Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Gabriel's Theorem

Valentin Nussli

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Dr. Sara Kališnik Hintz

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Abstract

This thesis is devoted to the study of quiver representations and the proof of Gabriel's Theorem. Quiver representations are collections of vector spaces and linear maps. Just like with decomposing numbers into products of primes, one aims to decompose quiver representations into the smallest possible building blocks, called indecomposable representations. Gabriel's Theorem specifies the quivers that have a finite number of indecomposable representations. Furthermore, it classifies these indecomposable representations. I prove a decomposition theorem by Krull, Remak, and Schmidt in Chapter 2. This theorem relies heavily on indecomposable representations, which form the centerpiece of Chapter 3. In this chapter I introduce a variety of mathematical tools to analyze quivers and their representations, culminating in the proof of Gabriel's Theorem. I also show how Gabriel's Theorem can be used in topological data analysis to characterize the persistence of topological features. To do this, I present the necessary quiver representation theory in Chapter 4 and I connect this theory with the topological aspects of persistent homology in Chapter 5. The thesis is based on [1] and [2].

Acknowledgements

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Contents

Acronyms and Abbreviations

w.l.o.g.	Without loss of generality
w.r.t	With respect to
point cloud	Finite metric space
$\bar{B}_r(x)$	Closed ball with center x and radius r
S^n	Sphere of dimension n, $\{(x_0, x_1, \ldots .x_n) \mid x_0^2 + x_1^2 + \ldots 1x_n^2 = 1\} \subseteq \mathbb{R}^{n+1}$
$\lvert X \rvert$	Cardinality of the set X
Q	Quiver with vertex set Q_0 and arrow set Q_1

Chapter 1

Introduction

Topology is a mathematical discipline focused on investigating the characteristics of spaces that remain unchanged under deformations. These deformations involve actions like bending, squishing, shrinking and expanding, whereas tearing is prohibited. Homotopy equivalence emerges as a formalism to precisely describe this equivalence between topological spaces.

A homotopy between two continuous maps $f, g\colon X \to Y$ is a continuous map $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. The map H describes a family of functions interpolating continuously between f and g as the parameter t varies from 0 to 1. A pair of continuous functions $f\colon X \to Y$ and $g\colon Y \to X$ are called homotopy equivalences if there exists a homotopy H between the maps $g \circ f$ and id_X, and there exists a homotopy K between the maps $f \circ g$ and id_Y. In this case, the topological spaces X and Y are called homotopy equivalent. The circle S^1 and the punctured disk $D' = D \setminus \{(0,0)\}$ are homotopy equivalent. The maps $f\colon D' \to S^1, x \mapsto \frac{x}{\|x\|}$ and $g\colon S^1 \to D', x \mapsto x$ are homotopy equivalences. Indeed, note that $f \circ g = \mathrm{id}_{S^1}$ and $H(x,t) = (1-t)\frac{x}{\|x\|} + tx$ is a homotopy between $g \circ f$ and $\mathrm{id}_{D'}$.

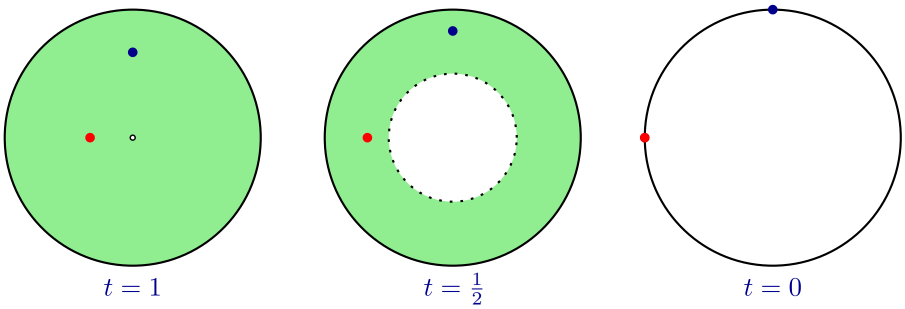


Figure 1.1: The punctured disk D' (left) and the circle S^1 (right) are homotopy equivalent. The red and blue points represent $H(x,t)$ for two points in D', where $H(x,t) = (1-t)\frac{x}{\|x\|} + tx$ denotes the homotopy between the maps $(g \circ f)(x) = \frac{x}{\|x\|}$ and $\mathrm{id}_{D'}$.

Properties of topological spaces that stay the same under deformations are called homotopy invariants. These invariants are the main objects of study in the field of algebraic topology. One such invariant is the presence of holes, voids and higher dimensional equivalents in a topological space and counting how many there are. In Figure 1.1, the punctured disk and the circle are homotopy equivalent. Observe that the volumes of the spaces differ, however, the middle hole is present throughout the deformation process. The mathematical formalization to measure holes and their higher dimensional analogues is called **homology**. Homology and other invariants measure the shape of topological spaces.

Recently, with a huge amount of data being generated, there have been efforts to adapt these methods to measure the shape of data. **Topological data analysis** offers one potential approach to address this challenge. However, since data is given by point clouds, its topology does not reveal any information besides the number of points. To fix this problem, we consider triangulations of point clouds that are called Čech complexes. To compute a Čech complex, we replace the points with balls of a certain radius. If two balls intersect, we connect the corresponding points by an edge. The non-trivial intersection of three balls results in a triangle and so on. A crucial

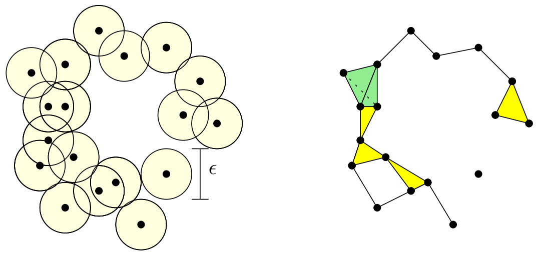


Figure 1.2: A point cloud with associated balls on the left and its associated Čech complex on the right.

characteristic is that the Čech complex corresponding to a smaller radius is included in the Čech complex associated with a larger radius. Figure 1.2 illustrates the construction of a Čech complex from a point cloud. Using the inclusion property of Čech complexes of increasing radii, we get a sequence of Čech complexes and inclusion maps between them, called a filtration. We then apply homology to this filtration and track over which parameter values the topological features persist. This yields a collection of intervals. Each interval $[a, b]$ represents a topological feature born at time a that disappears at time b. These intervals form a persistence module and they are visualized by a **persistence barcode**. Figure 1.3 shows the persistence barcode for the point cloud in Figure 1.2. Topological features that persist over a long parameter range are considered to be important with short-lived features as noise. This adaptation of homology to the setting of point clouds is called **persistent homology**.

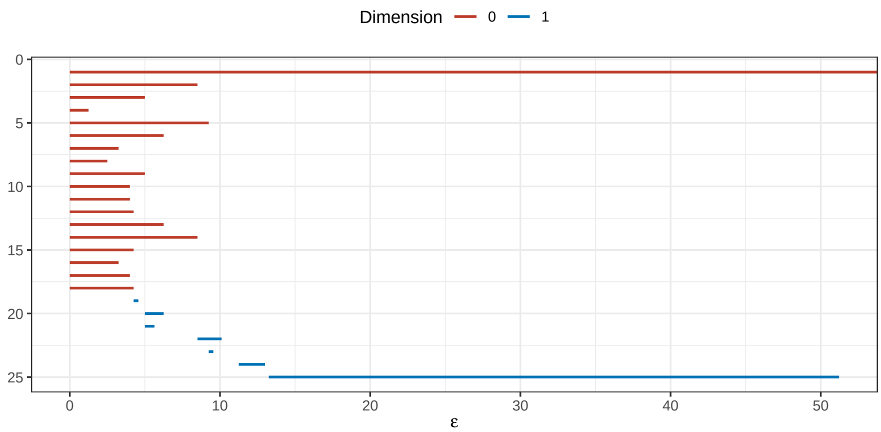


Figure 1.3: A persistence barcode coming from a point cloud.

The existence of persistence barcodes relies on a theorem from **quiver representation theory**, called Gabriel's Theorem. This discipline focuses on quivers and their corresponding representations. Quivers are directed graphs consisting of points and arrows connecting the points. A quiver representation is an allocation of vector spaces to the points and linear maps are allocated to the arrows. Figure 1.4 shows an example of a quiver and a possible representation. This quiver is called a linear quiver. Quiver representation theory is built on a purely algebraic foundation which offers distinct advantages. Firstly, the broad algebraic framework allows for versatile application in various contexts without the necessity of introducing novel concepts. This flexibility will be used to extend the theory of persistent homology to include zigzag persistent homology. Additionally, it is worth noting that quiver representation theory finds applications beyond topology, such as in the realms of Lie algebras and quantum groups [2]. However, these applications are beyond the scope of this thesis.

$$\underset{1}{\bullet} \xrightarrow{a} \underset{2}{\bullet} \xrightarrow{b} \cdots \xrightarrow{c} \underset{n}{\bullet} \qquad\qquad V_1 \xrightarrow{v_a} V_2 \xrightarrow{v_b} \cdots \xrightarrow{v_c} V_n$$

Figure 1.4: The linear quiver L_n (left) and a quiver representation (right).

The goal of this thesis is to prove Gabriel's Theorem. In Chapter 2, we introduce quiver representations and observe that they always possess a unique decomposition into the smallest possible building blocks called indecomposable representations. This naturally leads to the question of which quivers only have a finite number of indecomposable representations. Such quivers are called finite-type quivers. The answer to this question is given by Gabriel's Theorem, whose proof is at the center of Chapter 3. It turns out that the persistence modules are quiver representations of the linear quivers. Therefore, we can apply Gabriel's Theorem in the setting of persistence modules. This is the reason why a persistence barcode always exists. Chapter 4 focuses on the algebraic aspects of the quiver theory of A_n-type quivers, which are a broader generalization of the linear quivers. In Chapter 5, we introduce the topological aspects of persistent homology while using the theory from the previous chapters.

Chapter 2

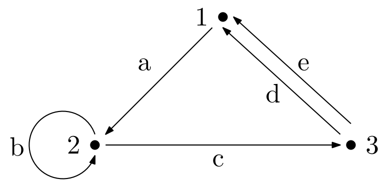
Quiver Representations

In this chapter, we introduce quiver representations and we consider their morphisms and direct sums of quiver representations. This culminates in a decomposition theorem by Krull, Remak, and Schmidt (Theorem 2.33). This chapter is primarily based on the book 'Persistence Theory: From Quiver Representations to Data Analysis' written by Steve Y. Oudot [1].

2.1 Quivers

Definition 2.1. *A **quiver** Q consists of two sets Q_0, Q_1 and two maps $h, t\colon Q_1 \to Q_0$. The elements of Q_0 are called the **vertices** of Q, while those of Q_1 are called the **arrows**. The **head map** h and the **tail map** t assign a head h_a and a tail t_a to every arrow $a \in Q_1$.*

Example 2.2. *Consider the quiver Q with vertices $Q_0 = \{1, 2, 3\}$ and arrows $Q_1 = \{a, b, c, d, e\}$.*



Its head map $h\colon Q_1 \to Q_0$ is given by $a \mapsto 2, b \mapsto 2, c \mapsto 3, d \mapsto 1, e \mapsto 1$ and its tail map $t\colon Q_1 \to Q_0$ is given by $a \mapsto 1, b \mapsto 2, c \mapsto 2, d \mapsto 3, e \mapsto 3$.

Q is a directed graph, where the elements in Q_0 are the vertices and for every $a \in Q_1$, the pair (t_a, h_a) is a directed edge in our graph. There are no restrictions on the sets Q_0 and Q_1, so there may be infinitely many points and edges and also multiple edges between two points, thus a quiver is graphically represented by a directed multigraph. We denote by $\bar{Q}$ the underlying undirected graph of Q.

Definition 2.3. *A quiver Q is called **finite** if both Q_0 and Q_1 are finite sets.*

Definition 2.4. *A quiver Q is called a **Dynkin** quiver if its underlying graph $\bar{Q}$ is one of the graphs in Figure 2.1.*

Dynkin quivers emerge in Gabriel's Theorem (Theorem 3.2). They also play a fundamental role in classifying semisimple Lie algebras [2, p. 29]. An important subset of the Dynkin quivers are the linear quivers.

Definition 2.5. *For each $n \in \mathbb{N}$, the following quiver is called the **linear** quiver L_n:*

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n-1}{\bullet} \longrightarrow \underset{n}{\bullet}$$

Definition 2.6. *A quiver Q is called **acyclic** if there is no oriented cycle in Q.*

Every Dynkin quiver is acyclic. An example of a non-acyclic quiver is the loop quiver.

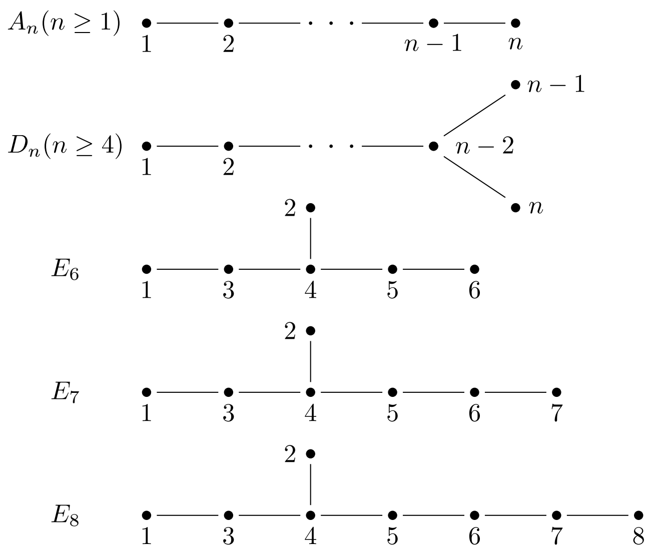
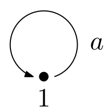


Figure 2.1: The Dynkin diagrams.

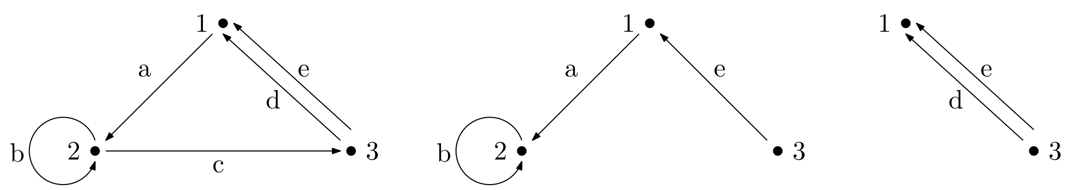
Definition 2.7. *The* ***loop quiver*** *consists of a single vertex that loops to itself.*



Sometimes, it is interesting to consider a part of a given quiver as it allows us to consider a property of the smaller quiver, and then extend it to the bigger quiver.

Definition 2.8. *Let $Q = (Q_0, Q_1)$ be a quiver and let $A_0 \subseteq Q_0$ and $A_1 \subseteq Q_1$ such that for all arrows $a \in A_1$ we have that $h(a) \in A_0$ and $t(a) \in A_0$. Then $A = (A_0, A_1)$ defines a* ***subquiver*** *of Q (A is a quiver). Moreover, if for all $i, j \in A_0$ we have that $\{a \in Q_1 \mid t(a) = i, h(a) = j\} \subseteq A_1$, then A is called a* ***full*** *subquiver with* ***support*** *A_0. In addition, if we say that we* ***delete a vertex*** *i from a quiver Q, then the resulting (sub)quiver is the full subquiver with support $Q_0 \setminus \{i\}$.*

Example 2.9. *Consider the quiver Q on the left. The quivers A (middle) and B (right) are both subquivers of Q. However, B is a full subquiver whereas A is not full.*



2.2 The Category of Quiver Representations

For a fixed quiver Q, we define quiver representations and morphisms between them. This is done in a way that turns them into a category. This category is called the category of quiver representations.

Definition 2.10. *A* ***quiver representation*** *of a quiver Q over a field k is a pair $\mathbb{V} = (V_i, v_a)$, which consists of a set of k-vector spaces $\{V_i \mid i \in Q_0\}$ together with a set of k-linear maps $\{v_a \colon V_{t_a} \to V_{h_a} \mid a \in Q_1\}$. We often abbreviate quiver representation to representation.*

There are no restrictions on the vector spaces and the maps. Thus the maps generally do not commute, i.e. if we have two different paths with the same starting point and end point, the compositions of the maps along the two paths need not be the same. Also, the vector spaces can be infinite-dimensional.

Definition 2.11. *A quiver representation $\mathbb{V}$ of a quiver Q is called **finite-dimensional** if the sum of the dimensions of the vector spaces $\sum_{i \in Q_0} \dim V_i$ is finite. If Q is a finite quiver with vertex set $Q_0 = \{1, 2, \ldots, n\}$ and $\mathbb{V}$ is a finite-dimensional representation, then we define the **dimension vector** $\underline{\dim}\mathbb{V}$ and its **dimension** $\dim \mathbb{V}$ as:*

$$\underline{\dim}\mathbb{V} = (\dim V_1, \ldots, \dim V_n)^T,$$
$$\dim \mathbb{V} = \|\underline{\dim}\mathbb{V}\|_1 = \sum_{i=1}^{n} \dim V_i.$$

Example 2.12. *A representation of the linear L_2-quiver*

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} \end{tikzcd}$$

is a set consisting of two vector spaces V_1, V_2 together with a linear map $v_a \colon V_1 \to V_2$.

Example 2.13. *[3] A representation of the loop quiver*

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[loop above, "a"] \end{tikzcd}$$

is a vector space V_1 together with an endomorphism $v_a \colon V_1 \to V_1$. If we restrict to the case where the field k is algebraically closed and V_1 is finite-dimensional, then we know from linear algebra that the matrix of v_a has a Jordan normal form (in a suitable basis)

$$\begin{pmatrix} J_{n_1,\lambda_1} & 0 & \cdots & 0 \\ 0 & J_{n_2,\lambda_2} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & J_{n_r,\lambda_r} \end{pmatrix},$$

where $J_{n,\lambda}$ denotes the $n \times n$-Jordan block:

$$\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & \emptyset & \\ & & \ddots & \ddots & \\ & \emptyset & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}.$$

Definition 2.14. *A representation $\mathbb{W} = (W_i, w_a)$ is a **subrepresentation** of a representation $\mathbb{V} = (V_i, v_a)$ if W_i is a subspace of V_i for all $i \in Q_0$ and if for all $a \in Q_1$, w_a is the restriction of the map v_a to the subspace W_{t_a} of the domain and the subspace W_{h_a} of the image, i.e. $w_a = v_a \mid_{W_{t_a}}^{W_{h_a}}$. We call $\mathbb{W}$ a **proper** subrepresentation of $\mathbb{V}$ if $0 \subsetneq \mathbb{W} \subsetneq \mathbb{V}$.*

Definition 2.15. *A representation of a quiver Q is called **simple** if it is non-trivial and it has no proper subrepresentations.*

Example 2.16. *We fix a vertex $i \in Q_0$ of a quiver Q. We define the representation $\mathbb{S}_i = (S_j^{(i)}, s_a^{(i)})$ to be*

$$S_j^{(i)} = \begin{cases} k & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} \quad \text{and } s_a^{(i)} = 0$$

for $j \in Q_0$ and $a \in Q_1$. This representation is simple. Indeed if $\mathbb{W} \subsetneq \mathbb{S}_i$ is a subrepresentation, then $W_j = 0$ for all $j \in Q_0$ and thus $\mathbb{W} = 0$ is trivial.

Our goal is not only to define quiver representations but also to compare and classify them. To be able to do this, we need to define morphisms between quiver representations.

Definition 2.17. *A **morphism** ϕ between two representations $\mathbb{V}, \mathbb{W}$ of a quiver Q is a set of k-linear maps $\{\phi_i \colon V_i \to W_i \mid i \in Q_0\}$ such that the following diagram commutes for each arrow $a \in Q_1$:*

$$\begin{tikzcd} V_{t_a} \arrow[r, "v_a"] \arrow[d, "\phi_{t_a}"'] & V_{h_a} \arrow[d, "\phi_{h_a}"] \\ W_{t_a} \arrow[r, "w_a"] & W_{h_a}. \end{tikzcd}$$

The morphism is called a ***monomorphism*** *if every linear map ϕ_i is injective, an* ***epimorphism*** *if every ϕ_i is surjective, and an* ***isomorphism*** *(denoted by $\cong$) if every ϕ_i is bijective. If $\phi\colon \mathbb{V} \to \mathbb{V}$ maps onto itself, we call it an* ***endomorphism****. We denote the set of morphisms from $\mathbb{V}$ to $\mathbb{W}$ by* $\mathrm{Hom}(\mathbb{V}, \mathbb{W})$*. If we equip* $\mathrm{Hom}(\mathbb{V}, \mathbb{W})$ *with pointwise multiplication and addition of linear maps, then it is a k-vector space.*

Example 2.18. *A morphism between two representations $\mathbb{V}, \mathbb{W}$ of the L_2-quiver is given by two linear maps $\phi_1 \colon V_1 \to W_1$, $\phi_2 \colon V_2 \to W_2$ such that the following diagram commutes*

$$\begin{tikzcd} V_1 \arrow[r, "v_a"] \arrow[d, "\phi_1"'] & V_2 \arrow[d, "\phi_2"] \\ W_1 \arrow[r, "w_a"] & W_2\,. \end{tikzcd}$$

Lemma 2.19. *Every isomorphism $\phi\colon \mathbb{V} \to \mathbb{W}$ is invertible, meaning that the map $\psi = \phi^{-1} \colon \mathbb{W} \to \mathbb{V}$ is a morphism of quiver representations (where $\psi_i = (\phi_i)^{-1}$). Thus the expressions isomorphism and invertible morphism are exchangeable.*

Proof. It is clear that ψ is pointwise well-defined. We need to check that for each arrow $a \in Q_1$ the following diagram commutes

$$\begin{tikzcd} V_i \arrow[r, "v_a"] \arrow[d, shift right, "\phi_i"'] & V_j \arrow[d, shift left, "\phi_j"] \\ W_i \arrow[r, "w_a"'] \arrow[u, shift right, "\psi_i"'] & W_j. \arrow[u, shift left, "\psi_j"] \end{tikzcd}$$

Now for each $w \in W_i$ we have $\phi_j v_a \psi_i(w) = w_a(w)$ since $\psi_i = (\phi_i)^{-1}$ and using that the diagram commutes with respect to the maps ϕ_i, ϕ_j. But then $v_a \psi_i(w) = \psi_j w_a(w)$. Thus the diagram commutes. $\square$

To turn quiver representations into a category, we need to define the composition of two morphisms in an associative way that guarantees the existence of an identity morphism. We can define such a composition by composing the maps ϕ_i at each point in our quiver.

Definition 2.20. *The* ***composition*** *of two morphisms $\phi\colon \mathbb{U} \to \mathbb{V}$ and $\psi\colon \mathbb{V} \to \mathbb{W}$ is given by the maps $(\psi \circ \phi)_i = \psi_i \circ \phi_i$ at each point $i \in Q_0$ in our quiver.*

This composition of morphisms is associative since it inherits the associativity of the composition of functions. Moreover, for each representation $\mathbb{V}$ we have the identity morphism $\mathbb{1}_{\mathbb{V}} \colon \mathbb{V} \to \mathbb{V}$ which is the identity on each V_i. Thus for a fixed quiver Q and a fixed field k, we get the **category of the quiver representations** of Q, denoted by $\mathrm{Rep}_{\mathrm{k}}(Q)$. If we restrict ourselves to the finite-dimensional representations we get the subcategory $\mathrm{rep}_{\mathrm{k}}(Q)$.

Example 2.21. *We give a description of the category $\mathrm{rep}_{\mathrm{k}}(Q)$ for the loop quiver Q. A morphism between two representations $\mathbb{V} = (V_1, v_a), \mathbb{W} = (W_1, w_a)$ of the loop quiver is a map $\phi_1 \colon V_1 \to W_1$ such that $w_a \phi_1 = \phi_1 v_a$. In Example 2.13 we saw that every representation is given by a map v_a which is in Jordan normal form in some suitable basis (if we restrict to an algebraically closed field k and to finite-dimensional representations). We also know that two finite-dimensional endomorphisms $A, B \in \mathrm{End}(V_1)$ are equivalent if and only if their Jordan normal forms are the same (up to reordering the Jordan blocks). Equivalence of matrices means that there exists a change of basis matrix $C \in \mathrm{End}(V_1)$ such that $A = C^{-1}BC$. Notice that such a change of basis represents an isomorphism between two representations given by the matrices A and B. Thus we have found that all representations are (up to isomorphism) given by Jordan normal forms and that two representations are isomorphic if and only if they have the same Jordan normal form (up to reordering the Jordan blocks). This is a complete description of all the isomorphism classes of the category* $\mathrm{rep}_{\mathrm{k}}(Q)$ *for the loop quiver Q and an algebraically closed field k.*

2.2.1 Direct Sums, Kernels and Cokernels

The categories $\mathrm{Rep}_k(Q)$ and $\mathrm{rep}_k(Q)$ have some properties that are useful towards our goal of classifying quiver representations. We list them here to give an overview and then we describe the second and third properties in detail.

ZO: Both categories contain a zero object, called the **trivial representation**, with all spaces and all maps equal to 0.

DS: We can combine representations to get new representations (of the same quiver). This is called a **direct sum**.

KI: Every morphism between representations (of the same quiver) $\phi\colon \mathbb{V} \to \mathbb{W}$ has a **kernel**, an **image** and a **cokernel**.

DS: The direct sum of two representations is defined for any representations $\mathbb{V}, \mathbb{W}$ to be the representation $\mathbb{V} \oplus \mathbb{W}$ with spaces $V_i \oplus W_i$ for $i \in Q_0$ and maps $v_a \oplus w_a = \left(\begin{smallmatrix} v_a & 0 \\ 0 & w_a \end{smallmatrix}\right)$ for $a \in Q_1$. This definition explicitly shows what happens on vector spaces and linear maps, which is helpful if we look at a given direct sum of representations. We give another equivalent definition of the direct sum, which is especially useful in proofs.

Definition 2.22. *[4, Def. 2.1] Let $\mathbb{V}_1, \ldots, \mathbb{V}_r$ be representations. A **direct sum** $\mathbb{V} = \mathbb{V}_1 \oplus \ldots \oplus \mathbb{V}_r$ is a representation $\mathbb{V}$ together with morphisms $\iota_i\colon \mathbb{V}_i \to \mathbb{V}$ and $\pi_i\colon \mathbb{V} \to \mathbb{V}_i$ for $1 \le i \le r$ such that $\sum_{i=1}^r \iota_i \pi_i = \mathrm{id}_{\mathbb{V}}$ and $\pi_i \iota_i = \mathrm{id}_{\mathbb{V}_i}$. We write $\mathbb{V}^r = \mathbb{V} \oplus \ldots \oplus \mathbb{V}$ for the direct sum of r copies of $\mathbb{V}$.*

Remark 2.23. *In the first definition of a direct sum, we can set the inclusion morphisms $\iota_{\mathbb{V},\mathbb{W}}$ to be pointwise inclusion of subspaces and the projection morphisms $\pi_{\mathbb{V},\mathbb{W}}$ are given pointwise by the projection onto a subspace. One can easily check that those morphisms satisfy the above conditions. Thus our second definition of a direct sum is equivalent to the first definition.*

Definition 2.24. *A non-trivial representation $\mathbb{V}$ is called **decomposable** if it is isomorphic to the direct sum of two non-trivial representations which are called **summands**. Else it is called **indecomposable**. The trivial representation is neither decomposable nor indecomposable.*

Example 2.25. *Again, we consider the loop quiver. Moreover, we assume that the field k is algebraically closed and we consider finite-dimensional representations in $\mathrm{rep}_k(Q)$. If v_a, w_a are matrices in Jordan normal form then $v_a \oplus w_a = \left(\begin{smallmatrix} v_a & 0 \\ 0 & w_a \end{smallmatrix}\right)$ is also a matrix in Jordan normal form and thus the direct sum is a representation of the loop quiver (see Example 2.21). In addition, if v_a has two or more Jordan blocks, then we can set v_a' to be the subrepresentation consisting only of the first (upper left) Jordan block and v_a'' to be all the other Jordan blocks (and in both cases restricting to the proper subspaces). Then v_a', v_a'' are actually summands and thus $\mathbb{V} = (V_1, v_a)$ is decomposable. This shows that the indecomposable representations of the loop quiver are given by the representations that only have one Jordan block [3]. We later give a more robust argument for this (see Example 2.39).*

KI: Every morphism of quiver representations has a kernel, an image, and a cokernel.

Definition 2.26. *Let $\phi\colon \mathbb{V} \to \mathbb{W}$ be a morphism of quiver representations. We define the **kernel** of ϕ to be $(\ker \phi)_i = \ker \phi_i$ for all $i \in Q_0$. Moreover, the **image** of ϕ is given by $(\mathrm{im}\, \phi)_i = \mathrm{im}\, \phi_i$ and the **cokernel** of ϕ is defined as $(\mathrm{coker}\, \phi)_i = \mathrm{coker}\, \phi_i$. The maps between the vector spaces are given by the induced subspace maps resp. quotient maps.*

Lemma 2.27. *For every morphism $\phi\colon \mathbb{V} \to \mathbb{W}$, $\ker \phi$ is a subrepresentation of $\mathbb{V}$, $\mathrm{im}\, \phi$ is a subrepresentation of $\mathbb{W}$ and $\mathrm{coker}\, \phi$ is a representation.*

Proof. Notice that for each point $i \in Q_0$, $\ker \phi_i$ is a subspace of V_i and $\mathrm{im}\, \phi_i$ is a subspace of W_i. It remains to show that the maps restrict well. For each arrow $a \in Q_1$ we have the commutative diagram

$$\begin{tikzcd} V_i \arrow[r, "v_a"] \arrow[d, "\phi_i"'] & V_j \arrow[d, "\phi_j"] \\ W_i \arrow[r, "w_a"] & W_j \end{tikzcd}$$

and we get that $v_a(\ker\phi_i) \subseteq \ker\phi_j$, which shows that $\ker\phi$ is a subrepresentation. Similarly, we get that $w_a(\operatorname{im}\phi_i) = \phi_j(\operatorname{im} v_a) \subseteq \operatorname{im}\phi_j$ and thus $\operatorname{im}\phi$ is a subrepresentation. Since $w_a(\operatorname{im}\phi_i) \subseteq \operatorname{im}\phi_j$, we have a well-defined map $\tilde{w_a}\colon W_i/\operatorname{im}\phi_i = \operatorname{coker}\phi_i \to \operatorname{coker}\phi_j = W_j/\operatorname{im}\phi_j$, which is given by $\tilde{w_a}(w + \operatorname{im}\phi_i) = w_a(w) + \operatorname{im}\phi_j$. This shows that $\operatorname{coker}\phi$ is a representation. $\square$

Remark 2.28. *A morphism ϕ is a monomorphism if and only if $\ker\phi = 0$, an epimorphism if and only if $\operatorname{coker}\phi = 0$, and an isomorphism if and only if ϕ is both a monomorphism and an epimorphism.*

Since both morphisms and the direct sum of quiver representations are defined pointwise, there are many properties of vector spaces that carry over to quiver representations. E.g. if a morphism $\phi\colon \mathbb{V} \to \mathbb{W}$ is a monomorphism, then for each $i \in Q_0$ we have $\dim V_i \leq \dim W_i$. One important property that does not carry over is **semisimplicity**: while each subspace $W \subseteq V$ is a summand (i.e. there exists a subspace $W^\perp$ s.t. $W \oplus W^\perp = V$), not all subrepresentations of a given representation $\mathbb{V}$ are summands of $\mathbb{V}$. The lack of semisimplicity makes the classification of quiver representations a lot harder since it does not suffice to find all representations with no proper subrepresentations (which, for example, is enough to classify the complex finite-dimensional group representations of a finite group).

Example 2.29. *We consider representations of the L_2-quiver from Example 2.12. Let $\mathbb{V} = k \xrightarrow{\mathbb{1}} k$ and $\mathbb{W} = 0 \xrightarrow{0} k$ be two such representations. Then $\mathbb{W}$ is a subrepresentation of $\mathbb{V}$, but it is not a summand. Indeed, if $\mathbb{U}$ is a subrepresentation of $\mathbb{V}$ such that $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$, then $\mathbb{U} = k \to 0$, which can only be the zero map. But then $\mathbb{U} \oplus \mathbb{W} = k \xrightarrow{0} k$ which is not isomorphic to $\mathbb{V}$.*

2.2.2 Properties of $\operatorname{Hom}(\mathbb{V}, \mathbb{W})$

We now decompose $\operatorname{Hom}(\mathbb{V}, \mathbb{W})$ and look at morphisms between indecomposable representations.

Lemma 2.30. *[4, Lemma 2.1.1] Let $\mathbb{V} = \mathbb{V}_1 \oplus ... \oplus \mathbb{V}_r$ and $\mathbb{W} = \mathbb{W}_1 \oplus ... \oplus \mathbb{W}_s$ be two direct sums of representations (of the same quiver). The decomposition of the representations induces vector space decompositions*

$$\bigoplus_{i=1}^{r} \operatorname{Hom}(\mathbb{V}_i, \mathbb{W}) \overset{(1)}{\cong} \operatorname{Hom}(\mathbb{V}, \mathbb{W}) \overset{(2)}{\cong} \bigoplus_{j=1}^{s} \operatorname{Hom}(\mathbb{V}, \mathbb{W}_j).$$

We associate the map $\phi \in \operatorname{Hom}(\mathbb{V}, \mathbb{W})$ with the collection of maps $(\phi_i)_{1\leq i\leq r}$ where $\phi_i \in \operatorname{Hom}(\mathbb{V}_i, \mathbb{W})$ and also with $(\phi_j)_{1\leq j\leq s}$ where $\phi_j \in \operatorname{Hom}(\mathbb{V}, \mathbb{W}_j)$ through the isomorphisms in (1) and (2).

Proof.

1. Let $\iota_i\colon \mathbb{V}_i \to \mathbb{V}$ and $\pi_i\colon \mathbb{V} \to \mathbb{V}_i$ be the maps from the decomposition of $\mathbb{V}$. For $1 \leq i \leq r$ we define the map $\tilde{\iota_i} = \pi_i^*\colon \operatorname{Hom}(\mathbb{V}_i, \mathbb{W}) \to \operatorname{Hom}(\mathbb{V}, \mathbb{W}), \phi \mapsto \phi\pi_i$ and the map $\tilde{\pi_i} = \iota_i^*\colon \operatorname{Hom}(\mathbb{V}, \mathbb{W}) \to \operatorname{Hom}(\mathbb{V}_i, \mathbb{W}), \phi \mapsto \phi\iota_i$. For $\phi \in \operatorname{Hom}(\mathbb{V}, \mathbb{W})$ we have

$$(\sum_{i=1}^{r} \tilde{\iota_i}\tilde{\pi_i})\phi = \sum_{i=1}^{r} \pi_i^*\iota_i^*\phi = \sum_{i=1}^{r} \phi\iota_i\pi_i = \phi(\sum_{i=1}^{r} \iota_i\pi_i) = \phi \operatorname{id}_{\mathbb{V}} = \phi.$$

   Thus we have $\sum_{i=1}^{r} \tilde{\iota_i}\tilde{\pi_i} = \operatorname{id}_{\operatorname{Hom}(\mathbb{V},\mathbb{W})}$. In addition, for $\phi \in \operatorname{Hom}(\mathbb{V}_i, \mathbb{W})$ we have

$$\tilde{\pi_i}\tilde{\iota_i}\phi = \iota_i^*\pi_i^*\phi = \phi\pi_i\iota_i = \phi \operatorname{id}_{\mathbb{V}_i} = \phi.$$

   So $\tilde{\pi_i}\tilde{\iota_i} = \operatorname{id}_{\operatorname{Hom}(\mathbb{V}_i,\mathbb{W})}$. This shows the first decomposition.

2. Let $\iota_j\colon \mathbb{W}_j \to \mathbb{W}$ and $\pi_j\colon \mathbb{W} \to \mathbb{W}_j$ be the maps from the decomposition of $\mathbb{W}$. For $1 \leq j \leq s$ we define the map $\tilde{\iota_j} = \iota_j^*\colon \operatorname{Hom}(\mathbb{V}, \mathbb{W}_j) \to \operatorname{Hom}(\mathbb{V}, \mathbb{W}), \phi \mapsto \iota_j\phi$ and we define the map $\tilde{\pi_j} = \pi_j^*\colon \operatorname{Hom}(\mathbb{V}, \mathbb{W}) \to \operatorname{Hom}(\mathbb{V}, \mathbb{W}_j), \phi \mapsto \pi_j\phi$. For $\phi \in \operatorname{Hom}(\mathbb{V}, \mathbb{W})$ we have

$$(\sum_{j=1}^{s} \tilde{\iota_j}\tilde{\pi_j})\phi = \sum_{j=1}^{s} \iota_j^*\pi_j^*\phi = \sum_{j=1}^{s} \iota_j\pi_j\phi = (\sum_{j=1}^{s} \iota_j\pi_j)\phi = \operatorname{id}_{\mathbb{W}} \phi = \phi.$$

Therefore, $\sum_{j=1}^{s} \tilde{\iota_j}\tilde{\pi_j} = \mathrm{id}_{\mathrm{Hom}(\mathbb{V},\mathbb{W})}$. In addition, for $\phi \in \mathrm{Hom}(\mathbb{V}, \mathbb{W}_j)$ we have

$$\tilde{\pi_j}\tilde{\iota_j}\phi = \pi_{j,*}\iota_{j,*}\phi = \pi_j\iota_j\phi = \mathrm{id}_{\mathbb{W}_j}\phi = \phi.$$

So $\tilde{\pi_j}\tilde{\iota_j} = \mathrm{id}_{\mathrm{Hom}(\mathbb{V},\mathbb{W}_j)}$. This shows the second decomposition. □

Lemma 2.31. *[5, Lemma p. 112] Let $\mathbb{U}, \mathbb{V}, \mathbb{W}$ be indecomposable representations (of the same quiver) and let $\phi\colon \mathbb{U} \to \mathbb{V}$ and $\psi\colon \mathbb{V} \to \mathbb{W}$ be morphisms such that $\psi\phi$ is an isomorphism. Then both ϕ and ψ are isomorphisms. In particular, the composition of a finite number of morphisms between indecomposable representations is invertible if and only if each morphism is invertible.*

Proof. Let $\mathbb{U}, \mathbb{V}, \mathbb{W}$ and $\phi\colon \mathbb{U} \to \mathbb{V}, \psi\colon \mathbb{V} \to \mathbb{W}$ be as in Lemma 2.31 (i.e. $\psi\phi$ is an isomorphism). We show that both ϕ and ψ are isomorphisms. We set $\sigma = (\psi\phi)^{-1}\psi\colon \mathbb{V} \to \mathbb{U}$ and observe that $\sigma\phi = \mathrm{id}_{\mathbb{U}}$ and thus ϕ is injective. We notice that $\mathbb{U} \cong \mathrm{im}\,\phi$ and we claim that $\mathbb{V} = \mathrm{im}\,\phi \oplus \ker\sigma$. Indeed, for any $v \in \mathbb{V}$ (meaning $v \in V_i$ for some $i \in Q_0$), we have that $\sigma(v - \phi\sigma(v)) = \sigma(v) - \mathrm{id}_{\mathbb{U}}\,\sigma(v) = 0$ and thus $v - \phi\sigma(v) \in \ker\sigma$. Also if $\phi(u) \in \ker\sigma$, then $u = \sigma\phi(u) = 0$ and thus $\mathrm{im}\,\phi \cap \ker\sigma = 0$. Thus we get the decomposition

$$v = \phi\sigma(v) + (v - \phi\sigma(v)) \in \mathrm{im}\,\phi + \ker\sigma.$$

This decomposition gives rise to the morphisms $\pi_\phi = \phi\sigma\colon \mathbb{V} \to \mathrm{im}\,\phi$ and $\pi_\sigma = \mathrm{id}_{\mathbb{V}} - \phi\sigma\colon \mathbb{V} \to \ker\sigma$. We denote by $\iota_\phi\colon \mathrm{im}\,\phi \to \mathbb{V}$ and $\iota_\sigma\colon \ker\sigma \to \mathbb{V}$ the (pointwise) inclusion morphisms. Now we have $\pi_\phi\iota_\phi = \mathrm{id}_{\mathrm{im}\,\phi}$ and $\pi_\sigma\iota_\sigma = \mathrm{id}_{\ker\sigma}$. Further, this decomposition yields that $\iota_\phi\pi_\phi = \phi\sigma$ and $\iota_\sigma\pi_\sigma = \mathrm{id}_{\mathbb{V}} - \phi\sigma$ and thus we get $\iota_\phi\pi_\phi + \iota_\sigma\pi_\sigma = \mathrm{id}_{\mathbb{V}}$. This shows that $\mathbb{V} = \mathrm{im}\,\phi \oplus \ker\sigma$. Since $\mathbb{V}$ is indecomposable and $\mathrm{im}\,\phi \cong \mathbb{U} \neq 0$ we get that $\ker\sigma = 0$ and therefore $\mathrm{im}\,\phi = \mathbb{V}$ and thus both ϕ and ψ are isomorphisms. The second claim follows by induction on the number of morphisms. □

Remark 2.32. *From Lemma 2.30 and Lemma 2.31 we have learned a lot about* $\mathrm{Hom}(\mathbb{V}, \mathbb{W})$. *The first lemma tells us that it suffices to consider* $\mathrm{Hom}(\mathbb{V}, \mathbb{W})$ *for indecomposable representations. Indeed, each morphism $\phi \in \mathrm{Hom}(\mathbb{V}, \mathbb{W})$ can be uniquely written as a block matrix $(\phi_{i,j})$, where $\phi_{i,j}\colon \mathbb{V}_j \to \mathbb{W}_i$ are morphisms between indecomposable representations. Such morphisms (between indecomposable representations) are characterized by the second lemma.*

2.3 The Krull-Remak-Schmidt Theorem

Towards our goal of classifying quiver representations, we can ask if a quiver representation can always be decomposed into a direct sum of finitely many indecomposable representations. We call such a decomposition a **Remak decomposition** and it turns out that such a decomposition always exists and it is unique up to isomorphism and permutation of the factors in the direct sum. This section is primarily based on the lecture notes 'Representations of quivers via reflection functors' written by Henning Krause [4].

Theorem 2.33 (**Krull, Remak, Schmidt**)**.** *Let Q be a finite quiver. Then for any $\mathbb{V} \in \mathrm{rep}_{\mathrm{k}}(Q)$ there are indecomposable representations $\mathbb{V}_1, \ldots, \mathbb{V}_r$ such that $\mathbb{V} \cong \mathbb{V}_1 \oplus \ldots \oplus \mathbb{V}_r$. Moreover, for indecomposable representations $\mathbb{W}_1, \ldots, \mathbb{W}_s$ such that $\mathbb{V} \cong \mathbb{W}_1 \oplus \ldots \oplus \mathbb{W}_s$, $r = s$ and there is a permutation σ such that $\mathbb{V}_i \cong \mathbb{W}_{\sigma(i)}$ for $1 \leq i \leq r$.*

Before we can prove Theorem 2.33, we observe a connection between the decomposability of a representation $\mathbb{V}$ and the structure and property of the ring of its endomorphisms $\mathrm{End}(\mathbb{V})$.

Remark 2.34. *Note that the composition of morphisms gives a ring structure on* $\mathrm{End}(\mathbb{V})$*, which is the reason why we generally refer to* $\mathrm{End}(\mathbb{V})$ *as a ring.* $\mathrm{Hom}(\mathbb{V}, \mathbb{W})$ *on the other hand does not have a ring structure (with respect to the composition of morphisms).*

Lemma 2.35 (**Fitting's Lemma**)**.** *[5, Lemma p. 113] Let $\mathbb{V}$ be a finite-dimensional representation and let ϕ be an endomorphism in* $\mathrm{End}(\mathbb{V})$.

1. *For large enough r, we have $\mathbb{V} = \mathrm{im}\,\phi^r \oplus \ker\phi^r$.*
2. *If $\mathbb{V}$ is indecomposable, then ϕ is either an automorphism or nilpotent.*

Proof. 1. Because $\mathbb{V}$ is finite-dimensional, we can choose r large enough, s.t. $\operatorname{im}\phi^r = \operatorname{im}\phi^{r+1}$. This is possible since all the maps ϕ_i are endomorphisms on finite-dimensional vector spaces and we can deduce the result pointwise: we have $\operatorname{im}\phi_i^{r_i+1} \subseteq \operatorname{im}\phi_i^{r_i} \subseteq \ldots \subseteq \operatorname{im}\phi_i^2 \subseteq \operatorname{im}\phi_i$. If all the inclusions were strict, then $\dim(\operatorname{im}\phi_i) > \dim(\operatorname{im}\phi_i^2) > \ldots$ is an infinite, strictly decreasing sequence of non-negative numbers, which is impossible since $\dim(\operatorname{im}\phi_i) < \infty$. This also shows that w.l.o.g. $r_i \leq \dim(\phi_i)$, where $\dim(\phi_i)$ is the dimension of the vector space on which ϕ_i operates. Thus we get the desired result for $r \geq \dim\mathbb{V}$. For the same r it also holds that $\ker\phi^{r+1} = \ker\phi^r$. Thus $\phi^r\colon \operatorname{im}\phi^r \to \operatorname{im}\phi^{2r}$ is an isomorphism and we denote by ψ its inverse. Let $\iota_1\colon \operatorname{im}\phi^r \to \mathbb{V}$ and $\iota_2\colon \ker\phi^r \to \mathbb{V}$ denote the inclusions. We put

$$\pi_1 = \psi\phi^r\colon \mathbb{V} \to \operatorname{im}\phi^r \text{and}\, \pi_2 = \mathrm{id}_{\mathbb{V}} - \psi\phi^r\colon \mathbb{V} \to \ker\phi^r.$$

This is well-defined since $\phi^r\pi_2 = \phi^r - \phi^r\psi\phi^r = \phi^r - \mathrm{id}_{\operatorname{im}\phi^r}\phi^r = 0$. Then $\iota_1\pi_1 + \iota_2\pi_2 = \mathrm{id}_{\mathbb{V}}$ and $\pi_1\iota_1 = \mathrm{id}_{\operatorname{im}\phi^r}, \pi_2\iota_2 = \mathrm{id}_{\ker\phi^r}$. Thus $\operatorname{im}\phi^r$ and $\ker\phi^r$ are summands and by the definition of the direct sum we have: $\mathbb{V} = \operatorname{im}\phi^r \oplus \ker\phi^r$.

2. If $\mathbb{V}$ is indecomposable, then one of the factors in the decomposition in part (1) needs to be 0. If $\ker\phi^r = 0$, then ϕ is an automorphism and if $\operatorname{im}\phi^r = 0$, then ϕ is nilpotent. □

Remark 2.36. *Notice that $\mathbb{V}$ needs to be finite-dimensional. Otherwise the decomposition in part (1) is not guaranteed, since there may not be any such r.*

Definition 2.37. *A ring is called **local** if the sum of two non-units is again a non-unit.*

Proposition 2.38. *[5, Proposition 3.1] A finite-dimensional representation $\mathbb{V}$ is indecomposable if and only if* $\operatorname{End}(\mathbb{V})$ *is local.*

Proof. $\underline{\Longrightarrow:}$ Let $\mathbb{V}$ be indecomposable and let $\phi, \phi' \in \operatorname{End}(\mathbb{V})$ such that $\phi + \phi'$ is invertible with inverse ρ. If ϕ is non-invertible, so is $\rho\phi$ and thus by Lemma 2.35, $\rho\phi$ is nilpotent, say $(\rho\phi)^r = 0$. The summation formula for geometric series yields:

$$(\mathrm{id}_{\mathbb{V}} - \rho\phi)(\mathrm{id}_{\mathbb{V}} + \rho\phi + \ldots + (\rho\phi)^{r-1}) = \mathrm{id}_{\mathbb{V}} - (\rho\phi)^r = \mathrm{id}_{\mathbb{V}}.$$

Therefore $\rho\phi' = \mathrm{id}_{\mathbb{V}} - \rho\phi$ is invertible and thus ϕ' is invertible, which shows that $\operatorname{End}(\mathbb{V})$ is local. $\underline{\Longleftarrow:}$ If $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$ is decomposable, then the endomorphisms $\iota_{\mathbb{U}}\pi_{\mathbb{U}}, \iota_{\mathbb{W}}\pi_{\mathbb{W}}$ have image $\mathbb{U}$ respectively $\mathbb{W}$ and thus are not invertible. But $\iota_{\mathbb{U}}\pi_{\mathbb{U}} + \iota_{\mathbb{W}}\pi_{\mathbb{W}} = \mathrm{id}_{\mathbb{V}}$ is invertible and therefore $\operatorname{End}(\mathbb{V})$ is not local. □

Example 2.39. *In Example 2.25 we have stated that the finite-dimensional indecomposable representations of the loop quiver are given by the endomorphisms v_a, which only have one Jordan block. Using Proposition 2.38 we can now prove that every indecomposable representation consists of a single Jordan block (proof by contrapositive). Indeed if v_a has two or more Jordan blocks, then we can define $\phi_1 = \left(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ to be the identity on the subspace belonging to the first Jordan block. Analogously, let $\phi_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right)$ be the identity on the subspace belonging to the other Jordan blocks. Then ϕ_1, ϕ_2 are both non-invertible, but $\phi_1 + \phi_2 = \mathrm{id}_{\mathbb{V}}$ is invertible. Thus* $\operatorname{End}(\mathbb{V})$ *is not local and therefore $\mathbb{V}$ is decomposable.*

We use Proposition 2.38 to consider a subspace of the vector space $\operatorname{Hom}(\mathbb{V}, \mathbb{W})$. To do this, we use the Lemmas 2.30 and 2.31 from Section 2.2.

Definition 2.40. *The **radical** of two representations $\mathbb{V}, \mathbb{W}$ is defined to be*

$$\operatorname{Rad}(\mathbb{V}, \mathbb{W}) = \left\{ \phi \in \operatorname{Hom}(\mathbb{V}, \mathbb{W}) \;\middle|\; \begin{array}{c} \text{for every } \mathbb{U} \text{ indecomposable, } \mathbb{U} \xrightarrow{\sigma} \mathbb{V} \text{ and } \mathbb{W} \xrightarrow{\tau} \mathbb{U}, \\ \tau\phi\sigma \text{ is non-invertible} \end{array} \right\}.$$

*A morphism $\phi \in \operatorname{Rad}(\mathbb{V}, \mathbb{W})$ is called **radical**.*

Lemma 2.41. *Let $\mathbb{V}, \mathbb{W}$ be two representations.*

1. $\operatorname{Rad}(\mathbb{V}, \mathbb{W})$ *is a subspace of* $\operatorname{Hom}(\mathbb{V}, \mathbb{W})$.

2. $\mathrm{Rad}(\mathbb{V}, \mathbb{W}_1 \oplus \mathbb{W}_2) \cong \mathrm{Rad}(\mathbb{V}, \mathbb{W}_1) \oplus \mathrm{Rad}(\mathbb{V}, \mathbb{W}_2)$.

3. $\mathrm{Rad}(\mathbb{V}_1 \oplus \mathbb{V}_2, \mathbb{W}) \cong \mathrm{Rad}(\mathbb{V}_1, \mathbb{W}) \oplus \mathrm{Rad}(\mathbb{V}_2, \mathbb{W})$.

4. If $\mathbb{V}, \mathbb{W}$ *are indecomposable, then* $\mathrm{Hom}(\mathbb{V}, \mathbb{W}) \setminus \mathrm{Rad}(\mathbb{V}, \mathbb{W})$ *is the set of isomorphisms* $\mathbb{V} \to \mathbb{W}$.

Proof. 1. Let $\alpha, \beta \in k$ and let $\phi_1, \phi_2 \in \mathrm{Rad}(\mathbb{V}, \mathbb{W})$ and let $\mathbb{U} \xrightarrow{\sigma} \mathbb{V}$ and $\mathbb{W} \xrightarrow{\tau} \mathbb{U}$ for $\mathbb{U}$ indecomposable as in the definition of $\mathrm{Rad}(\mathbb{V}, \mathbb{W})$. Then $\alpha \cdot \tau\phi_1\sigma, \beta \cdot \tau\phi_2\sigma$ are non-invertible and thus $\tau(\alpha \cdot \phi_1 + \beta \cdot \phi_2)\sigma = \alpha \cdot \tau\phi_1\sigma + \beta \cdot \tau\phi_2\sigma$ is non-invertible since $\mathrm{End}(\mathbb{U})$ is a local ring. Thus $\alpha \cdot \phi_1 + \beta \cdot \phi_2 \in \mathrm{Rad}(\mathbb{V}, \mathbb{W})$.

2. Let $\mathbb{W} = \mathbb{W}_1 \oplus \mathbb{W}_2$ and let $\phi = (\phi_1, \phi_2) \in \mathrm{Hom}(\mathbb{V}, \mathbb{W}_1) \oplus \mathrm{Hom}(\mathbb{V}, \mathbb{W}_2) \cong \mathrm{Hom}(\mathbb{V}, \mathbb{W})$. Let $\mathbb{U}$ be indecomposable and let $\sigma \in \mathrm{Hom}(\mathbb{U}, \mathbb{V})$ and $\tau = (\tau_1, \tau_2) \in \mathrm{Hom}(\mathbb{W}_1, \mathbb{U}) \oplus \mathrm{Hom}(\mathbb{W}_2, \mathbb{U})$. Then $\tau\phi\sigma = (\tau_1, \tau_2)(\phi_1, \phi_2)\sigma = \tau_1\phi_1\sigma + \tau_2\phi_2\sigma$. This results directly from the decomposition of $\mathrm{Hom}(\mathbb{V}, \mathbb{W}_1 \oplus \mathbb{W}_2)$ given in Lemma 2.30. If $\phi_i \in \mathrm{Rad}(\mathbb{V}, \mathbb{W}_i)$ for $i \in \{1, 2\}$, then $\tau_1\phi_1\sigma$ and $\tau_2\phi_2\sigma$ are non-invertible and thus $\tau\phi\sigma = \tau_1\phi_1\sigma + \tau_2\phi_2\sigma$ is non-invertible since $\mathrm{End}(\mathbb{U})$ is local (since $\mathbb{U}$ is indecomposable). Thus $\phi \in \mathrm{Rad}(\mathbb{V}, \mathbb{W})$. On the other hand, if we fix $i \in \{1, 2\}$ and if $\phi \in \mathrm{Rad}(\mathbb{V}, \mathbb{W})$, then we can choose $\tau = (\tau_1, \tau_2)$ such that $\tau_j = 0$ for $j \neq i$ (and τ_i arbitrary) and we get that $\tau_i\phi_i\sigma = \tau\phi\sigma$ is non-invertible. Therefore $\phi_i \in \mathrm{Rad}(\mathbb{V}, \mathbb{W}_i)$.

3. Let $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$ and let $\phi = (\phi_1, \phi_2) \in \mathrm{Hom}(\mathbb{V}_1, \mathbb{W}) \oplus \mathrm{Hom}(\mathbb{V}_2, \mathbb{W}) \cong \mathrm{Hom}(\mathbb{V}, \mathbb{W})$. Let $\mathbb{U}$ be indecomposable and let $\sigma = (\sigma_1, \sigma_2) \in \mathrm{Hom}(\mathbb{U}, \mathbb{V}_1) \oplus \mathrm{Hom}(\mathbb{U}, \mathbb{V}_2) \cong \mathrm{Hom}(\mathbb{U}, \mathbb{V})$ and $\tau \in \mathrm{Hom}(\mathbb{W}, \mathbb{U})$. Then $\tau\phi\sigma = \tau(\phi_1, \phi_2)(\sigma_1, \sigma_2) = \tau_1\phi_1\sigma + \tau_2\phi_2\sigma$. This results directly from the decomposition of $\mathrm{Hom}(\mathbb{V}_1 \oplus \mathbb{V}_2, \mathbb{W})$ given in Lemma 2.30. If $\phi_i \in \mathrm{Rad}(\mathbb{V}_i, \mathbb{W})$ for $i \in \{1, 2\}$, then $\tau\phi_1\sigma_1$ and $\tau\phi_2\sigma_2$ are non-invertible and thus $\tau\phi\sigma = \tau\phi_1\sigma_1 + \tau\phi_2\sigma_2$ is non-invertible since $\mathrm{End}(\mathbb{U})$ is local (since $\mathbb{U}$ is indecomposable). Thus $\phi \in \mathrm{Rad}(\mathbb{V}, \mathbb{W})$. On the other hand, if we fix $i \in \{1, 2\}$ and if $\phi \in \mathrm{Rad}(\mathbb{V}, \mathbb{W})$, then we can choose $\sigma = (\sigma_1, \sigma_2)$ such that $\sigma_j = 0$ for $j \neq i$ (and σ_i arbitrary) and we get that $\tau\phi_i\sigma_i = \tau\phi\sigma$ is non-invertible. Therefore $\phi_i \in \mathrm{Rad}(\mathbb{V}_i, \mathbb{W})$.

4. Let $\phi \in \mathrm{Hom}(\mathbb{V}, \mathbb{W})$ be an isomorphism. Then we can choose $\mathbb{U} = \mathbb{V}, \sigma = \mathrm{id}_{\mathbb{V}}, \tau = \phi^{-1}$ and see that $\tau\phi\sigma = \mathrm{id}_{\mathbb{V}}$ is invertible and thus $\phi \notin \mathrm{Rad}(\mathbb{V}, \mathbb{W})$. Now let $\phi \in \mathrm{Hom}(\mathbb{V}, \mathbb{W}) \setminus \mathrm{Rad}(\mathbb{V}, \mathbb{W})$. W.l.o.g we can assume that ϕ is non-invertible (else we are done). Now choose $\mathbb{U}$ indecomposable and $\sigma \in \mathrm{Hom}(\mathbb{U}, \mathbb{V})$ and $\tau \in \mathrm{Hom}(\mathbb{W}, \mathbb{U})$ such that $\tau\phi\sigma$ is invertible. Since $\mathbb{V}, \mathbb{W}$ are indecomposable, this is impossible. Indeed, since ϕ is non-invertible, it follows from Lemma 2.31 that $\tau\phi\sigma$ is non-invertible. □

Remark 2.42. *If* $\phi\colon \mathbb{V} \to \mathbb{W}$ *is radical and* $\phi = (\phi_{i,j})$ *is written as a block matrix of morphisms between indecomposable representations (see Lemma 2.30), then every such block* $\phi_{i,j}$ *is non-invertible. Indeed if* $\phi_{i,j}\colon \mathbb{V}_j \to \mathbb{W}_i$ *is invertible, then we can choose* $\mathbb{U} = \mathbb{W}_i$ *indecomposable,* $\tau\colon \mathbb{W} \to \mathbb{W}_i$ *the projection onto the summand* $\mathbb{W}_i$ *and* $\sigma = \iota_{\mathbb{V}_j}(\phi_{i,j})^{-1}\colon \mathbb{W}_i \to \mathbb{V}_j \to \mathbb{V}$ *where* $\iota_{\mathbb{V}_j}$ *is the inclusion of the summand* $\mathbb{V}_j$ *into* $\mathbb{V}$*. But then* $\tau\phi\sigma = \phi_{i,j}$ *is invertible, which is a contradiction to the choice of* ϕ.

Proof of Theorem 2.33. By induction on $\dim \mathbb{V}$ we can show that a Remak decomposition exists: for $\dim \mathbb{V} = 0$ all vector spaces are trivial and thus indecomposable. Therefore $\mathbb{V}$ is already indecomposable. For the induction step, let $\dim \mathbb{V} = n \geq 1$. If $\mathbb{V}$ is decomposable, then there exist proper summands $\mathbb{U}, \mathbb{W}$. Since both $\dim \mathbb{U}, \dim \mathbb{W} < n$ we know by the induction hypothesis that they each have a Remak decomposition. The direct sum of these decompositions gives a Remak decomposition for $\mathbb{V}$. For the proof of uniqueness, we order isomorphic summands together, i.e. let $\mathbb{V} = \mathbb{V}_1^{a_1} \oplus \ldots \oplus \mathbb{V}_r^{a_r}$ be a direct sum decomposition of $\mathbb{V}$ where the $\mathbb{V}_i$ are pairwise non-isomorphic representations and $a_i \geq 1$ for all $1 \leq i \leq r$. For $\mathbb{W}$ indecomposable, we can consider the number

$$\frac{\dim \mathrm{Hom}(\mathbb{V}, \mathbb{W}) - \dim \mathrm{Rad}(\mathbb{V}, \mathbb{W})}{\dim \mathrm{Hom}(\mathbb{W}, \mathbb{W}) - \dim \mathrm{Rad}(\mathbb{W}, \mathbb{W})}.$$

From Lemma 2.30 and Lemma 2.41 we know that for $\mathbb{W} = \mathbb{V}_i$ this number is equal to:

$$\begin{aligned}\frac{\sum_{j=1}^r (\dim \mathrm{Hom}(\mathbb{V}_j^{a_j}, \mathbb{V}_i) - \dim \mathrm{Rad}(\mathbb{V}_j^{a_j}, \mathbb{V}_i))}{\dim \mathrm{Hom}(\mathbb{V}_i, \mathbb{V}_i) - \dim \mathrm{Rad}(\mathbb{V}_i, \mathbb{V}_i)} &= \frac{\sum_{j=1}^r a_j \cdot \dim(\mathrm{Hom}(\mathbb{V}_j, \mathbb{V}_i) \setminus \mathrm{Rad}(\mathbb{V}_j, \mathbb{V}_i))}{\dim(\mathrm{Hom}(\mathbb{V}_i, \mathbb{V}_i) \setminus \mathrm{Rad}(\mathbb{V}_i, \mathbb{V}_i))} \\ &= \frac{a_i \cdot \dim(\mathrm{Hom}(\mathbb{V}_i, \mathbb{V}_i) \setminus \mathrm{Rad}(\mathbb{V}_i, \mathbb{V}_i))}{\dim(\mathrm{Hom}(\mathbb{V}_i, \mathbb{V}_i) \setminus \mathrm{Rad}(\mathbb{V}_i, \mathbb{V}_i))} = a_i.\end{aligned}$$

Here we used part (4) from Lemma 2.41 to see that $\dim \mathrm{Hom}(\mathbb{V}_j, \mathbb{V}_i) - \dim \mathrm{Rad}(\mathbb{V}_j, \mathbb{V}_i) = 0$ for $j \neq i$ since $\mathbb{V}_j$ and $\mathbb{V}_i$ are non-isomorphic. We notice that this number is independent of the decomposition of $\mathbb{V}$ and thus the decomposition is unique up to isomorphisms and reordering. $\square$

Remark 2.43. *In this section we restricted our focus to finite quivers and finite-dimensional representations. Moreover, we assumed that the quivers are connected. Those restrictions are sensible. Indeed, if a quiver Q is the disjoint union of two quivers Q' and Q'', then any representation of Q is the same as a pair of representations, one of Q' and one of Q'', and any morphism acts on each component of the representation separately. Therefore,* $\mathrm{rep}_{\mathrm{k}}(Q)$ *is isomorphic to the product category* $\mathrm{rep}_{\mathrm{k}}(Q') \times \mathrm{rep}_{\mathrm{k}}(Q'')$. *From the viewpoint of topological data analysis, restricting to finite quivers and finite-dimensional representations is somewhat justified since one looks at persistence modules, which generally are finite-dimensional representations of finite quivers. This is done in Chapter 5.*

Chapter 3

Gabriel's Theorem

In this chapter, we consider quivers that have a finite number of indecomposable representations.

Definition 3.1. *Let Q be a quiver and let k be a field. The quiver Q is of **finite-type** if it has finitely many isomorphism classes of indecomposable finite-dimensional representations.*

This leads to the question of whether there are quivers of finite-type and if so, which quivers are of finite-type. The answer to this question is given by Gabriel's Theorem. Proving Gabriel's Theorem is our main goal in this chapter. Sections 3.1 to 3.4 introduce important results, which help understand quiver representations. These sections are written with the goal of proving Gabriel's Theorem. Therefore, we generally only introduce what is needed to prove Gabriel's Theorem.

Theorem 3.2 (Gabriel, version 1). *Let Q be a finite connected quiver and let k be a field. Then Q is of finite type if and only if Q is a Dynkin quiver.*

The Krull-Remak-Schmidt Theorem guarantees that every finite-dimensional quiver representation admits a unique decomposition into indecomposable summands. Therefore, for a quiver of finite-type, it suffices to characterize the finitely many indecomposable representations. Using the indecomposable representations, one can then classify all quiver representations. The general outline of this chapter follows the book 'Persistence Theory: From Quiver Representations to Data Analysis' written by Steve Y. Oudot [1]. Some of the more technical details and proofs are based on the book 'Finite dimensional algebras and quantum groups' written by Bangming Deng [2].

Remark 3.3. *Note that Gabriel's Theorem is true for any field k. In addition, the classification of finite-type quivers happens on the level of the underlying graph, thus the orientations of the arrows do not matter.*

3.1 Dynkin and Euclidean Diagrams

In this section, we introduce the Tits form. Using the Tits form, we then divide all quivers into three distinct classes.

3.1.1 Tits Form

For a finite quiver, we look at a special quadratic form on $\mathbb{Z}^n$, called the Tits form. We later apply the Tits form to the dimension vectors of finite-dimensional representations.

Definition 3.4. *A vector in $\mathbb{Z}^n$ is called **positive** if it belongs to $\mathbb{N}_0^n \setminus \{0\}$. This means that every coordinate is non-negative and the vector is non-trivial (not all coordinates are 0). We write $x > 0$ if x is positive and we write $x > y$ if $x - y$ is positive. For example, the dimension vectors of non-trivial representations are positive. Dually, a vector $x \in \mathbb{Z}^n$ is called **negative** if $-x$ is positive. A vector $x \in \mathbb{Z}^n$ is called **sincere** if $x_i \neq 0$ for all $1 \leq i \leq n$. Finally,we denote by $e_i \in \mathbb{Z}^n$ the i-th coordinate vector, i.e. $(e_i)_j = \delta_{ij}$ (δ here denotes the Kronecker-delta).*

Definition 3.5. *The* ***Euler form*** *of a finite quiver Q is the bilinear form $\langle\cdot,\cdot\rangle_Q\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$, given by:*

$$\langle x, y\rangle_Q = \sum_{i\in Q_0} x_i y_i - \sum_{a\in Q_1} x_{t_a} y_{h_a}.$$

The symmetrization of the Euler form is called the ***symmetric Euler form*** *and is given by:*

$$(x, y)_Q = \langle x, y\rangle_Q + \langle y, x\rangle_Q.$$

Lemma 3.6. *If we view elements in $\mathbb{Z}^n$ as column vectors and if the finite number of edges (in $\bar{Q}$) joining the vertices i and j is denoted by $d_{ij} = d_{ji}$, then the symmetric Euler form can be expressed as:*

$$(x, y)_Q = x^T C_Q y,$$

where $C_Q = (c_{ij})_{i,j\in Q_0}$ is the symmetric matrix with the entries

$$c_{ij} = \begin{cases} 2 - 2|\{\text{loops at } i\}| = 2 - 2d_{ii} & \text{if } i = j, \\ -|\{\text{arrows between } i \text{ and } j\}| = -d_{ij} & \text{if } i \neq j. \end{cases}$$

Proof. We need to check that for all $1 \leq i, j \leq n$ we have: $(e_i, e_j)_Q = e_i^T C_Q e_j = c_{ij}$. For $i = j$ we have: $(e_i, e_i)_Q = 2(1\cdot 1 - d_{ii}) = 2 - dd_{ii}$. For $i \neq j$ we have:

$$\begin{aligned}(e_i, e_j)_Q &= \langle e_i, e_j\rangle_Q + \langle e_j, e_i\rangle_Q = (0 - |\{\text{arrows from } i \text{ to } j\}|) + (0 - |\{\text{arrows from } j \text{ to } i\}|) \\ &= -|\{\text{arrows between } i \text{ and } j\}| = -d_{ij}.\end{aligned}$$

□

Example 3.7. *We consider the L_2-quiver*

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet}.$$

The Euler form of L_2 is $\langle x, y\rangle_{L_2} = x_1y_1 + x_2y_2 - x_1y_2$. Its symmetric Euler form is given by $(x, y)_{L_2} = 2x_1y_1 + 2x_2y_2 - x_1y_2 - x_2y_1$.

Example 3.8. *The Euler form of the loop quiver is $\langle x, y\rangle_Q = x_1y_1 - x_1y_1 = 0$. Its symmetric Euler form is $(x, y)_Q = 0$.*

Definition 3.9. *For a finite quiver Q, the* ***Tits form*** *of Q is the quadratic form $q_Q\colon \mathbb{Z}^n \to \mathbb{Z}$ associated with the Euler form*

$$q_Q(x) = \langle x, x\rangle_Q = \frac{1}{2}(x, x)_Q = \frac{1}{2}x^T C_Q x = \frac{1}{2}\sum_{i,j} c_{ij}x_ix_j = \frac{1}{2}\sum_i (2 - 2d_{ii})x_i^2 - \sum_{i<j} d_{ij}x_ix_j.$$

The ***radical*** *of the quadratic form q_Q is the set $\operatorname{rad} q_Q = \{x \in \mathbb{Z}^n \mid (x,\cdot)_Q = 0\}$ and $x \in \operatorname{rad} q_Q$ is called a* ***radical vector***.

Definition 3.10. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a quadratic form.*

1. *q is called* ***positive definite*** *if $q(x) > 0$ for all non-zero $x \in \mathbb{Z}^n$.*
2. *q is called* ***positive semi-definite*** *if $q(x) \geq 0$ for all $x \in \mathbb{Z}^n$.*
3. *q is called* ***indefinite*** *if it takes on both positive and negative values, i.e. there exist $x, y \in \mathbb{Z}^n$ such that $q(x) > 0, q(y) < 0$.*

We use the same terminology for quadratic forms $\mathbb{Q}^n \to \mathbb{Q}$ respectively $\mathbb{R}^n \to \mathbb{R}$.

Example 3.11. *The Tits forms of the Dynkin quivers (respectively Dynkin graphs) are:*

$$A_n\colon q_Q(x) = \sum_{i\in Q_0} x_i^2 - \sum_{a\in Q_1} x_{t_a}x_{h_a} = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n-1} x_i x_{i+1} = \frac{1}{2}[x_1^2 + x_n^2 + \sum_{i=1}^{n-1}(x_i - x_{i+1})^2]$$

$$D_n\colon q_Q(x) = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n-2} x_i x_{i+1} - x_{n-2}x_n = \frac{1}{2}[x_1^2 + x_n^2 + (x_{n-2} - x_n)^2 + \sum_{i=1}^{n-2}(x_i - x_{i+1})^2]$$

$$E_6\colon q_Q(x) = \sum_{i=1}^{6} x_i^2 - x_1x_3 - x_2x_4 - x_3x_4 - x_4x_5 - x_5x_6$$

$$E_7\colon q_Q(x) = \sum_{i=1}^{7} x_i^2 - x_1x_3 - x_2x_4 - x_3x_4 - x_4x_5 - x_5x_6 - x_6x_7$$

$$E_8\colon q_Q(x) = \sum_{i=1}^{8} x_i^2 - x_1x_3 - x_2x_4 - x_3x_4 - x_4x_5 - x_5x_6 - x_6x_7 - x_7x_8$$

We observe that the Tits forms of the Dynkin graphs A_n and D_n are positive definite. We will later see that the Tits forms of the other Dynkin graphs are also positive definite, but this is not obvious from our calculations above.

Lemma 3.12. *Let Q be a finite quiver. Then neither its symmetric Euler form nor its Tits form depends on the orientations of the arrows in Q.*

Proof. Let Q be a finite quiver and let $a = (t_a, h_a) \in Q_1$ be an arrow. If we consider the quiver $\tilde{Q}$ resulting from Q by exchanging a with $\tilde{a} = (h_a, t_a)$ (reverse its direction), we get:

$$\begin{aligned}
(x,y)_{\tilde{Q}} &= \sum_{i\in\tilde{Q}_0} x_i y_i - \sum_{b\in\tilde{Q}_1} x_{t_b} y_{h_b} + \sum_{i\in\tilde{Q}_0} y_i x_i - \sum_{b\in\tilde{Q}_1} y_{t_b} x_{h_b} \\
&= \sum_{i\in Q_0} x_i y_i - \sum_{\substack{b\in Q_1\\ b\neq a}} x_{t_b} y_{h_b} - x_{h_a} y_{t_a} + \sum_{i\in Q_0} y_i x_i - \sum_{\substack{b\in Q_1\\ b\neq a}} y_{t_b} x_{h_b} - y_{h_a} x_{t_a} \\
&= \sum_{i\in Q_0} x_i y_i - \sum_{\substack{b\in Q_1\\ b\neq a}} x_{t_b} y_{h_b} - y_{h_a} x_{t_a} + \sum_{i\in Q_0} y_i x_i - \sum_{\substack{b\in Q_1\\ b\neq a}} y_{t_b} x_{h_b} - x_{h_a} y_{t_a} \\
&= \sum_{i\in Q_0} x_i y_i - \sum_{b\in Q_1} x_{t_b} y_{h_b} + \sum_{i\in Q_0} y_i x_i - \sum_{b\in Q_1} y_{t_b} x_{h_b} = (x,y)_Q.
\end{aligned}$$

The result for the Tits form is an immediate consequence: $q_Q(x) = (x,x)_Q = (x,x,)_{\tilde{Q}} = q_{\tilde{Q}}(x)$. □

3.1.2 Dynkin, Tame and Wild Quivers

We want to understand for which quivers Q, its Tits form q_Q is positive-definite respectively positive semi-definite. This divides all quivers into three distinct classes. The following lemma marks the first step towards this goal.

Lemma 3.13. *[2, Claim p. 53] Let Q be a finite connected quiver and let $y \in \operatorname{rad} q_Q$ be a positive radical vector. Then y is sincere and q_Q is positive semi-definite. Moreover for $x \in \mathbb{Z}^n$ we have*

$$q_Q(x) = 0 \Leftrightarrow x \in \mathbb{Q}y, \text{ i.e. } \exists a \in \mathbb{Q} \text{ s.t. } x = a\cdot y \Leftrightarrow x \in \operatorname{rad} q_Q.$$

Proof. We denote by e_i the i-th coordinate vector, i.e. $(e_i)_j = \delta_{ij}$. The assumption on y gives:

$$0 = (y, e_i)_Q = y^T C_Q e_i = \sum_{j=1}^{n} y_j c_{ji} = (2 - 2d_{ii})y_i - \sum_{\substack{j=1\\ j\neq i}}^{n} d_{ji} y_j \text{ for } 1 \le i \le n. \tag{3.1}$$

If $y_i = 0$, then $\sum_{j=1, j\neq 1}^{n} d_{ji} y_j = 0$ and since each term is non-negative (y is positive), we get that $y_j = 0$ for all j s.t. i and j are joined by an edge (in $\bar{Q}$). Using that $\bar{Q}$ is connected, we get that

$y = 0$. Indeed, let $i, j_1, \ldots, j_k, j$ be a path from i to j (meaning that there are edges in $\bar{Q}$ connecting the pairs $(i, j_1), (j_1, j_2), \ldots, (j_k, j)$). Since i and j_1 are connected by an edge, we have $y_{j_1} = 0$ by the observation above. Using the same observation again we get $y_{j_2} = 0$. We conclude the claimed result using the observation inductively on the set $i, j_1, \ldots, j_k, j$. This is a contradiction because y is positive and therefore sincere. We now show that q_Q is positive semi-definite. For $x \in \mathbb{Z}^n$ we have:

$$\begin{aligned} q_Q(x) = \frac{1}{2}(x,x)_Q &= \sum_i (2 - 2d_{ii}) y_i \frac{1}{2y_i} x_i^2 - \sum_{i<j} d_{ij} x_i x_j \\ &= \sum_i \sum_{j \neq i} d_{ij} y_j \frac{1}{2y_i} x_i^2 - \sum_{i<j} d_{ij} x_i x_j \\ &= \sum_{i<j} d_{ij} \frac{y_j}{2y_i} x_i^2 + \sum_{j<i} d_{ij} \frac{y_j}{2y_i} x_i^2 - \sum_{i<j} d_{ij} x_i x_j \\ &= \sum_{i<j} d_{ij} \frac{y_j}{2y_i} x_i^2 + \sum_{i<j} d_{ij} \frac{y_i}{2y_j} x_j^2 - \sum_{i<j} d_{ij} x_i x_j \\ &= \sum_{i<j} d_{ij} \frac{y_i y_j}{2} \left(\frac{x_i^2}{y_i^2} + \frac{x_j^2}{y_j^2} - 2\frac{x_i x_j}{y_i y_j}\right) \\ &= \sum_{i<j} d_{ij} \frac{y_i y_j}{2} \left(\frac{x_i}{y_i} - \frac{x_j}{y_j}\right)^2 \geq 0. \end{aligned}$$

In the second line, we used Equality (3.1). Now if $q_Q(x) = 0$, then $\frac{x_i}{y_i} = \frac{x_j}{y_j}$ whenever i and j are joined by an edge. By a similar argument as above, we use that $\bar{Q}$ is connected and conclude that $\frac{x_i}{y_i} = \frac{x_j}{y_j} = a \in \mathbb{Q}$ (for all i, j). Thus $x \in \mathbb{Q}y$. Now if $x \in \mathbb{Q}y$ we use that y is radical and get $(x, \cdot)_Q = a(y, \cdot)_Q = 0$. Therefore, $x \in \operatorname{rad} q_Q$ and $x \in \operatorname{rad} q_Q$ implies $q_Q(x) = (x,x)_Q = 0$. □

Definition 3.14. *A graph is called* ***Euclidean*** *if it is one of the graphs in Figure 3.1.*

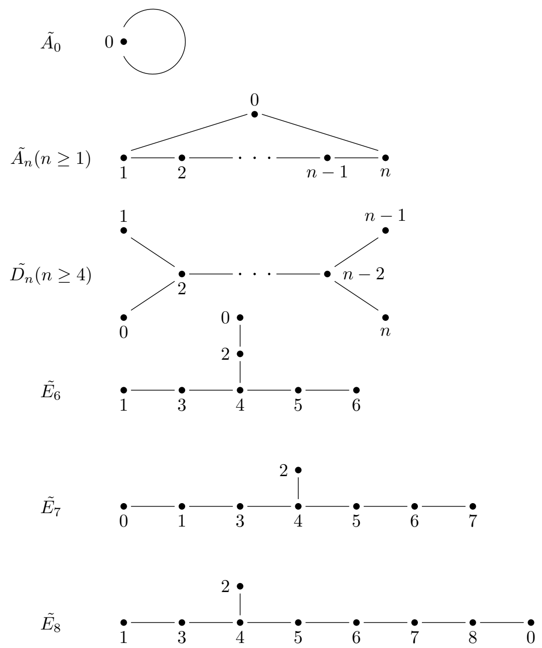
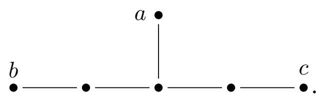


Figure 3.1: The Euclidean diagrams.

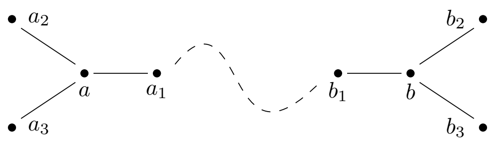
Lemma 3.15. *[2, Observation p. 52] Let $\bar{Q}$ be a graph that is neither Dynkin nor Euclidean. Then $\bar{Q}$ contains an Euclidean graph Γ as a subgraph.*

Proof. Note that if $\bar{Q}$ contains a loop, then $\Gamma = \tilde{A}_0$ is the desired subgraph. If $\bar{Q}$ contains a cycle, then the desired subgraph is of the type $\tilde{A}_n$. Thus w.l.o.g., we can assume that $\bar{Q}$ does not contain any loops or cycles. We also notice that $\bar{Q}$ has at least one **branch vertex** (= vertex connected to at least 3 different vertices by edges). Else $\bar{Q}$ is of type A_n, which is a Dynkin quiver. We do a proof by cases on the number of branch vertices.

Case 1. Let us consider the case that $\bar{Q}$ has exactly one branch vertex. There must be another vertex attached to the graph, or else we have the Dynkin graph D_4. If the additional vertex is attached to our branch vertex, we have the Euclidean graph $\tilde{D}_4$ as a subgraph and we are done. If the additional vertex is not attached to the branch vertex, then we actually have at least two additional vertices, which are attached to different branches. Else $\bar{Q} = D_n$ is a Dynkin graph (for some n). Thus $\bar{Q}$ contains a subgraph that looks like this:



But this is the Dynkin graph E_6, thus there needs to be another vertex d attached to a, b or c (notice that we only have one branch vertex!). If d is attached to a, then our desired subgraph is $\tilde{E}_6$. If there is no additional vertex attached to a, then by gradually attaching vertices to the ends at b or c we get the graphs $E_7, \tilde{E}_7, E_8$ or $\tilde{E}_8$. Thus we eventually get an Euclidean subgraph as claimed. Case 2. $\bar{Q}$ has at least two different branch vertices a and b as illustrated in the following picture:



Since $\bar{Q}$ is connected, we find a path between a_1 and b_1. By taking Γ to be this path (dotted line) and the 'forks' in the picture above, we get an Euclidean subgraph $\Gamma = \tilde{D}_n$ for some $n \geq 6$. $\square$

Theorem 3.16. *[2, Theorem 1.11] Let Q be a finite connected quiver and let q_Q be its Tits form.*

1. *Q is a Dynkin quiver if and only if q_Q is positive definite.*
2. *The underlying graph $\bar{Q}$ is Euclidean if and only if q_Q is positive semi-definite but not positive definite. In that case, there is a unique positive vector $\delta \in \mathbb{Z}^n$ s.t. $\operatorname{rad} q_Q = \mathbb{Z}\delta$.*

Proof. Step 1. *If $\bar{Q}$ is Euclidean, then q_Q is positive semi-definite and $\operatorname{rad} q_Q = \mathbb{Z}\delta$:* For each Euclidean graph, we explicitly give a positive radical vector δ (we enumerate the vertices as in Definition 3.14).

$\tilde{A}_0$: $\delta = (1)$,
$\tilde{A}_n$: $\delta = (1, 1, \ldots, 1)$,
$\tilde{D}_n$: $\delta = (1, 1, 2, \ldots, 2, 1, 1)$,
$\tilde{E}_6$: $\delta = (1, 1, 2, 2, 3, 2, 1)$,
$\tilde{E}_7$: $\delta = (1, 2, 2, 3, 4, 3, 2, 1)$,
$\tilde{E}_8$: $\delta = (1, 2, 3, 4, 6, 5, 4, 3, 2)$.

To check that δ is radical, we check that (see proof of Lemma 3.13):

$$0 = (\delta, e_i)_Q = (2 - 2d_{ii})\delta_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} d_{ji}\delta_j \text{ for } 1 \leq i \leq n. \tag{3.2}$$

If $\bar{Q}$ has no loops or multiple edges this is equivalent to (use $d_{ii} = 0$ and $d_{ij} \in \{0, 1\}$):

$$0 = (\delta, e_i)_Q = 2\delta_i - \sum_{\substack{j=1 \\ d_{ij} \neq 0}}^{n} \delta_j \text{ for } 1 \leq i \leq n. \tag{3.3}$$

We do the calculation for $\tilde{A_0}, \tilde{A_n}, \tilde{D_n}$. The other cases can also be checked by calculation.

$\tilde{A_0}$: $(\delta, e_1) \overset{(3.2)}{=} (2 - 2d_{11})\delta_1 = (2-2)\cdot 1 = 0$.

$\tilde{A_n}$: $(\delta, e_i) \overset{(3.3)}{=} 2\delta_i - \delta_{i-1} - \delta_{i+1} = 2 - 1 - 1 = 0$ (we consider the indices mod $n+1$).

$\tilde{D_n}$: For $i \in \{0, 1, n-1, n\}$: $(\delta, e_i) \overset{(3.3)}{=} 2\delta_i - 2 = 2 - 2 = 0$.
For $i \in \{3, \ldots, n-3\}$: $(\delta, e_i) \overset{(3.3)}{=} 2\delta_i - \delta_{i-1} - \delta_{i+1} = 2\cdot 2 - 2 - 2 = 0$.
For $i \in \{2, n-2\}$: $(\delta, e_i) \overset{(3.3)}{=} 2\delta_i - 2 - 1 - 1 = 2\cdot 2 - 2 - 1 - 1 = 0$.

From Lemma 3.13 it follows that q_Q is positive semi-definite. It also follows that $\operatorname{rad} q_Q = \mathbb{Q}\delta \cap \mathbb{Z}^n$. Since every δ above has $\delta_i = 1$ for some i, we know that for $a \in \mathbb{Q} \setminus \mathbb{Z}$ we have $a \cdot \delta_i = a \notin \mathbb{Z}$ and thus $\mathbb{Q}\delta \cap \mathbb{Z}^n = \mathbb{Z}\delta = \operatorname{rad} q_Q$. From Lemma 3.13 also follows that $q_Q(\delta) = 0$ and thus q_Q is not positive definite. The uniqueness of δ is a consequence of the property $\operatorname{rad} q_Q = \mathbb{Z}\delta$.

Step 2. *If Q is Dynkin, then q_Q is positive definite:* We first notice that for every Dynkin graph $\bar{Q}$, there exists an Euclidean graph $\bar{Q}'$ s.t. $\bar{Q}$ results from $\bar{Q}'$ by deleting the vertex 0 (and its incident edges). A_n results from $\tilde{A_n}$, D_n from $\tilde{D_n}$ and so on. We can extend $x \in \mathbb{Z}^n$ to $x' = (0, x) \in \mathbb{Z}^{n+1}$ and from step 1 we get that $q_{\bar{Q}}(x) = q_{\bar{Q}'}(x') \geq 0$. Thus $q_{\bar{Q}}$ is positive semi-definite. To see that it is actually positive definite, let $x \in \mathbb{Z}^n$ be such that $q_{\bar{Q}}(x) = q_{\bar{Q}'}(x') = 0$. From Lemma 3.13 we get that $x' = a \cdot \delta$ for $a \in \mathbb{Q}$ and δ defined in step 1. But since $x'_0 = 0$ we have that $a = 0$ (since $\delta_0 \neq 0$ for all the Euclidean graphs) and thus $x = 0$ is trivial.

Step 3. *If $\bar{Q}$ is neither Dynkin nor Euclidean, then q_Q is indefinite, i.e. $q_Q(x) < 0$ for some $X \in \mathbb{Z}^n$:* Let Γ be the proper Euclidean subgraph of $\bar{Q}$ (use Lemma 3.15 and let δ be the positive radical vector for Γ (see step 1).

- If $\Gamma_0 = \bar{Q}_0$ (the vertices coincide) and $\Gamma_1 \subsetneq \bar{Q}_1$ (edges do not coincide), we put $x = \delta \in \mathbb{N}^n$ and have $q_{\bar{Q}}(x) < q_\Gamma(x) = 0$. The inequality comes from the fact that there exists an edge $a = (i, j) \in \bar{Q}_1 \setminus \Gamma_1$. Notice that this yields an additional term $-x_i x_j < 0$ in $q_{\bar{Q}}(x)$.
- If $\Gamma_0 \subsetneq \bar{Q}_0$, then let i be a vertex in $\bar{Q}_0 \setminus \Gamma_0$ which is connected to Γ by an edge a (such a vertex exists since $\bar{Q}$ is connected). Now let δ' be an extension of δ ($\delta'_j = 0$ for $j \in \bar{Q}_0 \setminus \Gamma_0$) and define $x = 2\delta' + e_i$. We have $q_{\bar{Q}}(x) \leq 4q_\Gamma(\delta) + x_i^2 - x_{t_a}x_{h_a} \leq 0 + 1 - 2 = -1 < 0$. □

Definition 3.17. *A quiver Q is called a* ***tame*** *quiver if its underlying graph $\bar{Q}$ is Euclidean. If Q is neither a Dynkin quiver nor a tame quiver, it is called a* ***wild*** *quiver.*

This definition classifies all finite connected quivers into three different categories: Dynkin quivers, tame quivers, and wild quivers. Theorem 3.16 yields the same classification, but it considers the Tits forms q_Q rather than classifying the quivers based on underlying graphs $\bar{Q}$. The Tits form of the Dynkin quivers is positive definite, the Tits form of the tame quivers is positive semi-definite and the Tits form of the wild quivers is indefinite.

Remark 3.18. *In Gabriel's Theorem, there is no distinction between tame and wild quivers. But in the theory beyond Gabriel's Theorem, there are some interesting differences between the two categories, which is why we differentiate between the two categories.*

Example 3.19. *The loop quiver is a tame quiver since its underlying graph is the Euclidean graph $\tilde{A_0}$. This matches with our calculation in Example 3.8: $q_Q(x) = \frac{1}{2}(x, x)_Q = 0$, which is a positive semi-definite form.*

3.2 Weyl Groups and Root Systems

In this section, we introduce the Weyl group and the root system of a finite quiver. This section is based on the book 'Finite dimensional algebras and quantum groups' by B. Deng [2].

Definition 3.20. *Let Q be a finite quiver and let $i \in Q_0 = \{1, \ldots, n\}$ be a vertex such that Q has no loops at i, i.e. $d_{ii} = 0$. Then we say that the vector $e_i \in \mathbb{Z}^n$ is a* ***simple root****. We denote by Π_Q the set of all simple roots.*

Definition 3.21. *Let Q be a finite quiver and let i be a vertex such that Q has no loops at i. The **simple reflection** at the vertex i is the map*

$$\sigma_i\colon \mathbb{Z}^n \to \mathbb{Z}^n, x \mapsto x - (x, e_i)_Q e_i,$$

where $(x, y)_Q$ is the symmetric Euler form of Q and e_i is the i-th coordinate vector. Using that $(e_j, e_i)_Q = -d_{ij}$ for $j \neq i$ and $(e_i, e_i)_Q = 2 - 2d_{ii} = 2$ we get $\sigma_i(x)_j = x_j$ for $j \neq i$ and

$$\sigma_i(x)_i = x_i - \sum_{j=1}^{n} x_j(e_j, e_i)_Q = x_i - \sum_{\substack{j=1\\ j\neq i}}^{n} -d_{ji}x_j - 2x_i = \sum_{\substack{j=1\\ j\neq i}}^{n} d_{ji}x_j - x_i.$$

Example 3.22. *We consider the L_2-quiver. We have $(e_i, e_i)_Q = 2 - 2d_{ii} = 2$ for $i \in \{1, 2\}$. Also $(x, e_1)_Q = x_1(e_1, e_1)_Q + x_2(e_2, e_1)_Q = 2x_1 - d_{12}x_2 = 2x_1 - x_2$. Similarly, we get $(x, e_2)_Q = 2x_2 - x_1$. Thus we get the maps $\sigma_1, \sigma_2\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ given by:*

$$\begin{aligned}
\sigma_1((x_1, x_2)^T) &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 2x_1 - x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\\
\sigma_2((x_1, x_2)^T) &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ 2x_2 - x_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 - x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\end{aligned}$$

Lemma 3.23. *The simple reflections σ_i are automorphisms of order two and they preserve the symmetric Euler form $(\cdot,\cdot)_Q$ and the Tits form q_Q.*

Proof. We show that $\sigma_i^2 = \mathrm{id}_{\mathbb{Z}^n}$. Indeed, for $j \neq i$ we have $\sigma_i^2(x)_j = \sigma_i(x)_j = x_j$ and we also have

$$\sigma_i(\sigma_i(x))_i = \sum_{\substack{j=1\\ j\neq i}}^{n} d_{ji}\sigma_i(x)_j - \sigma_i(x)_i = \sum_{\substack{j=1\\ j\neq i}}^{n} d_{ji}x_j - \sigma_i(x)_i = \sum_{\substack{j=1\\ j\neq i}}^{n} d_{ji}x_j - (\sum_{\substack{j=1\\ j\neq i}}^{n} d_{ji}x_j - x_i) = x_i.$$

We now show that for any quiver Q and for all $x, y \in \mathbb{Z}^n$, we have $(\sigma_i(x), \sigma_i(y))_Q = (x, y)_Q$. First, we notice that

$$\sigma_i(e_k) = e_k - (e_i, e_k)_Q e_i = \begin{cases} e_i - (2 - 2d_{ii})e_i = -e_i & \text{if } k = i,\\ e_k + d_{ik}e_i & \text{if } k \neq i. \end{cases}$$

Thus we get $(\sigma_i(e_i), \sigma_i(e_i))_Q = (-e_i, -e_i)_Q = (e_i, e_i)_Q$ and for $k \neq i$ we get:

$$(\sigma_i(e_k), \sigma_i(e_i))_Q = (e_k + d_{ik}e_i, -e_i)_Q = -(e_k, e_i)_Q - d_{ik}(e_i, e_i)_Q = d_{ik} - 2d_{ik} = -d_{ik} = (e_k, e_i)_Q.$$

Also for $k \neq i, j \neq i$ we get:

$$\begin{aligned}
(\sigma_i(e_k), \sigma_i(e_l))_Q &= (e_k + d_{ik}e_i, e_l + d_{il}e_i)_Q = (e_k, e_l)_Q + d_{ik}(e_i, e_l)_Q + d_{il}(e_k, e_i)_Q + d_{ik}d_{il}(e_i, e_i)_Q\\
&= (e_k, e_l)_Q - d_{ik}d_{il} - d_{il}d_{ik} + 2d_{ik}d_{il} = (e_k, e_l)_Q.
\end{aligned}$$

Now, using linearity of σ_i and bilinearity of $(\cdot,\cdot)_Q$, we get that:

$$\begin{aligned}
(\sigma_i(x), \sigma_i(y))_Q &= (\sigma_i(\sum_j x_je_j), \sigma_i(\sum_k y_ke_k))_Q = \sum_j\sum_k x_jy_k(\sigma_i(e_j), \sigma_i(e_k))_Q\\
&= \sum_j\sum_k x_jy_k(e_j, e_k)_Q = (\sum_j x_je_j, \sum_k y_ke_k)_Q = (x, y)_Q.
\end{aligned}$$

Finally, this implies:

$$q_Q(\sigma_i(x)) = \frac{1}{2}(\sigma_i(x), \sigma_i(x))_Q = \frac{1}{2}(x, x)_Q = q_Q(x). \qquad \square$$

Definition 3.24. *For a finite quiver Q, we define the **Weyl group** $W(Q)$ to be the subgroup of the automorphism group $\mathrm{Aut}(\mathbb{Z}^n)$ generated by the simple reflections σ_i. The set*

$$\Phi_{\mathrm{re}}(Q) = \bigcup_{w\in W(Q)} w(\Pi_Q) = \bigcup_{\substack{w\in W(Q)\\ \alpha\in\Pi_Q}} w(\alpha)$$

*is called the set of **real roots** of Q.*

Because the symmetric Euler form is independent of the orientations of the arrows in Q, both the simple representations σ_i and the Weyl group $W(Q)$ are also independent of the orientations of the arrows in Q.

Definition 3.25. *For each $x = \sum_{i=1}^n x_i e_i \in \mathbb{Z}^n$ we define the* ***support of*** *x to be the set*

$$\operatorname{supp} x = \{i \in Q_0 \mid x_i \neq 0\}.$$

We say that $\operatorname{supp} x$ is ***connected*** *if the full subquiver of Q with vertex set $\operatorname{supp} x$ is connected.*

Definition 3.26. *For a finite quiver Q, the* ***fundamental set*** *$\mathcal{F}_Q$ is defined to be the set*

$$\mathcal{F}_Q = \{0 \neq x \in \mathbb{N}^n \mid (x, e_i)_Q \leq 0 \textit{ for all } e_i \in \Pi_Q, \textit{ and } \operatorname{supp} x \textit{ is connected}\}.$$

Then the set

$$\Phi_{\mathrm{im}}(Q) = \bigcup_{w \in W(Q)} w(\mathcal{F}_Q) \cup w(-\mathcal{F}_Q) = \bigcup_{\substack{w \in W(Q) \\ x \in \mathcal{F}_Q}} w(x) \cup w(-x)$$

is called the set of ***imaginary roots*** *of Q. Finally, the* ***root system*** *of Q is defined to be the set*

$$\Phi(Q) = \Phi_{\mathrm{re}}(Q) \cup \Phi_{\mathrm{im}}(Q).$$

An element $x \in \Phi(Q) \cap \mathbb{N}^n$ is called a ***positive root****. We denote by $\Phi^+(Q)$ (resp. $\Phi^+_{\mathrm{re}}(Q), \Phi^+_{\mathrm{im}}(Q)$) the set of all positive (resp. positive real, positive imaginary) roots.*

Example 3.27. *Let Q be the loop quiver. Then no simple root exists and thus $\Pi_Q = \emptyset$ and $W(Q) = \{\mathrm{id}\}$. We also have that $\mathcal{F}_Q = \mathbb{N}$ and thus $\Phi(Q) = \Phi_{\mathrm{im}}(Q) = \mathbb{Z} \setminus \{0\}$.*

For a better understanding of the definitions above, we prove some elementary properties of our root systems $\Phi_{\mathrm{re}}(Q), \Phi_{\mathrm{im}}(Q)$ and $\Phi(Q)$.

Lemma 3.28. *Let Q be a finite quiver.*

1. *If Q has no loops at $i \in Q_0$, then e_i is a real root.*
2. *If Q has a loop at $i \in Q_0$, then $e_i \in \mathcal{F}_Q$ and thus e_i is an imaginary root.*
3. *If $x \in \Phi_{\mathrm{re}}(Q)$, then $q_Q(x) = 1$.*
4. *If $x \in \Phi_{\mathrm{im}}(Q)$, then $q_Q(x) \leq 0$.*
5. *$\Phi_{\mathrm{re}}(Q) \cap \Phi_{\mathrm{im}}(Q) = \emptyset$.*
6. *If Q is a Dynkin or a tame quiver, then every root is either positive or negative. Therefore, we have $\Phi(Q) = \Phi^+(Q) \cup (-\Phi^+(Q))$.*

Proof.

1. Taking $w = \mathrm{id} \in w(Q)$, we get that $w(e_i) = e_i \in \Phi_{\mathrm{re}}(Q)$.
2. In this case, $\operatorname{supp} e_i$ is just one point and thus connected. Moreover, for all $e_j \in \Pi_Q$, we have:
$$(e_i, e_j)_Q = \begin{cases} 2 - 2d_{ii} \leq 0 & \text{if } j = i, \\ -d_{ij} \leq 0 & \text{if } j \neq i. \end{cases}$$
3. Each $w \in W(Q)$ is of the form $w = \sigma_{i_k} \dots \sigma_{i_1}$ for σ_{i_j} simple reflections (possibly $k = 0$ and $w = \mathrm{id}$). By Lemma 3.23, we have:
$$q_Q(x) = q_Q(\sigma_{i_k} \dots \sigma_{i_1}(e_j)) = q_Q(\sigma_{i_{k-1}} \dots \sigma_{i_1}(e_j)) = \dots = q_Q(e_j) = \frac{1}{2}(e_j, e_j)_Q = 1 - d_{jj} = 1.$$
4. We have $x = w(y)$ or $x = w(-y)$ for $w \in W(Q)$ and $y \in \mathcal{F}_Q$. In both cases, we have $q_Q(x) = q_Q(w(\pm y)) = q_Q(\pm y) = q_Q(y)$. Since $y \in \mathbb{N}^n$, we have:
$$q_Q(x) = q_Q(y) = \frac{1}{2}\sum_{i=1}^n y_i (y, e_i)_Q = \frac{1}{2}\sum_{d_{ii}=0} y_i \underbrace{(y, e_i)_Q}_{\leq 0} + \frac{1}{2}\sum_{d_{ii} \neq 0} y_i \sum_{j=1}^n y_j \underbrace{(e_j, e_i)_Q}_{\leq 0 \text{ by part 2}} \leq 0.$$

5. This immediately follows from parts 3 and 4.
6. Let x be a root and write $x = x^+ - x^-$ where x^+ is the positive part of x and x^- is the negative part of x, i.e. $x^+, x^- \geq 0$ and they have disjoint support. Then we have $(x^+, x^-)_Q \leq 0$ since they have disjoint support. We then have:

$$\begin{aligned}1 \geq q_Q(x) &= \frac{1}{2}(x^+ - x^-, x^+ - x^-)_Q = \frac{1}{2}[(x^+, x^+)_Q + (x^-, x^-)_Q - 2(x^+, x^-)_Q] \\ &= q_Q(x^+) + q_Q(x^-) - (x^+, x^-)_Q \geq q_Q(x^+) + q_Q(x^-) \geq 0.\end{aligned}$$

In the last inequality, we used that q_Q is positive semi-definite since Q is Dynkin or tame. We conclude that either $q_Q(x^+) = 0$ or $q_Q(x^-) = 0$. If Q is Dynkin, then either $x^+ = 0$ or $x^- = 0$ since q_Q is positive definite. If Q is tame, then we know from Lemma 3.13 that either x^+ or x^- is a radical vector and thus it is sincere. But then the other vector is trivial, which shows that x is either positive or negative. □

Remark 3.29. *Part (6) of Lemma 3.28 is true for any quiver Q. In addition, one can show that for any root $x \in \Phi(Q)$, $\operatorname{supp} x$ is connected. We do not prove these statements because they are not needed to prove Gabriel's Theorem. A proof can be found on page 7 in Chapter 1 of [6].*

An immediate consequence of this lemma is that $\Phi(Q) \subseteq \{0 \neq x \in \mathbb{Z}^n \mid q_Q(x) \leq 1\}$. We will later see that if Q is a Dynkin quiver, then these two sets are equal. Although the characterization $q_Q(x) \leq 1$ looks a lot simpler than the definition of the root systems $\Phi_{\text{re}}(Q)$ and $\Phi_{\text{im}}(Q)$, the latter give a more explicit description of the roots and are therefore useful in proofs. Moreover, the latter descriptions of $\Phi_{\text{re}}(Q)$ and $\Phi_{\text{im}}(Q)$ are closely related to the Weyl group, which plays an important role in the proof of Gabriel's Theorem (see Lemma 3.68 and Proposition 3.69).

3.2.1 Root Systems of Dynkin, Tame and Wild Quivers

We show an explicit characterization of the roots of a Dynkin quiver. In particular, we show that the root system $\Phi(Q)$ is finite for all Dynkin quivers. We also mention some results about the root systems of tame and wild quivers.

Lemma 3.30. *If Q is a Dynkin quiver, then $\Phi(Q)$ is finite.*

Proof. We prove that the set $\{x \in \mathbb{Z}^n \mid q_Q(x) = 1\}$ is finite. First, we can view q_Q also as a quadratic form on $\mathbb{Q}^n$ and $\mathbb{R}^n$. Since q_Q is positive definite on $\mathbb{Z}^n$ it is also positive definite on $\mathbb{Q}^n$. Indeed, if $0 \neq x = (\frac{a_i}{b_i})_{i=1}^n \in \mathbb{Q}^n$ such that $q_Q(x) \leq 0$, then we have that $q_Q(ax) \leq 0$, where a is the least common multiple of the b_i and $0 \neq ax \in \mathbb{Z}^n$. Now, we can take limits in $\mathbb{Q}^n$ to extend q_Q to $\mathbb{R}^n$ and get that q_Q is positive semi-definite on $\mathbb{R}^n$. However, since q_Q is positive definite on $\mathbb{Q}^n$, we know that its matrix C_Q is positive definite and symmetric (with values in $\mathbb{Q}$), thus it is invertible in $\mathbb{Q}$ and therefore also invertible in $\mathbb{R}$. Therefore, q_Q is also positive definite on $\mathbb{R}^n$ (since it is positive semi-definite and C_Q is invertible). Now consider the subset $S^1 = \{x = (x_i) \in \mathbb{R}^n \mid \|x\| = 1\}$. Here, $\|\cdot\|$ is the Euclidean norm. Since S^1 is compact and q_Q is positive definite and continuous, there exists $c > 0$ such that $q_Q(x) \geq c$ for all $x \in S^1$. Indeed, we have that $q_Q(S^1)$ is compact and thus $\inf_{x \in S^1}(q_Q(x)) = c > 0$. Therefore, for all $0 \neq x \in \mathbb{R}^n$, we have:

$$q_Q(x) = \|x\|^2 q_Q(\frac{x}{\|x\|}) \geq c\|x\|^2.$$

Therefore, $x \in \Phi(Q)$ implies $q_Q(x) = 1$, which again implies $\|x\|^2 \leq \sqrt{\frac{1}{c}}$. But there are only finitely many $x \in \mathbb{Z}^n$ such that the last inequality holds. □

Corollary 3.31. *If Q is a Dynkin quiver, then*

1. *the Weyl group $W(Q)$ is finite,*
2. *$\Phi(Q) = \{x \in \mathbb{Z}^n \mid q_Q(x) = 1\}$.*

Proof. 1. By the theorem above, $\Phi(Q) = \Phi_{\text{re}}(Q)$ is finite. For any $w \in W(Q)$, the induced map $\Phi_{\text{re}}(Q) \to \Phi_{\text{re}}(Q), w'(e_i) \mapsto w \circ w'(e_i)$ is a permutation on the set $\Phi(Q)$ of roots since w is an automorphism. Since $\Phi(Q)$ contains the basis $\{e_i \mid i \in Q_0\}$ of $\mathbb{Z}^n$, it follows that the permutation maps that are induced by $w_1, w_2 \in W(Q)$ are equal if and only if $w_1 = w_2$. Therefore, $W(Q)$ can be embedded into the permutation group of $\Phi(Q)$, which is finite since $\Phi(Q)$ is finite.

2. We have already shown that each $x \in \Phi(Q) = \Phi_{\text{re}}(Q)$ satisfies $q_Q(x) = 1$. Now let $x \in \mathbb{Z}^n$ be such that $q_Q(x) = 1$. We know from Lemma 3.28 that x is either positive or negative. Since $\Phi(Q) = \Phi^+(Q) \cup (-\Phi^+(Q))$, it suffices to show that x or $-x$ is a root. Thus w.l.o.g., we assume that x is positive. We use induction of the number $l_x = \sum_{i \in Q_0}$. If $l_x = 1$, then $x = e_i \in \Pi_Q \subseteq \Phi(Q)$. Now let $l_x \geq 2$. We claim that there exists an $i \in Q_0$ such that $0 < \sigma_i(x) < x$. Suppose this is not the case, i.e. for each $i \in Q_0$ we have either $0 \not< \sigma_i(x)$ or $\sigma_i(x) \not< x$. We consider the following formula:

$$2 = 2q_Q(x) = (x, x)_Q = \sum_{i \in Q_0} x_i (x, e_i)_Q.$$

It implies that there exists an $i_0 \in Q_0$ such that $x_{i_0} > 0$ and $(x, e_{i_0})_Q > 0$. Thus we get $\sigma_{i_0}(x) = x - (x, e_{i_0})_Q e_{i_0} < x$. But this forces $0 \not< \sigma_{i_0}(x)$, i.e. $x_{i_0} - (x, e_{i_0})_Q < 0$. On the other hand,

$$\begin{aligned} 0 \leq q_Q(x - x_{i_0} e_{i_0}) &= q_Q(x) + q_Q(x_{i_0} e_{i_0}) + (x, -x_{i_0} e_{i_0})_Q \\ &= 1 + x_{i_0}^2 - x_{i_0}(x, e_{i_0})_Q = 1 + x_{i_0}(x_{i_0} - (x, e_{i_0})_Q). \end{aligned}$$

This forces $x_{i_0} = 1$ and $x_{i_0} - (x, e_{i_0})_Q = -1$. But then $q_Q(x - x_{i_0} e_{i_0}) = 0$, i.e. $x = x_{i_0} e_{i_0} = e_{i_0}$ which contradicts $l_x \geq 2$. Therefore, there exists an $i \in Q_0$ such that $0 < \sigma_i(x) < x$. But then $q_Q(\sigma_i(x)) = q_Q(x) = 1$ and $l_{\sigma_i(x)} < l_x$ and thus by the induction hypothesis, we have $\sigma_i(x) \in \Phi(Q)$. We conclude that $x = \sigma_i(\sigma_i(x) \in \Phi(Q)$. □

Remark 3.32. *To get a sense of the size of the finite root system $\Phi(Q)$ for a Dynkin quiver Q, we list the sizes. If Q is a Dynkin quiver of type $A_n (n \geq 1), D_n (n \geq 4), E_6, E_7$ or E_8, then the number of roots in $\Phi(Q)$ is $n(n+1), 2n(n-1), 72, 126,$ or 240, respectively.*

We now have proven everything about the root system $\Phi(Q)$ and the Weyl group $W(Q)$ that we need to prove Gabriel's Theorem. Thus, if you are only interested in the proof of Gabriel's Theorem, you can continue reading Section 3.3. However, we have already done (almost) everything to prove a really nice theorem that connects different types of quivers and their root systems. Therefore, we spend the remaining part of this section proving the following theorem.

Theorem 3.33. *[2, Theorem 1.13] Let Q be a finite connected quiver without loops. Then*

1. *Q is a Dynkin quiver $\iff$ $\Phi(Q)$ is finite $\iff$ $\Phi_{\text{im}}(Q) = \emptyset$.*
2. *Q is a tame quiver $\iff$ $\Phi_{\text{im}}(Q) = \mathbb{Z}\delta \setminus \{0\}$ for the unique positive radical vector $\delta \in \mathbb{N}^n$ from Theorem 3.16.*
3. *Q is a wild quiver $\iff$ there exists $x \in \Phi^+(Q)$ such that $\operatorname{supp} x = Q$ and such that $(x, e_i)_Q < 0$ for all $i \in Q_0$.*

To prove this theorem, we need to consider the following lemma.

Lemma 3.34. *Let Q be a finite connected quiver without loops. Then Q is wild if and only if there exists a positive, sincere $x \in \mathbb{N}^n$ such that $C_Q x$ is negative and sincere, i.e. $(C_Q x)_i < 0$. Here, C_Q denotes the matrix belonging to the symmetric Euler form. This means that for all $x, y \in \mathbb{Z}^n$ we have $(x, y)_Q = x^T C_Q y$.*

Proof. The proof of this lemma can be found on page 59 of [2]. We skip it since it mainly consists of algebraic manipulations. □

Proof of Theorem 3.33. 1. We know that Q is a Dynkin quiver if and only if its Tits form q_Q is positive definite. We use this to show that Q is Dynkin if and only if $\mathcal{F}_Q = \emptyset$. Indeed, if Q is Dynkin, then $x \in \mathcal{F}_Q$ implies that $q_Q(x) = \frac{1}{2}(x,x)_Q = \frac{1}{2}\sum_{i=1}^n x_i(x,e_i) \leq 0$ and since q_Q is positive definite, this implies that $x = 0$ and thus $\mathcal{F}_Q = \emptyset$. We prove the other implication via contraposition. Therefore, we consider the cases for a tame resp. a wild quiver Q. If Q is a tame quiver, then we know from Theorem 3.16 that the unique smallest positive radical vector δ is in $\mathcal{F}_Q$. In the case that Q is a wild quiver, we know from the proof of Theorem 3.16 that there exists a positive vector δ (with connected support) such that $q_Q(\delta) < 0$. We show that $\delta \in \mathcal{F}_Q$. First, notice that if $x \in \mathbb{N}^n$ and $j \notin \operatorname{supp} x$, then $(x,e_j)_Q \leq 0$. Indeed $(x,e_j)_Q = \sum_{k\in\operatorname{supp} x} x_k(e_k,e_j)_Q = \sum_{k\in\operatorname{supp} x} -x_k d_{kj} \leq 0$. Moreover, using the case distinction from the proof of Theorem 3.16, we get that for $i \in \operatorname{supp} x$, we either have $(\delta,e_i)_Q = (\delta',e_i)_Q = 0$ or $(\delta,e_i)_Q = (e_j,e_i)_Q \leq 0$ (since Q has no loops). Using that $\mathcal{F}_Q = \emptyset \iff \Phi_{\mathrm{im}}(Q) = \emptyset$, we get that Q is Dynkin $\iff \Phi_{\mathrm{im}}(Q) = \emptyset$. From Lemma 3.30 we know that if Q is Dynkin, then $\Phi(Q)$ is finite. Notice that $x \in \mathcal{F}_Q$ implies $ax \in \mathcal{F}_Q$ for all $0 < a \in \mathbb{Z}$. Therefore, if $\Phi(Q)$ is finite, this implies that $\Phi_{\mathrm{im}}(Q) = \emptyset$ and therefore Q is Dynkin.

2. Let Q be a tame quiver. We show that $\Phi_{\mathrm{im}}(Q) = \operatorname{rad} q_Q \setminus \{0\} = \mathbb{Z}\delta \setminus \{0\}$. Let $0 \neq x \in \operatorname{rad} q_Q$. Then we have $(x,e_i)_Q = 0$ for all $i \in Q_0$ and from Lemma 3.13, we know that x is sincere and therefore $\operatorname{supp} x = Q$ is connected. Thus we have $x \in \Phi_{\mathrm{im}}(Q)$. Now let $0 \neq x \in \Phi_{\mathrm{im}}(Q)$. Then we have $q_Q(x) \leq 0$ but since q_Q is positive semi-definite (Q is tame), we have $q_Q(x) = 0$ and from Lemma 3.13, we know that $x \in \operatorname{rad} q_Q$. Finally, from Theorem 3.16, we know that $\operatorname{rad} q_Q = \mathbb{Z}\delta$. Note that if Q is Dynkin, we know that $\Phi_{\mathrm{im}}(Q) = \emptyset$. Therefore, it remains to show that for a wild quiver Q, we have that $\Phi_{\mathrm{im}}(Q) \neq \mathbb{Z}\delta \setminus \{0\}$. Let Q be wild and let x be as in Lemma 3.34. For any $y \in \mathbb{Z}^n$, there exists a suitably large positive integer m such that $y - mx \in \mathcal{F}_Q \subseteq \Phi_{\mathrm{im}}(Q)$. Indeed, since $\operatorname{supp} x$ is connected, we have that $\operatorname{supp}(y - mx)$ is connected (for m large enough). Moreover, we can find m large enough such that for all $i \in Q_0$ we have (note that Q_0 is finite):

$$(y - mx, e_i)_Q = (y,e_i)_Q - m(x,e_i)_Q = (y,e_i)_Q - m\underbrace{(C_Q x)_i}_{<0} \leq 0.$$

But then, $\Phi_{\mathrm{im}}(Q)$ cannot be equal to $\mathbb{Z}\delta \setminus \{0\}$ for a single δ. Otherwise we had that for any $y, z \in \mathbb{Z}^n$ and all m_1, m_2 large enough, there would be a_1, a_2 such that $y - m_1 x = a_1\delta$, $z - m_2 x = a_2\delta$. This yields

$$(m_2a_1 - m_1a_2)\delta = m_2y - m_2m_1x - m_1z + m_1m_1x = m_2y - m_1z.$$

However, this is not possible if y, z are linearly independent. Therefore, if we have that $\Phi_{\mathrm{im}}(Q) = \mathbb{Z}\delta \setminus \{0\}$, then Q can only be a tame quiver.

3. This follows immediately from Lemma 3.34. Indeed, notice that x is sincere if and only if $\operatorname{supp} x = Q$. In addition, $(x,e_i)_Q = (C_Q x)_i$. □

3.3 Reflection Functors

In this section, we introduce reflection functors, which are a useful tool to analyze and compare representations.

Definition 3.35. *A vertex $i \in Q_0$ is called a **sink** if there does not exist an edge starting at i, i.e. for all $a \in Q_1$ we have $t_a \neq i$. In contrast, a vertex $j \in Q_0$ is called a **source** if there is no edge ending at j, i.e. $h_a \neq j$. For any vertex $i \in Q_0$ we denote by $Q_1^i \subseteq Q_1$ the subset of all the arrows which are incident to i, i.e. $a \in Q_1^i$ if a either starts or ends at i.*

Definition 3.36. *Let $i \in Q_0$ be a vertex. We call s_i the **reflection** at vertex i. This means that s_iQ is the quiver obtained by reversing all the arrows that start or end at i.*

Example 3.37. *Consider the following quiver Q:*

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} & \underset{3}{\bullet} \arrow[l] & \underset{4}{\bullet} \arrow[l] \arrow[r] & \underset{5}{\bullet} \end{tikzcd}$$

The vertices 1 and 4 are sources, the vertices 2 and 5 are sinks and vertex 3 is neither a source nor a sink. We list all possible reflections of Q:

$$s_1Q\colon \begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l] & \underset{3}{\bullet} \arrow[l] & \underset{4}{\bullet} \arrow[l] \arrow[r] & \underset{5}{\bullet} \end{tikzcd}$$

$$s_2Q\colon \begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l] \arrow[r] & \underset{3}{\bullet} & \underset{4}{\bullet} \arrow[l] \arrow[r] & \underset{5}{\bullet} \end{tikzcd}$$

$$s_4Q\colon \begin{tikzcd} \underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} & \underset{3}{\bullet} \arrow[l] \arrow[r] & \underset{4}{\bullet} & \underset{5}{\bullet} \arrow[l] \end{tikzcd}$$

$$s_5Q\colon \begin{tikzcd} \underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} & \underset{3}{\bullet} \arrow[l] & \underset{4}{\bullet} \arrow[l] & \underset{5}{\bullet} \arrow[l] \end{tikzcd}$$

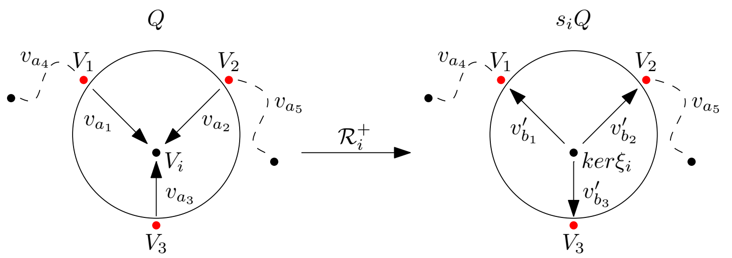
Definition 3.38. *Let Q be a quiver and let $i \in Q_0$ be a sink. Notice that for each $a \in Q_1^i$ we have $h_a = i$. We define a* ***covariant functor*** *$\mathcal{R}_i^+$ from the category $\operatorname{rep_k}(Q)$ to the category $\operatorname{rep_k}(s_iQ)$. For every representation $\mathbb{V} = (V_i, v_a) \in \operatorname{rep_k}(Q)$, we define a representation $\mathcal{R}_i^+\mathbb{V} = (V_i', v_a')$ in the category $\operatorname{rep_k}(\mathrm{s_iQ})$ as follows. For all $j \neq i$ we set $V_j' = V_j$, and define V_i' to be the kernel of the map*

$$\xi_i\colon \bigoplus_{a\in Q_1^i} V_{t_a} \to V_i, (x_{t_a})_{a\in Q_1^i} \mapsto \sum_{a\in Q_1^i} v_a(x_{t_a}).$$

For $\mathcal{R}_i^+\mathbb{V}$ to be a representation, we need to define linear maps belonging to the arrows in s_iQ. For each arrow $a \notin Q_1^i$ we set $v_a' = v_a$. For $a \in Q_1^i$, let b be the reverse arrow and we define v_b' to be the composition of maps

$$V_{t_b}' = V_i' = \ker \xi_i \hookrightarrow \bigoplus_{c\in Q_1^i} V_{t_c} \to V_{t_a} = V_{t_a}' = V_{h_b}'.$$

Here the map before the direct sum is the canonical inclusion and the map after the direct sum is the canonical projection onto the component V_{t_a}. To better understand what happens, we look at the following picture. The red vertices belong to the summands of the space $\bigoplus_{a\in Q_1^i} V_{t_a}$. Outside of the circle, everything stays the same.



Now let $\phi\colon \mathbb{V} \to \mathbb{W}$ be a morphism between two representations $\mathbb{V}, \mathbb{W} \in \operatorname{rep_k}(Q)$. We define the morphism $\phi' = \mathcal{R}_i^+\phi\colon \mathcal{R}_i^+\mathbb{V} \to \mathcal{R}_i^+\mathbb{W}$ by $\phi_j' = \phi_j$ for $j \neq i$ and ϕ_i' to be the restriction of the map

$$\bigoplus_{a\in Q_1^i} \phi_{t_a}\colon \bigoplus_{a\in Q_1^i} V_{t_a} \to \bigoplus_{a\in Q_1^i} W_{t_a}$$

to the subspace $V_i' = \ker \xi_i$.

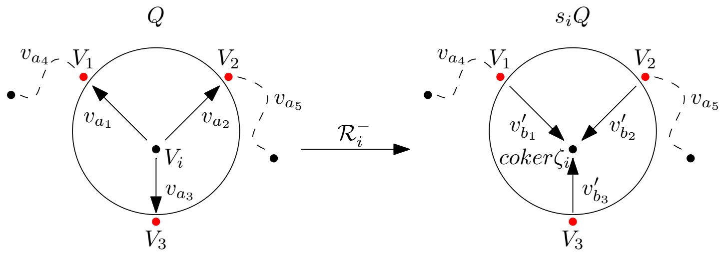
Definition 3.39. *Let Q be a quiver and let $i \in Q_0$ be a source. Notice that for each $a \in Q_1^i$ we have $t_a = i$. We define a* ***covariant functor*** *$\mathcal{R}_i^-$ from the category $\operatorname{rep_k}(Q)$ to the category $\operatorname{rep_k}(s_iQ)$. For every representation $\mathbb{V} = (v_i, v_a) \in \operatorname{rep_k}(Q)$, we define a representation $\mathcal{R}_i^-\mathbb{V} = (V_i', v_a')$ in the category $\operatorname{rep_k}(\mathrm{s_iQ})$ as follows. For all $j \neq i$ we set $V_j' = V_j$, and define V_i' to be the cokernel of the map*

$$\zeta_i\colon V_i \to \bigoplus_{a\in Q_1^i} V_{h_a}, x \mapsto (v_a(x))_{a\in Q_1^i}.$$

For $\mathcal{R}_i^- \mathbb{V}$ to be a representation, we need to define linear maps belonging to the arrows in s_iQ. For each arrow $a \notin Q_1^i$ we set $v'_a = v_a$. For $a \in Q_1^i$, let b be the reverse arrow and we define v'_b to be the composition of maps

$$V'_{t_b} = V'_{h_a} = V_{h_a} \hookrightarrow \bigoplus_{c \in Q_1^i} V_{h_c} \to \operatorname{coker} \zeta_i = V'_i = V'_{h_b}.$$

Here the map before the direct sum is the canonical inclusion and the map after the direct sum is the canonical quotient map (i.e. the quotient map modulo the image of ζ_i). Consider the following picture. The red vertices belong to the summands of the space $\bigoplus_{a \in Q_1^i} V_{h_a}$. Outside of the circle, everything stays the same.



Now let $\phi\colon \mathbb{V} \to \mathbb{W}$ be a morphism between two representations $\mathbb{V}, \mathbb{W} \in \operatorname{rep}_{\Bbbk}(Q)$. We define the morphism $\phi' = \mathcal{R}_i^- \phi\colon \mathcal{R}_i^- \mathbb{V} \to \mathcal{R}_i^- \mathbb{W}$ by $\phi'_j = \phi_j$ for $j \neq i$ and ϕ'_i to map induced by

$$\bigoplus_{a \in Q_1^i} \phi_{h_a} \colon \bigoplus_{a \in Q_1^i} V_{h_a} \to \bigoplus_{a \in Q_1^i} W_{h_a}$$

on the quotient space $V'_i = \operatorname{coker} \zeta_i$.

Remark 3.40. *Note that we consider the category $\operatorname{rep}_{\Bbbk}(Q)$ of finite-dimensional representations. By doing this, we ensure that the sums in the definitions of the maps ξ_i and ζ_i are finite sums (resp. there are only finitely many non-zero summands). Note that this is necessary for the well-definition of the maps ξ_i and ζ_i because else the order of summation may not be negligible. To fix this issue, one could also restrict to the case of finite quivers, as we sometimes do. Especially when considering dimension vectors of representations (which is important for Gabriel's Theorem) restricting to finite quivers is the right choice of restriction.*

Before giving an example for $\mathcal{R}_i^+$ and $\mathcal{R}_i^-$, we prove that they are covariant functors.

Lemma 3.41. *$\mathcal{R}_i^+$ and $\mathcal{R}_i^-$ are well-defined covariant functors.*

Proof. We show that $\mathcal{R}_i^+$ is a functor. Let ξ_i^W denote the map $\bigoplus_{a \in Q_1^i} W_{t_a} \to W_i$ from the definition of $\mathcal{R}_i^+ \mathbb{W}$. First notice that $\mathcal{R}_i^+ \mathbb{V}$ is indeed a representation of the quiver s_iQ. This means that for all arrows $b \in s_iQ_1$ we have that v'_b maps from V'_{t_b} to V'_{h_b}. This is true by observing the definition. Similarly, we need to check that $\phi' = \mathcal{R}_i^+ \phi$ is a morphism between the representations $\mathcal{R}_i^+ \mathbb{V}$ and $\mathcal{R}_i^+ \mathbb{W}$. First, we observe that all the maps ϕ'_j are well-defined, i.e. ϕ'_j maps from $\mathcal{R}_i^+ \mathbb{V}_j$ to $\mathcal{R}_i^+ \mathbb{W}_j$. This is clearly true for $j \neq i$ and in the case $j = i$, we notice that for $x = (x_{t_a})_{a \in Q_1^i} \in \ker \xi_i = V'_i$ we have:

$$\begin{aligned}
\xi_i^W\Big(\bigoplus_{a \in Q_1^i} \phi_{t_a}(x)\Big) &= \xi_i^W((\phi_{t_a}(x_{t_a}))_{a \in Q_1^i}) = \sum_{a \in Q_1^i} w_a(\phi_{t_a}(x_{t_a})) = \sum_{a \in Q_1^i} \phi_{h_a}(v_a(x_{t_a})) \\
&= \sum_{a \in Q_1^i} \phi_i(v_a(x_{t_a})) = \phi_i \sum_{a \in Q_1^i} (v_a(x_{t_a})) = \phi_i(0) = 0.
\end{aligned}$$

In the third equality we used that ϕ is a morphism (i.e. $\phi_{h_a} v_a = w_a \phi_{t_a}$). Thus we have that $\operatorname{im} \phi'_i \subseteq \ker \xi_i^W$ and this means that ϕ'_i is well-defined. We also need to check that ϕ' is a morphism, i.e. for each $b \in s_iQ_1$ we have $\phi'_{h_b} v'_b = w'_b \phi'_{t_b}$. Once again the case for $a \notin Q_1^i$ is clear since this part of the representations (and morphisms) stays the same (in the picture in the definition, this is the part on the outside of the circle). For $a \in Q_1^i$ (with inverse arrow b) we have that the following diagram commutes

$$\begin{tikzcd}
V_i' \arrow[r, "v_b'"] \arrow[d, "\phi_i'"'] & V_{h_b}' = V_{t_a} \arrow[d, "\phi_{h_b}' = \phi_{t_a}"] \\
W_i' \arrow[r, "w_b'"] & W_{h_b}' = W_{t_a}.
\end{tikzcd}$$

Indeed, for $x = (x_{t_c})_{c \in Q_1^i} \in \ker \xi_i = V_i'$ we have:

$$w_b' \phi_i'(x) = w_b'((\phi_{t_c}(x_{t_c}))_{c \in Q_1^i}) = \phi_{t_a}(x_{t_a}) = \phi_{h_b}'(x_{t_a}) = \phi_{h_b}'(v_b'(x_{t_c})_{c \in Q_1^i}) = \phi_{h_b}'(v_b'(x)).$$

So far, we have shown that $\mathcal{R}_i^+ \mathbb{V} \in \operatorname{rep}_{\mathrm{k}}(s_i Q)$ and $\mathcal{R}_i^+ \phi \colon \mathcal{R}_i^+ \mathbb{V} \to \mathcal{R}_i^+ \mathbb{W}$ are well-defined. Thus it remains to show that $\mathcal{R}_i^+$ preserves identity morphisms and composition of morphisms. Let $\mathbb{V} \in \operatorname{rep}_{\mathrm{k}}(Q)$ be a representation and let $\mathrm{id}_{\mathbb{V}}$ be the identity morphism. For $j \neq i$ we have that $(\mathcal{R}_i^+ \mathrm{id}_{\mathbb{V}})_j = \mathrm{id}_{\mathbb{V}_j}$ and for $j = i$ we have $(\mathcal{R}_i^+ \mathrm{id}_{\mathbb{V}})_i = \bigoplus_{a \in Q_1^i} \mathrm{id}_{V_{t_a}} |_{\ker \xi_i} = \mathrm{id}_{\ker \xi_i} = \mathrm{id}_{V_i}'$. Thus we have that $\mathcal{R}_i^+ \mathrm{id}_{\mathbb{V}} = \mathrm{id}_{\mathcal{R}_i^+ \mathbb{V}}$. Now let $\phi \colon \mathbb{U} \to \mathbb{V}$ and $\psi \colon \mathbb{V} \to \mathbb{W}$ be morphisms. We want to show that $\mathcal{R}_i^+(\psi \phi) = (\mathcal{R}_i^+ \psi)(\mathcal{R}_i^+ \phi)$. For $j \neq i$ we have that $(\mathcal{R}_i^+(\psi\phi))_j = (\psi\phi)_j = \psi_j \phi_j = (\mathcal{R}_i^+ \psi)_j (\mathcal{R}_i^+ \phi)_j$. For $j = i$ we have:

$$\begin{aligned}
(\mathcal{R}_i^+(\psi\phi))_i &= \bigoplus_{a \in Q_1^i} (\psi\phi)_{t_a} |_{\ker \xi_i^U}^{\ker \xi_i^W} = \bigoplus_{a \in Q_1^i} \psi_{t_a} \phi_{t_a} |_{\ker \xi_i^U}^{\ker \xi_i^W} = \bigoplus_{a \in Q_1^i} \psi_{t_a} |_{\ker \xi_i^V}^{\ker \xi_i^W} \phi_{t_a} |_{\ker \xi_i^U}^{\ker \xi_i^V} \\
&= (\bigoplus_{a \in Q_1^i} \psi_{t_a} |_{\ker \xi_i^V}^{\ker \xi_i^W})(\bigoplus_{a \in Q_1^i} \phi_{t_a} |_{\ker \xi_i^U}^{\ker \xi_i^V}) = (\mathcal{R}_i^+ \psi)_i (\mathcal{R}_i^+ \phi)_i.
\end{aligned}$$

This concludes the proof that $\mathcal{R}_i^+$ is a covariant functor. The proof that $\mathcal{R}_i^-$ is a functor is analogous. ☐

Example 3.42. *We consider the quiver Q from Example 3.37. Let $\mathbb{V}$ be a representation given by*

$$V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{v_d} V_5.$$

By applying $\mathcal{R}_5^+$ to $\mathbb{V}$, we get the representation $\mathcal{R}_5^+ \mathbb{V}$

$$V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \hookleftarrow \ker(v_d).$$

Notice that only the vector space at the vertex 5 and the incident edge between the vertices 4 and 5 change! This is true in general by looking at the pictures in the definition of $\mathcal{R}_i^+$ and $\mathcal{R}_i^-$. There, only the vector space and the maps inside the (imaginary) circle change. Now we apply $\mathcal{R}_5^-$ to $\mathcal{R}_5^+ \mathbb{V}$ (the vertex 5 is now a source!) and get:

$$V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{\operatorname{mod} \ker(v_d)} V_4 / \ker(v_d)$$

We conclude that $\mathcal{R}_5^- \mathcal{R}_5^+ \mathbb{V} \cong \mathbb{V}$ if and only $V_5 = \operatorname{im}(v_d)$. Indeed, by the first isomorphism theorem, if $V_5 = \operatorname{im}(v_d) \cong V_4 / \ker(v_d)$, then the vector spaces at vertex 5 coincide and the map v_d and the projection map $\operatorname{mod} \ker(v_d)$ are the same map! In the case that $\operatorname{im}(v_d) \subsetneq V_5$, we have that $V_5 \cong \operatorname{im}(v_d) \oplus k^r$ for some $r \in \mathbb{N}$. Moreover, the projection map $\operatorname{mod} \ker(v_d)$ is isomorphic to the restriction of v_d to its image $\operatorname{im}(v_d) \cong V_4 / \ker(v_d)$. Therefore, we have that $\mathbb{V} \cong \mathcal{R}_5^- \mathcal{R}_5^+ \mathbb{V} \oplus \mathbb{S}_5^r$, where $\mathbb{S}_5$ is the simple representation defined in Example 2.16. Now we apply $\mathcal{R}_2^+$ to $\mathbb{V}$ and get the top row of the following commutative diagram:

$$\begin{tikzcd}
V_1 & \ker(v_a + v_b) \arrow[l, hook'] \arrow[r] \arrow[d, hook] & V_3 & V_4 \arrow[l, "v_c"'] \arrow[r, "v_d"] & V_5 \\
& V_1 \oplus V_3 \arrow[ul, "\pi_1"] \arrow[ur, "\pi_3"'] & & &
\end{tikzcd}$$

The maps π_1 and π_3 denote the canonical projections of $V_1 \oplus V_3$ to its summands. Again we notice that only the vector space at the vertex 2 and the maps belonging to the edges a and b change. Now, by applying $\mathcal{R}_2^-$ to $\mathcal{R}_2^+ \mathbb{V}$ we get the middle row of the following commutative diagram, where the lower vertical arrow is the canonical map from $V_1 \oplus V_3$ onto $\frac{V_1 \oplus \mathbb{V}_3}{\ker(v_a + v_b)} = \operatorname{coker} \zeta_2$. By the first isomorphism theorem, we get an injective map $\phi_2 \colon \frac{V_1 \oplus \mathbb{V}_3}{\ker(v_a + v_b)} \to V_2$.

$$\begin{tikzcd}
 & V_2 & & & \\
V_1 \arrow[ur, "v_a"] \arrow[r] \arrow[dr, "{(-,0)}"'] & \frac{V_1 \oplus V_3}{\ker(v_a+v_b)} \arrow[u, "\phi_2"] & V_3 \arrow[ul, "v_b"'] \arrow[l] \arrow[dl, "{(0,-)}"] & V_4 \arrow[l, "v_c"'] \arrow[r, "v_d"] & V_5 \\
 & V_1 \oplus V_3 \arrow[u] & & &
\end{tikzcd}$$

We thus conclude that $\mathcal{R}_2^- \mathcal{R}_2^+ \mathbb{V} \cong \mathbb{V}$ *if and only if* $V_2 = \operatorname{im}(v_a + v_b)$. *Indeed, by the first isomorphism theorem, if* $V_2 = \operatorname{im}(v_a+v_b) \cong (V_1 \oplus \mathbb{V}_3)/\ker(v_a+v_b)$, *then the vector spaces at the vertex 2 coincide. Moreover, the map* $V_1 \to \frac{V_1 \oplus \mathbb{V}_3}{\ker(v_a+v_b)}$ *in the middle row and the map* v_a *are isomorphic. Also the map* $\frac{V_1 \oplus \mathbb{V}_3}{\ker(v_a+v_b)} \leftarrow V_3$ *in the middle row and the map* v_b *are isomorphic. Therefore, we have that* $\mathcal{R}_2^- \mathcal{R}_2^+ \mathbb{V} \cong \mathbb{V}$. *In the case that* $\operatorname{im}(v_a + v_b) \subsetneq V_2$, *we have that* $V_2 \cong \operatorname{im}(v_a + v_b) \oplus k^r$ *for some* $r \in \mathbb{N}$. *Moreover, the maps from* V_1 *resp.* V_3 *to* $\mathcal{R}_2^- \mathcal{R}_2^+ \mathbb{V} \cong \mathbb{V}$ *(in the middle row) are isomorphic the restriction of* v_a *to its image* $\operatorname{im}(v_a)$ *resp. to the restriction of* v_b *to its image* $\operatorname{im}(v_b)$. *Therefore, we have* $\mathbb{V} \cong \mathcal{R}_2^- \mathcal{R}_2^+ \mathbb{V} \oplus \mathbb{S}_2^r$.

3.3.1 Composition of Reflection Functors

Since $s_i s_i Q = Q$ (for a sink or source i) it is sensible to ask if the functors $\mathcal{R}_i^+$ and $\mathcal{R}_i^-$ are inverse to each other. This is not the case! Indeed, Example 3.42 is a counterexample in the cases where $V_5 \neq \operatorname{im}(v_d)$ resp. $v_2 \neq \operatorname{im}(v_a + v_b)$. Moreover, one can see that for the simple representation $\mathbb{S}_i$ we have $\mathcal{R}_i^+ \mathbb{S}_i = 0$ (for a sink i), respectively $\mathcal{R}_i^- \mathbb{S}_i = 0$ (for a source i). In the following example, we show this for an explicit quiver Q, however, the proof for a general quiver Q is analogous.

Example 3.43. *We again consider the quiver* Q *from Example 3.37. From Example 3.42 we get that* $(\mathcal{R}_5^+ \mathcal{S}_5)_5 = \ker(v_d) \subseteq V_4$ *but* $V_4 = 0$. *Thus we have that* $\mathcal{R}_5^+ \mathcal{S}_5 = 0$. *For the sink vertex 2, we get that* $(\mathcal{R}_2^+ \mathcal{S}_2)_2 = \ker(v_a + v_b) \subseteq V_1 \oplus V_3 = 0$ *and thus* $\mathcal{R}_2^+ \mathcal{S}_2 = 0$.

Even though $\mathcal{R}_i^+$ and $\mathcal{R}_i^-$ are not inverse to each other, we consider their compositions and observe what is missing to be equal to the identity. The following lemma marks a first step towards that.

Lemma 3.44. *Let* Q *be a quiver and let* $\mathbb{V}$ *be a finite-dimensional representation of* Q.

1. *Let* i *be a sink of* Q. *There exists a canonical monomorphism* $\iota_i \mathbb{V} \colon \mathcal{R}_i^- \mathcal{R}_i^+ \mathbb{V} \to \mathbb{V}$.
2. *Let* i *be a source of* Q. *There exists a canonical epimorphism* $\pi_i \mathbb{V} \colon \mathbb{V} \to \mathcal{R}_i^+ \mathcal{R}_i^- \mathbb{V}$.

Proof. Since $s_i s_i Q = Q$ we have that $\mathcal{R}_i^- \mathcal{R}_i^+ \mathbb{V}, \mathcal{R}_i^+ \mathcal{R}_i^- \mathbb{V} \in \operatorname{rep}_{\mathrm{k}}(s_i s_i Q) = \operatorname{rep}_{\mathrm{k}}(Q)$. We now prove that the maps are indeed morphisms (of representations) with the desired properties (monomorphism resp. epimorphism).

1. Let $(\iota_i \mathbb{V})_j = \mathrm{id}_{V_j}$ for $j \neq i$ and consider

$$\begin{aligned}
(\mathcal{R}_i^- \mathcal{R}_i^+ \mathbb{V})_i &= \operatorname{coker} \zeta_i = \operatorname{coker}(\ker \xi_i \to \bigoplus_{a \in (s_i Q)_1^i} V'_{h_a}) = \operatorname{coker}(\ker \xi_i \hookrightarrow \bigoplus_{a \in Q_1^i} V_{t_a}) \\
&= (\bigoplus_{a \in Q_1^i} V_{t_a}) / \ker \xi_i \overset{\phi}{\cong} \operatorname{im} \xi_i \subseteq V_i.
\end{aligned}$$

Using the canonical isomorphism ϕ, we set $(\iota_i \mathbb{V})_i \colon (\mathcal{R}_i^- \mathcal{R}_i^+ \mathbb{V})_i \xrightarrow{\phi} \operatorname{im} \xi_i \hookrightarrow V_i$. First notice that all the maps $(\iota_i \mathbb{V})_j$ are well-defined. Indeed, for $j \neq i$ we have $(\mathcal{R}_i^- \mathcal{R}_i^+ \mathbb{V})_j = V_j$ and thus $(\iota_i \mathbb{V})_j = \mathrm{id}_{V_j}$ is well-defined. For $j = i$, we have that $(\iota_i \mathbb{V})_i$ is well-defined by definition. Notice that all the maps are injective. Therefore, $\iota_i \mathbb{V}$ is a monomorphism, if we can show that it is a morphism. Let us now consider an arrow $a \in Q_1$. If $a \notin Q_1^i$ we have that the following commutative diagram.

$$\begin{tikzcd}
(\mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V})_{t_a} = V_{t_a} \arrow[r, "v_a"] \arrow[d, "\mathrm{id}_{V_{t_a}}"'] & V_{h_a} = (\mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V})_{h_a} \arrow[d, "\mathrm{id}_{V_{h_a}}"] \\
V_{t_a} \arrow[r, "v_a"] & V_{h_a}
\end{tikzcd}$$

For $a \in Q_1^i$ we have the following diagram.

$$\begin{tikzcd}
(\mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V})_{t_a} = V_{t_a} \arrow[r, "f"] \arrow[d, "\mathrm{id}_{V_{t_a}}"'] & \operatorname{im}\xi_i \cong (\mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V})_i \arrow[d, "(\iota_i\mathbb{V})_i"] \\
V_{t_a} \arrow[r, "v_a"] & V_i
\end{tikzcd}$$

Using the isomorphism $(\mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V})_i \overset{\phi}{\cong} \operatorname{im}\xi_i, (x_{t_c})_{c\in Q_1^i} + \ker\xi_i \mapsto \sum_{c\in Q_1^i} v_c(x_{t_c})$ we get that $f(x_{t_a}) = v_a(x_{t_a}) + \sum_{c\in Q_1^i, c\neq a} v_c(0) = v_a(x_{t_a}) \in \operatorname{im}\xi_i$. Therefore, the above diagram commutes. This shows that $\iota_i\mathbb{V}$ is a morphism (and thus a monomorphism).

2. Let $(\pi_i\mathbb{V})_j = \mathrm{id}_{V_j}$ for $j \neq i$. Now consider

$$(\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V})_i = \ker\xi_i = \ker(\bigoplus_{a\in(s_iQ)_1^i} V'_{t_a} \to \operatorname{coker}\zeta_i) = \ker(\bigoplus_{a\in Q_1^i} V_{h_a} \to \operatorname{coker}\zeta_i)$$
$$= \ker(\bigoplus_{a\in Q_1^i} V_{h_a} \to \bigoplus_{a\in Q_1^i} V_{h_a}/\operatorname{im}\zeta_i) \overset{\psi}{\cong} \operatorname{im}\zeta_i \subseteq \bigoplus_{a\in Q_1^i} V_{h_a}.$$

Using the canonical isomorphism ψ, we set $(\pi_i\mathbb{V})_i \colon V_i \xrightarrow{\zeta_i} \operatorname{im}\zeta_i \xrightarrow{\psi^{-1}} (\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V})_i$. First notice that all the maps $(\pi_i\mathbb{V})_j$ are well-defined. Indeed, for $j \neq i$ we have $(\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V})_j = V_j$ and thus $(\pi_i\mathbb{V})_j = \mathrm{id}_{V_j}$ is well-defined. For $j = i$, we have that $(\pi_i\mathbb{V})_i$ is well-defined by definition. Notice that all the maps are surjective. Therefore, $\pi_i\mathbb{V}$ is an epimorphism, if we can show that it is a morphism. Let us now consider an arrow $a \in Q_1$. If $a \notin Q_1^i$ we have that the following commutative diagram.

$$\begin{tikzcd}
V_{t_a} \arrow[r, "v_a"] \arrow[d, "\mathrm{id}_{V_{t_a}}"'] & V_{h_a} \arrow[d, "\mathrm{id}_{V_{h_a}}"] \\
(\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V})_{t_a} = V_{t_a} \arrow[r, "v_a"] & V_{h_a} = (\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V})_{h_a}
\end{tikzcd}$$

For $a \in Q_1^i$ we have the following diagram.

$$\begin{tikzcd}
V_i \arrow[r, "v_a"] \arrow[d, "(\pi_i\mathbb{V})_i"'] & V_{h_a} \arrow[d, "\mathrm{id}_{V_{h_a}}"] \\
(\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V})_i \cong \operatorname{im}\zeta_i \arrow[r, "g"] & V_{h_a} = (\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V})_{h_a}
\end{tikzcd}$$

Using the isomorphism $(\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V})_i \overset{\psi}{\cong} \operatorname{im}\zeta_i$ we get that $g((v_c(x))_{c\in Q_1^i}) = v_a(x) \in V_{h_a}$ (for $x \in V_i$). Therefore, the above diagram commutes. This shows that $\pi_i\mathbb{V}$ is a morphism (and thus an epimorphism). $\square$

Example 3.45. *We have already seen examples for the canonical monomorphism $\iota_i\mathbb{V}$ in Example 3.42. Indeed, for the sink vertex 5, we have an inclusion $\phi_5 \colon V_4/\ker(v_d) \hookrightarrow V_5$. By setting $\phi_i = \mathrm{id}_{V_i}$ for $i \in \{1,2,3,4\}$, we get that $\iota_5\mathbb{V} = (\phi_i)_{1\leq i\leq 5}$. For the sink vertex 2, we also have a canonical inclusion at the vertex 2, given by the map $\phi_2 \colon \frac{V_1\oplus V_3}{\ker(v_a+v_b)} \to V_2$. Now we set $\phi_i = \mathrm{id}_{V_i}$ for $i \in \{1,3,4,5\}$ and get that $\iota_2\mathbb{V} = (\phi_i)_{1\leq i\leq 5}$.*

To better understand the morphisms defined in Lemma 3.44, we observe, where $\iota_i\mathbb{V}$ fails to be surjective. Notice that this is described by the representation $\operatorname{coker}(\iota_i\mathbb{V})$. Analogously we observe, where $\pi_i\mathbb{V}$ fails to be injective by looking at the representation $\ker(\pi_i\mathbb{V})$.

Lemma 3.46. *Let Q be a quiver and let $\mathbb{V}$ be a finite-dimensional representation of Q.*

1. *Let i be a sink of Q. Then the representation $\operatorname{coker}(\iota_i\mathbb{V})$ is supported at the vertex i. This means that $\operatorname{coker}(\iota_i\mathbb{V})_j = 0$ for all $j \neq i$ and thus all maps $\tilde{v_a} = 0$ are trivial (for $a \in Q_1$). Moreover, we have that $\operatorname{coker}(\iota_i\mathbb{V})_i \cong \operatorname{coker}\xi_i$, where ξ_i is the map given in the definition of $\mathcal{R}_i^+\mathbb{V}$.*
2. *Let i be a source of Q. Then the representation $\ker(\pi_i\mathbb{V})$ is supported at the vertex i. This means that $\ker(\pi_i\mathbb{V})_j = 0$ for all $j \neq i$ and thus all maps $\tilde{v_a} = 0$ are trivial (for $a \in Q_1$). Moreover, we have that $\ker(\pi_i\mathbb{V})_i \cong \ker\zeta_i$, where ζ_i is the map given in the definition of $\mathcal{R}_i^-\mathbb{V}$.*

Proof. 1. For $j \neq i$ we have $(\iota_i\mathbb{V})_j = \mathrm{id}_{V_j}$ and thus $\operatorname{coker}(\iota_i\mathbb{V})_j = V_j/V_j = 0$. This shows that $\operatorname{coker}(\iota_i\mathbb{V})$ is supported at the vertex i. Moreover, in the proof of Lemma 3.44 we showed that we have $(\mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V})_i \cong \operatorname{im}\xi_i$. Thus $\operatorname{coker}(\iota_i\mathbb{V})_i \cong V_i/\operatorname{im}\xi_i = \operatorname{coker}\xi_i$.

2. For $j \neq i$ we have $(\pi_i\mathbb{V})_j = \mathrm{id}_{V_j}$ and thus $\ker(\pi_i\mathbb{V})_j = V_j/V_j = 0$. This shows that $\ker(\pi_i\mathbb{V})$ is supported at the vertex i. Moreover, in the proof of Lemma 3.44 we showed that we have $(\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V})_i \cong \operatorname{im}\zeta_i$. Thus $\ker(\pi_i\mathbb{V})_i = \ker(V_i \to \operatorname{im}\zeta_i) = \ker\zeta_i$. $\square$

Remark 3.47. *Since $\operatorname{coker}(\iota_i\mathbb{V})$ and $\ker(\pi_i\mathbb{V})$ are supported at the vertex i, they are both direct sums of copies of the simple representation $\mathbb{S}_i$.*

Example 3.48. *Again, we consider the quiver from Example 3.42 and the induced monomorphisms $\iota_2\mathbb{V}$ and $\iota_5\mathbb{V}$ that are described in Example 3.45. Using Lemma 3.46, we get the following representations.*

$$\mathcal{R}_5^-\mathcal{R}_5^+\mathbb{V}: \qquad V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{\mathrm{mod}\,\ker(v_d)} \frac{V_4}{\ker(v_d)} \cong \operatorname{im}(v_d)$$

$$\operatorname{coker}(\iota_5\mathbb{V}): \qquad 0 \xrightarrow{0} 0 \xleftarrow{0} 0 \xleftarrow{0} 0 \xrightarrow{0} \operatorname{coker}(v_d)$$

We observe that $\mathbb{V} \cong \mathcal{R}_5^-\mathcal{R}_5^+\mathbb{V} \oplus \operatorname{coker}(\iota_5\mathbb{V})$. Similarly, we get the representations.

$$\mathcal{R}_2^+\mathcal{R}_2^-\mathbb{V}: \qquad \begin{tikzcd}
 & V_2 & & & \\
V_1 \arrow[ur, "v_a"] \arrow[r] \arrow[dr, "{(-,0)}"'] & \frac{V_1\oplus V_3}{\ker(v_a+v_b)} \arrow[u, "\phi_2"] & V_3 \arrow[ul, "v_b"'] \arrow[l] \arrow[dl, "{(0,-)}"] & V_4 \arrow[l, "v_c"'] \arrow[r, "v_d"] & V_5 \\
 & V_1 \oplus V_3 \arrow[u] & & &
\end{tikzcd}$$

$$\operatorname{coker}(\iota_2\mathbb{V}): \qquad 0 \xrightarrow{0} \operatorname{coker}(v_a+v_b) \xleftarrow{0} 0 \xleftarrow{0} 0 \xrightarrow{0} 0$$

Again, we have $\mathbb{V} \cong \mathcal{R}_2^-\mathcal{R}_2^+\mathbb{V} \oplus \operatorname{coker}(\iota_2\mathbb{V})$. This shows that the representations $\operatorname{coker}(\iota_2\mathbb{V})$ and $\operatorname{coker}(\iota_5\mathbb{V})$ are supported at the vertex 2 resp. 5 and that they are of the form described in Lemma 3.46. In addition, we see that they are summands of $\mathbb{V}$.

We are now finally ready to investigate how $\mathcal{R}_i^+$, $\mathcal{R}_i^-$ and the compositions $\mathcal{R}_i^-\mathcal{R}_i^+$ and $\mathcal{R}_i^+\mathcal{R}_i^-$ operate on the summands of a representation $\mathbb{V}$. We have seen before that $\mathcal{R}_i^+$ and $\mathcal{R}_i^-$ are not inverse to each other. However, in Example 3.42, we have already seen that $\mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V}$ (if i is a sink) resp. $\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V}$ (if i is a source) is a summand of $\mathbb{V}$ and their ‘difference’ is a direct sum of copies of the simple representation $\mathbb{S}_i$. This suggests that $\mathcal{R}_i^-\mathcal{R}_i^+$ resp. $\mathcal{R}_i^+\mathcal{R}_i^-$ annihilates the summands isomorphic to $\mathbb{S}_i$. Since $\operatorname{coker}(\iota_i\mathbb{V})$ and $\ker(\pi_i\mathbb{V})$ are direct sums of copies of $\mathbb{S}_i$, we suspect that $\operatorname{coker}(\iota_i\mathbb{V})$ and $\ker(\pi_i\mathbb{V})$ are those missing copies of $\mathbb{S}_i$ in $\mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V}$ resp. $\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V}$. Note that we have already seen this in Example 3.48. It turns out that this is true in general, which is the content of the following theorem.

Theorem 3.49. *Let Q be a finite connected quiver and let $\mathbb{V}, \mathbb{W}$ be representations of Q. Then for any sink or source i we have:*

1. $\mathcal{R}_i^{\pm}(\mathbb{V} \oplus \mathbb{W}) \cong \mathcal{R}_i^{\pm}\mathbb{V} \oplus \mathcal{R}_i^{\pm}\mathbb{W}$.
2. *If i is a sink, then we have* $\mathbb{V} \cong (\mathcal{R}_i^{-}\mathcal{R}_i^{+}\mathbb{V}) \oplus \operatorname{coker}(\iota_i\mathbb{V})$.
3. *If i is a source, then we have* $\mathbb{V} \cong (\mathcal{R}_i^{+}\mathcal{R}_i^{-}\mathbb{V}) \oplus \ker(\pi_i\mathbb{V})$.
4. *If $\mathbb{V}$ is finite-dimensional and* $\operatorname{coker}(\iota_i\mathbb{V}) = 0$*, then* $\underline{\dim}\mathcal{R}_i^{+}\mathbb{V} = \sigma_i(\underline{\dim}\mathbb{V})$.
5. *If $\mathbb{V}$ is finite-dimensional and* $\ker(\pi_i\mathbb{V}) = 0$*, then* $\underline{\dim}\mathcal{R}_i^{-}\mathbb{V} = \sigma_i(\underline{\dim}\mathbb{V})$.

Proof. 1. We refer to this property as 'preserving direct sums'. We show that it is true for vector spaces, but also for morphisms. For the vector spaces, we observe that for $j \neq i$ we have $(\mathcal{R}_i^{\pm}(\mathbb{V} \oplus \mathbb{W}))_j = (\mathbb{V} \oplus \mathbb{W})_j = V_j \oplus W_j = \mathbb{V}_j \oplus \mathbb{W}_j = (\mathcal{R}_i^{\pm}\mathbb{V})_j \oplus (\mathcal{R}_i^{\pm}\mathbb{W})_j$. For $j = i$ we have

$$\begin{aligned}
(\mathcal{R}_i^{+}(\mathbb{V} \oplus \mathbb{W}))_i &= \ker(\xi_i^{V \oplus W} \colon \bigoplus_{a \in Q_1^i} (\mathbb{V} \oplus \mathbb{W})_{t_a} \to (\mathbb{V} \oplus \mathbb{W})_i) \\
&= \ker(\xi_i^{V \oplus W} \colon \bigoplus_{a \in Q_1^i} V_{t_a} \oplus W_{t_a} \to V_i \oplus W_i) \\
&= \ker(\xi_i^{V} \colon \bigoplus_{a \in Q_1^i} V_{t_a} \to V_i) \oplus \ker(\xi_i^{W} \colon \bigoplus_{a \in Q_1^i} W_{t_a} \to W_i) = (\mathcal{R}_i^{+}\mathbb{V})_i \oplus (\mathcal{R}_i^{+}\mathbb{W})_i.
\end{aligned}$$

Similarly, for $j = i$ we have

$$\begin{aligned}
&\operatorname{im}(\zeta_i^{V \oplus W} \colon (\mathbb{V} \oplus \mathbb{W})_i \to \bigoplus_{a \in Q_1^i} (\mathbb{V} \oplus \mathbb{W})_{h_a}) = \operatorname{im}(\zeta_i^{V \oplus W} \colon V_i \oplus W_i \to \bigoplus_{a \in Q_1^i} V_{h_a} \oplus W_{h_a}) \\
&\operatorname{im}(\zeta_i^{V} \colon V_i \to \bigoplus_{a \in Q_1^i} V_{h_a}) \oplus \operatorname{im}(\zeta_i^{W} \colon W_i \to \bigoplus_{a \in Q_1^i} W_{h_a}).
\end{aligned}$$

This yields the equation

$$\begin{aligned}
(\mathcal{R}_i^{-}(\mathbb{V} \oplus \mathbb{W}))_i = \operatorname{coker} \zeta_i^{V \oplus W} &= (\bigoplus_{a \in Q_1^i} (\mathbb{V} \oplus \mathbb{W})_{h_a}) / \operatorname{im} \zeta_i^{V \oplus W} \\
&= [(\bigoplus_{a \in Q_1^i} V_{h_a}) / \operatorname{im} \zeta_i^{V}] \oplus [(\bigoplus_{a \in Q_1^i} W_{h_a}) / \operatorname{im} \zeta_i^{W}] \\
&= \operatorname{coker} \zeta_i^{V} \oplus \operatorname{coker} \zeta_i^{W} = (\mathcal{R}_i^{-}\mathbb{V})_i \oplus (\mathcal{R}_i^{-}\mathbb{W})_i.
\end{aligned}$$

We now consider a pair of morphisms $\phi \colon \mathbb{V}_1 \to \mathbb{W}_1, \psi \colon \mathbb{V}_2 \to \mathbb{W}_2$. For $j \neq i$ we have $(\mathcal{R}_i^{\pm}(\phi + \psi))_j = (\phi + \psi)_j = \phi_j + \psi_j = (\mathcal{R}_i^{\pm}\phi)_j + (\mathcal{R}_i^{\pm}\psi)_j$. Now for $j = i$ we have

$$\begin{aligned}
(\mathcal{R}_i^{+}(\phi + \psi))_i &= \bigoplus_{a \in Q_1^i} (\phi + \psi)_{t_a}|_{\ker \xi_i^{V_1 \oplus V_2}} = \bigoplus_{a \in Q_1^i} \phi_{t_a} + \psi_{t_a}|_{\ker \xi_i^{V_1 \oplus V_2}} \\
&= \bigoplus_{a \in Q_1^i} \phi_{t_a}|_{\ker \xi_i^{V_1}} + \psi_{t_a}|_{\ker \xi_i^{V_2}} = \bigoplus_{a \in Q_1^i} \phi_{t_a}|_{\ker \xi_i^{V_1}} + \bigoplus_{a \in Q_1^i} \psi_{t_a}|_{\ker \xi_i^{V_2}} \\
&= (\mathcal{R}_i^{+}\phi)_i + (\mathcal{R}_i^{+}\psi)_i.
\end{aligned}$$

Here, we used that $\ker \xi_i^{V_1 \oplus V_2}$ splits into $\ker \xi_i^{V_1} \oplus \ker \xi_i^{V_2}$. We already showed this in the splitting part of the vector spaces. Similarly for $j = i$ we have

$$\begin{aligned}
(\mathcal{R}_i^{-}(\phi + \psi))_i &= \bigoplus_{a \in Q_1^i} (\phi + \psi)_{h_a}|_{\operatorname{coker} \zeta_i^{V_1 \oplus V_2}} = \bigoplus_{a \in Q_1^i} \phi_{h_a} + \psi_{h_a}|_{\operatorname{coker} \zeta_i^{V_1 \oplus V_2}} \\
&= \bigoplus_{a \in Q_1^i} \phi_{h_a}|_{\operatorname{coker} \zeta_i^{V_1}} + \psi_{h_a}|_{\operatorname{coker} \zeta_i^{V_2}} = \bigoplus_{a \in Q_1^i} \phi_{h_a}|_{\operatorname{coker} \zeta_i^{V_1}} + \bigoplus_{a \in Q_1^i} \psi_{h_a}|_{\operatorname{coker} \zeta_i^{V_2}} \\
&= (\mathcal{R}_i^{-}\phi)_i + (\mathcal{R}_i^{-}\psi)_i.
\end{aligned}$$

In the third equality, we used that the $\operatorname{coker} \zeta_i^{V_1 \oplus V_2}$ splits into $\operatorname{coker} \zeta_i^{V_1} \oplus \operatorname{coker} \zeta_i^{V_2}$ (already proven in the splitting part of the vector spaces).

2. To prove the direct sum decomposition, we first need to define two inclusion morphisms $\iota_i\mathbb{V}\colon \mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V} \to \mathbb{V}$ (already defined) and $\rho\colon \operatorname{coker}(\iota_i\mathbb{V}) \to \mathbb{V}$. We also need to define two projection morphisms $\pi_1\colon \mathbb{V} \to \mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V}$ and $\pi_2\colon \mathbb{V} \to \operatorname{coker}(\iota_i\mathbb{V})$. From Lemma 3.46 we have $\operatorname{coker}(\iota_i\mathbb{V})_i \cong \operatorname{coker}\xi_i$. We notice that the projection map $\rho_i'\colon V_i \to \operatorname{coker}\xi_i = V_i/(\operatorname{im}\xi_i)$ has a section, i.e. a map $\rho_i\colon \operatorname{coker}\xi_i \to V_i$ s.t. $\rho_i'\rho_i = \mathrm{id}_{\operatorname{coker}\xi_i}$. Indeed, let $\{v_j + \operatorname{im}\xi_i\}_{j\in I}$ be a basis of $\operatorname{coker}(\iota_i\mathbb{V})$. Using the surjectivity of ρ_i', we can choose $\tilde{v_j} \in V_i$ s.t. $\rho_i'(\tilde{v_j}) = v_j + \operatorname{im}\xi_i$ for all $j \in I$. Then the map induced by $v_j + \operatorname{im}\xi_i \mapsto \tilde{v_j}$ for $j \in I$ is a well-defined linear map and we have that $\rho_i'\rho_i(v_j + \operatorname{im}\xi_i) = \rho_i'(\tilde{v_j}) = v_j + \operatorname{im}\xi_i$. Using the map ρ_i, we get a morphism $\rho\colon \operatorname{coker}(\iota_i\mathbb{V}) \to \mathbb{V}$ by setting $\rho_j = 0$ for $j \neq i$ and for $j = i$ we set $\rho_i\colon \operatorname{coker}\xi_i \to V_i$. Notice that ρ defines a morphism since the following diagram commutes for every $a \in Q_1^i$ (for all other edges a from j_1 to j_2, the commutativity condition follows directly from $\rho_{j_1} = 0$ and $\rho_{j_2} = 0$).

$$\begin{tikzcd}
\operatorname{coker}\xi_i \arrow[d, hook, "\rho_i"'] & 0 \arrow[l, "0"'] \arrow[d, "\rho_{t_a}=0"] \\
V_i & V_{t_a} \arrow[l, "v_a"']
\end{tikzcd}$$

We also have projection morphisms $\pi_1\colon \mathbb{V} \to \mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V}$ and $\pi_2\colon \mathbb{V} \to \operatorname{coker}(\iota_i\mathbb{V})$ defined as follows. In the proof of Lemma 3.44 we have seen that $(\mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V})_i \cong \operatorname{im}\xi_i \subseteq V_i$. Thus we can view $\mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V}$ as a subrepresentation of $\mathbb{V}$ and we can define π_1 to be the projection morphism. We define $(\pi_2)_j\colon V_j \to 0$ to be the trivial map for $j \neq i$ and $(\pi_2)_i = \rho_i'\colon V_i \to \operatorname{coker}\xi_i = V_i/\operatorname{im}\xi_i$ is the canonical projection map. Now we check that this indeed yields a direct sum decomposition. We have that $\pi_1\iota_i\mathbb{V} = \mathrm{id}_{\mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V}}$ since $\mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V}$ can be viewed as a subrepresentation of $\mathbb{V}$. Moreover, we have

$$(\pi_2\rho)_j = \begin{cases} \rho_i'\rho_i = \mathrm{id}_{\operatorname{coker}\xi_i} & \text{if } j = i, \\ 0 = \mathrm{id}_0 = \mathrm{id}_{\operatorname{coker}(\iota_i\mathbb{V})_j} & \text{if } j \neq i. \end{cases}$$

This shows that $\pi_2\rho = \mathrm{id}_{\operatorname{coker}(\iota_i\mathbb{V})}$. It remains to show that $\iota_i\mathbb{V}\pi_1 + \rho\pi_2 = \mathrm{id}_{\mathbb{V}}$. Indeed, for $j \neq i$ we compute $(\iota_i\mathbb{V}\pi_1)_j = \mathrm{id}_{V_j}$ and $(\rho\pi_2)_j = 0$. We also have $(\iota_i\mathbb{V}\pi_1)_i\colon V_i \to \operatorname{im}\xi_i \hookrightarrow V_i$ and $(\rho\pi_2)_i\colon V_i \to \operatorname{coker}\xi_i \hookrightarrow V_i$. Thus for all $v = v_1 + v_2 \in \operatorname{im}\xi_i \oplus (\operatorname{im}\xi_i)^\perp = V_i$ we have

$$(\iota_i\mathbb{V}\pi_1 + \rho\pi_2)_i(v) = \iota_i(v_1) + \rho(v_2 + \operatorname{im}\xi_i) = v_1 + v_2 = v.$$

Therefore, we conclude that $\mathbb{V} \cong (\mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V}) \oplus \operatorname{coker}(\iota_i\mathbb{V})$.

3. Again, we need to define two inclusion morphisms and two projection morphisms. We have already defined the projection morphism $\pi_i\mathbb{V}\colon \mathbb{V} \to \mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V}$. Using that $\ker(\pi_i\mathbb{V})_i \cong \ker\zeta_i$ (Lemma 3.46), we can view $\ker(\pi_i\mathbb{V})$ as a subrepresentation of $\mathbb{V}$ and thus we can define the morphism $\pi_2\colon \mathbb{V} \to \ker(\pi_i\mathbb{V})$ to be the canonical projection onto the subrepresentation. We now define two inclusion morphisms $\iota_1\colon \mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V} \to \mathbb{V}$ and $\iota_2\colon \ker(\pi_i\mathbb{V}) \to \mathbb{V}$ as follows. We know from the proof of Lemma 3.46 that $(\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V})_i \cong \operatorname{im}\zeta_i$. We claim that the map $\zeta_i\colon V_i \to \operatorname{im}\zeta_i$ has a section, i.e. a map $\phi\colon \operatorname{im}\zeta_i \to V_i$ s.t. $\zeta_i\phi = \mathrm{id}_{\operatorname{im}\zeta_i}$. Similarly, as in part 2, we can define such a map on a basis of $\operatorname{im}\zeta_i$ using the surjectivity of ζ_i on the subspace $\operatorname{im}\zeta_i \cong (\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V})_i$. Using the map ϕ, we get a morphism $\iota_1\colon \mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V} \to \mathbb{V}$ by setting $(\iota_1)_j = \mathrm{id}_{V_j}$ for $j \neq i$ and for $j = i$, we use the map ϕ. This defines a morphism since the following diagram commutes for every $a \in Q_1^i$

$$\begin{tikzcd}
\operatorname{im}\zeta_i \arrow[r, "\pi_{V_{h_a}}"] \arrow[d, hook, "\phi"'] & V_{h_a} \arrow[d, "(\iota_1)_{h_a}=\mathrm{id}_{V_{h_a}}"] \\
V_i \arrow[r, "v_a"] & V_{h_a}.
\end{tikzcd}$$

Indeed, using that $v_a = \pi_{V_{h_a}}\zeta_i$ (where $\pi_{V_{h_a}}\colon \bigoplus_{c\in Q_1^i} V_{h_c} \to V_{h_a}$ is the projection map), we get that $v_a\phi = \pi_{V_{h_a}}\zeta_i\phi = \pi_{V_{h_a}}\mathrm{id}_{\operatorname{im}\zeta_i} = \pi_{V_{h_a}}|_{\operatorname{im}\zeta_i}$. Finally, using that $\ker(\pi_i\mathbb{V})_i \cong \ker\zeta_i$, we can view $\ker(\pi_i\mathbb{V})$ as a subrepresentation of $\mathbb{V}$ and thus we can define $\iota_2\colon \ker(\pi_i\mathbb{V}) \to \mathbb{V}$ to be the canonical inclusion morphism. Now we check that these four morphisms indeed yield a direct sum decomposition. We have that

$$(\pi_i\mathbb{V})(\iota_1) = \begin{cases} \pi_i\mathbb{V}\phi = \zeta_i\phi = \mathrm{id}_{\mathrm{im}\,\zeta_i} & \text{if } j = i, \\ \mathrm{id}_{V_j}(\iota_1)_j = \mathrm{id}_{V_j}\mathrm{id}_{V_j} = \mathrm{id}_{V_j} & \text{if } j \neq i. \end{cases}$$

This shows that $(\pi_i\mathbb{V})(\iota_1) = \mathrm{id}_{\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V}}$. We also have that $\pi_2\iota_2 = \mathrm{id}_{\ker(\pi_i\mathbb{V})}$ since $\ker(\pi_i\mathbb{V})$ is a subrepresentation of $\mathbb{V}$. It remains to show that $(\iota_1)(\pi_i\mathbb{V}) + \iota_2\pi_2 = \mathrm{id}_{\mathbb{V}}$. Indeed, for $j \neq i$ we compute $(\iota_1)_j(\pi_i\mathbb{V})_j = \mathrm{id}_{V_j}\mathrm{id}_{V_j} = \mathrm{id}_{V_j}$ and $(\iota_2\pi_2)_j = 0$. We have $(\iota_2\pi_2)_i\colon V_i \to \ker\zeta_i \hookrightarrow V_i$ and $(\iota_1)_i(\pi_i\mathbb{V})_i\colon V_1 \to \mathrm{im}\,\zeta_i \hookrightarrow V_i$ and thus for all $v = v_1 + v_2 \in \ker\zeta_i \oplus (\ker\zeta_i)^\perp = V_i$ we have

$$(\iota_1)_i(\pi_i\mathbb{V})_i(v) + (\iota_2\pi_2)_i(v) = \phi(v_2) + \iota_2(v_1) = v_2 + v_1 = v,$$

where we used that $(\ker\zeta_i)^\perp \cong V_i/\ker\zeta_i \cong \mathrm{im}\,\zeta_i$. We conclude that $\mathbb{V} \cong (\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V}) \oplus \ker(\pi_i\mathbb{V})$.

4. If $\mathrm{coker}(\iota_i\mathbb{V}) = 0$, then we have $\mathrm{coker}(\iota_i\mathbb{V})_i \cong \mathrm{coker}\,\xi_i = 0$ and thus we have that $\mathrm{im}\,\xi_i = V_i$. Using the dimension formula for finite-dimensional linear maps we get

$$\begin{aligned} \dim(\mathcal{R}_i^+\mathbb{V})_i &= \dim\ker\xi_i = \dim(\bigoplus_{a\in Q_1^i} V_{t_a}) - \dim\mathrm{im}\,\xi_i = \dim(\bigoplus_{a\in Q_1^i} V_{t_a}) - \dim V_i \\ &= \sum_{a\in Q_1^i} \dim V_{t_a} - \dim V_i \overset{(1)}{=} \sum_{\substack{j=1\\ j\neq i}} d_{ji}\dim V_j - \dim V_i = \sigma_i(\underline{\dim}\mathbb{V})_i, \end{aligned}$$

where d_{ji} is the number of arrows between the vertices i and j. In Equation (1), we used that there are no loops at i since i is a sink. Therefore, we have that $V_{t_a} \neq V_i$ for all $a \in Q_1^i$. For all $j \neq i$ we also have $\dim(\mathcal{R}_i^+\mathbb{V})_j = \dim V_j = \sigma_i(\underline{\dim}\mathbb{V})_j$ and thus we have $\underline{\dim}\mathcal{R}_i^+\mathbb{V} = \sigma_i(\underline{\dim}\mathbb{V})$.

5. If $\ker(\pi_i\mathbb{V}) = 0$, we then have $\ker(\pi_i\mathbb{V})_i = \ker\zeta_i = 0$ and from the dimension formula, we get:

$$\begin{aligned} \dim(\mathcal{R}_i^-\mathbb{V})_i &= \dim\mathrm{coker}\,\zeta_i = \dim(\bigoplus_{a\in Q_1^i} V_{h_a}) - \dim\mathrm{im}\,\zeta_i \\ &= \dim(\bigoplus_{a\in Q_1^i} V_{h_a}) - (\dim V_i - \dim\ker\zeta_i) = \sum_{a\in Q_1^i}\dim V_{h_a} - \dim V_i \\ &= \sum_{\substack{j=1\\ j\neq i}} d_{ji}\dim V_j - \dim V_i = \sigma_i(\underline{\dim}\mathbb{V})_i \end{aligned}$$

For $j \neq i$ we have $\dim(\mathcal{R}_i^-\mathbb{V})_j = \dim V_j = \sigma_i(\underline{\dim}\mathbb{V})_j$ and thus we get $\underline{\dim}\mathcal{R}_i^-\mathbb{V} = \sigma_i(\underline{\dim}\mathbb{V})$. □

3.3.2 Reflection Functors on Indecomposable Representations

We always have a decomposition into indecomposable representations by Theorem 2.33. By part (1) of Theorem 3.49, it suffices to observe how $\mathcal{R}_i^+$ and $\mathcal{R}_i^-$ operate on indecomposable representations.

Corollary 3.50. *Let Q be a finite connected quiver and $\mathbb{V}$ be an indecomposable representation.*

1. *If i is a sink, then we have two cases:*
   - *If $\mathbb{V} \cong \mathbb{S}_i$, then $\mathcal{R}_i^+\mathbb{V} = 0$ and $q_{s_iQ}(\underline{\dim}\mathcal{R}_i^+\mathbb{V}) = 0$.*
   - *If $\mathbb{V} \ncong \mathbb{S}_i$, then $\mathcal{R}_i^+\mathbb{V}$ is non-zero and indecomposable, $\mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V} \cong \mathbb{V}$ and the dimension vectors of $\mathbb{V}$ and $\mathcal{R}_i^+\mathbb{V}$ are related by $q_{s_iQ}(\underline{\dim}\mathcal{R}_i^+\mathbb{V}) = q_Q(\underline{\dim}\mathbb{V})$.*
2. *If i is a source, then we have two cases:*
   - *If $\mathbb{V} \cong \mathbb{S}_i$, then $\mathcal{R}_i^-\mathbb{V} = 0$ and $q_{s_iQ}(\underline{\dim}\mathcal{R}_i^-\mathbb{V}) = 0$.*
   - *If $\mathbb{V} \ncong \mathbb{S}_i$, then $\mathcal{R}_i^-\mathbb{V}$ is non-zero and indecomposable, $\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V} \cong \mathbb{V}$ and the dimension vectors of $\mathbb{V}$ and $\mathcal{R}_i^-\mathbb{V}$ are related by $q_{s_iQ}(\underline{\dim}\mathcal{R}_i^-\mathbb{V}) = q_Q(\underline{\dim}\mathbb{V})$.*

Proof. If $\mathbb{V} \cong \mathbb{S}_i$, we have already seen that $\mathcal{R}_i^+\mathbb{V} = 0$, respectively $\mathcal{R}_i^-\mathbb{V} = 0$. Then clearly $q_{s_iQ}(\underline{\dim}\mathcal{R}_i^{\pm}\mathbb{V}) = q_{s_iQ}(0) = 0$. Now let $\mathbb{V} \ncong \mathbb{S}_i$. If i is a sink, $\mathbb{V} \cong (\mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V}) \oplus \operatorname{coker}(\iota_i\mathbb{V})$. Since $\mathbb{V}$ is irreducible, one of those summands needs to be trivial and since $\operatorname{coker}(\iota_i\mathbb{V})$ is isomorphic to a direct sum of $\mathbb{S}_i$, we conclude that $\operatorname{coker}(\iota_i\mathbb{V}) = 0$ and $\mathbb{V} \cong \mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V}$. This also proves that $\mathcal{R}_i^+\mathbb{V}$ is indecomposable. Indeed, if $\mathcal{R}_i^+\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2$, then $\mathbb{V} \cong \mathcal{R}_i^-\mathbb{W}_1 \oplus \mathcal{R}_i^-\mathbb{W}_2$ is not indecomposable. Moreover, we have $q_{s_iQ}(\underline{\dim}\mathcal{R}_i^+\mathbb{V}) = q_{s_iQ}(\sigma_i(\underline{\dim}\mathbb{V})) = q_Q(\underline{\dim}\mathbb{V})$ (using Lemma 3.23). Analogously, if i is a source, we use that $\mathbb{V} \cong (\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V}) \oplus \ker(\pi_i\mathbb{V})$ and that $\ker(\pi_i\mathbb{V})$ is isomorphic to a direct sum of $\mathbb{S}_i$ and thus $\ker(\pi_i\mathbb{V}) = 0$ and $\mathbb{V} \cong \mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V}$. Using Lemma 3.23, we get that $q_{s_iQ}(\underline{\dim}\mathcal{R}_i^-\mathbb{V}) = q_{s_iQ}(\sigma_i(\underline{\dim}\mathbb{V})) = q_Q(\underline{\dim}\mathbb{V})$. □

Theorem 3.49 and Corollary 3.50 show that $\mathcal{R}_i^+$ and $\mathcal{R}_i^-$ are inverse everywhere except on the simple representation $\mathbb{S}_i$. To state this precisely, we consider the subcategory consisting of representations, which do not have a summand equal to $\mathbb{S}_i$.

Definition 3.51. *Let Q be a quiver and let $i \in Q_0$ be any vertex. We denote by* $\operatorname{Rep}_k(Q)\langle i\rangle$ *the full subcategory of* $\operatorname{Rep}_k(Q)$ *consisting of representations that do not have summands that are isomorphic to the simple representation $\mathbb{S}_i$. Analogously, we define the full subcategory* $\operatorname{rep}_k(Q)\langle i\rangle$ *of* $\operatorname{rep}_k(Q)$.

Corollary 3.52. *Let Q be a quiver and let $i \in Q_0$ be a sink or a source. Then $\mathcal{R}_i^+$ and $\mathcal{R}_i^-$ induce mutually inverse equivalences* $\operatorname{rep}_k(Q)\langle i\rangle \to \operatorname{rep}_k(s_iQ)\langle i\rangle$ *and* $\operatorname{rep}_k(s_iQ)\langle i\rangle \to \operatorname{rep}_k(Q)\langle i\rangle$. *In particular, there is a one-to-one correspondence between the isomorphism classes of indecomposable representations of Q and those of s_iQ.*

Proof. The first part follows directly from Corollary 3.50 and Theorem 3.49. Indeed, for a representation $\mathbb{V} = \mathbb{V}_1 \oplus \ldots \oplus \mathbb{V}_r \in \operatorname{Rep}_k(Q)\langle i\rangle$, where $\mathbb{V}_i$ are the indecomposable summands that are all non-isomorphic to $\mathbb{S}_i$. If i is a sink, we have:

$$\mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V} \cong (\mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V}_1) \oplus \ldots \oplus (\mathcal{R}_i^-\mathcal{R}_i^+\mathbb{V}_r) \cong \mathbb{V}_1 \oplus \ldots \oplus \mathbb{V}_r = \mathbb{V}.$$

Similarly, if i is a source, we have:

$$\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V} \cong (\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V}_1) \oplus \ldots \oplus (\mathcal{R}_i^+\mathcal{R}_i^-\mathbb{V}_r) \cong \mathbb{V}_1 \oplus \ldots \oplus \mathbb{V}_r = \mathbb{V}.$$

The one-to-one correspondence between the isomorphism classes of indecomposable representations in $\operatorname{Rep}_k(Q)$ and $\operatorname{Rep}_k(s_iQ)$ is given by:

$$\mathbb{S}_i \mapsto \mathbb{S}_i, \mathbb{V} \mapsto \begin{cases} \mathcal{R}_i^+\mathbb{V} & \text{if } i \text{ is a sink,} \\ \mathcal{R}_i^-\mathbb{V} & \text{if } i \text{ is a source.} \end{cases}$$

where $\mathbb{V} \in \operatorname{Rep}_k(Q)$ is indecomposable and $\mathbb{V} \ncong \mathbb{S}_i$. □

The subcategories $\operatorname{Rep}_k(Q)\langle i\rangle$ and $\operatorname{rep}_k(Q)\langle i\rangle$ are interesting because on those categories, $\mathcal{R}_i^-\mathcal{R}_i^+$ and $\mathcal{R}_i^+\mathcal{R}_i^-$ are the identity (up to isomorphism of representations). Moreover, if we classify all indecomposable representations in $\operatorname{Rep}_k(Q)\langle i\rangle$, we also classify all indecomposable representations in $\operatorname{Rep}_k(Q)$ since the only additional indecomposable representation is $\mathbb{S}_i$. However, the subcategory $\operatorname{Rep}_k(Q)\langle i\rangle$ has the crucial disadvantage that morphisms in $\operatorname{Rep}_k(Q)\langle i\rangle$ do (in general) not have kernels and cokernels. This means that for a morphism $\phi\colon \mathbb{V} \to \mathbb{W}$ between representations $\mathbb{V}, \mathbb{W} \in \operatorname{Rep}_k(Q)\langle i\rangle$ it can happen that one (or both) of the representations $\ker\phi$ or $\operatorname{coker}\phi$ is not in $\operatorname{Rep}_k(Q)\langle i\rangle$, meaning that the representation $\ker\phi$ or $\operatorname{coker}\phi$ has a summand that is isomorphic to $\mathbb{S}_i$.

Example 3.53. *We give an example, illustrating that the category* $\operatorname{Rep}_k(Q)\langle i\rangle$ *does (in general) not have kernels and cokernels. Let Q be the linear L_3-quiver and let $\phi\colon \mathbb{V} \to \mathbb{W}$ be the morphism represented by the following commutative diagram*

$$\begin{array}{lccccc} \mathbb{V}\colon & 0 & \xrightarrow{0} & k & \xrightarrow{\text{id}} & k \\ & \big\downarrow{\scriptstyle \phi_1=0} & & \big\downarrow{\scriptstyle \phi_2=\text{id}} & & \big\downarrow{\scriptstyle \phi_3=\text{id}} \\ \mathbb{W}\colon & k & \xrightarrow{\text{id}} & k & \xrightarrow{\text{id}} & k. \end{array}$$

Note that the representations $\mathbb{V}$ and $\mathbb{W}$ are indecomposable and thus do not have a summand that is isomorphic to a simple representation $\mathbb{S}_i$. The indecomposability of $\mathbb{V}$ and $\mathbb{W}$ is proven in Section 4.1. However, the representation $\operatorname{coker}\phi$ *is given by the following picture*

$$k \xrightarrow{0} 0 \xrightarrow{0} 0.$$

Therefore, $\operatorname{coker}\phi = \mathbb{S}_1$ *which shows that* $\mathrm{Rep}_{\mathrm{k}}(Q)\langle i\rangle$*, in general, does not have cokernels. Analogously, we consider the 'reverse diagram':*

$$\begin{tikzcd}
\tilde{\mathbb{V}}: & k \arrow[r, "\mathrm{id}"] \arrow[d, "\psi_1=\mathrm{id}"] & k \arrow[r, "\mathrm{id}"] \arrow[d, "\psi_2=\mathrm{id}"] & k \arrow[d, "\psi_3=0"] \\
\tilde{\mathbb{W}}: & k \arrow[r, "\mathrm{id}"] & k \arrow[r, "0"] & 0.
\end{tikzcd}$$

Here, we get $\ker\psi = \mathbb{S}_3$*, showing that* $\mathrm{Rep}_{\mathrm{k}}(Q)\langle i\rangle$ *does not necessarily have kernels.*

3.4 Coxeter Functors

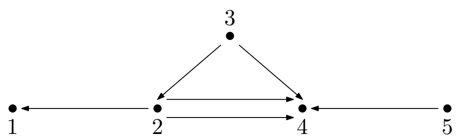
In this section, we introduce a notion of ordered vertices and we introduce the sequence of reflection functors belonging to this ordering. This notion of ordering coincides with the one given in [2], but is different from the one given in [1].

3.4.1 Admissible Orderings

We introduce a notion of a preferred ordering and show for which quivers such an ordering exists. The existence of such an ordering is closely related to the existence of cycles in the quiver.

Definition 3.54. *An ordering $i_1, i_2, \ldots, i_n$ of the vertices of a quiver Q is called* $(+)$-***admissible*** *if i_1 is a sink in Q and for each $2 \leq t \leq n$, i_t is a sink in $s_{i_{t-1}} \ldots s_{i_1} Q$. Dually, an ordering $i_1, i_2, \ldots, i_n$ is called* $(-)$-***admissible*** *if i_1 is a source in Q and for each $2 \leq t \leq n$, i_t is a source in $s_{i_{t-1}} \ldots s_{i_1} Q$. It is required that each vertex of Q appears exactly once in our ordering.*

Example 3.55. *We consider the following quiver Q:*



The ordering $1, 4, 2, 3, 5$ is $(+)$-admissible. Indeed, 1 is a sink in Q, 4 is a sink in $s_1 Q$, 2 is a sink in $s_4 s_1 Q$, 3 is a sink in $s_2 s_4 s_1 Q$ and 5 is a sink in $s_3 s_2 s_4 s_1 Q$. One can also check that $3, 2, 1, 5, 4$ is a $(-)$-admissible ordering.

Lemma 3.56. *There exists a $(+)$-admissible ordering of the vertices of a quiver Q if and only if Q is acyclic, i.e. there are no oriented cycles in Q.*

Proof. We prove one implication by induction on the number of vertices. Let n be the number of vertices of Q and suppose that Q is acyclic. Since the base case $n = 1$ is clear, we can assume that $n \geq 2$. Now let i_n be the starting vertex of an oriented path of maximal length. Since Q is acyclic there are no infinite ordered paths and thus such a path exists. Then i_n is a source, else we have a bigger ordered path. Now we remove i_n from the quiver (we also remove all its incident edges). By induction, the smaller quiver has an admissible ordering $i_1, \ldots, i_{n-1}$. Now $i_1, \ldots, i_{n-1}, i_n$ is an admissible ordering of the original quiver Q. Indeed, each incident edge $a = (i_n, i_t)$ once reverses direction by applying s_{i_t} and then reverses the direction back by applying s_{i_n}. Therefore, i_t is also a sink in $s_{i_{t-1}} \ldots s_{i_1} Q$ (notice that here, we apply the s_{i_j} to the whole quiver Q) and also i_n is a sink in $s_{i_{n-1}} \ldots s_{i_1} Q$. $\square$

In the induction step of the proof, we have chosen a vertex i_n. This hints at the fact that there is generally not a unique $(+)/(-)$-admissible ordering. A possible algorithm to choose a $(+)$-admissible ordering is the following. For an acyclic quiver Q, there always exists a sink i_1. Now we can delete the vertex i_1 (and its incident edges) and the resulting quiver is again acyclic, which allows us to pick a sink i_2. Doing this inductively yields an $(+)$-admissible ordering $i_1, \dots, i_n$.

Lemma 3.57. *Let $i_1, i_2, \dots, i_n$ be an ordering of the vertices of a quiver Q. Then $i_1, i_2, \dots, i_n$ is $(+)$-admissible if and only if $i_n, i_{n-1}, \dots, i_1$ is $(-)$-admissible. In particular, a quiver Q admits a $(+)$-admissible ordering if and only if it admits an $(-)$-admissible ordering. Therefore, we often just refer to an admissible ordering, meaning a $(+)$-admissible ordering with induced $(-)$-admissible ordering.*

Proof. Let $i_1, \dots, i_n$ be $(+)$-admissible. Then, i_n is a sink in $s_{i_{n-1}} \dots s_{i_1} Q$. But in the process of going from Q to $s_{i_{n-1}} \dots s_{i_1} Q$, every incident arrow of i_n is flipped once and thus i_n is a source in Q. Similarly, because i_{n-1} is a sink in $s_{i_{n-2}} \dots s_{i_1} Q$, we know the following for all $1 \leq k \leq n-2$. If a is an arrow (in Q) between i_{n-1} and i_k, then a must point away from i_{n-1} (since it points towards i_{n-1} in $s_{i_{n-1}} \dots s_{i_1} Q$). Therefore, a also points away from i_{n-1} in $s_{i_n} Q$. In addition, if b is an arrow (in Q) between i_{n-1} and i_n, then b must point away from i_{n-1} in $s_{i_n} Q$ since i_n is a source in Q. Repeating this argument inductively yields that $i_n, i_{n-1}, \dots, i_1$ is $(-)$-admissible. The other implication can be proven analogously. $\square$

Example 3.58. *We consider the quiver from Example 3.55. We observe that the sequence $5, 3, 2, 4, 1$ is a $(-)$-admissible ordering and that $4, 5, 1, 2, 3$ is a $(+)$-admissible ordering. In addition, this also shows that there can be multiple admissible orderings.*

Remark 3.59. *There is another way to think about admissible orderings. If a quiver Q is acyclic, then Q represents a partial order on the vertex set Q_0. Indeed, for two vertices i and j, we can define $i \leq j$ if and only if there exists a directed path from i to j (we say that there is a path from each vertex i to itself). This indeed defines a partial order since there are no oriented cycles in Q. Then we can choose a total order that extends our partial order and this yields an ordering of the vertices. It can be shown that this ordering is $(-)$-admissible. This point of view helps to understand why Q needs to be acyclic for it to admit an admissible ordering. Indeed, if Q would have an oriented cycle, then we would get a chain $i_1 \leq i_2 \leq \dots \leq i_r \leq i_1$ which would imply that $i_1 = i_2 = \dots = i_r$. Notice that an increasing ordering (with respect to our total order) yields a $(-)$-admissible ordering and a decreasing ordering yields a $(+)$-admissible ordering.*

Example 3.60. *We consider the quiver from Example 3.55. As explained in the remark above, we get the following partial order: $2 < 1, 2 < 4, 3 < 2, 3 < 4, 5 < 4$. We see that the $(+)$-admissible orderings $1, 4, 2, 3, 5$ and $4, 5, 1, 2, 3$ are total orders that extend our partial order (the total orderings are in decreasing order).*

We notice that each arrow in $s_{i_n} \dots s_{i_1} Q$ results from an arrow in Q by changing its orientation exactly twice and therefore, we have that $s_{i_n} \dots s_{i_1} Q = Q$. This motivates the following definition.

Definition 3.61. *Let Q be an acyclic quiver and let $i_1, \dots, i_n$ be a $(+)$-admissible ordering of the vertices of Q. The two following functors are called **Coxeter functors** (with respect to this ordering).*

$$\mathcal{C}^+ = \mathcal{R}^+_{i_n} \dots \mathcal{R}^+_{i_1} \colon \operatorname{Rep}_{\mathrm{k}}(Q) \to \operatorname{Rep}_{\mathrm{k}}(Q)$$
$$\mathcal{C}^- = \mathcal{R}^-_{i_1} \dots \mathcal{R}^-_{i_n} \colon \operatorname{Rep}_{\mathrm{k}}(Q) \to \operatorname{Rep}_{\mathrm{k}}(Q)$$

Lemma 3.62. *The Coxeter functors $\mathcal{C}^+$ and $\mathcal{C}^-$ do not depend on the choice of the admissible ordering of the vertices of a quiver Q.*

Proof. First, we claim that $\mathcal{R}^+_i \mathcal{R}^+_j = \mathcal{R}^+_j \mathcal{R}^+_i$ if i and j are sinks that are not adjacent to each other. Indeed, notice that $\mathcal{R}^+_i$ and $\mathcal{R}^+_j$ change different vector spaces and linear maps, and therefore the order of operation does not matter. Now let $i_1, \dots, i_n$ and $i'_1, \dots, i'_n$ be two $(+)$-admissible orderings of the vertices of Q. Let $i_1 = i'_m$. Then i_1 is a sink in Q and in $s_{i'_{m-1}} \dots s_{i'_1} Q$. Therefore, i_1 is not adjacent to any of the vertices $i'_1, \dots, i'_{m-1}$ and we get that:

$$\mathcal{R}^+_{i'_m} \mathcal{R}^+_{i'_{m-1}} \dots \mathcal{R}^+_{i'_1} = \mathcal{R}^+_{i'_{m-1}} \mathcal{R}^+_{i'_m} \dots \mathcal{R}^+_{i'_1} = \dots = \mathcal{R}^+_{i'_{m-1}} \dots \mathcal{R}^+_{i'_1} \mathcal{R}^+_{i'_m} = \mathcal{R}^+_{i'_{m-1}} \dots \mathcal{R}^+_{i'_1} \mathcal{R}^+_{i_1}.$$

Now let $i_2 = i'_k$. We denote by $\mathcal{R}^+_{i'_k} \dots \widehat{\mathcal{R}^+_{i_1}} \dots \mathcal{R}^+_{i'_1}$ the sequence $\mathcal{R}^+_{i'_k} \dots \mathcal{R}^+_{i'_1}$ with $\mathcal{R}^+_{i_1}$ omitted. If $\mathcal{R}^+_{i_1}$ does not occur in the sequence, we just leave the sequence as is. We know that i_2 is a sink in $s_{i_1}Q$ and in $s_{i'_{k-1}\dots s_{i'_1}}Q$. Therefore, i_2 is not adjacent to any of the vertices $i'_{k-1}, \dots, i'_1$ apart from maybe $i_1 = i'_m$. We do not know if i_1 is a vertex in the list of $i'_{k-1}, \dots, i'_1$, however, by possibly excluding $\mathcal{R}^+_{i_1}$ we get that

$$\mathcal{R}^+_{i'_k} \dots \widehat{\mathcal{R}^+_{i_1}} \dots \mathcal{R}^+_{i'_1} = \mathcal{R}^+_{i'_{k-1}} \dots \widehat{\mathcal{R}^+_{i_1}} \dots \mathcal{R}^+_{i'_1}\mathcal{R}^+_{i_2}.$$

Therefore, we can first move $\mathcal{R}^+_{i_1}$ to the right and then move $\mathcal{R}^+_{i_2}$ to the right and get

$$\mathcal{R}^+_{i'_n} \dots \mathcal{R}^+_{i'_1} = \mathcal{R}^+_{i'_n} \dots \widehat{\mathcal{R}^+_{i_1}} \dots \widehat{\mathcal{R}^+_{i_2}} \dots \mathcal{R}^+_{i'_1}\mathcal{R}^+_{i_2}\mathcal{R}^+_{i_1}.$$

Doing this inductively yields

$$\mathcal{R}^+_{i'_n} \dots \mathcal{R}^+_{i'_1} = \mathcal{R}^+_{i_n} \dots \mathcal{R}^+_{i_1}.$$

The case of the Coxeter functor $\mathcal{C}^-$ can be proven analogously. □

Since only acyclic quivers admit admissible orderings, we restrict our attention to acyclic quivers in the rest of this section. Moreover, in the acyclic case, we have an admissible ordering and because the Coxeter functors $\mathcal{C}^+$ and $\mathcal{C}^-$ are independent of the choice of admissible ordering, we assume w.l.o.g. that the vertices in $Q_0 = \{1, \dots, n\}$ already depict an admissible ordering, i.e. that $1, \dots, n$ is a $(-)$-admissible ordering. Now we use the results about reflection functors to better understand the Coxeter functors.

Corollary 3.63. *Let Q be a finite, connected, and acyclic quiver and let $\mathcal{C}^\pm$ be the Coxeter functors. Then for any indecomposable representation $\mathbb{V}$ of Q, either $\mathcal{C}^\pm\mathbb{V}$ is indecomposable or $\mathcal{C}^\pm\mathbb{V} = 0$. In the first case, we have $q_Q(\underline{\dim}\mathcal{C}^\pm\mathbb{V}) = q_Q(\underline{\dim}\mathbb{V})$, while in the second case we have $q_Q(\underline{\dim}\mathcal{C}^\pm\mathbb{V}) = 0$.*

Proof. This proof directly follows from Corollary 3.50. Since the case $\mathcal{C}^\pm\mathbb{V} = 0$ is trivial we can w.l.o.g. assume that $\mathcal{C}^\pm\mathbb{V} \neq 0$. From Corollary 3.50 we know that $\mathcal{R}^+_{i_1}\mathbb{V}$ is indecomposable. Using this argument inductively shows that $\mathcal{C}^+\mathbb{V}$ is indecomposable. Similarly, we can conclude that $\mathcal{C}^-\mathbb{V}$ is indecomposable. Moreover, we have

$$\begin{aligned} q_Q(\underline{\dim}\mathcal{C}^+\mathbb{V}) &= q_{s_{i_n}\dots s_{i_1}Q}(\underline{\dim}\mathcal{R}^+_{i_n}\dots\mathcal{R}^+_{i_1}\mathbb{V}) = q_{s_{i_{n-1}}\dots s_{i_1}Q}(\underline{\dim}\mathcal{R}^+_{i_{n-1}}\dots\mathcal{R}^+_{i_1}\mathbb{V}) \\ &= \dots = q_{s_{i_1}Q}(\underline{\dim}\mathcal{R}^+_{i_1}\mathbb{V}) = q_Q(\underline{\dim}\mathbb{V}). \end{aligned}$$

Analogously, it follows that $q_Q(\underline{\dim}\mathcal{C}^-\mathbb{V}) = q_Q(\underline{\dim}\mathbb{V})$. □

Example 3.64. *To see an example of the Coxeter functor, we refer to Section 4.2. There, it is demonstrated how the Coxeter functor operates on the linear L_n-quiver. Explicitly, it is calculated, what happens to the dimension vector of an indecomposable representation, when applying $\mathcal{C}^+$.*

Definition 3.65. *Let Q be an acyclic quiver and fix a $(+)$-admissible ordering $i_1, \dots, i_n$ of its vertices. The automorphism $c = \sigma_{i_n}\dots\sigma_{i_1} \in W(Q) \subseteq \mathrm{Aut}(\mathbb{Z}^n)$ is called a* ***Coxeter transformation****.*

The Coxeter transformation and the Coxeter functor are related by the following property.

Lemma 3.66. *Let Q be an acyclic quiver and let $\mathcal{C}^+$ be the Coxeter functor and c be the Coxeter transformation (w.r.t. any admissible ordering). Let $\mathbb{V}$ be an indecomposable representation of Q that is not isomorphic to the simple representation $\mathbb{S}_i$ for any $i \in Q_0$. We then have that $\underline{\dim}(\mathcal{C}^+)^r\mathbb{V} = c^r(\underline{\dim}\mathbb{V})$.*

Proof. This is an immediate consequence of $\underline{\dim}\mathcal{R}^+_i\mathbb{V} = \sigma_i(\underline{\dim})\mathbb{V}$ from Theorem 3.49. □

Remark 3.67. *The Coxeter transformation is independent of the choice of admissible ordering. However, since the Coxeter functor $\mathcal{C}^+$ is independent of the choice of admissible ordering and we only use the above property of the Coxeter transformation, it is not important that the Coxeter transformation is independent of the choice of admissible ordering. Therefore, we do not prove it.*

Lemma 3.68. *Let Q be a Dynkin quiver.*

1. *The Coxeter transformation c has no non-zero fixed vectors, that is $cx \neq x$ for all $0 \neq x \in \mathbb{Z}^n$.*
2. *For each positive $x \in \mathbb{Z}^n$, there exists an integer $r \geq 1$ such that $c^r(x)$ is not positive.*

Proof. First notice that since Q is Dynkin ($\implies$ acyclic), both the Coxeter functors $\mathcal{C}^+, \mathcal{C}^-$ and the Coxeter transformation c exist.

1. Suppose that $x \in \mathbb{Z}^n$ such that $cx = x$. Because $\sigma_{i_2}, \dots, \sigma_{i_n}$ do not change the i_1-th coordinate, we must have $\sigma_i(x)_i = x_i$ and thus $\sigma_i(x) = x$. Doing this inductively, yields that $\sigma_{i_j}(x) = x$ for all $1 \leq j \leq n$. From the definition of σ_i, we get that $(x, e_j)_Q = 0$ for all $1 \leq j \leq n$. This implies that $q_Q(x) = 0$ and thus $x = 0$.
2. We know from Corollary 3.31 that $W(Q)$ is finite. Therefore, there exists some $h \geq 1$ such that $c^h = \mathrm{id}$ (e.g. $h = |W(Q)|$ is the cardinality of $W(Q)$). If $x, cx, c^2x, \dots, c^{h-1}x$ are all positive, then $y = x + cx + c^2x + \ldots + c^{h-1}x$ is positive ($\implies$ $y \neq 0$) and $cy = y$ which contradicts part 1. But then one of the $cx, c^2x, \dots, c^{h-1}x$ is not positive. $\square$

Proposition 3.69. *Let Q be a Dynkin quiver, and let $\mathbb{V}$ be an indecomposable representation of Q. Then there is a finite r such that $\underbrace{\mathcal{C}^+ \dots \mathcal{C}^+}_{r \text{ times}} \mathbb{V} = 0$.*

Proof. Let $x = \underline{\dim}\mathbb{V}$. If $\mathbb{V} = \mathbb{S}_i$ (for some $i \in Q_0$), then we have $\mathcal{C}^+\mathbb{V} = 0$ since $\mathcal{R}_i^+\mathbb{S}_i = 0$. Thus let $\mathbb{V} \neq \mathbb{S}_i$ for all $i \in Q_0$. Then Lemma 3.68 yields an integer r such that the vector $c^r(x) = c^r(\underline{\dim}\mathbb{V}) = \underline{\dim}(\mathcal{C}^+)^r\mathbb{V}$ is not positive. Therefore, we have $(\mathcal{C}^+)^r\mathbb{V} = \underbrace{\mathcal{C}^+ \dots \mathcal{C}^+}_{r \text{ times}} \mathbb{V} = 0$. $\square$

Example 3.70. *In Section 4.2, Proposition 3.69 is shown in the example of the linear L_n-quiver. Then, we can take $r = n$.*

3.5 The Proof of Gabriel's Theorem

We first state the version of Gabriel's Theorem that we prove. This version of the theorem is superior to Theorem 3.2 since it gives a characterization of the isomorphism classes of the indecomposable representations for the Dynkin quivers. This section is based on [2].

Theorem 3.71 (Gabriel, version 2). *Let Q be a connected quiver and let k be an arbitrary field. Then:*

1. *Q is of finite-type if and only if Q is a Dynkin quiver.*
2. *When the equivalent conditions in (1) are satisfied, the correspondence $\mathbb{V} \mapsto \underline{\dim}\mathbb{V}$ induces a bijection between the set of isomorphism classes of indecomposable representations of Q and the set $\Phi^+(Q) = \Phi(Q) \cap \mathbb{N}^n$ of positive roots of Q.*

Proof. Step 1: First, we assume that Q is a Dynkin quiver. We show that Q is of finite-type and that (2) holds. For simplicity, we set $Q_0 = \{1, 2 \dots, n\}$ and we assume that $1, 2, \dots, n$ is a (+)-admissible ordering. Let $\mathbb{V}$ be an indecomposable representation of Q. Then $\underline{\dim}\mathbb{V}$ is positive. By Lemma 3.68, we know that there exists r such that $c^r(\underline{\dim}\mathbb{V})$ is non-positive which implies that $(\mathcal{C}^+)^r\mathbb{V} = 0$ by Proposition 3.69. Here, $c = \sigma_n \dots \sigma_2\sigma_1$ denotes the Coxeter transformation. We consider the sequence

$$(i_1, i_2, \dots, i_n, i_{n+}, \dots, i_{rn}) = (1, 2, \dots, n, 1, 2, \dots, n, 1, 2, \dots, n),$$

where the sequence $1, 2, \dots, n$ is repeated r times. Let $0 \leq t < rn$ such that $\sigma_{i_t} \dots \sigma_{i_1}(\underline{\dim}\mathbb{V})$ is positive but $\sigma_{i_{t+1}} \dots \sigma_{i_1}(\underline{\dim}\mathbb{V})$ is non-positive and thus 0. This is equivalent to saying that $\mathcal{R}^+_{i_{t+1}} \dots \mathcal{R}^+_{i_1}$ is the smallest subsequence of $(\mathcal{C}^+)^r$, which sends $\mathbb{V}$ to the trivial representation. Since $\mathcal{R}^+_{i_t} \dots \mathcal{R}^+_{i_1}\mathbb{V}$ is indecomposable and $\mathcal{R}^+_{i_{t+1}} \dots \mathcal{R}^+_{i_1}\mathbb{V} = 0$, we know from Corollary 3.50 that $\mathcal{R}^+_{i_t} \dots \mathcal{R}^+_{i_1}\mathbb{V} = \mathbb{S}_{i_{t+1}}$, where $\mathbb{S}_{i_{t+1}}$ is the simple representation (at the vertex i_{t+1}) of $s_{i_t} \dots s_{i_2}s_{i_1}Q$. Applying the same corollary inductively yields that

$$\mathbb{V} \cong \mathcal{R}^-_{i_1}\mathcal{R}^-_{i_2} \dots \mathcal{R}^-_{i_t}\mathcal{R}^+_{i_t} \dots \mathcal{R}^+_{i_1}\mathbb{V} = \mathcal{R}^-_{i_1}\mathcal{R}^-_{i_2} \dots \mathcal{R}^-_{i_t}\mathbb{S}_{i_{t+1}}.$$

Note that the sequence $\mathcal{R}^-_{i_1}\mathcal{R}^-_{i_2}\dots\mathcal{R}^-_{i_t}\mathcal{R}^+_{i_t}\dots\mathcal{R}^+_{i_1}$ is well-defined since for all $1 \le j \le t$, we have that i_j is a sink in $s_{i_{j-1}}\dots s_{i_1}Q$ and i_j is a source in $s_{i_{j+1}}\dots s_{i_t}s_{i_t}\dots s_{i_1}Q = s_{i_j}\dots s_{i_1}Q$ because $1,2,\dots,n$ is $(+)$-admissible. Moreover, Theorem 3.49 yields that

$$\underline{\dim}\mathbb{V} = \underline{\dim}\mathcal{R}^-_{i_1}\mathcal{R}^-_{i_2}\dots\mathcal{R}^-_{i_t}\mathbb{S}_{i_{t+1}} = \sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_t}(\underline{\dim}\mathbb{S}_{i_{t+1}}) = \sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_t}(e_{i_{t+1}}).$$

Therefore, all dimension vectors of indecomposable representations are (positive) roots of Q. Moreover, let $\mathbb{V},\mathbb{W}$ be two indecomposable representations such that $\underline{\dim}\mathbb{V} = \underline{\dim}\mathbb{W}$. We get that $\underline{\dim}\mathcal{R}^+_{i_t}\dots\mathcal{R}^+_{i_1}\mathbb{W} = \sigma_{i_t}\dots\sigma_{i_1}(\underline{\dim}\mathbb{W}) = \sigma_{i_t}\dots\sigma_{i_1}(\underline{\dim}\mathbb{V}) = e_{i_{t+1}} = \underline{\dim}\mathbb{S}_{i_{t+1}}$. But $\mathbb{S}_{i_{t+1}}$ is the only representation with dimension vector $e_{i_{t+1}}$ and therefore $\mathcal{R}^+_{i_t}\dots\mathcal{R}^+_{i_1}\mathbb{W} = \mathbb{S}_{i_{t+1}}$. Now, applying the mirrored sequence of reflection functors, we get

$$\mathbb{W} \cong \mathcal{R}^-_{i_1}\dots\mathcal{R}^-_{i_t}\mathcal{R}^+_{i_t}\dots\mathcal{R}^+_{i_1}\mathbb{W} \cong \mathcal{R}^-_{i_1}\dots\mathcal{R}^-_{i_t}\mathbb{S}_{i_{t+1}} \cong \mathbb{V}.$$

Therefore, we get a well-defined map

$$\phi\colon \mathcal{I} \to \Phi^+(Q), [\mathbb{V}] \mapsto \underline{\dim}\mathbb{V},$$

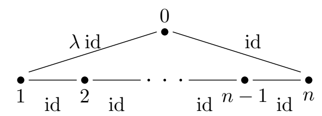
where $\mathcal{I}$ denotes the set of isomorphism classes of indecomposable representations $\mathbb{V}$ of Q and $[\mathbb{V}]$ denotes the isomorphism class of $\mathbb{V}$. Note that we have shown that ϕ is an injective map. Since $\Phi^+(Q)$ is finite (Lemma 3.30) this shows that Q is of finite-type. We now show that ϕ is surjective and thus proving (2). Let $x \in \Phi^+(Q)$ be a positive root of q_Q. From Lemma 3.68, we get an m such that $c^m(x)$ is not positive. Similarly as before, let $0 \le t < mn$ be minimal such that $\sigma_{i_t}\dots\sigma_{i_1}(x)$ is positive, but $\sigma_{i_{t+1}}\sigma_{i_t}\dots\sigma_{i_1}(x)$ is not positive. But since $\sigma_{i_{t+1}}\sigma_{i_t}\dots\sigma_{i_1}(x)$ is a root, it is either negative or positive. However, $\sigma_{i_{t+1}}$ only changes the i_{t+1}-th coordinate and thus we conclude that $\sigma_{i_t}\dots\sigma_{i_1}(x) = e_{i_{t+1}}$ and therefore we have $x = \sigma_{i_1}\dots\sigma_{i_t}(e_{i_{t+1}})$. But the representation $\mathbb{V} = \mathcal{R}^-_{i_1}\dots\mathcal{R}^-_{i_t}\mathbb{S}_{i_{t+1}}$ is indecomposable by Corollary 3.50 and has dimension vector x. This shows that ϕ is surjective.

Step 2: For any quiver Q and a subquiver Q', we have that $\mathrm{Rep}_k(Q')$ is a full subcategory of $\mathrm{Rep}_k(Q)$. Indeed, if $\mathbb{V}$ is a representation of Q', then we get a representation $\tilde{\mathbb{V}}$ of Q by adding zero spaces and zero maps to vertices and arrows in $Q \setminus Q'$. The subcategory is full because every morphism $\tilde{\phi}\colon \tilde{\mathbb{V}} \to \tilde{\mathbb{W}}$ induces a morphism $\phi\colon \mathbb{V} \to \mathbb{W}$. Indeed, if $i \in Q_0 \setminus Q_0'$, then $\tilde{\phi}_i = 0\colon 0 \to 0$ and therefore $\phi = (\tilde{\phi}_i)_{i \in Q_0'}$ is a morphism between representations of Q'. If two representations $\tilde{\mathbb{V}}, \tilde{\mathbb{W}}$ of Q are isomorphic, then the induced representations $\mathbb{V},\mathbb{W}$ of Q' are also isomorphic. Indeed, for an isomorphism $\tilde{\phi}\colon \tilde{\mathbb{V}} \to \tilde{\mathbb{W}}$ we have that the induced morphism $\phi\colon \mathbb{V} \to \mathbb{W}$ is in $\mathrm{Rep}_k(Q')$ (since the subcategory is full) and it is also an isomorphism. Therefore, if Q is of finite-type, then $Q' \subseteq Q$ is also of finite-type. But from Lemma 3.15, we know that each wild quiver Q has a tame subquiver Q'. Therefore, it suffices to show that all tame quivers are not of finite-type.

Step 3: We now go through the list of tame quivers resp. Euclidean graphs and show that they have infinitely many isomorphism classes of indecomposable representations.

$\tilde{A}_0$: Let Q be a quiver of type $\tilde{A}_0$. In Example 2.21 and Example 2.25 we have shown that (for k an algebraically closed field), the isomorphism classes of indecomposable representations are given by the Jordan normal blocks. For an arbitrary field k, we still get that the different Jordan blocks $J_{\lambda,n}$ for $\lambda \in k$ give pairwise non-isomorphic indecomposable representations. Indeed, from Example 2.21, we know that two representations $\mathbb{V} = k^n \xrightarrow{M} k^n, \mathbb{W} = k^n \xrightarrow{N} k^n$ are isomorphic if and only if there exists an invertible matrix A such that $NA = AM$. By a generalized version of the Jordan normal form for an arbitrary field k, this implies that M and N have the same Jordan normal form. If k is an infinite field, then for each $n \in \mathbb{N}$, we get an infinite number of isomorphism classes of indecomposable representations. If k is finite, we can vary over $n \in \mathbb{N}$ and still get an infinite number of isomorphism classes.

$\tilde{A}_n, n \ge 1$: Let Q be a quiver of type $\tilde{A}_n$. For an arbitrary $\lambda \in k$, we define a representation $\mathbb{V}_\lambda$ by setting $V_i = k$ for all vertices and by setting the maps $v_a = \mathrm{id}$ for all $a \in Q_1$, except one map $v_b = \lambda\,\mathrm{id}$. $\mathbb{V}_\lambda$ is given by the following picture.

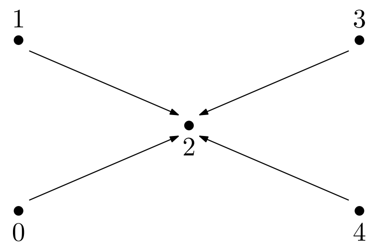


Since all the vector spaces are 1-dimensional and the quiver is connected, $\mathbb{V}_\lambda$ is indecomposable. Indeed if $\mathbb{V}_\lambda = \mathbb{V}_1 \oplus \mathbb{V}_2$ and $(\mathbb{V}_1)_i = k$ for some vertex i, then we have a vertex j such that the map between i and j is the identity, which implies that $(\mathbb{V}_1)_j = k$. Now, using that Q is connected, we conclude that $\mathbb{V} = \mathbb{V}_1$. Moreover, for a morphism $\phi\colon \mathbb{V}_{\lambda_1} \to \mathbb{V}_{\lambda_2}$ we have that $\phi_i = \phi_j$ for all vertices i, j. This can be shown using a similar argument as for the part that $\mathbb{V}_\lambda$ is indecomposable. But then we have the following commutative diagram.

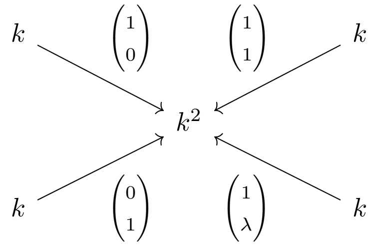
$$\begin{tikzcd}
k \arrow[r, "\lambda_1 \mathrm{id}"] \arrow[d, "\phi_0"'] & k \arrow[d, "\phi_0"] \\
k \arrow[r, "\lambda_2 \mathrm{id}"] & k
\end{tikzcd}$$

Therefore, if $\lambda_1 \neq \lambda_2$ we have that $\phi_0 = 0$ which shows that $\mathbb{V}_{\lambda_1} \cong \mathbb{V}_{\lambda_2}$ if and only if $\lambda_1 = \lambda_2$. More generally, for any $n \in \mathbb{N}$, we can define a representation by putting k^n at each vertex and by letting all maps be the identity, apart from one map which is given by a Jordan block $J_{\lambda,n}$. Similarly, as in the first case ($n = 1$), it can be shown that this defines an indecomposable representation since each Jordan block is indecomposable. The same argument as above also shows that different choices of $\lambda \in k$ give pairwise non-isomorphic indecomposable representations. Therefore, we have an infinite number of indecomposable representations.

$\underline{\tilde{D_n}, n \geq 4}$: We first consider the following quiver of type $\tilde{D}_4$:



For an arbitrary $\lambda \in k$ we consider the following representation, called $\mathbb{V}_\lambda$:



We claim that $\mathbb{V}_\lambda$ is indecomposable and that different choices of $\lambda \in k$ give pairwise non-isomorphic representations. Indeed, let $\mathbb{W}$ be a summand of $\mathbb{V}_\lambda$ and notice that $\mathbb{W}$ needs to have a non-trivial vector space at one of the outer vertices 0,1,3 or 4 (else $\mathbb{W}$ is trivial). But this implies that the vector spaces at all the other outer vertices must be equal to k (and thus $\mathbb{W} = \mathbb{V}$). To see this, consider the images of the four maps $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\\lambda\end{pmatrix}$. Notice that for any choice of three of the four maps, their images generate k^2. Also, for any partition into two pairs, the images of one pair (of maps) always generate k^2. Therefore, if we split the four copies of the vector spaces k (at the outer vertices) into two different summands of $\mathbb{V}$ (either split it 2 to 2 or 3 to 1), we have that one summand must have k^2 at the middle vertex. But then the other summand has 0 at the middle vertex, which is impossible because none of the maps is trivial. This shows that $\mathbb{V}_\lambda$ is indecomposable. Moreover, if there is an isomorphism $\phi\colon \mathbb{V}_{\lambda_1} \to \mathbb{V}_{\lambda_2}$, we have

$$\begin{pmatrix}1\\0\end{pmatrix} \phi_2 = \phi_2|_{k\oplus 0} = \phi_1 \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\alpha\\0\end{pmatrix},$$
$$\begin{pmatrix}0\\1\end{pmatrix} \phi_2 = \phi_2|_{0\oplus k} = \phi_0 \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\\beta\end{pmatrix}.$$

This implies that $\phi_2 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. But since $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \phi_3 = \phi_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ we have $\phi_2 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$. This yields $\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} \phi_4 = \phi_2 \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha\lambda_2 \end{pmatrix}$, which implies $\lambda_1 = \lambda_2$. More generally, for any $n \in \mathbb{N}$, one can show that the following representations are indecomposable and are pairwise non-isomorphic for different choices of $\lambda \in k$ (here $J_{\lambda,n}$ denotes the $n \times n$-Jordan block).

$$\begin{tikzcd}
k^n \arrow[dr, "\begin{pmatrix} \mathbb{1}_{k^n} \\ 0 \end{pmatrix}"] & & k^n \arrow[dl, "\begin{pmatrix} \mathbb{1}_{k^n} \\ \mathbb{1}_{k^n} \end{pmatrix}"'] \\
& k^{2n} & \\
k^n \arrow[ur, "\begin{pmatrix} 0 \\ \mathbb{1}_{k^n} \end{pmatrix}"'] & & k^n \arrow[ul, "\begin{pmatrix} \mathbb{1}_{k^n} \\ J_{\lambda,n} \end{pmatrix}"]
\end{tikzcd}$$

Therefore, for any field k, the above quiver has infinitely many isomorphism classes of indecomposable representations. We can extend this argument to the following quiver Q_{sp} of type $\tilde{D_n}$.

$$\begin{tikzcd}
\overset{1}{\bullet} \arrow[dr] & & & & & \overset{n-1}{\bullet} \arrow[dl] \\
& \underset{2}{\bullet} \arrow[r] & \underset{3}{\bullet} \arrow[r] & \cdot - \cdot - \cdot \arrow[r] & \underset{n-2}{\bullet} & \\
\underset{0}{\bullet} \arrow[ur] & & & & & \underset{n}{\bullet} \arrow[ul]
\end{tikzcd}$$

Indeed, for $n \in \mathbb{N}$, one can show that the following representations are indecomposable and pairwise non-isomorphic for different $\lambda \in k$.

$$\begin{tikzcd}
k^n \arrow[dr, "\begin{pmatrix} \mathbb{1}_{k^n} \\ 0 \end{pmatrix}"] & & & & & k^n \arrow[dl, "\begin{pmatrix} \mathbb{1}_{k^n} \\ \mathbb{1}_{k^n} \end{pmatrix}"'] \\
& k^{2n} \arrow[r, "\mathrm{id}"] & k^{2n} \arrow[r, dashed] & k^{2n} \arrow[r, "\mathrm{id}"] & k^{2n} & \\
k^n \arrow[ur, "\begin{pmatrix} 0 \\ \mathbb{1}_{k^n} \end{pmatrix}"'] & & & & & k^n \arrow[ul, "\begin{pmatrix} \mathbb{1}_{k^n} \\ J_{\lambda,n} \end{pmatrix}"]
\end{tikzcd}$$

Now we use Corollary 3.52 to conclude that any quiver of type $\tilde{D_n}$ has infinitely many isomorphism classes of indecomposable representations. Using the corollary, it remains to show that any quiver of type $\tilde{D_n}$ results from the quiver Q_{sp} by applying a sequence of $\mathcal{R}_i^+$ respectively $\mathcal{R}_i^-$ (meaning that we apply s_i to the quiver itself (but only if i is a sink or a source) and we apply $\mathcal{R}_i^{\pm}$ to representations). Indeed, one can proceed as follows:

1. First, we fix the direction of the arrow between 2 and 3 (if it does not have the right direction) by applying $\mathcal{R}_2^-\mathcal{R}_1^-\mathcal{R}_0^-$.
2. Then we fix the arrows $0 \to 2$ and $1 \to 2$ by possibly applying $\mathcal{R}_0^-$ resp. $\mathcal{R}_1^-$.
3. Now we sequentially fix the arrows $i \to i+1$ for $3 \leq i \leq n-3$ (i.e. the horizontal arrows). First notice that the sequence $\mathcal{R}_{i+1}^+ \dots \mathcal{R}_{n-3}^+\mathcal{R}_n^+\mathcal{R}_{n-1}^+\mathcal{R}_{n-2}^+$ is well-defined and it only changes the direction of the arrow $i \to i+1$. Thus applying this sequence over and over (for increasing i), we can fix the direction of the horizontal arrows.
4. Finally, we can fix the direction of the arrows $n-1 \to n-2$ and $n \to n-2$ by possibly applying $\mathcal{R}_{n-1}^-$ resp. $\mathcal{R}_n^-$.

To illustrate this algorithm, consider the following quiver Q:

$$\begin{tikzcd}
1 \arrow[dr] & & & & & n-1 \\
& 2 \arrow[dl] & 3 \arrow[l] \arrow[r] & \cdot - \cdot - \cdot \arrow[r] & n-2 \arrow[ur] \arrow[dr] & \\
0 & & & & & n
\end{tikzcd}$$

We have that $s_n s_{n-1} s_0 s_2 s_1 s_0 Q_{sp} = Q$ and therefore an indecomposable representation $\mathbb{V} \ncong \mathbb{S}_i$ of Q_{sp} yields an indecomposable representation $\mathcal{R}_n^- \mathcal{R}_{n-1}^- \mathcal{R}_0^- \mathcal{R}_2^- \mathcal{R}_1^- \mathcal{R}_0^- \mathbb{V}$ of Q. In general, by Corollary 3.52, there is a one-to-one correspondence between the isomorphism classes of indecomposable representations of Q_{sp} and any other quiver Q of type $\tilde{D_n}$ which shows that all quivers of type $\tilde{D_n}$ are not of finite-type.

<u>$\tilde{E_6}$:</u> We first reduce the problem to the following quiver Q_{sp} of type $\tilde{E_6}$.

$$\begin{tikzcd}
& & 0 \arrow[d] & & \\
& & 2 \arrow[d] & & \\
1 \arrow[r] & 3 \arrow[r] & 4 & 5 \arrow[l] & 6 \arrow[l]
\end{tikzcd}$$

Similarly as in the $\tilde{D_n}$-case, for any quiver Q of type $\tilde{E_6}$, we can find a sequence of reflection functors such that applying this sequence to Q_{sp} yields the quiver Q. Thus, using Corollary 3.52, it is enough to prove that Q_{sp} is not of finite-type. Now, instead of just giving a family of indecomposable representations, we show how we find such indecomposable representations. This is useful since this idea can then be extended to the cases $\tilde{E_7}$ and $\tilde{E_8}$. First, we note that by deleting the vertex 0, we get a Dynkin quiver Q' (see proof of Theorem 3.16). We also notice that the unique smallest positive radical vector $\delta = (1, 1, 2, 2, 3, 2, 1) \in \mathbb{N}^7$ yields a vector $\delta' = (1, 2, 2, 3, 2, 1) \in \mathbb{N}^6$ (take $\delta_i = \delta'_i$ for all $i \in Q'_0$) such that $q_{Q'}(\delta') = 1$ and thus δ' is a root of Q'. From part (1) we know that there exists an indecomposable representation $\mathbb{V}$ of Q' that has dimension vector δ'. Now, for any $\lambda \in k$, we get an indecomposable representation $\mathbb{V}_\lambda$ of Q by setting $V_0 = k$ and the map between V_0 and $V_2 = k^2$ to be $\begin{pmatrix} 1 \\ \lambda \end{pmatrix}$.

$$\begin{tikzcd}
& & k \arrow[d, "\begin{pmatrix} 1 \\ \lambda \end{pmatrix}"] & & \\
& & V_2 = k^2 \arrow[d, "v_b"] & & \\
V_1 = k \arrow[r, "v_a"] & V_3 = k^2 \arrow[r, "v_c"] & V_4 = k^3 & V_5 = k^2 \arrow[l, "v_d"'] & V_6 = k \arrow[l, "v_e"']
\end{tikzcd}$$

Since $\mathbb{V}$ is an indecomposable representation of Q', we conclude that $\mathbb{V}_\lambda$ is also indecomposable. Moreover, if $\phi \colon \mathbb{V}_{\lambda_1} \xrightarrow{\cong} \mathbb{V}_{\lambda_2}$, we have that (w.l.o.g) $\phi_i = \mathrm{id}_{V_i}$ for $i \in \{1, 2, 3, 4, 5, 6\}$. Thus, we get

$$\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} \phi_0 = \phi_2 \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} = \mathrm{id}_{V_2} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix},$$

which implies $\lambda_1 = \lambda_2$. Therefore, $\mathbb{V}_\lambda$ are indecomposable, pairwise non-isomorphic (for different choices of $\lambda \in k$) representations of Q. More generally, for any $n \in \mathbb{N}$, we get that the following representations are indecomposable and pairwise non-isomorphic.

$$\begin{tikzcd}
 & & k^n \arrow[d, "\begin{pmatrix} \mathbb{1}_{k^n} \\ J_{\lambda,n} \end{pmatrix}"] & & \\
 & & k^{2n} \arrow[d, "v_b \otimes \mathbb{1}_{k^n}"] & & \\
k^n \arrow[r, "v_a \otimes \mathbb{1}_{k^n}"] & k^{2n} \arrow[r, "v_c \otimes \mathbb{1}_{k^n}"] & k^{3n} & k^{2n} \arrow[l, "v_d \otimes \mathbb{1}_{k^n}"'] & k^n \arrow[l, "v_e \otimes \mathbb{1}_{k^n}"']
\end{tikzcd}$$

The tensor notation $v_\alpha \otimes \mathbb{1}_{k^n}$ comes from the fact that for all vertices $i \in \{1, 2, 3, 4, 5, 6\}$ we have that the vector space at vertex i is isomorphic to $V_i \otimes k^n$ and for all arrows $\alpha \in \{a, b, c, d, e\}$ we have that the map is given by $v_\alpha \otimes \mathbb{1}_{k^n}$. The maps are explicitly given by the block matrices

$$v_\alpha \otimes \mathbb{1}_{k^n} = \begin{pmatrix} v_\alpha & 0 & \dots & 0 \\ 0 & v_\alpha & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & v_\alpha \end{pmatrix} \quad \text{for } \alpha \in \{a, b, c, d, e\}.$$

<u>$\tilde{E}_7$ and $\tilde{E}_8$</u>: The cases for quivers of type $\tilde{E}_7$ and $\tilde{E}_8$ are analogous to the case of $\tilde{E}_6$-quivers. Indeed, by deleting the vertex 0, we get the roots $\delta' = (2, 2, 3, 4, 3, 2, 1)$ resp. $\delta' = (2, 3, 4, 6, 5, 4, 3, 2)$ of a quiver of type E_7 resp. E_8. Then, we can use the same arguments as above. $\square$

Remark 3.72. *In the proof of Gabriel's Theorem (step 1), we only needed that for a Dynkin quiver Q and any indecomposable representation $\mathbb{V}$, there exists a sequence of indices $i_1, \dots, i_s$ (possibly with repetitions) such that $\mathcal{R}^+_{i_s} \dots \mathcal{R}^+_{i_1}$ is well-defined and $\mathcal{R}^+_{i_s} \dots \mathcal{R}^+_{i_1} \mathbb{V} = 0$. This explains why one does not need to consider the Weyl group $W(Q)$ to prove the 'if part' of Gabriel's Theorem in the special case of the A_n-type quivers. Indeed, we mainly need the Weyl group to prove the existence of a finite number r such that $\underbrace{\mathcal{C}^+ \dots \mathcal{C}^+}_{r \text{ times}} \mathbb{V} = 0$ for any indecomposable representation $\mathbb{V}$ of a Dynkin quiver Q. In Chapter 4, we explicitly show how we can construct such a sequence of indices for A_n-type quivers.*

Chapter 4

The Special Case: A_n-type Quivers

In this chapter, we focus on the special case of A_n-type quivers ($n \geq 1$), a subset of the Dynkin quivers. These quivers are crucial in topological data analysis as we will see in Chapter 5. The theory in Chapters 2 and 3 is more explicit for A_n-type quivers. In this chapter we build indecomposable representations for A_n-type quivers, explore Coxeter functors on them, and establish the 'if part' of Gabriel's Theorem, demonstrating the finiteness of A_n-type quivers. This chapter is based on Appendix A of the book 'Persistence Theory: From Quiver Representations to Data Analysis' written by Steve Y. Oudot [1].

4.1 Interval Representations

We denote the set $\{b, b+1, \ldots, d\}$ by $[b, d]$. Recall that a quiver Q is of type A_n if its underlying graph $\bar{Q}$ is of the following form.

$$\underset{1}{\bullet} \text{———} \underset{2}{\bullet} \text{———} \cdots \text{———} \underset{n-1}{\bullet} \text{———} \underset{n}{\bullet}$$

An important example is the linear quiver L_n.

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n-1}{\bullet} \longrightarrow \underset{n}{\bullet}$$

A morphism $\phi = (\phi_i)_{1 \leq i \leq n}$ between two quiver representations $\mathbb{V}$ and $\mathbb{W}$ of an A_n-type quiver Q is given by the following commutative diagram.

$$\begin{tikzcd}
V_1 \arrow[r, dash, "v_1"] \arrow[d, "\phi_1"] & V_2 \arrow[r, dash, "v_2"] \arrow[d, "\phi_2"] & V_3 \arrow[r, dash, "v_3"] \arrow[d, "\phi_3"] & \cdots \arrow[r, dash, "v_{n-1}"] & V_n \arrow[d, "\phi_n"] \\
W_1 \arrow[r, dash, "w_1"] & W_2 \arrow[r, dash, "w_2"] & W_3 \arrow[r, dash, "w_3"] & \cdots \arrow[r, dash, "w_{n-1}"] & W_n
\end{tikzcd}$$

Lemma 4.1. *For all A_n-type quivers Q, the set of positive roots of q_Q is given by the vectors of the form $x = (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)^T$ with the first and last 1's occurring at positions $b \leq d$ in the range $[1, n]$.*

Proof. First, we recall that the Tits form q_Q is independent of the orientations of the arrows of Q, which allows us to simultaneously find the roots of all A_n-type quivers (for a fixed n). From Example 3.11, we know the Tits form for A_n-type quivers Q:

$$q_Q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{a \in Q_1} x_{t_a} x_{h_a} = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n-1} x_i x_{i+1} = \frac{1}{2}[x_1^2 + x_n^2 + \sum_{i=1}^{n-1} (x_i - x_{i+1})^2].$$

Since q_Q is positive definite (Q is Dynkin), we know from Corollary 3.31, that the roots of q_Q are exactly given by the set $\{x \in \mathbb{Z}^n \mid q_Q(x) = 1\}$. Thus for any positive root $x \in \mathbb{Z}^n$, we have $x_i \in \{0, 1\}$ for all coordinates/vertices i and that there can be at most two pairs of consecutive vertices/coordinates (x_i, x_{i+1}) with difference 1. This shows that the roots are of the desired form $x = (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)^T$. $\square$

We know from Gabriel's Theorem that for every A_n-type quiver Q, we have a bijection between the isomorphism classes of indecomposable representations and the positive roots of q_Q:

$$\phi\colon \mathcal{I} \to \Phi^+(Q), [\mathbb{V}] \mapsto \underline{\dim}\mathbb{V}.$$

Therefore, we use Lemma 4.1 to identify the indecomposable representations of the A_n-type quivers.

Proposition 4.2. *For all A_n-type quivers Q, each indecomposable representation is isomorphic to one of the representations in Figure 4.1 for some interval $[b,d] \subseteq [1,n]$.*

$$\underbrace{0 \overset{0}{\text{——}} \cdots \overset{0}{\text{——}} 0}_{[1,b-1]} \overset{0}{\text{——}} \underbrace{k \overset{\mathrm{id}}{\text{——}} \cdots \overset{\mathrm{id}}{\text{——}} k}_{[b,d]} \overset{0}{\text{——}} \underbrace{0 \overset{0}{\text{——}} \cdots \overset{0}{\text{——}} 0.}_{[d+1,n]}$$

Figure 4.1: The interval representation $\mathbb{I}_Q[b,d]$.

Definition 4.3. *The representations in Figure 4.1 are called* ***interval representations*** *and are denoted by $\mathbb{I}_Q[b,d]$.*

Proof of Proposition 4.2. Using Gabriel's Theorem, let $\mathbb{V}$ be the indecomposable representation of Q that belongs to the positive root $x = (0,\ldots,0,1,\ldots,1,0,\ldots,0)^T$ with the 1's occurring in the interval $[b,d]$ (see Lemma 4.1). The representation $\mathbb{V}$ has the vector space k at every vertex $i \in [b,d]$. The maps from and to 0 vector spaces are 0.

$$0 \overset{0}{\text{——}} \cdots \overset{0}{\text{——}} 0 \overset{0}{\text{——}} V_b = k \overset{v_b}{\text{——}} k \overset{v_{b+1}}{\text{——}} \cdots \overset{v_{d-1}}{\text{——}} k = V_d \overset{0}{\text{——}} 0 \overset{0}{\text{——}} \cdots \overset{0}{\text{——}} 0$$

Moreover, the maps between successive copies of k need to be isomorphisms, because otherwise $\mathbb{V}$ could be decomposed further. Indeed, if $k \overset{v_a}{\text{——}} k$ is not an isomorphism, then $v_a = 0$ and we have the decomposition $\mathbb{V} = \mathbb{V}_1 \oplus V_2$, where the representations $\mathbb{V}_1$ and $\mathbb{V}_2$ are as follows.

$$\mathbb{V}\colon \quad \cdots \overset{0}{\text{------}} V_b = k \overset{v_b}{\text{——}} k \text{ ------ } V_a = k \overset{v_a=0}{\text{——}} k = V_{a+1} \text{ ------ } k = V_d \overset{0}{\text{------}} \cdots$$

$$\mathbb{V}_1\colon \quad \cdots \overset{0}{\text{------}} V_b' = k \overset{v_b}{\text{——}} k \text{ ------ } V_a' = k \overset{0}{\text{——}} 0 = V_{a+1}' \overset{0}{\text{------}} 0 = V_d' \overset{0}{\text{------}} \cdots$$

$$\mathbb{V}_2\colon \quad \cdots \overset{0}{\text{------}} V_b'' = 0 \overset{0}{\text{——}} 0 \overset{0}{\text{------}} V_a'' = 0 \overset{0}{\text{——}} k = V_{a+1}'' \text{ ------ } k = V_d'' \overset{0}{\text{------}} \cdots$$

In addition, $\mathbb{V}$ is isomorphic to the interval representation $\mathbb{I}_Q[b,d]$. Indeed, for any isomorphism $k \overset{v_a}{\to} k$, consider the following commutative diagram.

$$\begin{tikzcd} k \arrow[r, "v_a=\alpha"] \arrow[d, "\phi_{t_a}=\alpha"'] & k \arrow[d, "\phi_{h_a}=\mathrm{id}"] \\ k \arrow[r, "\mathrm{id}"] & k \end{tikzcd}$$

Notice that this allows us to successively define the maps $\phi_b, \phi_{b+1}, \ldots, \phi_d$ so that the resulting morphism $\phi = (\phi_i)_{i=1}^n$ is an isomorphism between $\mathbb{V}$ and $\mathbb{I}_Q[b,d]$. $\square$

Now, we can use the Krull-Remak-Schmidt Theorem to decompose any representation of an A_n-type quiver.

Definition 4.4. *For an A_n-type quiver Q and a finite-dimensional representation $\mathbb{V}$, the following decomposition is called the* ***interval decomposition*** *of $\mathbb{V}$:*

$$\mathbb{V} \cong \bigoplus_{j=1}^{r} \mathbb{I}_Q[b_j, d_j].$$

We have now found all isomorphism classes of indecomposable representations of an A_n-type quiver Q. Since there are only finitely many possible choices for $b \leq d$ in $[1, n]$, we have also shown that A_n-type quivers are of finite-type. However, we have used Gabriel's Theorem, thus our argument above is a circular argument. Therefore, we want to prove the 'if part' of Gabriel's Theorem for A_n-type quivers, i.e. that quivers of type A_n are of finite-type and that the map $\phi \colon \mathcal{I} \to \Phi^+(Q), [\mathbb{V}] \mapsto \underline{\dim}\mathbb{V}$ is an isomorphism. To do this, we first observe the action of reflection functors and Coxeter functors on A_n-type quivers.

4.2 Coxeter Functors on A_n-Type Quivers

In Remark 3.72, we noticed the following: in the proof of Gabriel's Theorem we only needed that for a Dynkin quiver Q and any indecomposable representation $\mathbb{V}$, there exists a sequence of indices $i_1, \ldots, i_s$ (possibly with repetitions) such that $\mathcal{R}^+_{i_s} \ldots \mathcal{R}^+_{i_1}$ is well-defined and $\mathcal{R}^+_{i_s} \ldots \mathcal{R}^+_{i_1}\mathbb{V} = 0$. For an A_n-type quiver Q, we now construct such a sequence explicitly. We first consider the linear quiver L_n.

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{n-1}{\bullet} \arrow[r] & \underset{n}{\bullet} \end{tikzcd}$$

Proposition 4.5. *[1, Ex. A.19] For every finite-dimensional representation $\mathbb{V}$ of the linear quiver L_n, we have $\underbrace{\mathcal{C}^+ \ldots \mathcal{C}^+}_{n \text{ times}} \mathbb{V} = 0$. Here, $\mathcal{C}^+$ is the Coxeter functor belonging to the $(+)$-admissible ordering $n, n-1, \ldots, 1$.*

Proof. Notice that the natural ordering of the vertices is a $(-)$-admissible ordering and therefore, the ordering $n, n-1, \ldots, 1$ is $(+)$-admissible. Now, let $\mathbb{V}$ be an indecomposable representation of L_n and let $x = (x_1, x_2, \ldots, x_n)^T = \underline{\dim}\mathbb{V}$. We now apply the Coxeter functor $\mathcal{C}^+$ and by Theorem 3.49, we get:

$$\begin{aligned} \underline{\dim}\mathcal{R}^+_n\mathbb{V} = 0 &\text{ or } \sigma_n(x) = (x_1, x_2, \ldots, x_{n-1}, x_{n-1} - x_n)^T, \\ \underline{\dim}\mathcal{R}^+_{n-1}\mathcal{R}^+_n\mathbb{V} = 0 &\text{ or } \sigma_{n-1}\sigma_n(x) = (x_1, x_2, \ldots, x_{n-2} - x_n, x_{n-1} - x_n)^T. \\ &\vdots \\ \underline{\dim}\mathcal{R}^+_2 \ldots \mathcal{R}^+_{n-1}\mathcal{R}^+_n\mathbb{V} = 0 &\text{ or } \sigma_2 \ldots \sigma_{n-1}\sigma_n(x) = (x_1, x_1 - x_n, \ldots, x_{n-2} - x_n, x_{n-1} - x_n)^T. \\ \underline{\dim}\mathcal{R}^+_1\mathcal{R}^+_2 \ldots \mathcal{R}^+_{n-1}\mathcal{R}^+_n\mathbb{V} = 0 &\text{ or } \sigma_1\sigma_2 \ldots \sigma_{n-1}\sigma_n(x) = (-x_n, x_1 - x_n, \ldots, x_{n-2} - x_n, x_{n-1} - x_n)^T. \end{aligned}$$

Since dimension vectors are non-negative, we conclude that either $\mathcal{C}^+\mathbb{V} = \mathcal{R}^+_1\mathcal{R}^+_2 \ldots \mathcal{R}^+_{n-1}\mathcal{R}^+_n\mathbb{V} = 0$ or that $x_n = 0$. We can iterate this process successively and get:

$$\begin{aligned} \underline{\dim}\mathcal{C}^+\mathbb{V} &= 0 \text{ or } (0, x_1, x_2, \ldots, x_{n-2}, x_{n-1})^T, \\ \underline{\dim}\mathcal{C}^+\mathcal{C}^+\mathbb{V} &= 0 \text{ or } (0, 0, x_1, \ldots, x_{n-3}, x_{n-2})^T, \\ &\vdots \\ \underline{\dim}\underbrace{\mathcal{C}^+ \ldots \mathcal{C}^+}_{n-1 \text{ times}}\mathbb{V} &= 0 \text{ or } (0, 0, 0, \ldots, 0, x_1)^T, \\ \underline{\dim}\underbrace{\mathcal{C}^+ \ldots \mathcal{C}^+}_{n \text{ times}}\mathbb{V} &= 0. \end{aligned}$$

□

We now generalize the statement of Proposition 4.5 to be true for all A_n-type quivers.

Proposition 4.6. *For an A_n-type quiver Q and a finite-dimensional representation $\mathbb{V}$, there exists a sequence of reflection functors such that*

$$\underbrace{\mathcal{C}^+ \ldots \mathcal{C}^+}_{n \text{ times}}\mathcal{R}^+_1 \ldots \mathcal{R}^+_{i_r}\mathcal{R}^+_1 \ldots \mathcal{R}^+_{i_{r-1}} \ldots \mathcal{R}^+_1 \ldots \mathcal{R}^+_{i_1}\mathbb{V} = 0.$$

Proof. If Q is not the linear quiver L_n, let $i_1 < i_2 < \ldots < i_r$ be the heads of the backward arrows.

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{i_1}{\bullet} \longleftarrow \underset{i_1+1}{\bullet} \text{ ---- } \cdots \text{ ---- } \underset{i_r}{\bullet} \longleftarrow \underset{i_r+1}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n}{\bullet}$$

If we apply the sequence $s_1 \ldots s_{i_1}$ on Q, we get the same quiver, but the arrow between i_1 and $i_1 + 1$ has changed direction.

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{i_1}{\bullet} \longleftarrow \underset{i_1+1}{\bullet} \text{ ---- } \cdots \text{ ---- } \underset{i_r}{\bullet} \longleftarrow \underset{i_r+1}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n}{\bullet}$$

Also, notice that the sequence of reflection functors $\mathcal{R}_1^+ \ldots \mathcal{R}_{i_1}^+$ is well-defined. We can repeat this process for $i_2, \ldots, i_r$ and get that $s_1 \ldots s_{i_r} s_1 \ldots s_{i_{r-1}} \ldots s_1 \ldots s_{i_1} Q$ is the linear quiver L_n. In addition, from Corollary 3.50, we get that for any indecomposable representation $\mathbb{V}$ of Q, the representation $\mathcal{R}_1^+ \ldots \mathcal{R}_{i_r}^+ \mathcal{R}_1^+ \ldots \mathcal{R}_{i_{r-1}}^+ \ldots \mathcal{R}_1^+ \ldots \mathcal{R}_{i_1}^+ \mathbb{V}$ is either 0 or an indecomposable representation of L_n. Using Proposition 4.5, we conclude that

$$\underbrace{\mathcal{C}^+ \ldots \mathcal{C}^+}_{n \text{ times}} \mathcal{R}_1^+ \ldots \mathcal{R}_{i_r}^+ \mathcal{R}_1^+ \ldots \mathcal{R}_{i_{r-1}}^+ \ldots \mathcal{R}_1^+ \ldots \mathcal{R}_{i_1}^+ \mathbb{V} = 0. \qquad \square$$

4.3 Proof of Gabriel's Theorem for A_n-Type Quivers

We are now ready to prove the following theorem.

Theorem 4.7 (**Gabriel's Theorem for A_n-type quivers**). *Let Q be a quiver of type A_n and let k be an arbitrary field. Then:*

1. *Q is of finite-type.*
2. *The correspondence $\mathbb{V} \mapsto \underline{\dim}\mathbb{V}$ induces a bijection between the set of isomorphism classes of indecomposable representations of Q and the set $\Phi^+(Q) = \Phi(Q) \cap \mathbb{N}^n$ of positive roots of Q.*

Proof. The proof is analogous to Step 1 of the proof of Theorem 3.71. Indeed, we have shown that there exists a minimal sequence of reflection functors $\mathcal{R}_{i_{t+1}}^+ \ldots \mathcal{R}_{i_t}^+$, such that $\mathcal{R}_{i_{t+1}}^+ \ldots \mathcal{R}_{i_t}^+ \mathbb{V} = 0$ and $\mathcal{R}_{i_t}^+ \ldots \mathcal{R}_{i_t}^+ \mathbb{V} \neq 0$. The same argument as in the proof of Theorem 3.71 now yields a well-defined and injective map $\phi\colon \mathcal{I} \to \Phi^+(Q), [\mathbb{V}] \mapsto \underline{\dim}\mathbb{V}$. This shows that Q is of finite-type since we have shown that the set of positive roots $\Phi^+(Q)$ is finite for an A_n-type quiver. Notice that we do not need the result for a general Dynkin quiver (Lemma 3.30). In addition, proving the surjectivity of ϕ can be done explicitly in the case of A_n-type quivers. Indeed, we know that every positive root $x \in \Phi^+(Q)$ is of the form $x = (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)^T$ and we have that the interval representation $\mathbb{I}_Q[b, d]$ has this dimension vector for a suitable choice of $b \leq d$. It remains to show that the interval representations are indeed indecomposable. This can be either seen by direct observation or we can show that the endomorphism ring $\mathrm{End}(\mathbb{I}_Q[b, d])$ is local and then apply Proposition 2.38. Now, let $\psi \in \mathrm{End}(\mathbb{I}_Q[b, d])$ and observe the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r, dashed, dash, "0"] \arrow[d, "0"] & 0 \arrow[r, dash, "0"] \arrow[d, "0"] & k \arrow[r, dash, "\mathrm{id}"] \arrow[d, "\psi_b"] & k \arrow[r, dash, "\mathrm{id}"] \arrow[d, "\psi_{b+1}"] & \cdots \arrow[r, dash, "\mathrm{id}"] & k \arrow[r, dash, "0"] \arrow[d, "\psi_d"] & 0 \arrow[r, dashed, dash, "0"] \arrow[d, "0"] & 0 \arrow[d, "0"] \\
0 \arrow[r, dashed, dash, "0"] & 0 \arrow[r, dash, "0"] & k \arrow[r, dash, "\mathrm{id}"] & k \arrow[r, dash, "\mathrm{id}"] & \cdots \arrow[r, dash, "\mathrm{id}"] & k \arrow[r, dash, "0"] & 0 \arrow[r, dashed, dash, "0"] & 0
\end{tikzcd}$$

By commutativity, we have $\psi_b = \psi_{b+1} = \ldots = \psi_d = \alpha \in k$. Therefore, we can embed $\mathrm{End}(\mathbb{I}_Q[b, d])$ in the field k. However, since $\mathrm{End}(\mathbb{I}_Q[b, d])$ is a k-vector space, it follows that $\mathrm{End}(\mathbb{I}_Q[b, d]) = k$ which is local because fields are local. This shows that $\phi\colon \mathcal{I} \to \Phi^+(Q)$ is an isomorphism. $\square$

Remark 4.8. *We highlight the results from Chapter 3 that are needed in the proof above, and which results are not needed. We do this by sections:*

3.1: We have used the explicit form of the Tits form q_Q for an A_n-type quiver and the fact that the Tits form is independent of the orientation. We did not need radical vectors and results about tame and wild quivers.

3.2: We have used the central property of simple reflections: $\underline{\dim}\mathcal{R}_i^{\pm}\mathbb{V} = \sigma_i(\underline{\dim}\mathbb{V})$ (for indecomposable representations $\mathbb{V} \neq \mathbb{S}_i$). However, we did not use the Weyl group and it was sufficient to consider the set $\{x \in \mathbb{Z}^n \mid q_Q(x) = 1\}$ as our root system. Indeed, we were able to describe the roots explicitly, which showed that $\{x \in \mathbb{Z}^n \mid q_Q(x) = 1\}$ is finite.

3.3: The proof relied heavily on reflection functors. Therefore, the whole Section 3.3 is needed in the proof above.

3.4: We have used Coxeter functors. However, we did not use its connection to the Coxeter transformation and the Weyl group.

4.4 The Diamond Principle

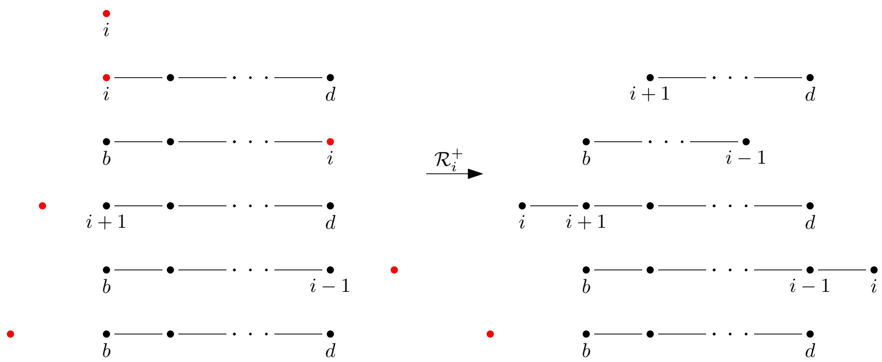
Let Q be an A_n-type quiver and let $i \in Q_0$ be a sink. Then, any representation $\mathbb{V}$ and its reflection $\mathbb{W} = \mathcal{R}_i^+\mathbb{V}$ can be expressed by the following diagram, because $\mathcal{R}_i^+$ only changes the vector space at the vertex i and its incident arrows resp. the maps corresponding to those arrows.

$$\begin{tikzcd}
 & & & V_i & & & \\
V_1 \arrow[r, dash] & \cdots \arrow[r, dash] & V_{i-1} \arrow[ur] & & V_{i+1} \arrow[ul] \arrow[r, dash] & \cdots \arrow[r, dash] & V_n \\
 & & & W_i \arrow[ul] \arrow[ur] & & &
\end{tikzcd}$$

Lemma 4.9. *We have the following transformation rules for the interval representations:*

$$\begin{aligned}
\mathcal{R}_i^+\mathbb{I}_Q[i,i] &= 0, \\
\mathcal{R}_i^+\mathbb{I}_Q[i,d] &= \mathbb{I}_Q[i+1,d] \ \textit{if } i < d, \\
\mathcal{R}_i^+\mathbb{I}_Q[b,i] &= \mathbb{I}_Q[b,i-1] \ \textit{if } b < i, \\
\mathcal{R}_i^+\mathbb{I}_Q[i+1,d] &= \mathbb{I}_Q[i,d] \ \textit{if } i+1 \leq d, \\
\mathcal{R}_i^+\mathbb{I}_Q[b,i-1] &= \mathbb{I}_Q[b,i] \ \textit{if } b \leq i-1, \\
\mathcal{R}_i^+\mathbb{I}_Q[b,d] &= \mathbb{I}_Q[b,d] \ \textit{otherwise.}
\end{aligned}$$

The following picture visualizes this result. The black interval represents the interval representation $\mathbb{I}_Q[b,d]$ and the red dot shows the position of the vertex i.



Proof. The result follows directly from the definition of the reflection functor $\mathcal{R}_i^+$. We have already seen the first rule and we prove the second and fifth rule, thus proving one case each, where the interval gets shorter resp. longer. The other cases are analogous. Using that $\mathcal{R}_i^+\mathbb{I}_Q[i,d]$ is indecomposable (Corollary 3.50) and thus isomorphic to an interval representation, it suffices to prove that $(\mathcal{R}_i^+\mathbb{I}_Q[i,d])_i = 0$. Indeed, we have:

$$(\mathcal{R}_i^+\mathbb{I}_Q[i,d])_i = \ker \xi_i = \ker(0 \oplus k \to k) = \ker(k \xrightarrow{\text{id}} k) = 0.$$

For the fifth rule, it suffices to prove that $(\mathcal{R}_i^+ \mathbb{I}_Q[b, i-1])_i = k$. Indeed, we have:

$$(\mathcal{R}_i^+ \mathbb{I}_Q[b, i-1])_i = \ker \xi_i = \ker(k \oplus 0 \to 0) = \ker(k \xrightarrow{0} k) = k. \qquad \square$$

Now, we understand how reflection functors operate on interval decomposition. Using this, we consider the above diagram and we construct a general setting, which behaves similarly to the transformation rules in Lemma 4.9.

Definition 4.10. *Let Q be an A_n-type quiver and let i be a sink. Given two finite-dimensional representations $\mathbb{V} \in \mathrm{rep}_{\mathrm{k}}(Q)$ and $\mathbb{W} \in \mathrm{rep}_{\mathrm{k}}(s_iQ)$, that differ only at the spaces V_i, W_i, and their incident maps, we get the following diagram, where the central rhombus is called a* ***diamond****:*

$$\begin{tikzcd} & & & V_i & & & \\ V_1 \arrow[r, dash] & \cdots \arrow[r, dash] & V_{i-1} \arrow[ur, "v_c"] & & V_{i+1} \arrow[ul, "v_d"'] \arrow[r, dash] & \cdots \arrow[r, dash] & V_n \\ & & & W_i \arrow[ul, "w_a"] \arrow[ur, "w_b"'] & & & \end{tikzcd}$$

The diamond is called ***exact*** *if* $\mathrm{im}\, f = \ker g$ *in the following sequence*

$$W_i \xrightarrow{f} V_{i-1} \oplus V_{i+1} \xrightarrow{g} V_i,$$

where $f \colon x \mapsto (w_a(x), w_b(x))$ and $g \colon (x, y) \mapsto v_c(x) + v_d(y)$.

Theorem 4.11 **(Diamond principle).** *Let Q be an A_n-type quiver and let i be a sink. Let $\mathbb{V} \in \mathrm{rep}_{\mathrm{k}}(Q)$ and $\mathbb{W} \in \mathrm{rep}_{\mathrm{k}}(s_iQ)$ be two finite-dimensional representations, that differ only at the spaces V_i, W_i, and their incident maps. If we suppose that the diamond is exact, then the interval decompositions of $\mathbb{V}$ and $\mathbb{W}$ are related to each other through the following matching rules:*

- *summands $\mathbb{I}_Q[i, i]$ and $\mathbb{I}_{s_iQ}[i, i]$ are unmatched,*
- *summands $\mathbb{I}_Q[b, i]$ are matched with summands $\mathbb{I}_{s_iQ}[b, i-1]$, and $\mathbb{I}_Q[b, i-1]$ with $\mathbb{I}_{s_iQ}[b, i]$,*
- *summands $\mathbb{I}_Q[i, d]$ are matched with summands $\mathbb{I}_{s_iQ}[i+1, d]$, and $\mathbb{I}_Q[i+1, d]$ with $\mathbb{I}_{s_iQ}[i, d]$,*
- *every other summand $\mathbb{I}_Q[b, d]$ is matched with the summand $\mathbb{I}_{s_iQ}[b, d]$.*

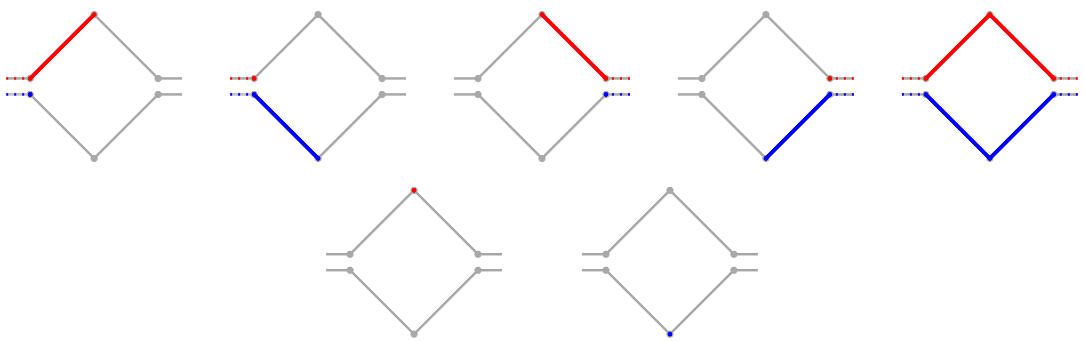


Figure 4.2: The matching rules from the diamond principle. The top row illustrates the second, third, and fourth matching rules. The bottom row shows the two unmatched cases.

The transformation result for the interval representations (Lemma 4.9) and the diamond principle look quite similar. Indeed, we prove that the results naturally imply each other.

Proof of the equivalence of Lemma 4.9 and the diamond principle. Step 1: We show Lemma 4.9 assuming the diamond principle. First, we prove the exactness of the diamond for the representations $\mathbb{V} = \mathbb{I}_Q[b, d]$ and $\mathbb{W} = \mathcal{R}_i^+ \mathbb{I}_Q[b, d]$.

$$\begin{tikzcd}
 & & & V_i & & & \\
V_1 \arrow[r, dash] & \cdots \arrow[r, dash] & V_{i-1} \arrow[ur, "v_c"] & & V_{i+1} \arrow[ul, "v_d"'] \arrow[r, dash] & \cdots \arrow[r, dash] & V_n \\
 & & & W_i = \ker \xi_i \arrow[ul, "w_a"] \arrow[ur, "w_b"'] & & &
\end{tikzcd}$$

Notice that $g = \xi_i$. Thus, we get $\operatorname{im} f = W_i = \ker \xi_i = \ker g$ and thus the diamond is exact. Now observe that the first matching in the diamond principle implies the first transformation in the lemma. The second matching implies the second and fourth transformations. The third matching implies the third and fifth transformations and the last matching implies the last transformation.

Step 2: We now assume Lemma 4.9. Let $\mathbb{V}, \mathbb{W}$ be as in the diamond principle and such that the diamond is exact. We claim that $\mathbb{W} \cong \mathbb{U} \oplus \mathbb{K}$, where $\mathbb{U} = \mathcal{R}_i^+ \mathbb{V}$ and $\mathbb{K} = \bigoplus_{j=1}^r \mathbb{I}_{s_i Q}[i,i] = \bigoplus_{j=1}^r \mathbb{S}_i$ is the representation of $s_i Q$ made from $r = \dim \ker f$ copies of the representation $\mathbb{I}_{s_i Q}[i,i] = \mathbb{S}_i$. Indeed, from the definition of $\mathcal{R}_i^+ \mathbb{V}$ and the exactness of the diamond, we have $U_j = V_j$ for all $j \neq i$ and $U_i = \ker \xi_i = \ker g = \operatorname{im} f$. Now let $C = \ker f^{\perp} \subseteq W_i$. From the first isomorphism theorem, we conclude that the map

$$f|_C \colon C \to \operatorname{im} f = U_i$$

is an isomorphism. We can also take an arbitrary isomorphism $h \colon \ker f \to K_i$ (here, we need $r = \dim \ker f$) and we can define an isomorphism of representations $\phi \colon \mathbb{W} \to \mathbb{U} \oplus \mathbb{K}$ as follows

$$\phi_j = \begin{cases} f|_C \oplus h = \begin{pmatrix} f|_C & 0 \\ 0 & h \end{pmatrix} & \text{if } j = i, \\ \mathbb{1}_{V_j} & \text{if } j \neq i. \end{cases}$$

One can explicitly check that this is an isomorphism of representations. Now, using the transformation rules in Lemma 4.9, we get the matching rules in the diamond principle. □

Using the equivalence of Lemma 4.9 and Theorem 4.11, one may attempt to proof the 'if part' of Gabriel's Theorem for A_n-type quivers using the diamond principle. It turns out that this is possible [1, Section 4.4.2.]. It also turns out that the diamond principle is helpful because it is easier to check if $\operatorname{im} f = \ker g$ rather than compute reflections of representations and check if they are isomorphic. We use the diamond principle in the setting of zigzag persistence in Section 5.3.

Chapter 5

Applications of Quiver Representation Theory to Persistent Homology

In this chapter, we explain the connection between topological data analysis and quiver representation theory. We first introduce simplicial complexes and homology groups. Using these concepts we establish a connection between the persistence of topological spaces and quiver representations. In Section 5.2 we focus on the topological aspect of this connection. Section 5.3 is devoted to the algebraic aspects of this connection. In particular, we show a connection between zigzag persistent homology and the diamond principle from Chapter 4.

5.1 Simplicial Complexes and Homology

In this section, we introduce simplicial complexes which are topological spaces that can be built from simple building blocks called simplices. We then introduce simplicial homology. This section roughly follows the outline of Chapters III.1 and IV.1 of the book 'Computational Topology: An Introduction' by Herbert Edelsbrunner and John Harer [7] and Chapter 1 of the paper 'Barcodes: The persistent topology of data' by Robert Ghrist [8].

Definition 5.1. *Let $A = \{v_0, v_1, \dots, v_n\} \subseteq \mathbb{R}^m$. We say that A is* ***in general position*** *if A is not contained in an affine hyperplane of dimension less than n. This is equivalent to the condition that the vectors $v_1 - v_0, \dots, v_n - v_0$ are linearly independent resp. to the condition that the subspace generated by A has dimension at least n.*

Definition 5.2. *Let $A = \{v_0, v_1, \dots, v_n\} \subseteq \mathbb{R}^m$ be points in general position. The* ***n-simplex*** *$\sigma = [v_0, v_1, \dots, v_n]$ is defined to be the convex hull of A, i.e. the smallest subset of $\mathbb{R}^m$ that contains all the points $v_0, v_1, \dots, v_n$. We can also represent the n-simplex using linear combinations:*

$$[v_0, \dots, v_n] = \{\sum_{i=0}^{n} \lambda_i v_i \mid \sum_{i=0}^{n} \lambda_i = 1, \forall\, 0 \leq j \leq n \colon \lambda_j \geq 0\}.$$

We say that the points v_i ***span*** *σ. The* ***dimension*** *of σ is $\dim \sigma = n$.*

Definition 5.3. *For A in general position and the associated n-simplex σ, the points $v_i \in A$ are called* ***vertices*** *and for each subset $B \subseteq A$, the convex hull of B, denoted by τ, is called a* ***face*** *of σ. Notice that B is also in general position and therefore, τ is a simplex itself. We often write $\tau \leq \sigma$. A face is called* ***proper*** *if $B \subsetneq A$ and is denoted by $\tau < \sigma$. Moreover, the* ***boundary*** *of σ is denoted by $\partial\sigma$ and it is the union of all proper faces of σ. The* ***interior*** *is given by $\sigma - \partial\sigma$.*

We observe that a point $x \in \sigma$ lies in the interior of σ if and only if all its coefficients λ_i are positive. Indeed, if some coefficient $\lambda_i = 0$, then we have that x lies in the face $[v_0, \dots, \widehat{v_i}, \dots, v_n]$,

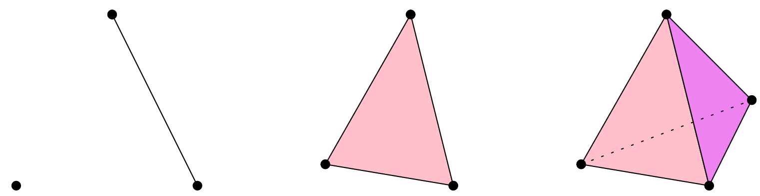


Figure 5.1: The simplices of dimensions 0, 1, 2, and 3 have special names and are called vertex, edge, triangle, and tetrahedron (from left to right).

where the hat indicates that the vertex v_i is omitted in the list of vertices. We conclude that every point $x \in \sigma$ lies in the interior of exactly one of the faces of σ. The vertices corresponding to this face are exactly the vertices v_j, for which $\lambda_j > 0$.

5.1.1 Geometric and Abstract Simplicial Complexes

We combine simplices to create a broader variety of topological spaces, called simplicial complexes. We do this in a way such that it is closed under taking faces and such that it has no improper intersections.

Definition 5.4. *A (geometric)* ***simplicial complex*** *K is a finite union of simplices such that for each simplex $\sigma \in K$ and any face $\tau \leq \sigma$, we have $\tau \in K$. Moreover, for two simplices $\sigma_1, \sigma_2 \in K$, we have that $\sigma_1 \cap \sigma_2$ is either empty of a face of both simplices σ_1 and σ_2. A* ***subcomplex*** *of K is a simplicial complex $L \subseteq K$. The* ***dimension*** *of K is given by the maximum dimension of its simplices.*

Definition 5.5. *For each $j \leq \dim K$, we define a particular subcomplex $K^{(j)}$, called the* ***j-skeleton*** *which consists of all simplices of dimension j or less. The 0-skeleton is usually called the* ***vertex set*** *$K^{(0)} = \operatorname{Vert} K$.*

A simplicial complex K is just a set of simplices and thus lacks a topology. We fix this by considering the topological space given by the union of the simplices together with the subspace topology inherited from the ambient Euclidean space $\mathbb{R}^m$ in which the simplices live. We denote this topological space by $|K|$. Because every point of a simplex belongs to the interior of one of its faces, we conclude that every point $x \in |K|$ belongs to the interior of exactly one simplex in K. This allows us to give the following description of $|K|$.

Definition 5.6. *Let K be a simplicial complex with vertices $v_0, v_1, \ldots, v_n$. Every point $x \in |K|$ belongs to the interior of exactly one simplex $\sigma = [v_0, v_1, \ldots, v_k] \in K$. Therefore, we have $x = \sum_{i=0}^k \lambda_i v_i$ with $\sum_{i=0}^k \lambda_i = 1$ and $\lambda_i > 0$ for all i. Setting $b_i(x) = \lambda_i$ for $0 \leq i \leq k$ and $b_i(x) = 0$ for $k+1 \leq i \leq n$, we have $x = \sum_{i=0}^n b_i(x) v_i$ and we call the $b_i(x)$ the* ***barycentric coordinates*** *of x in K.*

Remark 5.7. *If it is clear from the context that we consider a topological space, we often write K instead of $|K|$.*

It is often easier to describe or construct a simplicial complex abstractly instead of giving a full geometric description of its simplices.

Definition 5.8. *An* ***abstract simplicial complex*** *A is a finite collection of sets such that for all $\alpha \in A$ and all $\beta \subseteq \alpha$, we have $\beta \in A$. The sets in A are the* ***simplices*** *and the dimension of a simplex is $\dim \alpha = |\alpha| - 1$. The* ***dimension*** *of the complex A is given by the maximum dimension of its simplices. A* ***face*** *of $\alpha \in A$ is a non-empty subset $\beta \subseteq \alpha$ and it is* ***proper*** *if $\beta \subsetneq \alpha$. A* ***subcomplex*** *is an abstract simplicial complex $B \subseteq A$.*

We notice that this condition looks quite similar to the condition that for each simplex $\sigma \in K$ and each face $\tau \leq \sigma$, we have that $\tau \in K$. However, there is no explicit condition on the intersection of simplices. We now describe how one can switch from a geometric simplicial complex to an abstract simplicial complex (and vice versa).

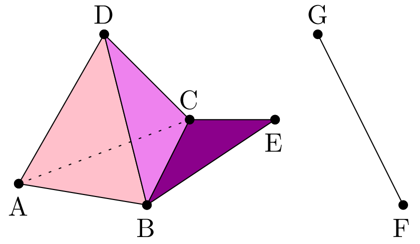


Figure 5.2: Simplicial complex with vertices A, B, C, D, E, F, G and simplices $[A, B]$, $[A, C]$, $[A, D]$, $[B, C]$, $[B, D]$, $[B, E]$, $[C, D]$, $[C, E]$, $[F, G]$, $[A, B, C]$, $[A, B, D]$, $[A, C, D]$, $[B, C, D]$, $[B, C, E]$ and $[A, B, C, D]$.

Definition 5.9. *Let K be a geometric simplicial complex. For each simplex $\sigma \in K$ we take its vertex set α. Those sets form an abstract simplicial complex A. We call A a* ***vertex scheme*** *of K. Analogously, we call K a* ***geometric realization*** *of A.*

Notice that for $\tau \leq \sigma$, we have an inclusion of vertex sets $\alpha_\tau \subseteq \alpha_\sigma$ and thus $\alpha_\tau \in A$ because $\tau \in K$. This proves that a vertex scheme is an abstract simplicial complex. We can also construct a geometric realization for an abstract simplicial complex if the dimension of the ambient space is high enough.

Theorem 5.10 (**Geometric Realization Theorem**)**.** *Every abstract simplicial complex of dimension d has a geometric realization in $\mathbb{R}^{2d+1}$ [7, p. 64].*

Since simplicial complexes are topological spaces, we can consider continuous maps between them. Favourable maps between simplicial complexes are the maps that respect the underlying simplicial complex structure.

Definition 5.11. *A* ***vertex map*** *is a function $\phi\colon \operatorname{Vert} K \to \operatorname{Vert} L$ such that it sends the vertices of every simplex in K to vertices of a simplex in L.*

Definition 5.12. *Every vertex map $\phi\colon \operatorname{Vert} K \to \operatorname{Vert} L$ can be extended to a continuous map $f\colon |K| \to |L|$ which is defined by:*

$$f(x) = \sum_{i=0}^{n} b_i(x)\phi(v_i).$$

This map is called the ***simplicial map*** *induced by ϕ. Note that f is linear one each simplex in K. Therefore, we say that f is* ***piecewise linear****. To abbreviate notation, we often write $f\colon K \to L$.*

Since linear maps are continuous and $f\colon K \to L$ is piecewise linear, it is indeed continuous. Per definition, simplicial maps map simplices to simplices. For a simplex $\sigma \in K$, we now consider the dimension of $f(\sigma)$ and observe how the faces of σ and $f(\sigma)$ are related.

Lemma 5.13. *Let σ be a p-simplex in K. Then $f(\sigma)$ is a simplex in L that has dimension less or equal than p. Moreover, we have*

- *If $f(\sigma)$ has dimension p, then the $(p-1)$-dimensional faces of σ map to the corresponding $(p-1)$-dimensional faces of $f(\sigma)$.*
- *If $f(\sigma)$ has dimension $p-1$, then exactly two of the $(p-1)$-dimensional faces of σ map to $f(\sigma)$ and all the other $(p-1)$-dimensional faces of σ map to faces of dimension $p-2$.*
- *If $f(\sigma)$ has dimension less than $p-1$, then the images of all faces of σ also have dimension less than $p-1$.*

Proof. Let $\sigma = [v_0, v_1, \ldots, v_p]$. We know that $f(\sigma) = [\phi(v_0), \phi(v_1), \ldots, \phi(v_p)]$ is a simplex in L, which has dimension less or equal than p since it is spanned by $p+1$ vertices.

- If $f(\sigma)$ has dimension p, then the vertices $\phi(v_0), \phi(v_1), \ldots, \phi(v_p)$ are pairwise different and the $(p-1)$-dimensional faces of σ map to the corresponding $(p-1)$-dimensional faces of $f(\sigma)$:

$$f([v_0, v_1, \ldots, \widehat{v_j}, \ldots, v_p] = [\phi(v_0), \phi(v_1), \ldots, \widehat{\phi(v_j)}, \ldots, \phi(v_p)].$$

- If $f(\sigma)$ has dimension $p-1$, then exactly two of the vertices in $\phi(v_0), \phi(v_1), \ldots, \phi(v_p)$ coincide. Let $\phi(v_i) = \phi(v_j)$. Then $f([v_0, v_1, \ldots, \hat{v_i}, \ldots, v_p]) = f([v_0, v_1, \ldots, \hat{v_j}, \ldots, v_p])$ and the images of all other $(p-1)$-dimensional faces have dimension $p-2$ since they contain the coinciding vertices $\phi(v_i)$ and $\phi(v_j)$.
- If $f(\sigma)$ has dimension less than $p-1$, then the images of all faces of σ also have dimension less than $p-1$. □

5.1.2 Homology

Homology groups are the mathematical description of holes and higher dimensional analogues in a topological space. We introduce homology for simplicial complexes (called simplicial homology with $\mathbb{Z}/2\mathbb{Z}$-coefficients) and develop the necessary theory that is used in persistent homology. Using the simplices of a simplicial complex, we create abelian groups. We then combine them, creating a so-called chain complex, which forms the foundation for homology.

Definition 5.14. *Let K be a simplicial complex and let $p \in \mathbb{Z}$. A **p-chain** c is a formal finite sum of p-simplices in K. We can write*

$$c = \sum a_i \sigma_i,$$

*where $a_i \in \mathbb{Z}/2\mathbb{Z}$ are the coefficients and σ_i are the p-simplices. Note that since the coefficients are either 0 or 1, we can assume w.l.o.g. that a p-chain is of the form $c = \sigma_1 + \ldots + \sigma_n$. We can add two p-chains together componentwise: if $c = \sum a_i \sigma_i$ and $c' = \sum b_i \sigma_i$, then $c + c' = \sum (a_i + b_i)\sigma_i$, where the coefficients are added modulo 2, i.e. $1 + 1 = 0$. The p-chains together with this addition form an abelian group, called the **group of p-chains**, denoted by $C_p = C_p(K)$. We have such a group for all $p \in \mathbb{Z}$, however, for p negative or bigger than $\dim K$, we have that $C_p(K) = 0$ is trivial.*

Definition 5.15. *The **boundary of a p-simplex** is defined to be the sum of its $(p-1)$-dimensional faces. If we write $\sigma = [v_0, v_1, \ldots, v_n]$ for the p-simplex which is spanned by the listed vertices, then the boundary is given by*

$$\partial_p \sigma = \sum_{j=0}^{p} [v_0, v_1, \ldots, \widehat{v_j}, \ldots, v_n].$$

*We extend this definition by linearity to define the boundary of a p-chain $c = \sum a_i \sigma_i$ to be $\partial_p c = \sum a_i \partial_p \sigma_i$. Therefore we get a map $\partial_p \colon C_p(K) \to C_{p-1}(K)$, which is called the **boundary map** for chains. Notice that the boundary map commutes with addition, i.e. $\partial_p(c+c') = \partial_p c + \partial_p c'$, therefore ∂_p is a homomorphism. The **chain complex** is the infinite sequence of chain groups connected by their boundary maps:*

$$\ldots \xrightarrow{\partial_{p+2}} C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K) \xrightarrow{\partial_{p-1}} \ldots$$

We often drop the index of the boundary map because it is implied by the dimension of the chain to which it is applied.

We now consider two special types of chains called cycles and boundaries. We observe how they are related to each other.

Definition 5.16. *A **p-cycle** is a p-chain with empty boundary, i.e. $\partial c = 0$. Because ∂ commutes with addition, we have that the p-cycles form an abelian subgroup of $C_p(K)$, denoted by $Z_p = Z_p(K)$. Notice that $Z_p = \ker \partial_p$.*

For $p = 0$, we have that each vertex is mapped to 0 and thus $Z_0 = \ker \partial_0 = C_0$. For higher p, it is generally not true that $Z_p = C_p$. However, one finds that a p-chain $c = \sigma_1 + \ldots + \sigma_n$ is a p-cycle if and only if each $(p-1)$-dimensional face of some σ_i occurs to an even number in the set of the $(p-1)$-dimensional faces of $\sigma_1, \ldots, \sigma_n$.

Definition 5.17. *A **p-boundary** is a p-chain that is the boundary of a $(p+1)$-chain, i.e. $c = \partial d$ for some $d \in C_{p+1}(K)$. Because ∂ commutes with addition, we have that the group of p-boundaries forms an abelian subgroup of $C_p(K)$ and it is denoted by $B_p = B_p(K)$. We have that $B_p = \operatorname{im} \partial_{p+1}$.*

We now look at the following lemma that connects cycles and boundaries. This lemma is the fundamental property that makes homology work.

Lemma 5.18 (**Fundamental Lemma of Homology**). *For each integer p and every $(p+1)$-chain d, we have $\partial_p \partial_{p+1} d = 0$.*

Proof. Because the boundary map is a homomorphism, it is enough to show $\partial_p \partial_{p+1} \tau = 0$ for every $(p+1)$-simplex $\tau = [v_0, \dots, v_{p+1}]$:

$$\begin{aligned}\partial_p \partial_{p+1} \tau = \partial_p(\sum_{j=0}^{p+1} [v_0, \dots, \widehat{v_j}, \dots, v_p]) &= \sum_{\substack{i,j=0 \\ i \neq j}}^{p+1} [v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_p] \\ &= \sum_{\substack{i,j=0 \\ i<j}}^{p+1} [v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j} \dots v_p] + \sum_{\substack{i,j=0 \\ j<i}}^{p+1} [v_0, \dots, \widehat{v_j}, \dots, \widehat{v_i}, \dots, v_p] = 0.\end{aligned}$$

□

Therefore, every p-boundary is also a p-cycle and we have that B_p is a subgroup of Z_p. This allows us to take the quotients Z_p/B_p. Note that two elements $c + B_p, c' + B_p$ are the same in Z_p/B_p if and only if their difference $c - c'$ is a boundary. We now use this connection between cycles and boundaries to define homology groups.

Definition 5.19. *The* ***p-th homology group*** *of a simplicial complex K is the quotient group, $H_p(K) = H_p = Z_p/B_p$. The* ***p-th Betti number*** *is the rank of this group, $\beta_p = \operatorname{rank} H_p$. A coset $[c] = c + B_p$ of H_p is called a* ***homology class****. Two cycles that are in the same homology class are called* ***homologous****, denoted by $c \sim c'$.*

Note that Z_n is an abelian group and thus the subgroup B_n is a normal subgroup. Therefore, $H_p = Z_p/B_p$ is actually a group and it is abelian since Z_n is. Note that H_p is actually a $\mathbb{Z}/2\mathbb{Z}$-vector space.

Lemma 5.20. *For every simplicial complex K, $H_p(K)$ is a finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-vector space for each integer p.*

Proof. Note that C_p is isomorphic to the free abelian group over all the p-simplices σ_i in K:

$$C_p = \bigoplus \mathbb{Z}/2\mathbb{Z} \cdot \sigma_i.$$

Because K consists of only finitely many simplices, C_p is finite-dimensional. Therefore the homomorphism ∂_p is a a $\mathbb{Z}/2\mathbb{Z}$-linear map between the two finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-vector spaces C_p and C_{p-1}. This shows that $Z_p = \ker \partial_p$ and $B_p = im\partial_{p+1}$ are $\mathbb{Z}/2\mathbb{Z}$-vector spaces and thus so is the quotient $H_p = Z_p/B_p$. □

5.1.3 Maps in Homology and Homotopy Invariance

We have now defined the homology groups $H_p(K)$ for a simplicial complex K. Next, we observe how simplicial maps behave when passing to homology.

Definition 5.21. *Let K and L be two simplicial complexes. We know that a simplicial map $f\colon K \to L$ takes each simplex of K linearly to a simplex in L. For every integer p, it induces a* ***map from the p-chains of K to the p-chains in L****. For a p-chain $c = \sum a_i \sigma_i$, we have that $f_\#(c) = \sum a_i \tau_i$, where $\tau_i = f(\sigma_i)$ if it has dimension p and $\tau_i = 0$ if $f(\sigma_i)$ has dimension less than p.*

Lemma 5.22. *For two simplicial complexes K and L, $f\colon K \to L$ a simplicial map and ∂_K and ∂_L the boundary maps of the two chain complexes of K and L, we have*

$$f_\# \circ \partial_K = \partial_L \circ f_\#.$$

Proof. We use Lemma 5.13. If $f(\sigma_i)$ has dimension p, then all $(p-1)$-dimensional faces of σ_i map to the corresponding $(p-1)$-dimensional faces of $\tau_i = f(\sigma_i)$. By linearity, we conclude that $f_\# \circ \partial_K(\sigma_i) = \partial_L \circ f_\#(\sigma_i)$. If $f(\sigma_i)$ has dimension less than p, then the $(p-1)$-dimensional faces of σ_i map to simplices of dimension less than $(p-1)$, with the possible exception of exactly two $(p-1)$-dimensional faces that map to the same simplex and therefore their images cancel. Therefore, we have $f_\#(\partial_K \sigma_i) = \partial_L f_\#(\sigma_i) = 0$. We conclude using the linearity of $f_\#, \partial_K$ and ∂_L. □

Lemma 5.23. *We have $f_\#(Z_p(K)) \subseteq Z_p(L)$ and $f_\#(B_p(K)) \subseteq B_p(L)$ for all simplicial maps $f\colon K \to L$. This induces a map in homology, denoted by $f_*\colon H_p(K) \to H_p(L)$.*

Proof. For a p-cycle c in K, the following map is well-defined:

$$f_*([c]_K) = f_*(c + B_p(K)) = f_\#(c) + B_p(L) = [f_\#(c)]_L.$$

This defines a well-defined map because $f_\#(c)$ is a p-cycle in L and if c and c' differ by a boundary in K, then their difference is mapped to a boundary in L. □

Remark 5.24. *The fact that simplicial maps induce maps in homology is often referred to as 'homology is functorial (for every p)'. This makes sense since the assignment*

$$K \mapsto H_p(K), (f\colon K \to L) \mapsto (f_*\colon H_p(K) \to H_p(L)$$

is a covariant functor from the category of simplicial complexes (with simplicial maps) to the category of $\mathbb{Z}/2\mathbb{Z}$-vector spaces (with $\mathbb{Z}/2\mathbb{Z}$-linear maps).

Considering induced maps in homology, one can ask, when (and if) such maps are isomorphisms. The following result answers a part of this question and explains, why homology is useful.

Theorem 5.25. *[9, Corollary 2.11] If a simplicial map $f\colon K \to L$ is a homotopy equivalence, then the induced maps $f_*\colon H_p(K) \to H_p(L)$ are isomorphisms for all $p \geq 0$.*

Therefore, if two simplicial complexes K and L are both triangulations of a space X, meaning that both K and L are homotopy equivalent to X, then we do not need to worry about which one is used since they yield isomorphic homology groups. This result is proven using a more general homology theory called singular homology. Singular homology can be applied to any topological space. In addition, singular homology can be defined with coefficients in any abelian group and not only for $\mathbb{Z}/2\mathbb{Z}$-coefficients. In the general case, homology groups are groups and not vector spaces. This explains why we talk about homology groups and not homology vector spaces (even though in our setting, they are always vector spaces). However, simplicial homology and singular homology are equivalent (on simplicial complexes) and therefore, we do not introduce singular homology here.

Example 5.26. *We have claimed that homology groups measure the holes and higher-dimensional analogues of a topological space. To explain this claim, we consider the homology of the sphere $S^n \subseteq \mathbb{R}^{n+1}$. In particular, we look at $H_n(S^n)$. For $n = 1$, S^1 is homotopy equivalent to the left simplicial complex in Figure 5.3. Its chain complex is ($\langle a, b, c\rangle$ denotes $\mathbb{Z}/2\mathbb{Z}\cdot a \oplus \mathbb{Z}/2\mathbb{Z}\cdot b \oplus \mathbb{Z}/2\mathbb{Z}\cdot c$).*

$$\ldots \longrightarrow 0 \xrightarrow{\partial_2} C_1 = \langle a, b, c\rangle \xrightarrow{\partial_1} C_0 = \langle v_0, v_1, v_2\rangle \xrightarrow{\partial_0} 0 \longrightarrow \ldots$$

We have $B_1 = im\partial_2 = 0, Z_1 = \ker \partial_1 = \langle a + b + c\rangle$. Therefore $H_1(S^1) = \mathbb{Z}/2\mathbb{Z}$ (using that homology is homotopy invariant). For $n = 2$, S^2 is homotopy equivalent to the right simplicial complex in Figure 5.3. Notice that this simplicial complex is not a single 3-dimensional simplex, but four 2-dimensional simplices. We denote them by $S = [v_0, v_1, v_2], T = [v_0, v_1, v_3], U = [v_0.v_2, v_3]$, $V = [v_1, v_2, v_3]$. Its chain complex is (all the other chain groups and boundary maps are zero).

$$0 \xrightarrow{\partial_3} C_2 = \langle S, T, U, V\rangle \xrightarrow{\partial_2} C_1 = \langle a, b, c, d, e, f\rangle \xrightarrow{\partial_1} C_0 = \langle v_0, v_1, v_2, v_3\rangle \xrightarrow{\partial_0} 0$$

We have $B_2 = im\partial_3 = 0$ and to compute $Z_2 = \ker \partial_2$, we look at its matrix (left) and compute its reduced row-echelon form (right) with $\mathbb{Z}/2\mathbb{Z}$-coefficients.

$$\partial_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We conclude that $Z_2 = \ker \partial_2 = \langle S + T + U + V \rangle$ and therefore $H_2(S^2) = \mathbb{Z}/2\mathbb{Z}$. It turns out that this is true for all $n \geq 1$, i.e. $H_n(S^n) = \mathbb{Z}/2\mathbb{Z}$. Therefore, the homology group H_n detects the single n-dimensional hole inside S^n. We conclude that in general, the p-th Betti number $\beta_p = \operatorname{rank} H_p = \dim H_p$ counts the number of p-dimensional holes in our simplicial complex.

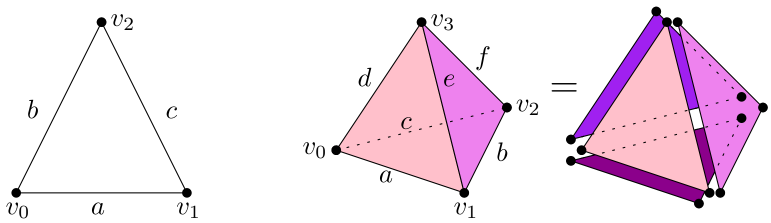


Figure 5.3: Two simplicial complexes that are homotopy equivalent to S^1 (left) resp. S^2 (right). The simplicial complex on the right has four 2-dimensional simplices and no 3-dimensional simplex.

5.2 Filtrations and Persistence

In this section, we use homology to finally show the connection between topological data analysis and quiver representation theory. First, we describe a method to construct a simplicial complex from a point cloud. We then apply homology to these simplicial complexes and obtain persistence modules, which are quiver representations of the linear L_n-quiver. We then analyze and visualize its interval decomposition. This section is based on Chapters III.1 and VII.1 of [7] and Chapter 2 of [8].

5.2.1 Point Cloud Triangulations

We describe a method to construct a simplicial complex from a point cloud.

Definition 5.27. *Let $\{x_i\} = X$ be a finite set of points in $\mathbb{R}^m$ and let $\epsilon > 0$. The **Čech complex** $\check{C}(X, \epsilon)$ is the abstract simplicial complex whose n-simplices are the sets $\{x_{i_0}, \ldots, x_{i_n}\} \subseteq X$ such that the closed balls of radius $\epsilon/2$, centered at the x_{i_k}, have a non-trivial intersection, i.e.*

$$\bigcap_{j=0}^{n} \bar{B}_{\epsilon/2}(x_{i_j}) \neq \emptyset.$$

Note that the Čech complex is an abstract simplicial complex. However, since there always exists a geometric realization, we can consider the Čech complex to be a geometric simplicial complex.

We now have a method to construct simplicial complexes from point clouds. However, it requires a choice of some parameter ϵ. For ϵ small enough, there are no non-trivial intersections, and thus $\check{C}(X, \epsilon)$ is a discrete set given by the vertices $\{x_i\}$. If ϵ is really big, all intersections are non-trivial, and therefore, $\check{C}(X, \epsilon)$ is a single high-dimensional simplex (and all of its faces). This leads to the question if there is a best-suited choice for ϵ, such that $\check{C}(X, \epsilon)$ is a good representation of our point cloud. Consider the point cloud and the Čech complexes in Figure 5.5. This point

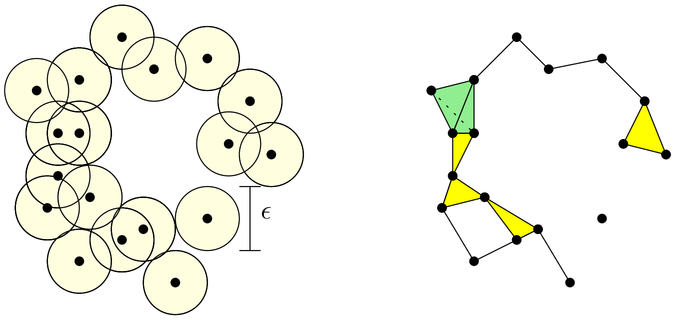


Figure 5.4: From a fixed set of points with balls of radius ϵ (left), one can construct the Čech complex $\check{C}(X, \epsilon)$, represented by its geometric realization (right).

cloud is a sample of points on an annulus. If ϵ is too small (left and middle), $\check{C}(X, \epsilon)$ does not yet represent the fact, that an annulus has a hole in the middle. For ϵ big enough (right), the hole suddenly appears. However, if ϵ is too big, then all balls intersect and the hole disappears. This example illustrates that there is generally no single preferred choice for the parameter ϵ and even if there was an optimal ϵ, we would already need to have a rough understanding of our point cloud to figure out the optimal value for ϵ. Instead of considering one value for ϵ, we can choose two values for ϵ and observe how the associated Čech complexes differ. We do this by considering simplicial maps between Čech complexes.

Lemma 5.28. *For $\epsilon_1 \leq \epsilon_2$ there exists an injective simplicial map $\iota\colon \check{C}(X, \epsilon_1) \hookrightarrow \check{C}(X, \epsilon_2)$. For any simplex $\sigma \in \check{C}(X, \epsilon_1)$, its image $\iota(\sigma)$ is σ itself, but now viewed in $\check{C}(X, \epsilon_2)$.*

Proof. For a simplicial complex L and a subcomplex K, the inclusion map $\operatorname{Vert} K \hookrightarrow \operatorname{Vert} L$ is a vertex map and thus, there exists a simplicial map $\iota\colon K \to L$. Moreover, for any simplex $\sigma = [v_0, v_1, \ldots, v_p]$, we have $\iota(\sigma) = [v_0, v_1, \ldots, v_p]$ and therefore this map is injective since ι is piecewise linear. Now we apply this to the simplicial complex $\check{C}(X, \epsilon_2)$ with subcomplex $\check{C}(X, \epsilon_1)$. $\square$

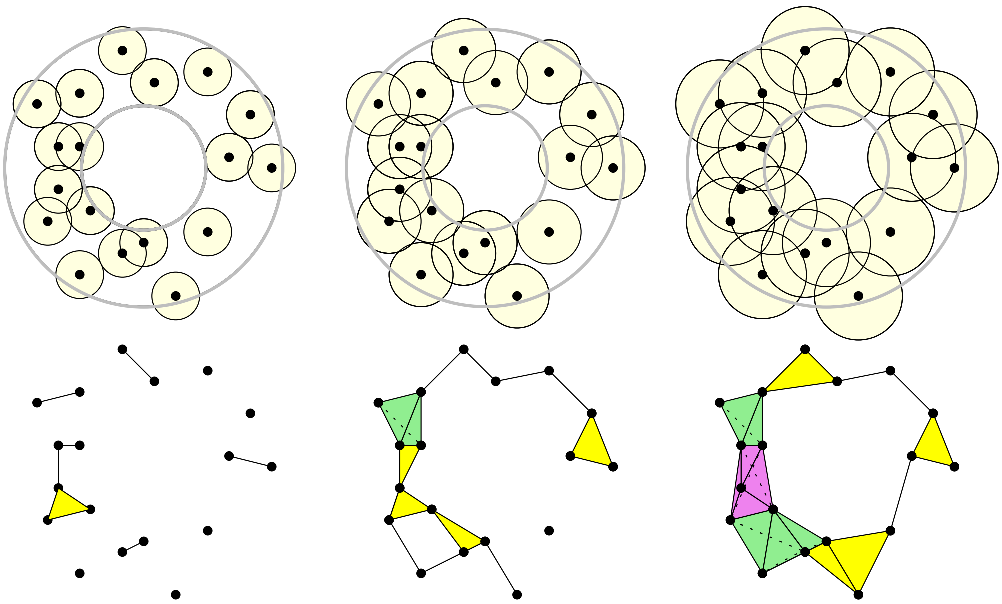


Figure 5.5: A sequence of Čech complexes for three different increasing choices of radius ϵ, coming from a point cloud that is a sample of an annulus. The 2-dimensional simplices are yellow, the 3-dimensional simplices are green and the 4-dimensional simplices are violet.

5.2.2 Persistence Vector Spaces and Persistence Barcodes

Using the inclusion maps from Lemma 5.28, we can construct a sequence of Čech complexes (for a sequence of radii $\epsilon_1 < \epsilon_2 < \ldots < \epsilon_n$).

$$\check{C}(X,\epsilon_1) \hookrightarrow \check{C}(X,\epsilon_2) \hookrightarrow \check{C}(X,\epsilon_3) \hookrightarrow \ldots \hookrightarrow \check{C}(X,\epsilon_n)$$

Definition 5.29. *A sequence of simplicial complexes $K_1 \hookrightarrow K_2 \hookrightarrow \ldots \hookrightarrow K_n$ is called a* ***filtration*** *of simplicial complexes.*

Notice that the sequence above is a filtration. Now, we apply the homology functor(s) H_*. This means that we do not just consider the homology of the Čech complexes $\check{C}(X,\epsilon_i)$, but for all the inclusion maps $\check{C}(X,\epsilon_i) \hookrightarrow \check{C}(X,\epsilon_{i+1})$, we consider the corresponding map in homology $H_*(\check{C}(X,\epsilon_i)) \to H_*(\check{C}(X,\epsilon_{i+1}))$. We know that all homology groups $H_*(\check{C}(X,\epsilon_i))$ are finite-dimensional vector spaces. Moreover, all maps between homology groups are linear maps. This yields the following sequence in homology

$$H_*(\check{C}(X,\epsilon_1)) \longrightarrow H_*(\check{C}(X,\epsilon_2)) \longrightarrow H_*(\check{C}(X,\epsilon_3)) \longrightarrow \ldots \longrightarrow H_*(\check{C}(X,\epsilon_n)).$$

As a sequence of vector spaces, this is a representation of the L_n-quiver!

Definition 5.30. *The representation $H_*(\check{C}(X,\epsilon_1)) \to H_*(\check{C}(X,\epsilon_2)) \to \ldots \to H_*(\check{C}(X,\epsilon_n))$ is called a* ***persistence module****. Note that it is a finite-dimensional representation since all the vector spaces are finite-dimensional.*

Remark 5.31. *By saying 'we apply the homology functor H_*', we mean that we apply the functor H_p for some integer $p \geq 0$. We use the notation H_* to illustrate that any choice of $p \geq 0$ is valid.*

Applying Gabriel's Theorem for A_n-type quivers (Theorem 4.7) yields that the persistence module is isomorphic to its interval decomposition $\bigoplus_{j=1}^{r} \mathbb{I}_Q[b_j, d_j]$.

Remark 5.32. *Note that this construction works for any filtration of simplicial complexes:*

$$K_1 \xrightarrow{f_1} K_2 \xrightarrow{f_2} K_3 \xrightarrow{f_3} \ldots \xrightarrow{f_{n-1}} K_n.$$

Indeed, applying homology yields a persistence module that can be decomposed using Gabriel's Theorem:

$$H_*(K_1) \xrightarrow{f_{1,*}} H_*(K_2) \xrightarrow{f_{2,*}} H_*(K_3) \xrightarrow{f_{3,*}} \ldots \xrightarrow{f_{n-1,*}} H_*(K_n).$$

Moreover, we can also consider simplicial complexes that are not finite-dimensional (they consist of an infinite number of simplices). In this case, the vector spaces may not be finite-dimensional and thus the persistence module may not be finite-dimensional. However, there exists a more powerful version of Gabriel's Theorem, which works in the infinite-dimensional case (see Chapter 1.1 of [1]).

As we go from $\check{C}(X,\epsilon_i)$ to $\check{C}(X,\epsilon_j)$ in a filtration (for $i < j$), we gain new homology classes or we lose some when they become trivial or merge with each other [7, p. 179]. To observe this, consider the images of the induced maps in homology, $f_p^{i,j} \colon H_p(\check{C}(X,\epsilon_i)) \to H_p(\check{C}(X,\epsilon_j))$.

Definition 5.33. *The* ***p-th persistent homology groups*** *are the images of the homomorphisms induced by inclusion, $H_p^{i,j} = \operatorname{im} f_p^{i,j}$, for $1 \leq i \leq j \leq n$. The corresponding* ***p-th persistent Betti numbers*** *are the ranks of these groups, $\beta_p^{i,j} = \operatorname{rank} H_p^{i,j}$.*

If we consider a homology class $[c]$ in $H_p(\check{C}(X,\epsilon_b))$ that is not in any of the p-th persistent homology groups $H_p^{i,b}$ for all $i < b$, then this homology class appears for the first time at parameter b. Moreover, there exists a parameter d for which this homology class is non-trivial for the last time. This means that $f_p^{b,j}[c] \neq 0$ for all $j \leq d$ and that $f_p^{b,d+1}[c] = 0$ (or $d = n$). In the persistence module, this homology class (which describes a topological feature) corresponds to the interval representation $\mathbb{I}[b,d]$. We conclude that each interval representation in the interval decomposition $\bigoplus_{j=1}^{r} \mathbb{I}_Q[b_j, d_j]$ corresponds to the evolution of a homology class and therefore describes the persistence of some topological feature. Given a parameterized family of spaces, those topological features that persist over a significant parameter range are to be considered as signal (resp. important features) with short-lived features as noise [8, p. 5]. This motivates the following definition.

Definition 5.34. *The **lifetime** of an interval representation* $\mathbb{I}_Q[b,d]$ *is given by* $d-b$. *Moreover,* b *is called the **birth time** and* d *is called the **death time**.*

We conclude that the p-th persistent Betti number $\beta_p^{i,j}$ counts the number of p-dimensional holes (features) that persist over the parameter interval $[i,j]$. Using the interval decomposition, we present a visualization of the p-th persistent homology groups and the corresponding intervals.

Definition 5.35. *If a persistence module has the interval decomposition* $\bigoplus_{j=1}^{r} \mathbb{I}_Q[b_j,d_j]$, *then its **persistence barcode** is the multiset of intervals* $\{[b_j,d_j]\}_{j=1}^{r}$. *The horizontal axis corresponds to the parameter* ϵ *and the vertical axis ranges over an arbitrary ordering of the homology generators of the p-th persistent homology groups of the persistence module.*

Figure 5.6 gives an example of the barcode of the persistence module coming from the point cloud from Figure 5.5. This point cloud is a sample of points from a planar annulus. We observe that there are 18 different 0-dimensional homology classes for ϵ small enough. They correspond to the 18 points in our sample. We also observe some small intervals in dimension 1 that represent noise. Moreover, we can see the big interval in dimension 1, which represents the 1-dimensional hole in the middle of the annulus. The long length of this interval corresponds to the fact that this middle hole is a significant feature of the planar annulus.

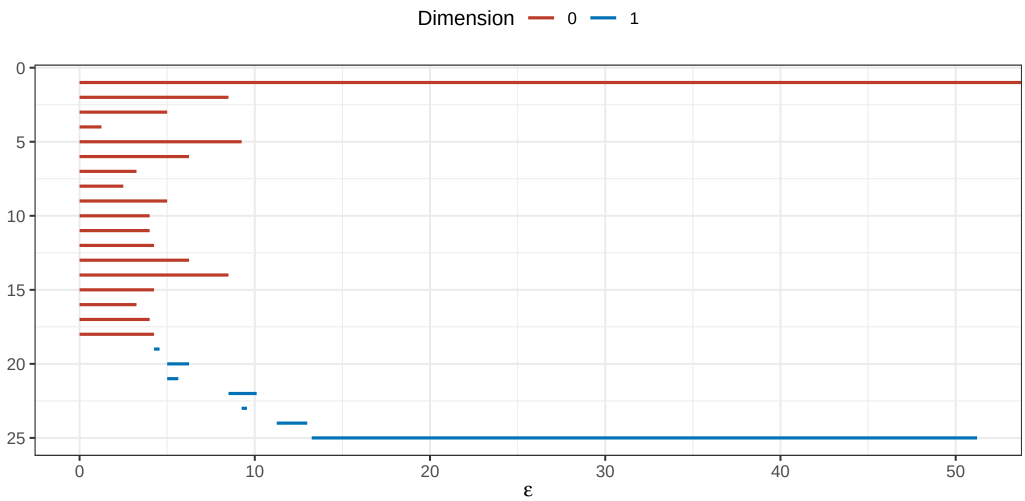


Figure 5.6: The persistence barcode for the point cloud from Figure 5.5.

Theorem 5.36. *[7, p. 181] The p-th persistent Betti number* $\beta_p^{i,j}$ *is equal to the number of intervals in the barcode spanning the whole interval* $[i,j]$ *(or more).*

This means that a barcode is not just a great tool to visualize the decomposition of a persistence module, it also encodes the entire information about persistent homology groups.

5.3 Zigzag Persistent Homology

In this section, we generalize the setting from persistent homology to introduce zigzag persistent homology. We use zigzag persistent homology to show a connection to the diamond principle from quiver representation theory. This section is based on the paper 'Zigzag Persistence' by Gunnar Carlsson and Vin Silva [10].

Similarly, as in Section 5.2, we start with a sequence of simplicial complexes.

Definition 5.37. *The following sequence of simplicial complexes is called a **zigzag diagram**:*

$$K_1 \xleftrightarrow{f_1} K_2 \xleftrightarrow{f_2} K_3 \xleftrightarrow{f_3} \dots \xleftrightarrow{f_{n-1}} K_n.$$

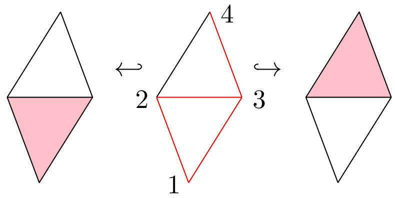


Figure 5.7: A zigzag diagram $K \hookleftarrow K \cap K' \hookrightarrow K'$. The red lines form a 1-cycle that is non-trivial in homology in all three complexes.

The major difference to the setting of persistent homology is that here, the maps f_i can go in either direction. Again, we apply the homology functors H_* and get a sequence in homology:

$$H_*(K_1) \xleftrightarrow{f_{1,*}} H_*(K_2) \xleftrightarrow{f_{2,*}} H_*(K_3) \xleftrightarrow{f_{3,*}} \dots \xleftrightarrow{f_{n-1,*}} H_*(K_n).$$

This is again a sequence of vector spaces with linear maps connecting them. However, since the maps $f_{i,*}$ can go from left to right or the other way around, this is a representation of an A_n-type quiver. Specifically, it is the A_n-type quiver that matches the direction of the arrows.

Definition 5.38. *The representation $H_*(K_1) \leftrightarrow H_*(K_2) \leftrightarrow \dots \leftrightarrow H_*(K_n)$ is called a **zigzag persistence module**.*

Using Gabriel's Theorem, this representation decomposes into its interval decomposition and yields a persistence barcode analogous to any persistence module. Therefore, the theory of zigzag persistence is a generalization of the persistence theory from Section 5.2. However, since we cannot observe homology classes from their birth time to their death time (because not all maps point in the same direction), understanding the intervals in the barcode is more involved. The following example illustrates this.

Example 5.39. *Let K and K' be the simplicial complexes shown in Figure 5.7 and let $K \cap K'$ be the simplex-wise intersection. This defines a zigzag diagram*

$$K \hookleftarrow K \cap K' \hookrightarrow K'.$$

Applying homology in dimension 1 (i.e. H_1), we get a zigzag persistence module

$$\mathbb{Z}/2\mathbb{Z} \cong H_1(K) \longleftarrow H_1(K \cap K') \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow H_1(K') \cong \mathbb{Z}/2\mathbb{Z}.$$

The 1-cycle $[1,2]+[1,3]+[2,3]+[3,4]$ (red lines) is non-trivial in homology in all three complexes. Therefore, one may think that this cycle corresponds to an interval representation $\mathbb{I}_Q[1,3]$ that spans all three vertices. However, this is not the case. For $H_1(K \cap K')$ we choose the following basis

$$\{[1,2]+[1,3]+[2,3], [2,3]+[2,3]+[3,4]\}.$$

This gives the following decomposition, which consists of the two interval representations $\mathbb{I}_Q[1,2]$ and $\mathbb{I}_Q[2,3]$.

$$\mathbb{Z}/2\mathbb{Z} \xleftarrow{[1,0]} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{[0,1]} \mathbb{Z}/2\mathbb{Z}$$

We now generalize the setting of Example 5.39 to observe two interesting zigzag persistence modules. For a finite collection of simplicial complexes $\mathcal{X} = \{K_i\}_{i=0}^n$, consider the following zigzag diagram

$$\begin{tikzcd}
 & K_0 \cap K_1 \arrow[dl] \arrow[dr] & & K_1 \cap K_2 \arrow[dl] \arrow[dr] & & \dots \arrow[dl] \arrow[dr] & \\
K_0 & & K_1 & & K_2 & & K_n.
\end{tikzcd}$$

Applying homology yields a zigzag persistence module

$$H_*(K_0) \longleftarrow H_*(K_0 \cap K_1) \longrightarrow H_*(K_1) \longleftarrow H_*(K_1 \cap K_2) \longrightarrow \dots \longrightarrow H_*(K_n).$$

We can also consider the zigzag diagram associated with the union of simplicial complexes

$$\begin{tikzcd}
 & K_0 \cup K_1 & & K_1 \cup K_2 & & \dots & \\
K_0 \arrow[ur] & & K_1 \arrow[ul] \arrow[ur] & & K_2 \arrow[ul] \arrow[ur] & & K_n. \arrow[ul]
\end{tikzcd}$$

We obtain another zigzag persistence module

$$H_*(K_0) \longrightarrow H_*(K_0 \cup K_1) \longleftarrow H_*(K_1) \longrightarrow H_*(K_1 \cup K_2) \longleftarrow \dots \longleftarrow H_*(K_n).$$

The motivation behind these zigzag persistence modules is the following: if the complexes in $\mathcal{X}$ correspond to different areas of a point cloud, these two zigzag persistence modules roughly correspond to the persistence of topological features over different parts of the point cloud. The following theorem gives a set of matching rules for the interval representations of their respective interval decomposition.

Theorem 5.40 (**Strong diamond principle**). *Let $\mathbb{V}^+ = H_*(\mathcal{X}_\cup)$ and $\mathbb{V}^- = H_*(\mathcal{X}_\cap)$ be the two zigzag persistence modules corresponding to the following zigzag diagrams.*

$$\mathcal{X}_\cup : \quad K_0 \longleftrightarrow \dots \longleftrightarrow K_{k-1} \hookrightarrow K_{k-1} \cup K_{k+1} \hookleftarrow K_{k+1} \longleftrightarrow \dots \longleftrightarrow K_n$$

$$\mathcal{X}_\cap : \quad K_0 \longleftrightarrow \dots \longleftrightarrow K_{k-1} \hookleftarrow K_{k-1} \cap K_{k+1} \hookrightarrow K_{k+1} \longleftrightarrow \dots \longleftrightarrow K_n$$

The interval decompositions of $\mathbb{V}^-$ and $\mathbb{V}^+$ are related to each other through the following matching rules:

- *summands $\mathbb{I}_Q[k,k]^{p+1}$ and $\mathbb{I}_{s_kQ}[k,k]^p$ are matched,*
- *summands $\mathbb{I}_Q[b,k]$ are matched with summands $\mathbb{I}_{s_kQ}[b,k-1]$, and $\mathbb{I}_Q[b,k-1]$ with $\mathbb{I}_{s_kQ}[b,k]$,*
- *summands $\mathbb{I}_Q[k,d]$ are matched with summands $\mathbb{I}_{s_kQ}[k+1,d]$, and $\mathbb{I}_Q[k+1,d]$ with $\mathbb{I}_{s_kQ}[k,d]$,*
- *every other summand $\mathbb{I}_Q[b,d]$ is matched with the summand $\mathbb{I}_{s_kQ}[b,d]$.*

The superscripts $p+1$ and p in the first matching rule denote a ± 1 shift of homological dimension. This means that $\mathbb{I}_Q[k,k]$ in the zigzag persistence module of dimension $p+1$ (of $\mathbb{V}^+$) is matched with $\mathbb{I}_{S_kQ}[k,k]$ in the zigzag persistence module of dimension p (of $\mathbb{V}^-$).

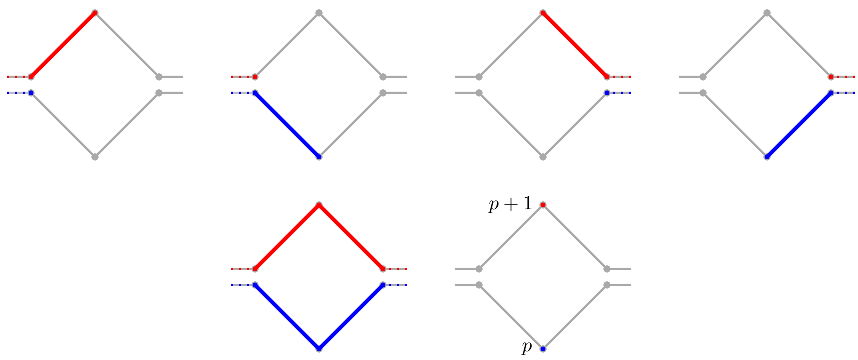


Figure 5.8: The matching rules from the strong diamond principle. The top row illustrates the second and third matching rules. The bottom row shows the first and last matching rules.

Proof of Theorem 5.40. For any $p \in \mathbb{N}_0$, the following diamond is exact:

$$\begin{tikzcd}
 & & H_p(K_{k-1} \cup K_{k+1}) & & \\
\dots \arrow[r, leftrightarrow] & H_p(K_{k-1}) \arrow[ur] & & H_p(K_{k+1}) \arrow[ul] \arrow[r, leftrightarrow] & \dots \\
 & & H_p(K_{k-1} \cap K_{k+1}) \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

This follows directly from the exactness of the Mayer-Vietoris sequence [9, p. 149ff]:

$$\dots \longrightarrow H_p(K_{k-1} \cap K_{k+1}) \longrightarrow H_p(K_{k-1}) \oplus H_p(K_{k+1}) \longrightarrow H_p(K_{k-1} \cup K_{k+1}) \longrightarrow \dots$$

Thus, the second, third, and fourth matching rules follow from the diamond principle (Theorem 4.11). It remains to prove the first matching rule. This follows from the Mayer-Vietoris sequence:

$$\begin{tikzcd}
 & \dots \arrow[r] & H_{p+1}(K_{k-1}) \oplus H_{p+1}(K_{k+1}) \arrow[r, "f"] & H_{p+1}(K_{k-1} \cup K_{k+1}) \arrow[dlll, "\partial"'] \\
H_p(K_{k-1} \cap K_{k+1}) \arrow[rr, "g"'] & & H_p(K_{k-1}) \oplus H_p(K_{k+1}) \arrow[r] & \dots
\end{tikzcd}$$

Using exactness and the first isomorphism theorem, we conclude that

$$\operatorname{coker}(f) = H_{p+1}(K_{k-1} \cup K_{k+1}) / \operatorname{im}(f) = H_{p+1}(K_{k-1} \cup K_{k+1}) / \ker(\partial) \cong \operatorname{im}(\partial) = \ker(g).$$

We notice that $\operatorname{coker}(f)$ is spanned by homology classes that do not come from homology classes in $H_{p+1}(K_{k-1})$ or $H_{p+1}(K_{k+1})$. These homology classes (of dimension $p+1$) correspond exactly to the representations $\mathbb{I}_Q[k,k]$. Similarly, elements in $\ker(g)$ are homology classes (of dimension p) that are trivial when observed in $H_p(K_{k-1})$ and $H_p(K_{k+1})$. Therefore $\ker(g)$ corresponds to the interval representations $\mathbb{I}_{s_k Q}[k,k]$. This proves the first matching rule. $\square$

We have proven the diamond principle using results about reflection functors. Therefore, using results from quiver representation theory, we have shown a result about the interval decomposition (and thus the barcode) of zigzag persistence modules.

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Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

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Titel der Arbeit:

Gabriel's Theorem

Verfasst von:
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Nussli	Valentin

Ich bestätige mit meiner Unterschrift:
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- Ich habe alle Methoden, Daten und Arbeitsabläufe wahrheitsgetreu und vollständig dokumentiert.
- Ich habe alle Personen erwähnt, welche die Arbeit wesentlich unterstützt haben.

Ich nehme zur Kenntnis, dass die Arbeit mit elektronischen Hilfsmitteln auf Eigenständigkeit überprüft werden kann.

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Zürich, 29.02.2024	V. Nussli

Bei Gruppenarbeiten sind die Namen aller Verfasserinnen und Verfasser erforderlich. Durch die Unterschriften bürgen sie grundsätzlich gemeinsam für den gesamten Inhalt dieser schriftlichen Arbeit.

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[1] z. B. ChatGPT, DALL E 2, Google Bard
[2] z. B. ChatGPT, DALL E 2, Google Bard
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