty} |\nu(E_j)|,$$

where the sup is taken over all sequences $\{E_j\}_{j=1}^{\infty}$ of pairwise disjoint elements in $\mathcal{M}$ with $\bigcup_{j=1}^{\infty} E_j = E$.

Note that the total variation is additive.

Definition 3.5 *The* ***support of a positive measure*** *μ on X is the set*

$$supp(\mu) = \{x \in X \colon \mu(U) > 0 \text{ for each neighborhood } U \text{ of } x\}.$$

Example 3.6 *We determine the support of two positive measures on $\mathbb{R}^n$.*

- *The support of the Lebesgue measure λ on $\mathbb{R}^n$ is $\mathbb{R}^n$, since $\forall x \in \mathbb{R}^n$, every neighborhood of x contains some open ball, which has positive Lebesgue measure, so any neighborhood of x has positive Lebesgue measure, and thus $x \in supp(\lambda)$.*
- *The support of the Dirac measure δ_a for $a \in \mathbb{R}^n$ is $\{a\}$, since every neighborhood of a contains a and has therefore positive measure, so $a \in supp(\delta_a)$. However, for any $x \neq a$, there exists a neighborhood of x that does not contain a, so this neighborhood has measure 0, and thus $x \notin supp(\delta_a)$*

By relying on the last definition, we can define the support of a signed measure.

Definition 3.7 *The* ***support of a signed measure*** *μ on X is $supp(\mu) = supp|\mu|$.*

We use the following notation for spaces of measures. Let A be a metric space. We denote by

- $M(A)$ the space of finite signed Borel measures on A;
- $M_+(A)$ the cone of positive measures on A;
- $FM(A)$ the space of finitely supported signed measures on A;
- $FM_+(A)$ the cone of finitely supported positive measures on A.

We equip $M(A)$ with the norm $||\mu|| = |\mu|(A)$, where $|\mu|$ is the total variation of μ.

3.2 Generalizations of Magnitude to Compact Metric Spaces

We introduce 4 possible generalizations of magnitude to infinite metric spaces. Let (A, d) be a compact metric space.

(1) The first notion of magnitude that we introduce is a naive generalization: $|A| := \sup_{B \subset A \text{ finite}} |B|$. In general, this definition does not coincide with the notion of magnitude we have seen for finite metric spaces, since in general, magnitude is not monotone with respect to inclusion. However, we have seen in in Corollary 2.42 that that is the case for positive definite metric spaces.

(2) Another option is to define magnitude by approximating a compact metric space using finite subspaces. Given a sequence $\{A_k\}_k$ of finite subspaces of A such that $\lim_{k\to\infty} A_k = A$ (where the limit is taken in d_H, see Definition 2.3), we set $|A| := \lim_{l\to\infty} |A_k|$. A priori, it is not clear that this definition does not depend on the choice of the sequence $\{A_k\}_k$.

(3) The next definition is motivated by the classical notion of 'transforming sums into integrals'.

Definition 3.8 *A* ***weight measure*** *for* (A, d) *is a finite signed measure* $\mu \in M$ *such that* $\int_A e^{-d(x,y)} d\mu = 1, \forall x \in A$.

If A possesses a weight measure, we define the magnitude as $|A| := \mu(A)$.

As in the case of finite metric spaces, this definition does not depend of a choice of the weight measure: Let μ and ν be two weight measures on A. Then we have

$$\mu(A) = \int_A 1 d\mu(x) = \int_A \left(\int_A e^{-d(x,y)} d\nu(y) \right) d\mu(x) =$$
$$= \int_A \left(\int_A e^{-d(x,y)} d\mu(x) \right) d\nu(y) = \int_A 1 d\nu(y) = \nu(A),$$

where for the third equality we have used Fubini's theorem. However, there is no general guarantee for the existence of a weight measure.

(4) We introduce our last definition for the magnitude based on a symmetric bilinear form on $M(A)$, by generalizing Proposition 2.41. For signed measures $\mu, \nu \in M(A)$ define

$$Z_A(\mu, \nu) = \int_A \int_A e^{-d(x,y)} d\mu(x) d\nu(y).$$

Linearity of the integral implies bilinearity of this map, whereas Fubini's theorem implies symmetry. We can now define the magnitude

$$|A| := \sup\{\frac{\mu(A)^2}{Z_A(\mu,\mu)} : \mu \in M(A), Z_A(\mu,\mu) \neq 0\}.$$

In this thesis, by magnitude of a compact metric space A, we mean magnitude in the sense of the last definition. The advantage of this convention is that we do not have to worry about existence, since the supremum of a set always exists if we set it to ∞ if the set is unbounded. We mainly work with positive definite metric spaces, in which case all four definitions are equivalent, so there is little source of confusion.

3.3 Equivalence of Definitions of Magnitude for Positive Definite Spaces

In this section, we show that all definitions above are equivalent for compact positive definite metric spaces. In the process, we also show our first continuity result, in the form of lower semicontinuity of magnitude (Theorem 3.21).

Definition 3.9 *A metric space (A, d) is **positive definite**, if all its finite subspaces are positive definite. For finite subspaces, positive definitenes is understood as in Definition 2.40. Similarly, a metric space is **positive semidefinite**, if all its finite subspaces are positive semidefinite.*

As in the case of finite spaces, magnitude is increasing with respect to inclusion in positive definite metric spaces:

Proposition 3.10 *For a compact, positive definite metric space A and $B \subset A$, we have $|B| \leq |A|$.*

Proof Let (A, d) be a positive definite metric space and $B \subset A$. Given a measure $\mu_B \in M(B)$ we can define $\mu_A \in M(A)$ as $\mu_A(E) = \mu_B(E \cap B), \forall E \subset A$. Then $\mu_A(A) = \mu_B(B)$ and

$$\begin{aligned}
Z_A(\mu_A, \mu_A) &= \int_A \int_A e^{-d(x,y)} d\mu_A(x) d\mu_A(y) = \\
&= \int_A \left(\int_B e^{-d(x,y)} d\mu_A(x) + \int_{A\setminus B} e^{-d(x,y)} d\mu_A(x) \right) d\mu_A(y) = \\
&= \int_A \left(\int_B e^{-d(x,y)} d\mu_A(x) \right) d\mu_A(y) = \\
&= \int_B \left(\int_B e^{-d(x,y)} d\mu_A(x) \right) d\mu_A(y) = Z_B(\mu_B, \mu_B).
\end{aligned}$$

The computation above implies $\frac{\mu(A)^2}{Z_A(\mu_A,\mu_A)} = \frac{\mu(B)^2}{Z_B(\mu_B,\mu_B)}$, and, in particular, $|A| = \sup\{\frac{\mu(A)^2}{Z_A(\mu,\mu)} : \mu \in M(A), Z_A(\mu,\mu) \neq 0\} \geq \frac{\mu(B)^2}{Z_B(\mu_B,\mu_B)}$. Since $\mu_B \in M(B)$ was arbitrary, we can take the supremum over all such μ_B and obtain the desired inequality. □

The modulus of continuity is useful in formulating our next results.

Definition 3.11 *Let (A,d) be a metric space. The **modulus of continuity** of an absolutely continuous function $f\colon A \to \mathbb{R}$ is the function*

$$\omega_f\colon (0,\infty) \to [0,\infty), w_f(\varepsilon) = \sup\{|f(x) - f(y)| : x,y \in A, d(x,y) < \varepsilon\}.$$

The following fact about the modulus of continuity of an exponential function will turn out useful:

Example 3.12 *For the exponential function $f\colon (0,\infty) \to \mathbb{R}, x \mapsto e^{-x}$, we have a simple upper bound for the modulus of continuity. Indeed, $\forall x,y \in \mathbb{R}$, the mean value theorem implies that $\exists c \in (0,\infty)$ such that $f'(c)(x-y) = f(x) - f(y)$. Therefore, $|e^{-x} - e^{-y}| = e^{-c}|x-y| \leq |x-y|$, so $\omega_f(\varepsilon) \leq \varepsilon$.*

The following lemmas are purely technical, but very useful in this section's proofs.

Lemma 3.13 *Let A, B be compact metric subspaces of some metric space X and $\mu \in M(A)$. Then $\forall \varepsilon > d_H(A,B), \exists \nu \in M(B)$ such that*

- *$\nu(B) = \mu(A)$,*
- *$||\nu|| \leq ||\mu||$ and*
- *for any uniformly continuous map $f\colon X \to \mathbb{R}$: $|\int_A f d\mu - \int_B f d\nu| \leq ||\mu||\omega_f(\varepsilon)$.*

Furthermore, if μ is positive, then ν can also be taken positive.

Proof Fix $\varepsilon > d_H(A,B)$. Then $\forall a \in A, \exists b \in B : d(a,b) < \varepsilon$. This means that $A \subset \bigcup\limits_{b\in B} B(b,\varepsilon)$. Compactness of A now implies that there exist $x_1, ..., x_N \in B$ such that $A \subset \bigcup\limits_{i=1}^{N} B(x_i,\varepsilon)$.

We define the sets $U_1 = B(x_1,\varepsilon)$ and $U_j = B(x_j,\varepsilon) \setminus \bigcup\limits_{i=1}^{j-1} B(x_i,\varepsilon)$.These also cover A and they are disjoint. Set $\nu := \sum\limits_{j=1}^{N} \mu(U_j \cap A)\delta_{x_j}$.

We show that this measure satisfies the desired properties.

- $\nu(B) = \sum\limits_{j=1}^{N} \mu(U_j \cap A)\delta_{x_j}(B) = \sum\limits_{j=1}^{N} \mu(U_j \cap A) = \mu(A)$, where we have used that $\forall j : x_j \in B$, so $\delta_{x_j}(B) = 1$ and that the sets U_j are disjoint, covering A.

- We have

$$||\nu|| = |\nu|(B) = \sum_{j=1}^{N} |\mu(U_j \cap A)\delta_{x_j}|(B) = \sum_{j=1}^{N} |\mu(U_j \cap A)| \leq$$
$$\leq |\mu|(A) = ||\mu||.$$

- For a uniformly continuous map $f \colon X \to \mathbb{R}$, we have

$$\left| \int_A f d\mu - \int_B f d\nu \right| = \left| \sum_{j=1}^{N} \int_{U_j \cap A} f d\mu - \sum_{j=1}^{N} \mu(U_j \cap A) f(x_j) \right| =$$
$$= \left| \sum_{j=1}^{N} \int_{U_j \cap A} (f(x) - f(x_j)) d\mu(x) \right| \leq \sum_{j=1}^{N} \int_{U_j \cap A} |f(x) - f(x_j)| d|\mu|(x) \leq$$
$$\leq \sum_{j=1}^{N} \int_{U_j \cap A} \omega_f(\varepsilon) d|\mu|(x) = \sum_{j=1}^{N} |\mu|(U_j \cap A) \omega_f(\varepsilon) = \omega_f(\varepsilon)|\mu|(A) =$$
$$= ||\mu|| \omega_f(\varepsilon).$$

Finally, if μ is positive, the ν we have defined above is also positive as a sum of positive measures. □

Lemma 3.14 *Let A, B be compact metric subspaces of some metric space X, $\mu \in M(A)$, $\varepsilon > 0$ and $\nu \in M(B)$ such that $\nu(B) = \mu(A)$, $||\nu|| \leq ||\mu||$ and for any uniformly continuous map $f \colon X \to \mathbb{R}$, $| \int_A f d\mu - \int_B f d\nu| \leq ||\mu|| \omega_f(\varepsilon)$. Then*

$$|Z_A(\mu, \mu) - Z_B(\nu, \nu)| \leq 2\varepsilon ||\mu||^2.$$

Proof We define the maps $f_\mu : A \to \mathbb{R}$, $f_\mu(x) = \int_A e^{-d(x,z)} d\mu(z)$ and $f_\nu : A \to \mathbb{R}$, $f_\nu(x) = \int_B e^{-d(x,z)} d\nu(z)$. Note that Fubini's theorem implies that $\int_B f_\mu d\nu = \int_A f_\nu d\mu$. For any $x \in A$, we have

$$|f_\mu(x) - f_\nu(x)| = |\int_A e^{-d(x,z)} d\mu(z) - \int_B e^{-d(x,z)} d\nu(z)| \leq ||\mu|| \omega_{e^{-d(x,\cdot)}}(\varepsilon) \leq ||\mu|| \varepsilon.$$

Therefore,

$$|Z_A(\mu, \mu) - Z_B(\nu, \nu)| = |\int_A f_\mu d\mu - \int_B f_\nu d\nu| \leq$$
$$\leq |\int_A f_\mu d\mu - \int_A f_\nu d\mu| + |\int_B f_\mu d\nu - \int_B f_\nu d\nu| \leq$$
$$\leq \int_A |f_\mu - f_\nu| d|\mu| + \int_B |f_\mu - f_\nu| d|\nu| \leq \int_A \varepsilon ||\mu|| d|\mu| + \int_B \varepsilon ||\mu|| d|\nu| =$$
$$= \varepsilon ||\mu|| \cdot |\mu|(A) + \varepsilon ||\mu|| \cdot |\nu|(A) = \varepsilon ||\mu||^2 + \varepsilon ||\mu|| \cdot ||\nu|| \leq 2||\mu||^2 \varepsilon.$$ □

Lemma 3.15 *Let A be a compact metric space, $\mu \in M(A)$ and $\varepsilon > 0$. Then there exists a finite subset $B \subset A$ and a measure $\nu \in M(B)$ as in Lemma 3.13 such that $|Z_A(\mu,\mu) - Z_B(\nu,\nu)| \leq 2\varepsilon||\mu||^2$. Furthermore, if μ is positive, then ν can also be chosen positive.*

Proof The balls $B(x,\varepsilon)$ for all $x \in A$ clearly form a covering of A. By compactness of A, there exists a finite subset B such that $A \subset \bigcup_{x\in B} B(x,\varepsilon)$. This implies $d(a,B) < \varepsilon, \forall a \in A$. Furthermore, $B \subset A$ implies $d(b,A) = 0, \forall b \in B$, so $d_H(A,B) < \varepsilon$. Thus, we can apply Lemma 3.13 to find a measure $\nu \in M(B)$ such that $\mu(A) = \nu(B)$, $||\nu|| \leq ||\mu||$ and $|\int_A f d\mu - \int_B f d\nu| \leq ||\mu||\omega_f(\varepsilon)$ for every absolutely continuous map $f\colon X \to \mathbb{R}$. The fact that if μ is positive, ν can also be chosen positive follows directly from the same statement in Lemma 3.13. Now, applying Lemma 3.14, we directly obtain the conclusion.□

Let us introduce a characterization of positive (semi-)definiteness of the metric space A based on the bilinear form Z_A.

Lemma 3.16 *A compact metric metric space A is positive semidefinite if and only if Z_A is a positive semidefinite bilinear form on $M(A)$. Furthermore, if Z_A is positive definite, then A is positive definite.*

Proof We start by showing the equivalence for B finite.

Let B be a finite space and μ a measure on B. Then,

$$\begin{aligned} Z_B(\mu,\mu) &= \int_B \int_B e^{-d(x,y)} d\mu(x) d\mu(y) = \\ &= \sum_{x,y\in B} e^{-d(x,y)} \mu(\{x\})\mu(\{y\}) = (\mu(\{x\}))^T_{x\in B} \zeta_B (\mu(\{y\}))_{y\in B} \end{aligned}$$

For an arbitrary vector $v = (v(x))_{x\in B}$ we can define a measure on B as $\mu(E) = \sum_{x\in E} v(x)$ for each $E \subset B$. Conversely, each measure μ on B gives a vector $v = (\mu(\{x\}))_{x\in B}$. Thus, the equality above shows that

$$v^T \zeta_B v \geq 0 \iff Z_B(\mu,\mu) \geq 0,$$

so B is positive semidefinite if and only if Z_B is positive semidefinite.

In the case of finite spaces, we also have that B is positive definite if and only if Z_B is positive definite, by the same reasoning as above with strict inequality instead of $\geq$.

Let us move on to the general case.

$\Longleftarrow$ We assume that Z_A is positive semidefinite. Let B be a finite subset of A. Given a measure μ_B on B, we extend it to $\mu \in M(A)$ by setting $\mu(E) = \mu_B(E \cap B)$ for any $E \subset A$. Then $Z_B(\mu_B,\mu_B) = Z_A(\mu,\mu) \geq 0$, so Z_B

is positive semidefinite and since B is finite, this implies that B is positive semidefinite. The definition of positive semidefiniteness of A now implies that the metric space A is positive semidefinite.

$\Longrightarrow$ Conversely, let us assume assume that A is positive semidefinite. Take an arbitrary measure $\mu \in M(A)$ and $\varepsilon > 0$. By Lemma 3.15, there exists a finite subset $B \subset A$ and a measure $\nu \in M(B)$ such that

$$|Z_A(\mu,\mu) - Z_B(\nu,\nu)| \leq 2\varepsilon||\mu||^2.$$

As a finite subset of A, B is positive semidefinite, so the computation we have started with implies $Z_B(\nu,\nu) \geq 0$. Thus, the inequality above gives $Z_A(\mu,\mu) \geq -2\varepsilon||\mu||^2$. Since, ε was arbitrary, it follows $Z_A(\mu,\mu) \geq 0$, so we are done.

If we assume that Z_A is positive definite, we can proceed as in the inverse implication, replacing $\geq$ with strict inequality. □

The fact that our definition of magnitude coincides with the one relying on weight measures follows immediately from the following theorem:

Theorem 3.17 *Let A be a compact, positive definite metric space. The supremum in $|A| = \sup\{\frac{\mu(A)^2}{Z_A(\mu,\mu)} : \mu \in M(A), Z_A(\mu,\mu) \neq 0\}$ is attained for a measure μ if and only if μ is a nonzero scalar multiple of some weight measure on A. In particular, if μ is a weight measure, then $|A| = \mu(A)$.*

Proof $\Longleftarrow$ Let μ be a weight measure for A. We observe that in this case, for any $\nu \in M(A)$, $Z_A(\mu,\nu) = \nu(A)$. Since A is positive definite, Lemma 3.16 implies that Z_A is a positive semidefinite bilinear form on $M(A)$, and thus it satisfies the Cauchy-Schwarz inequality. Hence, $\forall \nu \in M(A)$, we have $\nu(A) = Z_A(\mu,\nu) \leq \sqrt{Z_A(\mu,\mu)Z_A(\nu,\nu)} = \sqrt{\mu(A)Z_A(\nu,\nu)}$ with equality if and only if ν is a scalar multiple of μ. In particular, we have equality for $\mu = \nu$. Therefore,

$$\mu(A) = \frac{\mu(A)^2}{Z_A(\mu,\mu)} \geq \frac{\nu(A)^2}{Z_A(\nu,\nu)}, \forall \nu \in M(A).$$

This means that

$$\mu(A) = \sup\{\frac{\nu(A)^2}{Z_A(\nu,\nu)} : \nu \in M(A), Z_A(\nu,\nu) \neq 0\} = |A|.$$

For any $\lambda \in \mathbb{R} \setminus \{0\}$, we have $\frac{(\lambda\mu(A)^2)}{Z_A(\lambda\mu,\lambda\mu)} = \frac{\mu(A)^2}{Z_A(\mu,\mu)}$, so the supremum is also attained at $\lambda\mu$.

$\Longrightarrow$ Let us now suppose that the supremum in the definition of magnitude is attained at $\mu \in M(A)$ and take $\nu \in M(A)$ with $\nu(A) = 0$. By the choice of

μ, we have for any $t \in \mathbb{R}$,

$$\frac{\mu(A)^2}{Z_A(\mu,\mu)} \geq \frac{((\mu+t\nu)(A))^2}{Z_A(\mu+t\nu,\mu+t\nu)} = \frac{\mu(A)^2}{Z_A(\mu+t\nu,\mu+t\nu)}.$$

Therefore, $Z_A(\mu,\mu) \leq Z_A(\mu+t\nu,\mu+t\nu) = Z_A(\mu,\mu)+2tZ_A(\mu,\nu)+t^2Z_A(\nu,\nu)$ and so $0 \leq 2tZ_A(\mu,\nu)+t^2Z_A(\nu,\nu)$. By Lemma 3.16, $Z_A(\nu,\nu) \geq 0$, so the inequality above is only possible for all real values of t if $Z_A(\mu,\nu) = 0$. Fix $x,y \in A$. The above computation, for $\nu = \delta_x - \delta_y$ gives

$$0 = Z_A(\mu,\nu) = \int_A e^{-d(x,z)} d\mu(z) - \int_A e^{-d(y,z)} d\mu(z),$$

so the value $c := \int_A e^{-d(x,z)} d\mu(z)$ is constant.

If $c = 0$, then $Z_A(\mu,\mu) = \int_A c d\mu = 0$, which cannot be the case by assumption. Therefore, $\frac{\mu}{c}$ is a weight measure, which is precisely what we wanted to show. □

The following theorem shows that our definition of magnitude is equivalent to the one setting $|A|$ the supremum over the magnitude of all finite subspaces of A.

Theorem 3.18 *For a compact positive definite metric space A, we have*

$$|A| = \sup\{\frac{\mu(A)^2}{Z_A(\mu,\mu)} : \mu \in FM(A), \mu \neq 0\} = \sup\{|B| : B \subset A \text{ finite}\}.$$

Proof Let us start with motivating the second equality. For any $\mu \in FM(A)$, we can take B as the support of μ, and we have $\frac{\mu(A)^2}{Z_A(\mu,\mu)} \leq |B|$. For the other direction, let B be an arbitrary finite subset of A. Since A is positive-definite, B must also be positive-definite, so a weighting w exists. We can now define a finitely supported measure μ on A, by extending $\mu(\{x\}) = w(x)$ if $x \in B$ and $\mu(\{x\}) = 0$ to all sets. Then $\frac{\mu(A)^2}{Z_A(\mu,\mu)} = |B|$.

Let us prove the first equality. Since A is positive definite, we have for any $\mu \in FM(A)$, that $Z_A(\mu,\mu) = 0$ if and only if $\mu = 0$, since the integral is in fact a finite sum of non-negative entries. This means that

$$\{\frac{\mu(A)^2}{Z_A(\mu,\mu)} : \mu \in FM(A), \mu \neq 0\} \subset \{\frac{\mu(A)^2}{Z_A(\mu,\mu)} : \mu \in M(A), Z_A(\mu,\mu) \neq 0\}$$

and therefore $\sup\{\frac{\mu(A)^2}{Z_A(\mu,\mu)} : \mu \in FM(A), \mu \neq 0\} \leq |A|$.

It now remains to show the inequality

$$\sup\{\frac{\mu(A)^2}{Z_A(\mu,\mu)} : \mu \in FM(A), \mu \neq 0\} \geq |A|,$$

after which the theorem is proven.

Let $\mu \in M(A)$ with $Z_A(\mu,\mu) \neq 0$ and $\varepsilon > 0$. By Lemma 3.15, there exists a finite subset $B \subset A$ and a measure $\nu \in M(B)$ such that $\nu(B) = \mu(A)$, $||\nu|| \leq ||\mu||$ and $|Z_A(\mu,\mu) - Z_B(\nu,\nu)| \leq 2\varepsilon||\mu||^2$. This implies that

$$Z_B(\nu,\nu) \leq Z_A(\mu,\mu) + 2\varepsilon||\mu||^2$$

and hence

$$\frac{\nu(A)^2}{Z_A(\nu,\nu)} = \frac{\mu(A)^2}{Z_B(\nu,\nu)} \geq \frac{\mu(A)^2}{Z_A(\mu,\mu) + 2||\mu||^2\varepsilon}.$$

In particular, it holds that $\sup\{\frac{\nu(A)^2}{Z_A(\nu,\nu)} : \nu \in FM(A), \nu \neq 0\} \geq \frac{\mu(A)^2}{Z_A(\mu,\mu)+2||\mu||^2\varepsilon}$. Since ε was arbitrary, it now follows that

$$\sup\{\frac{\nu(A)^2}{Z_A(\nu,\nu)} : \nu \in FM(A), \nu \neq 0\} \geq \frac{\mu(A)^2}{Z_A(\mu,\mu)}.$$

Finally, taking the supremum over all $\mu \in M(A)$, we obtain the conclusion.□

Before we can formulate our first continuity result, we need the following definition:

Definition 3.19 *A map $f\colon X \to \overline{\mathbb{R}}$ is called* ***lower semicontinuous*** *at $x_0 \in X$ if $\forall y < f(x_0)$, there exists $\delta > 0$ such that $f(x) > y, \forall x$ with $d(x, x_0) < \delta$.*

We also give the symmetric definition for upper semicontinuity:

Definition 3.20 *A map $f\colon X \to \overline{\mathbb{R}}$ is called* ***upper semicontinuous*** *at $x_0 \in X$ if $\forall y > f(x_0)$, there exists $\delta > 0$ such that $f(x) < y, \forall x$ with $d(x, x_0) < \delta$.*

Note that a function is continuous if and only if it is both upper and lower semiconutinous.

The next proposition gives our first (semi-)continuity result for the magnitude. Based on it, we can also deduce the equivalence of our definition for the magnitude with the one based on approximating a space via finite subspaces, thus concluding this section.

Theorem 3.21 (Lower Semicontinuity of Magnitude) *The map $A \mapsto |A|$ with values in $[1, \infty]$ is lower semicontinuous with respect to the Gromov-Hausdorff*[1] *distance on the class of compact positive definite metric spaces.*

[1] See Definition 2.6

Proof Let A be a compact positive definite metric space. We first treat the case $|A| < \infty$. Let $\varepsilon > 0$. The definition of magnitude as a supremum implies that we can pick a signed measure μ on A with $Z_A(\mu,\mu) \neq 0$ and $|A| \leq \frac{\mu(A)^2}{Z_A(\mu,\mu)}(1+\varepsilon)$.

Let B be any other compact positive definite metric space. Since the Gromov-Hausdorff distance is a metric, this implies $d_{GH}(A,B) > 0$. By definition of the Gromov-Hausdorff distance, we can pick a metric space X and isometric embeddings $\varphi\colon A \to X$, $\psi\colon B \to X$ such that $0 < d_H(\varphi(A),\psi(B)) \leq 2d_{GH}(A,B)$. We can assume without loss of generality that $A = \varphi(A)$ and $B = \psi(B)$. Let us apply Lemma 3.13 for $\varepsilon = 2d_H(A,B)$ to deduce $\exists \nu \in M(B)$ such that $\nu(B) = \mu(A)$, $||\nu|| \leq ||\mu||$ and for any uniformly continuous map $f\colon X \to \mathbb{R}$, $|\int_A f d\mu - \int_B f d\nu| \leq ||\mu||\omega_f(2d_H(A,B))$. Now by Lemma 3.14, we obtain $|Z_A(\mu,\mu) - Z_B(\nu,\nu)| \leq 2 \cdot 2d_H(A,B)||\mu||^2$ and hence

$$|Z_A(\mu,\mu) - Z_B(\nu,\nu)| \leq 8||\mu||^2 d_{GH}(A,B).$$

Now we have

$$\begin{aligned} |B| \geq \frac{\nu(B)^2}{Z_B(\nu,\nu)} \geq \frac{\mu(A)^2}{Z_A(\mu,\mu) + 8||\mu||^2 d_{GH}(A,B)} \geq \\ \geq \frac{|A|}{1+\varepsilon} \cdot \left(1 + \frac{8||\mu||^2}{Z_A(\mu,\mu)} d_{GH}(A,B)\right)^{-1}, \end{aligned}$$

where in the last inequality we have used the choice of μ.

Hence if $d_{GH}(A,B) \leq \frac{Z_A(\mu,\mu)}{8||\mu||^2}\varepsilon$, the above inequality implies $|B| \geq \frac{|A|}{(1+\varepsilon)^2}$ and therefore lower semicontinuity is proven.

Let us now treat the case $|A| = \infty$. Fix $N \geq 1$. Again, by the definition of magnitude as a supremum, we can pick $\mu \in M(A)$ with $Z_A(\mu,\mu) \neq 0$, such that $\frac{\mu(A)^2}{Z_A(\mu,\mu)} \geq N$. For B distinct from A, the same computation as above gives $|B| \geq \frac{\mu(A)^2}{Z_A(\mu,\mu)+8||\mu||^2 d_{GH}(A,B)}$ and hence $|B| \geq N \cdot \left(1 + \frac{8||\mu||^2}{Z_A(\mu,\mu)} d_{GH}(A,B)\right)^{-1}$.

Thus, for $d_G H(A,B) \leq \frac{Z_A(\mu,\mu)}{8||\mu||^2}$, we get $|B| \geq \frac{N}{2}$, and lower semicontinuity is proven in this case as well. □

We can now prove our last equivalence:

Corollary 3.22 *Let A be a compact positive definite metric space and $\{A_k\}_k$ a sequence of compact subspaces of A such that $\lim_{k\to\infty} d_H(A_k, A) = 0$. Then $|A| = \lim_{k\to\infty} |A_k|$.*

Proof As discussed in Proposition 3.10, the magnitude is increasing with respect to inclusion, so $|A_k| \leq |A|$ for all k. Also, $A_k \subset A$ implies that

$d_{GH}(A_k, A) \leq d_H(A_k, A)$ (since we can take $X = A$ and the trivial embeddings for A and A_k into A in the definition of the Gromov-Hausdorff distance).

Lower semicontinuity of $A \mapsto |A|$ (Theorem 3.21) implies that $\forall \varepsilon > 0 \exists \delta > 0$ such that $\forall B \in B^{d_H}(A, \delta) : |A| - \varepsilon \leq |B|$. Now, $\lim_{k\to\infty} d_H(A, B) = 0$, so $\exists K \geq 1$ such that $\forall k \geq K : A_k \in B^{d_H}(A, \delta)$, so we obtain

$$|A| - \varepsilon \leq |A_k| \leq |A|, \forall k \geq K$$

and therefore we can conclude $|A| = \lim_{k\to\infty} |A_k|$. □

3.4 Magnitude of Compact Metric Spaces: Examples

Let us explicitly compute the magnitude for some spaces, as done by Willerton in [7]. It is important to note that there are no known methods for computing the magnitude of arbitrary spaces (not even for Euclidean subspaces). In fact, not even the magnitude of 'simple' spaces such as the 2-disc, or the cube can be computed with current methods.

Example 3.23 (Magnitude of Line Segment) *Let us compute the magnitude of a line segment $[a, b] \subset \mathbb{R}$. We denote λ the Lebesgue measure and δ_a, δ_b the Dirac delta measures supported at a, respectively b. We show that $\frac{1}{2}(\delta_a + \delta_b + \lambda)$ is a weight measure. For any $y \in [a, b]$, we have:*

$$\begin{aligned}
&\int_a^b e^{-d(x,y)}(d\delta_a + d\delta_b + d\lambda)(x) = \\
&= e^{-d(a,y)} + e^{-d(b,y)} + \int_a^y e^{-d(x,y)} d\lambda(x) + \int_y^b e^{-d(x,y)} d\lambda(x) = \\
&= e^{-(y-a)} + e^{-(b-y)} + \int_a^y e^{-(y-x)} d\lambda(x) + \int_y^b e^{-(b-y))} d\lambda(x) = \\
&= e^{a-y} + e^{y-b} + 1 - e^{a-y} - e^{y-b} + 1 = 2.
\end{aligned}$$

Therefore, $\frac{1}{2}(\delta_a + \delta_b + \lambda)$ is a weight measure and we can compute the magnitude:

$$|[a, b]| = \frac{1}{2} \int_a^b (d\delta_a + d\delta_b + d\lambda)(x) = \frac{1}{2}(1 + 1 + b - a) = 1 + \frac{b - a}{2}.$$

Note that the difficult part in the previous example was determining a weight measure. After that, the computation of the magnitude only consisted of routine computations.

We can generalize Example 2.13 about magnitude of finite homogeneous metric spaces to compact homogeneous metric spaces under some reasonable assumptions.

Theorem 3.24 (Speyer's Homogeneity Theorem) *Let X be a homogeneous compact metric space and μ an invariant measure on X. Then the integral $\int_X e^{-d(x,y)} d\mu(x)$ is independent of y. If this integral is non-zero and finite, then a weight measure on X is given by $\frac{\mu}{\int_X e^{-d(x,y)} d\mu(x)}$. In this case, the magnitude is $|X| = \frac{\int_X d\mu(x)}{\int_X e^{-d(x,y)} d\mu(x)}$.*

Proof The proof is similar to the finite case. We start by showing that $\int_X e^{-d(x,y)} d\mu(x)$ is independent of y. Let y, z be arbitrary points in X. Homogeneity implies that there exists an isometry $\varphi\colon X \to X$ satifying $\varphi(y) = z$. Then

$$\int_X e^{-d(x,y)} d\mu(x) = \int_X e^{-d(\varphi(x),\varphi(y))} d\mu(x) = \\ = \int_X e^{-d(\varphi(x),z)} d\mu(x) = \int_X e^{-d(x,z)} d\mu(x),$$

where in the last equality, we have used the invariance of μ. Therefore, $\int_X e^{-d(x,y)} d\mu(x)$ does not depend on the choice of $y \in X$.

If this quantity is non-zero and finite, we can immediately deduce that $\frac{\mu}{\int_X e^{-d(x,y)} d\mu(x)}$ is a weight measure on X and therefore the magnitude is

$$|X| = \frac{\mu}{\int_X e^{-d(x,y)} d\mu(x)}(X) = \frac{\int_X d\mu(x)}{\int_X e^{-d(x,y)} d\mu(x)},$$

where we are also using that if a weight measure exists, then the magnitude is the weight of the whole space (Theorem 3.17). □

Let us apply this to compute the magnitude of a circle.

Example 3.25 (Magnitude of Circle) *Let C_r be a circle of radius r and let λ denote the Lebesgue measure. Since C_r is homogeneous, the last theorem immediately gives the magnitude of the circle:*

$$|C_r| = \frac{\int_{C_r} d\lambda(x)}{\int_{C_r} e^{-d(x,y)} d\lambda(x)} = \frac{2\pi r}{\int_{C_r} e^{-d(x,y)} d\lambda(x)}$$

3.5 Maximum Diversity

Let us introduce an invariant similar to the magnitude, which is sometimes easier to work with. In particular, we prove a continuity result for the maximum diversity, which translates to continuity of magnitude in some special cases.

Definition 3.26 *The **maximum diversity** of a positive definite, compact metric space (A, d) is*

$$|A|_+ = \sup\{\frac{\mu(A)^2}{Z_A(\mu,\mu)} : \mu \in M_+(A), \mu \neq 0\}.$$

Note that for a positive measure $\mu \neq 0$ implies $Z_A(\mu, \mu) > 0$, so the definition above makes sense.

Lemma 3.27 *For a compact positive definite metric space A, we have*

$$|A|_+ \leq \exp(diam(A)).$$

Proof It suffices to show that $\forall \mu \in M_+(A), \mu \neq 0 : \frac{\mu(A)^2}{Z_A(\mu,\mu)} \leq \exp(\text{diam}(A))$. For any such μ, we have

$$e^{\text{diam}(A)} Z_A(\mu, \mu) = \int_A \int_A e^{\text{diam}(A)-d(x,y)} d\mu(x) d\mu(y) \geq \int_A \int_A 1 d\mu(x) d\mu(y) = \mu(A)^2.$$

Hence, $\frac{\mu(A)^2}{Z_A(\mu,\mu)} \leq \exp(\text{diam}(A))$ and we are done. □

Now let us prove an analogue to Theorem 3.18 for the maximum diversity.

Theorem 3.28 *For a compact positive definite metric space A, we have*

$$|A|_+ = \sup\{\frac{\mu(A)^2}{Z_A(\mu, \mu)} : \mu \in FM_+(A), \mu \neq 0\} = \sup\{|B|_+ : B \subset A \textit{ finite}\}.$$

Proof We clearly have

$$\sup\{\frac{\mu(A)^2}{Z_A(\mu, \mu)} : \mu \in FM_+(A), \mu \neq 0\} \subset \sup\{\frac{\mu(A)^2}{Z_A(\mu, \mu)} : \mu \in M_+(A), \mu \neq 0\}$$

and therefore

$$\sup\{\frac{\mu(A)^2}{Z_A(\mu, \mu)} : \mu \in FM_+(A), \mu \neq 0\} \leq |A|_+.$$

It now remains to show the inequality

$$\sup\{\frac{\mu(A)^2}{Z_A(\mu, \mu)} : \mu \in FM_+(A), \mu \neq 0\} \geq |A|_+,$$

after which the theorem is proven. Let $\mu \in M_+(A)$ with $\mu \neq 0$ and $\varepsilon > 0$. Now by lemma 3.15, there exists a finite subset $B \subset A$ and a positive measure $\nu \in M_+(B)$ such that $\nu(B) = \mu(A)$, $||\nu|| \leq ||\mu||$ and $|Z_A(\mu, \mu) - Z_B(\nu, \nu)| \leq 2\varepsilon ||\mu||^2$. This implies that

$$Z_B(\nu, \nu) \leq Z_A(\mu, \mu) + 2\varepsilon ||\mu||^2$$

and hence

$$\frac{\nu(A)^2}{Z_A(\nu, \nu)} = \frac{\mu(A)^2}{Z_B(\nu, \nu)} \geq \frac{\mu(A)^2}{Z_A(\mu, \mu) + 2||\mu||^2 \varepsilon}.$$

In particular, it holds that

$$\sup\{\frac{\mu(A)^2}{Z_A(\mu,\mu)} : \mu \in FM_+(A), \mu \neq 0\} \geq \frac{\mu(A)^2}{Z_A(\mu,\mu) + 2||\mu||^2\varepsilon}.$$

Since ε was arbitrary, it follows that

$$\sup\{\frac{\mu(A)^2}{Z_A(\mu,\mu)} : \mu \in FM_+(A), \mu \neq 0\} \geq \frac{\mu(A)^2}{Z_A(\mu,\mu)}.$$

Finally, taking the supremum over all $\mu \in M_+(A)$, we reach the conclusion.□

Note that in general, we have $|A|_+ \leq |A|$. However, if equality between the two invariants holds, then results about the maximum diversity automatically translate into results about magnitude. This motivates the following definition:

Definition 3.29 *A compact positive definite metric space A is* ***positively weighted*** *if $|A| = |A|_+$.*

We observe that a finite positive definite metric space is positively weighted if and only if it admits a positive weighting. The following lemma gives some sufficient conditions for a compact space to be positively weighted.

Lemma 3.30 *Let (A, d) be a compact, positive-definite metric space.*

(1) If A admits a non-negative weighting, then A is positively weighted.

(2) If every finite subset of A has a weighting with only nonnegative components, then A is positively weighted.

(3) If there exists an isometric embedding on A into $\mathbb{R}$ (where $\mathbb{R}$ is endowed with the standard metric) then A is positively weighted.

Proof (1) Let A be a compact positive definite metric space and $\mu \in M_+(A)$ a weight measure. By Theorem 3.17, $|A| = \frac{\mu(A)^2}{Z_A(\mu,\mu)}$ and therefore we get $|A| \leq |A|_+$ and hence equality.

(2) The assumption implies that for any finite subset $B \subset A$, we have $|B| = |B|_+$. Now, Theorem 3.18 and Theorem 3.28 imply $|A| = |A|_+$.

(3) By Proposition 2.44, every finite subspace of $\mathbb{R}$ is positive definite with positive weighting, so every finite subspace of A has a positive weighting and by the previous point we are done. □

Example 3.31 *Any homogeneous positive definite metric space, admitting a positive invariant measure is positively weighted. Indeed, let X be such a space. We can pick any non-zero invariant positive measure on X and by Theorem 3.24, there exists a positive weighting on X. Now by the lemma above, it follows that X is positively weighted.*

We want to show that the supremum in the definition of the maximum diversity is attained. For this, we will need the fact that the space $P(A)$ of probability measures is metrized by the so-called Wasserstein distance, which we define here.

Note that what we call the Wasserstein distance here is in the general equivalent to what is sometimes called the Wasserstein distance W_1, see Definition 6.4 and Remark 6.5 in [8] for details.

Definition 3.32 *The **Wasserstein distance** on the set of signed measures $M(A)$ is*

$$d_W(\mu,\nu) = \sup\left\{\int_A f d\mu - \int_A f d\nu \mid f\colon A \to \mathbb{R} \text{ is } 1-\text{Lipschitz}\right\}.$$

For a proof of the following proposition, see Corollary 6.13 in [8].[2]

Proposition 3.33 *For a compact metric space A, the space $P(A)$ of probability measures endowed with the weak-* topology inherited from $M(A)$ can be metrized by the Wasserstein distance.*

Let us show that the supremum in the definition of the maximum diversity is attained.

Proposition 3.34 *For any compact positive definite metric space A, the supremum in the definition of $|A|_+$ is attained for some measure $\mu \in M_+(A)$.*

Proof Let us set $P(A) = \{\mu \in M_+(A) : \mu(A) = 1\}$ the space of probability measures on A. $P(A)$ is compact in the weak-* and it is metrized by the Wasserstein distance[3] (see Corollary 5.4 in the Appendix [5], and Proposition 3.33).

We have $|A|_+ = \sup\left\{\frac{\mu(A)^2}{Z_A(\mu,\mu)} : \mu \in M_+(A), \mu \neq 0\right\} = \sup\limits_{\mu \in P(A)} \frac{1}{Z_A(\mu,\mu)}$, by homogeneity of the expression.

Take arbitrary $\mu,\nu \in P(A)$. As in Lemma 3.14, we define the maps $f_\mu : A \to \mathbb{R}$, $f_\mu(x) = \int_A e^{-d(x,z)} d\mu(z)$ and $f_\nu : A \to \mathbb{R}$, $f_\nu(x) = \int_A e^{-d(x,z)} d\nu(z)$. We show that these are 1-Lipschitz. For any $x, y \in A$, we have

$$\begin{aligned}|f_\mu(x) - f_\mu(y)| &= |\int_A (e^{-d(x,z)} - e^{-d(y,z)}) d\mu(z)| \leq \\ &\leq \int_A |e^{-d(x,z)} - e^{-d(y,z)}| d|\mu|(z) \leq d(x,y)||\mu|| = d(x,y),\end{aligned}$$

[2] In the cited work, in the case of compact spaces the weak topology is what we here call the weak-* topology.

[3] See Definition 3.32.

where the last equality follows since μ is a probability measure and therefore, $||\mu|| = \mu(A) = 1$. Similarly, f_ν is also 1-Lipschitz continuous.

Then we have

$$|Z_A(\mu,\mu) - Z_A(\nu,\nu)| = \left|\int_A f_\mu d\mu - \int_A f_\nu d\nu\right| \leq$$
$$\leq \left|\int_A f_\mu d\mu - \int_A f_\mu d\nu\right| + \left|\int_A f_\nu d\mu - \int_A f_\nu d\nu\right| \leq 2d_W(\mu,\nu).$$

Hence, the map $P(A) \to (0,\infty)$, $\mu \mapsto Z_A(\mu,\mu)$ is Lipschitz-continuous and in particular continuous. Therefore, the map $P(A) \to (0,\infty)$, $\mu \mapsto \frac{1}{Z_A(\mu,\mu)}$ is also continuous. Now by compactness of $P(A)$, it follows that this map attains its supremum, so $\exists \mu_0 \in P(A)$ such that $\frac{1}{Z_A(\mu_0,\mu_0)} = \sup_{\mu \in P(A)} \frac{1}{Z_A(\mu,\mu)} = |A|_+$. □

The following is a direct implication of the last proposition.

Corollary 3.35 *If A is positively weighted compact positive definite metric space, then there exists a positive weight measure on A.*

Proof By proposition 3.34, $\exists \mu \in M_+(A)$ such that $|A|_+ = \frac{\mu(A)^2}{Z_A(\mu,\mu)}$. By definition of positively weighted, we have therefore $|A| = \frac{\mu(A)^2}{Z_A(\mu,\mu)}$. Now by Theorem 3.17, it follows that μ is a scalar multiple of a weight measure on A and in particular, a positive weight measure exists. □

The following proposition shows that the maximum diversity is continuous.

Proposition 3.36 (Continuity of Maximum Diversity) *The map $A \mapsto |A|_+$ is continuous with respect to the Gromov-Hausdorff distance on the class of compact positive definite metric spaces.*

Proof Lower semicontinuity of this map follows as in the proof of Theorem 3.21. Therefore, it suffices to prove upper semicontinuity.

Take A, B with $d_{GH}(A,B) > 0$. We can assume without loss of generality that A and B are both subspaces of some metric space X such that $0 < d_H(A,B) < 2d_{GH}(A,B)$. By Proposition 3.34, $\exists \mu \in M_+(B)$ such that $|B|_+ = \frac{\mu(B)^2}{Z_B(\mu,\mu)}$. Let $\nu \in M_+(A)$ be as given by Lemma 3.13 for $\varepsilon = 2d_H(A,B)$. Now by Lemma 3.14, we have $|Z_A(\nu,\nu) - Z_B(\mu,\mu)| \leq 4d_H(A,B)||\nu||^2$. Therefore $|Z_A(\nu,\nu) - Z_B(\mu,\mu)| \leq 8d_{GH}(A,B)||\nu||^2$, and so

$$Z_B(\mu,\mu) \geq Z_A(\nu,\nu) - 8d_{GH}(A,B)||\nu||^2.$$

This implies

$$\frac{\mu(B)^2}{Z_B(\mu,\mu)} \leq \frac{\nu(A)^2}{Z_A(\nu,\nu) - 8d_{GH}(A,B)||\nu||^2}.$$

Since the measure ν is positive, we have $||\nu|| = |\nu|(A) = \nu(A)$ and so

$$|B|_+ = \frac{\mu(B)^2}{Z_B(\mu,\mu)} \leq \frac{\nu(A)^2}{Z_A(\nu,\nu) - 8d_{GH}(A,B)\nu(A)^2} =$$
$$= \left(\frac{Z_A(\nu,\nu)}{\nu(A)^2} - 8d_{GH}(A,B)\right)^{-1} = \frac{\nu(A)^2}{Z_A(\nu,\nu)} \cdot \left(1 - \frac{8\nu(A)^2}{Z_A(\nu,\nu)} d_{GH}(A,B)\right)^{-1} \leq$$
$$\leq |A|_+ \cdot \left(1 - \frac{8\nu(A)^2}{Z_A(\nu,\nu)} d_{GH}(A,B)\right)^{-1}$$

Thus, if $d_{GH}(A,B) \leq \frac{\varepsilon}{8|A|_+}$, for $0 < \varepsilon < 1$, then $|B|_+ \leq \frac{|A|_+}{1-\varepsilon}$ and we have upper semicontinuity and therefore continuity. □

If magnitude and maximum diversity are equal, we immediately obtain continuity of magnitude:

Corollary 3.37 *Magnitude is continuous with respect to the Gromov-Hausdorff distance on the class of positively weighted compact positive definite metric spaces.*

Proof This follows immediately from the previous proposition, since magnitude and maximal diversity are equal for positively weighted spaces. □

Corollary 3.38 *Magnitude is continuous with respect to the Gromov-Hausdorff distance on the class of compact subsets of* $\mathbb{R}$.

Proof By Lemma 3.30, compact subsets of $\mathbb{R}$ are positively weighted and therefore, by the previous corollary, the conclusion is immediate. □

Chapter 4

An Analytic Perspective on Magnitude

In this chapter, we develop a new perspective on magnitude. We start by viewing magnitude of compact metric spaces from the point of view of Hilbert spaces. Based on this, we develop a Fourier-analytic perspective on the magnitude of compact metric subspaces of the Euclidean space. These new tools will allow us to prove our final continuity result: continuity of magnitude for positive-definite subsets of $\mathbb{R}^n$ equipped with metrics coming from p-norms. This chapter is based on Section 4.3 in [3].

4.1 Magnitude and Hilbert Spaces

In this section, we give a characterization of magnitude relying on the theory of Hilbert spaces.

Let us start by introducing the following notation:

Definition 4.1 *For $X \subset \mathbb{R}^n$ and $0 < p < \infty$ we set $\mathbf{L_p}(\mathbf{X})$, the space of equivalence classes under equality almost everywhere of measurable functions $f\colon X \to \mathbb{R}$ such that $||f||_p := \left(\int_X |f(t)|^p dt\right)^{\frac{1}{p}} < \infty$.*

Definition 4.2 *A **positive definite kernel** on a space X is a function $K\colon X \times X \to \mathbb{C}$ such that for any finite $A \subset X$, the matrix $(K(a,b))_{a,b\in A} \in \mathbb{C}^{A\times A}$ is positive definite.*

We are mainly interested in the following example:

Example 4.3 *For a positive definite metric space (A, d), the map $A \times A \to \mathbb{R}$, $(x, y) \mapsto e^{-d(x,y)}$ is a positive definite kernel.*

Definition 4.4 *Given a positive definite kernel K on X, the **reproducing kernel Hilbert space (RKHS)** $\mathcal{H}$ on X with kernel K is the completion of the linear space on the functions $k_x(y) = K(x, y)$ with respect to the inner product given by $\langle k_y, k_y\rangle_{\mathcal{H}} = K(x, y)$.*

Let us remark that if $f \in \mathcal{H}$, then $\langle f, k_x \rangle_{\mathcal{H}} = f(x)$.

Based on the following theorem, we can give a characterization of magnitude using the RKHS. For a proof, see Theorem 5.3 in [5].

Theorem 4.5 (Riesz Representation Theorem for Hilbert Spaces) *Let Λ be a continuous linear funtional on a Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$. Then there exists a unique $g \in \mathcal{H}$ such that $\Lambda(f) = \langle f, g \rangle, \forall f \in \mathcal{H}$. Moreover, $||\Lambda|| = ||g||$.*

The following theorem allows us to determine the magnitude of a compact metric space via the norm in the RKHS.

Theorem 4.6 (Characterization of magnitude using RKHS) *Let (X, d) be a positive definite metric space and let $\mathcal{H}$ denote the RKHS of the map $K(x, y) = e^{-d(x,y)}$ on X. Let $A \subset X$ be a compact subset. Then $|A| < \infty \iff \exists h \in \mathcal{H}$ such that $h \equiv 1$ on A. In that case,*

$$|A| = \inf\{||h||^2_{\mathcal{H}} : h \in \mathcal{H}, h \equiv 1 \text{ on } A\}.$$

Furthermore, the infimum above is achieved for a unique function $h \in \mathcal{H}$.

Proof We start with the following useful computation: Let $B \subset X$ be finite, $w \in \mathbb{R}^B$ an arbitrary B-vector. Define $f_w = \sum_{b \in B} w_b e^{-d(\cdot,b)}$. Then

$$w^T \zeta_B w = \sum_{a,b \in B} w_a e^{-d(a,b)} w_b = \sum_{a,b \in B} w_a w_b \langle e^{-d(\cdot,a)}, e^{-d(\cdot,b)} \rangle_{\mathcal{H}} =$$
$$= \langle \sum_{a \in B} w_a e^{-d(\cdot,a)}, \sum_{b \in B} w_a e^{-d(\cdot,b)} \rangle_{\mathcal{H}} = ||f_w||^2_{\mathcal{H}}.$$

Let us assume that $|A| < \infty$ and let $B \subset A$ be finite, $w \in \mathbb{R}^B$. By Proposition 2.41, we have

$$|B| = \sup_{w' \in \mathbb{R}^B} \frac{\left(\sum_{b \in B} w'_b\right)^2}{w'^T \zeta_B w'}.$$

Therefore,

$$\left(\sum_{b \in B} w_b\right)^2 \leq |B| \cdot (w^T \zeta_B w) = |B| \cdot ||f_w||^2_{\mathcal{H}} \leq |A| \cdot ||f_w||^2_{\mathcal{H}},$$

where the equality follows from the computation above and the last inequality follows from monotonicity of magnitude with respect to inclusion in positive definite metric spaces (Proposition 3.10).

Therefore, the linear functional

$$f_w \mapsto \sum_{b \in B} w_b \text{ on } \{f_w : w \in \mathbb{R}^B, B \subset A \text{ finite}\} \subset \mathcal{H}$$

has norm at most $\sqrt{|A|}$. Let us also note that since A is positive definite, $\lim_{B \to A} |B| = |A|$, where the limit is taken over all finite subspaces $B \subset A$ with respect to the Hausdorff metric (see Proposition 3.28). This implies that the operator norm of the linear functional above is exactly $\sqrt{|A|}$.

By the Riesz Representation Theorem for Hilbert spaces (Theorem 4.5), we deduce $\exists h \in \mathcal{H}$ with such that

$$||h||_{\mathcal{H}} = \sqrt{|A|} \text{ and } \sum_{b \in B} w_b = \langle f_w, h \rangle_{\mathcal{H}} = \sum_{b \in B} w_b h(b), \text{ for all } w \in \mathbb{R}^B.$$

For any $a \in A$, let us now take $w_b = \delta_{a,b}$, i.e. $f_w = e^{-d(\cdot,a)}$. This implies $h(a) = 1$ and so $h \equiv 1$ on A. This means that the 'if' part in the theorem is proven. Even more, we know that in this case, there exists $h \in \mathcal{H}$ with $||h||^2_{\mathcal{H}} = |A|$ and $h \equiv 1$ on A.

Conversely, let us suppose $\exists h \equiv 1$ on A. Let $B \subset A$ be finite and $w \in \mathbb{R}^B$. The Cauchy-Schwarz inequality together with the computation we have started with imply

$$|\sum_{b \in B} w_b| = |\langle h, f_w \rangle| \leq ||h||_{\mathcal{H}} \cdot ||f_w||_{\mathcal{H}} = ||h||_{\mathcal{H}} \sqrt{w^T \zeta_B w}.$$

Therefore,

$$\frac{\left(\sum_{b \in B} w_b\right)^2}{w^T \zeta_B w} \leq ||h||^2_{\mathcal{H}}$$

and by Proposition 2.41 $|B| \leq ||h||^2_{\mathcal{H}}$. Since this holds for any finite subset of A, Theorem 3.28 now implies $|A| \leq ||h||^2_{\mathcal{H}}$ and in particular, $|A|$ is finite.

So, the if and only if part of the theorem is proven. Also, we have seen that if $|A|$ is finite, then for any $h \equiv 1$ on A, we have $|A| \leq ||h||^2_{\mathcal{H}}$. In the first part of the proof we have also seen that there exists some $h \equiv 1$ on A such that $|A| = ||h||^2_{\mathcal{H}}$. Therefore, we can immediately conclude

$$|A| = \inf\{||h||^2_{\mathcal{H}} : h \in \mathcal{H}, h \equiv 1 \text{ on } A\}.$$

Finally, uniqueness of h where it is attained follows from uniqueness in the Riesz Representation Theorem for Hilbert Spaces. ☐

Existence and uniqueness of the map h in the last theorem allows us to introduce the following definition.

Definition 4.7 *The unique function h which achieves the infimum in Theorem 4.6 is called the* ***potential function*** *of A.*

4.2 Magnitude on Subsets of $\mathbb{R}^n$

In this section, we introduce a Fourier-theoretic perspective on magnitude. The advantage is that Fourier theory allows us to give an explicit characterization of the RKHS, which in turn enables us to prove our final continuity result.

We work with the following convention for the Fourier transform:

Definition 4.8 *The **Fourier transform** of a function $f \in L_1(\mathbb{R}^n)$ is*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx.$$

We also need the following useful fact (which can be shown using a routine computation):

Lemma 4.9 (Fourier Tranform and Dilation) *For a function $f \in L_1(\mathbb{R}^n)$, the Fourier transform of the inverse dilation $\frac{1}{|\delta|^n} f(\frac{x}{\delta})$ is the dilation $\widehat{f}(\delta x)$.*

With the following generalization of the notion of norm, we can formulate our final continuity result.

Definition 4.10 *Let $p > 0$. A **p-norm** on a real vector space V is a map $|| \cdot || \colon V \to \mathbb{R}$ such that*

- $||v|| \geq 0, \forall v \in V$ *with equality if and only if* $v = 0$.
- $||tv|| = |t| \cdot ||v||, \forall t \in \mathbb{R}, \forall v \in V$.
- $||v + w||^p \leq ||v||^p + ||w||^p, \forall v, w \in V$.

Note that any p-norm $|| \cdot ||$ on $\mathbb{R}^n$ induces a distance $d_p(x, y) = ||x - y||^p$.

Example 4.11 *We note that for $p \geq 1$, $|| \cdot ||_p$ defines a norm on L_p, while for $0 < p < 1$, it defines a p-norm. Thus, in general, we can view L_p as endowed with the metric $d_p(f, g) = ||f - g||_p^{\min\{1,p\}}$.*

For the rest of this section, let $|| \cdot ||$ be a p-norm on $\mathbb{R}^n$ and d_p the induced metric, such that $(\mathbb{R}^n, d_p)$ is a positive definite metric space. Let $F_p \colon \mathbb{R}^n \to \mathbb{R}, F_p(x) = e^{-||x||^p}$ and let $\mathcal{B} := \{x \in \mathbb{R}^n : ||x|| = 1\}$ be the unit ball with respect to the fixed p-norm.

The following proposition is be useful in proving our last continuity result for magnitude. For a proof, see proposition 4.11(2) in [3].

Proposition 4.12 *For all $x \in \mathbb{R}^n$, the map $t \mapsto \widehat{F_p}(tx)$ is nonincreasing.*

We can now give an explicit expression for the RKHS $\mathcal{H}$ of $(\mathbb{R}^n, d)$, which will turn out vital in our last proof. For a proof see Theorem 10.12 in [9].

Proposition 4.13 (Explicit Characterization of the RKHS) *The RKHS of the map $K(x,y) = e^{-d_p(x,y)}$ on (R^n, d_p) is*

$$\mathcal{H} = \{f \in L_2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{1}{\widehat{F_p}(x)} |\hat{f}(x)|^2 dx < \infty\}$$

with norm given by $||f||^2_{\mathcal{H}} = \int_{\mathbb{R}^n} \frac{1}{\widehat{F_p}(x)} |\hat{f}(x)|^2 dx$.

For a subspace A of $\mathbb{R}^n$, we denote $t * A$ the space $\{t \cdot a : a \in A\} \subset \mathbb{R}^n$ with the metric d_p. Note that this is in general not the same as the space $tA = (A, td_p)$, in fact the space tA is isometric to the space $t^{1/p} * A$, equipped with d_p.

We now state and prove a continuity result for magnitude on subspaces of $\mathbb{R}^n$, equipped with the metric d_p.

Theorem 4.14 (Continuity of Magnitude for Compact Subsets of $\mathbb{R}^n$) *Let $\mathcal{K}_n$ be the class of compact subsets of $\mathbb{R}^n$ with non-empty interior, equipped with the Hausdorff distance d_H induced by d_p and suppose that $A \in \mathcal{K}_n$ is star-shaped. Then magnitude, as a map $\mathcal{K}_n \to \mathbb{R}$ is continuous at A.*

Proof We have already shown lower semicontinuity in Theorem 3.21, so it only remains to show upper semicontinuity.

We start by noting that $|A|$ is finite: by Proposition 4.13, it follows that there exist functions in $\mathcal{H}$ that only take the value 1 on A. Hence, by Theorem 4.6, $|A|$ is finite.

Let us prove $|t * A| \leq t^n |A|$. Let h be the potential function of A and for $t \geq 1$ let us denote h_t the map $h_t(x) = h(\frac{x}{t})$. Note that $h \equiv 1$ on A implies that $h_t \equiv 1$ on $t * A$. Therefore, by Theorem 4.6, we have $|t * A| \leq ||h_t||^2_{\mathcal{H}}$. Theorem 4.13 implies $||h_t||^2_{\mathcal{H}} = \int_{\mathbb{R}^n} \frac{1}{\widehat{F_p}(x)} |\widehat{h_t}(x)|^2$.

Observe that for any $t \geq 1$, $h_t(x) = t^n \cdot \frac{1}{t^n} h(\frac{x}{t})$ and therefore, by Lemma 4.9, $\widehat{h_t}(x) = t^n \cdot \widehat{h}(tx)$. Hence, we can make the substitution $y = tx$ in the integral above and obtain

$$\int_{\mathbb{R}^n} \frac{1}{\widehat{F_p}(x)} |\widehat{h_t}(x)|^2 = t^{-n} \int_{\mathbb{R}^n} \frac{1}{\widehat{F_p}(\frac{y}{t})} |t^n \widehat{h_t}(y)|^2 = t^n \int_{\mathbb{R}^n} \frac{1}{\widehat{F_p}(\frac{y}{t})} |\widehat{h_t}(y)|^2.$$

Now, $t \geq 1 \implies \frac{1}{t} \leq 1$, so by Proposition 4.12, $\widehat{F_p}(\frac{y}{t}) \geq \widehat{F_p}(y)$ and hence $\frac{1}{\widehat{F}(\frac{y}{t})} \leq \frac{1}{\widehat{F_p}(y)}$. So,

$$t^n \int_{\mathbb{R}^n} \frac{1}{\widehat{F_p}(\frac{y}{t})} |\widehat{h_t}(y)|^2 \leq t^n \int_{\mathbb{R}^n} \frac{1}{\widehat{F_p}(y)} |\widehat{h_t}(y)|^2 = t^n |A|,$$

where in the last equality we have used the assumption that h is a potential. This chain of inequalities implies $|t * A| \leq t^n |A|$.

By translation invariance, we can assume without loss of generality that $0 \in A$ and $r^{1/p} * \mathcal{B} \subset A$ for some $r > 0$. Let $\varepsilon > 0$ be arbitrary and let $B \in \mathcal{K}_n$ with $d_H(A, B) < \varepsilon$. By definition of the Hausdorff distance, it follows that $B \subset A + \varepsilon^{1/p} * \mathcal{B} \subset \left(1 + \left(\frac{\varepsilon}{r}\right)^{1/p}\right) * A$. Therefore,

$$|B| \leq \left| \left(1 + \left(\frac{\varepsilon}{r}\right)^{1/p}\right) * A \right| \leq \left(1 + \left(\frac{\varepsilon}{r}\right)^{1/p}\right)^n |A|$$

and we obtain upper semicontinuity. Together with the already established lower semicontinuity, we can conclude that magnitude is continuous at star-shaped sets in $\mathcal{K}_n$. □

Chapter 5

Appendix

In order to show that the supremum in the definition of maximum diversity is attained for some positive measure (Proposition 3.34), we rely on a well-known consequence of the Banach-Alaoglu Theorem, which we state in this section.

Let (A, d) be a compact metric space. We denote

$$P(A) = \{\mu \in M_+(A) : \mu(A) = 1\}$$

the space of probability measures on A and $C_b(A)$ the space of bounded, continuous functions on A.

Let E be a Banach space and E^* its continuous dual (the subspace of continuous linear functionals on E). We endow E^* with the norm $||f|| = \sup_{x \in E, ||x|| \leq 1} |f(x)|$.

For a proof of the following theorem, see [10], Theorem 7.54.

Theorem 5.1 (Riesz representation on $C_0(X)$) *Let X be a locally compact, σ-compact metric space and let $\Lambda \in (C_0(X))^*$ be a continuous linear functional on the space $C_0(X)$ of continuous functions on X that vanish at infinity. Then there exists a uniquely determined signed measure μ representing Λ. In other words, the map $\varphi\colon M(X) \to C_0(X)^*, \varphi(\mu) = \left(f \mapsto \int_X f d\mu\right)$ is an isometry.*

To every $x \in E$ we can associate a linear functional $\varphi_x\colon E^* \to \mathbb{R}$, defined by $f \mapsto f(x)$.

Definition 5.2 *The **weak-* topology** on E^* is defined as the coarsest topology on E^* such that the associated linear functional φ_x is continuous, for all $x \in E$.*

Since A is compact by assumption, it means that the spaces $C_0(A)$ and $C_b(A)$ are equal and the Riesz representation theorem (as stated above) implies that we can identify the space $M(A)$ of measures on A with the norm of total variation and the dual of the Banach space $(C_b(A), || \cdot ||_\infty)$. Under this identification, we can view the space $M(A)$ as endowed with the weak-* topology.

For a proof of the following see theorem 8.10 in [10].

Theorem 5.3 (Banach-Alaoglu Theorem) *The closed unit ball*

$$B_{E^*} := \{f \in E^* : ||f|| \leq 1\}$$

is compact in the weak-* topology.

The following corollary is useful in proving Proposition 3.34

Corollary 5.4 *For a compact metric space A, the space $P(A)$ of probability measures is compact in the weak-* topology on $M(A)$.*

Proof By Banach-Alaoglu, the unit ball $\{\mu \in M(A) : ||\mu|| \leq 1\} \subset M(A)$ is compact in the weak-* topology. We show that $P(A)$ is a closed subset of the unit ball.

Observe that $M_+(A) = \bigcap_{f \in C_0(A)^+} \{\mu \in M(A) : \int_A f d\mu \geq 0\}$, where by $C_0(A)^+$ we denote the set of nonnegative continuous functions that vanish at infinity. Each of the sets in this intersection is closed, therefore we deduce that $M_+(A)$ is closed in $M(A)$. Finally, $\{\mu \in M(A) : ||\mu|| = 1\}$ is closed, since it is the preimage of the closed set $\{1\}$ under the total variation norm (which is a continuous map).

By definition, $P(A) = M_+(A) \cap \{\mu \in M(A) : ||\mu|| = 1\}$, so it is closed as an interesection of closed sets. Also note that $P(A) \subset \{\mu \in M(A) : ||\mu|| \leq 1\}$, so $P(A)$ is a closed subset of a compact set and therefore compact. $\square$

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Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

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