Course No. 401-3462-00L

Functional Analysis II

"Some Fundamental Tools for PDE, Harmonic Analysis and Function Spaces Theory"

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1 The Fourier Transform of Tempered Distributions

1.1 The Fourier transforms of L^1 functions

Theorem-Definition 1.1. *Let $f \in L^1(\mathbb{R}^n, \mathbb{C})$. Define its Fourier transform $\hat{f}$ as follows:*

$$\forall \xi \in \mathbb{R}^n \quad \widehat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

We have that $\widehat{f} \in L^\infty(\mathbb{R}^n)$ and

$$\|\widehat{f}\|_{L^\infty(\mathbb{R}^n)} \le (2\pi)^{-\frac{n}{2}} \|f\|_{L^1(\mathbb{R}^n)}. \tag{1.1}$$

Moreover $\widehat{f} \in C^0(\mathbb{R}^n)$ and

$$\lim_{|\xi| \to +\infty} |\widehat{f}(\xi)| = 0. \tag{1.2}$$

We shall also sometimes denote the Fourier transform of f by $\mathcal{F}(f)$.

Remark 1.2. There are several possible normalizations for defining the Fourier transform of an L^1 function such as for instance

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2i\pi x\cdot\xi} f(x)\, dx.$$

None of them gives a full satisfaction. The advantages of the one we chose are the following:

i) $f \longmapsto \widehat{f}$ will define an isometry of L^2 as we will see in Proposition 1.5.

ii) With our normalization we have the convenient formula (see Lemma 1.11)

$$\forall k = 1 \ldots n \quad \partial_{\xi_k} \widehat{f} = i\, \xi_k\, \widehat{f}$$

but the less convenient formula (see theorem 1.62)

$$\widehat{g * f} = (2\pi)^{n/2}\ \widehat{g}\, \widehat{f}.$$

Proof of Theorem 1.1. The first part of the theorem that is inequality (1.1) is straightforward. We prove now that $\widehat{f} \in C^0(\mathbb{R}^n)$. Let $f_k \in C_0^\infty(\mathbb{R}^n)$ such that

$$f_k \longrightarrow f \text{ strongly in } L^1(\mathbb{R}^n).$$

It is clear that since $f_k \in C_0^\infty(\mathbb{R}^n)$ the functions $\widehat{f_k}$ are also C^∞. Inequality (1.1) gives

$$\|\widehat{f} - \widehat{f_k}\|_{L^\infty(\mathbb{R}^n)} \le (2\pi)^{-\frac{n}{2}}\ \|f - f_k\|_{L^1(\mathbb{R}^n)}.$$

Thus $\widehat{f}$ is the uniform limit of continuous functions and, as such, it is continuous. It remains to prove that $|f|(\xi)$ uniformly converges to zero as $|\xi|$ converge to infinity.

In Proposition 1.9 we shall prove that (1.2) holds if $f \in C_0^\infty(\mathbb{R}^n)$. Let $f \in L^1(\mathbb{R}^n)$, let $\varepsilon > 0$ and let $\varphi \in C_0^\infty$ such that

$$\|f - \varphi\|_{L^1(\mathbb{R}^n)} \leq \frac{\varepsilon}{2}\,(2\pi)^{\frac{n}{2}}. \tag{1.3}$$

There exists $R > 0$ such that

$$|\xi| > \mathbb{R} \Longrightarrow |\widehat{\varphi}(\xi)| \leq \frac{\varepsilon}{2}. \tag{1.4}$$

Combining (1.3) and (1.4) together with (1.1) applied to the difference $f - \varphi$, we obtain

$$\begin{aligned} |\xi| > R \Longrightarrow |\widehat{f}(\xi)| &\leq \|\widehat{f} - \widehat{\varphi}\|_{L^\infty} + |\widehat{\varphi}(\xi)| \\ &\leq \varepsilon. \end{aligned}$$

This implies (1.2) and Theorem 1.1 is proved. □

Exercise 1.3. Prove that for any $a \in \mathbb{R}_+^*$

$$\widehat{e^{-a|x|^2}} = \frac{1}{(2a)^{\frac{n}{2}}}\, e^{-\frac{|\xi|^2}{4a}}.$$

Prove that for any $a \in \mathbb{R}_+^*$

$$\widehat{f_a(x)} = a^n\, \widehat{f}(a\xi)$$

where $f_a(x) := f(\frac{x}{a})$ for any $x \in \mathbb{R}^n$.

It is then natural to ask among the functions which are continuous, bounded in L^∞ and converging uniformly to zero at infinity, which ones are the Fourier transform of an L^1 function. Unfortunately, there seems to be no satisfactory condition characterizing the space of Fourier transforms of $L^1(\mathbb{R}^n)$. We have nevertheless the following theorem.

Theorem 1.4. *(Inverse of the Fourier transform)*
Let $f \in L^1(\mathbb{R}^n;\mathbb{C})$ such that $\widehat{f} \in L^1(\mathbb{R}^n;\mathbb{C})$ then

$$\forall x \in \mathbb{R}^n \quad f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, \widehat{f}(\xi)\, d\xi.$$

Proof of Theorem 1.4. We can of course explicitly write

$$(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, \widehat{f}(\xi)\, d\xi = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, d\xi \int_{\mathbb{R}^n} e^{ix\cdot y}\, f(y)\, dy.$$

The problem at this stage is that we cannot a-priori reverse the order of integrations because the hypothesis for applying Fubini's theorem are not fullfilled:

$$(\xi, y) \longmapsto e^{i\xi(x-y)} f(y) \notin L^1(\mathbb{R}^n \times \mathbb{R}^n)$$

unless $f \equiv 0$.

The idea is to insert the Gaussian function $e^{-\frac{\varepsilon^2|\xi|^2}{4}}$ where ε is a positive number that we are going to take smaller and smaller. Introduce

$$I_\varepsilon(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\frac{\varepsilon^2|\xi|^2}{4}}\, d\xi \int_{\mathbb{R}^n} e^{-i\xi\cdot y}\, f(y)\, dy.$$

Now we have

$$(\xi, y) \longmapsto e^{-\frac{\varepsilon^2|\xi|^2}{4}}\ e^{i\xi(x-y)} f(y) \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$$

and we can apply Fubini's theorem.

We have in one hand

$$I_\varepsilon(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\ e^{-\frac{\varepsilon^2|\xi|^2}{4}}\ \widehat{f}(\xi)\, d\xi.$$

We can bound the integrand uniformly as follows:

$$\forall x, \xi \in \mathbb{R}^n \quad \left| e^{ix\cdot\xi}\ e^{-\frac{\varepsilon^2|\xi|^2}{4}}\ \widehat{f}(\xi) \right| \leq |\widehat{f} * \xi)|.$$

By assumption, the right-hand side of the inequality is integrable and we have moreover, for every x and ξ

$$\lim_{\varepsilon\Rightarrow 0} e^{ix\cdot\xi}\ e^{-\frac{\varepsilon^2|\xi|^2}{4}}\ \widehat{f}(\xi) = e^{ix\cdot\xi}\ \widehat{f}(\xi).$$

Hence dominated convergence theorem implies that for any $x \in \mathbb{R}$

$$\lim_{\varepsilon\Rightarrow 0} I_\varepsilon(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \widehat{f}(x)\, e^{ix\cdot\xi}\, d\xi. \tag{1.5}$$

Applying Fubini gives also

$$\begin{aligned} I_\varepsilon(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} f(y)\, dy \int_{\mathbb{R}_n} e^{-i(y-x)\cdot\xi}\ e^{-\frac{\varepsilon^2|\xi|^2}{4}}\ d\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(y)\, \mathcal{F}\big(e^{-\frac{\varepsilon^2}{4}|\xi|^2}\big)(y-x)\, dy \end{aligned}$$

using Exercise 1.3, we then obtain

$$I_\varepsilon(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(y)\ e^{\frac{-|y-x|^2}{\varepsilon^2}}\ \frac{2^{\frac{n}{2}}}{\varepsilon^n}\ dy.$$

One proves without much difficulties that for any Lebesgue point $x \in \mathbb{R}$ for f the following holds

$$\lim_{\varepsilon\Rightarrow 0} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(y)\ e^{-\frac{|y-x|^2}{\varepsilon^2}}\ \frac{2^{\frac{n}{2}}}{\varepsilon^n}\ dy = f(x).$$

Continuing this identity with (1.5) gives the theorem. □

The transformation

$$f \in L^1(\mathbb{R}^n) \longmapsto (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, f(\xi)\, d\xi$$

will be denoted $\overset{\vee}{f}$ or also $\mathcal{F}^{-1}(f)$.

Proposition 1.5. *Let f and $g \in L^1(\mathbb{R}^n;\mathbb{C})$. Then*

$$\int_{\mathbb{R}^n} f(x)\,\widehat{g}(x)\,dx^n = \int_{\mathbb{R}^n} \widehat{f}(x)\,g(x)\,dx^n.$$

Let $f \in L^1(\mathbb{R}^n;\mathbb{C})$ such that $\widehat{f} \in L^1(\mathbb{R}^n;\mathbb{C})$, then

$$\int_{\mathbb{R}^n} f(x)\,\overline{f(x)}\,dx^n = \int_{\mathbb{R}^n} \widehat{f}(\xi)\,\overline{\widehat{f}(\xi)}\,d\xi^n\ .$$

This last identity is called Plancherel identity.

Proof of Proposition 1.5. The proof of the first identity in Proposition 1.5 is a direct consequence of Fubini's theorem. The second identity can be deduced from the first one by taking $g := \mathcal{F}^{-1}(\overline{f})$ and by observing that

$$\mathcal{F}^{-1}(\overline{f}) = \overline{\mathcal{F}(f)}\,. \qquad \square$$

The second identity is an invitation to extend the Fourier transform as an isometry of L^2. The purpose of the present chapter is to extend the Fourier transform to an even larger class of distributions. To that aim we will first concentrate on looking at the Fourier transform in a "small" class of very smooth function with very fast decrease at infinity: the Schwartz space.

1.2 The Schwartz Space $\mathcal{S}(\mathbb{R}^n)$

The Schwartz functions are C^∞ functions whose successive derivatives decrease faster than any polynominal at infinity. We shall use below the following notations:

$$\forall\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n \qquad x^\alpha := x_1^{\alpha_1}\ldots x_n^{\alpha_n}$$
$$\forall\beta = (\beta_1,\ldots,\beta_n) \in \mathbb{N}^n \qquad \partial^\beta f := \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}}\cdots\frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}}\,(f)$$

and $|\alpha| := \sum \alpha_i$.

Definition 1.6. *The space of Schwartz functions is the following subspace of $C^\infty(\mathbb{R}^n;\mathbb{C})$:*

$$\mathcal{S}(\mathbb{R}^n) := \left\{ \begin{array}{l} \varphi \in C^\infty(\mathbb{R}^n;\mathbb{C}) \quad s.t. \\ \forall p \in \mathbb{N} \quad \mathcal{N}_p(\varphi) := \displaystyle\sup_{\substack{|\alpha|\le p\\ |\beta|\le p}} \|x^\alpha\,\partial^\beta\varphi\|_{L^\infty(\mathbb{R}^n)} < +\infty \end{array} \right\}.$$

The following obvious proposition holds

Proposition 1.7. $\mathcal{S}(\mathbb{R}^n)$ *is stable under the action of derivatives and the multiplication by polynomials in $\mathbb{C}[x_1,\ldots,x_n]$.*

We prove now the following proposition:

Proposition 1.8. *There exists $C_n > 0$ s.t. $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$*

$$\sum_{\substack{|\alpha| \leq p \\ |\beta| \leq p}} \|x^\alpha \, \partial^\beta \varphi\|_{L^1(\mathbb{R}^n)} \leq C_n \, \mathcal{N}_{p+n+1}(\varphi).$$

Proof of Proposition 1.8. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} |x^\alpha \, \partial^\beta \varphi(x)| \, dx^n &\leq \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^{n+1})} \, (1+|x|^{n+1}) \, |x^\alpha| \, |\partial^\beta \varphi|(x) \; dx^n \\
&\leq C_n \, \mathcal{N}_{p+n+1}(\varphi).
\end{aligned} \tag{1.6}$$

This concludes the proof of the proposition. □

The following proposition is fundamental in the theory of tempered distributions we are going to introduce later on.

Proposition 1.9. *Let φ be a Schwartz function on $\mathbb{R}^n$, then it's Fourier transform is also a Schwartz function. Moreover for any $p \in \mathbb{N}$ there exists $C_{n,p} > 0$ such that*

$$\mathcal{N}_p(\widehat{\varphi}) \leq C_{n,p} \, \mathcal{N}_{p+n+1}(\varphi).$$

Hence the Fourier transform is a one to one linear transformation from $\mathcal{S}(\mathbb{R}^n)$ into itself. We shall see in the next sub-chapter that it is also continuous for the topology induced by the ad-hoc Fréchet structure on $\mathcal{S}(\mathbb{R})$.

Before proving Proposition 1.9, we need to establish two intermediate elementary lemmas whose proofs are left to the reader. (They are direct applications respectively of the derivation with respect to a parameter in an integral as well as integration by parts. Both operations are justified due to the smoothness of the integrands as well as the fast decrease at infinity).

We have first

Lemma 1.10. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $\widehat{\varphi}$ is a C^1 function and*

$$\forall j = 1, \ldots, n \qquad \partial_{\xi_j} \widehat{\varphi}(\xi) = \mathcal{F}(-i x_j \, \varphi).$$

We have also the following lemma:

Lemma 1.11. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\forall j = 1, \ldots, n \qquad \widehat{\partial_{x_j} \varphi} = i \, \xi_j \, \widehat{\varphi}(\xi).$$

Observe that the two previous lemmas are illustrating the heuristic idea according to which Fourier transform exchanges derivatives or smoothness with decrease at infinity.

Proof of Proposition 1.9. By iterating Lemma 1.10 and Lemma 1.11, we obtain that $\widehat{\varphi} \in C^\infty$ and we have

$$|\xi^\alpha \, \partial_\xi^\beta \, \widehat{\varphi}(\xi)| = \left|\mathcal{F}\big(\partial_x^\alpha (x^\beta \varphi)\big)\right|.$$

Hence using inequality (1.1) we obtain

$$\begin{aligned}
\mathcal{N}_p(\widehat{\varphi}) &= \sup_{\substack{|\alpha| \le p \\ |\beta| \le p}} \|\xi^\alpha \, \partial_\xi^\beta \, \widehat{\varphi}\|_{L^\infty(\mathbb{R}^n)} \\
&= \sup_{\substack{|\alpha| \le p \\ |\beta| \le p}} \left\|\mathcal{F}\big(\partial_x^\alpha (x^\beta \varphi)\big)\right\|_{L^\infty(\mathbb{R}^n)} \\
&\le \sup_{\substack{|\alpha| \le p \\ |\beta| \le p}} (2\pi)^{-\frac{n}{2}} \|\partial_x^\alpha (x^\beta \varphi)\|_{L^1(\mathbb{R}^n)} \\
&\le C_{n,p} \sup_{\substack{|\alpha| \le p \\ |\beta| \le p+n+1}} \|x^\beta \, \partial_x^\alpha \, \varphi\|_{L^\infty(\mathbb{R}^n)} \le C_{n,p} \, \mathcal{N}_{p+n+1}(\varphi),
\end{aligned}$$

where we used (1.6). This concludes the proof of the proposition. □

We shall now use the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ in order to extend by duality the Fourier transform to the "dual" space to $\mathcal{S}(\mathbb{R}^n)$ as the first identity of proposition 1.5 is inviting to do. The idea behind is that $\mathcal{S}(\mathbb{R}^n)$ is a relatively small space and we expect the "dual" to be big and we would then extend Fourier to this larger space. Now the question is to give a precise meaning to the dual space to $\mathcal{S}(\mathbb{R}^n)$. The classical framework of Banach space is not sufficient since $(\mathcal{S}(\mathbb{R}^n), \mathcal{N}_p)$ is not complete. We have to build a topology out of the countable family of norms $(\mathcal{N}_p)_{p\in\mathbb{N}}$. This is the purpose of the next subsection.

1.3 Fréchet Spaces

Definition 1.12. *Let V be a $\mathbb{R}$ (or $\mathbb{C}$) vector space*

$$\mathcal{N} : V \to \mathbb{R}_+$$

is a pseudo-norm if

i) $\forall \lambda \in \mathbb{R}$ *(or $\mathbb{C}$)*, $\forall x \in V \quad \mathcal{N}(\lambda x) = |\lambda| \, \mathcal{N}(x)$

ii) $\forall x, y \in V \quad \mathcal{N}(x + y) \le \mathcal{N}(x) + \mathcal{N}(y)$.

In other words, a pseudo-norm is a norm without the non-degeneracy axiom.

Definition 1.13. *(Fréchet Space)*
Let V be a $\mathbb{R}$ or $\mathbb{C}$ vector space equipped with an increasing sequence of pseudo-norms

$$\mathcal{N}_p \le \mathcal{N}_{p+1}$$

such that the following non-degeneracy condition is satisfied

$$\left.\begin{array}{l} \mathcal{N}_p(x) = 0 \\ \forall p \in \mathbb{N} \end{array}\right\} \Longleftrightarrow x = 0.$$

Introduce on $V \times V$ the following distance:

$$\forall x, y \in V \quad d(x,y) = \sum_{p=0}^{+\infty} 2^{-p} \min\{1, \mathcal{N}_p(x-y)\}.$$

We say that $(V, (\mathcal{N}_p)_{p\in\mathbb{N}})$ defines a Fréchet space if (V,d) is a complete metric space.

Examples of Fréchet Spaces (left as exercise)

i) A Banach space $(V, \|\cdot\|)$ for the constant sequence of norms $\mathcal{N}_p(\cdot) := \|\cdot\|$ is Fréchet.

ii) The space of smooth functions $C^\infty(B^n_R(0))$ over the ball of $\mathbb{R}^n$ of center 0 and radius R is a Fréchet space for the sequence of C^p-norms

$$\forall p \in \mathcal{N} \quad \|f\|_{C^p} := \sup_{\substack{x \in B^n_1(0) \\ |\alpha| \le p}} |\partial^\alpha f|(x).$$

iii) The space $C^\infty(\mathbb{R}^n)$ of smooth functions over $\mathbb{R}^n$ is a Fréchet space for the sequence of C^p-norms over $B_{2^p}(0)$

$$\forall p \in \mathcal{N} \quad \|f\|_{C^p(B_{2^p}(0))} := \sup_{\substack{x \in B^n_{2^p}(0) \\ |\alpha| \le p}} |\partial^\alpha f|(x).$$

iv) We first recall the following classical notations.

$$\text{Let} \quad \varphi \in C^0(\mathbb{R}^n) \quad \text{then we define} \quad supp\ \varphi := \overline{\{x,\ \varphi(x) \neq 0\}}$$

Let K be a compact subset of $\mathbb{R}^n$. For any $p \in \mathbb{N}$ denote $C^p_K = \{\varphi \in C^p(K)$ and $supp\,\varphi \subset K\}$ and

$$\|\varphi\|_{C^p} = \sum_{|\alpha|\le p} \|\partial^\alpha \varphi\|_\infty \ .$$

$(C^p_K, \|\cdot\|_{C^p})$ is a Banach space.

v) Let K be a compact subset of $\mathbb{R}^n$. Denote

$$C^\infty_K = \{\varphi \in C^\infty,\ supp\ \varphi \subset K\}.$$

C^∞_K is a Fréchet space for the collection of pseudo-norms (norms in fact) $P_i(\cdot) = \|\cdot\|_{C^i}$.

Proof of the fact that C^∞_K is complete for d where $d(f,g) = \sum_{i=0}^\infty 2^{-i} \min\{P_i(f-g), 1\}$:

Let f_n be a Cauchy sequence in (C^∞_K, d), thanks to proposition 1.15

$$\forall i \quad \forall \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad P_i(f_n - f_m) \leq \varepsilon \quad \forall n, m \geq N.$$

$$\implies \forall\, \alpha : \sup_{x \in K} |\partial^\alpha f_n(x) - \partial^\alpha f_m(x)| \to 0 \quad \text{for} \quad n, m \to \infty.$$

In other words, for any mmulti-index α $\partial^\alpha f_n$ converges uniformly towards a continuous function ν_α which is clearly supported in K. Then the conclusion follows from the following classical result from Analysis 2 Let f_n be a sequence of C^1 functions on Ω and arbitrary open subset of $\mathbb{R}^n$

i) f_n converges everywhere to f .

ii) $\partial_i f_n$ converges uniformly to a continuous map g_i.

Then $f \in C^1$ and $\partial_i f = g_i$.

Applying iteratively this result to the situation above gives : $\partial^\alpha \nu_0 = \nu_\alpha$. Since the $\partial^\alpha f_n$ uniformly converge towards ν_α, $P_i(f_n - \nu_0)$ tends to zero for any i. This implies that ν_0 is in C^∞_K and the space is closed for d.

Remark 1.14. *Let Ω be an open set of $\mathbb{R}^n$. The space $C^\infty_0(\Omega)$ of compactly supported C^∞ function of Ω does not have such a simple topology but it is the union of the spaces $C^\infty_{K_j}$ where K_j is a sequence of compact sets such that $\cup_{j \in \mathbb{N}} K_j = \Omega$.*

vi) The space $L^q_{\text{loc}}(\mathbb{R}^n)$ of measurable functions of $\mathbb{R}^n$ which are L^q on every compact of $\mathbb{R}^n (q \in [1, \infty])$ is Fréchet for the family of pseudo-norms

$$\big(L^q(B_{2^p}(0))\big)_{p \in \mathbb{N}}.$$

vii)

$$\big(\mathcal{S}(\mathbb{R}^n), (\mathcal{N}_p)_{p \in \mathbb{N}}\big),$$

where $\mathcal{N}_p$ are the pseudo-norms defined in Definition 1.6 define a Fréchet Space. □

In practice the distance d is never really used and can also be replaced by

$$d_a(x, y) := \sum_{p \in \mathbb{N}} a_p \min\{1, \mathcal{N}_p(x - y)\},$$

where $a = (a_p)_{p \in \mathbb{N}}$ is an arbitrary sequence of positive number such that $\sum_{p \in \mathbb{N}} a_p < +\infty$. The following proposition happens to be very useful in the context of Fréchet space.

Proposition 1.15. *Let $F = (V, (\mathcal{N}_p)_{p \in \mathbb{N}})$ be a Fréchet space, then the following three assertions hold true:*

i) *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of elements from* V

$$f_n \underset{n\Rightarrow+\infty}{\overset{d}{\longrightarrow}} f \iff \forall p \in \mathbb{N} \quad \mathcal{N}_p(f_n - f) \underset{n\Rightarrow+\infty}{\longrightarrow} 0.$$

ii) $(f_n)_{n\in\mathbb{N}}$ *is a Cauchy sequence in* (F,d) *if and only if* $(f_n)_{n\in\mathbb{N}}$ *is a Cauchy sequence for all the pseudo-norms* $\mathcal{N}_p$.

iii) *Each of the pseudo-norm* $\mathcal{N}_p$ *is continuous in* (F,d).

Proof of Proposition 1.15. First we prove the assertion i):

$$\begin{aligned} f_n \underset{n\Rightarrow+\infty}{\overset{d}{\longrightarrow}} f &\Longrightarrow \forall p \in \mathbb{N} \ \min\{1, \mathcal{N}_p(f_n - f)\} \underset{n\Rightarrow+\infty}{\longrightarrow} 0 \\ &\iff \forall p \in \mathbb{N} \ \mathcal{N}_p(f_n - f) \underset{n\Rightarrow+\infty}{\longrightarrow} 0. \end{aligned}$$

We now prove the reciprocal of i):

Let $\varepsilon > 0$ and choose $Q \in \mathbb{N}$ such that

$$\sum_{p=Q}^{+\infty} 2^{-p} < \frac{\varepsilon}{2}.$$

Since $\mathcal{N}_p(f_n - f) \underset{n\to+\infty}{\longrightarrow} 0$ for every p there exists $N \in \mathbb{N}$ such that

$$\forall p < Q \ \text{ and } \ n \geq N \quad \mathcal{N}_p(f_n - f) \leq \frac{\varepsilon}{4}.$$

Thus $\forall n \geq N$:

$$\begin{aligned} &\sum_{p=0}^{+\infty} 2^{-p} \min\{1, \mathcal{N}_p(f_n - f)\} \\ &\leq \sum_{p=0}^{Q-1} 2^{-p}\, \mathcal{N}_p(f_n - f) + \sum_{p=Q}^{+\infty} 2^{-p} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This implies that $f_n \underset{n\to+\infty}{\overset{d}{\longrightarrow}} f$. This proves i).

The same arguments imply ii).

The proof of iii) is straightforward. Indeed, let $p \in \mathbb{N}$

$$d(f,g) \leq 2^{-p}\,\varepsilon \Longrightarrow \mathcal{N}_p(f-g) \leq \varepsilon.$$

This concludes the proof of Proposition 1.15. □

The following proposition extends a well-known fact in normed space topology.

Proposition 1.16. *Let $F = (V, (\mathcal{N}_p)_{p\in\mathbb{N}})$ and $G = (W, (\mathcal{M}_q)_{q\in\mathbb{N}})$ be two Fréchet spaces and let $L : V \to W$ be a linear map. The following three assertions are equivalent:*

i) *L is continuous at 0,*

ii) *L is continuous everywhere,*

iii) *$\forall q \in \mathbb{N} \quad \exists C_q > 0$ and $\exists p \in \mathbb{N}, \quad$ s.t. $\forall f \in F \quad \mathcal{M}_q(Lf) \leq C_q\, \mathcal{N}_p(f)$.*

Proof of Proposition 1.16. The implication ii) $\Longrightarrow$ i) is tautological. We are now proving i) $\Longrightarrow$ iii).

Since L is continuous at 0, for any neighbourhood $\mathcal{V}$ of $0 \in W$, there exists an open neighbourhood $\mathcal{U}$ of $0 \in V$ such that

$$L(\mathcal{U}) \subset G.$$

In other words, $\mathcal{U} \subset L^{-1}(\mathcal{V})$. Let $q \in \mathbb{N}$ and choose $\mathcal{V}_q = \mathcal{M}_q^{-1}([0,1))$. Since $\mathcal{M}_q$ is continuous in (W, d_G), due to Proposition 1.16, $\mathcal{V}_q$ is an open set containing 0. Because the topology in F is a metric topology, there exists $\alpha_q > 0$ such that

$$B_{\alpha_q}^{d_F}(0) \subset \mathcal{U}_q \subset L^{-1}(\mathcal{V}_q),$$

where $B_{\alpha_q}^{d_F}(0)$ denotes the ball of center $0 \in V$ and radius α_q for the Fréchet distance d_F. In other words, we have

$$\sum_{p\in\mathbb{N}} 2^{-p} \min\{1, \mathcal{N}_p(f)\} < \alpha_q \Longrightarrow \mathcal{M}_q\big(L(f)\big) < 1. \tag{1.7}$$

Let $p_0 \in \mathbb{N}$ such that

$$\sum_{p=p_0+1}^{+\infty} 2^{-p} \leq \frac{\alpha_q}{4}. \tag{1.8}$$

Since $\mathcal{N}_p$ is increasing with respect to p

$$\mathcal{N}_{p_0}(f) < \frac{\alpha_q}{4} \Longrightarrow \sum_{p\leq p_0} 2^{-p}\, \mathcal{N}_p(f) < \frac{\alpha_q}{2}. \tag{1.9}$$

Hence, combining (1.7), (1.8) and (1.9), we obtain for any $f \in V$

$$\mathcal{N}_{p_0}(f) < \frac{\alpha_q}{4} \Longrightarrow \mathcal{M}_q\big(L(f)\big) < 1$$

using the homogeneity of the two pseudo-norms $\mathcal{M}_{p_0}$ and $\mathcal{M}_q$, we have proved

$$\mathcal{M}_q\big(L(f)\big) \leq \frac{4}{\alpha_q}\, \mathcal{N}_{p_0}(f).$$

Hence we have proved the implication i) $\Longrightarrow$ iii).

In order to conclude the proof of Proposition 1.16, it suffices to establish the implication iii) $\Longrightarrow$ ii).

We assume iii) and we are going to prove that L is continuous. Since the topologies of both F and G are metric, it suffices to show that for any sequence $f_n \in V$ converging to $f \in V$ for d_F, then

$$\lim_{n\to+\infty} d_G\big(L(f_n),\, L(f)\big) = 0. \tag{1.10}$$

Because of Proposition 1.15 i) in order to establish (1.10), it suffices to prove

$$\forall q \in \mathbb{N} \quad \lim_{n\to+\infty} \mathcal{M}_q\big(L(f_n - f)\big) = 0. \tag{1.11}$$

Let $q \in \mathbb{N}$, because we are assuming iii), there exists $p_0 \in \mathbb{N}$ and $C_q > 0$ such that

$$\forall g \in V \quad \mathcal{M}_q\big(L(g)\big) \leq C_q\, \mathcal{N}_{p_0}(g).$$

Let $\varepsilon > 0$. Let N be large enough such that

$$\forall n \geq N \quad \mathcal{N}_{p_0}\,(f_n - f) \leq \frac{\varepsilon}{C_q},$$

then we have

$$\forall n \geq N \quad \mathcal{M}_q\,\big(L(f_n - f)\big) \leq \varepsilon.$$

This implies (1.11) and L is continuous everywhere. □

The following theorem is the extension of Fréchet spaces of the famous Banach-Steinhaus theorem for normed spaces.

Theorem 1.17. *(Banach-Steinhaus for Fréchet Spaces)*

Let $F = (V, (\mathcal{N}_p)_{p\in\mathbb{N}}$ and $G = (W, (\mathcal{M}_q)_{q\in\mathbb{N}})$ be two Fréchet spaces. Let L_n be a sequence of linear maps from V into W and assume that each L_n is continuous from F into G. Assume moreover that for any $f \in V$ the sequence $L_n f$ converges to a limit Lf in W. Then L defines a linear and continuous map.

Proof of Theorem 1.17. The linearity of L is straightforward. It remains to prove that L is continuous. For any $q \in \mathbb{N}$ and positive number A we introduce the following subset of V:

$$C_A^q := \{f \in V \text{ s.t. } \forall n \in \mathbb{N} \quad \mathcal{M}_q(L_n f) \leq A\}.$$

First, we observe that C_A^q is a closed set. Indeed, it is the intersection of closed sets

$$C_A^q = \bigcap_{n\in\mathbb{N}} (\mathcal{M}_q \circ L_n)^{-1}([0, A]).$$

We now claim that

$$\bigcup_{A\in\mathbb{R}_+^*} C_A^q = V. \tag{1.12}$$

Indeed, by assumption, $d_F(L_n f, Lf) \underset{n\to+\infty}{\longrightarrow} 0$, this implies that

$$\forall q \in \mathbb{N} \quad \sup_{n\in\mathbb{N}} \mathcal{M}_q(L_n f) < +\infty.$$

Thus if one takes $A > \sup_{n\in\mathbb{N}} \mathcal{M}_q(L_n f)$, one has that $f \in C^q_A$ and this proves the claim (1.12).

Obviously $A \geq A' \Longrightarrow C^q_{A'} \subset C^q_A$. Thus

$$V = \bigcup_{j\in\mathbb{N}} C^q_{2^j}.$$

By assumption (V, d_F) is a complete metric space to which we can apply Baire's theorem and there exists $j_0 \in \mathbb{N}$ such that $C^q_{2^{j_0}}$ has a non-empty interior:

$$\dot{C}^q_{2^{j_0}} \neq \emptyset.$$

Let $f_0 \in \dot{C}^q_{2^{j_0}}$, then there exists $\alpha > 0$ such that

$$B^{d_F}_\alpha(f_0) \subset C^q_{2^{j_0}} = \bigcap_{n\in\mathbb{N}} (\mathcal{M}_q \circ L_n)^{-1}([0, 2^{j_0}]).$$

In other words:

$$d_F(f, f_0) < \alpha \Longrightarrow \sup_{n\in\mathbb{N}} \mathcal{M}_q(L_n f) \leq 2^{j_0}. \tag{1.13}$$

Let $p_0 \in \mathbb{N}$ such that

$$\sum_{j=p_0+1}^{\infty} 2^{-j} < \frac{\alpha}{4}. \tag{1.14}$$

Since $\mathcal{N}_p$ is increasing with respect to p

$$\mathcal{N}_{p_0}(f - f_0) < \frac{\alpha}{4} \Longrightarrow \sum_{j=0}^{p_0} 2^{-j}\, \mathcal{N}_p(f - f_0) < \frac{\alpha}{2}.$$

Thus, because of (1.13) and (1.14), we deduce

$$\begin{aligned}\mathcal{N}_{p_0}(f - f_0) < \frac{\alpha}{4} &\Longrightarrow d_F\,(f, f_0) < \alpha \\ &\Longrightarrow \sup_{n\in\mathbb{N}} \mathcal{M}_q(L_n f) \leq 2^{j_0}.\end{aligned}$$

Since $\sup_{n\in\mathbb{N}} \mathcal{M}_q(L_n\, f_0) < 2^{j_0}$, we have

$$\mathcal{N}_{p_0}(h) < \frac{\alpha}{4} \Longrightarrow \sup_{n\in\mathbb{N}} \mathcal{M}_q(L_n h) \leq 2^{j_0+1}.$$

The homogeneity of the pseudo-norms gives then

$$\sup_{n\in\mathbb{N}} \mathcal{M}_q(L_n h) \leq \frac{4}{\alpha}\, 2^{j_0+1}\, \mathcal{N}_{p_0}(h).$$

Since $L_n\, h \underset{n\to+\infty}{\longrightarrow} Lh$ by continuity of $\mathcal{M}_q$, we deduce

$$\mathcal{M}_q(Lh) \le \frac{\alpha}{4}\, 2^{j_0+1}\, \mathcal{N}_{p_0}(h).$$

This holds for arbitrary $q \in \mathbb{N}$. Then, from the characterization of continuity given by Proposition 1.16 iii), we deduce that L is continuous. □

It is now time to define the dual of the Schwartz Space in the Fréchet Space theory.

1.4 The space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is from now on equipped with the Fréchet topology issued by the sequence of pseudo-norms $\mathcal{N}_p$ introduced in Definition 1.6.

Definition 1.18. *The space of tempered distributions denoted $\mathcal{S}'(\mathbb{R}^n)$ is the space of continuous and linear maps from $\mathcal{S}(\mathbb{R}^n)$ into $\mathbb{C}$.*

We have the following important characterization of tempered distributions: The action of a linear form T on $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ will be denoted either $T(\varphi)$ or $\langle T, \varphi\rangle$.

Proposition 1.19. *Let T be a linear map from $\mathcal{S}(\mathbb{R}^n)$ into $\mathbb{C}$. The following equivalence holds*

$$T \in \mathcal{S}'(\mathbb{R}^n) \iff \begin{array}{l} \exists C > 0 \text{ and } p \in \mathbb{N} \text{ such that} \\ \forall \varphi \in \mathcal{S}(\mathbb{R}^n) \quad |\langle T, \varphi\rangle| \le C\, \mathcal{N}_p(\varphi). \end{array} \tag{1.15}$$

The minimal $p \in \mathbb{N}$ for which (1.15) holds is called the order of the tempered distribution T.

The space of tempered distribution can be seen as a subspace of the more "coarse" space of general distributions.

Definition 1.20.

- *Let Ω be an arbitrary open subset of $\mathbb{R}^n$, a distribution T in $\Omega \subset \mathbb{R}^n$ is a linear function between $C_0^\infty(\Omega)$ and $\mathbb{R}$ with the following properties: $\forall\ K \subset \Omega$ compact $\exists\ p \in \mathbb{N}$ and a constant $C_K > 0$, such that*

$$\forall \varphi \in C_K^\infty : \ |\langle T, \varphi\rangle| \le C_K \sup_{|\alpha|\le p} \|\partial^\alpha \varphi\|_\infty.$$

  (p and C_K: depend on T and K.)

- *If there is $p \in \mathbb{N}$ such that for any compact subset K of Ω the above inequality holds for this fixed p, but where C can be depending on K, the minimal integer p such that this true is called the "order of T" and denoted ord(T). If no such a p exists then we say that T has infinite order.*

- *$\mathcal{D}'(\Omega)$ is denoting the space of Ω.* □

Observe that general distributions in $\mathcal{D}'(\mathbb{R}^n)$ don't always have an order. The $L^1_{\rm loc}$ function on $\mathbb{R}$ given by $t \longmapsto e^t$ is an element of $\mathcal{D}'(\mathbb{R})$ but cannot be an element of $\mathcal{S}'(\mathbb{R}^n)$ for that reason: indeed, one easily proves that for any $p \in \mathbb{N}$

$$\sup_{\varphi \in C^\infty_c(\mathbb{R})} \frac{\int_{\mathbb{R}} e^t \varphi(t)\, dt}{\mathcal{N}_p(\varphi)} = +\infty.$$

Consider $\varphi \geq 0$ compactly supported such that $\int_{\mathbb{R}} \varphi = 1$ and take for $k \in \mathbb{N}$ $\varphi_k(t) := \varphi(t-k)$. We have $\mathcal{N}_p(\varphi_k) \leq C k^p$ but

$$\lim_{k \to +\infty} \int_{\mathbb{R}} k^{-p}\, e^t\, \varphi_k(t) = +\infty.$$

Remark 1.21. *In other words the space of general distributions on an open subset Ω of $\mathbb{R}^n$, usually denoted $\mathcal{D}'(\Omega)$, is the space of linear maps from $C^\infty_0(\Omega)$ into $\mathbb{R}$ and continuous on each $C^\infty_{K_i}$ (viewed as a Fréchet Space) where K_i are compact subsets of $\mathbb{R}^n$ such that $K_i \subset K_{i+1}$ and $\Omega = \bigcup_{i \in \mathbb{N}} K_i$. We won't be working with $\mathcal{D}'(\mathbb{R}^n)$ (or even with $\mathcal{D}'(\Omega)$) further and we shall restrict to $\mathcal{S}'(\mathbb{R}^n)$. The main reason is that $\mathcal{D}'(\mathbb{R}^n)$ is not "compatible" with the Fourier transform, it is too large.*

Remark 1.22. *The space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ is exactly the subspace of general distributions on $\mathbb{R}^n$ made of elements of finite order. That is $T \in \mathcal{S}'(\mathbb{R}^n)$ if and only if $T \in \mathcal{D}'(\mathbb{R}^n)$ and there exists $p \in \mathbb{N}$ and $C \in \mathbb{R}^*_+$ such that*

$$\forall\, \varphi \in C^\infty_0(\mathbb{R}^n) \quad |\langle T, \varphi\rangle| \leq C\, \mathcal{N}_p(\varphi)\ .$$

The proof of this last fact is a consequence of the following density result which is left as a exercise : for any $p \in \mathbb{N}$ there holds

$$\forall\ \varphi \in \mathcal{S}(\mathbb{R}^n) \quad \exists\, (\varphi_k)_{k \in \mathbb{N}} \in \left(C^\infty_0(\mathbb{R}^n)\right)^{\mathbb{N}} \qquad s.\ t. \qquad \lim_{k \to +\infty} \mathcal{N}_p(\varphi_k - \varphi) = 0\ . \tag{1.16}$$

The proof of Proposition 1.19 follows from a direct application of the characterization of continuity in Fréchet space given by Proposition 1.16 iii). Indeed, $\mathbb{C}$ equipped with the modulus norm is interpreted as a Fréchet space with

$$\forall a \in \mathbb{C} \quad \mathcal{M}_q(a) := |a|.$$

Example of elements in $\mathcal{S}'(\mathbb{R}^n)$

i) We have for any $p \in [1, +\infty]$

$$L^p(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n).$$

Indeed, let $f \in L^p(\mathbb{R}^n)$, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ Hölder inequality gives

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} f(x)\, \varphi(x)\, dx\Big| &\leq \|f\|_{L^p}\, \|\varphi\|_{L^{p'}}, \\ &\leq \|f\|_{L^p} \left[\int_{\mathbb{R}^n} \frac{(1+|x|^{n+1})^{p'}}{(1+|x|^{n+1})^{p'}}\ |\varphi|^{p'}(x)\, dx\right]^{\frac{1}{p'}} \\ &\leq C_{n,p}\, \|f\|_{L^p}\, \mathcal{N}_{n+1}(\varphi).\end{aligned}$$

ii) Let $a \in \mathbb{R}^n$, the *Dirac Mass* $\delta_a : \varphi \in \mathcal{S}(\mathbb{R}^n) \longmapsto \varphi(a)$ is obviously a tempered distribution of order 0:

$$|\langle \delta_a, \varphi \rangle| \leq \mathcal{N}_0(\varphi).$$

More generally, let Ω be a bounded open set. Denote $\mathcal{M}(\overline{\Omega})$ the space *Signed Radon Measures on* $\overline{\Omega}$. This is the dual space to $C^0(\overline{\Omega})$ (see [3]).

Elements of $\mathcal{M}(\overline{\Omega})$ are tempered distributions of order 0.

example: Let $N \subset \mathbb{R}^n$ be a smooth oriented closed (compact without boundary) sub-manifold of $\mathbb{R}^n$ and denote by ω_N the volume of the induced metric on N. We have that $\varphi \mapsto \int_N \varphi\, \omega_N = \langle u, \varphi \rangle$ a signed Radon measure and a tempered distribution of orderord$(u) = 0$ since obviously

$$\left| \int_N \varphi\, \omega_N \right| \leq \|\varphi\|_\infty \cdot \int_N \omega_N \ .$$

iii) We shall now meet our first Calderón-Zygmund Kernel in this course.

The function $t \longmapsto \frac{1}{t}$ misses by "very little" to be an L^1 function. This is a measurable function which is only in the L^1-weak space (see the following chapters).

Nevertheless one can construct a tempered distribution out of $\frac{1}{t}$ that we shall denote $pv(\frac{1}{t})$ where pv stands for *principal value.* We proceed as follows. Observe that

$$\forall \varphi \in \mathcal{S}(\mathbb{R}) \quad \forall \varepsilon > 0 \qquad \int_{|t|>\varepsilon} \left| \frac{\varphi(t)}{t} \right| dt < +\infty.$$

Moreover

$$\lim_{\varepsilon \to \infty} \int_{|t|>\varepsilon} \frac{\varphi(t)}{t}\, dt = \left\langle pv\left(\frac{1}{t}\right), \varphi \right\rangle \in \mathbb{C} \tag{1.17}$$

exists. Indeed, we write

$$\int_{|t|>\varepsilon} \frac{\varphi(t)}{t}\, dt = \int_{|t|>1} \frac{\varphi(t)}{t}\, dt + \int_{-1}^{-\varepsilon} \frac{\varphi(t)}{t}\, dt + \int_{\varepsilon}^{1} \frac{\varphi(t)}{t}\, dt.$$

Using the fact that $\frac{1}{t}$ is odd, we have also

$$\int_{|t|>\varepsilon} \frac{\varphi(t)}{t}\, dt = \int_{|t|>1} \frac{\varphi(t)}{t}\, dt + \int_{\varepsilon<|t|<1} \frac{\varphi(t) - \varphi(0)}{t} \ .$$

Since φ in particular is Lipschitz, we have that $\frac{\varphi(t)-\varphi(0)}{t}$ is uniformly bounded in L^∞ which justifies the passage to the limit (1.17). Moreover we obviously have

$$\begin{aligned} \left| \left\langle pv\left(\frac{1}{t}\right), \varphi \right\rangle \right| &\leq c\big(\|t\,\varphi(t)\|_{L^\infty} + \|\varphi'\|_{L^\infty}\big) \\ &\leq c\,\mathcal{N}_1(\varphi). \end{aligned}$$

This proves that $pv(\frac{1}{t}) \in \mathcal{S}'(\mathbb{R})$.

One can also without too much difficulty establish that the order of $pv(\frac{1}{t})$ is exactly 1.

iv) The space $\mathbb{C}[x_1 \dots x_n]$ of complex polynomials in $\mathbb{R}^n$ is included in $\mathcal{S}'(\mathbb{R}^n)$.

more generally we define

Definition 1.23. *The space of slowly growing functions denoted $G(\mathbb{R}^n)$ is the subspace of C^∞ functions f in $\mathbb{R}^n$ such that*

$$\forall \beta = (\beta_1, \dots, \beta_n) \quad \exists m_\beta \in \mathbb{N} \;\; and \;\; C_\beta > 0$$

such that

$$|\partial^\beta f|(x) \leq C_\beta (1 + |x|)^{m_\beta}.$$

Exercise: Let $f \in G(\mathbb{R}^n)$ prove that the map

$$\varphi \in \mathcal{S}(\mathbb{R}^n) \longmapsto \int f(x)\,\varphi(x)\,dx$$

defines a tempered distribution that we shall simply denote by f.

Observe that $\mathbb{C}[x_1, \dots, x_n] \subset G(\mathbb{R}^n)$.

Proposition 1.24. *Let $f \in G(\mathbb{R}^n)$ the multiplication by f*

$$\begin{aligned} M_f \;\; \mathcal{S}(\mathbb{R}^n) &\longrightarrow \mathcal{S}(\mathbb{R}^n) \\ \varphi &\longrightarrow f\,\varphi \end{aligned}$$

is a continuous linear map from $\mathcal{S}(\mathbb{R}^n)$ into itself.

Proof of Proposition 1.24. Let $f \in G(\mathbb{R}^n)$, $q \in \mathbb{N}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have using mostly Leibnitz rule and triangular inequality

$$\begin{aligned}
\mathcal{M}_q(f\,\varphi) = &\sup_{\substack{|\alpha| \leq q \\ |\beta| \leq q}} \|x^\alpha\,\partial^\beta(f\,\varphi)\|_{L^\infty(\mathbb{R}^n)} \\
&\leq \sup_{\substack{|\alpha| \leq q \\ |\beta| \leq q}} \sum_{\gamma \leq \beta} C_{\gamma,\beta} \|x^\alpha\;\partial^\gamma \varphi\;\partial^{\beta-\gamma} f\|_{L^\infty(\mathbb{R}^n)} \\
&\leq \sup_{\substack{|\alpha| \leq q \\ |\beta| \leq q}} \sum_{\gamma \leq \beta} C_{\gamma,\beta} \|\;|x^\alpha|\;|\partial^\gamma \varphi|(x)\;(1+|x|)^{m_{\beta-\gamma}}\|_{L^\infty(\mathbb{R}^n)} \\
&\leq C_q \sum_{|\beta| \leq q} \mathcal{N}_{m_\beta + q}(\varphi) \leq C'_q\;\; \mathcal{N}_{q + \max\limits_{|\beta| \leq q} m_\beta}(\varphi).
\end{aligned}$$

This implies the proposition. $\square$

The following proposition is a direct consequence of Proposition 1.24.

Definition-Proposition 1.25. *Let $f \in G(\mathbb{R}^n)$ be a slowly increasing function for any $T \in \mathcal{S}'(\mathbb{R}^n)$, we define the multiplication of T by f as follows:*

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^n) \quad \langle f, T, \varphi \rangle := \langle T, f\,\varphi \rangle.$$

This multiplication denoted $f\,T$ is a tempered distribution.

1.5 The weak convergence of Distributions

Definition 1.26. *A sequence of tempered distributions $(T_k)_{k\in\mathbb{N}}$ is said to converge weakly if for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ the sequence $\langle T_k, \varphi\rangle$ converges in $\mathbb{C}$. From Banach Steinhaus theorem for Fréchet spaces we deduce that there exists $T \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\lim_{k\to+\infty} \langle T_k, \varphi\rangle = \langle T, \varphi\rangle.$$

The weak convergence of a sequence $(T_k)_{k\in\mathbb{N}}$ in $\mathcal{S}'(\mathbb{R}^n)$ towards an element $T \in \mathcal{S}'(\mathbb{R}^n)$ is denoted

$$T_k \rightharpoonup T \ \text{ in } \ \mathcal{S}'(\mathbb{R}^n).$$

Example:

$$u_j \in L^p,\ 1 \le p < \infty, \quad u_j \overset{w}{\rightharpoonup} u \quad \text{in} \quad L^p$$

that is

$$\forall f \in L^{p'}(\mathbb{R}^n) = (L^p(\mathbb{R}^n))^* : \ \int u_j f \to \int u f$$

this implies

$$u_j \to u \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

Observe that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^{p'}(\Omega)$, $p' \neq \infty$ (Exercise). [2]

Exercise: Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$ denote $\varphi_k(x) := 2^{kn}\,\varphi(2^k x)$. Prove that

$$\varphi_k \rightharpoonup \delta_0 \ \text{ in } \ \mathcal{S}'(\mathbb{R}^n).$$

1.6 The derivative of a tempered distribution

Definition-Proposition 1.27. *Let $T \in \mathcal{S}'(\mathbb{R}^n)$ for any $j = 1, \dots, n$ we denote by $\partial_{x_j} T$ the partial derivative of T along the direction x_j which is the following element of $\mathcal{S}'(\mathbb{R}^n)$*

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^n) \quad \langle \partial_{x_j} T, \varphi\rangle := -\langle T, \partial_{x_j}\varphi\rangle. \tag{1.18}$$

[2] We also have for any $(u_j)_{j\in\mathbb{N}} \in L^\infty$ such that

$$u_j \overset{w*}{\rightharpoonup} u \quad \text{in} \quad \sigma\left(L^\infty(\mathbb{R}^n), L^1(\mathbb{R}^n)\right)$$

then $u_j \rightharpoonup u$ in $\mathcal{S}'(\mathbb{R}^n)$.

Proof of Proposition 1.27. Let $T \in \mathcal{S}'(\mathbb{R}^n)$. It is clear that the map $\partial_{x_j} T$ defined by (1.18) is linear. Let $p \in \mathbb{N}$ and $c > 0$ such that

$$|\langle T, \varphi \rangle| \leq c\mathcal{N}_p(\varphi).$$

By (1.18) we have

$$\begin{aligned} |\langle \partial_{x_j} T, \varphi \rangle| = |\langle T, \partial_{x_j} \varphi \rangle| &\leq c\mathcal{N}_p(\partial_{x_j}) \\ &\leq c\mathcal{N}_{p+1}(\varphi). \end{aligned}$$

Hence from the characterization of tempered distributions given by Proposition (1.9), we deduce that $\partial_{x_j} T \in \mathcal{S}'(\mathbb{R}^n)$ and this concludes the proof of Proposition 1.27. □

More generally, by iterating proposition 1.27, we deduce that for any $T \in \mathcal{S}'(\mathbb{R}^n)$ and any $\alpha = (\alpha, \ldots, \alpha_n) \in \mathbb{N}^n$ the linear map on $\mathcal{S}(\mathbb{R}^n)$ given by

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^n) \quad \langle \partial^\alpha T, \varphi \rangle := (-1)^{|\alpha|} \langle T, \partial^\alpha, \varphi \rangle$$

is an element of $\mathcal{S}'(\mathbb{R}^n)$.
example:

i) Let T be a C^1-function, then, thanks to partial integration, the classical and the distributional derivatives coincide.

ii) We introduce the function given explicitly by

$$H_{\alpha,a} = \begin{cases} \alpha, & \text{for} \quad t > a \\ \\ 0, & \text{for} \quad t \leq a \end{cases}$$

and called Heaviside-Function.
Let $\varphi \in C_0^\infty(\mathbb{R})$, $supp\, \varphi \subset [-R, R]$ and $a \in [-R, R]$.

$$\begin{aligned} \langle H'_{\alpha,a}, \varphi \rangle = -\langle H_{\alpha,a},\, \varphi' \rangle &= -\int_{-R}^{R} \alpha 1_{x \geq a} \varphi' = -\alpha \int_a^R \varphi' \\ &= \alpha\varphi(a) = \langle \alpha\delta_a,\, \varphi \rangle. \end{aligned}$$

This implies

$$H'_{\alpha,a} = \alpha\delta_a \ .$$

For the second derivative we have

$$\langle H''_{\alpha,a}, \varphi \rangle = -\langle H'_{\alpha,a}, \varphi' \rangle = -\langle \alpha\delta_a, \varphi' \rangle = -\alpha\varphi'(a) = \langle \alpha\delta' a, \varphi \rangle.$$

This implies

$$H''_{\alpha,a} = \alpha\delta'_a \ .$$

iii) $\log(x) \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$. Let φ a function in $C_0^\infty([-R,R])$.

$$\begin{aligned}
\langle (\log|x|)', \varphi\rangle &= -\int \log|x|\,\varphi' \\
&= \lim_{\varepsilon\to 0} -\int_{\varepsilon\le|x|\le R} \log|x|\,\varphi' \\
&= \lim_{\varepsilon\to 0} -\left(\int_{-R}^{-\varepsilon} \log|x|\,\varphi' + \int_{\varepsilon}^{R} \log|x|\,\varphi'\right) \\
&= \lim_{\varepsilon\to 0}\left(-\log\varepsilon\,\varphi(-\varepsilon) + \int_{-R}^{-\varepsilon}\frac{\varphi(x)}{x} + \log\varepsilon\,\varphi(\varepsilon) + \int_{\varepsilon}^{R}\frac{\varphi(x)}{x}\right) \\
&= \lim_{\varepsilon\to 0}\left(\int_{|x|\ge\varepsilon}^{R}\frac{\varphi(x)}{x} + \log\varepsilon\,\underbrace{[\varphi(\varepsilon)-\varphi(-\varepsilon)]}_{=0(\varepsilon)}\right) \longrightarrow \left\langle pv\left(\frac{1}{x}\right), \varphi\right\rangle .
\end{aligned}$$

From these computations we deduce

$$\log|x|' = pv\left(\frac{1}{x}\right) \ .$$

1.7 The Support of a tempered Distribution

Definition-Proposition 1.28. *Let $T \in \mathcal{S}'(\mathbb{R}^n)$. There exists a maximal open sub subset of $\mathbb{R}^n$, ω, such that $\forall\varphi \in C_0^\infty(\omega)$ there holds $\langle T,\varphi\rangle = 0$. Where the property of being maximal has to be understood in the following sense: For any ω' open satisfying*

$$\omega \subset \omega' \quad \textit{and} \quad \forall\varphi\in C_0^\infty(\omega') \quad \langle T,\varphi\rangle = 0$$

then $\omega = \omega'$.

$\omega^c = \mathbb{R}^n\backslash\omega$ is called the support of T. □

We shall need two intermediate lemma for proving proposition 1.28.

Lemma 1.29. *Let $K \subset \mathbb{R}^n$, be a compact subset of $\mathbb{R}^n$ and let $U \subset \mathbb{R}^n$, open such that $K \subset U$. Then there exists $\Theta \in C_0^\infty(U)$ such that*

$0 \le \Theta \le 1$.

$\Theta \equiv 1$ on K.

Proof of lemma 1.29: Let $\varepsilon > 0$ such that $3\varepsilon < \mathrm{dist(K, U^c)}$. Introduce $\chi_\varepsilon(x)$ to be the characteristic function of the open set of points which are at a distance less than ε from K. That is

$$\chi_\varepsilon(x) := \begin{cases} 1 & \text{in case } \ \mathrm{dist(x,K)} < \varepsilon, \\ \\ 0 & \text{otherwise} \end{cases}$$

Let[3] $g \in C^\infty$ such that *supp* $g \subset B_1(0)$ and $\int_{\mathbb{R}^n} g = 1$. We then introduce

$$f(x) := (\chi_\varepsilon \star g_\varepsilon)(x) = \int_{\mathbb{R}_n} \chi_\varepsilon(y)\, g_\varepsilon(x-y)\, dy,$$

where

$$g_\varepsilon(x) := \frac{1}{\varepsilon^n} g_\varepsilon\left(\frac{x}{\varepsilon}\right).$$

Clearly $f \in C^\infty$, since $\chi_\varepsilon \in L^1(\mathbb{R}^n)$ and $g_\varepsilon \in C_0^\infty(\mathbb{R}^n)$. We are now proving

i) $f(x) \equiv 1$ on K

ii) $f(x) \equiv 0$ outside U

Proof of i):

$$supp\, g \subset B_1(0) \Rightarrow supp_x\, g_\varepsilon(x) \subset B_\varepsilon(0)\ ,$$

indeed

$$g_\varepsilon(x) \neq 0 \Leftrightarrow \left|\frac{x}{\varepsilon}\right| < 1$$

and

$$supp_y\, g_\varepsilon(x-y) \subset B_\varepsilon(x)\ ,$$

indeed

$$g_\varepsilon(x-y) \neq 0 \ \Leftrightarrow\ |x-y| < \varepsilon\ .$$

Let now $x \in K$: $g_\varepsilon(x-y) \neq 0$ implies $\mathrm{dist}(y, K) < \varepsilon$. This gives for such an $x \in K$

$$\int_{\mathbb{R}^n} \chi_\varepsilon(y)\, g_\varepsilon(x-y) = \int_{\mathbb{R}^n} g_\varepsilon(x-y) = \int_{\mathbb{R}^n} g(x) = 1.$$

Proof of ii): From *i)* we have: $supp_y\, g_\varepsilon(x-y) \subset B_\varepsilon(x)$. Since $x \in U^c$, from the choice of ε we have

$$\int_{\mathbb{R}_n} \chi_\varepsilon(y) g_\varepsilon(x-y)\, dy = 0, \quad \text{since} \quad B_\varepsilon(x) \cap K_\varepsilon = \{z \in \mathbb{R}^n, \mathrm{dist}\ (\mathrm{z}, \mathrm{K}) < \varepsilon\} = \emptyset.$$

Finally we take $\Theta \equiv f$. □

Lemma 1.30. (Existence of a partition of unity). *Let $K \subset \mathbb{R}^n, K$ be a compact subset of $\mathbb{R}^n$ and let $O_1, \ldots, O_P$ a finite covering of K by open sets, that is $K \subset \bigcup_{i=1}^P O_i, O_i$ open. Then there exists $\Theta_i \in C_0^\infty(O_i)$ such that $0 \leq \Theta_i \leq 1$ and $\sum_{i=1}^P \Theta_i = 1$ on K.*

Proof of lemma 1.30: Let $\varepsilon > 0$. From the previous lemma, for each O_i there exists f_i with the following properties :

[3] One can take for instance

$$g(x) = \begin{cases} e^{-\frac{1}{(|x|-1)^2(|x|+1)^2}} & \text{when } |x| < 1, \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

- $f_i \in C_0^\infty(O_i)$
- $0 \le f_i \le 1$
- $f_i \equiv 1$ on $\overline{(\tilde{O}_i \cap K)}$,

where
$$\tilde{O}_i := \{x \in O_i \mid \mathrm{dist\,(x, O_i^c)} > \varepsilon\}.$$
We now choose $\varepsilon > 0$ small enough in such a way that
$$K_\varepsilon := \{x \in \mathbb{R}^n, \mathrm{dist\ (x, K)} < \varepsilon\} \subset \bigcup_{\mathrm{i=1}}^{\mathrm{P}} \mathrm{\tilde{O}_i}$$
We consider f given by the previous lemma such that
$$f \equiv 1 \quad \text{on } K \quad \text{and} \quad f \equiv 0 \quad \text{in } U^c := \mathbb{R}^n \setminus K_\varepsilon\ .$$
Finally we denote $f_0 := 1 - f$. Observe that by construction
$$\sum_{j=0}^{P} f_j \ge 1 \quad \text{on } \mathbb{R}^n\ .$$
Let
$$\Theta_i(x) := \begin{cases} \dfrac{f_i(x)}{\sum_{j=0}^{P} f_j(x)} & \text{falls } x \in O_i \\ 0 & \text{otherwise} \end{cases}$$
$(\Theta_i)_{i=1\cdot P}$ is solving the expected requirements and the lemma is proved. □

Proof of proposition 1.28. Let
$$I := \{O | O \subset \mathbb{R}^n, O \text{ open} \quad \forall \varphi \in C_0^\infty(O)\ :\ \langle T, \varphi \rangle = 0\}$$
and denote $\omega := \bigcup_{O \in I} O$. Being a union of open sets, ω is open. Let $\varphi \in C_0^\infty(\omega)$. We claim that $\langle T, \varphi \rangle = 0$. Denote by K the support of φ. We have
$$K \subset \omega = \bigcup_{O \in I} O.$$
Since K is compact, one can extract from $(O)_{O \in I}$ a finite sub covering of K , Hence there exist $P \in \mathbb{N}$ and $\exists \quad O_1, \ldots, O_P \in I$ with $K \subset \bigcup_{i=1}^{P} O_i$. Thanks to the previous lemma we have a subordinated partition of unity $(\Theta_i)_{i=1,\ldots,P}$ with $\sum_{i=1}^{P} \Theta_i \equiv 1$ on K and *supp* $\Theta_i \subset O_i$. We decompose φ accordingly, that is
$$\varphi = \sum_{i=1}^{P} \varphi_i, \quad \text{on } K \text{ where} \quad \varphi_i = \Theta_i \varphi \in C_0^\infty\,(O_i).$$
Since φ is supported on K, the identity holds on the whole of $\mathbb{R}^n$. Since $O_i \in I$ we have $\langle T, \varphi_i \rangle = 0$. We deduce by linearity of T the desired identity $\langle T, \varphi \rangle = 0$. It follows moreover from the definition that ω is maximal. □

Notation 1.31. *We shall be denoting $\mathcal{E}'(\mathbb{R}^n)$ the subspace of $\mathcal{S}'(\mathbb{R}^n)$ with compact support.*

example

i) δ_0 : *supp* $\delta_0 = \{0\}$

ii) $\partial^\alpha \delta_0$: *supp* $\partial^\alpha \delta_0 = \{0\}$

Observe that for a compactly supported distribution, there is a natural extension of the duality from $\mathcal{S}(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$.

Definition-Proposition 1.32. *(Duality Extension)*
Let $T \in \mathcal{E}'(\mathbb{R}^n)$, and let $\theta \in C_0^\infty(\mathbb{R}^n)$, such that $\theta \equiv 1$ on suppT. We define $\forall \varphi \in C^\infty(\mathbb{R}^n)$

$$\langle T, \varphi \rangle_{\mathcal{E}', C^\infty} := \langle T, \theta\varphi \rangle_{\mathcal{S}', \mathcal{S}} \ .$$

$\langle T, \varphi \rangle_{\mathcal{E}', C^\infty}$ is independent of the choice of θ.

Proof of the proposition 1.32: Let $\theta, \theta' \in C_0^\infty(\mathbb{R}^n)$ with $\theta = \theta' = 1$ on *supp* T.

There holds

$$\langle T, \theta\varphi \rangle - \langle T, \theta'\varphi \rangle = \langle T, (\theta - \theta')\varphi \rangle = 0,$$

since

$$\theta - \theta' \equiv 0 \quad \text{on} \quad supp\, u,$$

which implies

$$\theta - \theta' \in C_0^\infty((supp\, u)^C) \ .$$

This concludes the proof of proposition 1.32. □

Proposition 1.33. *Let $T \in \mathcal{E}'(\mathbb{R}^n)$ and let p be the order of T. Consider moreover $\varphi \in C_0^\infty(\mathbb{R}^n)$, such that $\partial^\alpha \varphi = 0$ on supp T for any α such that $|\alpha| \leq p$. Then it holds*

$$\langle T, \ \varphi \rangle = 0.$$

□

Proof of Proposition 1.33: Let $K :=$ *supp* T. By definition K is compact. Denote $1_{K_{2\varepsilon}}$ the characteristic function of the set of points at a distance to K less or equal than 2ε

$$\mathbf{1}_{K_{2\varepsilon}}(x) = \begin{cases} 1 & x \in K_{2\varepsilon}(x), \\ \\ 0 & \text{otherwise} \end{cases}$$

where

$$K_{2\varepsilon} = \{z \ ; \ \text{dist(z, K)} \leq 2\varepsilon\} \ .$$

Let $\psi_\varepsilon = \mathbf{1}_{K_{2\varepsilon}} \star \chi_\varepsilon$, where

$$\chi \in C_0^\infty(B_1(0)), \ \int_{\mathbb{R}^n} \chi = 1 \quad \text{and} \quad \chi_\varepsilon(s) = \frac{1}{\varepsilon^n}\, \chi\left(\frac{s}{\varepsilon}\right) \ .$$

We have that $\psi_\varepsilon \in C^\infty(\mathbb{R}^n)$ and from the definition of the convolution operator we have

$$\mathrm{Supp}\,(\mathbf{1}_{K_{2\varepsilon}} \star \chi_\varepsilon) \subset \mathrm{Supp}\,(\mathbf{1}_{K_{2\varepsilon}}) + \mathrm{Supp}\,(\chi_\varepsilon) \subset \mathrm{Supp}\,(\mathbf{1}_{K_{3\varepsilon}}) \ .$$

Hence we deduce

$$\psi_\varepsilon \in C^\infty_0(\mathbb{R}^n) \ .$$

We claim that $\psi_\varepsilon \equiv 1$ on K_ε. Indeed, let $x \in K_\varepsilon$, there holds

$$\psi_\varepsilon(x) = \int_{\mathbb{R}^n} \mathbf{1}_{K_{2\varepsilon}}(y)\, \frac{1}{\varepsilon^n}\, \chi\left(\frac{x-y}{\varepsilon}\right)\, dy$$

Observe that $\mathrm{supp}\big(\chi\left(\frac{z}{\varepsilon}\right)\big) \subset B_\varepsilon(0)$. Hence, for $\chi\left(\frac{x-y}{\varepsilon}\right) \neq 0$ there need to be $|x-y| < \varepsilon$ which is implying $\mathrm{dist}(y, K_\varepsilon) < \varepsilon$ which itself implies $y \in K_{2\varepsilon}$. Hence

$$\psi_\varepsilon(x) = \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n}\, \chi\left(\frac{x-y}{\varepsilon}\right)\, dy = 1$$

which concludes the proof of the claim.

We decompose φ as follows

$$\varphi = \varphi\, \psi_\varepsilon + (1-\psi_\varepsilon)\varphi.$$

From the claim we just proved we deduce *supp* $(1-\psi_\varepsilon)\varphi \subset K^c_\varepsilon$. Hence $\langle T,\ (1-\psi_\varepsilon)\,\varphi\rangle = 0$ since $\mathrm{Supp}T = K$. Thus we have $\langle T,\ \varphi\rangle = \langle T,\ \varphi\psi_\varepsilon\rangle$.

We claim that for any $\alpha \in \mathbb{N}^n$ there exists of $C_\alpha > 0$ such that

$$\|\partial^\alpha \psi_\varepsilon\|_\infty \leq C_\alpha \varepsilon^{-|\alpha|} :$$

Indeed, we have on one hand

$$\partial^\alpha(\mathbf{1}_{K_{2\varepsilon}} \star \chi_\varepsilon) = \mathbf{1}_{K_{2\varepsilon}} \star \partial^\alpha \chi_\varepsilon \ ,$$

and on the other hand a direct computation gives

$$\partial^\alpha \chi_\varepsilon = \varepsilon^{-n-|\alpha|}(\partial^\alpha \chi) \Longrightarrow \|\partial^\alpha \chi_\varepsilon\|_1 = \varepsilon^{-|\alpha|} C_\alpha \ .$$

Combining these two facts we obtain

$$\begin{aligned}
|\partial^\alpha \psi_\varepsilon(x)|_\infty \ &= \left\| \int_{\mathbb{R}^n} \mathbf{1}_{K_{2\varepsilon}}(y)\, \frac{1}{\varepsilon^n}\, \partial^\alpha_x \chi\left(\frac{x-y}{\varepsilon}\right)\, dy \right\|_\infty \\
&\leq \|\mathbf{1}_{K_{2\varepsilon}}\|_\infty\ \|\partial^\alpha \chi_\varepsilon\|_1 \leq \varepsilon^{-|\alpha|} C_\alpha \ .
\end{aligned}$$

This implies the claim.

Let $x \in K_{3\varepsilon}$, we consider $y \in K$ such that $|x-y| \leq 4\varepsilon$. Taylor expansion at y gives for any γ with $|\gamma| \leq p$ gives the existence of ξ in the segment $[x,y]$ such that

$$\partial^\gamma \varphi(x) = \partial^\gamma \varphi(y) + \sum_{\substack{|\alpha|\leq p \\ \gamma<\alpha}} \partial^\alpha \varphi(y)\ \frac{h^{\alpha-\gamma}}{(\alpha-\gamma)!}\ + \sum_{|\beta|=p+1} \partial^\beta \varphi(\xi)\ \frac{h^{\beta-\gamma}}{(\beta-\gamma)!}$$

where

$$\frac{1}{\alpha!} = \frac{1}{\alpha_1! \dots \alpha_n!}$$

and

$$x - y = (h_1, \dots, h_n),\ h^\alpha = h_1^{\alpha_1} \cdots h_n^{\alpha_n}\ .$$

From the hypothesis we have for any $y \in K = \mathrm{Supp}(T)$ $\partial^\gamma \varphi(y) = 0 \quad \forall |\gamma| \leq p$. Combining this hypothesis with the Taylor expansion we obtain

$$\|\partial^\gamma \varphi\|_{L^\infty(K_{3\varepsilon})} \leq C_\varphi\ \varepsilon^{p+1-|\gamma|}\ .$$

Finally we bound

$$\begin{aligned}
|\langle T,\ \varphi\rangle| = |\langle T,\ \varphi\, \psi_\varepsilon\rangle| &\leq C \sum_{|\alpha|\leq p} \|\partial^\alpha(\varphi\, \psi_\varepsilon)\|_{L^\infty(K_{3\varepsilon})} \\
&\leq C \cdot \sum_{|\alpha|\leq p} \sum_{|\gamma|\leq|\alpha|} \|\partial^\gamma \varphi\|_{L^\infty(K_{3\varepsilon})} \|\partial^{\alpha-\gamma}\, \psi_\varepsilon\|_\infty C_\gamma \\
&\leq C \sum_{|\alpha|\leq p} C'' \varepsilon^{p+1-|\alpha|} \leq C_{\varphi,T}\, \varepsilon\ .
\end{aligned}$$

This holds for any arbitrary small ε hence we deduce $|\langle T,\ \varphi\rangle| = 0$. This concludes the proof of proposition 1.33 □

1.8 The Fourier transform of a tempered distribution

We define now the Fourier transform of a tempered distribution. This definition is motivated by the first identity in Proposition 1.5.

Definition-Proposition 1.34. *Let T be a tempered distribution. We define the Fourier transform of T that we denote by $\widehat{T}$ or $\mathcal{F}(T)$ to be the following linear map on $\mathcal{S}(\mathbb{R}^n)$*

$$\varphi \in \mathcal{S}(\mathbb{R}^n)\ \langle \widehat{T}, \varphi\rangle := \langle T, \widehat{\varphi}\rangle,$$

$\widehat{T}$ is a tempered distribution as well.

Proof of Proposition 1.34. Let $T \in \mathcal{S}'(\mathbb{R}^n)$ and let p be the order of T and $C > 0$ such that

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^n)\ \ |\langle T, \varphi\rangle| \leq c\, \mathcal{N}_p(\varphi).$$

Using Proposition 1.9 we then deduce

$$\begin{aligned}
\varphi \in \mathcal{S}(\mathbb{R}^n)\ \ |\langle \widehat{T}, \varphi\rangle| = |\langle T, \widehat{\varphi}\rangle|\ &\leq c\, \mathcal{N}_p(\widehat{\varphi}) \\
&\leq c^1\, \mathcal{N}_{p+n+1}(\varphi).
\end{aligned}$$

Using one more time the characterization of $\mathcal{S}'(\mathbb{R}^n)$ given by Proposition 1.19, we deduce that $\widehat{T}$ is a tempered distribution.

Example: Let $a \in \mathbb{R}^n$ we have

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^n)\ \ \langle \widehat{\delta_a}, \varphi\rangle = \langle \delta_a, \widehat{\varphi}\rangle = \widehat{\varphi}(a) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ia\cdot x}\, \varphi(x)\, dx.$$

Hence

$$\widehat{\delta}_a = (2\pi)^{-\frac{n}{2}}\ e^{-ia\cdot x} \in L^\infty(\mathbb{R}^n).$$

In other words, the Fourier transform exchanges the "most concentrated" measure into the "most dispersed" wave function. This phenomenon is known as the *Heisenberg Uncertainty Principle* in quantum mechanics. □

Example: More generally, given $\alpha = (\alpha, \ldots, \alpha_n) \in \mathbb{N}^n$ we have, using Lemma 1.10,

$$\begin{aligned}\langle \widehat{\partial_\alpha\, \delta_a}, \varphi\rangle &= (-1)^{|\alpha|}\, \langle \delta_a, \partial_a \widehat{\varphi}\rangle \\ &= (-1)^{|\alpha|} \langle \delta_a,\ \widehat{(-i)^{|\alpha|} x^\alpha \varphi}\rangle \\ &= (i)^{|\alpha|} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ia\cdot x}\, x^\alpha\, \varphi(x)\, dx.\end{aligned}$$

Hence we have established

$$\widehat{\partial_\alpha\, \delta_a} = (i)^{|\alpha|} (2\pi)^{-\frac{n}{2}}\ e^{-ia\cdot x} x^\alpha \in G. \tag{1.19}$$

Exercise: Prove that

$$\widehat{\mathbf{1}} = (2\pi)^{\frac{n}{2}}\ \delta_0$$

and more generally

$$\forall \alpha \in \mathbb{N}^n \quad \widehat{x^\alpha} = (2\pi)^{\frac{n}{2}}\ i^{|\alpha|}\, \partial^\alpha\, \delta_0.$$

□

Excercise: We shall now compute the Fourier transform of $pv(\frac{1}{t})$. First, we claim that

$$t\, pv\Big(\frac{1}{t}\Big) = 1 \ \text{ in } \ \mathcal{S}'(\mathbb{R}), \tag{1.20}$$

where the product by t has to be understood in the sense given by Proposition 1.25. Indeed,

$$\begin{aligned}\forall \varphi \in \mathcal{S}(\mathbb{R}) \quad \Big\langle tpv\big(\tfrac{1}{t}\big), \varphi\Big\rangle &= \big\langle pv\big(\tfrac{1}{t}\big),\ t\varphi(t)\big\rangle \\ &= \lim_{\varepsilon\to 0} \int_{|t|>\varepsilon} \varphi(t)\, dt = \int_{\mathbb{R}} \varphi(t)\, dt.\end{aligned}$$

This proves (1.20). The computation above of the Fourier transform of 1 gives then

$$\mathcal{F}\Big(t\, pv\big(\frac{1}{t}\big)\Big) = (2\pi)^{\frac{1}{2}}\, \delta_0.$$

Using now Proposition 1.40, we have

$$\frac{d}{dt}\, \widehat{pv\big(\frac{1}{t}\big)} = -i\, \widehat{t\, pv\big(\frac{1}{t}\big)} = -i\, \sqrt{2\pi}\, \delta_0.$$

Let $H(t)$ be the *Heaviside function* equal to the characteristic function of $\mathbb{R}_+$. An elementary calculus gives

$$\frac{d}{dt}\, H(t) = \delta_0.$$

Hence

$$\frac{d}{dt}\left[\widehat{pv\big(\frac{1}{t}\big)} + i\sqrt{2\pi}\, H(t)\right] = 0. \tag{1.21}$$

We shall now need the following lemma:

Lemma 1.35. *Let T be an element of $\mathcal{S}'(\mathbb{R})$ such that*

$$\frac{d}{dt}\, T = 0,$$

then T is the multiplication by a constant.

Proof of Lemma 1.35. Let $\varphi \in \mathcal{S}(\mathbb{R})$. It is not difficult to prove that if $\int_{-\infty}^{+\infty} \varphi(t)\, dt = 0$, then $t \longmapsto \int_{-\infty}^{t} \varphi(s) ds$ is still a Schwartz function. Hence since $\frac{d}{dt}\int_{-\infty}^{t} \varphi(s) ds = \varphi(t)$, we have by assumption of the lemma $\forall \varphi \in \mathcal{S}(\mathbb{R})$ such that $\int_{-\infty}^{+\infty} \varphi(s)\, ds = 0$

$$\langle T, \varphi\rangle = 0.$$

Let $\varphi \in \mathcal{S}(\mathbb{R})$ arbitrary. We have then

$$\left\langle T,\, \varphi(t) - e^{-t^2}\, \frac{\int_{-\infty}^{+\infty} \varphi(s)\, ds}{\int_{-\infty}^{+\infty} e^{-s^2}\, ds}\right\rangle = 0.$$

This gives

$$\langle T, \varphi\rangle = \int_{-\infty}^{+\infty} \frac{\langle T, e^{-t^2}\rangle}{\int_{-\infty}^{+\infty} e^{-s^2}\, ds}\, \varphi(t)\, dt.$$

Hence T is the multiplication by the constant $\frac{\langle T, e^{-t^2}\rangle}{\int_{-\infty}^{+\infty} e^{-s^2}\, ds}$. This concludes the proof of the lemma. □

Combining (1.21) and lemma 1.25, we obtain that there exists a constant $A \in \mathbb{C}$ such that

$$\widehat{pv\big(\frac{1}{t}\big)} = -i\, \sqrt{2\pi}\, H(t) + A.$$

Observe that for any even function $\varphi(t)$, one has

$$\left\langle \widehat{pv\big(\frac{1}{t}\big)},\ \check{\varphi} \right\rangle = 0.$$

It is not difficult to prove that a Schwartz function is even if and only if it's Fourier transform is even too. Hence for any even function we have

$$\int_{-\infty}^{+\infty} \big(-i\, \sqrt{2\pi}\, H(t) + A\big)\, \varphi(t)\, dt = 0,$$

this implies that $-i\sqrt{2\pi}\, H(t) + A$ is odd and we have proved that

$$\widehat{pv\big(\frac{1}{t}\big)} = -\frac{i}{2}\, \sqrt{2\pi}\, \mathrm{sign}(t).$$

As mentioned above, this function belongs to the family of Calderón-Zygmund multipliers that we are going to study more systematically in section 7.

Theorem 1.36. *The Fourier transformation realises an isomorphism $S' \to S'$, whose inverse is given by $\mathcal{F}^{-1} = \bar{\mathcal{F}}$ where $\bar{\mathcal{F}}$ is defined as follows*

$$\langle \bar{\mathcal{F}}(T),\, \varphi\rangle := \langle u,\, \bar{\mathcal{F}}\,(\varphi)\rangle.$$

and $\bar{\mathcal{F}}$ is the operation defined previously on $L^1(\mathbb{R}^n)$ (and A fortiori on $\mathcal{S}(\mathbb{R}^n)$)

$$\bar{\mathcal{F}} \;:\; f \in L^1(\mathbb{R}^n) \longmapsto (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, f(\xi)\, d\xi$$

will be denoted $\overset{\vee}{f}$ or also $\mathcal{F}^{-1}(f)$.

Consider moreover $T_j \in S' \to T \in S'$ in S', then $\hat{T}_j \to \hat{T}$ in S'. □

Proof of theorem 1.36.

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^n) \qquad \langle\, \bar{\mathcal{F}}\,\mathcal{F}(T),\, \varphi\rangle = (\langle \mathcal{F}(T),\, \bar{\mathcal{F}}\,(\varphi)\rangle = \langle T,\, \mathcal{F}\bar{\mathcal{F}}\,(\varphi)\rangle = \langle T,\, \varphi\rangle\ .$$

Moreover we have

$$\langle \hat{T}_j,\, \varphi\rangle = \langle T_j,\, \hat{\varphi}\rangle \to \langle T,\, \hat{\varphi}\rangle = \langle \hat{T},\, \varphi\rangle,$$

This implies

$$\hat{T}_j \to \hat{T} \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n)\ .$$

□

Theorem 1.37. *Let $T \in \mathcal{E}'(\mathbb{R}^n)$, then we have $\hat{T} \in G(\mathbb{R}^n)$ and $\forall \zeta \in \mathbb{R}^n$ there holds*

$$\hat{T}(\zeta) = (2\pi)^{-n/2}\, \langle T, e^{-ix\cdot\zeta}\rangle_{\mathcal{E}', C^\infty}\ .$$

where the duality $\mathcal{E}', C^\infty$ has to be understood in the sense of proposition 1.32. □

Before proving theorem 1.37 we establish the following lemma.

Lemma 1.38. *Let $k \in \mathbb{N}^*$ and $\varphi \;:\; x \in \mathbb{R}^n \;\longrightarrow\; \varphi_x \in \mathcal{S}(\mathbb{R}^n)$ such that for any p in $\mathbb{N}$ φ realizes a C^k map from $\mathbb{R}^n$ into the normed space $(\mathcal{S}(\mathbb{R}^n), \mathcal{N}_p)$ for every $p \in \mathbb{N}$. Then for any $T \in \mathcal{S}'(\mathbb{R}^n)$*

$$x \longrightarrow \langle T(y), \varphi_x(y)\rangle$$

is in $C^k(\mathbb{R}^n)$ and there holds

$$\forall \gamma = (\gamma_1 \cdots \gamma_n) \quad , \quad |\gamma| \le k \qquad \partial_x^\gamma \langle T(y), \varphi_x(y)\rangle = \langle T(y), \partial_x^\gamma \varphi_x(y)\rangle$$

Proof of lemma 1.38 It suffices to establish the lemma for $k = 1$. The assumption is saying that for any $x^0 \in \mathbb{R}^n$ there exists $\partial_{x_i}\varphi \in C^0(\mathbb{R}, \mathcal{S}(\mathbb{R}^n))$ such that

$$\lim_{h\to 0} \left\| \frac{\varphi_{x^0+h}(y) - \varphi_{x^0}(y) - \sum_{i=1}^n \partial_{x_i}\varphi_{x_0} h^i}{|h|} \right\|_{\mathcal{N}_p} = 0$$

In other words we have for any $p \in \mathbb{N}$

$$(1.22) \qquad \lim_{h\to 0} \sup_{\substack{|\alpha|\le p\\ |\beta|\le p}} \left\| \frac{y^\alpha \partial_y^\beta \varphi_{x^0+h}(y) - y^\alpha \partial_y^\beta \varphi_{x^0}(y) - \sum_{i=1}^n y^\alpha \partial_y^\beta \partial_{x_i}\varphi_{x^0} h^i}{|h|} \right\|_{L^\infty_y(\mathbb{R}^n)}$$

Hence, assuming T has order q there holds

$$\lim_{h\to 0} \left| \frac{\langle T(y), \varphi_{x^0+h}(y)\rangle - \langle T, \varphi_{x^0}(y)\rangle - \sum_{i=1}^n \langle T, \partial_{x_i}\varphi_{x^0}\rangle h^i}{|h|} \right|$$
$$\le \lim_{h\to 0} \sup_{\substack{|\alpha|\le q\\ |\beta|\le q}} \left\| \frac{y^\alpha \partial_y^\beta \varphi_{x^0+h}(y) - y^\alpha \partial_y^\beta \varphi_{x^0}(y) - \sum_{i=1}^n y^\alpha \partial_y^\beta \partial_{x_i}\varphi_{x^0} h^i}{|h|} \right\|_{L^\infty_y(\mathbb{R}^n)} = 0$$

Hence we have proved that $x \longrightarrow \langle T(y), \varphi_x(y)\rangle$ is differentiable at every point and the differential equals

$$\sum_{i=1}^n \langle T, \partial_{x_i}\varphi_x\rangle \, dx_i \ .$$

The continuity of each partial derivative $\partial_{x_i}\langle T, \varphi_x\rangle = \langle T, \partial_{x_i}\varphi_x\rangle$ at an arbitrary point x^0 is deduced from the following inequality

$$|\langle T, \partial_{x_i}\varphi_x\rangle - \langle T, \partial_{x_i}\varphi_{x^0}\rangle| \le C \ \|\partial_{x_i}\varphi_x - \partial_{x_i}\varphi_{x^0}\|_{\mathcal{N}_q}$$

combined with the hypothesis asserting the continuity of $x \longrightarrow \partial_{x_i}\varphi_x$ from $\mathbb{R}^n$ into $\mathcal{S}'(\mathbb{R}^n)$. This concludes the proof of lemma 1.38. □

Proof of theorem 1.37: Let

$$v := \langle T, \, e^{-ix\cdot\zeta}\rangle_{\mathcal{E}', C^\infty} = \langle T, \, \theta\, e^{-ix\cdot\zeta}\rangle_{\mathcal{S}', \mathcal{S}}$$

where $\theta \in C^\infty_0(\mathbb{R}^n)$ and $\theta \equiv 1$ on $supp\, T$.

Claim 1: $v \in G(\mathbb{R}^n)$.

Proof of claim 1: We have using lemma 1.38

$$\partial^\alpha_\zeta \ v = \langle T, \, \partial^\alpha_\zeta \,(e^{-ix\cdot\zeta})\rangle = \langle T, \, (-i)^{|\alpha|}\, x^\alpha e^{-ix\cdot\zeta}\rangle.$$

Since $T \in \mathcal{E}'(\mathbb{R}^n)$ there holds for some $p \in \mathbb{N}$

$$|\langle T, \, (-i)^{|\alpha|}\, x^\alpha e^{-ix\cdot\zeta}\rangle| \le \ C \sum_{|\beta|\le p} \| \, \partial^\beta_x \,(x^\alpha e^{-ix\cdot\zeta}) \, \|_{L^\infty\,(supp\, u)}$$

$$\le C' \,(1 + |\zeta|^p),$$

hence

$$|\partial^\alpha_\zeta \, v| \le \ C' \,(1 + |\zeta|^p)$$

which implies

$$v \in G(\mathbb{R}^n)\ .$$

Claim 2: $v = \hat{T}$.

Proof of claim 2: Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Since $v \in G(\mathbb{R}^n)$, its action on any element in $\mathcal{S}(\mathbb{R}^n)$ is given by the classical multiplication followed by the integration operation. Hence we have

$$\langle v,\, \varphi\rangle = \langle v(\zeta),\ \varphi(\zeta)\rangle = \int_{\mathbb{R}^n} v(\zeta)\ \varphi(\zeta)\, d\,\zeta$$

$$= \int_{\mathbb{R}^n} \langle u(x),\, e^{-ix\cdot\zeta}\rangle\ \varphi\,(\zeta)\, d\,\zeta$$

$$= \int_{\mathbb{R}^n} \left\langle u(x),\, e^{-ix\cdot\zeta}\, \varphi\,(\zeta)\right\rangle d\zeta = \left\langle u(x),\, \int_{\mathbb{R}^n} e^{-ix\cdot\zeta}\, \varphi(\zeta)\right\rangle$$

$$= (2\pi)^{n/2}\, \langle u(x),\ \hat{\varphi}(x)\rangle = (2\pi)^{n/2}\, \langle \hat{u},\ \varphi\rangle,$$

where the third inequality is using the first part of the proof of proposition 1.44 . Hence we have proved the claim 2 and this concludes the proof of theorem 1.37. □

We shall end this subsection by first proving the following important proposition known also under the name of "Schwartz Lemma" and then we will apply this proposition in order to establish a characterization of harmonic tempered distributions (theorem 1.41).

Proposition 1.39. *Let T be a tempered distribution supported at the origin that is to say $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\varphi \equiv 0$ in a neighborhood of 0, then $\langle T, \varphi\rangle = 0$. Then, there exists $p \in \mathbb{N}$ such that for any $\beta = (\beta_1, \ldots, \beta_n)$ satisfying $|\beta| \leq p$, there exists $c_\beta \in \mathbb{C}$ such that*

$$T = \sum_{|\beta|\leq p} C_\beta\, \partial^\beta \delta_0.$$

Proof of Proposition 1.39. Let p be the order of T. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we proceed to the Taylor expansion of φ to the order p at the origin: for any $\gamma \in \mathbb{N}^n$ and $|\gamma| \leq p$ there exists a_γ independent of φ such that

$$\varphi(x) = \sum_{|\gamma|\leq p} a_j\, \partial^\gamma\, \varphi(0)\, x^\gamma + R_p(x)$$

where

$$\lim_{|x|\to 0} \frac{|R_p(x)|}{|x|^p} = 0.$$

Moreover $\forall \gamma$, $|\gamma| \leq p$

$$\lim_{|x|\to 0} \frac{|\partial^\gamma R_p(x)|}{|x^{p-|\gamma|}|} = 0. \tag{1.23}$$

Let χ be a non-negative cut-off function in $C_c^\infty(B_1(0))$ such that χ is identically equal to one on $B_{\frac{1}{2}}(0)$. By assumption

$$\begin{aligned}\langle T, \varphi\rangle &= \langle T, \chi\,\varphi\rangle + \langle T, (1-\chi)\,\varphi\rangle \\ &= \langle T, \chi\,\varphi\rangle.\end{aligned}$$

We have, using the Taylor expansion of φ,

$$\langle T, \chi\varphi\rangle = \sum_{|\gamma|\le p} a_\gamma\,\partial^\gamma\varphi(0)\,\langle T, \chi(x)\,x^\gamma\rangle + \langle T, \chi(x)\,R_p(x)\rangle.$$

Observe that the functions $\chi(x)\,x^\gamma$ are Schwartz functions and hence $\langle T, \chi(x)\,x^\gamma\rangle$ are well-defined complex numbers. We claim that

$$\langle T, \chi(x)\,R_p(x)\rangle = 0. \tag{1.24a}$$

This claim implies obviously the proposition. Observe that this follows immediately from Proposition 1.33, but we choose to give a direct proof here. Let

$$\eta_\varepsilon(x) := 1 - \chi\big(\frac{x}{\varepsilon}\big)$$

where $0 < \varepsilon \ll 1$. By assumption we have

$$\begin{aligned}\langle T, \chi\,R_p\rangle &= \langle T, \chi\,R_p\,\eta_\varepsilon\rangle + \langle T, \chi\,R_p\,\chi_\varepsilon\rangle \\ &= \langle T,\,R_p\,\chi_\varepsilon\rangle,\end{aligned} \tag{1.24b}$$

where $\chi_\varepsilon(x) = \chi(x/\varepsilon)$. Since T is of order p, there exists $c > 0$ such that

$$|\langle T,\,P_p\,\chi_\varepsilon\rangle| \le C\,\mathcal{N}_p(R_p\,\chi_\varepsilon).$$

We have, using Leibnitz formula and triangular inequality,

$$\begin{aligned}\mathcal{N}_p(R_p\,\chi_\varepsilon) &= \sum_{\substack{|\alpha|\le p\\ |\beta|\le p}} \|x^\alpha\,\partial^\beta(R_p\,\chi_\varepsilon)\|_{L^\infty(\mathbb{R}^n)} \\ &\le \sum_{\substack{|\alpha|\le p\\ |\beta|\le p}} \sum_{\gamma\le\beta} C_\gamma\,\|x^\alpha\,\partial^{\beta-\gamma}\,R_p\,\partial^\gamma\,\chi_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \\ &\le C_p \sum_{|\beta|\le p}\sum_{\gamma\le\beta} \|\partial^{\beta-\gamma}\,R_p\,\partial^\gamma\,\chi_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\end{aligned} \tag{1.25}$$

We clearly have

$$|\partial^\gamma\,\chi_\varepsilon|(x) \le \frac{C_\gamma}{\varepsilon^{|\gamma|}}\,\mathbf{1}_{\beta_\varepsilon(0)}(x) \tag{1.26}$$

where $\mathbf{1}_{B_\epsilon(0)}(x)$ is the characteristic function of the ball centered at the origin and of radius ε. Because of (1.23) we have

$$\|\partial^{\beta-\gamma}\,R_p(x)\,\mathbf{1}_{B_\varepsilon(0)}(x)\|_{L^\infty(\mathbb{R}^n)} = o(\varepsilon^{p-|\beta-\gamma|}).$$

Combining this inequality with (1.25) and (1.26) we obtain

$$\mathcal{N}_p(R_p\,\chi_\varepsilon) = o\Big(\sum_{|\beta|\le p}\ \sum_{\gamma\le\beta} \varepsilon^{p-|\beta-\gamma|-|\gamma|}\Big).$$

Since $\gamma \le \beta$, we have $|\beta-\gamma| + |\gamma| = \sum \beta_i - \gamma_i + \sum \gamma_i = |\beta|$. Hence

$$\lim_{\varepsilon\to 0} \mathcal{N}_p(R_p\,\chi_\varepsilon) = 0$$

From (1.24b) we deduce (1.24a) and this concludes the proof of Proposition 1.39. $\square$

Proposition 1.40. *Let $T \in \mathcal{S}'(\mathbb{R}^n)$, then for any $\alpha = (\alpha_1,\dots,\alpha_n)$ and any $\beta = (\beta_1,\dots,\beta_n)$ we have respectively*

$$\partial^\alpha\,\widehat{T} = (-i)^{|\alpha|}\ \widehat{x^\alpha\,T}$$

and

$$\widehat{\partial^\beta T} = i^{|\beta|}\xi^\beta\ \widehat{T},$$

where the products $x^\alpha T$ and $\xi^\beta\widehat{T}$ have to be understood in the sense of Proposition 1.25.

Proposition 1.40 is a direct consequence of Lemma 1.10 and Lemma 1.11. We have the following theorem:

Theorem 1.41. *Let T be an harmonic tempered distribution that is an element of $\mathcal{S}'(\mathbb{R}^n)$ satisfying*

$$\forall\varphi\in\mathcal{S}(\mathbb{R}^n)\quad \langle\Delta T,\varphi\rangle = \langle T,\Delta\varphi\rangle = 0.$$

Then T is a polynomial.

Remark 1.42. *This result is a bit "counter-intuitive" since we know many more harmonic functions than polynomials. For instance in $\mathbb{R}^2$ every holomorphic function is harmonic but is not necessarily a polynomial (i.e. $f(z) = e^z$). This illustrates the difference between $\mathcal{S}'$ and $\mathcal{D}'$. $\mathcal{S}'$ being in a sense the space of distributions for which one can define a Fourier transform.*

Proof of Theorem 1.41. For any $\varphi\in\mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned} 0 = \langle\Delta T,\widehat{\varphi}\rangle &= \langle T,\Delta\,\widehat{\varphi}\rangle \\ &= -\langle T,\,\widehat{|x|^2\,\varphi}\rangle \qquad (1.27)\\ &= -\langle\widehat{T},|x|^2\varphi\rangle. \end{aligned}$$

Let $\psi\in\mathcal{S}(\mathbb{R}^n)$ such that ψ is identically 0 in a neighborhood of 0. Then $\psi(x)/|x|^2 = \varphi(x)$ is still an element of $\mathcal{S}(\mathbb{R}^n)$.

Then we deduce from (1.27) that for such a ψ we have $\langle \widehat{T}, \psi \rangle = 0$. In other words, the support of the Fourier transform of T is included in the origin. Applying Proposition 1.39 to $\widehat{T}$, we deduce the existence of $p \in \mathbb{N}$ and $C_\beta \in \mathbb{C}$ for any $\beta \in \mathbb{N}^n$ with $|\beta| \leq p$ such that

$$\widehat{T} = \sum_{|\beta| \leq p} c_\beta \, \partial^\beta \, \delta_0.$$

Using Proposition 1.40, we deduce that

$$T = \sum_{|\beta| \leq p} \frac{C_\beta (-i)^{|\beta|}}{(2\pi)^{\frac{n}{2}}} \, x^\beta.$$

This implies the theorem. □

1.9 Convolutions in $\mathcal{S}'(\mathbb{R}^n)$

1.9.1 The convolution of two Schwartz functions

Let φ and ψ be two Schwartz functions, we recall the classical definition of the convolution

$$\begin{aligned} \varphi * \psi(x) &:= \int_{\mathbb{R}} \varphi(x-y) \, \psi(y) \, dy \\ &= \int_{\mathbb{R}^n} \psi(x-y) \, \varphi(y) l dy. \end{aligned}$$

We have the following proposition

Proposition 1.43. *Let φ and ψ be two Schwartz Functions, then for any $p \in \mathbb{N}$*

$$\mathcal{N}_p(\varphi * \psi) \leq C_{p,n} \, \mathcal{N}_p(\varphi) \, \mathcal{N}_{p+n+1}(\psi). \tag{1.28}$$

and then $\varphi * \psi$ is also a Schwartz function.

Proof of Proposition 1.43.We have

$$\begin{aligned} \mathcal{N}_p(\varphi * \psi) = & \sup_{\substack{|\alpha| \leq p \\ |\beta| \leq p}} \left\| x^\beta \int_{\mathbb{R}^n} \varphi(x-y) \, \partial^\alpha \psi(y) \, dy \right\|_{L^\infty(\mathbb{R}^n)} \\ = & \sup_{\substack{|\alpha| \leq p \\ |\beta| \leq p}} \left\| \int_{\mathbb{R}^n} (x-y+y)^\beta \varphi(x-y) \, \partial^\alpha \psi(y) \, dy \right\|_{L^\infty(\mathbb{R}^n)}. \end{aligned}$$

Using the binomial formula, we obtain

$$\mathcal{N}_p(\varphi * \psi) \le \sup_{\substack{|\alpha| \le p \\ |\beta| \le p}} \sum_{\gamma \le \beta} C_{\beta,\gamma} \left\| \int_{\mathbb{R}^n} |x-y|^{\beta-\gamma} \, |\varphi(x-y)| \, |y|^\gamma \, |\partial^\alpha \psi|(y) \, dy \right\|_{L^\infty(\mathbb{R}^n)}$$

$$\le C_p \, \mathcal{N}_p(\varphi) \int \sum_{\substack{|\alpha| \le p \\ |\beta| \le p}} |y|^\gamma \, |\partial^\alpha \psi|(y) \, dy$$

$$\le C_p \, \mathcal{N}_p(\varphi) \, \mathcal{N}_{p+n+1}(\psi) \ .$$

This concludes the proof of proposition 1.43. □

1.9.2 Convolution of a tempered distribution with a Schwartz function

Definition-Proposition 1.44. *Let $T \in \mathcal{S}'(\mathbb{R}^n)$ and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$.*
For any $x \in \mathbb{R}^n$ we define

$$T \star \varphi(x) := \langle T(y), \varphi(x-y) \rangle_{\mathcal{S}'_y, \mathcal{S}_y}.$$

then

$$T \star \varphi \in C^\infty(\mathbb{R}^n), \tag{1.29}$$

and there holds

$$\forall \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \qquad \partial^\alpha (T \star \varphi) = T \star \partial^\alpha \varphi = \partial^\alpha T \star \varphi. \tag{1.30}$$

Moreover

$$\varphi \in \mathcal{S}(\mathbb{R}^n) \ \longrightarrow \ T \star \varphi \in C^\infty(\mathbb{R}^n) \ ,$$

is continuous between the Fréchet spaces $\mathcal{S}(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^n)$.

In case $T \in \mathcal{E}'(\mathbb{R}^n)$, and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then

$$T \star \varphi \in \mathcal{S}(\mathbb{R}^n) \ . \tag{1.31}$$

moreover

$$\varphi \in \mathcal{S}(\mathbb{R}^n) \longrightarrow T \star \varphi \in \mathcal{S}(\mathbb{R}^n)$$

is continuous map between Fréchet spaces.

In case $T \in \mathcal{E}'(\mathbb{R}^n)$, and $\varphi \in C^\infty_0(\mathbb{R}^n)$ then

$$T \star \varphi \in C^\infty_0(\mathbb{R}^n) \tag{1.32}$$

and

$$supp\, T \star \varphi \subset \ supp\, T + \ supp\, \varphi. \tag{1.33}$$

Proof of proposition 1.44. *Claim 1*: We claim that

$$x \in \mathbb{R}^n \mapsto \varphi(x-y) = \varphi_x(y) \in \mathcal{S}'(\mathbb{R}^n)$$

is C^∞ as a map from $\mathbb{R}^n$ into the normed spaces $(\mathcal{S}'(\mathbb{R}^n), \mathcal{N}_p)$ for any $p \in \mathbb{N}$. We fix $p \in \mathbb{N}$ arbitrary and we prove that the map is in $C^1(\mathbb{R}^n, (\mathcal{S}'(\mathbb{R}^n), \mathcal{N}_p))$. We have

$$\begin{aligned}
&\left\| \varphi_{x_0+h} - \varphi_{x_0} - \sum_{i=1}^n \partial_{x_i}\varphi_{x_0}\, h^i \right\|_{\mathcal{N}_p} \\
&= \sup_{\substack{|\alpha| \le p \\ |\beta| \le p}} \left\| y^\beta \left(\partial_y^\alpha \varphi(x_0+h-y) - \partial_y^\alpha \varphi(x_0-y) - \sum_{i=1}^n \partial_y^\alpha \partial_{x_i}\varphi(x_0-y)\, h^i \right) \right\|_{L^\infty_y(\mathbb{R}^n)}
\end{aligned}$$

Let $\varepsilon > 0$ and $R > 0$ that we shall fix depending on ε and φ later. We have the existence of $\xi_{x_0,h,\alpha,y}$ depending on x_0, h, α, y between y and $y-h$ such that

$$\begin{aligned}
&\sup_{\substack{|\alpha| \le p \\ |\beta| \le p}} \left\| y^\beta \left(\partial_y^\alpha \varphi(x_0+h-y) - \partial_y^\alpha \varphi(x_0-y) - \sum_{i=1}^n \partial_y^\alpha \partial_{x_i}\varphi(x_0-y)\, h^i \right) \right\|_{L^\infty_y(\mathbb{R}^n \setminus B_R(0))} \\
&= \sup_{\substack{|\alpha| \le p \\ |\beta| \le p}} \left\| y^\beta \left(-\sum_{i=1}^n \partial_{y_i}\partial_y^\alpha \varphi(x_0 - \xi_{x_0,h,\alpha,y})\, h^i - \sum_{i=1}^n \partial_y^\alpha \partial_{x_i}\varphi(x_0-y)\, h^i \right) \right\|_{L^\infty_y(\mathbb{R}^n \setminus B_R(0))} \\
&\le 2\,\mathcal{N}_{p+1}(\varphi)\, |h|\, R^{-1}
\end{aligned} \tag{1.34}$$

We now fix $R > 0$ large enough so that $2\,\mathcal{N}_{p+1}(\varphi)\, R^{-1} \le \varepsilon/2$. Now we bound

$$\begin{aligned}
&\sup_{\substack{|\alpha| \le p \\ |\beta| \le p}} \left\| y^\beta \left(\partial_y^\alpha \varphi(x_0+h-y) - \partial_y^\alpha \varphi(x_0-y) - \sum_{i=1}^n \partial_y^\alpha \partial_{x_i}\varphi(x_0-y)\, h^i \right) \right\|_{L^\infty_y(B_R(0))} \\
&\le C_p\, R^p \sum_{|\alpha| \le p} \left\| \partial_y^\alpha \varphi(x_0+h-y) - \partial_y^\alpha \varphi(x_0-y) - \sum_{i=1}^n \partial_y^\alpha \partial_{x_i}\varphi(x_0-y)\, h^i \right\|_{L^\infty_y(B_R(0))} \\
&\le C_p\, R^p \sum_{|\alpha| \le p} \left\| -\sum_{i=1}^n \partial_{y_i}\partial_y^\alpha \varphi(x_0 - \xi_{x_0,h,\alpha,y})\, h^i + \sum_{i=1}^n \partial_y^\alpha \partial_{y_i}\varphi(x_0-y)\, h^i \right\|_{L^\infty_y(B_R(0))} \\
&\le C_p\, R^p \sum_{|\alpha| \le p} \sum_{i=1}^n \left\| \partial_{y_i}\partial_y^\alpha \varphi(x_0 - \xi_{x_0,h,\alpha,y}) - \partial_y^\alpha \partial_{y_i}\varphi(x_0-y) \right\|_{L^\infty_y(B_R(0))}\ |h^i|
\end{aligned}$$

Since φ is C^∞ on $\mathbb{R}^n$, for any $|\alpha| \le p$ and any $i = 1 \cdots n$, $\partial_{y_i}\partial_y^\alpha \varphi$ is uniformly continuous on $B_{R+1}(x_0)$ and since $|\xi_{x_0,h,\alpha,y} - y| \le |h|$, for $|h| < \delta$ and δ small enough we deduce

$$\sup_{\substack{|\alpha| \le p \\ |\beta| \le p}} \left\| y^\beta \left(\partial_y^\alpha \varphi(x_0+h-y) - \partial_y^\alpha \varphi(x_0-y) - \sum_{i=1}^n \partial_y^\alpha \partial_{x_i}\varphi(x_0-y)\, h^i \right) \right\|_{L^\infty_y(B_R(0))} \le \frac{\varepsilon}{2}\, |h|$$

Combining the previous we obtain

$$\forall \varepsilon > 0 \quad , \quad \exists \delta > 0 \quad , \quad \text{s. t.} \quad \forall\ |h| < \delta$$

$$\left\| \varphi_{x_0+h} - \varphi_{x_0} - \sum_{i=1}^{n} \partial_{x_i}\varphi_{x_0}\, h^i \right\|_{\mathcal{N}_p} \le \varepsilon\ |h|$$

This implies that $x \in \mathbb{R}^n \mapsto \varphi(x-y) = \varphi_x(y)$ is differentiable everywhere as a map from $\mathbb{R}^n$ into $(\mathcal{S}'(\mathbb{R}^n), \mathcal{N}_p)$ moreover the differential is given by

$$x \in \mathbb{R}^n \mapsto \sum_{i=1}^{n} \partial_{x_i}\varphi(x-y)\, dx_i$$

which is,by iterating the argument above, continuous. Hence $x \in \mathbb{R}^n \mapsto \varphi(x-y) = \varphi_x(y)$ is in $C^1(\mathbb{R}^n, (\mathcal{S}'(\mathbb{R}^n), \mathcal{N}_p))$. By applying the argument above to each of the maps $x \in \mathbb{R}^n \mapsto \partial_{x_i}\varphi(x-y)$ we obtain that $x \in \mathbb{R}^n \mapsto \varphi(x-y) = \varphi_x(y)$ is in $C^2(\mathbb{R}^n, (\mathcal{S}'(\mathbb{R}^n), \mathcal{N}_p))$ and claim 1 follows by a straightforward induction.

Applying lemma 1.38 to $x \in \mathbb{R}^n \mapsto \varphi(x-y) = \varphi_x(y)$ and T we obtain (1.29) and (1.30).

In order to prove (1.31), because of (1.29) and (1.30) it suffices to prove that for any $\mathcal{T} \in \mathcal{E}'(\mathbb{R}^n)$, any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and any $\beta \in \mathbb{N}^n$ there holds

$$\left\| x^\beta \left\langle T(y), \varphi(x-y) \right\rangle \right\|_{L^\infty(\mathbb{R}^n)} < +\infty \ . \tag{1.35}$$

We write

$$x^\beta \left\langle T(y), \varphi(x-y) \right\rangle = \left\langle T(y), \varphi(x-y) \prod_{i=1}^{n} (x_i - y_i + y_i)^{\beta_i} \right\rangle \ .$$

By developing the expression $\prod_{i=1}^n (x_i - y_i + y_i)^{\beta_i}$ we obtain the existence of coefficients $c^\beta_{\alpha,\gamma} \in \mathbb{R}$ such that

$$\prod_{i=1}^{n} (x_i - y_i + y_i)^{\beta_i} = \sum_{|\alpha| \le |\beta|, |\gamma| \le |\beta|} c^\beta_{\alpha,\gamma} \prod_{i=1}^{n} (x_i - y_i)^{\alpha_i} \prod_{j=1}^{n} y_j^{\gamma_j} \ .$$

Observe that for any choice of α and γ in $\mathbb{N}^n$ we have that

$$\varphi_\alpha(y) := \varphi(y) \prod_{i=1}^{n} y_i^{\alpha_i} \in \mathcal{S}(\mathbb{R}^n) \quad \text{and} \quad \prod_{j=1}^{n} y_j^{\gamma_j}\, T \in \mathcal{E}'(\mathbb{R}^n) \ .$$

Hence we deduce (1.35) and (1.31) is proved. The fact that the operation $\varphi \in \mathcal{S}(\mathbb{R}^n) \longrightarrow T \star \varphi \in \mathcal{S}(\mathbb{R}^n)$ is continuous is left as an exercise.

Assuming now $T \in \mathcal{E}'(\mathbb{R}^n)$, and $\varphi \in C^\infty_0(\mathbb{R}^n)$ we prove (1.33). Let $x \in (\operatorname{supp} T + \operatorname{supp} \varphi)^c$. This implies that

$$\forall\, y \in \mathbb{R}^n \quad x - y \in \ \operatorname{supp}\varphi$$

$$\Longrightarrow \quad y\ \in -\ \operatorname{supp}\varphi + \ (\mathbb{R}^n \setminus (\operatorname{supp} T + \ \operatorname{supp}\varphi)) = \mathbb{R}^n \setminus \operatorname{supp} T$$

Hence for such an x one has $T \star \varphi(x) = 0$ which concludes the proof of proposition 1.44. $\square$

Proposition 1.45. *Let $T \in \mathcal{S}'(\mathbb{R}^n)$ and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then $T \star \varphi \in \mathcal{S}'(\mathbb{R}^n)$ and $\forall \psi \in \mathcal{S}(\mathbb{R}^n)$*

$$\langle T \star \varphi, \psi \rangle = \langle T,\, \check{\varphi} \star \psi \rangle,$$

where

$$\check{\varphi}(x) = \varphi(-x).$$

$\Box$

Proof of Proposition 1.45. Under the assumptions of the proposition we have that $T \star \varphi \in C^\infty(\mathbb{R}^n)$. Using the duality extension, and assuming first $\psi \in C_0^\infty(\mathbb{R}^n)$, we have

$$\langle T \star \varphi, \psi \rangle = \int_{x \in \mathbb{R}^n} T \star \varphi(x) \psi(x)\ dx^n = \int_{x \in \mathbb{R}^n} \langle T(y), \varphi(x-y)\psi(x) \rangle\ dx^n\ ,$$

Using the fact that $\psi \in C_0^\infty(\mathbb{R}^n)$, claim 1 in the proof of proposition 1.44 implies that for any $i \in \{1 \cdots n\}$

$$x \ \longrightarrow\ \int_{x_i}^{+\infty} \varphi(x-y)\ \psi(x)\ dx_i$$

is C^∞ from $\mathbb{R}^n$ into $(\mathcal{S}'(\mathbb{R}^n), \mathcal{N}_p)$ for any $p \in \mathbb{N}$. Applying lemma 1.38 we then deduce that

$$\frac{\partial}{\partial x_i} \left\langle T(y), \int_{x_i}^{+\infty} \varphi(x-y)\psi(x)\ dx_i \right\rangle = -\langle T(y), \varphi(x-y)\psi(x) \rangle$$
$$= \frac{\partial}{\partial x_i} \int_{x_i}^{+\infty} \langle T(y), \varphi(x-y)\psi(x)\ \rangle\ dx_i$$

Hence there exists $c_i \in \mathbb{R}$ such that

$$\left\langle T(y), \int_{x_i}^{+\infty} \varphi(x-y)\psi(x)\ dx_i \right\rangle = \int_{x_i}^{+\infty} \langle T(y), \varphi(x-y)\psi(x)\ \rangle\ dx_i + c_i\ .$$

Making x_i tend to $+\infty$ and using again the fact that $\psi \in C_0^\infty(\mathbb{R}^n)$ we obtain $c_i = 0$. Using one more time the fact that $\psi \in C_0^\infty(\mathbb{R}^n)$, we can make x_i converge to $-\infty$ to obtain

$$\left\langle T(y), \int_{-\infty}^{+\infty} \varphi(x-y)\psi(x)\ dx_i \right\rangle = \int_{-\infty}^{+\infty} \langle T(y), \varphi(x-y)\psi(x)\ \rangle\ dx_i$$

Integrating along the n directions and using proposition 1.43 we finally obtain

$$\langle T \star \varphi, \psi \rangle = \int_{x \in \mathbb{R}^n} \langle T(y), \varphi(x-y)\, \psi(x) \rangle\ dx^n = \left\langle T(y), \int_{x \in \mathbb{R}^n} \varphi(x-y)\, \psi(x) \right\rangle$$
$$= \left\langle T(y), \int_{x \in \mathbb{R}^n} \check{\varphi}(y-x)\, \psi(x) \right\rangle = \langle T(y), \check{\varphi} \star \psi(y) \rangle\ .$$

Let p be the order of T, combining the previous identity with (1.28) we obtain

$$\forall\ \psi \in C_0^\infty(\mathbb{R}^n) \quad |\langle T\star\varphi,\psi\rangle| \le C_T\,\mathcal{N}_p(\varphi)\,\mathcal{N}_{p+n+1}(\psi)\ .$$

Hence $T\star\varphi$ defines a finite order element in $\mathcal{D}'(\mathbb{R}^n)$ and thanks to the density property (1.16), it extends as an element in $\mathcal{S}'(\mathbb{R}^n)$ and proposition 1.45 is proved. □

Corollary 1.46. *Let $T \in \mathcal{S}'(\mathbb{R}^n)$. Then there exists a sequence $T_i \in C^\infty(\mathbb{R}^n)$, such that $T_i \to T$ in $\mathcal{S}'(\mathbb{R}^n)$.*

For proving the corollary we shall make use of the following lemma

Lemma 1.47. *Let $\chi \in C_0^\infty\big(B_1(0),\mathbb{R}_+\big)$, such that $\int_{\mathbb{R}^n}\chi(x)\ dx^n = 1$ and for any $\varepsilon > 0$ we denote*

$$\chi_\varepsilon := \frac{1}{\varepsilon^n}\,\chi\left(\frac{\cdot}{\varepsilon}\right)\ .$$

Then for any $\varphi\in\mathcal{S}(\mathbb{R}^n)$ there holds

$$\varphi\star\chi_\varepsilon\ \longrightarrow\varphi \qquad \text{in}\ \ \mathcal{S}(\mathbb{R}^n)\ ,$$

that is

$$\forall\, p\in\mathbb{N} \qquad \lim_{\varepsilon\to 0}\mathcal{N}_p(\varphi\star\chi_\varepsilon-\varphi)=0$$

Proof of Lemma 1.46. Let $p\in\mathbb{N}$, $\delta>0$ and $R>0$ to be fixed later. We bound

$$\sup_{\substack{|\alpha|\le p\\|\beta|\le p}}\ \left\|x^\beta\partial_x^\alpha\left(\varphi\star\chi_\varepsilon-\varphi\right)\right\|_{L^\infty(\mathbb{R}^n\setminus B_R(0))}\le R^{-1}\ \left(\mathcal{N}_{p+1}(\varphi\star\chi_\varepsilon)+\mathcal{N}_{p+1}(\varphi)\right)$$

Observe that

$$\left|\partial_x^\alpha(\varphi\star\chi_\varepsilon)\right|(x)=\left|\partial_x^\alpha\varphi\star\chi_\varepsilon\right|(x)\le\|\partial_x^\alpha\varphi\|_{L^\infty(B_\varepsilon(x))}\ \ \|\chi_\varepsilon\|_{L^1(\mathbb{R}^n)}\ .$$

For this reason there holds for ϵ small enough

$$\mathcal{N}_{p+1}(\varphi\star\chi_\varepsilon)\le\ 2\,\mathcal{N}_{p+1}(\varphi)\ .$$

Hence we have

$$\sup_{\substack{|\alpha|\le p\\|\beta|\le p}}\ \left\|x^\beta\partial_x^\alpha\left(\varphi\star\chi_\varepsilon-\varphi\right)\right\|_{L^\infty(\mathbb{R}^n\setminus B_R(0))}\le 3\,R^{-1}\ \mathcal{N}_{p+1}(\varphi)\ . \tag{1.36}$$

We choose $R>0$ such that $3\,R^{-1}\ \mathcal{N}_{p+1}(\varphi)\le\delta/2$. Since $\partial_x^\alpha\varphi$ is continuous on $\mathbb{R}^n$ it is uniformly continuous on $B_{R+\varepsilon}(0)$ and we deduce for any $\alpha\in\mathbb{N}^n$

$$\begin{aligned}&\lim_{\varepsilon\to0}\|\partial_x^\alpha\varphi\star\chi_\varepsilon-\partial_x^\alpha\varphi\|_{L^\infty(B_R(0))}\\&=\lim_{\varepsilon\to0}\left\|\int_{y\in\mathbb{R}^n}\left[\partial_x^\alpha\varphi(x-y)-\partial_x^\alpha\varphi(x)\right]\ \chi_\varepsilon(y)\ dy^n\right\|_{L^\infty(B_R(0))}\\&\le\lim_{\varepsilon\to0}\int_{y\in\mathbb{R}^n}\sup_{x\in B_R(0)}\|\partial_x^\alpha\varphi(x-y)-\partial_x^\alpha\varphi(x)\|_{L^\infty_y(B_\varepsilon(0))}\ \chi_\varepsilon(y)\ dy^n=0\ .\end{aligned}$$

Hence we can choose ε_0 such that

$$\text{(1.37)} \qquad \forall\ \varepsilon < \varepsilon_0 \qquad \sup_{\substack{|\alpha| \leq p \\ |\beta| \leq p}} \left\| x^\beta \partial_x^\alpha \left(\varphi \star \chi_\varepsilon - \varphi\right) \right\|_{L^\infty(B_R(0))} \leq \frac{\delta}{2} \ .$$

Combining (1.36) and (1.37) we obtain

$$\forall\ \varepsilon < \varepsilon_0 \qquad \mathcal{N}_p(\varphi \star \chi_\varepsilon - \varphi) \leq \delta \ .$$

This concludes the proof of Lemma 1.47 □

Proof of Corollary 1.46. Let $\chi \in C_0^\infty\big(B_1(0), \mathbb{R}_+\big)$, such that $\int_{\mathbb{R}^n} \chi = 1$. Let moreover $\varepsilon_i := i^{-1}$. We introduce

$$\chi_i(z) = \frac{1}{\varepsilon_i^n} \chi\left(\frac{z}{\varepsilon_i}\right) .$$

From proposition 1.44 and proposition 1.45 we have respectively

$$T \star \chi_i \in C^\infty.$$

and for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\langle T \star \chi_i, \varphi\rangle = \langle T, \check{\chi}_i \star \varphi\rangle \ \longrightarrow \langle T, \varphi\rangle$$

This concludes the proof of Corollary 1.46. □

This last proposition therefore shows that the convolution of a distribution with a Schwartz function is a "natural" operation in the following sense: We can prove properties of distributions by starting from smooth functions and then moving to the limit. Furthermore, one can see that with the distributions one has not defined a much too large object of generalized functions.

Next we consider translations. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ and let $a \in \mathbb{R}^n$. Then the translation τ_a is defined as follows $\tau_a\varphi(x) := \varphi(x - a)$. The same procedure is followed for tempered distributions.

Notation 1.48.

$$\forall\, T \in \mathcal{S}'(\mathbb{R}^n) : \langle \tau_a T, \varphi\rangle := \langle T, \tau_{-a}\varphi\rangle.$$

□

Proposition 1.49. $\forall\ T \in \mathcal{S}'(\mathbb{R}^n) \quad \forall\, \varphi \in \mathcal{S}(\mathbb{R}^n) \quad \forall a \in \mathbb{R}^n$ *there holds*

$$\tau_a(T \star \varphi) = (\tau_a T) \star \varphi = T \star \tau_a\varphi.$$

□

Proof of Proposition 1.49:

$$
\begin{aligned}
\tau_a(T \star \varphi(x)) = u \star \varphi(x-a) \quad &= \langle T(y),\ \varphi(x-a-y)\rangle \\
&= \langle T(y),\ \varphi(x-(y+a))\rangle \\
&= \langle T(y),\ \tau_{-a}\varphi(x-y)\rangle \\
&= \langle \tau_a T(y),\ \varphi(x-y)\rangle
\end{aligned}
$$

$\square$

Exercise Let $T \in \mathcal{E}'(\mathbb{R}^n)$ and $U : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$ be the following map: $U : \varphi \mapsto u \star \varphi$. Prove that $U\big|_{C_0^\infty}$ is continuous where $C_0^\infty(\mathbb{R}^n)$ is viewed as a sub-vector space of $C^\infty(\mathbb{R}^n)$ viewed as a Fréchet space.

Exercise Let $U \in C^0(C_0^\infty(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n))$ be linear and commuting with translations, that is for any $a \in \mathbb{R}^n$

$$U\tau_a\varphi = \tau_a U\varphi.$$

Then there is a $T \in \mathcal{E}'(\mathbb{R}^n)$ such that

$$U\varphi = T \star \varphi$$

1.9.3 Convolution of two distributions

Definition-Proposition 1.50. *Let $T \in \mathcal{E}'(\mathbb{R}^n)$ and let $\varphi \in C^\infty(\mathbb{R}^n)$. Then we define*

$$T \star \varphi(x) := \langle T(y),\ \varphi(x-y)\rangle_{\mathcal{E}', C^\infty}\ .$$

There holds

i) $T \star \varphi \in C^\infty$.

ii) $\partial^\alpha(u \star \varphi) = u \star \partial^\alpha\varphi = (\partial^\alpha u) \star \varphi.$ $\square$

Moreover the map which to $\varphi \in C^\infty(\mathbb{R}^n)$ assigns $T \star \varphi \in C^\infty(\mathbb{R}^n)$ is continuous as a map between Fréchet spaces.

Proof of proposition 1.50: .The proof is identical to the proof of proposition 1.44 after having inserted a cut-off function θ as in proposition 1.32 to extend the duality from $\mathcal{E}' \leftrightarrow \mathcal{S}$ to $\mathcal{E}' \leftrightarrow C^\infty$. The fact that the map

$$\varphi \in C^\infty(\mathbb{R}^n) \ \longrightarrow\ T \star \varphi \in C^\infty(\mathbb{R}^n)$$

is continuous as a map between Fréchet spaces is left as an exercise. $\square$

Definition-Proposition 1.51. (Convolution between $\mathcal{S}'$ and $\mathcal{E}'$)
Let $T \in \mathcal{S}'(\mathbb{R}^n)$ and let $S \in \mathcal{E}'(\mathbb{R}^n)$.

Then there exists $R \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\forall\, \varphi \in \mathcal{S}(\mathbb{R}^n) \qquad \langle R,\ \varphi\rangle = \langle T,\ \check{S} \star \varphi\rangle\ ,$$

where

$$\langle \check{S},\ \varphi\rangle := \langle S,\ \check{\varphi}\rangle\ .$$

We denote

$$R = T \star S\ .$$

We now define

$$\langle S \star T,\ \varphi\rangle_{\mathcal{S}',\mathcal{S}} := \langle S,\ \check{T} \star \varphi\rangle_{\mathcal{E}',C^\infty}\ . \tag{1.38}$$

With these notations there holds

$$T \star S = S \star T\ . \tag{1.39}$$

We have moreover

$$\frac{\partial}{\partial x_i}\,(T \star S) = \frac{\partial T}{\partial x_i} \star S = T \star \frac{\partial S}{\partial x_i}\ .$$

If both T and S in $\mathcal{E}'(\mathbb{R}^n)$ holds

$$supp\,(T \star S)\ \subset\ supp\,T + supp\,S\ ,$$

and

$$T \star S \in \mathcal{E}'(\mathbb{R}^n)\ .$$

□

Proof of Proposition-Definition 1.51. The fact that $R \in \mathcal{S}'(\mathbb{R}^n)$ follows from the fact that for a compactly supported distribution $\check{S}$ the map $\varphi \longrightarrow \check{S} \star \varphi$ is a continuous map from $\mathcal{S}(\mathbb{R}^n)$ into itself see (1.31).

The fact that (1.38) makes sense comes from (1.29) and the continuity of the map

$$\varphi \in \mathcal{S}(\mathbb{R}^n)\ \longrightarrow\ T \star \varphi \in C^\infty(\mathbb{R}^n)\ ,$$

as a map between Fréchet spaces .

We now prove (1.39). Introduce $\chi \in C^\infty_0(\mathbb{R}^n)$, $supp\,\chi \subset B_1(0)$ and $\int_{\mathbb{R}^n} \chi = 1$ as well as $\varepsilon_i \to 0$. Denote $\chi_i(z) = \frac{1}{\varepsilon_i^n}\chi(\frac{z}{\varepsilon_i})$. From the proof of corollary 1.46 we have

$$T_i := \chi_i \star T\ \longrightarrow\ T \quad \text{ in } \mathcal{S}'(\mathbb{R}^n) \qquad \text{and} \qquad S_i := \chi_i \star S\ \longrightarrow\ S \quad \text{ in } \mathcal{S}'(\mathbb{R}^n)$$

moreover $T_i \in C^\infty(\mathbb{R}^n)$ and $S_i \in C^\infty_0(\mathbb{R}^n)$ with

$$\text{supp}(S_i) \subset \text{supp}(S) + \text{supp}(\chi_i) \subset \text{supp}(S) + B_{\varepsilon_i}(0)\ .$$

There holds first

$$\begin{aligned}\langle T_i \star S_i, \varphi\rangle &= \int_{\mathbb{R}^n} T_i \star S_i(z)\varphi(z)dz^n \\ &= \int_{\mathbb{R}^n} dz^n \int_{\mathbb{R}^n} T_i(z-y)\, S_i(y)\varphi(z)\ dy^n \\ &= \int_{\mathbb{R}^n} dz^n \left[\int_{\mathbb{R}^n} T_i(y)S_i(z-y)dy^n\right]\ \varphi(z) = \langle S_i \star T_i,\, \varphi\rangle .\end{aligned}$$

We claim the following.
Claim 1):

$$\langle T_i \star S_i,\, \varphi\rangle = \langle T_i, \check{S}_i \star \varphi\rangle \longrightarrow \langle T, \check{S} \star \varphi\rangle = \langle T \star S,\, \varphi\rangle\ . \tag{1.40}$$

In order to establish the claim 1 we first prove
Claim 0):

$$\check{S}_i \star \varphi \longrightarrow \check{S} \star \varphi \in \mathcal{S}(\mathbb{R})\ .$$

We have

$$\begin{aligned}\check{S}_i \star \varphi(x) &= \langle \check{S}_i(y),\ \varphi(x-y)\rangle = \langle S_i(y),\ \check{\varphi}(x-y)\rangle \\ &= \langle S \star \chi_i\,(y),\ \varphi(x+y)\rangle \\ &= \langle S(y), \check{\chi}_i \star \varphi(x+y)\rangle\end{aligned}$$

Since $\partial^\alpha(\check{S}_i \star \varphi) = \check{S}_i \star \partial^\alpha\varphi$, it suffices to prove that $\forall\varphi \in \mathcal{S}(\mathbb{R}^n)$ and any $\beta \in \mathbb{N}^n$

$$\|x^\beta\ (\langle S(y), \check{\chi}_i \star \varphi(x+y)\rangle - \langle S(y), \varphi(x+y)\rangle)\,\|_{L^\infty(\mathbb{R}^n)} \to 0,$$

Let p be the order of S. Since $K := supp\, \check{v} \subset B_\rho(0)$ is compact, there exists a constant $C_S > 0$,such that

$$|\langle \check{S}, \psi\rangle|\ \le C_S \sum_{|\alpha|\le p} \|\partial^\alpha\psi\|_{L^\infty(K)}, \quad \forall\psi \in \mathcal{S}(\mathbb{R}^n).$$

Hence we have for any $x \in \mathbb{R}^n$

$$\begin{aligned}&|x^\beta\ (\langle S(y), \check{\chi}_i \star \varphi(x+y)\rangle - \langle S(y), \varphi(x+y)\rangle)\,| \\ &\le C_S \sum_{|\alpha|\le p} \|x^\beta \left(\check{\chi}_i \star \partial^\alpha_y\varphi(x+y) - \partial^\alpha_y\varphi(x+y)\right)\|_{L^\infty_y(K)},\end{aligned}$$

Let $\delta > 0$ and $R > 2\,\rho > 0$ to be fixed later on. For $x \in \mathbb{R}^n \setminus B_R(0)$ we bound

$$\begin{aligned}&\left\||x^\beta\ (\langle S(y), \check{\chi}_i \star \varphi(x+y)\rangle - \langle S(y), \varphi(x+y)\rangle)\,|\right\|_{L^\infty(\mathbb{R}^n\setminus B_R(0))} \\ &\le C_S \sum_{|\alpha|\le p} \left\|\|x^\beta \left(\check{\chi}_i \star \partial^\alpha_y\varphi(x+y) - \partial^\alpha_y\varphi(x+y)\right)\|_{L^\infty_y(B_\rho(0))}\right\|_{L^\infty_x(\mathbb{R}^n\setminus B_R(0))}\end{aligned}$$

Observe that for $|x| > R > 2\rho$ one has $2|x| > |x+y| > |x|/2$. One has also

$$|\check{\chi}_i \star \partial_y^\alpha \varphi(x+y)| \leq \|\check{\chi}_i\|_{L^1(\mathbb{R}^n)} \, \|\partial^\alpha \varphi\|_{L^\infty(B_{\rho+\varepsilon_i}(x))} \, .$$

Hence we have for any $|\alpha| \leq p$

$$\begin{aligned} &C_S \sum_{|\alpha|\leq p} \left\| \|x^\beta \left(\check{\chi}_i \star \partial_y^\alpha \varphi(x+y) - \partial_y^\alpha \varphi(x+y)\right) \|_{L_y^\infty(B_\rho(0))} \right\|_{L_x^\infty(\mathbb{R}^n \setminus B_R(0))} \\ &\leq C_p \, R^{-1} \, \mathcal{N}_{p+1}(\varphi) \, . \end{aligned}$$

We choose R such that $C_p \, R^{-1} \, \mathcal{N}_{p+1}(\varphi) < \delta/2$. R being now fixed, on $B_{R+\rho}(0)$ the convergence of $\check{\chi}_i \star \partial_y^\alpha \varphi$ towards $\partial_y^\alpha \varphi$ is uniform. Hence, for i large enough

$$C_S \sum_{|\alpha|\leq p} \left\| \|x^\beta \left(\check{\chi}_i \star \partial_y^\alpha \varphi(x+y) - \partial_y^\alpha \varphi(x+y)\right) \|_{L_y^\infty(B_\rho(0))} \right\|_{L_x^\infty(B_R(0))} \leq \delta/2$$

Combining the above, claim 0) is proved.

In order to prove the claim 1) we write

$$\langle T_i, \check{S}_i \star \varphi \rangle - \langle T, \check{S} \star \varphi \rangle = \langle T_i - T, \check{S} \star \varphi \rangle - \langle T_i, \check{S} \star \varphi - \check{S}_i \star \varphi \rangle$$

$$= \langle T_i - T, \check{S} \star \varphi \rangle - \langle T, \check{\chi}_i \star (\check{S} \star \varphi - \check{S}_i \star \varphi) \rangle$$

Since $T_i \to T$ in $\mathcal{S}'(\mathbb{R}^n)$ we have that $\langle T_i - T, \check{S} \star \varphi \to 0$. Let p be the order of T. We have

$$\left| \langle T, \check{\chi}_i \star (\check{S} \star \varphi - \check{S}_i \star \varphi) \rangle \right| \leq C \sum_{\substack{|\alpha| \leq p \\ |\beta| \leq p}} \left\| x^\beta \check{\chi}_i \star \partial^\alpha (\check{S} \star \varphi - \check{S}_i \star \varphi)(x) \right\|_\infty$$

Observe one more time that

$$\left| \check{\chi}_i \star \partial^\alpha (\check{S} \star \varphi - \check{S}_i \star \varphi)(x) \right| \leq \|\check{\chi}_i\|_{L^1(\mathbb{R}^n)} \, \left\| \partial^\alpha (\check{S} \star \varphi - \check{S}_i \star \varphi) \right\|_{L^\infty(B_{\varepsilon_i}(x))}$$

Hence

$$\sum_{\substack{|\alpha| \leq p \\ |\beta| \leq p}} \left\| x^\beta \check{\chi}_i \star \partial^\alpha (\check{S} \star \varphi - \check{S}_i \star \varphi)(x) \right\|_\infty \leq \; C \, \mathcal{N}_p(\check{S} \star \varphi - \check{S}_i \star \varphi)$$

Using claim 0) we obtain that $\mathcal{N}_p(\check{S} \star \varphi - \check{S}_i \star \varphi) \to 0$ and the above we deduce claim 1). Hence we have proved (1.39).

The last assertions of the proposition follow from (1.38), (1.30) and (1.33) and the details are left as an exercise.

□

Remark 1.52. *Attention! associativity does not hold in general.*

$\square$

Example: Consider in $\mathcal{S}(\mathbb{R})$. There holds

$$(1 \star \delta_0') \star H_{1,0} = 0 \ ,$$

indeed

$$\langle 1 \star \delta_0', \varphi \rangle = \langle \delta_0', \check{1} \star \varphi \rangle = -\langle \delta_0, (\check{1} \star \varphi)' \rangle = -(\check{1} \star \varphi)'(0) = 0.$$

On the other hand, there holds

$$\delta_0' \star H_{1,0} = \delta_0 \star \delta_0 = \delta_0 \in \mathcal{E}'(\mathbb{R})$$

and thus

$$1 \star (\delta_0' \star H_{1,0}) = 1 \star \delta_0 = 1.$$

(δ_0 is the neutral element of the convolution). We have shown

$$(1 \star \delta_0') \star H_{1,0} \neq 1 \star (\delta_0' \star H_{1,0}).$$

Theorem 1.53. *Assume $T, U, V \in \mathcal{S}'(\mathbb{R}^n)$ and that two of the 3 have compact support, then there holds*

$$T \star (U \star V) = (T \star U) \star V \ .$$

Exercise. Prove theorem 1.53.

Remark 1.54. *So far we have seen the following cases in which the convolution is defined between a distribution and a function or another distribution:*

- $T \in \mathcal{S}'$, $\varphi \in \mathcal{S}$,
- $T \in \mathcal{E}'$, $\varphi \in C^\infty$,
- $T \in \mathcal{S}'$, $S \in \mathcal{E}'$.

The question now arises as to whether there are other cases in which a convoolution is defined between two distributions.

In fact, one can define the convolution between $T \in \mathcal{S}'$ and $S \in \mathcal{S}'$, provided

$$\forall R > 0 \quad \exists\ \delta(R) > 0,$$

so that

$$(x \in supp\, T,\ y \in supp\, S,\ |x + y| \leq R) \Rightarrow (|x| \leq \delta(R),\ |y| \leq \delta(R))$$

One says that T and S have convolutive supports.

1.10 The use of convolutions to solve linear partial differential equations with constant coefficients

1.10.1 General Principles

Definition 1.55. *A convolution equation is an equation of the form $A \star u = f$, where $A \in \mathcal{E}'(\Omega)$ and $f \in \mathcal{E}'(\Omega)$ are given and $u \in \mathcal{D}'(\Omega)$ is unknown.* □

Example 1: (Partial Differential Equations) Let $A = \sum_{|\alpha| \le p} C_\alpha \partial^\alpha \delta_0, \ C_\alpha \in \mathbb{R}$ or $\mathbb{C}$

$$\begin{aligned} A \star u &= \sum_{|\alpha| \le p} C_\alpha \partial^\alpha \delta_0 \star u, \text{ where } \partial^\alpha \delta_0 \in \mathcal{E}' \text{ and } u \in \mathcal{D}' \\ &= \sum_{|\alpha| \le p} C_\alpha \delta_0 \star \partial^\alpha u = \sum_{|\alpha| \le p} C_\alpha \partial^\alpha u. \end{aligned}$$

Hence

$$A \star u = f \qquad \Longleftrightarrow \qquad \sum_{|\alpha| \le p} C_\alpha \partial^\alpha u = f \ .$$

Example 2: A discrete differential equations of the form $u(x+h) + u(x-h) - 2u(x) = f$ can be rewritten as follows:

$$(\delta_h + \delta_{-h} - 2\delta_0) \star u = f \ .$$

Definition 1.56. *Let $A \in \mathcal{E}'(\mathbb{R}^n)$. A solution $G \in \mathcal{S}'(\mathbb{R}^n)$ of the equation $A \star G = \delta_0$ is called the fundamental solution / Green's function / kernel of the convolution equation.* □

Theorem 1.57. *Let $A \in \mathcal{E}'(\mathbb{R}^n)$ and let $f \in \mathcal{E}'(\mathbb{R}^n)$. In addition, let G be a fundamental solution of the equation associated to A, i.e. $A \star G = \delta_0$.*

a) *Then $u := G \star f$ is a solution to the equation $A \star u = f$.*

b) *If u is a solution of $A \star u = f$ and $u \in \mathcal{E}'(\mathbb{R}^n)$, then $u = G \star f$ and this is the only solution, if there is one.*

Proof of the theorem 1.57:

Proof of a): Let $u = G \star f$. Then $A \star u = A \star (G \star f)$. Since $A, f \in \mathcal{E}'(\mathbb{R}^n)$, the associativity holds thanks to theorem 1.53, so

$$A \star u = (A \star G) \star f = \delta_0 \star f = f \ .$$

proof of b): Now let $u \in \mathcal{E}'(\mathbb{R}^n)$ be a solution of $A \star u = f$. We have $u = \delta_0 \star u = (A \star G) \star u$, and because of the associativity, which holds because of $A, u \in \mathcal{E}'(\mathbb{R}^n)$, one obtains

$$u = (G \star A) \star u = G \star (A \star u) = G \star f.$$

□

1.10.2 Solving $\Delta u = f$ for $f \in \mathcal{E}'(\mathbb{R}^n)$

(Model example for elliptic equations)

We introduce the following function

$$(1.41) \qquad G(x) = \begin{cases} \dfrac{1}{2\pi} \log|x| & n = 2, \\ -\dfrac{1}{|\partial B_1^n(0)|\,|x|^{n-2}(n-2)}, n > 2, & x \in \mathbb{R}^n. \end{cases}$$

Introduce the characteristic functions f the unit ball $\mathbf{1}_{B_1(0)}$ and the complement of the unit ball $\mathbf{1}_{\mathbb{R}\setminus B_1(0)}$

$$G = \mathbf{1}_{B_1(0)}\,G + \mathbf{1}_{\mathbb{R}\setminus B_1(0)}\,G \in L^1 + L^\infty(\mathbb{R}^n) \Longrightarrow G \in \mathcal{S}'(\mathbb{R}^n) \ ,$$

and

$$G \in C^\infty(\mathbb{R}^n \setminus \{0\}) \ .$$

We have the following lemma.

Lemma 1.58. *The tempered distribution defined by (1.41) satisfies*

$$\Delta G = \sum_{i=1}^n \frac{\partial^2 G}{\partial x_i^2} = \delta_0$$

Proof of lemma 1.58 A direct calculation of the derivatives gives

$$\Delta G = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n \setminus \{0\}).$$

That means that

$$supp\, \Delta G \subset \{0\}$$

Proposition 1.39 then yields the existence of $q \in \mathbb{N}$, so that holds

$$\Delta\, G = \sum_{|\alpha| \le q} C_\alpha \partial^\alpha \delta_0 \ ,$$

and

$$C_\alpha \in \mathbb{R} \quad \text{for all} \quad |\alpha| \le q \ .$$

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$, $supp\,\varphi \subset B_R(0)$. Observe that $E \in L^1(B_R(0))$. Thus we can write

$$\langle \Delta\, G,\, \varphi \rangle = \langle \Delta\, G,\, \varphi \rangle = \int_{\mathbb{R}^n} G\, \Delta\varphi = \int_{B_R(0)} G\, \Delta\varphi\; dx^n = \lim_{\varepsilon \to 0} \int_{B_R(0)\setminus B_\varepsilon(0)} G\, \Delta\varphi\; dx^n$$

Observe that

$$\int_{B_R(0)\setminus B_\varepsilon(0)} G\, \Delta\varphi\; dx^n = -\int_{\partial\, B_\varepsilon(0)} G\, \frac{\partial\varphi}{\partial r}\; dl_{\partial B_\varepsilon(0)} - \int_{B_R(0)\setminus B_\varepsilon(0)} \nabla\, G\, \nabla\varphi\; dx^n \ .$$

Since

$$\|G\|_{L^\infty(\partial B_\varepsilon(0))} \le \frac{C}{\varepsilon^{n-2}} \ \text{ for } n>2 \qquad \text{and} \qquad \|G\|_{L^\infty(\partial B_\varepsilon(0))} \le C\, \log \varepsilon^{-1} \ \text{ for } n=2\,.$$

This implies

$$\int_{\partial B_\varepsilon(0)} G\, \frac{\partial \varphi}{\partial r}\, dl_{\partial B_\varepsilon(0)} \le C_\varphi \|G\|_{L^\infty(\partial B_\varepsilon(0))} |\partial B_\varepsilon(0)| = o_\varepsilon(1) \to 0\ .$$

Moreover we have (for $n>2$), since $\Delta G = 0$ away from 0

$$\begin{aligned}
-\int_{B_R(0)\setminus B_\varepsilon(0)} \nabla G\, \nabla\varphi\, dx^n &= -\int_{\partial B_\varepsilon(0)} \frac{\partial G}{\partial r}\, \varphi\, dl_{\partial B_\varepsilon(0)} + \int_{B_R(0)\setminus B_\varepsilon(0)} \varphi \Delta\, G\, dx^n \\
&= +\int_{\partial B_\varepsilon(0)} \frac{n-2}{|\partial B_1^n|(n-2)|x|^{n-1}}\varphi,
\end{aligned}$$

A similar computation holds for $n=2$.

Finally we have obtained the following result

$$\langle \Delta G, \varphi\rangle = \lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(0)} \frac{1}{|\partial B_1^n|\ |x|^{n-1}}\varphi\ dl_{\partial B_\varepsilon(0)} = \varphi(0) = \langle \delta_0, \varphi\rangle\ .$$

This concludes the proof of Lemma 1.58. □

Combining Lemma 1.58 with the previous subsection we shall derive the following result.

Theorem 1.59. *Let*

$$G(x) = \begin{cases} \dfrac{1}{2\pi}\ \log|x| & n=2, \\[2ex] -\dfrac{1}{|\partial B_1^n(0)|\ |x|^{n-2}(n-2)} & n>2, \end{cases}$$

and let $f \in \mathcal{E}'(\mathbb{R}^n)$. Then $u = G\star f$ is a solution to $\Delta u = f$, $u \in \mathcal{S}'(\mathbb{R}^n)$, $u \in C^\infty(\mathbb{R}^n\setminus supp\, f)$ and u converges uniformly toward 0 at infinity. □

Proof of Theorem 1.59. Because of the previous subsection, the equation $\Delta u = f$ in $\mathcal{S}'(\mathbb{R}^n)$ can be rewritten as $A\star u = f$, where

$$A = \sum_{i=1}^n \partial^2_{x_i}\, \delta_0\ .$$

From theorem 1.57 $u := G\star f$ is a solution to this equation and $u \in \mathcal{S}'(\mathbb{R}^n)$.

We prove now the last part of the theorem, that is first $u \in C^\infty(\mathbb{R}^n\setminus \operatorname{supp} f)$ and then the uniform convergence of u towards 0 at infinity.

Let $\delta>0$ and let $\theta \in C_0^\infty(\mathbb{R}^n)$ with $\theta\equiv 1$ on $B_1^n(0)$ and $\theta \equiv 0$ on $\mathbb{R}^n\setminus B_2^n(0)$. Denote

$$\theta_\delta(x) = \theta\left(\frac{x}{\delta}\right)\ .$$

Now we decompose G as follows:

$$G = G_1^\delta + G_2^\delta \ ,$$

where $G_1^\delta = \theta_\delta G$ and $G_2^\delta = (1-\theta_\delta)G$. Observe that $G_1^\delta \in L^1(\mathbb{R}^n)$ and $G_2^\delta \in L^\infty(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$. By linearity of the convolution u can be written as follows

$$u = G \star f = G_1^\delta \star f + G_2^\delta \star f \ .$$

Further we have

$$supp\, G_1^\delta \star f \subset supp\, f + B_{2\delta}^n(0) \subset (\text{supp}\, f)_{2\delta} := \{x \in \mathbb{R}^n; \text{dist}(x, supp\, f) \leq 2\delta\} \ .$$

Now let $\varphi \in C_0^\infty(\mathbb{R}^n \backslash (\text{supp}\, f)_{2\delta})$, then the following holds

$$\langle u,\, \varphi\rangle = \langle G_1^\delta \star f\rangle + \langle G_2^\delta \star f,\, \varphi\rangle = \langle G_2^\delta \star f,\, \varphi\rangle \ .$$

From proposition 1.50 we have $G_2^\delta \star f \in C^\infty$ $\forall \delta > 0$. We deduce $u \in C^\infty(\mathbb{R}^n \backslash supp\, f)$.

Now we write

$$G_2^\delta \star f(x) = \langle f(y), G_2^\delta(x-y)\rangle_{\mathcal{E}', C^\infty}$$

and

$$|\langle f(y), G_2^\delta(x-y)\rangle| \leq C \sum_{|\alpha| \leq p} \|\partial_y^\alpha G_2^\delta(x-\cdot)\|_{L^\infty(supp\, f)}, \tag{1.42}$$

where $p = ord(f)$. Let $2 < R < \infty$, such that $\text{supp}\, f \subset B_R^n(0)$. There holds

$$\forall y \in B_R(0) \quad , \quad |x| > 2R$$

$$|\partial_y^\alpha G_2^\delta(x-y)| = C \left|\partial_y^\alpha \frac{1}{|x-y|^{n-2}}\right| \leq \frac{C_1}{(|x|-R)^{n-2+|\alpha|}} \longrightarrow 0$$

uniformly when $|x| \to +\infty$. Combining this fact with the bound (1.42) we obtain the uniform convergence of u toward 0 at infinity. This concludes the proof of theorem 1.59. $\square$ $\square$

1.10.3 The resolution of $\Box u = f$ in $\mathcal{E}'(\mathbb{R}^4)$

(Model for hyperbolic equations).

In $\mathbb{R}^4$, the differential operator

$$\Box u = \frac{\partial^2}{\partial t^2} u - \Delta u = \frac{\partial^2}{\partial t^2} u - \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2} = f,\ (x,t) = (x_1, x_2, x_3, t) \in \mathbb{R}^4$$

is called "wave operator". We introduce the *light cone*

$$t = \sqrt{x_1^2 + x_2^2 + x_3^2} = |x| \ .$$

We denote by T the integration along the light cone with respect to the volume form of the induced euclidian metric from $\mathbb{R}^4$, that is

$$\langle T,\, \varphi\rangle := \int_{lightcone} \varphi\, dvol_{lightcone} = \sqrt{2}\int_{\mathbb{R}^3} \varphi(x_1,x_2,x_3,|x|) dx_1 dx_2 dx_3\ .$$

Denote $\rho = \sqrt{x_1^2+x_2^2+x_3^2+t^2}$ and observe that $\rho = \sqrt{2}|x|$ along the light cone.

Let $\varphi \in \mathcal{S}'(\mathbb{R}^4)$. We compute

$$\begin{aligned}\left\langle \frac{T}{\rho},\, \varphi\right\rangle &:= \int_{lightcone} \frac{\varphi}{\rho} d\, vol_{lightcone}\\ &= \sqrt{2}\int_{\mathbb{R}^3} \frac{1}{\rho}\varphi(x_1,x_2,x_3,|x|)\, dx_1 dx_2 dx_3\\ &= \int_{\mathbb{R}^3} \frac{1}{|x|}\varphi(x_1,x_2,x_3,|x|)\, dx_1 dx_2 dx_3\ .\end{aligned}$$

We

$$\begin{aligned}\left|\left\langle \frac{T}{\rho},\, \varphi\right\rangle\right| &\le \|\varphi\|_\infty \int_{B_1^3(0)} \frac{1}{|x|} dx_1\, dx_2\, dx_3 + \sum_{|\beta|\le 3} \|x^\beta \varphi\|_\infty \int_{B_1^3(0)} \frac{1}{|x|^4} dx_1\, dx_2\, dx_3\\ &= 4\,\pi\, \|\varphi\|_\infty \int_0^1 \frac{r^2 dr}{r} + 4\,\pi \sum_{|\beta|\le 3} \|x^\beta\varphi\|_\infty \int_1^{+\infty} \frac{r^2 dr^4}{r} \le C\, \mathcal{N}_3(\varphi)\end{aligned}$$

This implies that T/ρ defines an element of $\mathcal{S}'(\mathbb{R}^4)$.

Proposition 1.60. *$S := T/\rho$ is a solution in $\mathcal{S}'(\mathbb{R}^4)$ of $\Box S = 4\pi\,\delta_0$.*

Proof of Proposition 1.60: Let $\varphi \in \mathcal{S}(\mathbb{R}^4)$, and for $r > 0$ and $t \in \mathbb{R}$ write

$$\overline{\varphi}(r,t) := \frac{1}{4\pi r^2}\int_{\partial B_r(0)} \varphi(y,t)\, \mathrm{d}\sigma(y) = \frac{1}{4\pi}\int_{\mathbb{S}^2} \varphi(ry,t)\, \mathrm{d}\sigma(y).$$

In polar coordinates we have that

$$\Box\varphi = \partial_t^2\varphi - \partial_r^2\varphi - \frac{2}{r}\partial_r\varphi - \frac{1}{r^2}\Delta_{\mathbb{S}^2}\varphi.$$

Thus we can compute, writing the integral in polar coordinates,

$$\langle S, \Box\varphi\rangle = \int_0^\infty \int_{\mathbb{S}^2} \frac{\partial_t^2\varphi(ry,r) - \partial_r^2\varphi(ry,r) - \frac{2}{r}\partial_r\varphi(ry,r) - \frac{1}{r^2}\Delta_{\mathbb{S}^2}\varphi(ry,r)}{r}\, \mathrm{d}\sigma(y)\, r^2 \mathrm{d}r.$$

Notice that, by the divergence theorem and the fact that $\mathbb{S}^2$ is closed,

$$\int_{\mathbb{S}^2} \Delta_{\mathbb{S}^2}\varphi\, \mathrm{d}\sigma(y) = 0.$$

Hence we get

$$\begin{aligned}\langle S, \Box\varphi\rangle &= \int_0^\infty \int_{\mathbb{S}^2} \frac{\partial_t^2\varphi(ry,r) - \partial_r^2\varphi(ry,r) - \frac{2}{r}\partial_r\varphi(ry,r)}{r}\,\mathrm{d}\sigma(y)\, r^2\mathrm{d}r \\ &= \int_0^\infty \int_{\mathbb{S}^2} r\partial_t^2\varphi(ry,r) - r\partial_r^2\varphi(ry,r) - 2\partial_r\varphi(ry,r)\,\mathrm{d}\sigma(y)\,\mathrm{d}r \\ &= \int_0^\infty r\partial_t^2\overline{\varphi}(r,r) - r\partial_r^2\overline{\varphi}(r,r) - 2\partial_r\overline{\varphi}(r,r)\,\mathrm{d}r.\end{aligned}$$

Now observe that $\frac{\mathrm{d}}{\mathrm{d}r}(r\overline{\varphi}_t(r,r)) = \overline{\varphi}_t(r,r) + r\overline{\varphi}_{tt}(r,r) + r\overline{\varphi}_{rt}(r,r)$ and $\frac{\mathrm{d}}{\mathrm{d}r}(r\overline{\varphi}_r(r,r)) = \overline{\varphi}_r(r,r) + r\overline{\varphi}_{rt}(r,r) + r\overline{\varphi}_{rr}(r,r)$. Integrating by parts and observing that both $r\overline{\varphi}_t(r,r)$ and $r\overline{\varphi}_r(r,r)$ vanish at 0 and at ∞, we get

$$\begin{aligned}\langle \Box S, \varphi\rangle = \langle S, \Box\varphi\rangle &= \int_0^\infty -\partial_t\overline{\varphi}(r,r) + \partial_r\overline{\varphi}(r,r) - 2\partial_r\overline{\varphi}(r,r)\,\mathrm{d}r \\ &= \int_0^\infty -\partial_t\overline{\varphi}(r,r) - \partial_r\overline{\varphi}(r,r)\,\mathrm{d}r = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}r}\left(\overline{\varphi}(r,r)\right)\,\mathrm{d}r \\ &= \overline{\varphi}(0,0) = \frac{1}{4\pi}\int_{\mathbb{S}^2} \varphi(0\cdot y, 0)\,\mathrm{d}\sigma(y) = \varphi(0,0) = \langle \delta_{(0,0)}, \varphi\rangle.\end{aligned}$$

This concludes the proof of Proposition 1.60. $\square$

Theorem 1.61. *Let $f \in \mathcal{E}'(\mathbb{R}^4)$. Then*

$$u := \underbrace{\frac{T}{4\pi\,\rho}}_{\in\mathcal{S}'(\mathbb{R}^4)} \star \underbrace{f}_{\in\mathcal{E}'(\mathbb{R}^4)}$$

is a solution of $\Box u = f$. Moreover

$$\textit{supp}\, u_0 \subset \{(x,t) \in \mathbb{R}^4;\ \exists\ (x_0,t_0) \in \textit{supp}\, f \quad \textit{such that} \quad |x - x_0| = |t - t_0|\}.$$

(Light cone centred at (x_0,t_0)). Moreover u is the unique solution to $\Box u = f$ null in the past : whose support is included in a half space of the form $\{(x,t);\ t > t_0\}$.

Proof of theorem 1.61: The equation $\Box\, u = f$ can be rewritten in the form

$$A \star u = f, \quad \text{where} \quad A = \partial_t^2\delta_0 - \sum_{i=1}^3 \partial_{x_i}^2\delta_0 \in \mathcal{E}'(\mathbb{R}^4)\ .$$

Since $A, f \in \mathcal{E}'(\mathbb{R}^4)$, using theorem 1.53, we have

$$\begin{aligned}\Box u &= \Box\left(\frac{T}{4\pi\,\rho}\star f\right) \\ &= A\star\left(\frac{T}{4\,\pi\,\rho}\star f\right) = \left(A\star\frac{T}{4\pi\,\rho}\right)\star f = \delta_0\star f = f\end{aligned}$$

This shows that u solves $\Box\, u = f$. We have moreover

$$\operatorname{supp} u \subset \operatorname{supp} \frac{T}{4\,\pi\rho} \; + \operatorname{supp} f,$$

this means that for any (x,t) in supp$u$ there existts

$$(y,s) \in supp\; \frac{T}{4\,\pi\,\rho} \quad \text{and} \quad (x_0,t_0) \in supp\, f \qquad ,$$

such that

$$(x,t) = (y,s) + (x_0,t_0)\ .$$

This implies that $(x,t)-(x_0,t_0) \in$ light cone with origin 0. In other words $|x-x_0| = |t-t_0|$, which means that $(x,t) \in$ light cone with origin (x_0,t_0).

Assume there exists another solution $\tilde{u}$, supported in $\{(x,t);\, t > t_0'\}$ for some t_0'. Denote $w := u - \tilde{u}$, then we have

$$\operatorname{supp} w \subset \{(x,t);\, t > t_0''\}$$

and

$$w = \delta_0 \star w = \left(\frac{T}{4\,\pi\,\rho} \star \Box\,\delta_0\right) \star w.$$

Let now $\Theta \in C_0^\infty(\mathbb{R}^4)$ with $\Theta \equiv 1$ on $B_1(0)$ and $\Theta \equiv 0$ on $B_2(0)^c$. Denote $\Theta_i(x) = \Theta(x/i),\ i \in \mathbb{N}$. Then we have $\Theta_i \equiv 1$ on $B_i(0)$ and $\Theta_i \equiv 0$ on $B_{2i}(0)^c$. This gives

(1)
$$\begin{aligned} &(\underbrace{\Theta_i \frac{T}{4\pi\,\rho}}_{\in\mathcal{E}'} \star \underbrace{\Box\,\delta_0}_{\in\mathcal{E}'}) \star \underbrace{w}_{\in\mathcal{S}'} \\ &= \Theta_i \frac{T}{4\pi\,\rho} \star (\Box\,\delta_0 \star w) \\ &= \Theta_i \frac{T}{4\pi\,\rho} \star \Box\, w = 0. \end{aligned}$$

Moreover there holds

$$\Box\left(\Theta_i \frac{T}{4\pi\rho}\right) = \Box\left(\frac{T}{4\pi\rho}\right) = \delta_0 \qquad \text{in } \mathcal{D}'(B_i(0))\ ,$$

and

$$\Box\left(\Theta_i \frac{T}{4\pi\,\rho}\right) = 0 \qquad \text{in } \mathcal{D}'(B_{2_i}(0)^c)\ :$$

Thus finally

$$\Box\left(\Theta_i \frac{T}{4\pi\rho}\right) = \delta_0 + h_i,$$

where $\operatorname{supp} h_i \subset B_{2_i}(0) \backslash B_i(0) \cap$ light cone with origin 0.

Let now $\varphi \in C_0^\infty(\mathbb{R}^4)$ with $supp\,\varphi \subset B_R^4(0)$. Since

(2)
$$\langle(\Theta_i \frac{T}{4\pi\,\rho} \star \Box\delta_0) \star w,\, \varphi\rangle = \langle \Box\Theta_i \frac{T}{4\pi\,\rho},\, \check{w} \star \varphi\rangle$$

and

$$\operatorname{supp} \check{w} \subset \{(x,t),\, t < -t_0''\}\ .$$

This implies

$$\operatorname{supp} \check{w} \star \varphi \subset \{(x,t), t \leq -t_0'' + R\}\ .$$

Thus

$$\langle h_i, \check{w} \star \varphi \rangle_{\mathcal{E}', C^\infty} = 0$$

for i large enugh. Combining the above we have for i large enough

$$\begin{aligned}
\underset{(1),(2)}{0 \underset{\uparrow}{=}} & \left\langle \Box\, \Theta_i \, \frac{T}{4\pi\,\rho},\, \check{w} \star \varphi \right\rangle = \langle \delta_0 + h_i,\, \check{w} \star \varphi \rangle \\
&= \langle \delta_0, \check{w} \star \varphi \rangle \\
&= \langle \delta_0, \langle w(-y),\, \varphi(x-y) \rangle \rangle \\
&= \langle w, \varphi \rangle\ .
\end{aligned}$$

Hence we have proved that $w = u - \tilde{u} = 0$ in $\mathcal{D}'(\mathbb{R}^4)$. This holds as well in $\mathcal{S}'(\mathbb{R}^4)$ since $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$. This concludes the proof of theorem 1.61. □

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We have covered two model cases both for linear elliptic and hyperbolic equations, however, at this stage, the following questions still remain open:

i) How does one find the fundamental solution?

ii) What if f is no longer in $\mathcal{E}'(\mathbb{R}^n)$?
How to define $G \star f$ (resp. $E \star f$) for general f?

iii) What regularity properties does $f \star G$ (resp. $E \star f$) have with respect to the regularity of f?

iv) What about partial differential equations in bounded domains?

1.11 Convolutions and Fourier Transforms

Theorem 1.62. *Assume either $u \in \mathcal{E}'(\mathbb{R}^n)$ and $v \in \mathcal{S}'(\mathbb{R}^n)$ or $u \in L^1(\mathbb{R}^n)$ and $v \in L^1(\mathbb{R}^n)$. then we have*

$$\widehat{u \star v} = (2\pi)^{n/2}\ \hat{u}\, \hat{v}\ .$$

Proof of theorem 1.62. We consider first $u \in L^1(\mathbb{R}^n)$ and $v \in L^1(\mathbb{R}^n)$.

$$\widehat{u \star v}\ (\zeta) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\zeta} \int_{\mathbb{R}^n} u(x-y) v(y) dy^n\ dx^n.$$

Since $e^{-ix\cdot\zeta}\ u(x-y)\ v(y) \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ we can apply the theorem of Fubini to deduce

$$\widehat{u \star v}\,(\zeta) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} v(y) \int_{\mathbb{R}^n} e^{-ix\cdot\zeta}\, u(x-y)\, dy^n\, dx^n$$

$$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} v(y) \int_{\mathbb{R}^n} e^{-i(z+y)\cdot\zeta}\, u(z)\, dz^n\, dy^n$$

$$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-iy\cdot\zeta}\, v(y)\, dy^n \int_{\mathbb{R}^n} e^{-iz\cdot\zeta}\, u(z)\, dz^n$$

$$= (2\pi)^{n/2}\ \hat{v}\,(\zeta)\,\hat{u}\,(\zeta)\ .$$

Consider now $u, v \in \mathcal{E}'$, then from proposition 1.51 we know that $u \star v \in \mathcal{E}'$ and $\operatorname{supp} u \star v \subset \operatorname{supp} u + \operatorname{supp} v$. Applying now theorem 1.37 we have

$$\widehat{u \star v}\,(\zeta) = (2\pi)^{-n/2}\,\langle u \star v(x),\ e^{-ix\cdot\zeta}\rangle = (2\pi)^{-n/2}\,\langle u(y),\ \check{v}_x \star e^{-ix\cdot\zeta}\ (y)\rangle\ .$$

We have moreover, since $\hat{v} \in G(\mathbb{R}^n)$ and $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \check{v} \star e^{-ix\cdot\zeta},\ \varphi\rangle = \langle e^{-ix\cdot\zeta},\ v \star \varphi\rangle$$

$$= \int_{\mathbb{R}^n} e^{-ix\cdot\zeta} \int_{\mathbb{R}^n} v(x-y)\,\varphi(y)\, dy^n\ dx^n$$

$$= \int_{\mathbb{R}^n} e^{-i(y+z)\cdot\zeta} \int_{\mathbb{R}^n} v(z)\,\varphi(y)\, dy^n\ dz^n$$

$$= \hat{v}\ \hat{\varphi} \in \mathcal{S}(\mathbb{R}^n)\ .$$

Consider now more generally $v \in \mathcal{S}'(\mathbb{R}^n)$.

Claim : There is a sequence $v_j \in \mathcal{E}'$ such that $v_j \to v$ in $\mathcal{S}'(\mathbb{R}^n)$.

Proof of the claim: Let $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\psi \equiv 1$ on $B_1(0)$. We denote $v_j := \psi\,(x/j)\ v$. Then

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^n) \qquad \left\langle \psi\left(\frac{x}{j}\right)\ v, \varphi\right\rangle = \left\langle v, \psi\left(\frac{x}{j}\right)\varphi\right\rangle$$

and then

$$|\langle v_j - v,\ \varphi\rangle| = \left|\left\langle \psi\left(\frac{x}{j}\right)\ v - v,\ \varphi\right\rangle\right| = \left|\left\langle v,\ \psi\left(\frac{x}{j}\right)\varphi - \varphi\right\rangle\right|$$

$$\leq C\ \mathcal{N}_p\left(\psi(\frac{x}{j})\cdot\varphi - \varphi\right) \leq j^{-1}\,\mathcal{N}_{p+1}\left(\psi(\frac{x}{j})\cdot\varphi - \varphi\right) = o_j(1)\ .$$

Hence $v_j \to v$ in $\mathcal{S}'(\mathbb{R}^n)$ and obviously $v_j \in \mathcal{E}'(\mathbb{R}^n)$. This concludes the proof of the claim.

Consider now $u \in \mathcal{E}'(\mathbb{R}^n)$ and $v \in \mathcal{S}'(\mathbb{R}^n)$ as well as $v_j \in \mathcal{E}'(\mathbb{R}^n)$ a sequence which converges towards v in $\mathcal{S}'(\mathbb{R}^n)$. We have already established that $\widehat{u \star v_j} = (2\pi)^{n/2}\,\hat{u}\,\hat{v}_j$ and we know from theorem 1.37 that $\hat{u} \in G(\mathbb{R}^n)$. Since $\hat{v}_j \to v$ in $\mathcal{S}'$ we have using proposition 1.24

$$\widehat{u \star v_j} = (2\pi)^{n/2}\,\hat{u}\;\hat{v}_j \to (2\pi)^{n/2}\,\hat{u}\;\hat{v} \qquad \text{in } \mathcal{S}' \ .$$

Using proposition 1.51, we have

$$\forall\,\varphi \in \mathcal{S}(\mathbb{R}^n) \qquad \langle \widehat{u \star v_j}, \varphi\rangle = \langle u \star v_j, \hat{\varphi}\rangle = \langle v_j \star u, \hat{\varphi}\rangle = \langle v_j, \check{u} \star \hat{\varphi}\rangle \to \langle v, \check{u} \star \hat{\varphi}\rangle$$

This concludes the proof of theorem 1.62. □

1.12 The use of the Fourier transform for solving Cauchy-Problems in $\mathcal{S}'(\mathbb{R}^n)$

This subsection is devoted to the solvability question of partial differential equations with initial conditions.

First of all, we shall be considering the following natural question : Is there any sufficient condition on

$$A \in \mathcal{E}'(\mathbb{R}^n)\ .$$

that guaranties the existence of exactly one $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfying $A \star u = f$ for any given $f \in \mathcal{S}'(\mathbb{R}^n)$?

The following theorem is giving an answer to this question assuming A is supported at the origin.

Theorem 1.63. *Let $A \in \mathcal{E}'(\mathbb{R}^n)$ of the form*

$$A = \sum_{|\alpha|\le m} a_\alpha\,\partial^\alpha\,\delta_0,$$

so that the Fourier transform of A, $\hat{A} = \sum_{|\alpha|\le m} b_\alpha\,x^\alpha$, satisfies the following condition

$$\hat{A}(\xi) \neq 0 \quad \text{for all } \xi \in \mathbb{R}^n\ . \tag{1.43}$$

If A satisfies (1.43) then[4] *for any arbitrary $f \in \mathcal{S}'(\mathbb{R}^n)$ there exists exactly one $u \in \mathcal{S}'(\mathbb{R}^n)$, such that $A \star u = f$.*

[4] If A satisfies the slightly different condition, $m = 2p$

$$\forall \xi \in \mathbb{R}^n \setminus \{0\} \qquad \sum_{|\alpha|=m} b_\alpha\,\xi^\alpha \ge c\,|\xi|^m\ ,$$

for some $c > 0$, one says that A is **strongly elliptic**. This last condition is very important in many applications from geometry and physics. From a strictly analysis perspective it is a condition related to interior regularisation effect and non-degeneracy and uniqueness for prescribed boundary problems.

Proof of theorem 1.63: We consider the Fourier transform applied to the equality $A \star u = f$. We have thanks to proposition 1.51 $\hat{f} \in \mathcal{S}'(\mathbb{R}^n)$. We have moreover thanks to theorem 1.62 $\widehat{A \star u} = (2\pi)^{n/2} \hat{A} \cdot \hat{u}$. Since $A \in \mathcal{E}'(\mathbb{R}^n)$ we deduce from theorem 1.37 $\hat{A} \in O_M\ (\mathbb{R}^n)$, which itself implies thanks to proposition 1.25 $\hat{A} \cdot \hat{u} \in \mathcal{S}'(\mathbb{R}^n)$.

Hence the equation

$$A \star u = f$$

is equivalent to $(2\pi)^{n/2} \hat{A} \cdot \hat{u} = \hat{f}$ and it posses a unique solution given formally by

$$\hat{u} = (2\pi)^{-n/2} \frac{\hat{f}}{\hat{A}} \ . \tag{1.44}$$

Because of (1.19) $\hat{A}$ is a polynomial that never vanishes and consequently its inverse is a slowly increasing function : $\hat{A}^{-1} \in G(\mathbb{R}^n)$. Hence $\frac{\hat{f}}{\hat{A}} \in \mathcal{S}'(\mathbb{R}^n)$ and (1.44) makes sense. The formula implies uniqueness. □

Example 1.64. *The Bessel operator.* Let $A = -\Delta\delta_0 + \delta_0$.

$$\hat{A}(\xi) = (2\pi)^{-n/2} \left(|\xi|^2 + 1\right) \ .$$

Thanks to the previous theorem, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ the unique solution of

$$-\Delta u + u = f \quad \text{ in } \mathcal{S}'(\mathbb{R}^n)$$

is given by

$$u := \mathcal{F}^{-1}\left(\frac{\hat{f}}{|\xi|^2 + 1}\right)$$

Example 1.65. *A degenerate case : the Poisson equation.*

Let $f \in \mathcal{E}'(\mathbb{R}^n)$. We aim at solving again the Poisson equation with right-hand-side equal to f but with the mean of the Fourier transform this time. We look for $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\Delta u = f \ , \tag{1.45}$$

that can be rewritten as

$$\Delta\delta_0 \star u (= \delta_0 \star \Delta u = \Delta u) = f, \quad f \in \mathcal{E}'(\mathbb{R}^n).$$

We restrict to the case $n \geq 3$. After application of the Fourier Transform we obtain

$$-|\xi|^2 \, \hat{u} = \hat{f}$$

. Since $f \in \mathcal{E}'(\mathbb{R}^n)$, $\hat{f}$ is C^∞ and hence

$$-\frac{\hat{f}}{|\xi|^2} \in L^1_{loc}(\mathbb{R}^n)$$

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi \equiv 1$ on $B_1^n(0)$. Then we have

$$-\frac{\hat{f}}{|\xi|^2} = -\chi(\xi)\,\frac{\hat{f}}{|\xi|^2} - (1-\chi(\xi))\frac{\hat{f}}{|\xi|^2} \in L^1(\mathbb{R}^n) + G(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n),$$

Hence

$$u = \bar{\mathcal{F}}\left(-\frac{\hat{f}}{|\xi|^2}\right)$$

is a solution to the Poisson equation (1.45). Let v be another solution in $\mathcal{S}'(\mathbb{R}^n)$. Then $u - v$ solves the Laplace equation

$$\Delta(u-v) = 0\ ,$$

and then, thanks to theorem 1.41, $u-v$ is an harmonic polynomial. Hence the space of solution to (1.45) in $\mathcal{S}'(\mathbb{R}^n)$ is given by

$$u = \bar{\mathcal{F}}\left(-\frac{\hat{f}}{|\xi|^2}\right) + P(x) \quad \text{where } P \text{ is an harmonic polynomial}\ .$$

We now compare this expression with theorem 1.59, we must have

$$G \star f = (2\pi)^n\ \bar{\mathcal{F}}\left(-\frac{\hat{f}}{|\xi|^2}\right)\ . \tag{1.46}$$

Since $G \in \mathcal{S}'(\mathbb{R}^n)$ and $f \in \mathcal{E}'(\mathbb{R}^n)$, we have thanks to theorem 1.62

$$(2\pi)^{n/2}\,\hat{G}\,\hat{f} = -\ \frac{\hat{f}}{|\xi|^2}\ .$$

Since this holds for any $f \in \mathcal{E}'(\mathbb{R}^n)$, one deduces (for $f = \delta_0$) from the explicit expression of the Green Function G given by theorem 1.59 the following Lemma

Lemma 1.66. *For any $n > 2$*

$$\mathcal{F}\left(\frac{1}{|x|^{n-2}}\right) = (2\pi)^{-n/2}\ (n-2)\,\frac{|S^{n-1}|}{|\xi|^2}\ . \tag{1.47}$$

Exercise 1.67. *Compute in $\mathbb{R}^2$ the Fourier transform $\mathcal{F}(\log|x|)$.*

Example 1.68. *The Heat Equation:*

Let $f \in \mathcal{S}'(\mathbb{R}^n)$. We are looking for a solution $u \in C^\infty(\mathbb{R}_+^*, \mathcal{S}')$ from the following problem

$$\begin{cases} \partial_t u - \Delta u = 0 \\ \\ u(0,x) = f(x)\ . \end{cases}$$

where the rigorous way the initial condition $u(0,x) = f(x)$ has to be understood has to be specified. We proceed first to some computations. One considers the Fourier transform in x and, assuming that $u \in \mathcal{S}'$

$$(\partial_t + |\xi|^2)\hat{u} = 0, \quad \hat{u}(0,\xi) = \hat{f}(\xi).$$

It follows immediately

$$\hat{u}(t,\xi) = e^{-t\,|\xi|^2}\hat{f}(\xi).$$

We now rewrite this identity as follows:

$$\hat{u}(t,\xi) = \hat{H}(t,\xi)\cdot\hat{f}(\xi),$$

where

$$\hat{H}(t,\xi) := e^{-t\,|\xi|^2},$$

That means

$$u = (2\pi)^{-/2}\, H \star f.$$

We now calculate $H(t,x)$. Recall from exercise 1.3 for any $a \in \mathbb{R}_+^*$

$$\widehat{e^{-a|x|^2}} = \frac{1}{(2a)^{\frac{n}{2}}}\; e^{-\frac{|\xi|^2}{4a}}.$$

We apply this identity for $a = \dfrac{1}{4t}$ and it follows

$$e^{-|x|^2/4t} = \bar{\mathcal{F}}\mathcal{F}\left(e^{-|x|^2/4t}\right) = (2\,t)^{n/2}\ \bar{\mathcal{F}}\left(e^{-|\xi|^2 t}\right)$$

$$\Rightarrow H(t,x) = \left(\frac{1}{2t}\right)^{n/2} e^{-|x|^2/4t}.$$

The formal computations above are leading (exercise) to the following result.

Lemma 1.69. *For any $f \in \mathcal{S}'(\mathbb{R}^n)$ there exists $u \in C^\infty_{loc}(\mathbb{R}^*_+, \mathcal{S}(\mathbb{R}^n)$ solving*

$$\partial_t u - \Delta u = 0 \quad \text{in } \mathbb{R}^*_+ \times \mathbb{R}^n\ ,$$

moreover

$$\lim_{t\to 0} u(t,\cdot) = f \quad \text{weakly in } \mathcal{S}'(\mathbb{R}^n) \tag{1.48}$$

in other words

$$\forall\,\varphi \in \mathcal{S}(\mathbb{R}^n) \qquad \lim_{t\to 0}\int_{\mathbb{R}^n} u(t,x)\ \varphi(x)\ dx^n =< f,\varphi >_{\mathcal{S}',\mathcal{S}}$$

The solution is unique and there holds

$$\forall t > 0 \qquad u(t,x) = \left(\frac{1}{4\pi t}\right)^{n/2} e^{-|x|^2/4t} \star f\ .$$

Example 1.70. *The Wave Equation*

For any choice of $f, g \in \mathcal{E}'$ we are looking for a solution u in a sense which has to be precised of

$$\left\{ \begin{array}{l} \Box u = \partial_t^2 u - \Delta u = 0 \ , \\ \\ u(0,x) = f(x) \ , \\ \\ \partial_t u(0,x) = g(x), \end{array} \right. \tag{1.49}$$

We perform formal computations first. The Fourier transform of the equation (1.49) is giving

$$\left\{ \begin{array}{l} (\partial_t^2 + |\xi|^2)\hat{u} = 0, \\ \\ \hat{u}(0,\xi) = \hat{f}(\xi), \\ \\ \partial_t \hat{u}(0,\xi) = \hat{g}(\xi) \end{array} \right.$$

This leads to the following solution

$$\hat{u}(t,\xi) = C(\xi)\ \sin(t|\xi|) + C'(\xi)\ \cos(t|\xi|).$$

With

$$\partial_t \hat{u}(t,\xi) = C(\xi)\,|\xi|\ \cos(t\,|\xi|) - C'(\xi)\,|\xi|\sin(t\,|\xi|)$$

It follows now

$$C(\xi)\,|\xi| = \hat{g}(\xi)$$

and

$$C'(\xi) = \hat{f}(\xi).$$

This gives

$$\hat{u}(t,\xi) = \hat{g}(\xi)\ \frac{\sin(t\,|\xi|)}{|\xi|} + \hat{f}(\xi)\cos(t\,|\xi|)\ . \tag{$\star$}$$

We have

$$\forall\, t \in \mathbb{R} \qquad \frac{\sin(t\,|\xi|)}{|\xi|} \in L^\infty(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \quad \text{and} \quad \cos(t\,|\xi|) \in L^\infty(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$$

After these formal computations we can now develop an argument. Since f and g are both assumed to be in $\mathcal{E}'(\mathbb{R}^n)$, thanks to theorem 1.37 $\hat{f} \in G(\mathbb{R}^n)$ and $\hat{g} \in G(\mathbb{R}^n)$ and thanks to proposition 1.25

$$\forall\, t \in \mathbb{R} \qquad \hat{g}(\xi)\ \frac{\sin(t\,|\xi|)}{|\xi|} \in \mathcal{S}'(\mathbb{R}^n) \quad \text{and} \quad \hat{f}(\xi)\cos(t\,|\xi|) \in \mathcal{S}'(\mathbb{R}^n)$$

Thanks now to proposition 1.51 and theorem 1.62 there holds

$$\begin{aligned} \forall\, t \in \mathbb{R} \qquad & \overline{\mathcal{F}}\left(\hat{g}(\xi)\ \frac{\sin(t\,|\xi|)}{|\xi|} + \hat{f}(\xi)\cos(t\,|\xi|)\right) \\ &= (2\pi)^{-n/2}\, g \star \overline{\mathcal{F}}\left(\frac{\sin(t\,|\xi|)}{|\xi|}\right) + (2\pi)^{-n/2}\, f \star \overline{\mathcal{F}}\left(\cos(t\,|\xi|)\right) \end{aligned}$$

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and let $u(t,x) := \overline{\mathcal{F}}\left(\hat{g}(\xi)\,\frac{\sin(t\,|\xi|)}{|\xi|} + \hat{f}(\xi)\cos(t\,|\xi|)\right)$. We have

$$\begin{aligned}
&\partial^2_{t^2}\langle u(t,x),\varphi(x)\rangle - \langle \Delta u(t,x),\varphi(x)\rangle \\
&= \partial^2_{t^2}\langle \hat{u}(t,\xi),\overline{\mathcal{F}}(\varphi)(\xi)\rangle + \langle |\xi|^2\,\hat{u}(t,\xi),\overline{\mathcal{F}}(\varphi)(\xi)\rangle \\
&= \partial^2_{t^2}\left\langle \hat{g}(\xi)\,\frac{\sin(t\,|\xi|)}{|\xi|} + \hat{f}(\xi)\cos(t\,|\xi|),\overline{\mathcal{F}}(\varphi)(\xi)\right\rangle \\
&+ \left\langle |\xi|^2\,\hat{g}(\xi)\,\frac{\sin(t\,|\xi|)}{|\xi|} + |\xi|^2\,\hat{f}(\xi)\cos(t\,|\xi|),\overline{\mathcal{F}}(\varphi)(\xi)\right\rangle
\end{aligned}$$

We claim (exercise) that

$$\langle u(t,x),\varphi(x)\rangle = \left\langle \hat{g}(\xi)\,\frac{\sin(t\,|\xi|)}{|\xi|} + \hat{f}(\xi)\cos(t\,|\xi|),\overline{\mathcal{F}}(\varphi)(\xi)\right\rangle \in C^2(\mathbb{R})$$

and that

$$\begin{aligned}
&\partial^2_{t^2}\left\langle \hat{g}(\xi)\,\tfrac{\sin(t\,|\xi|)}{|\xi|} + \hat{f}(\xi)\cos(t\,|\xi|),\overline{\mathcal{F}}(\varphi)(\xi)\right\rangle \\
&= -\left\langle |\xi|^2\,\hat{g}(\xi)\,\frac{\sin(t\,|\xi|)}{|\xi|} + |\xi|^2\,\hat{f}(\xi)\cos(t\,|\xi|),\overline{\mathcal{F}}(\varphi)(\xi)\right\rangle
\end{aligned}$$

Hence we have proved

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^n) \qquad \partial^2_{t^2}\langle u(t,x),\varphi(x)\rangle - \langle \Delta u(t,x),\varphi(x)\rangle = 0$$

Exercise 1.71. *Prove that*

$$\langle u(0,x),\varphi(x)\rangle = \langle f(x),\varphi(x)\rangle \quad \text{and} \quad \partial_t\langle u(t,x),\varphi(x)\rangle|_{t=0} = \langle g(x),\varphi(x)\rangle\ .$$

Exercise 1.72. *Prove that for $n=3$ and $t \neq 0$*

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^3) \qquad \left\langle (2\pi)^{-3/2}\,\mathcal{F}\left(\frac{\sin(t\,|\xi|)}{|\xi|}\right),\varphi\right\rangle = \frac{1}{4\pi\,t}\int_{\partial B_{|t|}(0)} \varphi(x)\ dvol_{\partial B_{|t|(0)}}$$

and

$$\begin{aligned}
&\forall \varphi \in \mathcal{S}(\mathbb{R}^3) \qquad \left\langle (2\pi)^{-3/2}\,\mathcal{F}\left(\cos(t\,|\xi|)\right),\varphi\right\rangle = \frac{d}{dt}\left(\frac{1}{4\pi\,t}\int_{\partial B_{|t|}(0)} \varphi(x)\ dvol_{\partial B_{|t|(0)}}\right) \\
&= \ldots
\end{aligned}$$

We claim that the solution u is unique in the class of solutions which are compactly supported for every $t \in \mathbb{R}$.

By linearity, it suffices to prove that any solution u in this class for $f=0$ and $g=0$ is identically equal to zero. Let $\chi \in C^\infty_0(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n}\chi(x)\ dx^n = 1$ and for ay $\epsilon \in (0,1)$ we denote $\chi_\epsilon(x) := \epsilon^{-n}\,\chi(x/\epsilon)$. We have seen

$$\chi_\epsilon \to \delta_0 \qquad \text{in } \mathcal{S}'(\mathbb{R}^n)\ .$$

Denote $u_\epsilon := u \star \chi_\epsilon$. We have that $u_\epsilon(t,\cdot) \in C_0^\infty(\mathbb{R}^n)$ for every $t \in \mathbb{R}$. We have $u_\epsilon(x,t) =< u(t,y), \chi_\epsilon(x-y) >_{<\mathcal{S}',\mathcal{S}>}$. Hence for any $x \in \mathbb{R}^n$ there holds by assumption

$$\partial^2_{t^2}\langle u(t,y), \chi_\epsilon(x-y)\rangle = \langle \Delta_y u(t,y), \chi_\epsilon(x-y)\rangle = \langle u(t,y), \Delta_y \chi_\epsilon(x-y)\rangle$$

$$= \langle u(t,y), \Delta_x \chi_\epsilon(x-y)\rangle = \Delta_x \langle u(t,y), \chi_\epsilon(x-y)\rangle$$

Hence $u_\epsilon(t,\cdot)$ is a classical solution to the wave equation and in particular it is C^2 in x and t. Since we are assuming $f = 0$ and $g = 0$ we have for any $\epsilon > 0$

$$\forall x \in \mathbb{R}^n \qquad \lim_{t\to 0}\langle u(t,y), \chi_\epsilon(x-y)\rangle = 0 \qquad \text{and} \qquad \lim_{t\to 0}\partial_t\langle u(t,y), \chi_\epsilon(x-y)\rangle = 0 \ .$$

Introduce

$$E(u_\epsilon) := \frac{1}{2}\int_{\mathbb{R}^n} |\partial_t u_\epsilon|^2 + |\nabla_x u_\epsilon|^2 \ dx^n$$

which is finite since $u_\epsilon(t,\cdot) \in C_0^\infty(\mathbb{R}^n)$. Since $u_\epsilon(t,\cdot)$ is a classical solution to the wave equation and in particular it is C^2 in x and t we have

$$\partial_t E\ (u_\epsilon) = \int_{\mathbb{R}^n} (\partial_t\, u_\epsilon)(\partial_t^2\, u_\epsilon) + \sum (\partial_t\, \partial_{x_i}\, u_\epsilon)(\partial_{x_i} u_\epsilon)\ dx^n$$

$$= \int_{\mathbb{R}^n} (\partial_t\, u_\epsilon)\,(\partial_t^2\, u_\epsilon - \Delta\, u_\epsilon)\ dx^n = 0$$

Hence E is constant for all time. Since $E(u_\epsilon(0,\cdot)) = 0$ and $u_\epsilon(0,\cdot) = 0$ we have

$$\forall\ \epsilon > 0 \quad \forall\ (t,x) \in \mathbb{R}\times\mathbb{R}^n \qquad u_\epsilon(t,x) = 0\ .$$

This implies that u is identically equal zero which concludes the proof of the uniqueness.

To summarise we have proved the following result

Theorem 1.73. *Let $f \in \mathcal{E}'(\mathbb{R}^n)$ and $g \in \mathcal{E}'(\mathbb{R}^n)$. Then there exists a map $t \in \mathbb{R} \to u(t,x) \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^n) \qquad \langle u(t,x), \varphi(x)\rangle \in C^2(\mathbb{R})\ ,$$

and for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ there holds

$$\partial^2_{t^2}\langle u(t,x), \varphi(x)\rangle - \langle \Delta u(t,x), \varphi(x)\rangle = 0 \tag{1.50}$$

together with

$$\langle u(0,x), \varphi(x)\rangle = \langle f(x), \varphi(x)\rangle \quad \text{and} \quad \partial_t\langle u(t,x), \varphi(x)\rangle|_{t=0} = \langle g(x), \varphi(x)\rangle\ . \tag{1.51}$$

If $n = 3$,

$$u(t,x) := R(t,x) \star f + \partial_t R(t,x) \star g$$

is a solution of (1.50)-(1.51) where for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\langle R(t,x), \varphi\rangle = \fint_{\partial B_{|t|(0)}} \varphi(x)\ dvol_{\partial B_{|t|(0)}}\ .$$

Moreover $u(t,x)$ is unique among the solutions which are compactly supported for every $t \in \mathbb{R}$.

2 Hilbert-Sobolev Spaces

2.1 Definition and Fundamental Properties

Definition 2.1. *(Hilbert-Sobolev Spaces of integer order). Let $m \in \mathbb{N}$ and let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then, u is in the Sobolev space $H^m(\mathbb{R}^n)$ if for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ with $|\alpha| = \sum_i |\alpha_i| \leq m$, we have $\partial^\alpha u \in L^2(\mathbb{R}^n)$ and we denote*

$$\|u\|_{H^m} := \left[\sum_{|\alpha| \leq m} \int_{\mathbb{R}^n} |\partial^\alpha u|^2 \, dx^n \right]^{1/2} .$$

Remark 2.2. *Let $u \in H^m(\mathbb{R}^n)$, i.e., for all α, $|\alpha| \leq m$, we have $\partial^\alpha u \in L^2$. This implies*

$$\xi^\alpha \, \hat{u} = C \, \mathcal{F}(\partial^\alpha u) \in L^2(\mathbb{R}^n) \ ,$$

that is

$$\int_{\mathbb{R}^n} |\xi^\alpha|^2 \, |\hat{u}|^2 < \infty \quad \forall \alpha \quad \textit{with} \quad |\alpha| \leq m.$$

This, in turn, implies

$$\int_{\mathbb{R}^n} (1 + |\xi|^2)^m \, |\hat{u}|^2(\xi) \, d\xi^n < \infty,$$

that is

$$(1 + |\xi|^2)^{m/2} \, \hat{u} \in L^2.$$

Conversely, let $u \in \mathcal{S}'(\mathbb{R}^m)$, so that $(1 + |\xi|^2)^{m/2} \, \hat{u} \in L^2$. Then, it follows that for all α with $|\alpha| \leq m$, $\xi^\alpha \, \hat{u} \in L^2$, from which it follows again that for all α with $|\alpha| \leq m$, $\partial^\alpha \, u \in L^2$.

From these considerations, the following proposition follows:

Proposition 2.3. *$u \in H^m(\mathbb{R}^n) \iff (1 + |\xi|^2)^{m/2} \, \hat{u} \in L^2(\mathbb{R}^n)$.*

Definition 2.4. *Let $s \in \mathbb{R}$. $H^s(\mathbb{R}^n)$ is the space of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ for which*

$$(1 + |\xi|^2)^{s/2} \ \hat{u} \in L^2(\mathbb{R}^n) \ .$$

Note that $(1+|\xi|)^{s/2} \in G(\mathbb{R}^n)$ and $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$. From this, it follows that $(1+|\xi|^2)^{s/2}\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$.

$$\|u\|_{H^s} := \|(1+|\xi|^2)^{s/2}\ \hat{u}\|_{L^2}.$$

Remark 2.5. *For $s \in \mathbb{N}$, the definitions 2.1 and 2.4 agree, and from the remark on Definition 2.1, it is also clear that the two norms*

$$\|u\|_{H^s} = \|(1+|\xi|^2)^{s/2}\ \hat{u}\|_{L^2}$$

and

$$\|u\|_{H^s} = \left[\sum_{|\alpha|\leq m} \int_{\mathbb{R}^n} |\partial^\alpha u|^2\ dx^n\right]^{1/2}$$

are equivalent.

Proposition 2.6. *The mapping*

$$(.,.)_s : H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{C}$$

$$(u,\ v) \mapsto \int_{\mathbb{R}^n} (1+|\xi|^2)^s\ \hat{u}\ \bar{\hat{v}}$$

is an inner product on $H^s(\mathbb{R}^n)$. Furthermore, $(H^s(\mathbb{R}^n), (.,.)_s)$ is complete, that is, $(H^s(\mathbb{R}^n), (.,.)_s)$ is a Hilbert space.

Proof of Proposition 2.6.

$$\int_{\mathbb{R}^n} (1+|\xi|^2)^s\ \hat{u}\,\bar{\hat{v}} = \int_{\mathbb{R}^n} (1+|\xi|^2)^{s/2}\ \hat{u}(1+|\xi|^2)^{s/2}\ \bar{\hat{v}}.$$

By the Hölder inequality, it immediately follows

$$|(u,v)_s| \leq \|u\|_{H^s}\,\|v\|_{H^s}.$$

It is also true that $(u,u)_s = \|u\|^2_{H^s}$. From this, it follows that $(u,u)_s$ is zero if and only if $u = 0$, and positive otherwise. This shows that $(.,.)_s$ is positive definite. Furthermore, from the definition of $(.,.)_s$, it is immediately clear that the following holds:

$$(u, \lambda v)_s = \bar{\lambda}\,(u,v)_s$$

and

$$(\lambda u, v)_s = \lambda(u,v)_s.$$

Thus, it is shown that $(.,.)_s$ is an inner product.

Now, let us turn to the question of completeness.

Let L denote the following mapping

$$\begin{aligned} L : H^s &\to L^2 \\ u &\mapsto (1+|\xi|^2)^{s/2}\ \hat{u}. \end{aligned}$$

L is obviously linear and bijective. Furthermore, let L' be the following mapping

$$\begin{aligned} L' : L^2 &\to H^s \\ v &\mapsto \mathcal{F}^{-1}\left((1+|\xi|^2)^{-s/2}\, v\right). \end{aligned}$$

Now, it follows that $L^{-1} = L'$ and L is moreover an isometry between H^s and L^2. Since $(L^2, \|.\|_2)$ is complete ($\to$ cf. Analysis 4), it follows that $(H^s, \|.\|_{H^s})$ is a Hilbert space, where $\|.\|_{H^s}$ is the norm induced by $(.,.)_s$.

Proposition 2.7. *The subspace of smooth compactly supported functions is dense in $H^s(\mathbb{R}^n)$ for any $s \in \mathbb{R}$:*

$$\overline{C_0^\infty(\mathbb{R}^n)}^{H^s} = H^s(\mathbb{R}^n)\ .$$

Proof of Proposition 2.7.

We first prove that $\overline{\mathcal{S}(\mathbb{R}^n)}^{H^s} = H^s(\mathbb{R}^n)$.

Proof of Claim 1: We first show that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. We know from (1.6) that $\mathcal{S}(\mathbb{R}^n)$ embeds (continuously) into L^1 . Because of the continuity of $\mathcal{N}_0$ it also embeds continuously into L^∞. Hence, using (1.6), for any $p \in [1, +\infty]$ and any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ there holds

$$\int_{\mathbb{R}^n} |\varphi(x)|^p\ dx^n \le C_n\, \mathcal{N}_{n+1}(\varphi)\ \mathcal{N}_0^{p-1}(\varphi)$$

which implies the continuity of the embedding of $\mathcal{S}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for any $p \in [1, +\infty]$. Furthermore, we have $\mathcal{S}(\mathbb{R}^n) \supset C_0^\infty(\mathbb{R}^n)$ and we moreover claim that $\overline{C_0^\infty}^{L^2} = L^2$. Indeed, for any $u \in L^2(\mathbb{R}^n)$ we have on one hand

$$\lim_{R\to+\infty} \int_{R^n} |\mathbf{1}_{B_R(0)}\ u - u|^2\ dx^n = 0$$

where $\mathbf{1}_{B_R(0)}$ is the characteristic function of $B_R(0)$. On the other hand, Let $\chi \in C_0^\infty(B_1(0))$ such that $\int_{\mathbb{R}^n} \chi(x)\ dx^n = 1$, then for any $R > 0$ we have

$$\int_{\mathbb{R}^n} |\chi_\varepsilon \star u_R - u_R|^2\ dx^n = \int_{\mathbb{R}^n} |[(2\pi)^{n/2}\, \hat{\chi}(\varepsilon\, \xi) - 1]\, \hat{u}(\xi)|^2\ d\xi^n$$

where $\chi_\varepsilon(x) := \varepsilon^{-n}\chi(x/\varepsilon)$. Observe that since $\int_{\mathbb{R}^n} \chi(x)\ dx^n = 1$

$$\hat{\chi}(0) = (2\pi)^{-n/2}\ ,$$

Hence we have

$$[(2\pi)^{n/2}\,\hat{\chi}(\varepsilon\,\xi)-1]\,\hat{u}(\xi)\ \longrightarrow\ 0 \qquad \text{almost everywhere,}$$

and obviously

$$|[(2\pi)^{n/2}\,\hat{\chi}(\varepsilon\,\xi)-1]\,\hat{u}(\xi)|^2 \le [1+(2\pi)^{n/2}\,\|\hat{\chi}\|_\infty]\ |\hat{u}| \qquad \text{almost everywhere.}$$

Hence dominated convergence implies that for any R

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}|\chi_\varepsilon\star u_R-u_R|^2\ dx^n=0\ .$$

A diagonal argument gives a sequence $\varepsilon_R\to 0$ such that

$$\chi_{\varepsilon_R}\star u_R\ \longrightarrow u \quad \text{strongly in } L^2(\mathbb{R}^n) \quad \text{as } R\to+\infty\ .$$

From (1.32) we have $\chi_{\varepsilon_R}\star u_R\in C_0^\infty(\mathbb{R}^n)$. Thus we have proved that

$$\overline{C_0^\infty(\mathbb{R}^n)}^{L^2}=L^2(\mathbb{R}^n)\ .$$

Now consider $L^{-1}:L^2\,|_{\mathcal{S}}\to\mathcal{S}$. For $u\in\mathcal{S}$, $\hat{u}\in\mathcal{S}(\mathbb{R}^n)$, and since $(1+|\xi|^2)^{-s/2}\in G(\mathbb{R}^n)$, we have also $(1+|\xi|^2)^{-s/2}\,\hat{u}\in\mathcal{S}(\mathbb{R}^n)$.

From the proof of Proposition 2.6, it is also known that L^{-1} is a bijective isometry. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, it follows that $L^{-1}\big(\mathcal{S}(\mathbb{R}^n)\big)=\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$, that is,

$$\overline{\mathcal{S}\,(\mathbb{R}^n)}^{H^s}=H^s(\mathbb{R}^n)\ .$$

We aim at proving something more refined that is $\overline{C_0^\infty(\mathbb{R}^n)}^{H^s}=H^s(\mathbb{R}^n)$. We first claim that $\mathcal{S}(\mathbb{R}^n)$ embeds continuously in $H^s(\mathbb{R}^n)$ for any $s\in\mathbb{R}$. This comes from the fact that each of the maps

$$\varphi\ \longrightarrow\ \hat{\varphi}\ \longrightarrow\ (1+|\xi|^2)^{s/2}\,\hat{\varphi}\ \longrightarrow\ \mathcal{F}^{-1}\left((1+|\xi|^2)^{s/2}\,\hat{\varphi}\ \right)$$

is continuous from $\mathcal{S}(\mathbb{R}^n)$ into itself (proposition 1.9 and proposition 1.24) and the embedding $\mathcal{F}^{-1}\left((1+|\xi|^2)^{s/2}\,\hat{\varphi}\ \right)\in\mathcal{S}(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ is continuous. Hence we have proved that for any $s\in\mathbb{R}$, $\exists\ p_{s,n}\in\mathbb{N}$, $\exists\ C_{s,n}>0$, such that $\forall\varphi\in\mathcal{S}(\mathbb{R}^n)$

$$\|\varphi\|_{H^s}\le C_{s,n}\,\mathcal{N}_{p_{s,n}}(\varphi)\ . \tag{2.52}$$

Now let $u\in H^s$ and $\varepsilon>0$. There exists a $\varphi\in\mathcal{S}$ such that $\|u-\varphi\|_{H^s}\le\varepsilon/2$. Further, let $\theta\in C_0^\infty(\mathbb{R}^n)$ such that $\theta\equiv 1$ on $B_1(0)$. Denote $\varphi_j(x):=\theta(x/j)\ \varphi$. Clearly $\varphi_j\in C_0^\infty$ moreover one verifies (exercise)

$$\mathcal{N}_p(\varphi_j-\varphi)\ \underset{j\to\infty}{\longrightarrow}\ 0\ .$$

Choose j_0 such that

$$\mathcal{N}_p(\varphi_{j_0}-\varphi)\le\frac{\varepsilon}{2\,C_s}\ ,$$

which implies because of (2.52)

$$\|\varphi_{j_0} - \varphi\|_{H^s} \leq \varepsilon/2 \ .$$

Thus, we have

$$\|u - \varphi_{j_0}\|_{H^s} \leq \|u - \varphi\|_{H^s} + \|\varphi_{j_0} - \varphi\|_{H^s} \leq \varepsilon.$$

This shows that $C_0^\infty(\mathbb{R}^n)$ is dense in $\big(H^s, (.,.)_s\big)$ and this concludes the proof of proposition 2.7. □

We are now proving the following theorem.

Theorem 2.8. *Let $s \in \mathbb{R}$, then*

$$(H^s(\mathbb{R}^n))^\star = H^{-s}(\mathbb{R}^n) \ .$$

Proof of Theorem 2.8.
Claim 1:

$$|\langle u,\, \varphi\rangle| \leq (2\pi)^{-n}\|u\|_{H^{-s}}\|\varphi\|_{H^s} \qquad \forall \varphi \in \mathcal{S}$$

Proof of Claim 1:

$$\begin{aligned}
\mathcal{F}\mathcal{F}\,\check{\varphi}(x) &= (2\pi)^{-n/2}\,\mathcal{F}\int_{\mathbb{R}^n} e^{-iy\cdot\xi}\,\check{\varphi}(y)\,dy^n \\
&= (2\pi)^{-n/2}\mathcal{F}\int_{\mathbb{R}^n} e^{-iy\cdot\xi}\,\varphi(-y)\,dy^n \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\int_{\mathbb{R}^n} e^{-iy\cdot\xi}\varphi(-y)\,dy^n\,d\xi^n \\
&= (-2\pi)^{-n}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\int_{\mathbb{R}^n} e^{i\,z\cdot\xi}\,\varphi(z)\,dz^n\,d\xi^n \\
&= \ \mathcal{F}\bar{\mathcal{F}}\,\varphi(x) = \ \varphi(x).
\end{aligned}$$

From this it follows that

$$\begin{aligned}
\langle u,\, \varphi\rangle &= \langle u,\, \mathcal{F}\mathcal{F}\,\check{\varphi}\rangle \\
&= \langle \mathcal{F}\,u,\, \mathcal{F}\,\check{\varphi}\rangle_{\mathcal{S}',\mathcal{S}} \\
&= \langle (1+|\xi|^2)^{-s/2}\,\mathcal{F}(u),\, (1+|\xi|^2)^{s/2}\,\mathcal{F}\,(\check{\varphi})\rangle_{\mathcal{S}',\mathcal{S}}.
\end{aligned}$$

We have

$$(1+|\xi|^2)^{s/2} \in G(\mathbb{R}^n) \quad \text{ and } \quad \mathcal{F}\,(\check{\varphi}) \in \mathcal{S}(\mathbb{R}^n) \ .$$

Hence

$$(1+|\xi|^2)^{s/2}\,\mathcal{F}\,(\check{\varphi}) \in \mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$$

Moreover since $u \in H^{-s}(\mathbb{R}^n)$

$$(1+|\xi|^2)^{-s/2}\, \mathcal{F}(u) \in L^2(\mathbb{R}^n)\ .$$

Thus, $\langle u,\, \varphi\rangle$ can be rewritten as follows:

$$\langle u,\, \varphi\rangle = \int_{\mathbb{R}^n} (1+|\xi|^2)^{-s/2}\ \hat{u}(1+|\xi|^2)^{s/2}\ \hat{\check{\varphi}}\, d\xi^n$$

$$\leq \|u\|_{H^{-s}} \|\check{\varphi}\|_{H^s} = \|u\|_{H^{-s}} \|\varphi\|_{H^s},$$

where the last inequality holds due to the Cauchy-Schwarz inequality.

Then the mapping

$$\begin{aligned} U : \mathcal{S} &\to \mathbb{C} \\ \varphi &\mapsto \langle u,\, \varphi\rangle_{\mathcal{S}',\mathcal{S}} \end{aligned}$$

is a linear, continuous mapping from $\mathcal{S}(\mathbb{R}^n)$, a dense subset of $H^s(\mathbb{R}^n)$, into $\mathbb{C}$. This implies that U can be uniquely extended to a linear continuous mapping defined on the entire space $H^s(\mathbb{R}^n)$.

Now let $L \in (H^s(\mathbb{R}^n))^\star$ be given. The goal is to find a $u \in H^s(\mathbb{R}^n)$ such that $\langle u,\, \cdot\rangle = L(\cdot)$.

We consider the following mapping:

$$M : L^2 \to \mathbb{C}$$

$$f \mapsto L\Big(\mathcal{F}^{-1}\big((1+|\xi|^2)^{-s/2} f\big)\Big) = L(w)\ ,$$

where

$$w := \mathcal{F}^{-1}\big((1+|\xi|^2)^{-s/2} f\big)\ .$$

Observe that M is well defined since $\mathcal{F}$ realizes an isometry from $L^2(\mathbb{R}^n)$ into itself and therefore, for any $f \in L^2(\mathbb{R}^n)$ there exists $u \in L^2(\mathbb{R}^n)$ such that $f = \hat{u}$ and then $w \in H^s(\mathbb{R}^n)$ by definition. Take $f \in \mathcal{S}(\mathbb{R}^n)$. Since $(1+|\xi|^2)^{s/2}\, \hat{w} = f \in L^2$, it follows that $w \in H^s(\mathbb{R}^n)$ and we have

$$\|f\|_{L^2} = \|w\|_{H^s}\ .$$

Since $f \in \mathcal{S}(\mathbb{R}^n)$, it also follows that $w \in \mathcal{S}(\mathbb{R}^n)$. Furthermore, we have

$$|M(f)| = |L(w)| \leq C\|w\|_{H^s} = C\|f\|_{L^2}\ ,$$

which means

$$M \in (L^2(\mathbb{R}^n))^\star\ .$$

Since $L^2(\mathbb{R}^n) = (L^2(\mathbb{R}^n))^\star$, there exists $g \in L^2$ such that

$$\begin{aligned} M(f) &= \int_{\mathbb{R}^n} g(\xi)\, f(\xi)\, d\xi^n \\ &= \int_{\mathbb{R}^n} (1+|\xi|^2)^{s/2}\, g(\xi)\ (1+|\xi|^2)^{-s/2} f(\xi)\, d\xi^n \\ &= \big\langle \mathcal{F}\big((1+|\xi|^2)^{s/2}\, g(\xi)\big),\ \mathcal{F}^{-1}\big((1+|\xi|^2)^{-s/2}\, f(\xi)\big)\big\rangle_{\mathcal{S}',\mathcal{S}}\ . \end{aligned}$$

Note that

$$\begin{aligned} (1+|\xi|^2)^{-s/2}\, \mathcal{F}\mathcal{F}\big((1+|\xi|^2)^{s/2}\, g(\xi)\big) &= (1+|\xi|^2)^{-s/2}\ \big((1+|\xi|^2)^{s/2}\, g(\xi)\big)^\vee \\ &= \check{g}(\xi) \in L^2 \\ &\Longrightarrow \mathcal{F}\big((1+|\xi|^2)^{s/2}\, g(\xi)\big) \in H^{-s}(\mathbb{R}^n)\ . \end{aligned}$$

It now follows that

(1) $$L(w) = M(f) = \langle \mathcal{F}\big((1+|\xi|^2)^{s/2} g(\xi),\, w\rangle_{\mathcal{S}',\mathcal{S}} = \langle u,\, w\rangle_{\mathcal{S}',\mathcal{S}}\ ,$$

where

$$u = \mathcal{F}\big((1+|\xi|^2)^{s/2}\, g(\xi)\big),\ u \in H^{-s}(\mathbb{R}^n)\ .$$

This holds for any f in $\mathcal{S}(\mathbb{R}^n)$ and

$$w := \mathcal{F}^{-1}\big((1+|\xi|^2)^{-s/2} f\big)\ .$$

Let $w \in \mathcal{S}$, so there exists an $f \in \mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ such that

$$(1+|\xi|^2)^{-s/2}\, f = \mathcal{F}(w)\ .$$

Hence

$$w \mapsto f = (1+|\xi|^2)^{s/2}\, \mathcal{F}(w)$$

is a bijection from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ and (1) holds for all $w \in \mathcal{S}(\mathbb{R}^n)$. It follows that

$$L(w) = \langle u,\, w\rangle \quad \forall w \in \mathcal{S}.$$

From what has already been proven, namely the fact that $\langle u,\, \cdot\rangle_{\mathcal{S}',\mathcal{S}}$ can be uniquely extended to a linear, continuous mapping from H^s to $\mathbb{C}$, it follows that

$$L(\cdot) = \langle u,\, \cdot\rangle_{H^{-s},H^s}.$$

This concludes the proof of theorem 2.8 . □

2.2 Comparison of H^s with other spaces

Lemma 2.9. *For all $s \geq 0$ and $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$, the following holds:*

$$(1) \qquad (1+|\xi|^2)^s \leq 4^s \left[(1+|\xi-\eta|^2)^s + (1+|\eta|^2)^s\right]$$

For all $s \in \mathbb{R}$ and $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$, the following holds:

$$(2) \qquad (1+|\xi|^2)^s \leq 2^{|s|}\,(1+|\eta|^2)^s\,(1+|\eta-\xi|^2)^{|s|}$$

Proof of Lemma 2.9. Let $s \geq 0$. For all $a, b \in \mathbb{R}_+$, the following inequality holds:

$$(3) \qquad (a+b)^s \leq 2^s(a^s+b^s)$$

Furthermore, we have

$$\begin{aligned}(1+|\xi|^2) &\leq (1+2|\xi-\eta|^2+2|\eta|^2)\\ &\leq 2(1+|\xi-\eta|^2+1+|\eta|^2)\end{aligned}$$

For $a := 1+|\xi-\eta|^2$ and $b := 1+|\eta|^2$, inequality (3) immediately gives the desired inequality (1).

Observe now

$$(2) \quad \Longleftrightarrow \quad (1+|\xi|^2)^s\,(1+|\eta|^2)^{-s} \leq 2^{|s|}\,(1+|\xi-\eta|^2)^{|s|}.$$

Thus, it is enough to prove the case $s \geq 0$ (one may swap ξ and η if necessary). By homogeneity, it suffices to prove the case $s = 1$. (The case $s = 0$ is trivial.)

So let $s = 1$. As already seen, we have

$$\begin{aligned}(1+|\xi|^2) &\leq 2+2|\xi-\eta|^2+2|\eta|^2\\ &\leq 2+2|\xi-\eta|^2+2|\eta|^2+2|\eta|^2|\xi-\eta|^2\\ &= 2(1+|\xi-\eta|^2)\,(1+|\eta|^2).\end{aligned}$$

This concludes the proof of Lemma 2.9. □

Theorem 2.10. *Let $s > \frac{n}{2}+k$, where $k \in \mathbb{N}$. Then $H^s(\mathbb{R}^n)$ embeds continuously into $C^k(\mathbb{R}^n)$: there exists a constant $C_s > 0$ such that*

$$\|u\|_{C^k(\mathbb{R}^n)} \leq C_s\,\|u\|_{H^s(\mathbb{R}^n)}\ .$$

Furthermore, for all $u \in H^s(\mathbb{R}^n)$ and for $|\alpha| \leq k$ we have $|\partial^\alpha u|(x) \to 0$ uniformly as $|x| \to \infty$.

Finally, for $s > \frac{n}{2}$, then $H^s(\mathbb{R}^n)$ is an algebra and there holds

$$(2.53) \qquad \forall\, u, v \in H^s(\mathbb{R}^n) \qquad \|u\,v\|_{H^s(\mathbb{R}^n)} \leq C_s\,\|u\|_{H^s(\mathbb{R}^n)}\,\|v\|_{H^s(\mathbb{R}^n)}\ .$$

Proof of Theorem 2.10. Let $k \in \mathbb{N}$ and $s > \frac{n}{2} + k$. Then for $|\alpha| \leq k$, with $s' = s - k$, we write

$$\xi^\alpha \, \hat{u} = \xi^\alpha \, (1 + |\xi|^2)^{s'/2} \, \hat{u}(1 + |\xi|^2)^{-s'/2} \ ,$$

Observe that since $s' = s - k > n/2$ we have

$$(1 + |\xi|^2)^{-s'/2} \in L^2(\mathbb{R}^n) \ ,$$

By assumptions

$$(1 + |\xi|^2)^{s/2} \, \hat{u} \in L^2(\mathbb{R}^n) \ ,$$

Hence for $|\alpha| \leq k$, $|\alpha| + s' \leq s$ and then

$$\xi^\alpha (1 + |\xi|^2)^{s'/2} \, \hat{u} \in L^2 \ ,$$

Using Cauchy Schwartz inequality we have then
(2.54)

$$\begin{aligned} \|\partial^\alpha u\|_{L^\infty(\mathbb{R}^n)} & \leq \|\xi^\alpha \, \hat{u}\|_{L^1(\mathbb{R}^n)} \leq \|\xi^\alpha (1 + |\xi|^2)^{s'/2} \, \hat{u}\|_{L^2(\mathbb{R}^n)} \; \|(1 + |\xi|^2)^{-s'/2}\|_{L^2(\mathbb{R}^n)} \\ & \leq C_s \; \|u\|_{H^s(\mathbb{R}^n)} \end{aligned}$$

This shows that $u \in C^k(\mathbb{R}^n)$ and, moreover, using theorem 1.1 we have that $\partial^\alpha u$ is continuous and converges uniformly to zero at infinity.

Now let $s > \frac{n}{2}$ and $u, v \in H^s(\mathbb{R}^n)$. We claim that $uv \in H^s$. Applying theorem 1.62 is giving

$$(1 + |\xi|^2)^{s/2} \, \widehat{u\,v} = (1 + |\xi|^2)^{s/2} (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{u}(\eta) \, \hat{v}(\zeta - \eta) \, d\eta,$$

Now we can estimate further using (1) from Lemma V.2.1:

$$\begin{aligned} |(1 + |\xi|^2)^{s/2} \, \widehat{u\,v}(\xi)| \leq C & \int_{\mathbb{R}^n} |\hat{u}\,(\eta)\,(1 + |\eta|^2)^{s/2} \mid |\hat{v}\,(\xi - \eta)| \, d\eta^n \\ & + C \int_{\mathbb{R}^n} |\hat{u}\,(\eta)| \; |\hat{v}\,(\xi - \eta)\,(1 + |\xi - \eta|^2)^{s/2} \mid d\eta^n \ , \end{aligned}$$

From the first part of the proof we have that $|\hat{u}|$ and $|\hat{v}|$ are both in $L^1(\mathbb{R}^n)$. Hence, using Young inequality[5] : $L^1 \star L^2 \hookrightarrow L^2$ we have that

$$\begin{aligned} \|(1 + |\xi|^2)^{s/2} \, \widehat{u\,v}(\xi)\|_{L^2(\mathbb{R}^n)} & \leq C \, \|\hat{u}\,(\xi)\,(1 + |\xi|^2)^{s/2}\|_{L^2(\mathbb{R}^n)} \, \|\hat{v}\|_{L^1(\mathbb{R}^n)} \\ & \leq C \; \|\hat{u}\|_{L^1(\mathbb{R}^n)} \; \|\hat{v}\,(\xi)\,(1 + |\xi|^2)^{s/2}\|_{L^2(\mathbb{R}^n)} \end{aligned}$$

and (2.53) follows from (2.54) for $\alpha = 0$. This concludes the proof of theorem 2.10.
□

We introduce the spaces of Hölder functions $C^{k,\alpha}(\overline{\Omega})$ where Ω is an arbitrary open subset of $\mathbb{R}^n$. Precisely :

[5] See Analysis 3 & 4 and next section.

Definition 2.11. *Let Ω is an arbitrary open subset of $\mathbb{R}^n$. Let $0 < \alpha \leq 1$ and $k \in \mathbb{N}$. We denote by $C^{k,\alpha}(\overline{\Omega})$ the subspace of functions in $C^k(\overline{\Omega})$ such that*

$$\forall |\beta| \leq k \qquad \sup_{x \neq y} \frac{|\partial^\beta u(x) - \partial^\beta u(y)|}{|x-y|^\alpha} < +\infty$$

and we denote

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} := \|u\|_{C^k(\overline{\Omega})} + \sup_{|\beta|=k} \sup_{x \neq y} \frac{|\partial^\beta u(x) - \partial^\beta u(y)|}{|x-y|^\alpha}$$

Exercise 2.12. *Prove that for any $k \in \mathbb{N}$ and $\alpha \in (0,1]$ the quantity $\|\cdot\|_{C^{k,\alpha}(\overline{\Omega})}$ defines a norm and that $C^{k,\alpha}(\overline{\Omega})$ is a Banach space.*

Exercise 2.13. *Let Ω be an open bounded subset of $\mathbb{R}^n$. Prove that for any $k \in \mathbb{N}$ $\alpha, \beta \in (0,1]$ with $\beta < \alpha$ the canonical embedding*

$$C^{k,\alpha}(\overline{\Omega}) \hookrightarrow C^{k,\beta}(\overline{\Omega})$$

is compact.

Theorem 2.14. *Let $s = \dfrac{n}{2} + k + \alpha$, where $k \in \mathbb{N}$ and $\alpha \in (0,1)$. Then it holds that $H^s(\mathbb{R}^n)$ embeds continuously into $C^{k,\alpha}(\mathbb{R}^n)$.*

Proof of Theorem 2.14: First, let us take $k = 0 : s = \dfrac{n}{2} + \alpha, \alpha \in (0,1)$ and assume that $u \in H^s(\mathbb{R}^n)$. Then we write

$$\begin{aligned} u(x+h) - u(x) &= < \delta_{x+h} - \delta_x, u > = \langle \overline{\mathcal{F}}(\delta_{x+h} - \delta_x), \mathcal{F}(u) \rangle \\ &= \left(\int_{\mathbb{R}^n} e^{ix\cdot\xi}\, \hat{u}(\xi) \cdot (e^{i\xi\cdot h} - 1)\, d\xi^n \right) (2\pi)^{-n} \end{aligned}$$

Hence

$$|u(x+h) - u(x)| \leq C \int_{\mathbb{R}^n} |(1+|\xi|^2)^{s/2}\, \hat{u}(\xi)| (1+|\xi|^2)^{-s/2} |e^{i\xi\cdot h} - 1|,$$

Using the Cauchy-Schwarz inequality we get:

$$|u(x-h) - u(x)| \leq C \|u\|_{H^s} \left[\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} |e^{i\xi\cdot h} - 1|^2 \right]^{1/2}$$

$$\leq C \|u\|_{H^s} \left(\left[\int_{|\xi| \leq \frac{1}{|h|}} (1+|\xi|^2)^{-s} |e^{i\xi\cdot h} - 1|^2 \right]^{1/2} + \left[\int_{|\xi| \geq \frac{1}{|h|}} (1+|\xi|^2)^{-s} |e^{i\xi\cdot h} - 1|^2 \right]^{1/2} \right).$$

We first bound

$$\int_{|\xi|\leq\frac{1}{|h|}} (1+|\xi|^2)^{-s}|e^{i\xi\cdot h}-1|^2$$

$$\leq C_{s,n}\int_{|\xi|\leq\frac{1}{|h|}} (1+|\xi|^2)^{-n/2-\alpha}|\xi|^2|h|^2|\xi|^{n-1}\ d|\xi|$$

$$\leq C_{s,n}\int_{|\xi|\leq\frac{1}{|h|}} |\xi|^{-n-2\,\alpha}|\xi|^2|h|^2|\xi|^{n-1}\ d|\xi| = C_{s,n}\ |h|^2\ \left[\frac{t^{2-2\alpha}}{2-2\alpha}\right]_0^{1/|h|} \leq\ C'_{s,n}\ |h|^{2\alpha}\ .$$

Then we bound

$$\int_{|\xi|\geq\frac{1}{|h|}} (1+|\xi|^2)^{-s}|\xi|^{n-1}\ d|\xi| \leq C|h|^{2\alpha}\ .$$

Combining the three previous estimates is giving

$$|u(x+h)-u(x)| \leq C_{s,n}\ \|u\|_{H^s}\ |h|^{\alpha}.$$

Thus, the claim follows for $k=0$. For $k\neq 0$, we have

$$s=\frac{n}{2}+k+\alpha>\frac{n}{2}+k.$$

Let $u\in H^s(\mathbb{R}^n)$, then by Theorem 2.10 $u\in C^k$.

To show the claim $u\in C^{k,\alpha}$, we perform the above calculations for the derivatives with $|\alpha|\leq k$. This concludes the proof of theorem 2.14. □

One can ask why are $\alpha=0$ and $\alpha=1$ excluded? For example, if we perform the above calculation for $\alpha=1$, we get

$$|u(x+h)-u(x)| \leq C|h|(\log\frac{1}{|h|})^{1/2} \not\Rightarrow u\in C^{0,1}$$

For $\alpha=0$ and $k=0$ the following result gives a clear answer.

Theorem 2.15. *There exists a function $u\in H^{n/2}(\mathbb{R}^n)$ such that $u\notin L^\infty(\mathbb{R}^n)$.*

Proof of Theorem 2.15. Consider

$$\hat{u}(\xi) := \frac{(1+|\xi|^2)^{-n/2}}{1+\log(1+|\xi|^2)}$$

It holds that

$$\|u\|^2_{H^{n/2}} = \int_{\mathbb{R}^n} |\hat{u}|^2(\xi)(1+|\xi|^2)^{n/2}\ d\xi^n$$

$$= C\int_{\mathbb{R}} \frac{(1+|\xi|^2)^{-n/2}}{(1+\log(1+|\xi|^2)^2}\ |\xi|^{n-1}\, d|\xi| < \infty.$$

Thus, $u \in H^{n/2}$.

We claim that $u \notin L^\infty(\mathbb{R}^n)$. Observe first $\hat{u} \notin L^1(\mathbb{R}^n)$. Hence, since $\hat{u}(\xi) > 0$,

$$\int e^{-\varepsilon|\xi|} \ \hat{u}(\xi) \ d\xi^n \underset{\varepsilon \to 0}{\longrightarrow} \infty \ . \tag{2.55}$$

We have

$$\int_{\mathbb{R}^n} e^{-\varepsilon|\xi|^2} \hat{u} \ d\xi^n = \int_{\mathbb{R}^n} \mathcal{F}(e^{-\varepsilon|\xi|^2})\, u(x)\, dx^n \ .$$

Exercise 1.3 gives

$$\mathcal{F}\left(e^{-\varepsilon|\zeta|^2}\right) = \frac{1}{(2\varepsilon)^{\frac{n}{2}}}\, e^{-\frac{|x|^2}{4\varepsilon}}$$

Observe that

$$\int_{\mathbb{R}^n} \left|\mathcal{F}\left(e^{-\varepsilon|\zeta|^2}\right)\right| \ dx^n = \int_{\mathbb{R}^n} \frac{1}{(2\varepsilon)^{\frac{n}{2}}}\, e^{-\frac{|x|^2}{4\varepsilon}} \ dx^n = 2^{n/2} \int_{\mathbb{R}^n} e^{-|y|^2} \ dy^n$$

Assuming $u \in L^\infty(\mathbb{R}^n)$, we bound

$$\int \mathcal{F}(e^{-\varepsilon|\xi|^2})\, u(x) \ dx^n \leq \|u\|_\infty \int \mathcal{F}(e^{-\varepsilon|\xi|^2}) \ dx^n = 2^{n/2}\, \|u\|_\infty \int_{\mathbb{R}^n} e^{-|y|^2} \ dy^n \ .$$

This is contradicting (2.55). Hence $u \notin L^\infty(\mathbb{R}^n)$ and this concludes the proof of theorem 2.15. □

Proposition 2.16. *If $s > \sigma$, then $H^s(\mathbb{R}^n) \subset H^\sigma(\mathbb{R}^n)$.*

Proof of Proposition 2.16.

$$\begin{aligned}\|u\|^2_{H^\sigma} &= \int_{\mathbb{R}^n} (1+|\xi|^2)^\sigma \, |\hat{u}(\xi)|^2 \ d\xi^n \\ &\leq \sup_{\xi \in \mathbb{R}^n} (1+|\xi|^2)^{\sigma - s} \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 \ d\xi^n \leq \|u\|^2_{H^s}.\end{aligned}$$

□

Proposition 2.17. *For all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, for any $u \in H^s(\mathbb{R}^n)$ then $\varphi\, u \in H^s(\mathbb{R}^n)$.*

Proof of Proposition 2.17.

$$(1+|\xi|^2)^s \, |\widehat{\varphi\, u}\,(\xi)|^2 = (1+|\xi|^2)^s \ \left|\int_{\mathbb{R}^n} \hat{\varphi}(\xi - \eta) \ \hat{u}(\eta)\, d\eta^n\right|^2 (2\pi)^{-n}.$$

Using equation (2) from Lemma 2.9 we get

$$(1+|\xi|^2)^{s/2} \ |\widehat{\varphi\, u}\,(\xi)| \leq C_s \int_{\mathbb{R}^n} |\hat{\varphi}(\xi - \eta)|(1+|\xi - \eta|^2)^{|s|/2} \ |\hat{u}(\eta)| \ (1+|\eta|^2)^{s/2} \ d\eta^n \ ,$$

we have on one hand since $u \in H^s(\mathbb{R}^n)$

$$|\hat{u}(\eta)| \ (1 + |\eta|^2)^{s/2} \in L^2 \ ,$$

and on the other hand

$$\hat{\varphi}(\xi) \ (1 + |\xi|^2)^{|s|/2} \in \mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \ ,$$

Using again Young inequality $L^1 \star L^2 \hookrightarrow L^2$ we obtain

$$\|(1+|\xi|^2)^{s/2} \, |\widehat{\varphi u}\,(\xi)|\|_{L^2(\mathbb{R}^n)} \leq C_s \left\| |\hat{u}(\eta)| \ (1 + |\eta|^2)^{s/2} \right\|_{L^2(\mathbb{R}^n)} \left\| \hat{\varphi}(\xi) \ (1 + |\xi|^2)^{|s|/2} \right\|_{L^1(\mathbb{R}^n)} \ .$$

This concludes the proof of proposition 2.16. □

2.3 Solving Cauchy Problems for Elliptic Partial Differential Equations in Hilbert-Sobolev Spaces

From now on, let

$$f \in \mathcal{S}'(\mathbb{R}^n) \quad \text{or} \quad f \in \mathcal{S}'(\mathbb{C}),$$

$$u \in \mathcal{S}'(\mathbb{R}^n) \quad \text{or} \quad \mathcal{S}'(\mathbb{C}) \ ,$$

Let $m \in \mathbb{N}^*$ and consider $C_\alpha \in \mathbb{C}$ for any $|\alpha| \leq m$ and assume

$$\exists \ \alpha_0 \qquad |\alpha_0| = m \quad , \qquad C_{\alpha_0} \neq 0.$$

We are now studying linear operators of the form

$$Lu := \sum_{|\alpha| \leq m} C_\alpha \, \partial^\alpha \, u = f,$$

Definition 2.18. *The operator L is called elliptic if*

$$\sum_{|\alpha|=m} C_\alpha \, \xi^\alpha = 0 \quad \Longleftrightarrow \quad \zeta = 0.$$

Example: $L = \Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$, that is, $c_i = 1, \ 1 \leq i \leq n$. Obviously

$$\sum_{i=1}^n \xi_i^2 = 0 \Leftrightarrow \xi = 0,$$

so $L = \Delta$ is elliptic.

On the countrary $L = \partial_t^2 - \Delta$ is not elliptic.

2.3.1 Cauchy Problems in $\mathbb{R}^n$

For $f \in \mathcal{S}'(\mathbb{R}^n)$ we consider the equation

$$-\Delta u + u = f \qquad \Longleftrightarrow \qquad (|\xi|^2 + 1)\,\hat{u} = \hat{f}\ .$$

Theorem 2.19. $\forall s \in \mathbb{R}$ *The map*

$$\begin{aligned} H^{s+2}(\mathbb{R}^n) &\longrightarrow H^s(\mathbb{R}^n) \\ u &\longrightarrow -\Delta u + u \end{aligned}$$

is a continuous isomorphism, and there holds

$$\|u\|_{H^{s+2}(\mathbb{R}^n)} = \| - \Delta u + u\|_{H^s(\mathbb{R}^n)}.$$

Proof of Theorem 2.19. Let $u \in H^{s+2}$, then we have

$$\int_{\mathbb{R}^n} (1 + |\xi|^2)^{s+2}\,|\hat{u}|^2(\xi)\ d\xi^n < \infty.$$

Furthermore, using

$$\mathcal{F}(-\Delta u + u)(\xi) = (|\xi|^2 + 1)\,\hat{u}(\xi)$$

we have

$$\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |(-\Delta u + u)^\wedge|^2\ d\xi^n = \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s+2}\,|\hat{u}|^2(\xi)\ d\xi^n < \infty\ .$$

It follows that

$$\|u\|_{H^{s+2}} = \| - \Delta u + u\|_{H^s}\ .$$

Now suppose that $f \in H^s$. Then $u := \mathcal{F}^{-1}[(1 + |\xi|^2)^{-1}\,\hat{f}(\xi)]$ is in $H^{s+2}(\mathbb{R}^)$ and it solves $-\Delta u + u = f$. This shows the surjectivity of the above map.It is clearly also injective and continuous. This concludes the proof of Theorem 2.19. □

2.3.2 Cauchy-Problem in $\mathbb{R}^n_+$ (Half-space)

We introduce the following notations

$$\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n,\ x_n > 0\}$$

$$x = (x_1, \ldots, x_n) = (x',\, x_n)\ ,$$

where

$$x' = (x_1, \ldots, x_{n-1})\ .$$

Then we define

$$H^s(\mathbb{R}^n_+) := H^s(\mathbb{R}^n)/_{\sim},$$

where

$$u \sim v \Longleftrightarrow supp(u-v) \subset \mathbb{R}^n_-$$

and

$$\|u\|_{H^s(\mathbb{R}^n_+)} := \inf_{v\sim u} \|v\|_{H^s(\mathbb{R}^n)} \ .$$

For $f \in H^s(\mathbb{R}^n_+)$ we aim at solving a problem of the form

$$\text{(2.56)} \qquad \begin{cases} -\Delta u + u = f \qquad \text{in } \mathbb{R}^n_+ \quad \text{i.e.} \\ \\ \forall\, \varphi \in \mathcal{S}(\mathbb{R}^n) \quad \text{and} \quad \mathrm{Supp}(\varphi) \subset \mathbb{R}^n_+ \quad \displaystyle\int_{\mathbb{R}^n} [-\Delta\varphi + \varphi]\ u\ dx^n = 0\ . \end{cases}$$

prescribing $u(x',0) = g(x')$ where g is a fixed given tempered Distribution on $\mathbb{R}^{n-1} = \partial\mathbb{R}^n_+$. Two main questions come then naturally to the reader

i) What is the subspace of $\mathcal{S}'(\mathbb{R}^{n-1})$ in which we can arbitrary choose g so that there exists exactly one solution $u \in H^{s+2}(\mathbb{R}^n_+)$ to this problem ?

ii) what does it means "prescribing" $u(x',0) = g(x')$ on $\mathbb{R}^{n-1} = \partial\mathbb{R}^n_+$?

Giving satisfying answers to these two questions is the main goal of this subsection.

Theorem 2.20. *Let $s > \frac{1}{2}$. Then the linear mapping*

$$T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{n-1}) \quad \textit{(trace)}$$

$$\varphi(x', x_n) \mapsto \varphi(x', 0)$$

can be extended to a surjective, linear, continuous map $T : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$, that is, there exists a constant $C_{s,n}$ depending only on s and n such that, for all $\varphi \in H^s(\mathbb{R}^n)$, we have

$$\text{(2.57)} \qquad \|T\varphi\|_{H^{s-1/2}(\mathbb{R}^{n-1})} \le C_{s,n}\ \|\varphi\|_{H^s(\mathbb{R}^n)}.$$

Remark 2.21. *It is important to insist on the fact that the assumption $s > 1/2$ is necessary : arbitrary L^2 functions on $\mathbb{R}^n$ have no trace in the sense that $T(\varphi_k)$ might not necessarily converge in $\mathcal{S}'(\mathbb{R}^{n-1})$ for a sequence $\varphi_k \in \mathcal{S}(\mathbb{R}^n)$ coonverging strongly in $L^2(\mathbb{R}^n)$. Constructing counter-examples is left as an exercise.*

Remark 2.22. *Observe that the drop of exponent $s \to s - 1/2$ is concomitant to the drop of dimension $n \to n-1$ in a consistent way as illustrated by the following diagram*

$$\begin{array}{ccc} H^s(\mathbb{R}^n) = H^{n/2+\alpha}(\mathbb{R}^n) & \hookrightarrow & C^{o,\alpha}(\mathbb{R}^n), \quad \alpha \in (0,1) \\ \downarrow T & & \downarrow T \\ H^{n/2-1/2+\alpha}(\mathbb{R}^{n-1}) & \hookrightarrow & C^{o,\alpha}(\mathbb{R}^{n-1}). \end{array}$$

Proof of Theorem 2.20.

Claim 1: There exists a constant $C > 0$ such that for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\|T\varphi\|_{H^{s-1/2}(\mathbb{R}^{n-1})} \le C\|\varphi\|_{H^s(\mathbb{R}^n)}.$$

Proof of Claim 1: Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then we have

$$\begin{aligned}
\varphi(x',0) &= (2\pi)^{-n/2}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} e^{ix'\cdot\xi'}\hat{\varphi}(\xi',\xi_n)\,d(\xi')^{n-1}\,d\xi_n \\
&= (2\pi)^{-n/2}\int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'}\left[\int_{\mathbb{R}}\hat{\varphi}(\xi',\xi_n)\,d\xi_n\right]\,d(\xi')^{n-1} \\
&= (2\pi)^{-(n-1)/2}\int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'}\tilde{\varphi}(\xi',0)\,d(\xi')^{n-1}\ ,
\end{aligned}$$

where $(\star\star)$

$$\begin{aligned}
\tilde{\varphi}(\xi',0) &= (2\pi)^{-1/2}\int_{\mathbb{R}}\hat{\varphi}(\xi',\xi_n)\,d\xi_n \\
&= (2\pi)^{-1/2}\int_{\mathbb{R}}(1+|\xi|^2)^{s/2}\hat{\varphi}(\xi',\xi_n)(1+|\xi|^2)^{-s/2}\,d\xi_n
\end{aligned}$$

Thus, using the Cauchy-Schwarz inequality, we obtain

$$(\star)\quad \begin{aligned}
(1+|\xi'|^2)^{s-1/2}|\tilde{\varphi}(\xi',0)|^2 \ \le\ & \frac{(1+|\xi'|^2)^{s-1/2}}{2\pi}\int_{\mathbb{R}}(1+|\xi|^2)^{s}|\hat{\varphi}(\xi',\xi_n)|^2\,d\xi_n \\
& \times\int_{\mathbb{R}}(1+|\xi|^2)^{-s}\,d\xi_n.
\end{aligned}$$

we bound

$$\begin{aligned}
\int_{\mathbb{R}}(1+|\xi|^2)^{-s}\,d\xi_n &= \int_{\mathbb{R}}\frac{1}{(1+|\xi'|^2+|\xi_n|^2)^s}\,d\xi_n \quad (<\infty, \text{since } s>1/2) \\
&= \int_{\mathbb{R}}\frac{\sqrt{1+|\xi'|^2}}{(1+|\xi'|^2)^s(1+\lambda^2)^s}\,d\lambda \\
&= C_1(1+|\xi'|^2)^{-s+1/2} \quad \text{with} \quad C_1 = \int_{\mathbb{R}}(1+\lambda^2)^{-s}\,d\lambda,
\end{aligned}$$

where in the second-to-last step, the following variable transformation was used:

$$\xi_n = (1+|\xi'|^2)^{1/2}\lambda$$

$$d\xi_n = (1+|\xi'|^2)^{1/2}d\lambda.$$

Integration of $(\star)$ with respect to ξ' finally yields

$$(\star\star\star) \qquad \|(1+|\xi'|^2)^{s-1/2}|\tilde{\varphi}(\xi',0)|^2\|_{L^1} \leq \; C\,\|\varphi\|^2_{H^s(\mathbb{R}^n)}.$$

From $(\star\star)$ it is further clear that $\widehat{T\varphi} = \tilde{\varphi}$. Thus $(\star\star\star)$ is proving the claim.

Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ we have proved (2.57). It remains to prove for the surjectivity of T.

We are now proving the following claim.

Claim 2: There exists $C > 0$ such that for all $g \in H^{s-1/2}(\mathbb{R}^{n-1})$, it holds

$$\exists \quad \varphi \in H^s(\mathbb{R}^n) \qquad \text{s. t.} \quad T(\varphi) = g$$

and moreover

$$\|\varphi\|_{H^s(\mathbb{R}^n)} \leq C\,\|g\|_{H^{s-1/2}(\mathbb{R}^{n-1})}\,.$$

*Proof of Claim 2:*First we consider the case $g \in \mathcal{S}(\mathbb{R}^{n-1})$. From the proof of Claim 1, we know that the Fourier transform with respect to the first $n-1$ variables of φ is exactly $\tilde{\varphi}$. Thus, we seek $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\tilde{g}(\xi') = (2\pi)^{-1/2} \int_{\mathbb{R}} \hat{\varphi}(\xi', \xi_n)\, d\xi_n.$$

We now define

$$\hat{\varphi}(\xi) := C_1\, \frac{1}{(1+|\xi|^2)^{1/2}}\; e^{-\frac{|\xi_n|^2}{1+|\xi'|^2}}\, \tilde{g}(\xi')\,,$$

where C_1 will be chosen later. We leave as an exercise the proof that $\hat{\varphi}$ and hence φ are in $\mathcal{S}(\mathbb{R}^n)$. Then we have

$$\begin{aligned}\int_{\mathbb{R}} \hat{\varphi}(\xi)\, d\xi_n &= C_1\, \tilde{g}(\xi') \int_{\mathbb{R}} \frac{e^{-\frac{|\xi_n|^2}{1+|\xi'|^2}}}{(1+|\xi'|^2+|\xi_n|^2)^{1/2}}\, d\xi_n \\ &= C_1\; \tilde{g}(\xi')\; C_0.\end{aligned}$$

Now choose C_1 such that $C_1 \cdot C_0 = (2\pi)^{1/2}$ and from the first part of the proof that

$$T\,\varphi = g.$$

Observe that

$$\begin{aligned}\int_{\mathbb{R}^n} (1+|\xi|^2)^s\, |\hat{\varphi}|^2\; d\xi^n &\leq |C_1|^2 \int_{\mathbb{R}^{n-1}} |\tilde{g}|^2(\xi') \left[\int_{\mathbb{R}} \frac{e^{-2\frac{|\xi_n|^2}{1+|\xi'|^2}}}{(1+|\xi|^2)^{1-s}} d\xi_n\right] d(\xi')^{n-1} \\ &\leq |C_1|^2 \int_{\mathbb{R}^{n-1}} (1+|\xi'|^2)^{s-1/2}\; |\tilde{g}|^2(\xi') \left[\int_{\mathbb{R}} \frac{e^{-2\lambda^2}}{(1+\lambda^2)^{1-s}} d\lambda\right] d(\xi')^{n-1}\end{aligned}$$

and we obtain

$$\|\varphi\|_{H^s(\mathbb{R}^n)} \leq C\,\|g\|_{H^{s-1/2}(\mathbb{R}^{n-1})}\,.$$

Let now $g \in H^{s-1/2}(\mathbb{R}^{n-1})$, since $\mathcal{S}(\mathbb{R}^{n-1})$ is dense in $H^{s-1/2}(\mathbb{R}^{n-1})$ we choose $g_k \in \mathcal{S}(\mathbb{R}^{n-1})$ such that

$$g_k \longrightarrow g \qquad \text{in } H^{s-1/2}(\mathbb{R}^{n-1}) \ .$$

We consider the sequence $\varphi_k \in \mathcal{S}(\mathbb{R}^n)$ given by

$$\hat{\varphi}_k(\xi) := C_1 \, \frac{1}{(1+|\xi|^2)^{1/2}} \, e^{-\frac{|\xi_n|^2}{1+|\xi'|^2}} \, \tilde{g}_k(\xi') \ ,$$

Because of the previous estimates, φ_k is a Cauchy sequence in $H^s(\mathbb{R}^n)$ and converges to a limit $\varphi \in H^s(\mathbb{R}^n)$ by the continuity of the operator T we just constructed we have

$$T\,\varphi = g.$$

Hence T is surjective and this concludes the proof of Claim 2 as well as the proof of theorem 2.20. $\square$

Proposition 2.23. *Let $s > \frac{1}{2}$ and let $\varphi \in H^s(\mathbb{R}^n)$ such that supp $\varphi \subset \mathbb{R}^n_+$. Then it holds*

$$T\,\varphi \equiv 0 \ .$$

From this proposition one deduces immediately the following corollary.

Corollary 2.24. *If $u \sim_{H^s} v$, then $Tu = Tv$. Consequently the trace is well-defined on*

$$H^s(\mathbb{R}^n_+) \quad \textit{for} \quad s > \frac{1}{2} \ .$$

Proof of Proposition 2.23. Let $h > 0$, and denote

$$\tau_h\,\varphi(x) = \varphi(x', x_n - h) \ .$$

First we claim

Claim 1: $\tau_{-h}\,\varphi \to \varphi$ in H^s.

Proof of Claim 1 We have

$$\|\tau_{-h}\,\varphi - \varphi\|^2_{H^s} = \int_{\mathbb{R}^n} (1+|\xi|^2)^s \left|\hat{\varphi}(\xi)\left(e^{i\zeta_n \cdot h} - 1\right)\right|^2 \, d\xi^n,$$

where

$$(1+|\xi|^2)^s \left|\hat{\varphi}(\xi)\left(e^{i\xi_n\,h} - 1\right)\right|^2 \to 0 \quad \text{everywhere}$$

and since $\varphi \in H^s(\mathbb{R}^n)$

$$(1+|\xi|^2)^s \left|\hat{\varphi}(\xi)\left(e^{i\xi_n\,h} - 1\right)\right|^2 \leq 4\,(1+|\xi|^2)^s\,|\hat{\varphi}(\xi)|^2 \in L^1(\mathbb{R}^n) \ ,$$

claim 1 follows by calling upon the dominated convergence theorem, .

Further, let $\chi \in C_0^\infty$ with $\operatorname{supp}\chi \subset B_1(0)$ and $\int \chi(x)\, dx^n = 1$. As in previous subsections, we denote

$$\chi_\varepsilon(x) := \frac{1}{\varepsilon^n}\chi\left(\frac{x}{\varepsilon}\right) .$$

We claim

Claim 2:

$$\forall u \in H^s(\mathbb{R}^n) \qquad \chi_\varepsilon \star u \longrightarrow \qquad \text{in } H^s(\mathbb{R}^n)$$

Proof of Claim 2 We have on one hand

$$\|\chi_\varepsilon \star u - u\|^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|\xi|^2)^s\, |\hat{\chi_\varepsilon}(\xi) - 1|^2\, |\hat{u}|^2(\xi)\, d\xi^n .$$

On the other hand

$$\hat{\chi_\varepsilon}(\xi) = \hat{\chi}(\varepsilon\, \xi) .$$

Since $\chi \in C_0^\infty(B_1(0))$, $\hat{\chi} \in \mathcal{S}(\mathbb{R}^n)$. Moreover, since $\int \chi dx^n = 1$, $\hat{\chi}(0) = 1$. Hence

$$\forall \xi \in \mathbb{R}^n \qquad (1+|\xi|^2)^s\, |\hat{\chi_\varepsilon}(\xi) - 1|^2\, |\hat{u}|^2(\xi) \longrightarrow 0$$

We have moreover

$$(1+|\xi|^2)^s\, |\hat{\chi_\varepsilon}(\xi) - 1|^2\, |\hat{u}|^2(\xi) \le (1+|\xi|^2)^s\, [\|\hat{\chi}\|_\infty + 1]\, |\hat{u}|^2(\xi) \ \in L^1(\mathbb{R}^n) .$$

Dominated convergence again is implying claim 2.

For $h > 0$ the support of φ_{-h} is included in $\{\xi_n \ge h\}$. Hence for $0 < \varepsilon < h$ we have $\operatorname{Supp}(\chi_\varepsilon \star \varphi_{-h}) \subset \{\xi_n \ge h - \varepsilon\}$. Since $\chi_\varepsilon \star \varphi_{-h} \in C^\infty(\mathbb{R}^n)$ we have then

$$\forall\ 0 < \varepsilon < h \qquad T(\chi_\varepsilon \star \varphi_{-h}) \equiv 0 \tag{2.58}$$

Because of claim 1 and claim 2 we have

$$\forall\, h \in \mathbb{R} \quad \chi_\varepsilon \star \varphi_{-h} \ \longrightarrow\ \varphi_{-h} \quad \text{in } H^s(\mathbb{R}^n) \quad \text{and} \quad \varphi_{-h} \ \longrightarrow\ \varphi \quad \text{in } H^s(\mathbb{R}^n) .$$

Hence using a diagonal argument we can find $\varepsilon_h \to 0$ such that $0 < \varepsilon_h < h$ and such that

$$\chi_{\varepsilon_h} \star \varphi_{-h} \ \longrightarrow\ \varphi \quad \text{in } H^s(\mathbb{R}^n) \qquad \text{as } h \to 0 .$$

From (2.58) moreover we have $T(\chi_{\varepsilon_h} \star \varphi_{-h}) \equiv 0$. Using the continuity of T established in the first part of the proof of the proposition we can pass to the limit in this last identity in order to obtain $T(\varphi) = 0$. this concludes the proof of proposition 2.23 □

Theorem 2.25. *Let $s + 2 > \frac{1}{2}$. The mapping*

$$L : H^{s+2}(\mathbb{R}^n_+) \to H^{s+3/2}(\mathbb{R}^{n-1}) \times H^s(\mathbb{R}^n_+)$$

$$u \mapsto (Tu, -\Delta u + u)$$

is a continuous isomorphism, meaning there exists a constant $C > 0$ such that

$$\|u\|_{H^{s+2}(\mathbb{R}^n_+)} \le C\left[\|Tu\|_{H^{s+\frac{3}{2}}(\mathbb{R}^{n-1})} + \|-\Delta u + u\|_{H^s(\mathbb{R}^n_+)}\right] .$$

Remark 2.26. *This theorem states that the problem*

$$\text{(2.59)} \qquad \begin{cases} -\Delta u + u = f \in H^s\ (\mathbb{R}^n_+) \\ \\ Tu = g \in H^{s+3/2}\ (\mathbb{R}^{n-1}) \end{cases}$$

has a unique solution $u \in H^{s+2}\ (\mathbb{R}^n)$ *for* $s+2 > 1/2$.

Proof of Theorem 2.25. We first consider the following problem. Let $g \in \mathcal{S}(\mathbb{R}^{n-1})$ we are looking for a solution w of the following problem

$$\text{(2.60)} \qquad \begin{cases} -\Delta w + w = 0 \qquad \text{on } \mathbb{R}^n_+ \\ \\ Tw = g \qquad \text{on } \partial\,\mathbb{R}^n_+\ . \end{cases}$$

In the following, we shall denote by $\tilde{w}$ the Fourier transform of w with respect to the first $n-1$ variables x'.

Assume there exists a solution to the problem (2.60), applying formally[6] the Fourier transform with respect to the first $n-1$ variables and restricting to the hyperplane $\{x_n = 0\}$, since everything is smooth we obtain that $\tilde{w}(\xi', x_n)$ is a classical solution to the following problem

$$\begin{cases} -\dfrac{\partial^2}{\partial x_n^2}\tilde{w} + \left(|\xi'|^2 + 1\right)\tilde{w} = 0 \qquad \text{on } \mathbb{R}^n_+ \\ \tilde{w}(\xi', 0) = \tilde{g}(\xi') \qquad \text{on } \partial\,\mathbb{R}^n_+\ . \end{cases}$$

Explicit computations give

$$\text{(2.61)} \qquad \tilde{w}(\xi', x_n) = e^{-x_n\sqrt{|\xi'|^2+1}}\tilde{g}(\xi')\ .$$

We consider[7] $\psi \in \mathcal{S}(\mathbb{R})$ such that $\psi(t) := e^{-t}$, $t \geq 0$ and we introduce the function $\tilde{w}(\zeta', x_n)$ given by

$$\tilde{w}(\xi', x_n) := \psi\left(x_n\sqrt{|\xi'|^2+1}\right)\ \tilde{g}(\xi').$$

For every $\xi' \in \mathbb{R}^{n-1}$ the map $x_n \to \tilde{w}(\xi', x_n)$ is obviously in $\mathcal{S}(\mathbb{R})$ and we can take

[6] We have no information of such a solution that would permit to justify the use of Fourier with respect to the $n-1$ first variables.

[7] Such a function ψ can be constructed as follows:

$$\psi(t) := e^{-t}\,\chi * \mathbf{1}_{t>-2}$$

where $\mathbf{1}_{t>-2}$ is the characteristic function of the set of reals larger than -2 and χ is a function in $C_0^\infty((-1,1))$ such that $\int_{\mathbb{R}}\chi(t)\,dt = 1$.

its Fourier transform. We denote

$$\varphi(\xi',\xi_n) := (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-i\,\xi_n\,x_n}\,\tilde{w}(\xi',x_n)\,dx_n$$

$$= (2\pi)^{-1/2}\,\tilde{g}(\xi')\int_{\mathbb{R}}\psi\left(x_n\,\sqrt{|\xi'|^2+1}\right)\,e^{-i\xi_n\,x_n}\,dx_n$$

$$= (2\pi)^{-1/2}\,\tilde{g}(\xi')\int_{\mathbb{R}}\psi(z)\;e^{-i\,\xi_n\,\frac{z}{\sqrt{|\xi'|^2+1}}}\,\frac{1}{\sqrt{|\xi'|^2+\lambda}}\,dz$$

$$= \tilde{g}(\xi')\;\hat{\psi}\left(\frac{\xi_n}{\sqrt{|\xi'|^2+1}}\right)\frac{1}{\sqrt{|\xi'|^2+1}}$$

and thus

$$\int_{\mathbb{R}^n}(1+|\xi|^2)^{s+2}|\varphi|^2(\xi)\;d\xi^n = \int_{\mathbb{R}^n}\frac{(1+|\xi'|^2+|\xi_n|^2)^{s+2}}{|\xi'|^2+1}\;|\tilde{g}(\xi')|^2\left|\hat{\psi}\left(\frac{\xi_n}{\sqrt{|\xi'|^2+1}}\right)\right|^2\;d\xi^n$$

$$\le C_s\;\int_{\mathbb{R}^n}\frac{(1+|\xi'|^2)^{s+2}}{\sqrt{|\xi'|^2+1}}|\tilde{g}(\xi')|^2\;\int_{\mathbb{R}}|\hat{\psi}(z)|^2\;dz$$

$$+C_s\;\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}\frac{|\xi_n|^{2\cdot(s+2)}}{|\xi'|^2+1}\left|\hat{\psi}\left(\frac{\xi_n}{\sqrt{|\xi'|^2+1}}\right)\right|^2\;|\tilde{g}(\xi')|^2\,d\xi_n\,d(\xi')^{n-1}$$

$$\le C_s\;\|\hat{\psi}\|^2_{L^2(\mathbb{R})}\;\|g\|^2_{H^{s+3/2}}$$

$$+\int_{\mathbb{R}^{n-1}}(1+|\xi'|^2)^{s+3/2}\;|\hat{g}(\xi')|^2\;d(\xi')^{n-1}\;\int_{\mathbb{R}}|z|^{2(s+2)}\;|\hat{\psi}(z)|^2\,dz$$

Hence we have proven the existence of a constant $C_s>0$ depending only on s such that

$$\|\mathcal{F}^{-1}(\varphi)\|_{H^{s+2}(\mathbb{R}^n)}\le C_s\;\|g\|_{H^{s+3/2}(\mathbb{R}^{n-1})}\;.$$

Observe that since we are still working under the assumption $g\in\mathcal{S}(\mathbb{R}^{n-1})$, $\varphi\in\mathcal{S}(\mathbb{R}^n)$. Going backwards in the argument we observe that $w:=\mathcal{F}^{-1}(\varphi)$ solves (2.60) and there holds

$$\|w\|_{H^{s+2}(\mathbb{R}^n)}\le C_{s,n}\;\|g\|_{H^{s+3/2}(\mathbb{R}^{n-1})}\;. \tag{2.62}$$

Using one more time the density of $\mathcal{S}(\mathbb{R}^{n-1})$ in $H^{s+3/2}(\mathbb{R}^{n-1})$ and the continuity of the trace operation from $H^{s+2}(\mathbb{R}^n)$ into $H^{s+3/2}(\mathbb{R}^{n-1})$ given by theorem 2.20 we deduce that (2.60) admits a solution w satisfying (2.62) for any $g\in H^{s+3/2}(\mathbb{R}^{n-1})$.

Consider now $f\in H^s(\mathbb{R}^n)$ and $g\in H^{s+3/2}(\mathbb{R}^{n-1})$ arbitrary. From (2.19) there exists a unique $w_f\in H^{s+2}(\mathbb{R}^n)$ solving

$$-\Delta w_f+w_f=f\qquad\text{in }\mathcal{S}'(\mathbb{R}^n)\qquad\text{and}\qquad\|w_f\|_{H^{s+2}(\mathbb{R}^n)}\le C_{s,n}\;\|f\|_{H^s(\mathbb{R}^n)}$$

From theorem 2.20 there exists a constant $C_{s,n}$ depending only on s and n such that

$$\|Tw_f\|_{H^{s+3/2}(\mathbb{R}^{n-1})}\le C_{s,n}\;\|w_f\|_{H^{s+2}(\mathbb{R}^n)}\le C'_{s,n}\;\|f\|_{H^s(\mathbb{R}^n)}$$

From the first part of the proof we have the existence of $w' \in H^{s+2}(\mathbb{R}^n)$ solving

$$
(2.63) \qquad \begin{cases} -\Delta w' + w' = 0 \\ \\ T w' = g - T w_f \qquad \text{on } \partial\,\mathbb{R}^n_+ \ , \end{cases}
$$

moreover
(2.64)

$$
\|w'\|_{H^{s+2}(\mathbb{R}^n)} \le C_{s,n}\ \|g - T w_f\|_{H^{s+3/2}(\mathbb{R}^{n-1})} \le C_{s,n}\ \|g\|_{H^{s+3/2}(\mathbb{R}^{n-1})} + C'_{s,n}\ \|f\|_{H^s(\mathbb{R}^n)} \ .
$$

By linearity $w := w' + w_f$ solves

$$
(2.65) \qquad \begin{cases} -\Delta w + w = f \qquad \text{in } \mathbb{R}^n_+ \\ \\ T w = g \qquad \text{on } \partial\,\mathbb{R}^n_+ \ . \ , \end{cases}
$$

moreover

$$
(2.66) \qquad \|w\|_{H^{s+2}(\mathbb{R}^n)} \le C_{s,n}\ \|g\|_{H^{s+3/2}(\mathbb{R}^{n-1})} + C'_{s,n}\ \|f\|_{H^s(\mathbb{R}^n)} \ .
$$

By linearity, the uniqueness of the solution to (2.60) will follow if we can prove that any solution $w \in H^{s+2}(\mathbb{R}^n_+)$ of

$$
(2.67) \qquad \begin{cases} -\Delta w + w = 0 \qquad \text{in } \mathbb{R}^n_+ \\ \\ T w = 0 \qquad \text{on } \partial\,\mathbb{R}^n_+ \ . \ , \end{cases}
$$

is zero. Let w be a solution in $H^{s+2}(\mathbb{R}^n_+)$ of (2.67). We introduce

$$
\mathring{w} := \begin{cases} w(x', x_n) \qquad \text{for } x_n \ge 0 \\ -\,w(x', -x_n) \qquad \text{for } x_n \le 0 \end{cases}
$$

We claim that $\mathring{w}$ is a solution of

$$
(2.68) \qquad -\Delta \mathring{w} + \mathring{w} = 0 \qquad \text{in } \mathcal{S}'(\mathbb{R}^n) \ .
$$

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Introduce a smooth even "cut-off" function on $\mathbb{R}$ ($\chi(t) = \chi(-t)$) that we assume to be supported in $[-1, +1]$ and such that χ is equal to 1 on $[-1/2, 1/2]$. For any $0 < \varepsilon < 1$ we introduce $\chi_\varepsilon(t) := \chi(t/\varepsilon)$. We write

$$
\int_{\mathbb{R}^n} [-\Delta\varphi + \varphi]\, \mathring{w}\ dx^n = \int_{\mathbb{R}^n} [-\Delta\varphi + \varphi]\, \chi_\varepsilon(x_n)\, \mathring{w}\ dx^n + \int_{\mathbb{R}^n} [-\Delta\varphi + \varphi]\, (1 - \chi_\varepsilon(x_n))\, \mathring{w}\ dx^n
$$

Observe first that

$$
\left| \int_{\mathbb{R}^n} [-\Delta\varphi + \varphi]\, \chi_\varepsilon\, \mathring{w}\ dx^n \right| \le \|\chi\|_\infty\ \| - \Delta\varphi + \varphi\|_{L^2(\mathbb{R}^n)}\ \|\mathring{w}\|_{L^2(|x_n| \le \varepsilon)}
$$

Since $w \in L^2(\mathbb{R}^n)$

$$
\|\mathring{w}\|_{L^2(|x_n| \le \varepsilon)} = 2\, \|w\|_{L^2(0 \le x_n \le \varepsilon)} \ \longrightarrow\ 0 \qquad \text{as } \varepsilon \to 0 \ .
$$

Hence

$$\int_{\mathbb{R}^n} [-\Delta\varphi + \varphi]\, \mathring{w}\ dx^n = o_\varepsilon(1) + \int_{\mathbb{R}^n} [-\Delta\varphi + \varphi]\, (1 - \chi_\varepsilon)\, \mathring{w}\ dx^n\ . \tag{2.69}$$

Now we write

$$\begin{aligned}\int_{\mathbb{R}^n} [-\Delta\varphi + \varphi]\, (1 - \chi_\varepsilon)\, \mathring{w}\ dx^n &= \int_{\mathbb{R}^n} [-\Delta(\varphi\, (1 - \chi_\varepsilon)) + \varphi\, (1 - \chi_\varepsilon)]\, \mathring{w}\ dx^n \\ &- \int_{\mathbb{R}^n} \nabla\varphi \cdot \nabla\chi_\varepsilon\, \mathring{w}\ dx^n - \int_{\mathbb{R}^n} \varphi\, \Delta\chi_\varepsilon\, \mathring{w}\ dx^n\end{aligned} \tag{2.70}$$

First we observe that

$$\begin{aligned}&\int_{\mathbb{R}^n} [-\Delta(\varphi\, (1 - \chi_\varepsilon)) + \varphi\, (1 - \chi_\varepsilon)]\, \mathring{w}\ dx^n \\ &= \int_{\mathbb{R}^n_+} [-\Delta(\varphi\, (1 - \chi_\varepsilon)) + \varphi\, (1 - \chi_\varepsilon)](x)\ w(x)\ dx^n \\ &- \int_{\mathbb{R}^n_-} [-\Delta(\varphi\, (1 - \chi_\varepsilon)) + \varphi\, (1 - \chi_\varepsilon)](x)\ w(x, -x_n)\ dx^n\end{aligned}$$

Using the fact that for any $\psi \in \mathcal{S}(\mathbb{R}^n)$ $-\Delta_x(\psi)(x', -x_n) = -\Delta_x(\psi(x', -x_n))$, from (2.56), we deduce that

$$\begin{aligned}&- \int_{\mathbb{R}^n_-} [-\Delta(\varphi\, (1 - \chi_\varepsilon)) + \varphi\, (1 - \chi_\varepsilon)](x)\ w(x, -x_n)\ dx^n \\ &= - \int_{\mathbb{R}^n_+} [-\Delta(\varphi(x', -x_n)\, (1 - \chi_\varepsilon(x', -x_n))) + \varphi(x', -x_n)\, (1 - \chi_\varepsilon(x', -x_n))](x)\ w(x)\ dx^n = 0\ .\end{aligned}$$

Hence finally, using that w solves $-\Delta w + w = 0$ on $\mathbb{R}^n_+$ we obtain

$$\int_{\mathbb{R}^n} [-\Delta(\varphi\, (1 - \chi_\varepsilon)) + \varphi\, (1 - \chi_\varepsilon)](x)\ \mathring{w}(x)\ dx^n = 0\ . \tag{2.71}$$

We have moreover

$$\begin{aligned}\int_{\mathbb{R}^n} \nabla\varphi \cdot \nabla\chi_\varepsilon\, \mathring{w}\ dx^n &= -\frac{1}{\varepsilon} \int_{-\varepsilon}^{-\varepsilon/2} \chi'\left(\frac{t}{\varepsilon}\right)\ dt \int_{x_n = t} T_{-t}(w(x', -t))\, \partial_{x_n}\varphi\ d(x')^{n-1} \\ &+ \frac{1}{\varepsilon} \int_{\varepsilon/2}^{\varepsilon} \chi'\left(\frac{t}{\varepsilon}\right)\ dt \int_{x_n = t} T_t(w(x', t))\, \partial_{x_n}\varphi\ d(x')^{n-1}\end{aligned}$$

where for any $t \in \mathbb{R}$, we denote $T_t(w(x', t)) := T(\tau_{(0,-t)} w)$ the trace on $x_n = t$ of w. Using the continuity of the trace together with the fact that

$$\tau_{(0,-t)} w \ \longrightarrow\ w \qquad \text{in } H^{s+2}(\mathbb{R}^n)$$

the continuity of T from $H^{s+2}(\mathbb{R}^n)$ into $H^{s+3/2}(\mathbb{R}^{n-1})$ together with the continuity of the embedding $H^{s+3/2}(\mathbb{R}^{n-1}) \hookrightarrow L^2(\mathbb{R}^{n-1})$ we have

$$\lim_{t \to 0} \|T_t(w(x', t))\|_{L^2(\mathbb{R}^{n-1})} = 0 \tag{2.72}$$

Hence we deduce

$$\left|\frac{1}{\varepsilon}\int_{-\varepsilon}^{-\varepsilon/2}\chi'\left(\frac{t}{\varepsilon}\right)\ dt\int_{x_n=t}T_{-t}(w(x',-t))\,\partial_{x_n}\varphi\ d(x')^{n-1}\ \right|$$
$$\leq o_\varepsilon(1)\ \|\chi'\|_\infty\ \sup_{t\in[-\varepsilon,-\varepsilon/2]}\|\partial_{x_n}\varphi(x',t)\|_{L^2(\mathbb{R}^{n-1})}$$

and we bound

$$\int_{\mathbb{R}^{n-1}}|\partial_{x_n}\varphi(x',t)|^2\ d(x')^{n-1}=\int_{\mathbb{R}^{n-1}}\frac{1}{(1+|x'|^n)^2}\ d(x')^{n-1}\|(1+|x'|^n)\,\partial_{x_n}\varphi(x',t)\|_\infty^2$$
$$\leq\ C_n\ \mathcal{N}_{n+1}(\varphi)^2$$

Hence we deduce

$$\int_{\mathbb{R}^n}\nabla\varphi\cdot\nabla\chi_\varepsilon\,\mathring{w}\ dx^n=o(1)\ . \tag{2.73}$$

Now we write, using the fact that $\chi''(t)=\chi''(-t)=\chi''(|t|)$

$$\int_{\mathbb{R}^n}\varphi\,\Delta\chi_\varepsilon\,\mathring{w}\ dx^n=\frac{1}{\varepsilon^2}\int_{\mathbb{R}^n}\varphi(x',x_n)\ \chi''\left(\frac{|x_n|}{\varepsilon}\right)\ \mathring{w}\ dx^n$$
$$=\frac{1}{\varepsilon^2}\int_{\mathbb{R}^n_+}[\varphi(x',x_n)-\varphi(x',-x_n)]\ \chi''\left(\frac{|x_n|}{\varepsilon}\right)\ w(x',x_n)\ dx^n$$
$$=\frac{2}{\varepsilon^2}\int_{|x_n|<\varepsilon}dx_n\int_{\mathbb{R}^{n-1}}|x_n|\ \partial_{x_n}\varphi(x',t_{x_n}(x'))\ \chi''\left(\frac{|x_n|}{\varepsilon}\right)\ w(x',x_n)\ d(x')^{n-1}$$

where $t_{x_n}(x')\in[-x_n,x_n]$. We then bound using (2.72)

$$\left|\int_{\mathbb{R}^n}\varphi\,\Delta\chi_\varepsilon\,\mathring{w}\ dx^n\right|$$
$$\leq 2\,\|\chi''\|_\infty\ \sup_{|x_n|\leq\varepsilon}\|\partial_{x_n}\varphi(x',t_{x_n}(x'))\|_{L^2(\mathbb{R}^{n-1})}\ \sup_{|t|\leq\varepsilon}\|T_t(w(x',t))\|_{L^2(\mathbb{R}^{n-1})}$$
$$\leq o_\varepsilon(1)\ \|\|\partial_{x_n}\varphi(x',x_n))\|_{L^\infty_{x_n}([-\varepsilon,\varepsilon])}\|_{L^2(\mathbb{R}^{n-1})}\ .$$

We write

$$\|\|\partial_{x_n}\varphi(x',x_n))\|_{L^\infty_{x_n}([-\varepsilon,\varepsilon])}\|^2_{L^2(\mathbb{R}^{n-1})}=\int_{\mathbb{R}^{n-1}}\|\partial_{x_n}\varphi(x',x_n))\|^2_{L^\infty_{x_n}([-\varepsilon,\varepsilon])}\ d(x')^{n-1}$$
$$\leq\int_{\mathbb{R}^{n-1}}\frac{d(x')^{n-1}}{(1+|x'|^n)^2}\ \|(1+|x'|^n)\,\partial_{x_n}\varphi(x',x_n)\|^2_{L^\infty(\mathbb{R}^n)}\leq C_n\ \mathcal{N}_n(\varphi)^2$$

Combining the two previous bounds we finally obtain

$$\int_{\mathbb{R}^n}\varphi\,\Delta\chi_\varepsilon\,\mathring{w}\ dx^n=o_\varepsilon(1) \tag{2.74}$$

Combining (2.69), (2.70), (2.71), (2.73) and (2.74) we obtain the claim (2.68). Hence $w=0$ and the solution to (2.63) in $H^{s+2}(\mathbb{R}^n_+)$ is unique. This concludes the proof of the theorem 2.25. □

3 Fundamental Properties of L^p spaces

3.1 Hölder, Minkowski and Young inequalities

For any open set $\Omega \subset \mathbb{R}^n$ we refer to Analysis 3 for the definition $L^p(\Omega)$. We recall the Hölder inequality.

Theorem 3.1. *Let $1 \le p \le +\infty$ and Ω an open set of $\mathbb{R}^n$. We introduce $p' \in [1,\infty]$ given by*[8]

$$1 = \frac{1}{p} + \frac{1}{p'} \ .$$

Let $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$. Then $f\, g \in L^1(\Omega)$ and

$$\int_\Omega |f\, g|(x)\ dx^n \le \|f\|_{L^p(\Omega)}\ \|g\|_{L^{p'}(\Omega)} \ . \tag{3.75}$$

where

$$\|f\|_{L^p(\Omega)} := \left[\int_\Omega |f(x)|^p\ dx^n\right]^{1/p} \qquad \text{and} \qquad \|g\|_{L^{p'}(\Omega)} := \left[\int_\Omega |g(x)|^{p'}\ dx^n\right]^{1/p'}$$

Proof of theorem 3.1 The cases $p = 1, \infty$ are straightforward. Hence we restrict to the case $1 < p < +\infty$. The concavity of the logarithmic function gives for any x such that $|f(x)| > 0$ and $|g(x)| > 0$

$$\frac{1}{p}\ \log|f(x)|^p + \frac{1}{p'}\ \log|g(x)|^{p'} \le \log\left[\frac{1}{p}\,|f(x)|^p + \frac{1}{p'}\,|g(x)|^{p'}\right]$$

This implies

$$|f\, g|(x) \le \frac{1}{p}\,|f(x)|^p + \frac{1}{p'}\,|g(x)|^{p'}$$

Obviously this last inequality extends to the case $|f(x)| = 0$ or $|g(x)| = 0$ and we deduce

$$\int_\Omega |f\, g|(x)\ dx^n \le \frac{1}{p}\int_\Omega |f(x)|^p\ dx^n + \frac{1}{p'}\int_\Omega |g(x)|^{p'}\ dx^p$$

Applying this inequality to the pair (tf, g) instead of (f, g) is giving

$$\int_\Omega |f\, g|(x)\ dx^n \le \frac{1}{p}\,t^{p-1}\int_\Omega |f(x)|^p\ dx^n + \frac{1}{p'}\,t^{-1}\int_\Omega |g(x)|^{p'}\ dx^p$$

Choosing $t := \|f\|^{-1}_{L^p(\Omega)}\,\|g\|^{p'/p}_{L^{p'}(\Omega)}$ is giving (3.75) and this concludes the proof of theorem 3.1. $\square$

A corollary of this theorem is the *Littlewood inequality* whose proof is left as an exercise.

[8] We are adopting in the statement of theorem 3.1 the notation convention

$$\text{“}\frac{1}{\infty} = 0\text{''}$$

Corollary 3.2. *Let Ω be an open subset of $\mathbb{R}^n$. Let $p_0 \in [1,+\infty]$ and $p_1 \in [1,+\infty]$. Let $t \in (0,1)$ and denote $p_t \in [1,+\infty]$ given by*

$$\frac{1}{p_t} := \frac{1-t}{p_0} + \frac{t}{p_1} \ .$$

Then for any $f \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$, $f \in L^{p_t}(\Omega)$ and there holds

$$\|f\|_{L^{p_t}(\Omega)} \le \|f\|^{(1-t)}_{L^{p_0}(\Omega)} \ \|f\|^{t}_{L^{p_1}(\Omega)} \ . \tag{3.76}$$

We are now proving the following theorem which is a classical result from Functional Analysis 1.

Theorem 3.3. *Let $1 \le p \le +\infty$ and Ω an open set of $\mathbb{R}^n$. Then $L^p(\Omega)$ is a vector space and $\|\cdot\|_{L^p(\Omega)}$ defines a norm, moreover $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ is a Banach space.*

Proof of theorem 3.3. It is clear that $\|\cdot\|_{L^1(\Omega)}$ and $\|\cdot\|_{L^\infty(\Omega)}$ define norms. We now prove that $L^p(\Omega)$ is a vector space and $\|\cdot\|_{L^p(\Omega)}$ defines a norm for $1 < p < +\infty$. Let f and g in $L^p(\Omega)$. We first claim that $f + g \in L^p(\Omega)$. We have for any $x \in \Omega$

$$|f(x)+g(x)|^p \le 2^p \ (\max\{|f(x)|, |g(x)|\})^p \le 2^p \ (|f(x)|^p + |g(x)|^p) \ .$$

integrating this inequality on Ω is giving the integrability of $|f(x)+g(x)|^p$ on Ω. We now prove that $\|\cdot\|_{L^p(\Omega)}$ defines a norm. We have using Hölder inequality

$$\begin{aligned}
\int_\Omega |f(x)+g(x)|^p \ dx^n &= \int_\Omega |f(x)+g(x)|^{p-1} \, |f(x)+g(x)| \ dx^n \\
&\le \int_\Omega |f(x)+g(x)|^{p-1} \, |f(x)| \ dx^n + \int_\Omega |f(x)+g(x)|^{p-1} \, |g(x)| \ dx^n \\
&\le \|f+g\|^{p-1}_{L^p(\Omega)} \, \|f\|_{L^p(\Omega)} + \|f+g\|^{p-1}_{L^p(\Omega)} \, \|g\|_{L^p(\Omega)}
\end{aligned}$$

This implies

$$\|f+g\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)} \ ,$$

from which we deduce that $\|\cdot\|_{L^p(\Omega)}$ defines a norm.

We now prove that $(L^\infty(\Omega), \|\cdot\|_{L^\infty(\Omega)})$ is complete. Let f_n be a Cauchy sequence in L^∞. For every $k \in \mathbb{N}$ there is a integer N_k such that

$$\forall \ m,n \ge N_k \qquad \|f_n - f_m\|_{L^\infty(\Omega)} \le \frac{1}{k} \ . \tag{3.77}$$

Denote, for any $n, m \ge N_k$

$$E_k(m,n) := \left\{ x \in \Omega \quad \text{s. t. } |f_n(x) - f_m(x)| > \frac{1}{k} \right\}$$

It is clear from (3.77) that $E_k(m,n)$ has Lebesgue measure zero. Let

$$E_k := \bigcup_{m,n \ge N_k} E_k(m,n) \ .$$

A countable union of measure zero set has measure zero. Hence E_k has measure zero as well. Finally we denote

$$E := \bigcup_{k\in\mathbb{N}} E_k \ .$$

For the same reason has above E has also Lebesgue measure zero. For any $x \in \Omega \setminus E$ there holds

$$\forall\, m, n \geq N_k \qquad |f_n(x) - f_m(x)| \leq \frac{1}{k} \ , \tag{3.78}$$

hence $f_n(x)$ is a Cauchy sequence. We denote by $f(x)$ its limit. The fact that $f(x)$ is measurable is left as an exercise. Passing to the limit in (3.78), we have for any $n \geq N_k$

$$\forall\, x \in \Omega \setminus E \qquad |f(x) - f_n(x)| \leq \frac{1}{k} \ . \tag{3.79}$$

Since this holds for any $x \in \Omega \setminus E$ and since E has zero measure, we deduce that for such a n

$$\operatorname{esssup}\{|f(x)| \ ; \ x \in \Omega\} \leq \frac{1}{k} + \|f_n\|_{L^\infty(\Omega)} < +\infty \ .$$

hence f is in $L^\infty(\Omega)$ and thanks to (3.79), $f_n \to f$ in $L^\infty(\Omega)$. This implies that $(L^\infty(\Omega), \|\cdot\|_{L^\infty(\Omega)})$ is a Banach space.

We consider now the case $1 \leq p < +\infty$. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L^p(\Omega)$. It suffices to prove that there exists a subsequence which converges to an element in $L^p(\Omega)$. We first extract a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that

$$\forall\, k \in \mathbb{N} \qquad \left\| f_{n_{k+1}} - f_{n_k} \right\|_{L^p(\Omega)} \leq 2^{-k} \ .$$

Introduce

$$g_k(x) := \sum_{j=1}^{k} |f_{n_{j+1}}(x) - f_{n_j}(x)| \ .$$

Using the triangular inequality there holds

$$\forall\, k \in \mathbb{N} \qquad \|g_k\|_{L^p(\Omega)} \leq \sum_{j=0}^{k} 2^{-j} \leq 2 \ .$$

The sequence $g_k(x)^p$ is increasing hence, using the Beppo Levi monotone convergence theorem, we deduce that g^p converges in $L^1(\Omega)$ to a limit g^p and we have

$$\forall x \in \Omega \qquad g_k(x) \leq g(x) \qquad \text{and} \qquad \|g\|_{L^p(\Omega)} < +\infty \ .$$

Observe that

$$\forall k \leq l \quad |f_{n_l}(x) - f_{n_k}(x)| \leq g_{l-1}(x) - g_k(x) \leq g(x) - g_k(x) \ .$$

Since g_k^p converges strongly in $L^1(\Omega)$ towards g^p then $g_k(x)$ converges almost everywhere towards $g(x)$ and we deduce that, for almost every $x \in \Omega$ $f_{n_k}(x)$ is converging to a limit that we denote $f(x)$. Moreover there holds

$$\forall k \in \mathbb{N} \quad |f(x) - f_{n_k}(x)|^p \leq g^p(x) \ .$$

By dominated convergence we deduce that f_{n_k} converges strongly to f in $L^p(\Omega)$. Hence we have proved that for any $1 \leq p \leq +\infty$ the spaces $(L^p(\Omega), \| \cdot \|_{L^p(\Omega)})$ are complete and this concludes the proof of theorem 3.3. $\Box$

We have introduced the convolution between an arbitrary element in $\mathcal{S}'(\mathbb{R}^n)$ and an arbitrary element in $\mathcal{E}'(\mathbb{R}^n)$. We are going to extend this operation between an element in $L^1(\mathbb{R}^n)$ and an element in $L^p(\mathbb{R}^n)$ for any $1 \leq p \leq +\infty$. This is the famous Young inequality (see a proof in [1] theorem 4.15).

Theorem 3.4. *Let $1 \leq p \leq +\infty$. For any $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, for almost every $x \in \mathbb{R}^n$ the function $y \in \mathbb{R}^n \longrightarrow f(x-y)\, g(y)$ is integrable on $\mathbb{R}^n$. We denote*

$$(f \star g)(x) := \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy^n \ .$$

Moreover $f \star g \in L^p(\mathbb{R}^n)$ and

$$\|f \star g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^1(\Omega)} \ \|g\|_{L^p(\Omega)}$$

The Young inequality can be extended as follows.

Theorem 3.5. *Let $1 \leq p \leq +\infty$ and $1 \leq q \leq +\infty$. For any $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ then $f \star g$ is in $L^r(\mathbb{R}^n)$ where*

$$\frac{1}{r} := \frac{1}{p} + \frac{1}{q} - 1 \geq 0$$

and there holds

$$\|f \star g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\Omega)} \ \|g\|_{L^q(\mathbb{R}^n)}$$

The proofs of the Young inequalities theorem 3.4 and theorem 3.5 are given in the series. We shall be proving a generalization of these ineqalities in the framework of Lorentz spaces in chapter 6 using the notions of decreasing rearrangements (see theorem 6.43).

3.2 Reflexivity and Duals of L^p-Spaces

3.2.1 The uniform convexity and reflexivity of $L^p(\Omega)$ for $1 < p < +\infty$

We recall the definition of a uniform convex normed space

Definition 3.6. *A Banach space $(E, \| \cdot \|)$ is said to be uniformly convex if*

$$\forall\, x, y \in B_1^E(0) \quad \forall \varepsilon > 0 \quad \exists\, \delta > 0 \quad s.\ t.\ \|x - y\| > \varepsilon \implies \left\| \frac{x+y}{2} \right\| \leq 1 - \delta$$

We recall the notion of reflexive spaces.

Definition 3.7. *Let E be a Banach space and let J be the canonical injection of E into $(E^*)^*$ given by*

$$\forall\, x \in E \quad , \quad \forall\, l \in E^* \qquad \langle J_E(x), l\rangle_{(E^*)^*,E^*} := \langle l, x\rangle_{E^*,E} \ .$$

E is called reflexive if J is surjective, that is

$$(E^*)^* = J_E(E) \ .$$

An important property of reflexive Banach spaces is the following

Proposition 3.8. *Let $(E, \|\cdot\|_E)$ be a reflexive Banach space then $(E^*, \|\cdot\|_{E^*})$ is also reflexive*

Proof of Proposition 3.8. We denote again J_E the canonical isomorphism from definition 3.7 from E into $(E^*)^*$. Let $\varphi \in ((E^*)^*)^*$ the map

$$x \in E \ \longrightarrow \ \langle \varphi, J_E(x)\rangle_{((E^*)^*)^*,(E^*)^*}$$

is obviously continuous and linear on E. Hence there exists $l \in E^*$ such that

$$\forall\, x \in E \quad \forall \varphi \in ((E^*)^*)^* \qquad \langle \varphi, J_E(x)\rangle_{((E^*)^*)^*,(E^*)^*} = \langle l, x\rangle_{E^*,E} = \langle J_E(x), l\rangle_{(E^*)^*,E^*}$$

Since J_E is surjective we then have proved

$$\forall \sigma \in (E^*)^* \qquad \langle \varphi, \sigma\rangle_{((E^*)^*)^*,(E^*)^*} = \langle \sigma, l\rangle_{(E^*)^*,E^*} \ .$$

This shows that J_{E^*} is also surjective and then E^* is reflexive. This concludes the proof of proposition 3.8. □

We shall also make use of another proposition (see [1] proposition 3.20 for a proof).

Proposition 3.9. *Any closed linear subspace of a reflexive Banach space is also reflexive.*

We recall a classical theorem, called "Milman-Pettis Theorem" and seen in Functional Analysis 1 (see also [1] for a proof).

Theorem 3.10. *Every uniformly convex Banach space E is reflexive.*

We are now going to prove the uniform convexity of $L^p(\Omega)$ for any $1 < p < +\infty$ where Ω is an arbitrary open set of $\mathbb{R}^n$. First we prove it for $2 \le p < +\infty$: this will be the consequence of the so called *Clarkson's first inequality.*

Lemma 3.11. *Let $2 \le p < +\infty$ then*

$$\forall\, f, g \in L^p(\Omega) \qquad \left\|\frac{f+g}{2}\right\|^p_{L^p(\Omega)} + \left\|\frac{f-g}{2}\right\|^p_{L^p(\Omega)} \le \frac{1}{2}\ \left[\|f\|^p_{L^p(\Omega)} + \|g\|^p_{L^p(\Omega)}\right] \ .$$

Proof of Lemma 3.11. Observe that for any $q \geq 1$ there holds

$$\forall\, x, y \geq 0 \qquad x^q + y^q \leq (x+y)^q$$

Let $a, b \in \mathbb{R}$ and choose $x := \left|\frac{a+b}{2}\right|^2$, $y := \left|\frac{a-b}{2}\right|^2$ and $q := p/2$. This gives

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \leq \left[\left|\frac{a+b}{2}\right|^2 + \left|\frac{a-b}{2}\right|^2\right]^{p/2} = \left[\frac{a^2}{2} + \frac{b^2}{2}\right]^{p/2} \leq \frac{|a|^p + |b|^p}{2}\ ,$$

where in the last inequality we have used the convexity of $t \to t^{p/2}$ on $\mathbb{R}_+$. Replacing a by $f(x)$ and b by $g(x)$, integrating with respect to x is giving the desired inequality and the lemma 3.11 is proved. □

From the Clarkson's first inequality we deduce the uniform convexity of $L^p(\Omega)$ for $2 \leq p < +\infty$. Now we prove that the reflexivity extends to $L^p(\Omega)$ for $1 < p < +\infty$.

Theorem 3.12. *Let Ω be an open set of $\mathbb{R}^n$ and $1 < p < +\infty$, then $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ is reflexive.*

Proof of theorem 3.12. The reflexivity of $L^p(\Omega)$ for $2 \leq p < +\infty$ is a consequence of the uniform convexity property implied by lemma 3.11 thanks to theorem 3.10.

For any $p \in (1, +\infty)$ we denote $p' := p/(p-1)$. Let J be the map which to $f \in L^p(\Omega)$ assigns the following linear map from $L^{p'}(\Omega)$ into $\mathbb{C}$ given by

$$\forall\, g \in L^{p'}(\Omega) \qquad J(f)(g) := \int_\Omega f(x)\, g(x)\ dx^n\ .$$

Thanks to Hölder inequality we deduce that $J(f)$ defines a continuous linear map and therefore $J(f) \in (L^{p'}(\Omega))^*$. Again, thanks to Hölder inequality we have that

$$\|J(f)\|_{(L^{p'}(\Omega))^*} := \inf_{\|g\|_{L^{p'}(\Omega)} \leq 1} J(f)(g) \leq \|f\|_{L^p(\Omega)}$$

Introduce the map $g(x)$ which is equal to zero when $f(x) = 0$ and

$$g(x) := |f(x)|^{p-2}\, f(x)/\|f\|^{p-1}_{L^p(\Omega)}$$

otherwize. One has

$$\|g\|_{L^{p'}(\Omega)} = 1 \quad \text{ and } \quad J(f)(g) = \left[\int_\Omega |f(x)|^p\ dx^n\right]^{1/p}\ .$$

Hence

$$\|J(f)\|_{(L^{p'}(\Omega))^*} = \|f\|_{L^p(\Omega)}$$

which implies that J realizes an isometry from $L^p(\Omega)$ into $(L^{p'}(\Omega))^*$ and consequently in particular $J(L^p(\Omega))$ is closed in $(L^{p'}(\Omega))^*$.

Consider now $1 < p \leq 2$. Since $L^{p'}(\Omega)$ is reflexive it follows from proposition 3.8 that $(L^{p'}(\Omega))^*$ is also reflexive. Using this time proposition 3.9 we deduce that $J(L^p(\Omega))$ is reflexive and since J is an isometry we deduce that $L^p(\Omega)$ is reflexive and this concludes the proof of the theorem 3.12. □

3.2.2 The Dual of $L^p(\Omega)$ for $1 \leq p < +\infty$

We identify now the dual of $L^p(\Omega)$ for any $1 < p < +\infty$. This is the subject of the following "Riesz representation type" theorem.

Theorem 3.13. *Let $1 < p < +\infty$ and let $l \in (L^p(\Omega))^*$. Then there exists $g \in L^{p'}(\Omega)$ such that*

$$\forall\, f \in L^p(\Omega) \qquad \langle l, f\rangle = \int_{\mathbb{R}^n} f(x)\, g(x)\ dx^n\ ,$$

moreover

$$\|g\|_{L^{p'}(\Omega)} = \|l\|_{(L^p(\Omega))^*}\ .$$

Proof of Theorem 3.13 We consider again the operator J from $L^{p'}(\Omega)$ into $(L^p(\Omega))^*$ given by

$$\forall\, f \in L^p(\Omega)\ ,\ \forall\, g \in L^{p'}(\Omega) \qquad \langle J(g), f\rangle_{(L^p(\Omega))^*, L^p(\Omega)} := \int_{\mathbb{R}^n} f(x)\, g(x)\ dx^n\ .$$

We have using the same argument as in the proof of theorem 3.12

$$\|J(g)\|_{(L^p(\Omega))^*} := \sup_{\|f\|_{L^p(\Omega)} \leq 1} \int_{\mathbb{R}^n} f(x)\, g(x)\ dx^n = \|g\|_{L^{p'}(\Omega)}\ .$$

We claim that J is surjective in $(L^p(\Omega))^*$. J realizes an isometry. Let $J(L^{p'}(\Omega))$, this is a closed subspace of $(L^p(\Omega))^*$. We claim that $F = J(L^{p'}(\Omega))$ is dense in $(L^p(\Omega))^*$. In order to prove this claim we shall be using the following lemma from Functional Analysis 1, which is a direct consequence of Hahn Banach theorem, and whose proof can be found in [1] (corollary 1.8)

Lemma 3.14. *Let F be a closed linear subspace of a Banach space $(E, \|\cdot\|_E)$ such that $E \neq F$. Then there exists $l \in E^*$ such that $l \neq 0$ and*

$$\forall\, x \in F \qquad \langle l, x\rangle_{E^*, E} = 0\ .$$

Proof of theorem 3.13 continued Using the previous lemma, assuming $J(L^{p'}(\Omega)) \neq (L^p(\Omega))^*$ there exists $\varphi \in ((L^p(\Omega))^*)^*$ such that $\varphi \neq 0$ and

$$\forall\ g \in L^{p'}(\Omega) \qquad \langle \varphi, J(g)\rangle = 0$$

Since $L^p(\Omega)$ is reflexive, there exists $f \in L^p(\Omega)$ such that

$$\forall\, l \in (L^p(\Omega))^* \qquad \langle \varphi, l\rangle_{((L^p(\Omega))^*)^*, (L^p(\Omega))^*} = \langle l, f\rangle_{(L^p(\Omega))^*, L^p(\Omega)}\ .$$

We have in particular

$$\forall\ g \in L^{p'}(\Omega) \qquad 0 = \langle \varphi, J(g)\rangle = \int_\Omega f(x)\, g(x)\ dx\ .$$

We choose g such that $g(x) = 0$ whenever $f(x) = 0$ and $g(x) := |f(x)|^{p-2}\, f(x)$ otherwize. From the previous identity we obtain $f \equiv 0$ and hence $\varphi = 0$ which is a contradiction. This concludes the proof of the Riesz theorem 3.13. $\square$

We now consider the cases $p = 1$ and $p = +\infty$. First we have the following Riesz representation theorem which says $(L^1(\Omega))^* = L^\infty(\Omega)$.

Theorem 3.15. *Let Ω be an open set in $\mathbb{R}^n$ and let $l \in (L^1(\Omega))^*$ then there exists a unique function $g \in L^\infty(\Omega)$ such that*

$$\forall f \in L^1(\Omega) \qquad \langle l, f\rangle_{(L^1(\Omega))^*, L^1(\Omega)} = \int_\Omega f(x)\, g(x)\; dx^n \ . \tag{3.80}$$

Moreover

$$\|g\|_{L^\infty(\Omega)} = \|l\|_{(L^1(\Omega))^*}$$

Proof of theorem 3.15. First we establish the uniqueness of g. It suffices to show that

$$\forall f \in L^1(\Omega) \qquad \int_\Omega f(x)\, g(x)\; dx^n = 0 \quad \Longrightarrow \quad g \equiv 0 \ . \tag{3.81}$$

We denote $\Omega_k := \Omega \cap B_{2^k}(0)$ and we consider the functions $f_n(x) := 0$ if $g(x) = 0$ and $f_k(x) := \frac{g(x)}{|g(x)|}\, \mathbf{1}_{\Omega_k}(x)$ whenever $g(x) \neq 0$, where $\mathbf{1}_{\Omega_k}(x)$ is the characteristic function of Ω_k. The hypothesis is then implying

$$\forall k \in \mathbb{N} \qquad \int_{\Omega \cap B_{2^k}(0)} |g(x)|\; dx^n = 0$$

from which we deduce that $g(x) \equiv 0$ and we have proved (3.81).

Let $l \in (L^1(\Omega))^*$. We establish the existence of g such that (3.80) holds. Let $\theta \in L^2(\Omega)$ such that $\theta := 2^{-kn}$ on $\Omega_k \setminus \Omega_{k-1}$ for $k \geq 1$. We consider the map

$$T \ : \ h \in L^2(\Omega) \ \longrightarrow \ \mathbb{C} \qquad \text{s. t.} \qquad T(h) := \langle l, \theta\, h\rangle_{(L^1(\Omega))^*, L^1(\Omega)} \ .$$

Thanks to theorem 3.13 there exists $v \in L^2(\Omega)$ such that

$$T(h) = \int_\Omega v(x)\, h(x)\; dx^n \ .$$

It is then natural to introduce $g(x) := v(x)/\theta(x)$. We are going to prove that g is a solution to our problem. Observe that, since θ is bounded from below by a positive number on each of the Ω_k, $g \in L^2(\Omega_k)$ for any $k \in \mathbb{N}$. Hence we have for any $k \in \mathbb{N}$

$$\begin{aligned} &\forall h \in L^2(\Omega) \qquad \langle l, \mathbf{1}_{B_{2^k}(0)}\; h\rangle_{(L^1(\Omega))^*, L^1(\Omega)} = \left\langle l, \theta\, \mathbf{1}_{B_{2^k}(0)}\; \tfrac{h}{\theta}\right\rangle_{(L^1(\Omega))^*, L^1(\Omega)} \\ &\int_\Omega v(x)\, \mathbf{1}_{B_{2^k}(0)}\, \frac{h(x)}{\theta(x)}\; dx^n = \int_{\Omega_k} g(x)\; h(x)\; dx^n \end{aligned} \tag{3.82}$$

We claim that $g \in L^\infty(\Omega)$ and that we have

$$\|g\|_{L^\infty(\Omega)} \leq \|l\|_{(L^1(\Omega))^*} \ . \tag{3.83}$$

Let $K > 0$ and introduce

$$\omega_K := \{x \in \Omega \ ; \ |g(x)| > K\} \ .$$

In order to establish the claim it suffices to prove that ω_K has zero measure. We choose for any x at which $g(x) \neq 0$, $h := \mathbf{1}_{\omega_K}\ g(x)/|g(x)|$ and $h(x) = 0$ otherwize, where we denote by $\mathbf{1}_{\omega_K}$ the characteristic function of ω_K. Observe that $h \in L^\infty(\Omega)$, hence, obviously, $\mathbf{1}_{B_{2^k}(0)}\ h \in L^1(\Omega)$ and from (3.82) we obtain

$$K\,|\Omega_k\cap\omega_K| \leq \int_{\Omega_k\cap\omega_K} |h(x)|\ dx^n \leq \|l\|_{(L^1(\Omega))^*}\ \|\mathbf{1}_{B_{2^k}(0)}\ h\|_{L^1(\Omega)} = \|l\|_{(L^1(\Omega))^*}\ |\Omega_k\cap\omega_K|$$

Hence for $K > \|l\|_{(L^1(\Omega))^*}$ the measure of $|\Omega_k \cap \omega_K|$ is zero for any $k \in \mathbb{N}$ and this implies the claim (3.83).

Let $f \in L^1(\Omega)$. We denote by $T_k(f)(x) := f(x)\ \frac{\inf\{k,|f(x)|\}}{|f(x)|}$ if $|f(x)| \neq 0$ and $T_k(f)(x) = 0$ if $f(x) = 0$. By dominated convergence we have that

$$\mathbf{1}_{B_{2^k}(0)}\ T_k(f)\ \longrightarrow\ f \qquad \text{strongly in } L^1(\Omega)\ .$$

We have from 3.82

$$\langle l, \mathbf{1}_{B_{2^k}(0)}\ T_k(f)\rangle_{(L^1(\Omega))^*,L^1(\Omega)} = \int_\Omega g(x)\,\mathbf{1}_{B_{2^k}(0)}\ T_k(f)(x)\ dx^n$$

Passing to the limit in both sides of the equality we obtain

$$\langle l, f\rangle_{(L^1(\Omega))^*,L^1(\Omega)} = \int_\Omega g(x)\, f(x)\ dx^n\ . \tag{3.84}$$

It remains to prove that

$$\|g\|_{L^\infty(\Omega)} = \|l\|_{(L^1(\Omega))^*}\ . \tag{3.85}$$

From (3.84) there holds

$$\forall\ f \in L^1(\Omega) \qquad \left|\langle l, f\rangle_{(L^1(\Omega))^*,L^1(\Omega)}\right| \leq \|g\|_{L^\infty(\Omega)}\ \|f\|_{L^1(\Omega)}\ .$$

hence $\|l\|_{(L^1(\Omega))^*} \leq \|g\|_{L^\infty(\Omega)}$. Combining this inequality with (3.83) we obtain (3.85) and this concludes the proof of theorem 3.15. □

We have seen in serie 1 using Hahn Banach that there exists $l \in (L^\infty(\Omega))^*\setminus L^1(\Omega)$. Hence we have the following result.

Proposition 3.16. *Let Ω be an open set of $\mathbb{R}^n$ then $L^1(\Omega)$ is not reflexive.*

3.3 Separability of $L^p(\Omega)$ for $1 \leq p < +\infty$ and approximability properties.

We recall the classical notion from Topology.

Definition 3.17. *We say that a metric space E is separable if there exists a subset $D \subset E$ that is countable and dense.*

We then have the following theorem

Theorem 3.18. *Let Ω be an open subset in $\mathbb{R}^n$ and let $1 \leq p < +\infty$. Then $L^p(\Omega)$ is separable.*

Regarding $L^\infty(\Omega)$ the answer is negative.

Theorem 3.19. *Let Ω be an open subset in $\mathbb{R}^n$. Then $L^\infty(\Omega)$ is not separable.*

The theorem 3.18 and theorem 3.19 are proved in the series.

Regarding the approximability property of any element in $L^p(\mathbb{R}^n)$ by elements in $C_0^\infty(\mathbb{R}^n)$ we have already seen in chapter 2 using the Fourier Transform that

$$\overline{C_0^\infty(\mathbb{R}^n)}^{L^2} = L^2(\mathbb{R}^n) \ .$$

see the proof of proposition 2.7. We are extending this result to any $1 \leq p < +\infty$. Precisely we have

Theorem 3.20. *Let Ω be an open set of $\mathbb{R}^n$ and let $1 \leq p < +\infty$. Then $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$.*

Remark 3.21. *The theorem 3.20 obviously does not extend to the case $p = +\infty$ since the limits for the uniform convergence of smooth functions are continuous and $C^0(\overline{\Omega})$ is strictly included in $L^\infty(\Omega)$.*

In order to prove the theorem 3.20 we shall need the following lemma.

Lemma 3.22. *Let $\chi \in C_0^\infty(B_1(0))$ such that $\int_{\mathbb{R}^n} \chi(x)\, dx^n = 1$. For any $\varepsilon > 0$ we denote $\chi_\varepsilon(\cdot) := \varepsilon^{-n}\, \chi(\varepsilon^{-1}\cdot)$. Let $1 \leq p < +\infty$ then*

$$\forall\, f \in L^p(\mathbb{R}^n) \qquad f \star \chi_\varepsilon \longrightarrow f \quad \text{strongly in } L^p(\mathbb{R}^n) \ . \tag{3.86}$$

Proof of lemma 3.22. First of all we claim that any function $f \in L^p(\mathbb{R}^n)$ can be strongly approximated in $L^p(\mathbb{R}^n)$ by continuous and compactly supported functions. The result for $p = 1$ has been proved in analysis 3 [2]. We are proving the claim now for $1 < p < +\infty$. We denote

$$f_k(x) := 0 \quad \text{if } f(x) = 0 \quad \text{and} \quad f_k(x) := \mathbf{1}_{B_{2^k}(0)}\, f(x)\, \frac{\max\{|f(x)|, k\}}{|f(x)|} \quad \text{if } f(x) \neq 0 \, .$$

where $\mathbf{1}_{B_{2^k}(0)}$ is denoting the characteristic function of the ball $B_{2^k}(0)$. From the dominated convergence theorem we have

$$f_k(x) \longrightarrow f(x) \qquad \text{strongly in } L^p(\Omega) \ .$$

Let $\varepsilon > 0$. There exists $k_\varepsilon \in \mathbb{N}$ such that

$$\forall\, k \geq k_\varepsilon \qquad \|f - f_k\|_{L^p(\mathbb{R}^n)} \leq \frac{\varepsilon}{2} \ . \tag{3.87}$$

Since f_{k_ε} is supported in a bounded set $B_{2^{k_\varepsilon}}(0)$ and since $\|f_{k_\varepsilon}\|_{L^\infty(\mathbb{R}^n)} \leq k_\varepsilon$ we have $f_{k_\varepsilon} \in L^q(\mathbb{R}^n)$ for any $q \in [1, +\infty]$. Hence, using the claim proved for $p = 1$, for any $\delta > 0$ there exists $g_\delta \in C^0_c(\Omega)$ such that

$$\|f_{k_\varepsilon} - g_\delta\|_{L^1(\Omega)} \leq \delta \ .$$

By replacing g_δ by

$$\tilde{g}_\delta(x) := \mathbf{1}_{B_{2k_\varepsilon}(0)}\ g_\delta(x)\ \frac{\max\{|g_\delta(x)|, k_\varepsilon\}}{|g_\delta(x)|} \qquad \text{if } g_\delta(x) \neq 0$$

and $\tilde{g}_\delta(x) = 0$ if $g_\delta(x) = 0$. We have for any x

$$|f_{k_\varepsilon}(x) - g_\delta(x)| \geq |\mathbf{1}_{B_{2k_\varepsilon}(0)}\ (f_{k_\varepsilon}(x) - g_\delta(x))| = |f_{k_\varepsilon}(x) - \mathbf{1}_{B_{2k_\varepsilon}(0)}\ g_\delta(x)|$$

Using that the map from π_k from $\mathbb{C}$ into $B_k(0)$ given by $\pi_k(y) = y$ if $|y| \leq k$ and $\pi_k(y) = k\, y/|y|$ for $|y| \geq k$ is a Lipschitz contraction (i.e $\|\nabla_y \pi_k(y)\|_\infty \leq 1$) we have for any x

$$|f_{k_\varepsilon}(x) - g_\delta(x)| \geq |f_{k_\varepsilon}(x) - \mathbf{1}_{B_{2k_\varepsilon}(0)}\ \pi_k(g_\delta(x))| = |f_{k_\varepsilon}(x) - \tilde{g}_\delta(x)|\ .$$

Hence we have

$$\|f_{k_\varepsilon} - \tilde{g}_\delta\|_{L^1(\mathbb{R}^n)} \leq \delta \quad , \quad \tilde{g}_\delta \in C^0_0(\mathbb{R}^n) \quad \text{and} \quad \|\tilde{g}_\delta\|_{L^\infty(\mathbb{R}^n)} \leq k\ .$$

Hence we have

$$\|f_{k_\varepsilon} - \tilde{g}_\delta\|_{L^p(\mathbb{R}^n)} \leq \|f_{k_\varepsilon} - \tilde{g}_\delta\|^{1/p}_{L^1(\mathbb{R}^n)}\ \|f_{k_\varepsilon} - \tilde{g}_\delta\|^{1-1/p}_{L^\infty(\mathbb{R}^n)} \leq \delta^{1/p}\,(2\,k_\varepsilon)^{1-1/p}\ . \tag{3.88}$$

We then choose δ such that $\delta^{1/p}\,(2\,k_\varepsilon)^{1-1/p} = \varepsilon/2$. Combining (3.87) and (3.88) is giving finally the existence of $\tilde{g}_\delta \in C^0_0(\mathbb{R}^n)$ such that

$$\|f - \tilde{g}_\delta\|_{L^p(\mathbb{R}^n)} \leq \varepsilon\ .$$

This implies the claim that any function $f \in L^p(\mathbb{R}^n)$ can be strongly approximated in $L^p(\mathbb{R}^n)$ by continuous and compactly supported functions.

Let $g \in C^0_0(\mathbb{R}^n)$, g is uniformly continuous and then for any $\delta > 0$ there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$

$$\forall\, x \in \mathbb{R}^n \qquad |\chi_\varepsilon \star g(x) - g(x)| \leq \frac{1}{\varepsilon^n} \int_{B_\varepsilon(0)} \|\chi\|_\infty\, |g(x-y) - g(x)|\ dy^n \leq \delta\ .$$

This implies that

$$\forall\, g \in C^0_0(\mathbb{R}^n) \qquad \lim_{\varepsilon \to 0} \|\chi_\varepsilon \star g - g\|_{L^\infty(\mathbb{R}^n)} = 0$$

Since g has compact support and since $\mathrm{Supp}(\chi_\varepsilon \star g) \subset \mathrm{Supp}(g) + B_\varepsilon(0)$, we deduce

$$\forall\, g \in C^0_0(\mathbb{R}^n) \quad \forall 1 \leq q \leq +\infty \qquad \lim_{\varepsilon \to 0} \|\chi_\varepsilon \star g - g\|_{L^q(\mathbb{R}^n)} = 0\ . \tag{3.89}$$

Let $f \in L^p(\mathbb{R}^n)$ and let $\delta > 0$. From the first part of the proof, there exists $g_\delta \in C^0_0(\mathbb{R}^n)$ such that

$$\|f - g_\delta\|_{L^p(\mathbb{R}^n)} \leq \delta\ ,$$

Hence we have for any $\varepsilon > 0$, using Young inequality

$$\|f - \chi_\varepsilon \star f\|_{L^p(\mathbb{R}^n)} \le \|(f - g_\delta) - \chi_\varepsilon \star (f - g_\delta)\|_{L^p(\mathbb{R}^n)} + \|\chi_\varepsilon \star g_\delta - g_\delta\|_{L^p(\mathbb{R}^n)}$$

$$\le [1 + \|\chi\|_{L^1(\mathbb{R}^n)}] \; \|f - g_\delta\|_{L^p(\mathbb{R}^n)} + \|\chi_\varepsilon \star g_\delta - g_\delta\|_{L^p(\mathbb{R}^n)} \; .$$

Because of (3.89), for ε small enough we deduce

$$\|f - \chi_\varepsilon \star f\|_{L^p(\mathbb{R}^n)} \le [2 + \|\chi\|_{L^1(\mathbb{R}^n)}] \; \delta$$

This implies (3.86) and lemma 3.22 is proved. □

Proof of theorem 3.20. From the proof of lemma 3.22, for any $\delta > 0$ there exists $g_\delta \in C^0_0(\mathbb{R}^n)$ such that

$$\|f - g_\delta\|_{L^p(\mathbb{R}^n)} \le \delta/2 \; .$$

For any $\varepsilon > 0$ we write

$$\|f - \chi_\varepsilon \star g_\delta\|_{L^p(\mathbb{R}^n)} \le \|f - g_\delta\|_{L^p(\mathbb{R}^n)} + \|\chi_\varepsilon \star g_\delta - g_\delta\|_{L^p(\mathbb{R}^n)}$$

The map $g_\delta \in C^0_0(\mathbb{R}^n)$ being fixed, from (3.89), for ε small enough we have

$$\|g - \chi_\varepsilon \star g_\delta\|_{L^p(\mathbb{R}^n)} \le \delta/2 \; .$$

Combining the three previous inequalities we have found $g := \chi_\varepsilon \star g_\delta \in C^\infty_0(\mathbb{R}^n)$ such that

$$\|f - g\|_{L^p(\mathbb{R}^n)} \le \delta \; .$$

This holds for any $\delta > 0$ hence this concludes the proof of theorem 3.20. □

3.4 Riesz-Thorin interpolation theorem and the Fourier Transform of an L^p function for $1 \le p \le 2$.

The interpolations of operators is an important method in Functional Analysis. We shall now prove a first result in the theory for linear operators and show how this can be applied to prove new inequalities. This result is known as *Riesz Thorin Interpolation Theorem* or *Riesz Convexity Theorem*.

Theorem 3.23. *Let $1 \le p_j \le +\infty$ and $1 \le q_j \le +\infty$ for $i = 0, 1$ with $\max\{q_0, q_1\} > 1$. Let Ω be an open set of $\mathbb{R}^n$ and T be a bounded linear operator from $L^{p_0}(\Omega) + L^{p_1}(\Omega)$ into $L^{q_0}(\Omega) + L^{q_1}(\Omega)$ such that*

$$T \; : \; L^{p_j}(\Omega) \longrightarrow L^{q_j}(\Omega) \qquad \textit{continuously.}$$

Let $t \in (0, 1)$ and denote

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \textit{and} \qquad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

with the usual convention $1/\infty = 0$. Then

$$T \; : \; L^{p_t}(\Omega) \longrightarrow L^{q_t}(\Omega) \qquad \textit{continuously}$$

and there holds

$$|||T|||_{L^{p_t}(\Omega)\to L^{q_t}(\Omega)} \le |||T|||^{(1-t)}_{L^{p_0}(\Omega)\to L^{q_0}(\Omega)} \; |||T|||^{t}_{L^{p_1}(\Omega)\to L^{q_1}(\Omega)}$$

Remark 3.24. *Observe that $L^{p_t}(\Omega)$ is in the domain of definition of T since for $1 \le p_0 < p_t < p_1 \le +\infty$ we have that $L^{p_t}(\Omega) \subset L^{p_0}(\Omega) + L^{p_1}(\Omega)$. Indeed we write for $\lambda > 0$ to be fixed later*

$$f(x) := f(x)\,\mathbf{1}_{|f(x)|\le\lambda} + f(x)\,\mathbf{1}_{|f(x)|>\lambda}$$

and we have respectively

$$\int_\Omega |f(x)\,\mathbf{1}_{|f(x)|\le\lambda}|^{p_1}\ dx^n \le\ \lambda^{p_1-p_t}\ \int_\Omega |f(x)\,\mathbf{1}_{|f(x)|\le\lambda}|^{p_t}\ dx^n$$

hence

$$\|f(x)\,\mathbf{1}_{|f(x)|\le\lambda}\|_{L^{p_1}(\Omega)} \le \lambda^{1-p_t/p_1}\,\|f(x)\,\mathbf{1}_{|f(x)|\le\lambda}\|^{p_t/p_1}_{L^{p_t}(\Omega)}$$

and

$$\lambda^{p_t}\ |\{x\in\Omega\ ;\ |f(x)|>\lambda\}| \le \int_\Omega |f(x)\,\mathbf{1}_{|f(x)|>\lambda}|^{p_t}\ dx^n < +\infty$$

which gives

$$|\{x\in\Omega\ ;\ |f(x)|>\lambda\}|^{\frac{p_t-p_0}{p_t}} \le \lambda^{-(p_t-p_0)}\ \left[\int_\Omega |f(x)\,\mathbf{1}_{|f(x)|>\lambda}|^{p_t}\ dx^n\right]^{\frac{p_t-p_0}{p_t}}$$

moreover

$$\int_\Omega |f(x)\,\mathbf{1}_{|f(x)|>\lambda}|^{p_0}\ dx^n \le \left[\int_\Omega |f(x)\,\mathbf{1}_{|f(x)|>\lambda}|^{p_t}\ dx^n\right]^{\frac{p_0}{p_t}}\ |\{x\in\Omega\ ;\ |f(x)|>\lambda\}|^{1-\frac{p_0}{p_t}}$$

$$\le \lambda^{p_0-p_t}\ \int_\Omega |f(x)\,\mathbf{1}_{|f(x)|>\lambda}|^{p_t}\ dx^n$$

Thus

$$\|f(x)\,\mathbf{1}_{|f(x)|>\lambda}\|_{L^{p_0}(\Omega)} \le \lambda^{1-p_t/p_0}\,\|f(x)\,\mathbf{1}_{|f(x)|>\lambda}\|^{p_t/p_0}_{L^{p_t}(\Omega)}$$

Hence we have for any $\lambda > 0$

$$\|f(x)\,\mathbf{1}_{|f(x)|>\lambda}\|_{L^{p_0}(\Omega)} + \|f(x)\,\mathbf{1}_{|f(x)|\le\lambda}\|_{L^{p_1}(\Omega)}$$

$$\le \lambda^{1-p_t/p_0}\,\|f(x)\|^{p_t/p_0}_{L^{p_t}(\Omega)} + \lambda^{1-p_t/p_1}\,\|f(x)\|^{p_t/p_1}_{L^{p_t}(\Omega)}$$

We then choose $\langle = \|f(x)\|_{L^{p_t}(\Omega)}$ and we deduce

$$\|f(x)\,\mathbf{1}_{|f(x)|>\lambda}\|_{L^{p_0}(\Omega)} + \|f(x)\,\mathbf{1}_{|f(x)|\le\lambda}\|_{L^{p_1}(\Omega)} \le 2\,\|f(x)\|_{L^{p_t}(\Omega)}$$

Then we have proved that for any $f \in L^{p_t}(\Omega)$ there exists $g \in L^{p_0}(\Omega)$ and $h \in L^{p_1}(\Omega)$ such that $f = g + h$ and

$$\|g\|_{L^{p_0}(\Omega)} + \|h\|_{L^{p_1}(\Omega)} \le C\,\|f\|_{L^{p_t}(\Omega)}\ .$$

Hence $L^{p_t}(\Omega)$ embeds continuously into the Banach space[9] $L^{p_0}(\Omega) + L^{p_1}(\Omega)$ given by

$$L^{p_0}(\Omega) + L^{p_1}(\Omega) := \left\{f \in L^1_{loc}(\Omega)\ ;\ \exists (g,h) \in L^{p_0}(\Omega)\times L^{p_1}(\Omega)\quad s.\ t.\ f = g+h\right\}$$

and

$$\|f\|_{L^{p_0}(\Omega)+L^{p_1}(\Omega)} := \inf\left\{\|g\|_{L^{p_0}(\Omega)} + \|h\|_{L^{p_1}(\Omega)}\ ;\ f = g+h\right\}\ .$$

[9]Recall from Functional Analysis 1 that the sum of two Banach spaces which both embed continuously in an Hausdorff topological space - here that would be the Fréchet space $L^1_{loc}(\Omega)$ - is again a Banach space.

Proof of theorem 3.23 The goal is to show that for any $f \in L^{p_t}(\Omega)$ and $g \in L^{q'_t}(\Omega)$, where $1/q'_t = 1 - 1/q_t$ there holds
(3.90)
$$\int_\Omega T(f)(x)\, g(x)\, dx^n \le |||T|||^{(1-t)}_{L^{p_0}(\Omega)\to L^{q_0}(\Omega)}\, |||T|||^{t}_{L^{p_1}(\Omega)\to L^{q_1}(\Omega)}\, \|f\|_{L^{p_t}(\Omega)}\, \|g\|_{L^{q'_t}(\Omega)} \ .$$

Observe that we can assume that $p_t < +\infty$. Indeed, if $p_t = +\infty$ then $p_0 = p_1 = +\infty$ then we have that T is mapping continuously $L^\infty(\Omega)$ to $L^{q_0}(\Omega) \cap L^{q_1}(\Omega)$ that is

$$\forall\, f \in L^\infty(\Omega) \qquad \left\{ \begin{array}{l} \|T(f)\|_{L^{q_0}(\Omega)} \le |||T|||_{L^\infty(\Omega)\to L^{q_0}(\Omega)} \\ \\ \|T(f)\|_{L^{q_1}(\Omega)} \le |||T|||_{L^\infty(\Omega)\to L^{q_1}(\Omega)} \end{array} \right.$$

We have by Littlewood inequality (corollary 3.2)

$$\begin{array}{l} \|T(f)\|_{L^{q_t}(\Omega)} \le \|T(f)\|^{1-t}_{L^{q_0}(\Omega)}\, \|T(f)\|^{t}_{L^{q_1}(\Omega)} \\ \\ \quad \le |||T|||^{(1-t)}_{L^\infty(\Omega)\to L^{q_0}(\Omega)}\, |||T|||^{t}_{L^\infty(\Omega)\to L^{q_1}(\Omega)}\, \|f\|_{L^\infty(\Omega)} \end{array}$$

and the theorem is proved in this particular case.

Observe that we can moreover assume $1 < q_t$. Indeed, if $q_t = 1$ for instance, this imposes $q_0 = q_1 = 1$ which is excluded by the hypothesis .

The proof for general $p_t < +\infty$ and $1 < q_t < +\infty$ is based on a "complex interpolation" strategy. It is convenient to introduce $\alpha_j := 1/p_j$ and $\beta_j := 1/q_j$ and

$$\forall\, z \in \mathbb{C} \qquad \alpha(z) := (1-z)\alpha_0 + z\alpha_1 \quad \text{ and } \quad \beta(z) := (1-z)\beta_0 + z\beta_1$$

Let f and g be two step functions on Ω of the form

$$f := \sum_{k=1}^m a_k\, \mathbf{1}_{E_k} \qquad \text{and} \qquad g := \sum_{l=1}^r b_l\, \mathbf{1}_{F_l}$$

where $a_k, b_l \in \mathbb{C}$ and E_k and F_l are measurable subsets of Ω. We first aim at proving (3.90) for these kinds of functions. Without loss of generality we fix $\|f\|_{L^{p_t}(\Omega)} = 1$. We introduce θ_k and ϕ_l such that

$$a_k = |a_k|\, e^{i\,\theta_k} \quad \text{ and } \quad b_l = |b_l|\, e^{i\,\phi_l} \ .$$

We define
$$f_z := \sum_{k=1}^m |a_k|^{\alpha(z)/\alpha(t)}\, e^{i\,\theta_k}\, \mathbf{1}_{E_k}$$
and
$$g_z := \sum_{l=1}^r |b_l|^{(1-\beta(z))/(1-\beta(t))}\, e^{i\,\phi_l}\, \mathbf{1}_{F_l} \ .$$

We introduce

$$F(z) := \int_\Omega T(f_z)\, g_z\, dx^n = \sum_{k=1}^m \sum_{l=1}^r |a_k|^{\alpha(z)/\alpha(t)}\, |b_l|^{(1-\beta(z))/(1-\beta(t))}\, \gamma_{kl}$$

where

$$\gamma_{kl} := e^{i\,(\theta_k+\phi_l)}\ \int_\Omega T(\mathbf{1}_{E_k})\ \mathbf{1}_{F_l}\ dx^n\ .$$

The function F is holomorphic. We write $z = x + iy$ and we restrict to the strip $S := \{z = x + iy \in \mathbb{C}\ ;\ 0 \le x \le 1\}$. We have for any $k \in \{1 \cdots m\}$

$$|a_k|^{\alpha(z)/\alpha(t)} = |a_k|^{\alpha_0/\alpha(t)}\ |a_k|^{x(\alpha_1-\alpha_0)/\alpha(t)}\ |a_k|^{i\,y\,(\alpha_1-\alpha_0)/\alpha(t)}\ .$$

Hence

$$\left||a_k|^{\alpha(z)/\alpha(t)}\right| = |a_k|^{x(\alpha_1-\alpha_0)/(t\,\alpha_0+(1-t)\alpha_1)}$$

Since $x \in [0,1]$ we have

$$\||a_k|^{\alpha(z)/\alpha(t)}\|_{L^\infty(S)} < +\infty$$

and similarly for any $l = 1 \cdots r$

$$\||b_l|^{(1-\beta(z))/(1-\beta(t))}\|_{L^\infty(S)} < +\infty\ .$$

Thus we deduce

$$\|F\|_{L^\infty(S)} < +\infty\ .$$

We shall make use of the following lemma

Lemma 3.25. *Let F be and holomorphic uniformly bounded function on $S := \{z = x + iy \in \mathbb{C}\ ;\ 0 \le x \le 1\}$. Let $M_0 > 0$ and $M_1 > 0$ such that*

$$\|F(iy)\|_{L^\infty_y(\mathbb{R})} \le M_0 \qquad \textit{and} \qquad \|F(1+iy)\|_{L^\infty_y(\mathbb{R})} \le M_1\ ,$$

then

$$\forall x \in [0,1] \qquad \|F(x+iy)\|_{L^\infty_y(\mathbb{R})} \le M_0^{(1-x)}\ M_1^x\ .$$

Proof of Lemma 3.25. The proof is more or less a direct application of the *Maximum Principle.* We replace F by the function

$$\tilde F(z) := \frac{F(z)}{M_0^{(1-z)}\ M_1^z}\ .$$

The new function $\tilde F$ is again holomorphic, bounded and we have by assumption

$$\|\tilde F\|_{L^\infty(\partial S)} \le 1\ .$$

Let

$$\tilde F_k(z) := \tilde F(z)\ e^{\frac{z^2-1}{k}}$$

Since $\tilde F$ is bounded, we have

$$|\tilde F_k(z)| \le \|\tilde F\|_{L^\infty(S)}\ e^{-\frac{y^2}{k}}\ e^{\frac{x^2-1}{k}}\ .$$

Hence

$$\|\tilde F_k(x+iy)\|_{L^\infty_x([0,1])} \longrightarrow 0 \qquad \text{uniformly as } y \to +\infty\ .$$

Applying the Maximum Principle (see the Complex Analysis course in 3rd semester) on sufficiently large rectangles we obtain

$$\forall k \qquad \|\tilde{F}_k\|_{L^\infty(S)} \leq 1 \ .$$

Passing to the limit $k \to +\infty$ we deduce $\|\tilde{F}\|_{L^\infty(S)} \leq 1$ and the lemma is proved. □

Proof of theorem 3.23 continued. Using the previous lemma we have

$$\forall\, z \in S \qquad |F(z)| \leq \left[\sup_{y\in\mathbb{R}} \left|\int_\Omega T(f_{iy})\, g_{iy}\ dx^n\right|\right]^{1-x} \ \left[\sup_{y\in\mathbb{R}} \left|\int_\Omega T(f_{1+iy})\, g_{1+iy}\ dx^n\right|\right]^{x}$$

We have

$$f_{iy} := \sum_{k=1}^m |a_k|^{\alpha(iy)/\alpha(t)}\ e^{i\,\theta_k}\ \mathbf{1}_{E_k} \qquad \text{and} \qquad g_{iy} := \sum_{l=1}^r |b_l|^{(1-\beta(iy))/(1-\beta(t))}\ e^{i\,\phi_l}\ \mathbf{1}_{F_l}\ .$$

Hence in particular almost everywhere

$$|f_{iy}|^{p_0} = |f|^{p_t} \qquad \text{and} \qquad |g_{iy}|^{q_0'} = |g|^{q_t'}\ .$$

Thus

$$\|f_{iy}\|_{L^{p_0}(\Omega)} = \|f\|^{p_t/p_0}_{L^{p_t}(\Omega)} \qquad \text{and} \qquad \|g_{iy}\|_{L^{q_0'}(\Omega)} = \|g\|^{q_t'/q_0'}_{L^{q_t'}(\Omega)}$$

Using the fact that T is continuous from $L^{p_0}(\Omega)$ into $L^{q_0}(\Omega)$ we have

$$\begin{aligned}\left|\int_\Omega T(f_{iy})\, g_{iy}\ dx^n\right| &\leq \|\!|T|\!\|_{L^{p_0}(\Omega)\to L^{q_0}(\Omega)}\ \|f_{iy}\|_{L^{p_0}(\Omega)}\ \|g_{iy}\|_{L^{q_0'}(\Omega)}\\ &\leq \|\!|T|\!\|_{L^{p_0}(\Omega)\to L^{q_0}(\Omega)}\ \|f\|^{p_t/p_0}_{L^{p_t}(\Omega)}\ \|g\|^{q_t'/q_0'}_{L^{q_t'}(\Omega)}\ .\end{aligned}$$

Similarly we have

$$\begin{aligned}\left|\int_\Omega T(f_{1+iy})\, g_{1+iy}\ dx^n\right| &\leq \|\!|T|\!\|_{L^{p_1}(\Omega)\to L^{q_1}(\Omega)}\ \|f_{1+iy}\|_{L^{p_0}(\Omega)}\ \|g_{1+iy}\|_{L^{q_1'}(\Omega)}\\ &\leq \|\!|T|\!\|_{L^{p_1}(\Omega)\to L^{q_1}(\Omega)}\ \|f\|^{p_t/p_1}_{L^{p_t}(\Omega)}\ \|g\|^{q_t'/q_1'}_{L^{q_t'}(\Omega)}\ .\end{aligned}$$

Hence we have

$$\begin{aligned}\left|\int_\Omega T(f)\, g\ dx^n\right| &= \left|\int_\Omega T(f_t)\, g_t\ dx^n\right|\\ &\leq \|\!|T|\!\|^{(1-t)}_{L^{p_0}(\Omega)\to L^{q_0}(\Omega)}\ \|\!|T|\!\|^{t}_{L^{p_1}(\Omega)\to L^{q_1}(\Omega)}\ \|f\|_{L^{p_t}(\Omega)}\ \|g\|_{L^{q_t'}(\Omega)}\ .\end{aligned}$$

Since step functions are dense in $L^{q_t'}(\Omega)$ we deduce

$$\|T(f)\|_{L^{q_t}(\Omega)} \leq \|\!|T|\!\|^{(1-t)}_{L^{p_0}(\Omega)\to L^{q_0}(\Omega)}\ \|\!|T|\!\|^{t}_{L^{p_1}(\Omega)\to L^{q_1}(\Omega)}\ \|f\|_{L^{p_t}(\Omega)} \tag{3.91}$$

Hence the result is proved if f is a step function.

Let f be an arbitrary function in $L^{p_t}(\Omega)$ and, since we are considering the case $p_t < +\infty$, there exists f_k step functions such that

$$\lim_{k\to+\infty} \|f - f_k\|_{L^{p_t}(\Omega)} = 0 \ .$$

Because of remark 3.24, $L^{p_t}(\Omega)$ embeds continuously into $L^{p_0}(\Omega)+L^{p_1}(\Omega)$ and therefore there exists $g_k \in L^{p_0}(\Omega)$ and $h_k \in L^{p_1}(\Omega)$ such that

$$\|g_k\|_{L^{p_0}(\Omega)} + \|h_k\|_{L^{p_1}(\Omega)} \longrightarrow 0 \qquad \text{and} \qquad f - f_k = g_k + h_k \ .$$

Using the hypothesis on T we have that

$$T(f_k) - T(f) \longrightarrow 0 \qquad \text{in} \ \ L^{q_0}(\Omega) + L^{q_1}(\Omega) \ .$$

Hence in particular

$$T(f_k) \longrightarrow T(f) \qquad \text{almost everywhere} \ \ .$$

Using (3.91) for f_k we have

$$\limsup_{k\to+\infty} \|T(f_k)\|_{L^{q_t}(\Omega)} \le |||T|||^{(1-t)}_{L^{p_0}(\Omega)\to L^{q_0}(\Omega)} \ |||T|||^{t}_{L^{p_1}(\Omega)\to L^{q_1}(\Omega)} \ \|f\|_{L^{p_t}(\Omega)}$$

Since $1 \le q_t < +\infty$, we have

$$|T(f_k)|^{q_t} \longrightarrow |T(f)|^{q_t} \quad \text{almost everywhere} \qquad \text{and} \qquad \limsup_{k\to+\infty} \int_\Omega |T(f_k)|^{q_t}(x)\, dx^n < +\infty$$

Using Fatou Lemma we conclude that $T(f) \in L^{q_t}(\Omega)$ and there holds

$$\|T(f)\|_{L^{q_t}(\Omega)} \le |||T|||^{(1-t)}_{L^{p_0}(\Omega)\to L^{q_0}(\Omega)} \ |||T|||^{t}_{L^{p_1}(\Omega)\to L^{q_1}(\Omega)} \ \|f\|_{L^{p_t}(\Omega)}$$

This concludes the proof of the theorem 3.23

We present now two applications of the Riesz-Thorin interpolation theorem. First we recall that the Fourier transform realizes an isometry from L^2 into itself moreover we have proved (1.1)

$$\forall f \in L^1(\mathbb{R}^n) \qquad \|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \le (2\pi)^{-n/2}\, \|f\|_{L^1(\mathbb{R}^n)}$$

Combining these two facts with the Riesz-Thorin interpolation theorem we obtain the famous *Hausdorff Young inequality*

Theorem 3.26. *Let $p \in [1,2]$ and denote by $t \in [0,1]$ the number such that*

$$\frac{1}{p} = (1-t) + \frac{t}{2} = 1 - \frac{t}{2} \qquad i.e \quad p = \frac{2}{2-t} \ ,$$

then $\mathcal{F}$ realizes a continuous mapping from $L^p(\mathbb{R}^n)$ into $L^{p'}(\mathbb{R}^n)$ and there holds

$$\forall\, f \in L^p(\mathbb{R}^n) \qquad \|\hat{f}\|_{L^{p'}(\mathbb{R}^n)} \le (2\pi)^{\frac{(t-1)n}{2}}\, \|f\|_{L^p(\mathbb{R}^n)} \ .$$

The second application of the theorem is a proof of theorem 3.5.

Proof of theorem 3.5. From theorem 3.4 we know that for any $g \in L^p(\mathbb{R}^n)$ and $p \in [1, +\infty]$ the convolution wit g is a continuous linear operator from $L^1(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ and there holds

$$\forall\, f \in L^1(\mathbb{R}^n) \qquad \|g \star f\|_{L^p(\mathbb{R}^n)} \le \|g\|_{L^p(\mathbb{R}^n)}\ \|f\|_{L^1(\mathbb{R}^n)}\ ,$$

moreover, the convolution with g is obviously mapping $L^{p'}(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ and there holds

$$\forall\, f \in L^{p'}(\mathbb{R}^n) \qquad \|g \star f\|_{L^\infty(\mathbb{R}^n)} \le \|g\|_{L^p(\mathbb{R}^n)}\ \|f\|_{L^{p'}(\mathbb{R}^n)}\ .$$

Hence, using Riesz Thorin theorem, we deduce that for any $t \in [0, 1]$ and $p_t \in [1, p']$ given by

$$\frac{1}{p_t} := (1-t) + t\left(1 - \frac{1}{p}\right) = 1 - \frac{t}{p}$$

there holds

$$\forall\, f \in L^{p_t}(\mathbb{R}^n) \qquad \|g \star f\|_{L^{q_t}(\mathbb{R}^n)} \le \|g\|_{L^p(\mathbb{R}^n)}\ \|f\|_{L^{p_t}(\mathbb{R}^n)}$$

where

$$\frac{1}{q_t} = \frac{1-t}{p} = \frac{1}{p} + \frac{1}{p_t} - 1$$

Since p is arbitrary in $[1, +\infty]$ and $p_t \in [1, p']$ is also arbitrary this concludes proof of theorem 3.5. $\square$

4 Riesz Potentials and Sobolev Embeddings

4.1 The Marcinkiewicz Interpolation Theorem

Definition 4.1. *Let $1 \leq p, q \leq \infty$ and let T be a mapping from $L^p(\mathbb{R}^n)$ to the space of measurable functions. For $1 \leq q \leq \infty$, we say that the mapping T is of strong type (p, q) – or simply of type (p, q) – if*

$$\|Tf\|_{L^q} \leq C \, \|f\|_{L^p} \, ,$$

where the constant C is independent of $f \in L^p(\mathbb{R}^n)$. For the case of $q < \infty$, we say that T is of weak type (p, q) if

$$\mu(\{x \in \mathbb{R}^n \; : \; |Tf(x)| > \alpha\}) \leq C \left(\frac{1}{\alpha} \, \|f\|_{L^p} \right)^q \, ,$$

where the constant C is independent of f and $\alpha > 0$. For $q = \infty$, we say that T is of weak type (p, ∞) if T is of type (p, ∞).

Remark 4.1. *Observe that for $q < \infty$, and for any measurable function g we have trivialy*

$$\sup_{\alpha < +\infty} \alpha^q \; |\{x \; : \; |g(x)| > \alpha\}| \leq \|g\|_{L^q}^q \quad . \tag{4.92}$$

Applying this inequality to $g = Tf$ we obtain the fact that T being of type (p, q) is also of weak type (p, q). Let Ω be an open subset of $\mathbb{R}^n$. The space of measurable functions g on Ω satisfying

$$|g|_{q,\infty} := \left[\sup_{\alpha < +\infty} \alpha^q \; |\{x \in \Omega \; : \; |g(x)| > \alpha\}| \right]^{1/q}$$

is called the weak L^q Marcinkiewicz space and denoted $L^{q,\infty}(\Omega)$. $L^{q,\infty}(\Omega)$ is strictly larger than $L^q(\Omega)$. Indeed, for any $x_0 \in \Omega$, $|x - x_0|^{n/q}$ is in $L^{q,\infty}(\Omega)$ and not in $L^q(\Omega)$. It is a quasi-Banach space (see the next chapter) and for $1 < q$ it defines a Banach space in the sense that the quasi-norm $| \cdot |_{q,\infty}$ is equivalent to a norm for which the space $L^{q,\infty}(\Omega)$ is complete. For $q > 1$ functions in $L^{q,\infty}(\mathbb{R}^n)$ define tempered distributions (see chapter 6). These spaces are important in PDE and potential theory, harmonic analysis... because it contains the important family of Riesz operator $|x|^{n-\beta}$ in an optimal way : $|x|^{n-\beta} \in L^{n/(n-\beta),\infty}(\mathbb{R}^n)$ but $|x|^{n-\beta} \notin L^{n/(n-\beta)}(\mathbb{R}^n)$.

We are now proving a new interpolation theorem for which we are considering applications in the following subsections and chapters.

Theorem 4.2 (Marcinkiewicz Interpolation Theorem- The L^p case)**.** *Let $1 < r \leq \infty$ and suppose that T is a subadditive operator from $L^1 + L^r(\mathbb{R}^n)$ to the space of measurable functions, i.e., for all $f, g \in L^1 + L^r(\mathbb{R}^n)$, the following pointwise estimate holds:*

$$\text{for a. e. } x \in \mathbb{R}^n \qquad |T(f + g)|(x) \leq |Tf|(x) + |Tg|(x) \, . \tag{4.93}$$

Moreover, assume that T is of weak type $(1,1)$ and also of weak type (r,r). Then, for $1 < p < r$, we have that T is of type (p,p) meaning that

$$\|Tf\|_{L^p} \leq C\,\|f\|_{L^p}\,,$$

for all $f \in L^p(\mathbb{R}^n)$.

Proof of theorem 4.2.

To simplify the presentation we restrict to the case $r < +\infty$. As in the proof of theorem 5.5, for an arbitrary parameter $\alpha > 0$, we introduce the following function

$$f_1(x) := \begin{cases} f(x) & \text{if} \quad |f(x)| > \alpha \\ \\ 0 & \text{if} \quad |f(x)| \leq \alpha \end{cases},$$

and we denote $f_2(x) := f(x) - f_1(x)$ in such a way that $|f_2(x)| \leq \alpha$. The sub-additivity of T gives then $|Tf(x)| \leq |Tf_1(x)| + |Tf_2(x)|$ and from this we deduce that

$$\{x\ ;\ |Tf(x)| > \alpha\} \subset \{x\ ;\ |Tf_1(x)| > \alpha/2\} \cup \{x\ ;\ |Tf_2(x)| > \alpha/2\} \quad .$$

Hence, using (5.3) and (5.4), we bound $d_{Tf}(\alpha) = |\{x\ ;\ |Tf(x)| > \alpha\}|$ as follows

$$\begin{aligned} d_{Tf}(\alpha) &\leq d_{Tf_1}(\alpha/2) + d_{Tf_2}(\alpha/2) \\ &\leq \frac{2\,C_1}{\alpha}\|f_1\|_{L^1} + \frac{2^r\,C_r^r}{\alpha^r}\|f_2\|_{L^r}^r \qquad (4.94) \\ &\leq \frac{2\,C_1}{\alpha}\int_{E_\alpha} |f(y)|\;dy + \frac{2^r\,C_r^r}{\alpha^r}\int_{\mathbb{R}^n\setminus E_\alpha} |f(y)|^r\;dy \end{aligned}$$

where E_α denotes as usual the set $\{x\ ;\ |f(x)| > \alpha\}$,

$$C_1 = \sup \frac{|Tf|_{L^1_w}}{\|f\|_{L^1}}\,,$$

and

$$C_r = \sup \frac{|Tf|_{L^{r,\infty}}}{\|f\|_{L^r}}\,,$$

and where we have also applied inequality (4.92).

Expressing now the L^p norm of Tf by the mean of lemma 5.2 and combining it with (4.94) we get, using Fubini in the third line,

$$\begin{aligned} &\frac{1}{2^r}\int_{\mathbb{R}^n} |Tf(x)|^p\;dx = \frac{p}{2^r}\int_0^{+\infty} \alpha^{p-1}\,d_{Tf}(\alpha)\;d\alpha \\ &\leq p\,C_1\int_0^{+\infty} \alpha^{p-2}\,d\alpha\int_{E_\alpha} |f(y)|\;dy + p\,C_r^r\int_0^{+\infty} \alpha^{p-1-r}\,d\alpha\int_{\mathbb{R}^n\setminus E_\alpha} |f(y)|^r\;dy \\ &= p\,C_1\int_{\mathbb{R}^n} |f(y)|\;dy\int_0^{|f(y)|} \alpha^{p-2}\;d\alpha + p\,C_r^r\int_{\mathbb{R}^n} |f(y)|^r\;dy\int_{|f(y)|}^{+\infty} \alpha^{p-1-r}\;d\alpha \\ &\leq 2^r\,p\left(\frac{C_1}{p-1} + \frac{C_r^r}{p-r}\right)\int_{\mathbb{R}^n} |f(y)|^p\;dy \quad , \end{aligned}$$

which proves the theorem. □ There is a stronger version of this theorem

Theorem 4.3 (Marcinkiewicz Interpolation Theorem- The $L^p - L^q$ case)**.** *Let $1 \le p_i \le q_i \le +\infty$ for $i = 0, 1$ with $p_0 < p_1$ and $q_0 \ne q_1$ Let T is a sub additive operator from $L^{p_0} + L^{p_1}(\mathbb{R}^n)$ to the space of measurable functions, i.e., for all $f, g \in L^1 + L^r(\mathbb{R}^n)$, the following pointwise estimate holds:*

$$\text{for a. e. } x \in \mathbb{R}^n \qquad |T(f+g)|(x) \le |Tf|(x) + |Tg|(x)\,. \tag{4.95}$$

Moreover, assume that T is of weak type (p_0, q_0) and also of weak type (p_1, q_1). Let $t \in (0,1)$ and denote

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \text{and} \qquad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

Then T is of strong type (p_t, q_t), that is

$$\|Tf\|_{L^{q_t}(\mathbb{R}^n)} \le C\, \|f\|_{L^{p_t}(\mathbb{R}^n)}\,,$$

for all $f \in L^{p_t}(\mathbb{R}^n)$ and $C > 0$ is independent of t.

See a proof in Appendix B of [4].

4.2 The Hardy-Littlewood-Sobolev Theorem for fractional integration and the L^p theory for the fractional Laplacians $(-\Delta)^{\alpha/2}$

In this part we are interested with the operator

$$I_\alpha \;:\; f \in \mathcal{S}(\mathbb{R}^n) \;\longrightarrow\; \frac{1}{|x|^{n-\alpha}} \star f \in \mathcal{S}'(\mathbb{R}^n)$$

where $\alpha \in (0, n)$. Observe that $\frac{1}{|x|^{n-\alpha}} \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$. Indeed

$$\frac{1}{|x|^{n-\alpha}} = \frac{1}{|x|^{n-\alpha}}\, \mathbf{1}_{B_1(0)} + \frac{1}{|x|^{n-\alpha}}\, \mathbf{1}_{\mathbb{R}^n \setminus B_1(0)}$$

Thanks to Young inequality (remember that $\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ continuously) we have that for any $f \in \mathcal{S}(\mathbb{R}^n)$

$$\left\| \frac{1}{|x|^{n-\alpha}} \star f \right\|_{L^1 \cap L^\infty(\mathbb{R}^n)} \le C_{\alpha,n}\, \|f\|_{L^1 \cap L^\infty(\mathbb{R}^n)}$$

The purpose of the following theorem is to show that I_α extends to a continuous operator on some $L^p(\mathbb{R}^n)$ spaces into to some $L^q(\mathbb{R}^n)$. Precisely we have the following.

Theorem 4.4. *Let $0 < \alpha < n$ and let $1 \le p < n/\alpha$. Denote*

$$\frac{1}{q} := \frac{1}{p} - \frac{\alpha}{n}\,. \tag{4.96}$$

Assume first $p > 1$ then, I_α extends as a continuous operator from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, that is, there exists $C_{p,\alpha,n} > 0$ such that

$$\forall\, f \in L^p(\mathbb{R}^n) \qquad \left\| \frac{1}{|x|^{n-\alpha}} \star f \right\|_{L^q(\mathbb{R}^n)} \le C_{p,\alpha,n}\ \|f\|_{L^p(\mathbb{R}^n)} \ .$$

Moreover, for any $f \in L^1(\mathbb{R}^n)$ the map $I_\alpha(f) \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ satisfies

$$\left| \frac{1}{|x|^{n-\alpha}} \star f \right|_{L^{q,\infty}(\mathbb{R}^n)} := \sup_{\lambda>0}\ \lambda\ \left|\{x \in \mathbb{R}^n\ ;\ |I_\alpha(f(x))| > \lambda\}\right|^{\frac{1}{q}} \le C_{\alpha,n}\ \|f\|_{L^p(\mathbb{R}^n)} \ .$$

where $C_{\alpha,n} > 0$ is independent of f.

Remark 4.5. *Recall from Serie 4 that*

$$\mathcal{F}\left(\frac{1}{|x|^{n-\alpha}}\right)(\xi) = 2^{\alpha-\frac{n}{2}}\ \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}\ |\xi|^{-\alpha}$$

Hence there exists $\gamma_{\alpha,n} > 0$ such that

$$\forall\, f \in \mathcal{S}(\mathbb{R}^n) \qquad \mathcal{F}\left(I_\alpha(f)\right)(\xi) := \mathcal{F}\left(\frac{1}{|x|^{n-\alpha}} \star f\right)(\xi) = \gamma_{\alpha,n}\ |\xi|^{-\alpha}\ \hat{f}$$

Recall that

$$\forall\, f \in \mathcal{S}(\mathbb{R}^n) \qquad \mathcal{F}\left(-\Delta f\right)(\xi) = |\xi|^2\ \hat{f}\ .$$

This justify the notation (common in the literature)

$$I_\alpha = \gamma_{\alpha,n}\ (-\Delta)^{-\alpha/2}\ .$$

The fractional laplacian $(-\Delta)^{\alpha/2}$ plays a central role in several areas of mathematics going from stochastic processes to the geometric analysis of "free boundaries"...etc.

Proof of theorem 4.4. We first aim at proving that for any $1 \le p < n/\alpha$ the operator I_α is of weak type (p,q) where q is given by (4.96). Let $\mu > 0$ to be fixed later. We proceed to the decomposition

$$\frac{1}{|x|^{n-\alpha}} = \frac{1}{|x|^{n-\alpha}}\,\mathbf{1}_{B_\mu(0)} + \frac{1}{|x|^{n-\alpha}}\,\mathbf{1}_{\mathbb{R}^n\setminus B_\mu(0)}\ .$$

We denote respectively

$$K_1(x) := \frac{1}{|x|^{n-\alpha}}\,\mathbf{1}_{B_\mu(0)} \quad , \quad K_\infty(x) := \frac{1}{|x|^{n-\alpha}}\,\mathbf{1}_{\mathbb{R}^n\setminus B_\mu(0)} \qquad \text{and} \qquad K = K_1 + K_\infty = \frac{1}{|x|^{n-\alpha}}\ .$$

Let $f \in L^p(\mathbb{R}^n)$ such that $\|f\|_{L^p(\mathbb{R}^n)} = 1$. Since $K_1 \in L^1(\mathbb{R}^n)$, $K_1 \star f \in L^p(\mathbb{R}^n)$ and since $K_\infty \in L^q(\mathbb{R}^n)$ for any $q > n/(n-\alpha)$ (i.e. $q^{-1} < 1-\frac{\alpha}{n}$), since $\frac{1}{p'} = 1-\frac{1}{p} < 1-\frac{\alpha}{n}$, we have that $K_\infty \in L^{p'}(\mathbb{R}^n)$ and $K_\infty \star f \in L^\infty(\mathbb{R}^n)$. Thus

$$\forall\ f \in L^p(\mathbb{R}^n) \qquad K \star f \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)\ .$$

We have obviously for any $\lambda > 0$

$$\{x \in \mathbb{R}^n\ ;\ |K \star f(x)| > 2\lambda\} \subset \{x \in \mathbb{R}^n\ ;\ |K_1 \star f(x)| > \lambda\} \cup \{x \in \mathbb{R}^n\ ;\ |K_\infty \star f(x)| > \lambda\}$$

and consequently

$$\begin{aligned} |\{x \in \mathbb{R}^n\ ;\ |K \star f(x)| > 2\lambda\}| &\leq |\{x \in \mathbb{R}^n\ ;\ |K_1 \star f(x)| > \lambda\}| \\ &+ |\{x \in \mathbb{R}^n\ ;\ |K_\infty \star f(x)| > \lambda\}|\ . \end{aligned} \tag{4.97}$$

Using Young inequality we have first

$$\begin{aligned} \lambda^p\ |\{x \in \mathbb{R}^n\ ;\ |K_1 \star f(x)| > \lambda\}| &\leq \int_{|K_1 \star f(x)| > \lambda} |K_1 \star f(x)|^p\ dx^n \\ &\leq \|K_1 \star f\|^p_{L^p(\mathbb{R}^n)} \leq \|K_1\|^p_{L^1(\mathbb{R}^n)}\ \|f\|^p_{L^p(\mathbb{R}^n)} \end{aligned}$$

and we compute

$$\|K_1\|_{L^1(\mathbb{R}^n)} = \int_{|x| \leq \mu} |x|^{-n+\alpha}\ dx^n = c_{n,\alpha}\ \mu^\alpha\ .$$

Hence finally we obtain

$$|\{x \in \mathbb{R}^n\ ;\ |K_1 \star f(x)| > \lambda\}| \leq c^p_{n,\alpha}\ \mu^{p\,\alpha}\, \lambda^{-p}\ \|f\|^p_{L^p(\mathbb{R}^n)} = c^p_{n,\alpha}\ \mu^{p\,\alpha}\, \lambda^{-p}\ .$$

For the contribution issued from $K_\infty \star f$ we proceed as follows

$$\|K_\infty \star f\|_{L^\infty(\mathbb{R}^n)} \leq \|K_\infty\|_{L^{p'}(\mathbb{R}^n)}\ \|f\|_{L^p(\mathbb{R}^n)} = \|K_\infty\|_{L^{p'}(\mathbb{R}^n)}\ .$$

We compute

$$\|K_\infty\|_{L^{p'}(\mathbb{R}^n)} = \left[\int_{\mathbb{R}^n} |x|^{(-n+\alpha)\,p'}\ dx^n\right]^{1/p'} = \tilde{c}_{n,\alpha,p}\ \mu^{-n/q}$$

For λ fixed we choose $\tilde{c}_{n,\alpha,p}\ \mu^{-n/q} = \lambda$ so that

$$\begin{aligned} |\{x \in \mathbb{R}^n\ ;\ |K \star f(x)| > \lambda\}| &\leq \lambda^{-p}\ \|K_1\|^p_{L^1(\mathbb{R}^n)} = c^p_{n,\alpha}\ \left(\frac{\mu^\alpha}{\lambda}\right)^p \\ = C_{n,\alpha,p}\ \left(\frac{\lambda^{-\frac{\alpha\, q}{n}}}{\lambda}\right)^p &= C_{n,\alpha,p}\ \lambda^{-q}\ . \end{aligned}$$

Hence by linearity we have finally proved

$$\forall\ f \in \mathcal{S}(\mathbb{R}^n) \qquad |K \star f|_{L^{q,\infty}(\mathbb{R}^n)} \leq C_{n,\alpha,p}\ \|f\|_{L^p(\mathbb{R}^n)}\ . \tag{4.98}$$

Thus is weak (p,q) for every $p \in [1, n/\alpha)$. Let $1 < p < n/\alpha$. We choose $p_0 = 1$ and $p < p_1 < n/\alpha$. We have that T is of weak type $(1, n/(n-\alpha))$ and that T is of weak type (p_1, q_1) where

$$\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}\ .$$

Let t such that
$$\frac{1}{p} = 1 - t + \frac{t}{p_1} \ .$$
We have
$$\frac{1}{q} = (n-\alpha)\,\frac{1-t}{n} + \frac{t}{q_1} = (n-\alpha)\,\frac{1-t}{n} + \frac{t}{p_1} - t\,\frac{\alpha}{n} = \frac{1}{p} - \frac{\alpha}{n} \ .$$
Using Marcinkiewicz Interpolation theorem 4.3. This concludes the proof of theorem 4.4. $\Box$

4.3 Sobolev Inequalities

An application of the Hardy-Littlewood-Sobolev Theorem for fractional integration is the following Sobolev inequality.

Theorem 4.6. *Let $1 < p < n$ and let $1 < p^* < +\infty$ given by*
$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$$
then there exists $C_{p,n} > 0$ such that
$$\forall\, f \in \mathcal{S}(\mathbb{R}^n) \qquad \|f\|_{L^{p^*}(\mathbb{R}^n)} \le C_{s,n}\ \|\nabla f\|_{L^p(\mathbb{R}^n)} \ .$$

This inequality is an example of a vast family of inequalities called Sobolev inequalities on bounded or unbounded domains (see [1]).

Proof of theorem 4.6. Let G be the fundamental solution to the laplacian in $\mathbb{R}^n$ given by theorem 1.59. We have
$$\forall\, f \in \mathcal{S}(\mathbb{R}^n) \qquad f = \delta_0 \star f = \Delta G \star f = -\sum_{j=1}^{n} \partial_{x_j} G \star \partial_{x_j} f \ .$$

Hence

$$\forall\ x \in \mathbb{R}^n \qquad |f(x)| \le \sum_{j=1}^{n} |\partial_{x_j} G| \star |\partial_{x_j} f|(x) \tag{4.99}$$

Observe from the explicit expression of G given by theorem 1.59 we obtain the existence of $C_n > 0$ such that

$$\forall\ x \in \mathbb{R}^n \qquad |\nabla G|(x) \le \frac{C_n}{|x|^{n-1}} \ . \tag{4.100}$$

Combining (4.99), (4.100) and Hardy-Littlewood-Sobolev Theorem for fractional integration we obtain theorem 4.6. $\Box$

5 The Hardy-Littlewood Maximal Function

5.1 Definition

The Lebesgue measure on $\mathbb{R}^n$ will be denoted by μ. By *measurable function* or *measurable set* in this book we implicitly mean *measurable function with respect to* μ or *measurable set with respect to* μ unless we precise the underlying measure. Integration along a variable x in $\mathbb{R}^n$ with respect to the Lebesgue measure on $\mathbb{R}^n$ will be simply denoted by dx.

If E is a measurable set, we denote by χ_E it's characteristic function.

Definition 5.1. *For a measurable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, we define its associated distribution function by*

$$d_f(\alpha) = \mu(\{x \in \mathbb{R}^n \ : \ |f(x)| > \alpha\}) \,,$$

with $\alpha \geq 0$.

With these notations we establish the following lemma.

Lemma 5.2. *For a measurable function f and $0 < p < \infty$, we have*

$$\|f\|^p_{L^p} = p \int_0^\infty \alpha^{p-1} d_f(\alpha)\, d\alpha \,. \tag{5.1}$$

Proof of lemma 5.2. From elementary calculus, we get

$$|f(x)|^p = p \int_0^{|f(x)|} \alpha^{p-1}\, d\alpha = p \int_0^\infty \alpha^{p-1} \chi_{\{x\,:\,\alpha<|f(x)|\}}\, d\alpha \,.$$

By integration over $\mathbb{R}^n$ and Fubini's theorem, it then follows

$$\|f\|^p_{L^p} = p \int_0^\infty \alpha^{p-1} \left(\int_{\mathbb{R}^n} \chi_{\{x\,:\,|f(x)|>\alpha\}}\, dx \right) d\alpha = p \int_0^\infty \alpha^{p-1} d_f(\alpha)\, d\alpha \,.$$

□

For every x in $\mathbb{R}^n$ and every $r > 0$ we denote by $B_r(x)$ the euclidian ball of center x and radius r.

Definition 5.3. *For a locally integrable function $f \in L^1_{loc}(\mathbb{R}^n)$, we define its associated Hardy-Littlewood maximal function at the point x by*

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y)|\, dy \in \mathbb{R}_+ \cup \{+\infty\} \quad . \tag{5.2}$$

We now prove the following elementary proposition.

Proposition 5.4. *Let f be a locally integrable function, then Mf is measurable function into $[0, +\infty]$. Moreover, if $f \in L^1(\mathbb{R}^n)$ then $Mf(x)$ is finite almost everywhere.*

Proof of Proposition 5.4. For any measurable function in L^1_{loc} one easily check that the map
$$(r,x) \longrightarrow A_r f(x) = \frac{1}{\mu(B_r(x))}\int_{B_r(x)} |f(x)|\,dy$$
is continuous. It implies in one hand that, for a fixed r, $A_r f(x)$ is measurable and it also implies, in the other hand, that taking the supremum at a point x among the real radii, $r \in \mathbb{R}$, coincide with the supremum among rational radii, $r \in \mathbb{Q}$. Since the supremum function of countably many measurable functions is measurable (1.1.2 in [?]), we deduce that $Mf(x)$ is measurable. The second part of the statement in proposition 5.4 is a direct consequence of Lebesgue-Besicovitch differentiation theorem (1.7.1 in [?]). It also follows from Theorem 5.5 below. □

From the Lebesgue-Besicovitch differentiation theorem (1.7.1 in [?]) we deduce the pointwise estimate $|f(x)| \leq |Mf(x)|$ which holds almost everywhere for any locally integrable function. Therefore, for every $p \in [1,+\infty]$, and for every function f in $L^p(\mathbb{R}^n)$, we obtain the identity
$$\|f\|_{L^p(\mathbb{R}^n)} \leq \|Mf\|_{L^p(\mathbb{R}^n)} \quad .$$
The following important result gives the reverse estimate when $p > 1$ and "almost" but not quite the reverse estimate when $p = 1$.

5.2 Hardy-Littlewood Maximal Function Theorem

Theorem 5.5 (Hardy-Littlewood Maximal Function Theorem)**.** *Let* $1 < p \leq \infty$ *and* $f \in L^p(\mathbb{R}^n)$*. Then, we have*

$$(5.3) \qquad \|Mf\|_{L^p} \leq 2\left(\frac{5^n\,p}{p-1}\right)^{1/p} \|f\|_{L^p} \quad .$$

Moreover, for $f \in L^1(\mathbb{R}^n)$ *and* $\alpha > 0$*, we have*

$$(5.4) \qquad \mu(\{x\,:\,Mf(x) > \alpha\}) \leq \frac{5^n}{\alpha}\,\|f\|_{L^1} \quad .$$

Remark 5.1. *The last identity (5.4) is saying that the maximal function of an* L^1 *function is in the space* L^1-*weak (denoted also* $L^1_w(\mathbb{R}^n)$*). This space is given by the subset of measurable functions on* $\mathbb{R}^n$ *satisfying*

$$(5.5) \qquad |f|_{L^1_w} = \sup_{\alpha>0}\{\alpha\ \mu(\{x \in \mathbb{R}^n\ :\ |f(x)| > \alpha\})\} \quad .$$

L^1-*weak functions do not define a-priori distributions. A typical example of a function in* L^1_w *is* $|x|^{-n}$ *in* $\mathbb{R}^n$*.* $|\cdot|_{L^1_w}$ *defines a quasi-norm on* L^1_w *- the triangle inequality is satisfied modulo a constant, which is* 2 *in the present case - and* L^1_w *is complete for this quasi-norm which makes* L^1_w *to be a quasi-Banach space by definition. However it is very important to remember that* L^1_w *cannot be made to be a Banach space with a norm equivalent to the quasi norm given by* $|\cdot|_{L^1_w}$*. If it would be the case Calderón-Zygmund theory, and this book in particular, would dramatically shrink to almost nothing ! We discuss this fact later in this chapter when we come to the Singular Integral Operators.*

The proof of the Hardy-Littlewood Maximal Function Theorem that we are giving uses the following famous covering lemma.

Lemma 5.6 (Vitali's Covering Lemma)**.** *Let E be measurable subset of $\mathbb{R}^n$ and let $\mathcal{F} = \{B_j\}_{j\in J}$ be a family of euclidian balls with uniformly bounded diameter i.e., $\sup_j diam(B_j) = R < \infty$, such that $E \subset \bigcup_j B_j$. Then, there exists an at most countable subfamily $\{B_{j_k}\}_{k\in\mathbb{N}}$ of disjoint balls satisfying*

$$\mu(E) \leq 5^n \sum_{k=1}^{\infty} \mu(B_{j_k})\,. \tag{5.6}$$

Proof of lemma 5.6. For any $i \in \mathbb{N}$ we denote

$$\mathcal{F}_i = \left\{B_j \in \mathcal{F} \ ; \ 2^{-i-1}R < \text{diam } B_j \leq 2^{-i}R\right\} \quad .$$

We shall now extract our sub-covering step by step in $\mathcal{F}_i$ by induction on i.

- Denote by $\mathcal{G}_0$ a maximal disjoint collection of balls in $\mathcal{F}_0$.
- Assuming $\mathcal{G}_0, \cdots, \mathcal{G}_k$ have been selected, we choose $\mathcal{G}_{k+1}$ to be a maximal collection of balls in $\mathcal{F}_{k+1}$ such that each ball in this collection is disjoint from the balls in $\cup_{i=0}^{k}\mathcal{G}_i$.

We claim now that $\mathcal{G} = \cup_{i=0}^{\infty}\mathcal{G}_i$ is a suitable solution to the lemma.

It is by construction a sub-family of $\mathcal{F}$ made of disjoint balls. Let B_j be in $\mathcal{F}$. There exists $i \in \mathbb{N}$ such that $B_j \in \mathcal{F}_i$. If B_j would intersect none of the balls in $\mathcal{G}_i$ it would contradict the fact that $\mathcal{G}_i$ has been chosen to be maximal. Hence, for any $B_j \in \mathcal{F}_i$ there exist $B \in \mathcal{G}_i$ such that $B \cap B_j \neq \emptyset$. Since the ratio between the two diameters of respectively B and B_j is contained in $(2^{-1}, 2)$, the concentric ball $\widehat{B}$ to B having a radius 5 times larger than the one of B contains necessarily B_j. This proves that $E \subset \cup_{B\in\mathcal{G}}\widehat{B}$ and this finishes the proof of the lemma. □

Proof of theorem 5.5. We first consider the case $p = 1$ and prove (5.4). Let

$$E_\alpha = \{x \in \mathbb{R}^n \ ; \ Mf(x) > \alpha\} \quad .$$

By definition, for any $x \in E_\alpha$ there exists an euclidian ball B_x of center x such that

$$\int_{B_x} |f(y)| \ dy > \alpha\mu(B_x) \quad . \tag{5.7}$$

Since f is assumed to be in L^1, the size of the balls B_x is controlled as follows : $\mu(B_x) \leq \alpha^{-1}\|f\|_{L^1}$. Hence the family $\{B_x\}_{x\in E_\alpha}$ realizes a covering of E_α by balls of uniformly bounded radii. We are then in the position to apply Vitali's covering lemma 5.6. Let $(B_k)_{k\in K}$ be an at most countable sub-family to $\{B_x\}$ given by this lemma 5.6. (B_k) are disjoint balls satisfying

$$\sum_{k\in K} \mu(B_k) \geq \frac{1}{5^n}\mu(E_\alpha) \quad .$$

Combining this last inequality and (5.7) gives

$$\|f\|_{L^1(\mathbb{R}^n)} \geq \int_{\cup_{k\in K}^{+\infty} B_k} |f(y)|\, dy > \alpha \sum_{k\in K} \mu(B_k) \geq \frac{\alpha}{5^n}\mu(E_\alpha) \quad .$$

This is proves the desired inequality (5.4).

We establish now (5.3) for $1 < p < +\infty$ (the case $p = +\infty$ being straightforward). Define

$$f_1(x) := \begin{cases} f(x) & \text{if} \quad |f(x)| \geq \alpha/2 \\ 0 & \text{if} \quad |f(x)| < \alpha/2\ . \end{cases}$$

This definition implies the following inequalities $|f(x)| \leq |f_1(x)| + \alpha/2$ and also $|Mf(x)| \leq |Mf_1(x)| + \alpha/2$ which hold for almost every $x \in \mathbb{R}^n$. Hence we have

$$E_\alpha = \{x \in \mathbb{R}^n\ ;\ Mf(x) > \alpha\} \subset \{x \in \mathbb{R}^n\ ;\ Mf_1(x) > \alpha/2\} \quad . \tag{5.8}$$

Observe that, for any $\alpha > 0$, $f_1 \in L^1(\mathbb{R}^n)$. Indeed

$$\int_{\mathbb{R}^n} |f_1(y)|\, dy \leq \left(\frac{2}{\alpha}\right)^{p-1} \int_{\mathbb{R}^n} |f(y)|^p\, dy < +\infty \quad .$$

Thus we can apply identity (5.4) to f_1 and this gives, using (5.8),

$$\begin{aligned} \mu(E_\alpha) &\leq \mu\left(\{x \in \mathbb{R}^n\ ;\ Mf_1(x) > \alpha/2\}\right) \leq \tfrac{2\cdot 5^n}{\alpha}\|f_1\|_{L^1} \\ &\leq \tfrac{2\cdot 5^n}{\alpha} \int_{\{x\ :\ |f(x)|\geq\alpha/2\}} |f(y)|\, dy \quad . \end{aligned} \tag{5.9}$$

Next, we deduce from Lemma 5.2 that

$$\begin{aligned} \|Mf\|_{L^p}^p &= p\int_0^\infty \alpha^{p-1}\mu(E_\alpha)\, d\alpha \\ &\overset{(5.9)}{\leq} p\int_0^\infty \alpha^{p-1}\left(\frac{2\cdot 5^n}{\alpha}\int_{\{x\,:\,|f(x)|\geq\alpha/2\}} |f(x)|\, dx\right) d\alpha \\ &= p\int_0^\infty \alpha^{p-1}\left(\frac{2\cdot 5^n}{\alpha}\int_{\mathbb{R}^n} \chi_{\{x\,:\,|f(x)|\geq\alpha/2\}}|f(x)|\, dx\right) d\alpha\, . \end{aligned}$$

Using Fubini's theorem it follows

$$\begin{aligned} \|Mf\|_{L^p}^p &\leq 2\cdot 5^n\, p\int_{\mathbb{R}^n} |f(x)|\left(\int_0^{2|f(x)|} \frac{\alpha^{p-1}}{\alpha}\, d\alpha\right) dx \\ &= \frac{2C\,p}{p-1}\int_{\mathbb{R}^n} |f(x)|\, 2^{p-1}|f(x)|^{p-1}\, dx \quad , \end{aligned}$$

since $p > 1$ by assumption. Thus we arrive at the desired result

$$\|Mf\|_{L^p} \leq 2\left(\frac{5^n\, p}{p-1}\right)^{1/p} \|f\|_{L^p} \quad .$$

□

Remark 5.7. *The best constant in the previous theorem, both in* (5.3) *and in* (5.4)*, is far from being known. For* $1 < p < \infty$*, a remarkable result by Stein is that the optimal constant stays bounded as* n *goes to infinity. Whether this holds or not for the optimal constant in* (5.4) *is still an open problem. However, one can easily replace* 5^n *with* 2^n*. Indeed, observe that the constant* 5 *in Vitali's covering theorem can be replaced with* $3+3\epsilon$ *for every* $\epsilon > 0$ *(just using* $(1+\epsilon)$ *in place of* 2 *when comparing the radii of the balls). Moreover, here we are interested in a disjoint family of balls whose dilations cover just the set of* centers *of the original family: this allows to replace* 5^n *with* $(2+2\epsilon)^n$ *for every* ϵ*.*

5.3 The limiting case $p = 1$.

It is important to emphasize that inequality (5.3) does not extend to the limiting case $p = 1$: the maximal operator M is not bounded from $L^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Assume f is a non zero integrable function on $\mathbb{R}^n$ then Mf is not integrable on $\mathbb{R}^n$. Indeed, for a non zero f there exists an euclidian ball $B_r(0)$ such that

$$\int_{B_r(0)} |f(y)|\, dy = \eta \neq 0 \quad .$$

Let x be an arbitrary point in $\mathbb{R}^n \backslash B_r(0)$. For such a point x one has $B_r(0) \subset B_{2|x|}(x)$, hence, it follows that

$$\begin{aligned} Mf(x) &= \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y)|\, dy \\ &\geq \frac{1}{\mu(B_{2|x|}(x))} \int_{B_{2|x|}(x)} |f(y)|\, dy \\ &\geq \frac{1}{\mu(B_{2|x|}(x))} \int_{B_r(0)} |f(y)|\, dy \geq \frac{C\,\eta}{|x|^n}\,, \end{aligned}$$

showing that the integrability of Mf fails at infinity.

Even worth, the integrability of the function f does not ensure the local integrability of Mf. We illustrate this fact by the following example: For $n = 1$ consider the positive function

$$f(t) = \frac{1}{t(\log t)^2} \chi_{(0,1)}\,,$$

which is integrable on $[0, 1/2]$.For $t \in (0, 1/2)$, let $B_t(t) = (0, 2t)$ and we have

$$\begin{aligned} Mf(t) &\geq \frac{1}{2t} \int_0^{2t} \frac{1}{t(\log t)^2}\, dt \\ &= \frac{1}{2t} \left(-\frac{1}{\log t} \right)\Big|_0^{2t} = -\frac{1}{2t(\log 2t)}\,. \end{aligned}$$

This directly gives that Mf is not integrable over the interval $[0, 1/2]$.

If we assume "slightly" more than the integrability of f one can reach the local integrability of Mf. Denote by $L^1 \log L^1(\mathbb{R}^n)$ the following Orlicz space

$$L^1 \log L^1(\mathbb{R}^n) = \left\{ f \in L^1(\mathbb{R}^n) \quad ; \quad \int_{\mathbb{R}^n} |f|(y)\, \log \left(e + \frac{|f(y)|}{\|f\|_{L^1}} \right)\, dy < +\infty \right\} \quad .$$

This space is of particular interest for applications due to the fact in particular that the $L^1 \log L^1$ control of a non-negative integrable function f can be interpreted as an "entropy control" of the probability f - assuming it has been normalized in such a way that $\int_{\mathbb{R}^n} f = 1$ -. Back to real-variable function space theory *per se*, we shall probably see in the next chapter that $L^1 \log L^1$ coincide with the non-homogeneous Hardy space for non-negative functions which makes also $L^1 \log L^1$ particularly interesting.

Observe that a norm can be assigned to this subspace of integrable functions by taking the Luxembourg norm :

$$\|f\|_{L^1 \log L^1} := \|f\|_{L^1} + \inf \left\{ t > 0 \ ; \ \int_{\mathbb{R}^n} \frac{|f(y)|}{t} \ \log^+ \frac{|f(y)|}{t} \ dy \right\} \quad .$$

Theorem 5.8. *Let f be a measurable function in $L^1 \log L^1(\mathbb{R}^n)$, then $Mf \in L^1_{loc}(\mathbb{R}^n)$ and for any measurable subset A of finite Lebesgue measure the following inequality holds*

$$\int_A |Mf|(y) \ dy \leq C_n \int_{\mathbb{R}^n} |f|(y) \ \log \left(e + \mu(A) \ \frac{|f(y)|}{\|f\|_{L^1}} \right) \ dy \quad , \tag{5.10}$$

where $C_n > 0$ only depends on n.

Proof of theorem 5.8. From lemma 5.2 we express the L^1 norm of Mf as follows

$$\int_A |Mf|(y) \ dy \leq \int_0^{+\infty} \mu \left(\{ x \in A; \ |Mf|(x) > \alpha \} \right) \ d\alpha \quad .$$

Denote μ^A the restriction of the Lebesgue measure to A and use again the notation $E_\alpha = \{x \in \mathbb{R}^n; \ |Mf|(x) > \alpha\}$. Let $\delta > 0$ to be chosen later on. We write

$$\begin{aligned} \int_A |Mf|(y) \ dy \ &\leq \int_0^\delta \mu^A(E_\alpha) \ d\alpha + \int_\delta^{+\infty} \mu^A(E_\alpha) \ d\alpha \\ &\leq \delta \ \mu(A) + 2 \int_{\delta/2}^{+\infty} \mu(E_{2\alpha}) \ d\alpha \quad . \end{aligned} \tag{5.11}$$

Applying inequality (5.9) to (5.11) gives

$$\begin{aligned} \int_A |Mf|(y) \ dy \ &\leq \delta \ \mu(A) + 2 \cdot 5^n \int_{\delta/2}^{+\infty} \frac{d\alpha}{\alpha} \int_{\{x \ ; \ |f(x)| > \alpha\}} |f(y)| \ dy \\ &\leq \delta \ \mu(A) + 2 \cdot 5^n \int_{\mathbb{R}^n} |f(y)| \ \log^+ \frac{2|f(y)|}{\delta} \ dy \quad , \end{aligned}$$

where $\log^+ \cdot = \max\{0, \log \cdot\}$. Choosing $\delta = \int_{\mathbb{R}^n} |f(y)| \ dy / 2\mu(A)$ gives inequality (5.10) and theorem 5.8 is proved. $\square$

A converse of theorem 5.8 will be given in the next subsection - see theorem 7.4 - once we will have at our disposal the Calderón-Zygmund decomposition.

6 Quasi-normed vector spaces

6.1 Definition and examples

In the following, $\mathbb{K}$ will denote either $\mathbb{R}$ or $\mathbb{C}$ (since the theory below works equally well for real or complex coefficients).

Definition 6.1. *A* topological vector space *over $\mathbb{K}$ is a $\mathbb{K}$-vector space V with a topology τ such that*

- *the sum, i.e. $+ : V \times V \to V$, is continuous,*
- *the multiplication by scalar, i.e. $\cdot : \mathbb{K} \times V \to V$, is continuous,*
- *the topology τ is Hausdorff.*

Example 6.2. *A normed vector space $(V, \| \|)$ is a topological vector space with the topology induced by the canonical distance, namely $d(x, y) := \|x - y\|$.*

Definition 6.3. *Let V be a $\mathbb{K}$-vector space. A* quasi-norm *on V is a function $| \cdot | : V \to [0, \infty)$ such that*

- *$|x| = 0$ if and only if $x = 0$,*
- *for all $\lambda \in \mathbb{K}$ and all $x \in V$ we have $|\lambda x| = |\lambda||x|$,*
- *there exists a constant $C \geq 1$ such that, for all $x, y \in V$, we have*
$$|x + y| \leq C(|x| + |y|).$$

The couple $(V, | |)$ is called a quasi-normed vector space.

Remark 6.4. *For $C = 1$ this is exactly the definition of a norm. In general, we use the notation $| \cdot |$ in place of $\| \cdot \|$ to recall that we are in presence of a quasi-norm. Notice that the last property in the definition, which replaces the usual triangle inequality, does not allow to say that the function $d(x, y) := |x - y|$ is a distance any longer! Nonetheless, we will see that a quasi-norm induces a canonical topology and that this topology is always metrizable (by means of a highly nontrivial construction of a true distance function d).*

Example 6.5. *Given $f : \mathbb{R}^n \to \mathbb{K}$ measurable, let $|f|_{L^{1,\infty}} := \sup_{\alpha>0} \alpha\mu\{|f| > \alpha\}$ and let $L^{1,\infty}(\mathbb{R}^n)$ be the set of all functions f such that $|f|_{L^{1,\infty}} < \infty$. Notice that, by Chebyshev–Markov inequality, $L^1(\mathbb{R}^n) \subseteq L^{1,\infty}(\mathbb{R}^n)$ and $|f|_{L^{1,\infty}} \leq \|f\|_{L^1}$. Also, $| \cdot |_{L^{1,\infty}}$ is a quasi-norm (with $C = 2$): given two functions $F, g : \mathbb{R}^n \to \mathbb{K}$, for any $\alpha > 0$ we have*

$$\mu(\{|f + g| > \alpha\}) \leq \mu\Big(\Big\{|f| > \frac{\alpha}{2}\Big\}\Big) + \mu\Big(\Big\{|g| > \frac{\alpha}{2}\Big\}\Big) \leq 2|f|_{L^{1,\infty}} + 2|g|_{L^{1,\infty}}$$

(since $\{|f + g| > \alpha\} \subseteq \{|f| > \frac{\alpha}{2}\} \cup \{|g| > \frac{\alpha}{2}\}$). Hence, $|f + g|_{L^{1,\infty}} \leq 2|f|_{L^{1,\infty}} + 2|g|_{L^{1,\infty}}$. The second requirement in the definition is satisfied since, for $\lambda \neq 0$, $\alpha\mu(\{|\lambda f| > \alpha\}) = |\lambda| \frac{\alpha}{|\lambda|}\mu(\{|f| > \frac{\alpha}{|\lambda|}\})$, while the first one is trivial.

In terms of this quasi-norm, Hardy–Littlewood maximal inequality (for $p = 1$) says that $|f|_{L^{1,\infty}} \leq 5^n\|f\|_{L^1}$.

6.2 The topology of quasi-normed vector spaces

Theorem 6.6. *A quasi-normed vector space $(V, |\,|)$ has a unique vector space topology such that*

$$B_\alpha(0) := \{x \in V : |x| < \alpha\}, \quad \alpha > 0$$

is a local basis of neighborhoods of 0.

The above requirement should be compared with the situation of a normed vector space, where $B_\alpha(0)$ is the standard ball of radius α and center 0. Notice that the theorem is not asserting that $B_\alpha(0)$ is an open set in this canonical topology (which could be false in general)!

Proof of Theorem 6.6 If such a topology τ exists, then the sets

$$B_\alpha(y) := \{x \in V : |y - x| < \alpha\}, \quad \alpha > 0$$

form a local basis of neighborhoods of y for any $y \in V$: this is because the translation by y, namely the map $x \mapsto x + y$, is continuous and has continuous inverse $x \mapsto x - y$ (with respect to τ), hence it is a homeomorphism and carries a local basis of neighborhoods of 0 into a local basis at y. So the open sets of τ must be the sets

$$U \subseteq V \text{ such that } \forall y \in U\ \exists \alpha > 0 \text{ s.t. } B_\alpha(y) \subseteq U. \tag{6.12}$$

This shows that, if τ exists, it is necessarily unique. To show existence, let us declare that the open sets are the ones satisfying (6.12). They define a topology, since the axioms for a topology are clearly satisfied. Let us check that the sets $B_\alpha(0)$ form a local basis at 0: since every open set contains one such set by definition, it suffices to check that $B_\alpha(0)$ includes an open set U containing 0. Let

$$U := \{x \in V : \exists \delta > 0 \text{ s.t. } B_\delta(x) \subseteq B_\alpha(0).$$

Clearly, $0 \in U$ and $U \subseteq B_\alpha(0)$. In order to show that U satisfies 6.12, given $x \in U$ let $\delta > 0$ such that $B_\delta(x) \subseteq B_\alpha(0)$. We claim that $B_\sigma(x) \subseteq U$, with $\sigma := \frac{\delta}{2C}$ (which will conclude the proof that U is open in τ).

Indeed, if $y \in B_\sigma(x)$ then $B_\sigma(y) \subseteq B_\delta(x) \subseteq B_\alpha(x)$, since

$$|z - x| \le C(|z - y| + |y - x|) < 2C\sigma = \delta \quad \text{for all } z \in B_\sigma(y).$$

This shows that $y \in U$ (by definition of U), i.e. that $B_\sigma(x) \subseteq U$, which is what we wanted. In order to show that τ is Hausdorff, given $x \neq y$ it suffices to observe that $B_\alpha(x) \cap B_\alpha(y) = \emptyset$, where $\alpha := \frac{|x-y|}{2} > 0$: indeed, we just proved that $B_\alpha(x)$ and $B_\alpha(y)$ are neighborhoods of x and y respectively (being τ clearly translation invariant).

Finally, we have to check that the operations are continuous. If $x + y = z$ and U is an open neighborhood of z, then there exists $\alpha > 0$ such that $B_{2C\alpha}(z) \subseteq U$. Hence, given $x' \in B_\alpha(x)$ and $y' \in B_\alpha(y)$, we have

$$|x' + y' - z| = |(x' - x) + (y' - y)| \le C\,|x' - x| + C\,|y' - y| < 2C\alpha,$$

so that the sum maps $B_\alpha(x) \times B_\alpha(y)$ to a subset of U. Since $B_\alpha(x)$ and $B_\alpha(y)$ contain open neighborhoods of x and y respectively, this shows that the sum is continuous. The continuity of the multiplication by scalar is similar and is left to the reader. $\square$

6.3 The metrizability of quasi-normed vector spaces

The metrizability of quasi-normed vector spaces was proved independently by Tosio Aoki and Stefan Rolewicz between 1941 and 1957.

Theorem 6.7. *(Aoki-Rolewicz)*

The canonical topology of a quasi-normed vector space $(V, |\,|)$ is metrizable. In fact, it is induced by a translation-invariant distance $d(x,y) := \Lambda(x-y)$, for a suitable function $\Lambda : V \to [0,\infty)$ satisfying $\Lambda(z) = \Lambda(-z)$, $\Lambda(z+w) \le \Lambda(z) + \Lambda(w)$ and vanishing only at 0.

Remark 6.8. *In general, one cannot hope to have a distance induced by a norm (meaning that Λ is a norm, i.e. it also satisfies $\Lambda(\alpha x) = |\alpha|\,\Lambda(x)$ for $\alpha \in \mathbb{K}$): in this case (V,τ) would be a locally convex topological vector space, but we will see in Remark 6.17 that this fails for $L^{1,\infty}(\mathbb{R}^n)$.*

We will deduce Aoki–Rolewicz theorem from the following lemma.

Lemma 6.9. *Let $0 < p \le 1$ be defined by $2^{1/p} := 2C$. Given $x_1, \dots, x_n \in V$ we have*

$$|x_1 + \cdots + x_n|^p \le 4(|x_1|^p + \cdots + |x_n|^p).$$

Proof of Lemma 6.9. This proof illustrate the utility of decomposing dyadically a range of values. This idea will turn out to be fruitful also later in the course. Define $H : V \to [0,\infty)$ by the following formula:

$$H(x) := \begin{cases} 0 & \text{if } x = 0 \\ 2^{j/p} & \text{if } 2^{(j-1)/p} < |x| \le 2^{j/p}. \end{cases}$$

Notice that $|x| \le H(x) \le 2^{1/p}|x|$. We show, by induction on n, that

$$|x_1 + \cdots + x_n| \le 2^{1/p}(H(x_1)^p + \cdots + H(x_n)^p)^{1/p}. \tag{6.13}$$

By the observation just made, (6.13) clearly implies the statement. Also, (6.13) holds for the base case $n = 1$. We now show that it holds for a generic n, assuming it holds for $n-1$. By symmetry, we can assume that

$$|x_1| \ge |x_2| \ge \cdots \ge |x_n|\,,$$

which implies that $H(x_1) \ge H(x_2) \ge \cdots \ge H(x_n)$. We distinguish two cases.

i) There exists an index $1 \le i_0 < n$ such that $H(x_{i_0}) = H(x_{i_0+1})$: let $2^{j_0/p}$ be the common value of H at x_{i_0} and x_{i_0+1} and notice that, since

$$|x_{i_0} + x_{i_0+1}| \le C(|x_{i_0}| + |x_{i_0+1}|) \le 2C \cdot 2^{j_0/p} = 2^{(j_0+1)/p},$$

we have $H(x_{i_0} + x_{i_0+1}) \le 2^{(j_0+1)/p}$. This gives

$$H(x_{i_0} + x_{i_0+1})^p \le 2^{j_0+1} = H(x_{i_0})^p + H(x_{i_0+1})^p$$

and so, grouping $x_1 + \cdots + x_n = x_1 + \cdots + x_{i_0-1} + (x_{i_0} + x_{i_0+1}) + x_{i_0+2} + \dots$ and using induction,

$$\begin{aligned} |x_1 + \cdots + x_n| &\le 2^{1/p}(H(x_1)^p + \cdots + H(x_{i_0} + x_{i_0+1})^p + \cdots + H(x_n)^p)^{1/p} \\ &\le 2^{1/p}(H(x_1)^p + \cdots + H(x_{i_0})^p + H(x_{i_0+1})^p + \cdots + H(x_n)^p)^{1/p}. \end{aligned}$$

ii) We have a strictly decreasing sequence $H(x_1) > H(x_2) > \cdots > H(x_n)$: in this case we must have $H(x_i) \leq 2^{-(i-1)/p} H(x_1)$ for all i. Also, iterating the approximate triangle inequality we obtain

$$\begin{aligned}
|x_1 + \cdots + x_n| &\leq C(|x_1| + |x_2 + \cdots + x_n|) \\
&\leq \max\{2C|x_1|, 2C|x_2 + \cdots + x_n| \\
&\leq \max\{2C|x_1|, (2C)^2 |x_2|, (2C)^2 |x_3 + \cdots + x_n| \\
&\leq \cdots \leq \max_i (2C)^i |x_i| \\
&\leq \max_i 2^{i/p} H(x_i) \\
&\leq 2^{1/p} H(x_1)
\end{aligned}$$

and (6.13) trivially follows. □

Proof of Theorem 6.7. For all $x \in V$ we define

$$\Lambda(x) := \inf \sum_{i=1}^{n} |x_i|^p, \quad x = \sum_{i=1}^{n} x_i, \ n \geq 1,$$

meaning that the infimum is taken over all possible representations of x as a finite sum of elements of V. Since a possible choice is $n = 1$ and $x_1 = x$, we trivially have $\Lambda(x) \leq |x|^p$. Moreover, the previous lemma gives

$$|x|^p = |x_1 + \cdots + x_n|^p \leq 4(|x_1|^p + \cdots + |x_n|^p)$$

for all such possible representations, so $\Lambda(x) \geq \frac{1}{4}|x|^p$. In particular, this implies that Λ vanishes only at 0. From the definition it is clear that $\Lambda(-x) = \Lambda(x)$.

Also, $\Lambda(x+y) \leq \Lambda(x)+\Lambda(y)$: given $\epsilon > 0$, if $x = x_1+\cdots+x_m$ and $y = y_1+\cdots+y_n$ are chosen so that $\sum_{i=1}^m |x_i|^p < \Lambda(x) + \epsilon$ and $\sum_{j=1}^n |y_j|^p < \Lambda(y) + \epsilon$, then (being $x + y = \sum_i x_i + \sum_j y_j$)

$$\Lambda(x + y) \leq \sum_{i=1}^{m} |x_i|^p + \sum_{j=1}^{n} |y_j|^p < \Lambda(x) + \Lambda(y) + 2\epsilon.$$

Hence, defining $d : V \times V \to [0, \infty)$ by $d(x, y) := \Lambda(x - y)$ gives a distance on V. This induces the same topology as the quasi-metric since

$$B_{r^{1/p}}(x) \subseteq \{y \in V : d(x, y) < r\} \subseteq B_{(4r)^{1/p}}(x)$$

for all $x \in V$ and all $r > 0$. □

Remark 6.10. The space $L^p(E)$, with $0 < p < 1$, is a quasi-normed vector space, with quasi-norm

$$|f|_{L^p} := \Big(\int_E |f|^p \Big)^{1/p},$$

which yields a constant $2^{1/p-1}$ in the approximate triangle inequality. The construction Aoki–Rolewicz metric is reminiscent of the distance

$$d(f, g) := \int_E |f - g|^p$$

on $L^p(E)$ (for $0 < p < 1$), which induces the same topology as the quasi-norm but is built in a nonlinear way.

6.4 Lorentz spaces

We will now see important concrete examples of quasi-normed vector spaces, namely Lorentz spaces, which refine the classical Lebesgue spaces in terms of control over the integrability of a function. Standard estimates such as Sobolev's embedding or Young's inequality can be slightly (but crucially for some applications) improved using these more refined spaces.

6.5 The space $L^{p,\infty}$

Definition 6.11. *Let $E \subseteq \mathbb{R}^n$ be a set of positive measure. Given $1 \leq p < \infty$ and a measurable function $f : E \to \mathbb{K}$, we let*

$$|f|_{L^{p,\infty}} := \sup_{\alpha>0} \alpha\mu(\{|f| > \alpha\})^{1/p}$$

and we define $L^{p,\infty}(E)$ to be the set of all functions $f : E \to \mathbb{K}$ with $|f|_{L^{p,\infty}} < \infty$. We also let $|f|_{L^{\infty,\infty}} := \|f\|_{L^\infty}$, so that $L^{\infty,\infty}(E) = L^\infty(E)$. The space $L^{p,\infty}$ is called weak L^p *(however, it is totally unrelated to the weak topology on the L^p space!).*

Remark 6.12. *Notice that this specializes to Example 6.5 when $p = 1$. Again, we have $L^p(E) \subset \|f\|_{L^p})$. For $p < \infty$, this inclusion is strict in general: take e.g. $E := \mathbb{R}^n$ and $f(x) := |x|^{-n/p}$, which lies in $L^{p,\infty}(\mathbb{R}^n) \setminus L^p(\mathbb{R}^n)$ (the inclusion is actually always strict for subsets of $\mathbb{R}^n$, as can be seen taking $|x - x_0|^{-n/p}$ with x_0 a density point for E).*

Remark 6.13. *Using the inequality $(\alpha + \beta)^{1/p} \leq \alpha^{1/p} + \beta^{1/p}$ and arguing as in Example 6.5, we see that $L^{p,\infty}(E)$ is a quasi-normed vector space, with $C = 2$.*

Definition 6.14. *A quasi-normed vector space is called* quasi-Banach *if every $|\ |$-Cauchy sequence converges to a (necessarily unique) limit in the canonical topology, or equivalently converges with respect to the quasi-norm.*

Remark 6.15. *Notice that a sequence is Cauchy with respect to the quasi-norm if and only if it is Cauchy with respect to the Aoki-Rolewicz distance. The same holds for convergence.*

Proposition 6.16. *The space $L^{p,\infty}(E)$ is a quasi-Banach space.*

Proof. Omitted. □

Remark 6.17. *The space $L^{1,\infty}(R^n)$ is not locally convex, meaning that it does not possess a local basis of neighborhoods of 0 made of open convex sets. This rules out the possibility of finding a* norm *equivalent to its quasi-norm, which is the main difficulty in Calderón–Zygmund theory for singular convolution kernels (so that, as we will see, not all kernels in $L^{1,\infty}(\mathbb{R}^n)$ but only those with enough cancellation and regularity give rise to the important $L^1 \to L^{1,\infty}$ bound). Let us see this failure of convexity when $n = 1$, for simplicity.*

For all integers $m \geq 2$ and $1 \leq k \leq m$ let

$$f_{m,k}(x) := \frac{1}{\log m}|x - \frac{k}{m}|^{-1}.$$

Observe that $f_{m,k} \in L^{1,\infty}(\mathbb{R})$*, with* $|f_{m,k}|_{L^{1,\infty}} \le \frac{2}{\log m}$*, so that* $f_{m,k} \to 0$ *in* $L^{1,\infty}(\mathbb{R})$ *as* $m \to \infty$ *(uniformly in the index* k*). On the other hand, the arithmetic mean of* $f_{m,1}, \dots, f_{m,m}$ *is pointwise bounded from below on* $(0,1)$*:*

$$F_m := \frac{f_{m,1}(x) + \cdots + f_{m,m}(x)}{m} \ge \frac{1}{\log m} \sum_{j=1}^{m} \frac{1}{j} \ge c > 0,$$

since if $\frac{k_0}{m} < x < \frac{k_0+1}{m}$ *then the left-hand side is at least*

$$\frac{1}{m \log m}\Big(\frac{m}{k_0} + \cdots + \frac{m}{1} + \frac{m}{1} + \cdots + \frac{m}{m-k_0}\Big)$$

(the first part being not present if $k_0 = 0$*). So* $|F_m|_{L^{1,\infty}} \ge c$*, implying that* F_m *cannot converge to* 0*. This however should hold if* $L^{1,\infty}(\mathbb{R})$ *were locally convex!*

6.6 Decreasing rearrangement

In order to define all the Lorentz spaces $L^{p,q}$ we have to introduce the notion of decreasing rearrangement.

Definition 6.18. *Given* $f : E \to \mathbb{K}$ *measurable, we define its* decreasing rearrangement $f_* : [0,+\infty] \to [0,+\infty]$ *as*

$$f_*(t) := \inf\{0 \le \lambda \le +\infty : \mu(\{|f| > \lambda\}) \le t\}, \tag{6.14}$$

with the convention that $0 \cdot \infty = \infty \cdot 0 = 0$ *(as it is customary in measure theory).*

Remark 6.19. *The infimum in* (6.14) *is actually always a minimum: if* $\lambda_1 \ge \lambda_2 \ge \dots$ *are values such that* $\mu(\{|f| > \lambda_i\}) \le t$ *and* $\lambda_\infty := \lim_{i\to\infty} \lambda_i$*, then we still have* $\mu(\{|f| > \lambda_\infty\}) \le t$ *(since the last set is the increasing union of the sets* $\{|f| > \lambda_i\}$*). Hence,* $\mu(\{|f| > f_*(t)\}) \le t$*.*

Remark 6.20. *Define* $d_f(\lambda) := \mu(\{|f| > \lambda\})$ *(as a function from* $[0,+\infty]$ *to itself), which is called* distribution function*, or* tail distribution *in probability theory. It is clear that* d_f *and* f_* *are decreasing and* d_f *is right-continuous. Also* f_* *is right-continuous: given* $0 \le t_0 < +\infty$*, setting* $\bar\lambda := \lim_{t\to t_0^+} f_*(t)$ *we have*

$$\mu(\{|f| > \bar\lambda\}) = \lim_{t\to t_0^+} \mu(\{|f| > f_*(t)\}) \le \lim_{t\to t_0^+} t = t_0,$$

where the first equality holds since we have a decreasing union of sets with finite measure. Hence, $f_*(t) \le \bar\lambda$*. Since the converse inequality also holds (being* f_* *decreasing), the claim follows. One can show that* d_f *and* f_* *are "pseudo-inverses" of each other:*

- *as already said,* $d_\lambda \circ f_*(t) \le t$ *and, assuming* $0 < t, f_*(t) < +\infty$*, equality holds if and only if* $f_*(t') > f_*(t)$ *for all* $t' < t$*;*
- *similarly with* f_* *and* d_f *interchanged.*

Proposition 6.21. *The functions f and f_*, although defined on different domains, have the same distribution function (meaning that $d_f = d_{f_*}$) and the same decreasing rearrangement (meaning that $f_* = (f_*)_*$).*

Proof. Fix $0 \le \lambda \le +\infty$ and notice that, given $0 \le t < +\infty$,

$$\mu(\{|f| > \lambda\}) \le t \Leftrightarrow \lambda \ge f_*(t) \Leftrightarrow \{f_* > \lambda\} \subseteq [0,t) \Leftrightarrow \mu(\{f_* > \lambda\}) \le t.$$

The penultimate equivalence follows from the fact that f_* is decreasing, while the last one follows from the right-continuity of f_* (so that one cannot have $\{f_* > \lambda\} = [0,t]$). Both statements now follow from this chain of equivalences (observe that $f_*(+\infty) = (f_*)_*(+\infty) = 0$). □

Corollary 6.22. *For any measurable $f : E \to \mathbb{K}$, we have $|f|_{L^{p,\infty}} = |f_*|_{L^{p,\infty}}$ for all $1 \le p \le \infty$. Also, we have $\|f\|_{L^p} = \|f_*\|_{L^p}$ since*

$$\|f\|_{L^p}^p = \int_0^\infty p\lambda^{p-1} d_f(\lambda)\, d\lambda = \int_0^\infty p\lambda^{p-1} d_{f_*}(\lambda)\, d\lambda = \|f_*\|_{L^p}^p$$

for $1 \le p < \infty$ and $\|f\|_{L^\infty} = \inf\{\lambda : d_f(\lambda) = 0\} = \|f_*\|_{L^\infty}$.

The following two lemmas are very useful in practice, for instance when approximating a function by mollification or by simple functions.

Lemma 6.23. *If $|f_k| \to |f_\infty|$ pointwise a.e., or more generally if $|f_\infty| \le \liminf_{k\to\infty} |f_k|$ a.e., then $d_{f_\infty} \le \liminf_{k\to\infty} d_{f_k}$ and $(f_\infty)_* \le \liminf_{k\to\infty} (f_k)_*$.*

Proof. Let $N \subset E$ be a negligible subset such that $|f_\infty| \le \liminf_{k\to\infty} |f_k|$ everywhere on $E \setminus N$. Given $0 \le \lambda \le +\infty$, if $x \notin N$ has $|f_\infty(x)| > \lambda$ then $|f_k(x)| > \lambda$ eventually, so

$$\chi_{\{|f_\infty| > \lambda\} \setminus N} \le \liminf_{k\to\infty} \chi_{\{|f_k| > \lambda\} \setminus N}.$$

Integrating and applying Fatou's lemma gives the first claim. Now let $0 \le t \le +\infty$ and

$$\lambda_k := (f_k)_*(t), \qquad \bar\lambda := \liminf_{k\to\infty} \lambda_k.$$

Passing to a subsequence, we can assume that $\bar\lambda = \lim_{k\to\infty} \lambda_k$. Notice that the hypothesis still holds (in both versions). Again, if $|f_\infty(x)| > \bar\lambda$ (and $x \notin N$) then $|f_k(x)| > \lambda_k$ eventually, so as before we obtain

$$\mu(\{|f_\infty| > \bar\lambda\}) \le \liminf_{k\to\infty} \mu(\{|f_k| > \lambda_k\}) \le t$$

by Remark 6.19. By definition of decreasing rearrangement, it follows that $(f_\infty)_*(t) \le \bar\lambda = \liminf_{k\to\infty} (f_k)_*(t)$. □

Lemma 6.24. *If $|f_k| \uparrow |f_\infty|$ pointwise a.e. (meaning that $|f_\infty|$ is the increasing limit of $|f_k|$), then $d_{f_k} \uparrow d_{f_\infty}$ and $(f_k)_* \uparrow (f_\infty)_*$ everywhere.*

Proof. Let $N \subset E$ be a negligible subset such that $|f_k| \uparrow |f_\infty|$ everywhere on $E \setminus N$. For every $0 \leq \lambda \leq +\infty$, since $\{|f_\infty| > \lambda\} \cap E$ is the increasing union of the sets $\{|f_n| > \lambda\} \cap E$, we get

$$d_{f_\infty}(\lambda) = \mu(\{|f_\infty| > \lambda\}) = \lim_{k\to\infty} \mu(\{|f_k| > \lambda\}) = \lim_{k\to\infty} d_{f_k}(\lambda).$$

Given $0 \leq t \leq +\infty$, we set $\lambda_k := (f_k)_*(t)$ (for $k \in \mathbb{N} \cup \{\infty\}$) and $\bar{\lambda} = \lim_{k\to\infty} \lambda_k$. This limit exists and is at most λ_∞ as

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\infty.$$

We also have

$$\begin{aligned}
\mu(\{|f_\infty| > \bar{\lambda}\}) &= \lim_{k\to\infty} \mu(\{|f_k| > \bar{\lambda}\}) \\
&= \lim_{k\to\infty} \mu(\{|f_k| > \lambda_k\}) \\
&\leq \liminf_{k\to\infty} \mu(\{|f_k| > \lambda_k\}) \leq t,
\end{aligned}$$

so $\lambda_\infty = (f_\infty)_*(t) \leq \bar{\lambda}$. We conclude that $\lambda_\infty = \bar{\lambda}$, i.e. $(f_k)_*(t) \uparrow (f_\infty)_*(t)$. □

6.7 The Lorentz spaces $L^{p,q}$

Definition 6.25. *Given $1 \leq p < \infty$ and $1 \leq q \leq \infty$, we set*

$$|f|_{L^{p,q}}^q := \int_0^\infty t^{q/p} f_*(t)^q \, \frac{dt}{t}$$

and we call $L^{p,q}(E)$ the set of all measurable functions $f : E \to \mathbb{K}$ with $|f|_{L^{p,q}} < \infty$. We also set $|f|_{L^{\infty,q}} := \|f\|_{L^\infty}$ (so that $L^{\infty,q}(E) = L^\infty(E)$).

Remark 6.26. *As we will see, even if f_* is hit by the exponent q, the first exponent p is the dominant one.*

Proposition 6.27. *The quantity $|\cdot|_{L^{p,q}}$ is a quasi-norm.*

Proof. It suffices to show that $(f+g)_*(t) \leq f_*(\frac{t}{2}) + g_*(\frac{t}{2})$. Actually, if $0 \leq s, s', t \leq +\infty$ and $s + s' \leq t$, it always holds that $(f+g)_*(t) \leq f_*(s) + g_*(s')$, since

$$\mu(\{|f+g| > f_*(s) + g_*(s')\}) \leq \mu(\{|f| > f_*(s)\}) + \mu(\{|g| > g_*(s')\}) \leq s + s' \leq t.$$

□

Remark 6.28. *It follows that $L^{p,q}(E)$ is a quasi-normed vector space for all exponents $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. Again, one can show that it is always a quasi-normed vector space. For $p > 1$, as opposed to the case of $L^{1,\infty}$, we will see that the quasi-norm admits an equivalent* norm*, giving thus rise to a genuine Banach space.*

Remark 6.29. *The Lorentz quasi-norm* $|\ |_{L^{p,q}}$ *measures the integrability of the function, rather than the regularity. In the language of probability, it depends only on the law of* f*, since it is defined in terms of* f_* *(which in turn depends only on* d_f*). Rearranging the places where the values are attained, thus possibly making the function very irregular, does not alter the* $L^{p,q}$*-quasinorm. One can define it in the same way on general measure spaces. What we just observed can be made precise as follows: if* $h : E \to E'$ *is a measure-preserving map between two measure spaces, then* $|f \circ h|_{L^{p,q}} = |f|_{L^{p,q}}$ *for any* $f : E' \to \mathbb{K}$ *(since* $d_f = d_{f\circ h}$ *and thus* $f_* = (f \circ h)_*$*).*

The definition of the $L^{p,q}$-quasinorm when $q < \infty$ suggests the following equivalent definition when $q = \infty$.

Proposition 6.30. *For* $1 \le p < \infty$ *we have* $|f|_{L^{p,\infty}} = \sup_{0\le t\le+\infty} t^{1/p} f_*(t)$*.*

Proof. ($\le$): given $\lambda > 0$ with $d_f(\lambda) > 0$, set $t := d_f(\lambda) - \epsilon$ (where $\epsilon > 0$ is arbitrary and will tend to 0). Letting $\lambda' := f_*(t)$, being $d_f(\lambda') \le t = d_f(\lambda) - \epsilon$ we must have $\lambda' > \lambda$. Hence,

$$\lambda(d_f(\lambda) - \epsilon)^{1/p} \le \lambda' t^{1/p} = f_*(t)t^{1/p} \le \sup_{0\le t\le+\infty} t^{1/p} f_*(t)$$

and the inequality follows letting $\epsilon \downarrow 0$ and then taking the supremum over λ.

($\ge$): analogous. □

Similarly, the $L^{p,q}$-quasinorm can be expressed in terms of the distribution function.

Proposition 6.31. *For all* $1 \le p < \infty$ *and* $1 \le q < \infty$ *we have*

$$|f|_{L^{p,q}} = p^{1/q}\Big(\int_0^\infty \lambda^{q-1} d_f(\lambda)^{q/p}\, d\lambda\Big)^{1/q}.$$

Proof. We start with the trivial observation that one has $f_*(t) > \lambda$ if and only if $d_f(\lambda) > t$, thanks to Remark 6.19. This, together with Fubini, gives

$$\begin{aligned}
|f|^q_{L^{p,q}} &= \int_0^\infty t^{q/p-1} \int_0^{f_*(t)} q\lambda^{q-1}\, d\lambda\, dt \\
&= q \int_{\{(t,\lambda): f_*(t)>\lambda\}} t^{q/p-1}\lambda^{q-1}\, dt\, d\lambda \\
&= q \int_0^\infty \int_0^{d_f(\lambda)} t^{q/p-1}\lambda^{q-1}\, dt\, d\lambda \\
&= p \int_0^\infty d_f(\lambda)^{q/p}\lambda^{q-1}\, d\lambda.
\end{aligned}$$

□

Proposition 6.32. *If* $|f_k| \to |f_\infty|$ *pointwise a.e., or more generally if* $|f_\infty| \le \liminf_{k\to\infty} |f_k|$ *a.e., then*

$$|f_\infty|_{L^{p,q}} \le \liminf_{k\to\infty} |f_k|_{L^{p,q}}.$$

If $f_k \to f_\infty$ and $|f_k| \uparrow |f_\infty|$, then

$$|f_k - f_\infty|_{L^{p,q}} \to 0,$$

provided that $f_\infty \in L^{p,q}(E)$ and $1 \leq p, q < \infty$. In particular, simple functions are dense in $L^{p,q}(E)$ if $1 \leq p, q < \infty$

Proof. The first part follows immediately from Lemma 6.23 and Fatou. The second part follows from the pointwise convergence $(f_k)_* \to (f_\infty)_*$ given by Lemma 6.24, together with the dominated convergence theorem. □

Proposition 6.33. *We have*

(1) $L^{p,p}(E) = L^p(E)$,

(2) $L^{p,q}(E) \subseteq L^{p,r}(E)$ if $q < r$,

(3) $L^{p,q}(E) \subseteq L^{t,u}(E)$ if $\mu(E) < \infty$ and $p > t$ (regardless of q and u).

Proof. (1) From the definition of the $L^{p,p}$-quasinorm and Corollary 6.22 we have $|f|^p_{L^{p,p}} = \|f_*\|^p_{L^p} = \|f\|^p_{L^p}$.

(2) We assume $p < \infty$ without loss of generality. We first deal with the case $r = \infty$: since f_* is decreasing, we deduce

$$t^{1/p} f_*(t) = \Big(\frac{q}{p}\int_0^t s^{q/p-1} f_*(t)^q \, ds\Big)^{1/q} \leq \Big(\frac{q}{p}\int_0^t s^{q/p-1} f_*(s)^q \, ds\Big)^{1/q} \leq \Big(\frac{q}{p}\Big)^{1/q} |f|_{L^{p,q}}$$

for all $0 \leq t < +\infty$. Taking the supremum over t, we deduce that $|f|_{L^{p,\infty}}$ is estimated by $|f|_{L^{p,q}}$ and the inclusion follows. If $r < \infty$, notice that

$$\begin{aligned}
|f|_{L^{p,r}} &= \Big(\int_0^\infty s^{r/p} f_*(s)^r \, \frac{ds}{s}\Big)^{1/r} \\
&\leq \Big(\int_0^\infty s^{q/p} f_*(s)^q \, \frac{ds}{s}\Big)^{1/r} \sup_{0 \leq s \leq +\infty} s^{(r-q)/(pr)} f_*(s)^{(r-q)/r} \\
&= |f|^{q/r}_{L^{p,q}} |f|^{(r-q)/r}_{L^{p,\infty}} \\
&\leq C(p,q,r) |f|^{q/r}_{L^{p,q}} |f|^{(r-q)/r}_{L^{p,r}}
\end{aligned}$$

by the previous case. Dividing both sides by $|f|^{(r-q)/r}_{L^{p,r}}$ and raising to the power $\frac{r}{q}$, the claim follows.

(3) From the definition of f_* it follows that $f_*(s) = 0$ for all $s \geq \mu(E)$. In view of (2), it suffices to deal with the case $u = 1$, $q = \infty$. If $p < \infty$ we have

$$|f|_{L^{t,u}} = \int_0^{\mu(E)} s^{1/t} f_*(s) \, \frac{ds}{s} \leq \Big(\int_0^{\mu(E)} s^{1/t-1/p} \, \frac{ds}{s}\Big) \sup_{0 \leq s \leq \mu(E)} s^{1/p} f_*(s).$$

Since $\frac{1}{t} - \frac{1}{p} > 0$, the first integral is a finite constant, while the supremum equals $|f|_{L^{p,\infty}}$ by Proposition 6.30. If $p = \infty$, it suffices to bound $f_*(s)$ by $\|f\|_{L^\infty}$ right after the first equality. □

Remark 6.34. *The inclusion $L^{p,q}(E) \subseteq L^{p,r}(E)$ is always strict: assuming $0 \in E$ is a density point without loss of generality, it is easy to check that*

- $|x|^{-n/p} \in L^{p,\infty}(E) \setminus \bigcup_{q<\infty} L^{p,q}(E)$,
- $|x|^{-n/p} \log(|x|^{-1})^{-\alpha} \chi_{B_{1/2}}(x) \in L^{p,q}(E)$ *if and only if* $\alpha q > 1$, *for all* $\alpha > 0$.

We now turn to the promised fact that $L^{p,q}$ is normable for $p > 1$.

Theorem 6.35 (normability of $L^{p,q}$). *For all $1 < p \leq \infty$ the $L^{p,q}$-quasinorm has an equivalent norm, for all $1 \leq q \leq \infty$.*

Lemma 6.36. *Define $f_{**} : (0, +\infty) \to [0, +\infty]$ by*

$$f_{**}(t) := \frac{1}{t} \int_0^t f_*(s)\, ds.$$

This modification of the decreasing rearrangement satisfies

$$f_{**}(t) = \frac{1}{t} \sup\{ \int_F |f| \,;\, F \subseteq E, \mu(F) \leq t \}. \tag{6.15}$$

Proof. The statement holds if f is a nonnegative simple function, namely $f = \sum_{i=1}^N \lambda_i \chi_{A_i}$ with $\lambda_1 \geq \lambda_2 \geq \dots$ and $A_i \cap A_j = \emptyset$: indeed, it is easy to check that both sides of (6.15) equal

$$\frac{1}{t} \sum_{i=1}^k \lambda_i \mu(A_i) + \delta \mu(A_{i+1})$$

where k is such that $\sum_{i=1}^k \mu(A_i) \leq t < \sum_{i=1}^{k+1} \mu(A_i)$ ($k = N$ if $t \geq \sum_{i=1}^N \mu(A_i)$) and $\delta := t - \sum_{i=1}^k \mu(A_i)$. In general, we approximate $|f|$ pointwize from below with nonnegative simple functions f_k. By Lemma 6.24 and the monotone convergence theorem, both sides of (6.15) converge from below to the desired quantities. □

Corollary 6.37. *We have $(f+g)_{**} \leq f_{**} + g_{**}$.*

Proof. This immediately follows from the inequality $\int_F |f+g| \leq \int_F |f| + \int_F |g|$ and the last lemma. □

Lemma 6.38 (Hardy's inequality). *Given $1 < p < \infty$, $1 \leq q < \infty$ and $f : (0, +\infty) \to [0, +\infty]$, it holds*

$$\Big(\int_0^\infty \Big(\frac{1}{x} \int_0^x f(t)\, dt \Big)^p dx \Big)^{1/p} \leq p' \Big(\int_0^\infty f(x)^p\, dx \Big)^{1/p}$$

and more generally

$$\Big(\int_0^\infty x^{q/p-1} \Big(\frac{1}{x} \int_0^x f(t)\, dt \Big)^q dx \Big)^{1/q} \leq p' \Big(\int_0^\infty x^{q/p-1} f(x)^q\, dx \Big)^{1/q}.$$

Proof. We argue by duality. In order to show the first inequality, let $g \geq 0$ with $\|g\|_{L^{p'}} = 1$. We get

$$\begin{aligned}
\int_0^\infty \Big(\frac{1}{x}\int_0^x f(t)\,dt\Big)g(x)\,dx &= \int_0^\infty \Big(\int_0^1 f(sx)\,dt\Big)g(x)\,dx \\
&= \int_0^1 \Big(\int_0^\infty f(sx)g(x)\,dx\Big)\,ds \\
&\leq \int_0^1 \|f(s\cdot)\|_{L^p}\|g\|_{L^{p'}}\,ds \\
&= \int_0^1 s^{-1/p}\|f\|_{L^p}\,ds \\
&= p'\|f\|_{L^p}.
\end{aligned}$$

The proof of the second inequality is identical, working rather with the measure space $X := ((0,\infty), x^{q/p-1}\,dx)$ and using the duality $(L^q(X))^* = L^{q'}(X)$, observing that we still have $\|f(s\cdot)\|_{L^q(X)} = s^{-1/p}\|f\|_{L^q(X)}$. □

Proof of Theorem 6.35. We assume without loss of generality that $1 < p < \infty$. We let

$$\|f\|_{L^{p,q}} := \Big(\int_0^\infty t^{q/p} f_{**}(t)^q\,\frac{dt}{t}\Big)^{1/q}$$

for $1 \leq q < \infty$ and $\|f\|_{L^{p,\infty}} := \sup_{0<t<\infty} t^{1/p} f_{**}(t)$, i.e. we are merely replacing f_* with f_{**} in the definitions. From Corollary 6.37 it follows that this is a norm (when $q < \infty$ we also use Minkowski's inequality for $L^q(X)$, where X is the same measure space as in the previous proof). Finally, since f_* is decreasing, we have $f_{**} \geq f_*$ and thus $\|f\|_{L^{p,q}} \geq |f|_{L^{p,q}}$. Conversely, by Hardy's inequality applied to f_*,

$$\|f\|_{L^{p,q}} \leq p'|f|_{L^{p,q}}.$$

This shows that the norm $\|\ \|_{L^{p,q}}$ is equivalent to the quasi-norm $|\ |_{L^{p,q}}$. □

Remark 6.39. *By Fatou's lemma, the conclusions of Lemmas 6.24 and 6.23 are still true with f_{**} in place of f_*. Hence, Proposition 6.32 still holds with $|\ |_{L^{p,q}}$ replaced with $\|\ \|_{L^{p,q}}$.*

The dual spaces of Lorentz spaces are the expected ones, for $p > 1$.

Theorem 6.40 (Dual spaces)**.** *For $1 < p < \infty$ and $1 \leq q < \infty$ we have*

$$(L^{p,q}(E))^* = L^{p',q'}(E),$$

where duality is represented by integration.

Proof. Omitted. □

6.8 Functional inequalities for Lorentz spaces

Theorem 6.41 (Hölder's inequality)**.** *Assume that $f \in L^{p_1,q_1}(E)$ and $g \in L^{p_2,q_2}(E)$ with*

$$\begin{aligned} 1 < p_1, p_2, p < \infty, &\qquad 1 \le q_1, q_2, q \le \infty, \\ \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, &\qquad \frac{1}{q_1} + \frac{1}{q_2} \ge \frac{1}{q}. \end{aligned}$$

Then $fg \in L^{p,q}(E)$, with $\|fg\|_{L^{p,q}} \le C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}$ (where C depends on p_1, p_2, q_1, q_2).

Proof. Thanks to Proposition 6.33, we can replace q_1 and q_2 with possibly higher exponents and assume, without loss of generality, that $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$. Given $0 \le t_1, t_2 \le +\infty$, notice that

$$\mu(\{|f| > f_*(t_1)\}) \le t_1, \quad \mu(\{|g| > g_*(t_2)\}) \le t_2,$$

so that, since $|fg| > f_*(t_1)g_*(t_2)$ implies either $|f| > f_*(t_1)$ or $|g| > g_*(t_2)$, we infer

$$\mu(\{|fg| > f_*(t_1)g_*(t_2)\}) \le t_1 + t_2$$

and thus

$$(fg)_*(t_1 + t_2) \le f_*(t_1)g_*(t_2).$$

This, together with the classical Hölder's inequality for Lebesgue spaces with exponents $\frac{q_1}{q}$ and $\frac{q_2}{q}$ (on the measure space $(0, +\infty)$), gives

$$\begin{aligned} |fg|_{L^{p,q}} &= \|t^{1/p-1/q}(fg)_*(t)\|_{L^q} \\ &\le \|t^{1/p-1/q} f_*\Big(\frac{t}{2}\Big) g_*\Big(\frac{t}{2}\Big)\|_{L^q} \\ &= C'\|t^{1/p_1-1/q_1} f_*(t)\ t^{1/p_2-1/q_2} g_*(t)\|_{L^q} \\ &\le C'\|t^{1/p_1-1/q_1} f_*(t)\|_{L^{q_1}} \|t^{1/p_2-1/q_2} g_*(t)\|_{L^{q_2}} \\ &= C'|f|_{L^{p_1,q_1}} |g|_{L^{p_2,q_2}}. \end{aligned}$$

□

Remark 6.42. *Of course, Hölder's inequality works also if $(p_1, q_1) = (\infty, \infty)$ (or similarly if $(p_2, q_2) = (\infty, \infty)$), since in this case it reduces to the inequality*

$$\|fg\|_{L^{p,q}} \le \|f\|_{L^\infty}\|g\|_{L^{p,q}} \le C\|f\|_{L^{\infty,\infty}}\|g\|_{L^{p_2,q_2}}.$$

Theorem 6.43 (Young's inequality)**.** *Assume that $f \in L^{p_1,q_1}(\mathbb{R}^n)$ and $g \in L^{p_2,q_2}(\mathbb{R}^n)$ with*

$$\begin{aligned} 1 < p_1, p_2, p < \infty, &\qquad 1 \le q_1, q_2, q \le \infty, \\ \frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p}, &\qquad \frac{1}{q_1} + \frac{1}{q_2} \ge \frac{1}{q}. \end{aligned}$$

Then the convolution $f * g$ is a.e. defined (meaning that the integral defining $f * g$ exists a.e.) and $f * g \in L^{p,q}(\mathbb{R}^n)$, with $\|fg\|_{L^{p,q}} \le C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}$ (where C depends on p_1, p_2, q_1, q_2).

Remark 6.44. *In some cases, this improves the classical Young's inequality for Lebesgue spaces: for instance, it gives $L^{3/2} * L^{3/2} \subseteq L^{3,1}$ rather than just $L^{3/2} * L^{3/2} \subseteq L^3$.*

The proof, due to O'Neil, is now given.

Lemma 6.45. *If $f, g \geq 0$ are measurable functions on $\mathbb{R}^n$ and $f \leq \alpha \chi_{E_0}$, then*

(1) $(f * g)_{**} \leq \alpha \mu(E_0) g_{**}$,

(2) $\|(f * g)_{**}\|_{L^\infty} \leq \alpha \mu(E_0) g_{**}(\mu(E_0))$.

Proof. Given $0 < t < +\infty$ and $F \subseteq \mathbb{R}^n$ with $\mu(F) \leq t$, then by (6.15)

$$\begin{aligned} t^{-1} \int_F f * g &\leq \alpha t^{-1} \int_F \int_{E_0} g(x-y)\, dy\, dx \\ &= \alpha \int_{E_0} t^{-1} \int_{F-y} g(x)\, dx\, dy \\ &\leq \alpha \int_{E_0} g_{**}(t)\, dy \\ &= \alpha \mu(E_0) g_{**}(t), \end{aligned}$$

so that taking the supremum over F and using (6.15) the first claim follows. Similarly, notice that

$$\begin{aligned} \alpha t^{-1} \int_F \int_{E_0} g(x-y)\, dy\, dx &= \alpha t^{-1} \int_F \int_{x-E_0} g(y)\, dy\, dx \\ &\leq \alpha t^{-1} \mu(F) \mu(E_0) g_{**}(\mu(E_0)) \\ &\leq \alpha \mu(E_0) g_{**}(\mu(E_0)), \end{aligned}$$

as $\mu(x - E_0) = \mu(E_0)$. This gives the second claim. □

Lemma 6.46. *For $f, g \geq 0$ and $0 < t < +\infty$, we have*

$$(f * g)_{**}(t) \leq t f_{**}(t) g_{**}(t) + \int_t^\infty f_* g_*.$$

Proof. We can assume that f is simple and finite, so we can write

$$f = \sum_{i=1}^N \alpha_i \chi_{E_i}$$

with $\alpha_i \geq 0$ and $\mathbb{R} =: E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_N \supsetneq E_{N+1} := \emptyset$. Possibly adding artificially a set with measure t, we can assume that $t = \mu(E_{i_0})$ (with $1 \leq i_0 \leq N$). Using the previous lemma we have

$$(f * g)_{**}(t) \leq \sum_{i=1}^{i_0-1} \alpha_i \mu(E_i) g_{**}(\mu(E_i)) + \sum_{i=i_0}^{N} \alpha_i \mu(E_i) g_{**}(t). \tag{6.16}$$

Observe that f_* equals $\sum_{i=1}^{j} \alpha_i$ on the set $[\mu(E_{j+1}, \mu(E_j))$. The first sum in (6.16) equals

$$\sum_{i=1}^{i_0-1} \alpha_i \int_0^{\mu(E_i)} g_* = \sum_{i=1}^{i_0-1} \sum_{j=i}^{N} \int_{\mu(E_{j+1})}^{\mu(E_j)} \alpha_i g_* = \sum_{j=1}^{N} \sum_{i=1}^{\min\{j,i_0-1\}} \int_{\mu(E_{j+1})}^{\mu(E_j)} \alpha_i g_*$$

and the contribution for $j < i_0$ is precisely

$$\sum_{j=1}^{i_0-1} \int_{\mu(E_{j+1})}^{\mu(E_j)} \sum_{i=1}^{j} \alpha_i g_* = \sum_{j=1}^{i_0-1} \int_{\mu(E_{j+1})}^{\mu(E_j)} f_* g_* = \int_{\mu(E_{i_0})}^{\mu(E_1)} f_* g_* = \int_t^{\infty} f_* g_*.$$

On the other hand, the contribution for $j \geq i_0$ is

$$\begin{aligned} \sum_{j=i_0}^{N} \sum_{i=1}^{\min\{j,i_0-1\}} \int_{\mu(E_{j+1})}^{\mu(E_j)} \alpha_i g_* &= \sum_{j=i_0}^{N} \sum_{i=1}^{i_0-1} \alpha_i (\mu(E_j) g_{**}(\mu(E_j)) - \mu(E_{j+1}) g_{**}(\mu(E_{j+1}))) \\ &= \sum_{i=1}^{i_0-1} \alpha_i \mu(E_{i_0}) g_{**}(\mu(E_{i_0})) = \sum_{i=1}^{i_0-1} \alpha_i \mu(E_{i_0}) g_{**}(t), \end{aligned}$$

where $\mu(E_{N+1}) g_{**}(E_{N+1})$ has to be replaced with 0. Finally, notice that

$$\sum_{i=1}^{i_0-1} \alpha_i \mu(E_{i_0}) g_{**}(t) + \sum_{i=i_0}^{N} \alpha_i \mu(E_i) g_{**}(t) = \Big(\int_{E_{i_0}} f_* \Big) g_{**}(t) = t f_{**}(t) g_{**}(t)$$

by (6.15). □

Proof of Young's inequality. We assume $q < \infty$. The case $q = \infty$ (where $q_1 = q_2 = \infty$) is far easier and left to the reader. It is clear that

$$\begin{aligned} \|t^{1/p-1/q+1} f_{**}(t) g_{**}(t)\|_{L^q} &= \|t^{1/p_1-1/q_1} f_{**}(t) \; t^{1/p_2-1/q_2} g_{**}(t)\|_{L^q} \\ &\leq \|t^{1/p_1-1/q_1} f_{**}(t)\|_{L^{q_1}} \|t^{1/p_2-1/q_2} g_{**}(t)\|_{L^{q_2}} \\ &= \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}} \end{aligned}$$

(assuming without loss of generality that $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$). Moreover, changing variables $t = \frac{1}{u}$, $s = \frac{1}{r}$ and using Hardy's inequality,

$$\begin{aligned} &\Big(\int_0^{\infty} t^{q/p-1} \Big(\int_t^{\infty} f_*(s) g_*(s) \, ds \Big)^q \, dt \Big)^{1/q} \\ &= \Big(\int_0^{\infty} u^{q/p'-1} \Big(\frac{1}{u} \int_0^{u} r^{-2} f_*(r^{-1}) g_*(r^{-1}) \, dr \Big)^q \, du \Big)^{1/q} \\ &\leq C \Big(\int_0^{\infty} u^{q/p'-1} u^{-2q} f_*(u^{-1})^q g_*(u^{-1})^q \, du \Big)^{1/q} \\ &= C \Big(\int_0^{\infty} t^{q+q/p-1} f_*(t)^q g_*(t)^q \, dt \Big)^{1/q} \\ &= C \|t^{1/p_1-1/q_1} f_*(t) \; t^{1/p_2-1/q_2} g_*(t)\|_{L^q}, \end{aligned}$$

which can be estimated by $|f|_{L^{p_1,q_1}}|g|_{L^{p_2,q_2}}$ as before. The inequality follows from the fact that

$$\|f*g\|_{L^{p,q}} \le \|t^{1/p-1/q+1}f_{**}(t)g_{**}(t) + t^{1/p-1/q}\int_t^\infty f_* g_*\|_{L^q}$$

by the previous lemma. □

Let us now see an important consequence when $n \ge 2$.

Corollary 6.47 (improved Sobolev's embedding). *We have the continuous embedding $W^{1,p}(\mathbb{R}^n) \subseteq L^{p^*,p}(\mathbb{R}^n)$ for all $1 < p < d$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.*

Sketch of proof. By mollification and cut-off, it suffices to show that $\|f\|_{L^{p^*,p}} \le C\|f\|_{W^{1,p}}$ whenever $f \in C_c^\infty(\mathbb{R}^n)$ (since, by Lemma 6.23 and Fatou's lemma, the $L^{p,q}$-quasinorm is lower semicontinuous under pointwise convergence a.e.). We have

$$f = G * \Delta f,$$

where G is Green's function for the Laplacian. Recall that, up to a multiplicative constant, G equals $\log|x|$ if $n = 2$ and $|x|^{2-n}$ if $n \ge 3$. In all cases, commuting a derivative with the convolution, we get

$$f = \sum_{i=1}^n \frac{\partial G}{\partial x_i} * \frac{\partial f}{\partial x_i}$$

(this is legitimate since $G \in W^{1,q}_{loc}(\mathbb{R}^n)$ for any $q < \frac{n}{n-1}$) and, observing that $\frac{\partial G}{\partial x_i}$ equals $\frac{x_i}{|x|^n}$ up to a multiplicative constant, we get $|\frac{\partial G}{\partial x_i}| \in L^{n/(n-1),\infty}(\mathbb{R}^n)$. The claim follows from Young's inequality for Lorentz spaces. □

Remark 6.48. *The improved Sobolev's embedding also holds for $p = 1$, although this is not immediately clear from this proof. Instead, it can be shown using the coarea formula and the isoperimetric inequality. Assuming without loss of generality $f \in C_c^\infty$ nonnegative,*

$$\begin{aligned} |f|_{L^{1^*,1}} &= 1^* \int_0^\infty \mu(\{f > \lambda\})^{1/1^*}\, d\lambda \\ &\le C\int_0^\infty \mathcal{H}^{n-1}(\{f = \lambda\})\, d\lambda \\ &= C\int |\nabla f|, \end{aligned}$$

where the first equality is Proposition 6.31, the inequality is the isoperimetric inequality for the set $\{f > \lambda\}$ (which is a smooth bounded domain for a.e. λ; notice that $1/1^* = (n-1)/n$) and the last equality is the coarea formula.

Proposition 6.49. *In spite of the fact that $W^{1,n}(\mathbb{R}^n) \not\subseteq L^\infty(\mathbb{R}^n)$, a function $f \in L^1_{loc}(\mathbb{R}^n)$ with weak gradient in the Lorentz space $L^{n,1}(\mathbb{R}^n)$ has a continuous representative and satisfies*

$$\|f - c(f)\|_{L^\infty} \le C\|\nabla f\|_{L^{n,1}}$$

for a suitable constant function $c(f)$.

Proof. The main point is that, if $f \in C_c^\infty(\mathbb{R}^n)$, the same proof as Corollary 6.47 gives

$$\|f\|_{L^\infty} \le C\|\nabla f\|_{L^{n,1}}.$$

Instead of Young's inequality, we just use this version of Hölder: $L^{n,1} \cdot L^{n/(n-1),\infty} \subseteq L^1$ (same proof as Theorem 6.41). This allows to say that

$$|f(x)| \le \int \Big|\frac{\partial G}{\partial x_i}\Big|(y)\Big|\frac{\partial f}{\partial x_i}\Big|(x-y)\,d\mu(y) \le \|\nabla G\|_{L^{n/(n-1),\infty}}\|\nabla f\|_{L^{n,1}}$$

for all x. The rest of the work is to reduce to this situation.

Notice first that the convolution with a nonnegative function $\rho_\epsilon \in C_c^\infty(\mathbb{R}^n)$, with support in $B_\epsilon(0)$ and $\int \rho_\epsilon = 1$, satisfies

$$\|\nabla f - \nabla(\rho_\epsilon * f)\|_{L^{p,q}} \le \sup_{|h|\le\epsilon} \|\nabla f - \nabla f(\cdot + h)\|_{L^{p,q}} \tag{6.17}$$

for all $1 < p \le \infty$, $1 \le q \le \infty$: indeed, being $f \in W^{1,1}_{loc}(\mathbb{R}^n)$, $\rho_\epsilon * f$ is smooth and its gradient equals $\rho_\epsilon * \nabla f$, which can be thought as a pointwise limit of convex combinations of functions $\nabla f(\cdot + h)$, with $|h| \le \epsilon$ (e.g. approximating the convolution with a finite sum as for a Riemann integral). The claim follows from Remark 6.39 and the fact that $\|\ \|_{L^{p,q}}$ is a norm invariant under translations in $\mathbb{R}^n$.

As a consequence, if $1 \le p, q < \infty$ then $\nabla(\rho_\epsilon * f) \to \nabla f$: in fact, $g(\cdot + h) \to g$ as $h \to 0$ when $g = \chi_E$ is a characteristic function (with $\mu(E) < \infty$) because $\mu(E\Delta(E-h)) \to 0$ and $(\chi_E - \chi_{E-h})_* = \chi_{[0,\mu(E\Delta(E-h)))}$, so by Corollary 6.32 this holds also for a generic $g \in L^{p,q}(\mathbb{R}^n)$ and the claim follows from (6.17). So there exist smooth functions f_k such that $f_k \to f$ in $L^1_{loc}(\mathbb{R}^n)$ and $\nabla f_k \to \nabla f$ in $L^{n,1}(\mathbb{R}^n)$.

For any $R \ge 1$, the embedding $W^{n,1}(\mathbb{R}^n) \subset L^{2n}(\mathbb{R}^n)$ and Poincaré's inequality give

$$\|f_k - c_{k,R}\|_{L^{2n}(B_{2R})} \le CR^{1/2}\|\nabla f_k\|_{L^n(B_{2R})} \le CR^{1/2}\|\nabla f_k\|_{L^{n,1}(B_{2R})}$$

(with $c_{k,R} := \fint_{B_{2R}} f_k$) and thus, as the proof of Proposition 6.33(3) shows, we get

$$f_k - c_{k,R}\|_{L^{n,1}(B_{2R})} \le CR\|\nabla f_k\|_{L^{n,1}}.$$

Finally, choosing a smooth cut-off function ϕ_R with $\phi = 1$ on B_R, $\phi_R = 0$ outside B_{2R} and $|\nabla\phi_R| \le \frac{2}{R}$,

$$\|\nabla(\phi_R(f_k - c_{k,R}))\|_{L^{n,1}} \le \frac{2}{R}\|f_k - c_{k,R}\|_{L^{n,1}(B_{2R})} + \|\nabla f_k\|_{L^{n,1}} \le C\|\nabla f_k\|_{L^{n,1}}$$

and thus, by the initial part of the proof,

$$\|f_k - c_{k,R}\|_{L^\infty(B_R)} \le \|\phi_R(f_k - c_{k,R})\|_{L^\infty} \le C\|\nabla f_k\|_{L^{n,1}}.$$

The constants $c_{k,R}$ are obviously equibounded (in k, R), since this inequality gives in particular

$$|\fint_{B_1} f_k - c_{k,R}| \le C\|\nabla f_k\|_{L^{n,1}}.$$

Hence, letting $R \to \infty$ along a suitable sequence depending on k, we get $\|f_k - c_k\|_{L^\infty} \le C\|\nabla f_k\|_{L^{n,1}}$ (with $\sup_k |c_k| < \infty$). Letting $k \to \infty$, again along a subsequence, we get the statement. $\square$

6.9 Dyadic characterization of some Lorentz spaces and another proof of Lorentz–Sobolev embedding

In this part we show that, when $q \le p$, the $L^{p,q}$-norm of a function f can be measured in terms of a dyadic decomposition of f according to its values.

In the sequel, $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth function supported in the annulus $\bar{B}_2(0) \setminus B_{1/2}(0)$ and such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}t) = 1, \qquad \text{for all } t \in \mathbb{R}\setminus\{0\}. \tag{6.18}$$

In order to construct φ, take for instance any $\psi \in C_c^\infty(B_2)$ such that $\psi = 1$ on B_1, and set $\varphi(t) := \psi(t) - \psi(2t)$. For any $t \in \mathbb{R}\setminus\{0\}$, it holds

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}t) = \lim_{N\to\infty} \sum_{j=-N}^{N} (\psi(2^{-j}t) - \psi(2^{-(j-1)}t)) = \lim_{N\to\infty} (\psi(2^{-N}t) - \psi(2^{N+1}t)) = 1;$$

the sum is well defined and the first equality holds, since at most two terms in the sum are nonzero: if $2^k \le t \le 2^{k+1}$, then $\varphi(2^{-j}t) = 0$ for $j \ne k, k+1$ since $\varphi(2^{-j}\cdot)$ is supported in the annulus $\bar{B}_{2^{j+1}} \setminus B_{2^{j-1}}$.

Given $f : \mathbb{R}^n \to \mathbb{R}$, we split it according to its values: we set

$$f_j := f\,\varphi(2^{-j}|f|),$$

so that the piece f_j vanishes at x if $|f|(x)$ is not in the range $(2^{j-1}, 2^{j+1})$. Notice that, thanks to (6.18),

$$f = \sum_{j\in\mathbb{Z}} f_j$$

where the sum is actually finite at each point (since at most two terms are nonzero). *This decomposition should not be confused with the Littlewood–Paley decomposition, encountered later in the course, which involves the phase space rather than the values of f!*

Lemma 6.50. *For $1 < p < \infty$ and $1 \le q \le p$ we have*

$$C^{-1}\|f\|_{L^{p,q}} \le \Big(\sum_{j\in\mathbb{Z}} \|f_j\|_{L^p}^q\Big)^{1/q} \le C\|f\|_{L^{p,q}}$$

for some C depending on p, q.

Proof. Since $f_j \le 2^{j+1}\chi_{|f|>2^{j-1}}$, we have

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \|f_j\|_{L^p}^q &\le 2^q \sum_{j\in\mathbb{Z}} 2^{qj}\mu(\{|f| > 2^{j-1}\})^{q/p} \\
&= 8^q \sum_{j\in\mathbb{Z}} 2^{qj}\mu(\{|f| > 2^{j+1}\})^{q/p} \\
&\le 8^q \sum_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \lambda^{q-1}\mu(\{|f| > \lambda\})^{q/p}\,d\lambda \\
&= 8^q \int_0^\infty \lambda^{q-1}\mu(\{|f| > \lambda\})\,d\lambda.
\end{aligned}$$

Conversely, using the subadditivity of $t \mapsto t^{q/p}$ (true as $q \le p$),

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \lambda^{q-1}\mu(\{|f|>\lambda\})^{q/p}\,d\lambda &\le 2^{q-1}\sum_{j\in\mathbb{Z}} 2^{qj}\mu(\{|f|>2^j\})^{q/p} \\
&= 2^{q-1}\sum_{j\in\mathbb{Z}} 2^{qj}\Big(\sum_{k\ge j}\mu(\{2^k<|f|\le 2^{k+1}\})\Big)^{q/p} \\
&\le 2^{q-1}\sum_{j\in\mathbb{Z}}\sum_{k\ge j} 2^{qj}\mu(\{2^k<|f|\le 2^{k+1}\})^{q/p} \\
&\le 2^{q}\sum_{k\in\mathbb{Z}} 2^{qk}\mu(\{2^k<|f|\le 2^{k+1}\})^{q/p}.
\end{aligned}$$

For a given $x \in E$ with $f(x) \neq 0$, if $k \in \mathbb{Z}$ is such that $2^k < |f(x)| \le 2^{k+1}$ then $2^k \le |f(x)| = |f_k(x) + f_{k+1}(x)|$, so

$$2^{pk}\mu(\{2^k<|f|\le 2^{k+1}\}) \le \int |f_k+f_{k+1}|^p \le 2^{p-1}\int |f_k|^p + 2^{p-1}\int |f_{k+1}|^p.$$

Hence, raising to the power $\frac{q}{p}$,

$$2^q\sum_{k\in\mathbb{Z}} 2^{qk}\mu(\{2^k<|f|\le 2^{k+1}\})^{q/p} \le 4^q\sum_{k\in\mathbb{Z}}\Big(\int |f_k|^p + \int |f_{k+1}|^p\Big)^{q/p} \le 2\cdot 4^q\sum_{k\in\mathbb{Z}}\|f_k\|_{L^p}^q.$$

The claim now follows from Proposition 6.31. □

We now present an alternative proof of the Lorentz–Sobolev embedding $W^{1,p}(\mathbb{R}^n) \subset L^{p^*,p}(\mathbb{R}^n)$ for $n \ge 2$ and $1 \le p < n$.

Given $f \in C_c^\infty(\mathbb{R}^n)$, we apply the classical Sobolev embedding to the pieces f_j to get

$$\|f\|_{L^{p^*,p}}^p \le C\sum_{j\in\mathbb{Z}}\|f_j\|_{L^{p^*}}^p \le C\sum_{j\in\mathbb{Z}}\|\nabla f_j\|_{L^p}^p.$$

Since $\nabla f_j = \varphi(2^{-j}f)\nabla f + 2^{-j}f\varphi'(2^{-j}f)\nabla f$ is bounded by $|\nabla f|\chi_{\{2^{j-1}<|f|<2^{j+1}\}}$ up to constants (being $|2^{-j}f| \le 2$ on the support of $\varphi'(2^{-j}f)$), we finally get

$$\sum_{j\in\mathbb{Z}}\|\nabla f_j\|_{L^p}^p \le C\sum_{j\in\mathbb{Z}}\int |\nabla f|^p\chi_{\{2^{j-1}<|f|<2^{j+1}\}} \le C\int |\nabla f|^p.$$

(as it is customary, in the above estimates the value of C can change from line to line). The conclusion follows as in the previous proof.

7 The Calderón-Zygmund decomposition

7.1 Calderón-Zygmund convolution operators

The Calderón-Zygmund decomposition of an integrable function is the key ingredient for proving the continuity of the sub-linear Maximal Operator M in L^p spaces and the continuity of Calderón-Zygmund Operators in L^p Spaces as well. The later being the starting point to the analysis of elliptic PDE in L^p and more generally in non Hilbertian Sobolev or Besov Spaces.

We adopt the following denomination : A cube of size $\delta > 0$ in $\mathbb{R}^n$ is a closed set of the form $C = \prod_{i=1}^n [a_i, a_i + \delta]$ where (a_i) is an arbitrary sequence of n real numbers.

Theorem 7.1 (Calderón-Zygmund Decomposition). *Let $f \in L^1(\mathbb{R}^n)$ with $f \geq 0$ and let $\alpha > 0$. Then there exists an at most countable family of cubes $(C_k)_{k\in K}$ having disjoint interiors such that*

(i) The average of f on all cubes is bounded from below and above by

$$\alpha < \frac{1}{\mu(C_k)} \int_{C_k} f(x)\, dx \leq 2^n \alpha\,. \tag{7.1}$$

(ii) On the complement Ω^c of the union $\Omega = \bigcup_{k\in K} C_k$, we have

$$f(x) \leq \alpha \qquad a.e.\,. \tag{7.2}$$

(iii) There exists a constant $C = C(n)$ depending only on the dimension n such that

$$\mu(\Omega) \leq \frac{C}{\alpha}\, \|f\|_{L^1}\,. \tag{7.3}$$

Remark 7.1. *An alternative way to look at the result is the following. The Calderón-Zygmund Decomposition of threshold $\alpha > 0$ is a non-linear decomposition of any function $f \in L^1$ of the form $f = g + b$ where g and b are two functions respectively in $L^1 \cap L^\infty(\mathbb{R}^n)$ and in $L^1(\mathbb{R}^n)$ satisfying*

i) $\exists\ (C_k)_{k\in K}$ *a family of disjoint cubes of $\mathbb{R}^n$ such that*

$$b = \sum_{k\in K} b_k \qquad \text{with } b_k \equiv 0 \text{ in } \mathbb{R}^n \setminus C_k \quad .$$

ii) *For all $k \in K$ hold the two following conditions*

$$\int_{C_k} b_k(y)\ dy = 0 \qquad \text{and} \qquad \frac{1}{\mu(C_k)} \int_{C_k} |b_k(y)|\ dy \leq\ 2^{n+1}\,\alpha \quad .$$

iii) *g satisfies the following pointwise inequalities*

$$\begin{cases} |g(x)| = |f(x)| \leq \alpha & \text{for a.e. } x \in \mathbb{R}^n \setminus \cup_{k\in K} C_k \quad , \\[2mm] |g(x)| \leq 2^n\, \alpha & \text{for a.e. } x \in \cup_{k\in K} C_k \end{cases}$$

iv) *The L^2 norm of g is controlled as follows*

$$\|g\|^2_{L^2(\mathbb{R}^n)} \leq 2^{2n}\ \alpha\ \|f\|_{L^1(\mathbb{R}^n)} \quad .$$

v) *The Lebesgue measure of the so called "bad set" $\Omega = \cup_{k\in K} C_k$ satisfies*

$$\mu(\Omega) = \sum_{k\in K} \mu(C_k) \leq \frac{1}{\alpha}\ \|f\|_{L^1(\mathbb{R}^n)} \quad .$$

The link between our construction in the proof of theorem 7.1 (applied to $|f|$) and the decomposition $f = g + b$ satisfying $i)\cdots v)$ is made by taking

$$b_k := \left(f - \frac{1}{\mu(C_k)} \int_{C_k} f(y)\ dy \right)\ \chi_{C_k} \quad ,$$

and i)...v) follow from simple estimates. It is worth remembering that Calderón-Zygmund decomposition is not unique.

Example 7.2. *Consider the function $f = \chi_{[0,1]}$, the characteristic function of the segment $[0,1]$ in $\mathbb{R}$. A Calderón-Zygmund decomposition of f with threshold 2^{-i-1} is given by $g = 2^{-i}\chi_{[0,2^i]}$ and the set Ω is made of a unique cube : $[0,2^i]$. $b = 0$ outside $[0,2^i]$ and $b = \chi_{[0,1]} - 2^{-i}\chi_{[0,2^i]}$ has indeed average 0 on the unique cube of the decomposition.*

Proof of theorem 7.1.

We divide $\mathbb{R}^n$ into a mesh of equal cubes chosen large enough such that their volume is larger or equal than $\|f\|_{L_1}/\alpha$. Thus, for every cube C_0 in this mesh, we have

$$\frac{1}{\mu(C_0)} \int_{C_0} f(x)\, dx \leq \alpha\,. \tag{7.4}$$

Every cube C^0 from the initial mesh is decomposed into 2^n equal disjoint cubes with half of the side-length. For the resulting cubes, there are now two possibilities: Either (7.4) still holds or (7.4) is violated. Cubes of the first case are called the *good cubes*, the set of *good cubes* is denoted by $\mathcal{C}_1^g$, and the set of non *good cubes*, the *bad cubes*, is denoted by $\mathcal{C}_1^b$. In a next step, we decompose all cubes in $\mathcal{C}_1^g$ into equal disjoint cubes with half side-length and leave the cubes in $\mathcal{C}_1^b$ unchanged. The resulting cubes for which an estimate of the form (7.4) still holds are denoted by $\mathcal{C}_2^g$ - they are called *good cubes* as well - and the remaining ones by $\mathcal{C}_2^b$. Then, we proceed as before dividing the cubes in $\mathcal{C}_2^g$ and leaving the cubes in $\mathcal{C}_2^b$ unchanged. – Repeating this procedure for each cube in the initial mesh, we can define $\Omega = \bigcup_{k\in K} C_k$ as the union of all cubes which violate in some step of the decomposition process an estimate of the form (7.4). (These are precisely those cubes with an upper index b for *bad*.)

Note that for a cube C_i^b in $\mathcal{C}_i^b$ obtained in the i-th step, we have

$$\frac{1}{\mu(C_i^b)} \int_{C_i^b} f(x)\, dx > \alpha\,. \tag{7.5}$$

Since $2^n\mu(C_i^b) = \mu(C_{i-1}^g)$, where C_{i-1}^g is any cube in $\mathcal{C}_{i-1}^g$, we then deduce

$$\alpha < \frac{1}{\mu(C_i^b)} \int_{C_i^b} f(x)\, dx \leq \frac{2^n}{\mu(C_{i-1}^g)} \int_{C_{i-1}^g} f(x)\, dx \leq 2^n\, \alpha\,.$$

This shows (i) of the theorem.

In order to show (ii), we note that by Lebesgue's differentiation theorem, almost everywhere the following holds

$$f(x) = \lim_{d\to 0} \frac{1}{\mu(C_{x,d})} \int_{C_{x,d}} f(y)\, dy\,,$$

where $C_{x,d}$ denotes a cube containing $x \in \mathbb{R}^n$ with diameter d. By construction of the decomposition, there exists for every $x \in \Omega^c$ a diameter $d_x > 0$ such that all cubes $C_{x,d}$ with diameter $d < d_0$ satisfy an estimate of the form (7.4). This implies directly that $f(x) \leq \alpha$ for a.e. $x \in \Omega^c$.

The last part (iii) of the theorem can be established as follows:

$$\mu(\Omega) = \sum_{k\in K} \mu(C_k) \overset{(7.5)}{<} \frac{1}{\alpha} \int_\Omega f(x)\, dx \leq \frac{1}{\alpha}\, \|f\|_{L^1}\,.$$

□

7.2 An application of Calderón-Zygmund decomposition

The following theorem gives a statement which is close to a converse to theorem 5.8. The proof of this theorem we give is an interesting application of the Calderón-Zygmund decomposition.

Theorem 7.3. *Let f be an integrable function on $\mathbb{R}^n$ supported on an euclidian ball B. Then $Mf \in L^1(B)$ if and only if $f \in L^1 \log L^1(B)$.*

The proof of theorem 7.3 is using the following lemma.

Lemma 7.4. *Let f be a locally integrable function on $\mathbb{R}^n$. Let B be an open euclidian ball of $\mathbb{R}^n$ such that $Mf \in L^1(B)$ then $f \in (L^1 \log L^1)_{loc}(B)$.*

Proof of lemma 7.4. Let ω be an open subset strictly included in B - i.e. $\overline{\omega} \subset B$. Denote by f_ω the restriction of f to ω. It is clear that the inequality $Mf(x) \geq Mf_\omega(x)$ holds for almost every $x \in \mathbb{R}^n$. Hence, for every $\beta > 0$ the following holds

$$\mu\left(\{x\ ;\ Mf(x) > \beta\}\right) \geq \mu\left(\{x\ ;\ Mf_\omega(x) > \beta\}\right) \quad . \tag{7.6}$$

In order to show that $f_\omega \in L^1 \log L^1(\mathbb{R}^n)$, we use the following "reverse" inequality to (5.9) for the Hardy-Littlewood maximal function : there exists a constant c depending only on n such that

$$\mu(\{x \in \Omega\ :\ Mf_\omega(x) > c\,\alpha\}) \geq \frac{1}{2^n\alpha} \int_{\{x\in\mathbb{R}^n\,:\,|f_\omega(x)|>\alpha\}} |f_\omega(x)|\, dx \tag{7.7}$$

where $\Omega = \cup_{k\in K} C_k$ is the union of bad cubes for a Calderón-Zygmund decomposition of mesh α applied to f_ω on $\mathbb{R}^n$ and given by the previous theorem 7.1.

Proof of inequality (7.7). For any $\alpha > 0$ theorem 7.1 gives, for the function f_ω, a family of cubes $(C_k)_{k\in K}$ of disjoint interiors such that (see (7.1))

$$\left\{ \begin{array}{ll} 2^n\alpha \geq \frac{1}{\mu(C_k)} \int_{C_k} |f_\omega(x)|\, dx \geq \alpha & \text{and} \\ \forall x \in \mathbb{R}^n \setminus \Omega \quad |f_\omega(x)| \leq \alpha & . \end{array} \right. \tag{7.8}$$

Thus, if $x \in C_k$, it follows that $Mf_\omega(x) > c\,\alpha$, where the constant $c > 0$ is an adjustment which permits to pass from cubes to balls in the definition of the maximal function. As a direct consequence, we have that

$$\mu(\{x \in \Omega \,:\, Mf_\omega(x) > c\,\alpha\}) \geq \sum_{k=1}^{\infty} \mu(C_k) \overset{(7.8)}{\geq} \frac{1}{2^n\alpha} \int_\Omega |f(x)|\, dx \quad .$$

Since $|f_\omega(x)| \leq \alpha$, for $x \in \mathbb{R}^n \setminus \Omega$, the desired inequality (7.7) is established.

Let $\delta > 0$ such that for every cube C

$$\mu(C) \leq \delta \quad \text{and} \quad C \cap \omega \neq \emptyset \quad \Longrightarrow C \cap \mathbb{R}^n \setminus B = \emptyset \quad . \tag{7.9}$$

δ has been chosen in such a way that, for any $\alpha > \alpha_0 = \int_\omega f/\delta$, the bad set Ω is included in B - this lower bound on α ensures indeed the fact that the mesh of the starting cubes in the associated Calderón-Zygmund decomposition is less than δ . Hence we deduce using (7.6), for any $\alpha > \alpha_0$, that

$$\mu(\{x \in B \,:\, Mf(x) > c\,\alpha\}) \geq \frac{1}{2^n\alpha} \int_{\{x\in\omega \;;\; |f(x)|>\alpha\}} |f(x)|\, dx \quad .$$

Using the previous estimate we compute

$$\begin{aligned} \|Mf\|_{L^1(B)} = \int_B Mf(x)\, dx &\overset{(5.1)}{\geq} \int_{c\alpha_0}^{\infty} \mu(\{x \in B \,:\, Mf(x) > \alpha\})\, d\alpha \\ &\overset{(7.7)}{\geq} c \int_{\alpha_0}^{\infty} \left(\frac{1}{2^n\alpha} \int_{\{x\in\omega\,:\,|f(x)|>\alpha\}} |f(x)|\, dx \right) d\alpha \\ &= c \int_\omega |f(x)| \left(\int_{\alpha_0}^{\max\{\alpha_0, |f(x)|\}} \frac{1}{\alpha}\, d\alpha \right) dx \\ &= c \int_\omega |f(x)| \log^+ \frac{|f(x)|}{\alpha_0}\, dx\,. \end{aligned}$$

This proves the lemma. □

Proof of theorem 7.3.

One direction in the equivalence has been established in theorem 5.8. It suffices then to establish that $Mf \in L^1(B)$ and f supported in B imply that $f \in L^1 \log L^1(B)$.

Let's take to simplify the presentation B to be the unit ball of center the origin $B := B_1(0)$. First we show the following statement

$$
(7.10) \quad f \equiv 0 \text{ in } \mathbb{R}^n \setminus B_1(0) \quad \text{and} \quad Mf \in L^1(B_1(0)) \quad \Longrightarrow Mf \in L^1(B_2(0)) \quad .
$$

Once we will have proved this implication, using the previous lemma, we will deduce that $f \in L^1 \log L^1(B)$ and this will finish the proof of theorem 7.3.

Proof of (7.10). Let x be a point in $B_2(0) \setminus B_1(0)$. Since every point in $B_1(0)$ is closer to $x/|x|^2$ than to x, for $|x| > 1$, one obtains that $B_R(x) \cap B_1(0) \subset B_R(x/|x|) \cap B_1(0)$. We then deduce

$$
\int_{B_R(x)} |f(y)| \, dy \le \int_{B_R(x/|x|^2)} |f(y)| \, dy \quad ,
$$

which implies that $Mf(x) \le Mf(x/|x|^2)$ for $|x| > 1$. Thus

$$
\int_{B_2(0)\setminus B_1(0)} Mf(x) \, dx \le 2^{2n} \int_{B_1(0)\setminus B_{1/2}(0)} Mf(y) \, dy \quad .
$$

This last inequality implies (7.10) and theorem 7.3 is then proved. □

7.3 Singular Integral Operators over L^p

Singular integral operators are special cases of Calderón-Zygmund type operators. They are the "historical" ones : the first one introduced by Calderón and Zygmund in the 50's-60's corresponding to the principal values of singular integrals. They are the key notion giving access to the L^p theory (and more generally to the non hilbertian theory) of elliptic operators. Roughly speaking a typical question relevant to the theory of Singular Integral Operators is the following : if the L^p norm of the laplacian of a function is in L^p is it true or not that every second derivatives of this function are in L^p ?

This question is answered easily in the case $p = 2$ by the mean of Fourier transform but requires a more sophisticated analysis for being considered for $p \neq 2$. Of course the interest and the use of Singular Integral Operators goes much far beyond the resolution to this question and we will see applications of them all along this book.

A singular integral operator is formally a linear mapping of the form $T \ : \ f \to K \star f$ where K is the kernel which misses to be in L^1 or even L^1_{loc} from "very little". If K would be in L^1 then the continuity of T from L^p into itself would be a simple consequence of Young's inequality on convolutions. Usually the pointwise expression of the Kernel K is only in L^1-weak :

$$
\sup_{\alpha > 0} \alpha \; \mu\left(\{x \; ; \; |K(x)| > \alpha\}\right) < +\infty \quad .
$$

A typical example of such a convolution operator is the one which to $f = \Delta u$ assigns the second derivative of u along the i and j directions : $\partial_{x_i}\partial_{x_j} u$ (modulo harmonic functions of course). This operator is given formally for $i \neq j$ by

$$
\partial_{x_i}\partial_{x_j} u = C_n \int_{\mathbb{R}^n} \frac{(x_i - y_i)\,(x_j - y_j)}{|x-y|^{n+2}} \, f(y) \, dy \quad .
$$

It is a convolution type operator T of kernel $K(x) = C_n\ x_i x_j/|x|^{n+2}$. K is in L^1-weak but it is not *a priori* a distribution and this makes the use of the convolution operation and the definition of T problematic or *singular*. Calderón-Zygmund operators of the first generation share the same difficulty. The reason why the Calderón-Zygmund Kernels K can be made to be a distribution is a cancellation property. In the previous example the cancellation property happens to be (recall that we look at the case$i \neq j$)

$$\int_{S^{n-1}} \frac{x_i x_j}{|x|^{n+2}}\ dy = 0 \quad .$$

Because of this later fact, for a smooth given compactly supported function f, it is not difficult to show that

$$\lim_{\epsilon \to 0} C_n \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{(x_i - y_i)\,(x_j - y_j)}{|x-y|^{n+2}}\ f(y)\ dy \tag{7.11}$$

exists for every x. This singular integral is the convolution between f and the distribution called *Principal Value of* K denoted $PV(K)$.

One of the spectacular result of Calderón-Zygmund theory says the following : the limit (7.11) $PV(K) \star f(x)$ exists almost everywhere whenever f is in $L^p(\mathbb{R}^n)$ for $p \in [1, +\infty]$ and is also in $L^p(\mathbb{R}^n)$ if f is in $L^p(\mathbb{R}^n)$ for $p \in (1, +\infty)$.

Another example of Singular Integral Operator is the Hilbert Transform on $\mathbb{R}$ - which corresponds in Fourier space by multiplying $\widehat{f}(\xi)$ by the sign of ξ - that is : $f \to f * \frac{1}{i\pi\, x}$. This *singular* integral has to be understood as being the limit of the following process

$$\lim_{\epsilon \to 0} \frac{1}{i\pi} \int_{|y|>\epsilon} \frac{f(x-y)}{y}\ dy \quad . \tag{7.12}$$

at least when f is smooth and compactly supported, since x^{-1} is odd, one easily check that this limit exists everywhere. It is equal to the convolution between f and the *Principal Value of* x^{-1}, $PV(1/x)$. Here again Calderón-Zygmund theory will tell us that the limit (7.12) $PV(x^{-1}) \star f$ exists almost everywhere whenever f is in $L^p(\mathbb{R}^n)$ for $p \in [1, +\infty]$ and is also in $L^p(\mathbb{R}^n)$ if f is in $L^p(\mathbb{R}^n$ for $p \in (1, +\infty)$.

In a way which is reminiscent to the L^p-theory of the maximal operator in the previous sections, the Hilbert transform and more generally Calderón-Zygmund operator won't map L^1 functions into L^1 functions but to L^1-weak functions only. In analogy with the previous section again, Calderón-Zygmund operator will however send $L^1 \log L^1$ functions into L^1. The parallel with the results obtained for the maximal operator in the previous section has some limit since, as we will see, L^∞ functions won't be map by Calderón-Zygmund operators to L^∞ functions but to $\cap_{p<+\infty} L^p_{loc}(\mathbb{R}^n)$ functions only.

Here again the Calderón-Zygmund decomposition will be the key instrument in the proofs. This use of Calderón-Zygmund decomposition is also known under the name of the *real variable method* of Calderón and Zygmund.

Let us finish the introduction to this very important section by making the following amusing remark. If the L^1-weak would have been a Banach space for a norm $\|\cdot\|_\star$ equivalent to the quasi-norm L^1_w - (5.5) -, then the L^p theory of Calderón-Zygmund operator would be trivially true without any assumption on the Kernel K except that it is in L^1-weak and that $T\ :\ f \to K\star f$ sends L^2 into L^2. Indeed, for any finite set of k points $a_1,\cdots,a_k$ in $\mathbb{R}^n$ and any family of k reals $\lambda_1\cdots\lambda_k$ one would have using the triangular inequality

$$\left\|\sum_{i=1}^k K(x-a_i)\ \lambda_i\right\|_\star \le \|K\|_\star \sum_{i=1}^k |\lambda_i| \quad ,$$

and we would directly deduce that T sends L^1 into L^1_w. The Marcinkiewicz interpolation theorem 4.2 would then imply that T is continuous from L^p into L^p for any $p\in(1,2]$ and the continuity for $p\in[2,+\infty)$ would be obtained by a simple duality argument.

We shall see three different formulations of the continuity of a Singular Integral Operator in L^p spaces, each of these formulations are based on different assumptions on the Kernel K.

7.3.1 A "primitive" formulation

In this subsection we prove the following "primitive" formulation of the L^p-continuity of Calderón-Zygmund convolution operator. The sense we give to the adjective "primitive" here should not be interpreted as something pejorative about this formulation, which has the clear pedagogical advantage to bring us progressively to more elaborated ones in the next subsections. In this formulation the difficulties caused by the singular nature of the convolution does not appear since the kernel K is "artificially" assumed to be in L^2.

Theorem 7.5. *Let $K\in L^2(\mathbb{R}^n)$ and assume the following:*

(i) The Fourier transform $\widehat{K}$ of K is bounded in L^∞

$$\|\widehat{K}\|_{L^\infty} < +\infty \quad . \tag{7.13}$$

(ii) The function K satisfies the so-called Hörmander condition : there exists $0 < B < +\infty$ such that

$$\int_{2\|y\|\le\|x\|} \big|K(x-y)-K(x)\big|\,dx \le B\,, \qquad \forall y\neq 0 \quad . \tag{7.14}$$

Moreover, let T be the well-defined convolution operator on $L^1(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$, with $1<p<\infty$, given pointwise by

$$Tf(x) = K\star f(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\,dy \quad . \tag{7.15}$$

Then, there exists a constant $C_p = C(n,p,\|K\|_\infty,B)$ – independent of the L^2-norm of K – such that

$$\|Tf\|_{L^p} \le C_p\,\|f\|_{L^p} \quad . \tag{7.16}$$

Moreover there exists a constant $C_1 = C(n, \|K\|_\infty, B)$ – independent of the L^2-norm of K – such that for any $f \in L^1(\mathbb{R}^n)$

$$\sup_{\alpha>0} \alpha \ \mu(\{x \in \mathbb{R}^n \ ; \ |K \star f(x)| > \alpha\}) \leq C_1 \ \|f\|_{L^1} \tag{7.17}$$

Remark 7.2. *a) Note that T is a densely defined linear operator on $L^p(\mathbb{R}^n)$. More precisely, the operator is well-defined on the dense linear subset $L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ of $L^p(\mathbb{R}^n)$ and from (7.16) we can deduce that T can be extended to all of $L^p(\mathbb{R}^n)$ by this.*

b) In the previous theorem, the kernel K is assumed to be in $L^2(\mathbb{R}^n)$. This happens to be "artificial" in the following sense : it permits to make the convolution operator T well defined on $L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, for $1 < p < \infty$ indeed by Young's inequality we have

$$\|Tf\|_{L^2} \leq \|K\|_{L^2} \|f\|_{L^1} \quad .$$

However the final crucial estimate leading to the continuity of T from L^p into L^p is independent of the L^2 norm of K.

c) Observe that the Hörmander condition (7.14) holds, for instance, whenever K is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ and there exists $C > 0$ such that

$$\forall x \in \mathbb{R}^n \setminus \{0\} \qquad |\nabla K|(x) \leq \frac{C}{|x|^{n+1}} \quad .$$

This comes from the following estimate : Let $y \neq 0$ and denote $v = y/|y|$, then the following holds

$$\begin{aligned}
&\int_{2\|y\|\leq\|x\|} \big|K(x-y) - K(x)\big| \, dx \\
&= \int_{2\|y\|\leq\|x\|} \left|\int_0^{|y|} \tfrac{\partial K}{\partial v}(x + t\, v) \ dt\right| \ dx \\
&\leq \int_0^{|y|} dt \int_{2\|y\|\leq\|x\|} |\nabla K|(x+tv) \ dx \\
&\leq |y| \int_{\|y\|\leq\|z\|} |\nabla K|(z) \ dz \leq C_n \tfrac{|y|}{|y|} = C_n \quad .
\end{aligned} \tag{7.18}$$

where we have proceeded to the change of variable $z = x + tv$.

Proof of theorem 7.5 The proof is divided in the following three steps: First, we show that the convolution operator T is of strong type $(2,2)$. In a second step, we establish that T is of weak type $(1,1)$ - i.e. inequality (7.17), which is the most difficult part of the proof. Finally we obtain the inequality (7.16) from Marcinkiewicz's interpolation theorem and a duality argument.

First step: Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then for the Fourier transform $\widehat{Tf}$ of $Tf \in L^2(\mathbb{R}^n)$, we have

$$\|\widehat{Tf}\|_{L^2} = \|\widehat{K \star f}\|_{L^2} = \|\widehat{K}\,\widehat{f}\|_{L^2} \overset{(7.13)}{\leq} \|K\|_\infty \, \|f\|_{L^2} \quad .$$

Since $\|\widehat{Tf}\|_{L^2} = \|Tf\|_{L^2}$ by Plancherel's theorem, we then obtain

$$\|Tf\|_{L^2} \leq \|K\|_\infty \, \|f\|_{L^2} \, . \tag{7.19}$$

This shows that T is of type $(2,2)$, which also implies that T is of weak type $(2,2)$ as we mentioned in remark 4.1, precisely

$$\forall \alpha > 0 \qquad \mu(\{x\,:\,|Tf(x)| > \alpha\}) \leq \frac{\|K\|_\infty^2}{\alpha^2}\,\|f\|_{L^2}^2\,. \tag{7.20}$$

Second step: Let $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$. We apply the Calderón-Zygmund Decomposition 7.1 of threshold α to f. The resulting family of disjoint "bad cubes" will be denoted by $\{C_k\}_{k\in K}$ and we write $\Omega = \bigcup_{k=1}^{\infty} C_k$ for their union.

Now, we define

$$g(x) = \begin{cases} f(x) & \text{for } x \in \Omega^c \\ \dfrac{1}{\mu(C_k)} \displaystyle\int_{C_k} f(y)\,dy & \text{for } x \in C_k\,. \end{cases} \tag{7.21}$$

Following remark 7.1 C-Z Decomposition permits to write f as sum of a "good" and a "bad" function, namely $f = g + b$ - "good" and "bad" stand for the fact that there is a better control, namely L^∞, on g than on b - where

$$b = \sum_{k\in K} b_k \quad , \tag{7.22}$$

with

$$b_k(x) = \left(f(x) - \frac{1}{\mu(C_k)}\int_{C_k} f(y)\,dy\right)\chi_{C_k}(x) \quad .$$

From the linearity of the convolution operator T and the triangular inequality we have for all $x \in \mathbb{R}^n$

$$|Tf(x)| \leq |Tg(x)| + |Tb(x)| \quad . \tag{7.23}$$

Hence we deduce

$$\begin{aligned} \mu(\{x\,:\,|Tf(x)| > \alpha\}) \;&\leq\; \mu(\{x\,:\,|Tg(x)| > \alpha/2\}) \\ &\quad + \mu(\{x\,:\,|Tb(x)| > \alpha/2\}) \quad . \end{aligned} \tag{7.24}$$

In order to get an estimate for the first term on the right-hand side of (7.24), we first use the fact that g is an element of $L^2(\mathbb{R}^n)$- see remark 7.1 iv) - with the following control

$$\|g\|_{L^2(\mathbb{R}^n)}^2 \leq 2^{2n}\,\alpha\,\|f\|_{L^1(\mathbb{R}^n)} \quad .$$

As a consequence, we can apply (7.20) to $g \in L^2(\mathbb{R}^n)$ in order to get the following estimate for the first term on the right-hand side of (7.24):

$$\begin{aligned} \mu(\{x\,:\,|Tg(x)| > \alpha/2\}) \;&\leq\; \frac{4\|K\|_\infty^2}{\alpha^2}\|g\|_{L^2}^2 \\ &\leq\; 2^{2n+2}\,\frac{\|K\|_\infty^2}{\alpha}\,\|f\|_{L^1(\mathbb{R}^n)} \quad . \end{aligned} \tag{7.25}$$

Next, we estimate the second term on the right hand-side of (7.24). – For this purpose, we expand each cube C_k in the Calderón-Zygmund decomposition by the factor $2\sqrt{n}$ leaving its center c_k fixed. The new bigger cubes are denoted by $\tilde{C}_k$ and its union by $\tilde{\Omega} = \bigcup_{k\in K} \tilde{C}_k$. It is easy to see that $\Omega \subset \tilde{\Omega}$, $\tilde{\Omega}^c \subset \Omega^c$ and $\mu(\tilde{\Omega}) \leq (2\sqrt{n})^n \,\mu(\Omega)$. Moreover, for $x \notin \tilde{C}_k$, we have

$$\|x - c_k\| \geq 2\,\|y - c_k\|\,, \qquad \text{for all } y \in C_k \quad . \tag{7.26}$$

Now, let c_k denote the center of the cube C_k. Then, we can write

$$\begin{aligned} Tb(x) = \sum_{k\in K} Tb_k(x) \;&=\; \sum_{k\in K} \int_{C_k} K(x-y) b_k(y)\, dy \\ &=\; \sum_{k\in K} \int_{C_k} \big(K(x-y) - K(x-c_k)\big) b_k(y)\, dy\,, \end{aligned}$$

being a direct consequence of the fact that for all C_k

$$\int_{C_k} b_k(y)\, dy = \int_{C_k} \left(f(y) - \frac{1}{\mu(C_k)} \int_{C_k} f(z)\, dz \right) dy = 0 \quad ,$$

- condition ii) in remark 7.1 -. This then leads to

$$\begin{aligned} \int_{\tilde{\Omega}^c} |Tb(x)|\, dx \;&\leq\; \sum_{k\in K} \int_{\tilde{\Omega}^c} \left(\int_{C_k} \big|K(x-y) - K(x-c_k)\big|\, \big|b_k(y)\big|\, dy \right) dx \\ &\leq\; \sum_{k\in K} \int_{\tilde{C}_k^c} \left(\int_{C_k} \big|K(x-y) - K(x-c_k)\big|\, \big|b_k(y)\big|\, dy \right) dx \\ &=\; \sum_{k\in K} \int_{C_k} \left(\int_{\tilde{C}_k^c} \big|K(x-y) - K(x-c_k)\big|\, dx \right) \big|b_k(y)\big|\, dy\,. \end{aligned}$$

Setting $\bar{x} = x - c_k$, $\bar{y} = y - c_k$ and using (7.26), the integral in parenthesis ca be bounded this way

$$\int_{\tilde{C}_k^c} \big|K(x-y) - K(x-c_k)\big|\, dx \leq \int_{2\|\bar{y}\| \leq \|\bar{x}\|} \big|K(\bar{x}-\bar{y}) - K(\bar{x})\big|\, d\bar{x} \quad .$$

The assumption (7.14) of the theorem hence implies that

$$\int_{\tilde{\Omega}^c} |Tb(x)|\, dx \leq B \sum_{k\in K} \int_{C_k} |b_k(y)|\, dy \leq C\, \|f\|_{L^1} \quad . \tag{7.27}$$

At this stage, we are ready to give the following estimate for the second term in (7.24):

$$\begin{aligned} \mu(\{x\in\mathbb{R}^n \,:\, |Tb(x)| > \frac{\alpha}{2}\}) \;&\leq\; \mu(\{x\in\tilde{\Omega}^c \,:\, |Tb(x)| > \alpha/2\}) + \mu(\tilde{\Omega}) \\ &\overset{(7.27)}{\leq}\; \frac{2C}{\alpha}\, \|f\|_{L^1} + (2\sqrt{n})^n\, \mu(\Omega) \\ &\overset{(7.3)}{\leq}\; \frac{2C}{\alpha}\, \|f\|_{L^1} + \frac{C}{\alpha}\, \|f\|_{L^1} \leq \frac{C}{\alpha}\, \|f\|_{L^1}\,. \end{aligned} \tag{7.28}$$

Where C only depends on n, $\|K\|_\infty$ and B. Combining (7.25) with (7.28), we end up with the existence of a constant $C_1 > 0$ such that

$$\mu(\{x : |Tf(x)| > \alpha\}) \leq \frac{C_1}{\alpha} \|f\|_{L^1}, \tag{7.29}$$

showing (7.17) and hence that the convolution operator T is of weak type $(1,1)$.

Third step: Note that we have already shown the inequality (7.16) in the case of $p = 2$ in (7.19). – Putting $r = 2$ in Marcinkiewicz Interpolation Theorem 4.2 and using the fact that T is of weak type $(1,1)$, respectively $(2,2)$, by (7.20), respectively (7.29), we conclude that

$$\|Tf\|_{L^p} \leq C \|f\|_{L^p}, \tag{7.30}$$

for $1 < p < 2$ and where C only depends on n, p, $\|K\|_\infty$ and B the constant in the Hörmander condition.

For the case $2 < p < \infty$, we will use a *duality argument*. – Consider the dual space $L^{p'}(\mathbb{R}^n)$ of $L^p(\mathbb{R}^n)$ with $1/p + 1/p' = 1$. We easily see that $1 < q < 2$. Consider now $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. Since L^p is itself the dual space to $L^{p'}$ and since $L^1 \cap L^{p'}$ is dense in $L^{p'}$, the L^p-norm of Tf is given by the following expression:

$$\|Tf\|_{L^p} = \sup_{\substack{g \in L^1 \cap L^{p'} \\ \|g\|_{L^{p'}} \leq 1}} \left| \int_{\mathbb{R}^n} Tf(x) g(x)\, dx \right|. \tag{7.31}$$

We calculate

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} Tf(x) g(x)\, dx \right| &= \left| \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(x-y) f(y)\, dy \right) g(x)\, dx \right| \\
&= \left| \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(x-y) g(x)\, dx \right) f(y)\, dy \right|,
\end{aligned}$$

where Fubini's theorem was applied because of $K \in L^2(\mathbb{R}^n)$ and the assumptions on g and f. For the first integral, we conclude from (7.30) that it is an element of $L^{p'}(\mathbb{R}^n)$. Using Hölder's inequality, we end up with

$$\begin{aligned}
\sup_{\substack{g \in L^1 \cap L^{p'} \\ \|g\|_{L^{p'}} \leq 1}} \left| \int_{\mathbb{R}^n} Tf(x) g(x)\, dx \right| &\leq \int_{\mathbb{R}^n} \left| \left(\int_{\mathbb{R}^n} K(x-y) g(x)\, dx \right) f(y) \right| dy \\
&\overset{(7.30)}{\leq} C \|g\|_{L^{p'}} \|f\|_{L^p} \leq C \|f\|_{L^p}.
\end{aligned}$$

This establishes the theorem. □

7.3.2 A singular integral type formulation

In the present formulation of the L^p continuity for convolution type Calderón-Zygmund Operator we will skip the too strong assumption that the kernel K is in L^2 and will assume only a L^1-weak type pointwise control of K + a cancellation property together, still with the Hörmander condition. We will be then facing the heart of the matter : how can we deal with the singular integral $K \star f$ when f is only assumed to be in L^p ?

Theorem 7.6. *Let $K : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a measurable function such that there exists $A, B > 0$ for which the following holds*

$$|K(x)| \leq \frac{A}{\|x\|^n} \quad , \qquad \forall\, x \neq 0 \quad . \tag{7.32a}$$

$$\int_{2\|y\|\leq\|x\|} \big|K(x-y) - K(x)\big|\, dx \leq B \quad , \qquad \forall\, x \neq 0 \quad . \tag{7.32b}$$

$$\int_{\partial B_r(0)} K(x)\, dx = 0 \quad , \qquad \text{for a. e. } r > 0 \quad . \tag{7.32c}$$

For $\varepsilon > 0$ and $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$, we set

$$T_\varepsilon f(x) = \int_{\|y\|\geq\varepsilon} f(x-y) K(y)\, dy \quad . \tag{7.33}$$

Then, for any $1 < p < +\infty$ there exists a positive constant C such that for any $\varepsilon > 0$ and any $f \in L^p(\mathbb{R}^n)$,

$$\|T_\varepsilon f\|_{L^p} \leq C\, \|f\|_{L^p} \,, \tag{7.34}$$

where the constant $C = C(p, n, A, B)$ is independent of ε and f. Moreover, there exists $Tf \in L^p(\mathbb{R}^n)$ such that

$$T_\varepsilon f \longrightarrow Tf \quad \text{in } L^p \qquad (\varepsilon \longrightarrow 0) \quad . \tag{7.35}$$

For any $f \in L^1(\mathbb{R}^n)$ there exists a measurable function Tf in L^1-weak such that

$$T_\varepsilon f \longrightarrow Tf \quad \text{in } L^1_w \tag{7.36}$$

and there exists a constant positive $C(n, A, B)$ independent of f and ϵ such that

$$\sup_{\alpha>0} \alpha\ \mu(\{x \in \mathbb{R}^n\ ;\ |Tf(x)| > \alpha\}) \leq C(n, A, B)\ \|f\|_{L^1} \tag{7.37}$$

Remark 7.3. *The singular integral defined in (7.33) is, for a fixed ϵ, absolutely convergent. To see this, note that due to (7.32a) we have that $K \in L^{p'}(\mathbb{R}^n \setminus B_\varepsilon)$, where $1 < p'$ is the Hölder conjugate exponent of p. From Young's inequality, it then follows that $\|T_\varepsilon f\|_\infty \leq \|f\|_{L^p} \|K\|_{L^{p'}}$.*

A substantial part of the proof of theorem 7.6 will be to derive from the assumptions (7.32a), (7.32b) and (7.32c) an L^∞ bound for the Fourier transform of $K_\varepsilon(y) := K(y)\ \chi_{\mathbb{R}^n \setminus B_\varepsilon(0)}$ independent of ε. This estimate will permit us to invoke theorem 7.5 at some point in our proof. Precisely the following lemma holds.

Lemma 7.7. *Let $K : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a measurable function such that*

$$|K(x)| \leq \frac{A}{\|x\|^n} \quad , \qquad \text{for } x \neq 0 \quad . \tag{7.38a}$$

$$\int_{2\|y\|\leq\|x\|} \big|K(x-y) - K(x)\big|\, dx \leq B \quad , \qquad \text{for } y \neq 0 \quad . \tag{7.38b}$$

$$\int_{\partial B_r(0)} K(x)\, dx = 0 \quad , \qquad \text{for a. e. } \ r > 0 \quad . \tag{7.38c}$$

Moreover, for every $\varepsilon > 0$, we define

$$K_\varepsilon(x) = \begin{cases} K(x) & \text{if } \|x\| \geq \varepsilon \\ 0 & \text{if } \|x\| < \varepsilon\,. \end{cases} \tag{7.39}$$

Then, there exists a constant $C = C(n, A, B)$, independent of ε, such that

$$\|\widehat{K_\varepsilon}\|_\infty \leq C\,. \tag{7.40}$$

Before to prove this L^∞ bound we would like to show first how the hypothesis relative to the cancellation property (7.32c) is essential. How cancellation property can lead to decisive improvements in the estimates will be a leitmotiv in this book - see in particular the chapter on Hardy spaces and the integrability by compensation phenomenon.

Example 7.8. *Consider the function on $\mathbb{R}$ given by $K(t) = \frac{1}{|t|}$ It is not difficult to check that K satisfies hypothesis (7.38a) and (7.38b) but the cancellation assumption (7.38c) is violated. we now prove that for this function K the conclusion of lemma 7.7 fails. We have*

$$\begin{aligned} \widehat{K_\varepsilon}(\xi) := & \lim_{r\to 0} \int_{\varepsilon<|t|<r} e^{2\pi i t\,\xi}\ \tfrac{dt}{|t|} = \lim_{r\to 0} \int_{\varepsilon<|t|<r} cos(2\pi t\,\xi)\ \tfrac{dt}{|t|} \quad , \\ = & \ 2sgn(\xi)\ \int_{\varepsilon|\xi|}^{+\infty} \tfrac{\cos 2\pi s}{s}\ ds \quad , \end{aligned}$$

where we have used the parity and the imparity respectively of $cos(2\pi t\,\xi)/|t|$ and $sin(2\pi t\,\xi)/|t|$. Now, since $\int_0^{+1} \cos s/s\ ds = +\infty$ we deduce that $\widehat{K_\varepsilon}(\xi)$ goes to $+\infty$ as ε goes to zero for non zero ξ.

Observe that a change of sign for K that would ensure the cancellation property (7.38c) - by taking $1/t$ instead of $1/|t|$ - would lead to the integral $\int_0^{+\infty} \sin s/s$, which converges, instead of the previous integral $\int_0^{+\infty} \cos s/s$ which diverges. This illustrate the importance of the cancellation assumption (7.38c)

Proof of lemma 7.7.

For any $0 < \varepsilon < R$ Denote $K_{\varepsilon,R} := K(x)\ \chi_{B_R(0)\setminus B_\varepsilon(0)}$. For a fixed ξ such that $\varepsilon < |\xi|^{-1} < R$, we write

$$\begin{aligned} \widehat{K_{\varepsilon,R}}(\xi) &= \int_{\varepsilon<|x|<R} e^{2\pi i\,x\cdot\xi}\ K(x)\,dx \\ &= \int_{\varepsilon<|x|<|\xi|^{-1}} e^{2\pi i\,x\cdot\xi}\ K(x)\,dx + \int_{|\xi|^{-1}<|x|<R} e^{2\pi i\,x\cdot\xi}\ K(x)\,dx \\ &= I_1 + I_2 \quad . \end{aligned}$$

We bound I_1 first. Using the cancellation assumption (7.38c), we have

$$I_1 = \int_{\varepsilon<|x|<|\xi|^{-1}} (e^{2\pi i\,x\cdot\xi} - 1)\ K(x)\,dx \quad .$$

Hence we deduce the following bound, using this time assumption (7.38a)

$$|I_1| \leq 2\pi\,|\xi| \int_{\varepsilon<|x|<|\xi|^{-1}} |x|\ |K(x)|\ dx \leq C_n\ A \quad .$$

In order to bound I_2 we introduce $z = \xi/2|\xi|^2$. Observe that the choice of z has been made in such a way that $\exp(2\pi\, iz\cdot\xi) = -1$, hence a change of variable $x \to x+z$ will generate a minus sign in front of the integral and formally we would have

$$\int_{\mathbb{R}^n} e^{2\pi i\,x\cdot\xi}\ K(x)\,dx = \frac{1}{2}\int_{\mathbb{R}^n} e^{2\pi i\,x\cdot\xi}\ K(x) - K(x-z)\,dx$$

which would put us in position to make use of the Hörmander condition (7.38b). The only difficulty is to keep track of the domains of integrations that we precise now.

$$2I_2 = \int_{|\xi|^{-1}<|x|<R} e^{2\pi i\,x\cdot\xi}\ K(x)\,dx - \int_{|\xi|^{-1}<|x-z|<R} e^{2\pi i\,x\cdot\xi}\ K(x-z)\ dx$$

We write

$$\begin{aligned}\int_{|\xi|^{-1}<|x-z|<R} e^{2\pi i\,x\cdot\xi}\ K(x-z)\ dx &= \int_{|\xi|^{-1}<|x|<R} \cdots\ dx\\ -\int_{|x-z|<|\xi|^{-1}<|x|} \cdots\ dx &- \int_{|x|<R<|x-z|} \cdots\ dx\\ +\int_{|x|<|\xi|^{-1}<|x-z|} \cdots\ dx &+ \int_{|x-z|<R<|x|} \cdots\ dx \quad .\end{aligned}$$

The following elementary inclusions are longer to state than to prove...

$$\{x\ ;\ |x-z| < |\xi|^{-1} < |x|\} \subset \{x\ ;\ |x-z| < |\xi|^{-1} < |x-z| + |z|\}$$

$$\{x\ ;\ |x| < R < |x-z|\} \subset \{x\ ;\ |x-z| - |z| < R < |x-z|\}$$

$$\{x\ ;\ |x| < |\xi|^{-1} < |x-z|\} \subset \{x\ ;\ |x-z| - |z| < |\xi|^{-1} < |x-z|\}$$

$$\{x\ ;\ |x-z| < R < |x|\} \subset \{x\ ;\ |x-z| < R < |x-z| + |z|\} \quad .$$

Using these inclusions and the fact that $|z| = 1/2|\xi|$, we can bound I_2 in the following way

$$\begin{aligned}2|I_2| \leq &\int_{|\xi|^{-1}<|x|<R} |K(x) - K(x-z)|\,dx\\ &+\int_{\frac{1}{2}|\xi|^{-1}<|x|<\frac{3}{2}|\xi|^{-1}} |K(x)|\ dx + \int_{R-\frac{1}{2}|\xi|^{-1}<|x|<R+\frac{1}{2}|\xi|^{-1}} |K(x)|\ dx\end{aligned} \tag{7.41}$$

Since $|z| = \frac{1}{2}|\xi|^{-1}$ we can invoke the Hörmander condition (7.38b) and bound the first integral in the right-hand-side of (7.41) by B. For the second integral we use (7.38a) and bound it by a constant $C_n\ A$ and the third integral is treated in the same way using the fact that $|\xi|^{-1} < R$ which implies that the quotient of $R + \frac{1}{2}|\xi|^{-1}$ by $R - \frac{1}{2}|\xi|^{-1}$ is bounded by 3. Hence I_2 is bounded by $B + 4C_n\ A$. So we have proved that $|\widehat{K_{\varepsilon,R}}(\xi)|$ is uniformly bounded by a constant depending only on n, A and B, which is the desired result. $\square$

Proof of theorem 7.6.

Combining lemma 7.7 and theorem 7.5 we obtain (7.34) and (7.37) where Tf is replaced by $T_\varepsilon f$. It remains to show the L^p convergence (7.35), the L^1_w convergence (7.36) and inequality (7.37) for Tf itself.

We consider first a smooth function $f \in C^\infty_0(\mathbb{R}^n)$ and using the cancellation property (7.32c) we write

$$\begin{aligned} T_\varepsilon f(x) &= \int_{1\leq\|y\|} f(x-y)K(y)\,dy + \int_{\varepsilon\leq\|y\|\leq 1} f(x-y)K(y)\,dy \\ &= \int_{\mathbb{R}^n} f(x-y)K_1(y)\,dy + \int_{\varepsilon\leq\|y\|\leq 1} \big(f(x-y)-f(x)\big)K(y)\,dy\,. \end{aligned} \tag{7.42}$$

Because of the regularity of f, using assumption (7.32a), we have the following bound which holds for every x in $\mathbb{R}^n$ and $y \neq 0$

$$\big|\big(f(x-y)-f(x)\big)K(y)\big| \leq \|\nabla f\|_\infty \|y\|\,|K(y)| \overset{(7.32a)}{\leq} \|\nabla f\|_\infty \frac{A}{\|y\|^{n-1}} \quad . \tag{7.43}$$

Hence, inserting the bound (7.43) in (7.42) we can define for every x the limit

$$Tf(x) := \lim_{\varepsilon\to 0} T_\varepsilon f(x) = \int_{\mathbb{R}^n} f(x-y)K(y)\,dy \quad . \tag{7.44}$$

Observe that at this stage Tf is a distribution obtained by the convolution between a smooth compactly supported function and the principal value of K, $p.v.K$, which is an order 1 distribution. However using (7.43) again we have

$$\begin{aligned} \forall x \in \mathbb{R}^n \qquad |Tf(x) - T_\varepsilon f(x)| &\leq \int_{B_\varepsilon(0)} |f(x-y) - f(x)|\;|K(y)|\;dy \\ &\leq C_n \|\nabla f\|_\infty\; A\;\varepsilon \quad . \end{aligned} \tag{7.45}$$

Thus $T_\varepsilon f$ converges uniformly to Tf and hence in $L^p_{loc}(\mathbb{R}^n)$ for any $p \geq 1$. Let $R > 1$ such that $f \equiv 0$ in $\mathbb{R}^n \setminus B_R(0)$. For $|x| > 4R$

$$K(y)[f(x-y) - f(x)] = K(y)\; f(x-y)$$

is supported in $B_R(x)$ and one has $|K(y)\; f(x-y)| \leq 2^n \|f\|_\infty A/|x|^n$. Hence the bound (7.45) can be completed by a behavior at infinity as follows :

$$\begin{aligned} \forall x \in \mathbb{R}^n \qquad |Tf(x) - T_\varepsilon f(x)| &\leq \int_{B_\varepsilon(0)} |f(x-y)|\;|K(y)|\;dy \\ &\leq C_n\; A\; \frac{\varepsilon^n}{|x|^n} \|f\|_\infty \quad . \end{aligned} \tag{7.46}$$

This later inequality implies that $T_\varepsilon f \to Tf$ in $L^p(\mathbb{R}^n)$ for any $p > 1$ and that $|T_\varepsilon f - Tf|_{L^1_w}$ converges to zero.

Let us take now $f \in L^p(\mathbb{R}^n)$ for $p \geq 1$. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, using inequalities (7.16) and (7.17) for $K_\varepsilon \star g$ - where g is a difference between f and a finer and finer approximation of it in C_0^∞ for the L^p norm - a classical diagonal argument implies that, for $p > 1$, $T_\varepsilon f$ converges strongly in L^p and that, for $p = 1$, T_ε is Cauchy for the quasi-norm L^1_w. This concludes the proof of theorem 7.6. $\square$

Remark 7.4. *The exact cancellation assumption (7.38c) can be relaxed in the statement of theorem 7.6 by requiring only the existence of a constant $C > 0$ such that for any $0 < r < R < +\infty$*

$$\left| \int_{B_R(0)\setminus B_r(0)} K(x)\, dx \right| \leq C \quad . \tag{7.47}$$

*Under this weakened assumption however the convergence of $T_\varepsilon f$ to Tf does not necessarily hold in L^p or even almost everywhere but in the distributional sense only (see a counterexample in [**?**]). The nature of this convergence nevertheless is not a main point in the theory the most important one being given by the inequalities (7.34) and (7.37) which still hold under the weakest assumption (7.47).*

7.3.3 The L^p theory for Calderón-Zygmund convolution operators: the case of homogeneous kernels

It is interesting to look at the case of homogeneous kernels which correspond to operators of special geometric interest - such as Hilbert Transform for instance. The following result is obtained as a corollary of theorem 7.6 and has the advantage to provide a "translation", in the special case of homogeneous Kernels, of general assumptions on K that imply (7.32a), (7.32b) and (7.32c). Precisely we consider kernels K of the form

$$K(x) = \frac{\Omega(x)}{\|x\|^n} \quad , \tag{7.48}$$

where Ω is an homogeneous function of degree 0, i.e., $\Omega(\delta x) = \Omega(x)$, for $\delta > 0$. In other words, the function Ω is radially constant and therefore completely determined by its values on the sphere S^{n-1}. Note also that K is homogeneous of degree $-n$, i.e., $K(\delta x) = \delta^{-n} K(x)$.

Proposition 7.9. *Let $K : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a measurable function given by $K(x) = \Omega(x)/\|x\|^n$ where Ω is an homogeneous function of degree 0 satisfying*

i)

$$\int_{S^{n-1}} \Omega(x)\, d\sigma(x) = 0 \quad . \tag{7.49}$$

ii) If we set

$$\omega(\delta) = \sup_{\substack{\|x-y\|\leq\delta \\ x,y\in S^{n-1}}} \left|\Omega(x) - \Omega(y)\right| \quad ,$$

the following integral is finite:

$$\int_0^1 \frac{\omega(\delta)}{\delta}\, d\delta < \infty\,. \tag{7.50}$$

Then K satisfies the conditions (7.32a)–(7.32c) and theorem 7.6 can be applied to K.

Remark 7.5. *Observe that the so called Dini condition ii) implies that Ω is continuous on S^{n-1}. Moreover observe that if Ω is assumed to be Hölder continuous, $C^{0,\alpha}(S^{n-1})$, for some exponent $1 > \alpha > 0$ then the Dini condition ii) is automatically satisfied.*

Proof of proposition 7.9.

The conditions (7.32a), respectively (7.32c), follow directly from (7.50), respectively (7.49) and integration in polar coordinates. In order to establish (7.32b), we first observe that

$$\begin{aligned} \int_{2\|y\|\le\|x\|} \big|K(x-y)-K(x)\big|\,dx \;&\le\; \int\limits_{2\|y\|\le\|x\|} \frac{\big|\Omega(x-y)-\Omega(x)\big|}{\|x-y\|^n}\,dx \\ &+ \int\limits_{2\|y\|\le\|x\|} |\Omega(x)| \left|\frac{1}{\|x-y\|^n}-\frac{1}{\|x\|^n}\right| dx\,. \end{aligned} \tag{7.51}$$

Since Ω is bounded due to (7.50) and as a consequence of the mean value theorem

$$\left|\frac{1}{\|x-y\|^n}-\frac{1}{\|x\|^n}\right| \le \frac{C\|y\|}{\|x\|^{n+1}}\,,$$

we conclude by integration in polar coordinates that the second integral on the right-hand side of (7.51) is finite. Note also that

$$\begin{aligned} \big|\Omega(x-y)-\Omega(x)\big| \;&=\; \left|\Omega\left(\frac{x-y}{\|x-y\|}\right)-\Omega\left(\frac{x}{\|x\|}\right)\right| \\ &\le\; \omega\left(\left\|\frac{x-y}{\|x-y\|}-\frac{x}{\|x\|}\right\|\right) \end{aligned}$$

by definition of the function ω. Moreover, if $2\|y\|\le\|x\|$, then $1/\|x-y\|^n \le C/\|x\|^n$ and also

$$\left\|\frac{x-y}{\|x-y\|}-\frac{x}{\|x\|}\right\| \le C\,\frac{\|y\|}{\|x\|}\quad .$$

Inserting these estimates in the first integral on the right-hand side of (7.51), we obtain

$$\begin{aligned} \int\limits_{2\|y\|\le\|x\|} \frac{\big|\Omega(x-y)-\Omega(x)\big|}{\|x-y\|^n}\,dx \;&\le\; C \int\limits_{2\|y\|\le\|x\|} \frac{\omega\left(C\frac{\|y\|}{\|x\|}\right)}{\|x\|^n}\,dx \\ &\le\; C\int_{2\|y\|}^{\infty} \frac{\omega\left(C\frac{\|y\|}{r}\right)}{r}\,dr \quad . \end{aligned}$$

Changing coordinates $\delta = C\|y\|/r$ and using (7.50), we deduce that the last integral is finite showing that (7.32b) holds and proposition 7.9 is proved. □

7.3.4 A multiplier type formulation

It is useful to explicit sufficient conditions on $\widehat{K}$ only that implies the strong type (p,p) (for $1 < p < +\infty$) and the weak type $(1,1)$ properties of the corresponding convolution operator T. Such results are called *multiplier theorems* - $m(\xi) := \widehat{K}(\xi)$ is the *multiplier* associated to T. We shall give more and more sophisticated multiplier theorem in this book that will play a crucial role in characterizing real-variable function spaces using the Fourier transform. Multiplier theorems are moreover the basic tools in the analysis of pseudo-differential operators. Here is maybe the most elementary one that we will deduce from the previous sections.

Theorem 7.10. *Let m be a C^∞ function on $\mathbb{R}^n$ satisfying :*

$$\begin{aligned} &\forall\, l \in \mathbb{N} \quad \exists\, C_l > 0 \quad s.\ t. \quad \forall \xi \in \mathbb{R}^n \\ &|\nabla^l m|(\xi) \le C_l\ |\xi|^{-l} \quad . \end{aligned} \tag{7.52}$$

Let $p \in [1, +\infty)$. Define T_m on $L^p \cap L^2$ by

$$\forall f \in L^p \cap L^2(\mathbb{R}^n) \qquad \forall \xi \in \mathbb{R}^n \qquad \widehat{T_m f}(\xi) := m(\xi)\ \widehat{f}(\xi) \quad .$$

Then for $p \in (1, +\infty)$ there exists $C_{p,m} > 0$ such that for any $f \in L^p \cap L^2$

$$\|T_m f\|_{L^p} \le C_{p,m}\, \|f\|_{L^p}\,, \tag{7.53}$$

and there exists $C_{1,m} > 0$ such that for any $f \in L^1 \cap L^2$

$$\sup_{\alpha > 0} \alpha\ \mu(\{x \in \mathbb{R}^n\ ;\ |T_m f(x)| > \alpha\}) \le C_{1,m}\ \|f\|_{L^1} \quad . \tag{7.54}$$

Hence T_m extends continuously as a linear operator of strong type (p,p) - $1 < p < +\infty$ - and weak type $(1,1)$.

Remark 7.6. *It is important to compare at this stage already, before to proceed to the proof of theorem 7.10 itself, the difference between the assumption (7.52) and the assumptions we made on K in the previous subsections. Take for instance the condition $|\nabla K|(x) \le C/|x|^{n+1}$ that implies the Hörmander condition (7.14) - as it is established in remark 7.2 c) - would hold if, for instance, we would assume $\nabla^{n+1} m$ to be in L^1. Observe that this later condition is just "at the border" to be implied, but is not implied, by our assumption (7.52). As it will be seen later in the book, assumption (7.52) is however very relevant to the theory.*

Proof of theorem 7.10. Theorem 7.10 will a direct consequence of theorem 7.5 once we will have proved that assumption (7.52) implies the Hörmander condition (7.14) for $K := \widehat{m}$ - Observe that (7.52) contains (7.13) already.

In order to establish the Hörmander condition we cannot afford to be as little cautious as we were in establishing the bound (7.18). We shall use a more refined argument based on dyadic decomposition in the Fourier variable ξ - the phase space. This techniques is making use of the Littlewood-Paley decomposition presented in chapter ??. Precisely let $\psi \in C_0^\infty(B_2(0))$ be a smooth non negative function with compact support in the ball $B_2(0)$ such that ψ equals identically 1 on $B_1(0)$ and let $\phi(\xi) := \psi(\xi) - \psi(2\xi)$. It follows from this definition that $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and that

$$1 \equiv \sum_{k\in\mathbb{Z}} \phi(2^{-k}\xi) \qquad \text{on } \mathbb{R}^n \quad .$$

For $k \in \mathbb{Z}$ we denote

$$m_k(\xi) := \phi(\xi)\ m(2^{-k}\xi) \quad .$$

Observe that with this notation

$$m(\xi) = \sum_{k\in\mathbb{Z}} m_k(2^k\xi) \quad .$$

Denoting $K_k(x) := \widehat{m_k}(x)$, we have :

$$K(x) := \widehat{m}(\xi) = \sum_{k\in\mathbb{Z}} 2^{-k\,n} K_k(2^{-k}x) \quad . \tag{7.55}$$

Using now the assumption (7.52) on m and the definition of m_k, it is not difficult to see that

$$\forall l \in \mathbb{N} \qquad \exists\, C_l > 0 \quad \text{s.t.} \quad \forall k \in \mathbb{Z} \qquad \|\nabla^l m_k\|_{L^\infty(\mathbb{R}^n)} \le C_l \quad . \tag{7.56}$$

Moreover, since the m_k are supported in the fixed compact set $B_2(0) \setminus B_{1/2}(0)$, we deduce that every H^s norm of m_k is bounded independently of k.

Take $s > n/2$, we then have the existence of C, independent of k such that

$$\int_{\mathbb{R}^n} (1+|x|^2)^{s/2} |K_k(x)|^2\ dx = \|m_k\|_{H^s}^2 \le C \quad .$$

Hence, using Cauchy-Schwarz, we deduce the following bound

$$\begin{aligned}
&\int_{|x|>|y|} |K_k(x)|\ dx\\
&\le \left[\int_{|x|>|y|} \frac{1}{(1+|x|^2)^{s/2}}\ dx\right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^n} (1+|x|^2)^{s/2} |K_k(x)|^2\ dx\right]^{\frac{1}{2}}\\
&\le \frac{C}{(1+|y|)^{-n/2+s/4}}
\end{aligned} \tag{7.57}$$

where C is possibly a new constant but again independent of k.

Similarly as before, $\xi\, m_k(\xi)$ is a function supported in the fixed compact set $B_2(0) \setminus B_{1/2}(0)$ and, hence, (7.56) implies that

$$\forall\, l \in \mathbb{N} \qquad \exists\, C_l > 0 \quad \text{s.t.} \quad \forall k \in \mathbb{Z} \qquad \|\nabla^l(\xi\, m_k(\xi))\|_{L^\infty(\mathbb{R}^n)} \le C_l \quad . \tag{7.58}$$

Hence for the same reasons as above we obtain a uniform bound, independent of k, for ∇K_k. Precisely there exists $C > 0$ such that for every $k \in \mathbb{Z}$

$$\int_{\mathbb{R}^n} |\nabla K_k(x)|\ dx \le C < +\infty \quad . \tag{7.59}$$

Let now $y \in \mathbb{R}^n$ and denote $v = y/|y|$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} \big|K_k(x-y) - K_k(x)\big|\, dx &= \int_{\mathbb{R}^n} \left| \int_0^{|y|} \frac{\partial K_k}{\partial v}(x + t\,v)\; dt \right| \; dx \\
&\le |y| \int_{\mathbb{R}^n} |\nabla K_k|(z)\; dz \le C\; |y| \quad .
\end{aligned}
\tag{7.60}
$$

Consider again $y \in \mathbb{R}^n \setminus \{0\}$ and let k_0 be the largest integer less than $\log_2 |y|$: $k_0 = [\log_2 |y|]$. Using (7.57), we obtain

$$
\begin{aligned}
&\int_{|x|>2|y|} \left| \sum_{k \le k_0} 2^{-n\,k} \left[K_k(2^{-k}(x+y) - K_k(2^{-k}x)\right] \right| \; dx \\
&\quad \le 2 \sum_{k \le k_0} \int_{|z|>2^{-k}|y|} |K_k(z)|\; dz \le \sum_{k \le k_0} \frac{C}{(1 + 2^{-k}|y|)^{\alpha}}
\end{aligned}
$$

where $\alpha = -n/2 + s/4 > 0$. Hence, we have in one hand

$$
\begin{aligned}
\int_{|x|>2|y|} \left| \sum_{k \le k_0} 2^{-n\,k} \left[K_k(2^{-k}(x+y) - K_k(2^{-k}x)\right] \right| &\le C \sum_{k \le k_0} 2^{\alpha(k-k_0)} \\
&\le \frac{C}{1 - 2^{\alpha}} \quad .
\end{aligned}
\tag{7.61}
$$

In the other hand, using (7.60), we have

$$
\begin{aligned}
&\int_{|x|>2|y|} \left| \sum_{k \ge k_0} 2^{-n\,k} \left[K_k(2^{-k}(x+y) - K_k(2^{-k}x)\right] \right| \\
&\le \sum_{k \ge k_0} \int_{|z|>2^{-k}|y|} \left|K_k(z + 2^{-k}y) - K_k(z)\right| \; dz \\
&\le C \sum_{k \ge k_0} 2^{-k}|y| \le 2C \sum_{k \ge k_0} 2^{k_0 - k} \le 4C
\end{aligned}
\tag{7.62}
$$

Combining (**??**), (7.61) and (7.62) gives

$$
\int_{|x|>2|y|} |K(x+y) - K(x)|\; dx \le B < +\infty \quad ,
\tag{7.63}
$$

where B is independent of $y \in \mathbb{R}^n \setminus \{0\}$. This is the Hörmander condition (7.14) and theorem 7.10 is proved. □

7.3.5 Applications: The L^p theory of the Riesz Transform and the Laplace and Bessel Operators

In this subsection we apply to the Riesz Transform and the Laplace Operator the L^p continuity of the convolution type Calderón-Zygmund operators that we proved above.

For $j = 1, \ldots, n$, we now consider the kernels $K_j(x) = \Omega_j(x)/\|x\|^n$ with

$$\Omega_j(x) = c_n \frac{x_j}{\|x\|}\,, \tag{7.64}$$

where

$$c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}} \quad .$$

Observe first that $\Omega_j = c_n x_j$ is smooth on S^{n-1} and moreover, since Ω_j is an odd function the cancellation property

$$\int_{S^{n-1}} \Omega_j(x)\, d\sigma(x) = 0$$

also holds. Hence proposition 7.9 can be applied to the kernels K_j. For any $1 \le p < \infty$, any $j = 1 \cdots n$ and any $f \in L^p(\mathbb{R}^n)$ the following limit exists (in L^p or L^1_w when $p = 1$)

$$R_j f(x) = \lim_{\varepsilon \to 0} R_{j,\varepsilon} f(x) \quad , \tag{7.65}$$

where

$$\begin{aligned} R_{j,\varepsilon} f(x) &= \int_{\varepsilon \le \|y\|} f(x-y) K_j(y)\, dy \\ &= c_n \int_{\varepsilon \le \|y\|} f(x-y) \frac{y_j}{\|y\|^{n+1}}\, dy\,. \end{aligned}$$

Definition 7.11. Riesz Transform *For any function $f \in L^p(\mathbb{R}^n)$, $1 \le p < +\infty$, the $\mathbb{R}^n$ valued measurable map given almost everywhere by*

$$Rf(x) := (R_1 f(x), \cdots, R_n f(x)) \quad ,$$

is called the Riesz transform of f.

Theorem 7.6 implies the following proposition

Proposition 7.12. *For any $1 < p < +\infty$ and any $f \in L^p(\mathbb{R}^n)$*

$$\|Rf\|_{L^p} \le C_{n,p} \|f\|_{L^p} \quad . \tag{7.66}$$

Moreover, for any $f \in L^1(\mathbb{R}^n)$

$$\sup_{\alpha > 0} \alpha\ \mu(\{x \in \mathbb{R}^n\ ;\ |Rf(x)| > \alpha\}) \le C_n \|f\|_{L^1} \quad . \tag{7.67}$$

We now derive the multiplier $m(\xi) = (m_1(\xi), \cdots, m_n(\xi))$ corresponding to the Riesz transform. Precisely we establish the following result.

Proposition 7.13. *The following holds*

$$\widehat{R_j f}(\xi) = \frac{i\,\xi_j}{|\xi|}\,\widehat{f}(\xi) = m_j(\xi)\,\widehat{f}(\xi) \tag{7.68}$$

i.e. the multiplier corresponding to R_j is

$$m_j(\xi) = i\frac{\xi_j}{|\xi|} \quad .$$

Remark 7.7. *Observe that the multipliers $m_j(\xi)$ of the components R_j of the Riesz transform R satisfy the main assumption (7.52) of theorem 7.10. Hence combining the previous proposition together with the theorem 7.10 provides a new proof of proposition 7.12.*

Proof of proposition 7.13. For a C_0^∞ function f we have that

$$K_j \star f = c_n\ PV\left(\frac{x_j}{|x|^{n+1}}\right) \star f = -\frac{c_n}{n-1}\frac{\partial}{\partial x_j}|x|^{-n+1} \star f \quad .$$

Hence

$$m_j(\xi) = 2i\pi\frac{c_n}{n-1}\ \xi_j\ \widehat{|x|^{-n+1}} \quad . \tag{7.69}$$

In order to identify m_j it remains to compute the Fourier transform of $|x|^{-n+1}$. Denoting $d\sigma^{n-1}$ the canonical volume form on the $n-1$ sphere, one has for $\xi \neq 0$:

$$\begin{aligned}\widehat{|x|^{-n+1}}(\xi) &= \lim_{\delta\to 0}\frac{\widehat{e^{-\pi\,\delta\,|x|^2}}}{|x|^{n-1}}(\xi)\\ &= \int_0^{+\infty}\int_{S^{n-1}} e^{-\pi\,\delta\,\rho^2}\ e^{2\pi\,i\ \rho\ \zeta\cdot\xi}\ d\sigma^{n-1}(\zeta)\ d\rho \quad .\end{aligned}$$

Denote $S^{n-1}_\xi := \{\zeta \in S^{n-1}\ ;\ \zeta\cdot\xi \geq 0\}$. Using this notation, the previous identity becomes

$$\begin{aligned}\widehat{|x|^{-n+1}}(\xi) &= \int_0^{+\infty}\int_{S^{n-1}_\xi} e^{-\pi\,\delta\,\rho^2}\ e^{2\pi\,i\,\rho\,\zeta\cdot\xi}\ d\sigma^{n-1}(\zeta)\ d\rho\\ &\quad + \int_0^{+\infty}\int_{S^{n-1}\setminus S^{n-1}_\xi} e^{-\pi\,\delta\,\rho^2}\ e^{2\pi\,i\ \rho\ \zeta\cdot\xi}\ d\sigma^{n-1}(\zeta)\ d\rho\\ &= \int_0^{+\infty}\int_{S^{n-1}_\xi} e^{-\pi\,\delta\,\rho^2}\ \left[e^{2\pi\,i\ \rho\ \zeta\cdot\xi} - e^{-2\pi\,i\ \rho\ \zeta\cdot\xi}\right]\ d\sigma^{n-1}(\zeta)\ d\rho\\ &= \int_{S^{n-1}_\xi} d\sigma^{n-1}(\zeta)\int_{\mathbb{R}} e^{-\pi\,\delta\,\rho^2}\ e^{2\pi\,i\,\rho\,|\xi|\,\alpha}\ d\rho \quad ,\end{aligned}$$

where $\alpha := \zeta\cdot\xi/|\xi|$. Using the fact that the Fourier transform of $e^{-\pi\,\delta\,t^2}$ is equal at the point τ to $\delta^{-1/2}\ e^{-\pi\,(\tau/\sqrt{\delta})^2}$, we obtain

$$\widehat{|x|^{-n+1}}(\xi) = \int_{S^{n-1}_\xi}\frac{1}{\sqrt{\delta}}\ e^{-\pi\ \left(\frac{|\xi|\,\alpha}{\sqrt{\delta}}\right)^2}\ d\sigma^{n-1}(\zeta) \quad . \tag{7.70}$$

We interpret $\alpha = z_1$ as being the first coordinate of a positive orthonormal basis containing the unit vector $\xi/|\xi|$ as first vector. We have

$$d\sigma^{n-1} = \sum_{i=1}^n (-1)^{i-1}\ z_i\ dz_1\cdots dz_{i-1}\wedge dz_{i+1}\cdots dz_n \quad .$$

We decompose $d\sigma^{n-1}$ is the following way : $d\sigma^{n-1} = dz_1 \wedge d\sigma^{n-2} + z_1\, dz_2 \cdots dz_n$

$$
\begin{aligned}
\int_{S_\xi^{n-1}} \frac{1}{\sqrt{\delta}}\; e^{-\pi\,\left(\frac{|\xi|\,\alpha}{\sqrt{\delta}}\right)^2}\; d\sigma^{n-1}(\zeta) &= \int_{S^{n-2}} d\sigma^{n-2} \int_0^1 \frac{1}{\sqrt{\delta}} e^{-\pi\,\left(\frac{|\xi|\,z_1}{\sqrt{\delta}}\right)^2}\; dz_1 \\
&+ \int_{S_\xi^{n-1}} \frac{z_1}{\sqrt{\delta}}\; e^{-\pi\,\left(\frac{|\xi|\,z_1}{\sqrt{\delta}}\right)^2}\; dz_2 \cdots dz_n \quad .
\end{aligned}
\tag{7.71}
$$

Since , as δ goes to zero, $\frac{z_1}{\sqrt{\delta}}\; e^{-\pi\,\left(\frac{|\xi|\,z_1}{\sqrt{\delta}}\right)^2}$ is converging to zero uniformly on any compact subset of $S_\xi^{n-1} \setminus \{\xi/|\xi|\}$, we obtain that

$$
\int_{S_\xi^{n-1}} \frac{1}{\sqrt{\delta}}\; e^{-\pi\,\left(\frac{|\xi|\,\alpha}{\sqrt{\delta}}\right)^2}\; d\sigma^{n-1}(\zeta) = |S^{n-2}|\; |\xi|^{-1} \int_0^{\frac{|\xi|}{\sqrt{\delta}}} e^{-\pi t^2}\; dt + o_\delta(1) \quad ,
\tag{7.72}
$$

where $|S^{n-2}|$ denotes the volume of the $n-2$ unit sphere which is equal to $2\,\pi^{(n-1)/2}/\Gamma((n-1)/2)$ - Γ is the Euler Gamma Function. Recall that

$$
\int_0^{+\infty} e^{-\pi t^2}\; dt = \frac{1}{2} \quad .
$$

Hence, combining (7.69) and (7.72) we obtain that

$$
m_j(\xi) = 2i\, \frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \frac{c_n}{n-1}\, \frac{\xi_j}{|\xi|} = \frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}\; c_n \frac{\xi_j}{|\xi|} = i\, \frac{\xi_j}{|\xi|} \quad .
$$

where we have used that $\Gamma(z+1) = z\,\Gamma(z)$. We have proved proposition 7.13. □

Let $f \in C_0^2(\mathbb{R}^n)$ and note that the Fourier transform of its second order partial derivatives are given by

$$
\widehat{\partial_k \partial_j f}(\xi) = (i\,\xi_k)(i\,\xi_j)\, \widehat{f}(\xi) = -\xi_k \xi_j\, \widehat{f}(\xi)\,.
$$

In particular, we have for the Fourier transform of the Laplace operator $\widehat{\Delta f}(\xi) = -\|\xi\|^2\, \widehat{f}(\xi)$. This enables us to write the following:

$$
\begin{aligned}
\widehat{\partial_k \partial_j f}(\xi) \quad &= \quad -\xi_k \xi_j\, \widehat{f}(\xi) = \frac{i\,\xi_k}{\|\xi\|} \frac{i\,\xi_j}{\|\xi\|}\, \widehat{\Delta f}(\xi) \\
&\overset{(7.68)}{=} \quad \frac{i\,\xi_k}{\|\xi\|} \widehat{R_j(\Delta f)}(\xi) \overset{(7.68)}{=} \big(R_i(\widehat{R_j(\Delta f)})\big)(\xi)\,.
\end{aligned}
$$

Thus, we get

$$
\partial_i \partial_j f = R_i\big(R_j(\Delta f)\big)\,.
\tag{7.73}
$$

From (7.66), it then follows for $1 < p < +\infty$ that

$$
\begin{aligned}
\|\partial_i \partial_j f\|_{L^p} \quad &= \quad \big\|R_i\big(R_j(\Delta f)\big)\big\|_{L^p} \\
&\leq \quad C_{n,p}\, \|R_j(\Delta f)\|_{L^p} \leq C_{n,p}^2\, \|\Delta f\|_{L^p}\,,
\end{aligned}
$$

Using the density of $C_0^\infty(\mathbb{R}^n)$ in the Sobolev space $W^{2,p}(\mathbb{R}^n)$, we have proved the following result.

Proposition 7.14. *Let $1 < p < +\infty$. There exists a positive constant $C_p > 0$ such that, for any function f in the Sobolev Space $W^{2,p}(\mathbb{R}^n)$ the following identity holds*

$$\|\nabla^2 f\|_{L^p(\mathbb{R}^n)} \leq C_p \, \|\Delta f\|_{L^p(\mathbb{R}^n)} \quad .$$

where $\nabla^2 f$ denotes the Hessian matrix of f.

The previous result can be improved when the operator Δ is made inhomogeneous and more coercive by adding $-id$ to it. Precisely, the following result which says that the inverse of the *Bessel Operator*, given by $(\Delta - id)^{-1}$, is continuous from $L^p(\mathbb{R}^n)$ into $W^{2,p}(\mathbb{R}^n)$ is a direct application of theorem 7.10.

Proposition 7.15. *Let $1 < p < +\infty$. Let f be an L^p function on $\mathbb{R}^n$. Then there exists a unique tempered Distribution u in $\mathcal{S}'(\mathbb{R}^n)$ such that $(\Delta - id)u = f$ moreover u belongs to the Sobolev Space $W^{2,p}(\mathbb{R}^n)$ and the following inequality holds*

$$\|u\|_{W^{2,p}} \leq C_p \, \|f\|_{L^p(\mathbb{R}^n)} \quad .$$

Proof of Proposition 7.15. A tempered Distribution f being given and $\widehat{f}$ being its Fourier transform, $-(1+|\xi|^2)^{-1}\widehat{f}(\xi)$ is the Fourier transform of the only tempered Distribution solution to

$$\Delta u - u = f \qquad \text{in } \mathcal{S}'(\mathbb{R}^n) \quad .$$

It is straightforward to check that the multipliers $-(1+|\xi|^2)^{-1}$, $-i\xi_j\,(1+|\xi|^2)^{-1}$ and $\xi_k\,\xi_j\;(1+|\xi|^2)^{-1}$ satisfy the assumption (7.52) of theorem 7.10 and hence proposition 7.15 follows. □

7.3.6 The cases $p = 1$ and $p = +\infty$

As for the sub-linear maximal operator, Calderón-Zygmund convolution operators are usually not bounded from L^1 into L^1. The following proposition illustrates this fact.

Proposition 7.16. *Let R be the Riesz Transform and let $f \in L^1(\mathbb{R}^n)$ such that $f \geq 0$ on $\mathbb{R}^n$ and $f \not\equiv 0$ then the measurable function Rf is not in $L^1(\mathbb{R}^n)$.*

Proof of proposition 7.16. Since f is in $L^1(\mathbb{R}^n)$, $\widehat{f}$ is a continuous function and moreover $\widehat{f}(0) = \int_{\mathbb{R}^n} f(x)\,dx > 0$.

$m_j(\xi) = \xi_j/|\xi|$ is discontinuous at the origin and hence, since $\widehat{f}$ is continuous at the origin and since $\widehat{f}(0) \neq 0$, $m_j(\xi)\,\widehat{f}(\xi)$ is also discontinuous at the origin.

Assuming $Rf \in L^1(\mathbb{R}^n)$ this implies that $\widehat{Rf}$ is continuous too on $\mathbb{R}^n$ and in particular at 0, which contradicts the previous assertion. □

Lemma 7.17. *There exists $f \in L^1(\mathbb{R}^2)$ such that, for any $u \in \mathcal{S}'(\mathbb{R}^2)$ satisfying*

$$\Delta u = f \ \text{ in } \ \mathcal{S}'(\mathbb{R}^2), \tag{7.74}$$

then $\nabla^2 \notin L'_{\text{Loc}}(\mathbb{R}^2)$.

Proof of Lemma 7.17. We choose

$$f(x) := \frac{\mathbf{1}_{D^2_{1/2}(x)}}{|x|^2\,\mathrm{Log}^2|x|}\ ,$$

where $\mathbf{1}_{D^2_{1/2}}(x)$ is the characteristic function of the disc of radius $1/2$ and centered at the origin. One easily verifies that $f \in L^1(\mathbb{R}^2)$. We are now looking for an axially symmetric solution of (7.74) in $\mathcal{S}'(\mathbb{R}^2)$. That is, we look for $u(x) = v(|x|)$ and we use the conventional notation $r = |x|$. V should then satisfy

$$\ddot{V} + \frac{\dot{V}}{r} = \frac{\mathbf{1}_{[0,1/2](r)}}{r^2\,\mathrm{Log}^2_{1/2}} \ \text{ in } \ \mathbb{R}^*_+$$

or, in other words,

$$\frac{d}{dr}(r\,\dot{V}) = \frac{\mathbf{1}_{[0,1/2](r)}}{r\,\mathrm{Log}^2\,r}\ .$$

For this to be satisfied, it suffices

$$\dot{V}(r) = \begin{cases} \dfrac{1}{r\,\mathrm{Log}\,r^{-1}} & \text{for } \ r \in \left(0, \frac{1}{2}\right] \\[2ex] \dfrac{1}{r\,\mathrm{Log}\,r^2} & \text{for } \ r > \dfrac{1}{2} \end{cases}\ .$$

This holds in particular if

$$V(r) = \begin{cases} +\,\mathrm{Log}\left[\dfrac{1}{\mathrm{Log}\,r^{-1}}\right] \text{ for } \ r \in \left(0, \frac{1}{2}\right] \\[2ex] 1 + \dfrac{\mathrm{Log}\,r}{\mathrm{Log}\,2} - \mathrm{Log}\,\mathrm{Log}\,2 \text{ for } \ r > \dfrac{1}{2} \end{cases}\ .$$

Observe that $u(x) := v(|x|) \in \mathcal{S}'(\mathbb{R}^2)$. By construction, we have

$$\Delta u(x) = f(x) \ \text{ in } \ \mathcal{S}'(\mathbb{R}^2\backslash\{0\})\ .$$

Let $\chi(x)$ be a cut-off function in $C^\infty_c B_1(0))$ with $\chi \equiv 1$ on $B_{1/2}(0)$. Denote $\chi_\varepsilon(x) = \chi(\frac{x}{\varepsilon})$. For any $\varphi \in \mathcal{S}(\mathbb{R}^2)$ one has

$$\int \varphi[\Delta u - f(x)]\,dx + \int \chi_\varepsilon \varphi[\Delta u - f(x)]\,dx.$$

Since $f \in L^1(\mathbb{R}^2)$

$$\lim_{\varepsilon\to 0} \int \chi_\varepsilon\varphi\ f(x)\,dx = 0. \tag{7.75}$$

We write

$$\begin{aligned} \int_{\mathbb{R}^2} \chi_\varepsilon\varphi(x)\,\Delta u(x)\,dx = &-\int_{\mathbb{R}^2} \nabla\chi_\varepsilon\ \nabla u\ \varphi(x)\,dx \\ &+ \int_{\mathbb{R}^2} \chi_\varepsilon\ \nabla_u \cdot \nabla\varphi(x)\,dx. \end{aligned} \tag{7.76}$$

Oberseve that for $|x| < \frac{1}{2}$

$$\nabla u = \dot{V}(r)\,\frac{\partial}{\partial r} = \frac{1}{r\,\mathrm{Log}\,r^{-1}}\,\frac{\partial}{\partial r}\,.$$

Since

$$\int_{B_\varepsilon(0)} |\nabla u|^2\,dx = 2\pi\int_0^\varepsilon \frac{dr}{r(\mathrm{Log}\,r^{-1})^2} = \frac{2\pi}{\mathrm{Log}\,\varepsilon^{-1}},$$

we have

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon} |\nabla u|^2\,dx = 0.$$

Hence this last fact implies

$$\lim_{\varepsilon\to 0}\left|\int_{\mathbb{R}^2}\chi_\varepsilon\,\nabla u\cdot\nabla\varphi(x)\,dx\right| \le \lim_{\varepsilon\to 0}\ \|\nabla\varphi\|_\infty\ \|\chi\|_\infty\,\varepsilon\left[\int_{B_\varepsilon(0)}|\nabla u|^2\,dx\right]^{\frac{1}{2}} = 0.$$

Moreover we have also

$$\lim_{\varepsilon\to 0}\left|\int_{\mathbb{R}^n}\nabla\chi_\varepsilon\cdot\nabla u\,\varphi\right| \le \lim_{\varepsilon\to 0}\ \left[\int_{B_\varepsilon(0)}|\nabla u|^2\,dx\right]^{\frac{1}{2}} = 0.$$

Hence we have proved

$$\Delta u = f \ \text{ in } \ \mathcal{S}'(\mathbb{R}^2).$$

A classical computation gives for $|x| < \frac{1}{2}$

$$\sum_{i,j=1}^{2}\frac{x_i\,x_j}{r^2}\ \frac{\partial^2 u}{\partial x_i,\,\partial x_j} = \frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2\,\mathrm{Log}\,r^{-1}} + \frac{1}{r^2(\mathrm{Log}\,r)^2}.$$

Hence

$$\int_{B_{1/2}}\left|\sum_{i,j=1}^{2}\frac{x_i\,x_j}{r^2}\ \frac{\partial^2 u}{\partial x_i\,\partial x_j}\right| dx = +\infty$$

and we cannot have that $\Delta^2 u \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$.

This being established, if we make a slightly stronger integrability assumption on the function f such as $f \in L^1\,\log L^1(\mathbb{R}^n)$, then, in the similar way to the case of the maximal sub-linear operator, Tf is in L^1_{loc}.

Theorem 7.18. *Let T be a convolution operator satisfying the assumptions of either theorem 7.5, theorem 7.6, theorem 7.10 or proposition 7.9. Let f be a measurable function in $L^1\log L^1(\mathbb{R}^n)$, then $Tf \in L^1_{loc}(\mathbb{R}^n)$ and for any measurable subset A of finite Lebesgue measure the following inequality holds*

$$\int_A |Tf(y)|\ dy \le C_T\ \int_{\mathbb{R}^n}|f(y)|\ \log\left(e + \mu(A)\ \frac{|f(y)|}{\|f\|_{L^1}}\right)\ dy\,, \tag{7.77}$$

where $C_T > 0$ only depends on T.

Proof of theorem 7.18. We use the notations from the proof of theorem 7.5. For any positive number α we proceed to the Calderón-Zygmund decomposition of f : $f = g_\alpha + b_\alpha$. -we add the subscript α in order to insists on the fact that the result of the decomposition depends on α. Let $\delta > 0$ to be fixed later and write

$$\begin{aligned} \int_A |Tf|(x)\ dx \ &= \int_0^\delta \mu\left(\{x \in A\ ;\ |Tf(x)| > \alpha\}\right)\ d\alpha \\ &+ \int_\delta^{+\infty} \mu\left(\{x \in A\ ;\ |Tf(x)| > \alpha\}\right)\ d\alpha \quad . \end{aligned} \tag{7.78}$$

We use the decomposition $f = g_\alpha + b_\alpha$ in order to deduce :

$$\begin{aligned} \mu(\{x\ :\ |Tf(x)| > \alpha\}) \ &\le \mu(\{x\ :\ |Tg_\alpha(x)| > \alpha/2\}) \\ &+\mu(\{x\ :\ |Tb_\alpha(x)| > \alpha/2\}) \quad . \end{aligned} \tag{7.79}$$

We recall that, given $f \in L^1_{loc}(\mathbb{R}^n)$, its Hardy–Littlewood maximal function is

$$Mf(x) := \sup_{r>0} \fint_{B_r(x)} |f|(y)\, dy.$$

We have, using the embedding $L^2(\mathbb{R}^n) \hookrightarrow L^{2,\infty}(\mathbb{R}^n)$

$$\int_\delta^{+\infty} \mu\Big(\Big\{x \in A : |Tg_\alpha(x)| > \frac{\alpha}{2}\Big\}\Big)\, d\alpha \le c_n \int_\delta^{+\infty} \|g_\alpha\|^2_{L^2(\mathbb{R}^n)}\ \frac{d\alpha}{\alpha^2}. \tag{7.80}$$

We decompose

$$\int_{\mathbb{R}^n} |g_\alpha|(x)^2 dx = \int_{\mathbb{R}^n \setminus \Omega_\alpha} |g_\alpha|^2(x)\, dx + \int_{\Omega_\alpha} |g_\alpha|^2(x)\, dx,$$

where Ω_α is the "bad set" away from which $g_\alpha \equiv f$. Recall moreover that

$$\sup_{x \in \Omega_\alpha} |g_\alpha|(x) \le 2^n \alpha$$

and

$$|f(x)| \le \alpha \ \text{ in } \ \mathbb{R}^n \setminus \Omega_\alpha.$$

Combining these facts with (7.80) give:

$$\begin{aligned} \int_\delta^{+\infty} \mu\Big(\Big\{x \in A : |Tg_\alpha(x)| < \frac{\alpha}{2}\Big\}\Big)\, d\alpha \le C_n \int_\delta^{+\infty} \frac{d\alpha}{\alpha^2} \int_{|f| \le \alpha} |f|^2(x)\, dx \\ + \int_\delta^{+\infty} 2^{2n}\, \frac{\alpha^2}{\alpha^2}\, \mu(\Omega_\alpha)\, d\alpha. \end{aligned} \tag{7.81}$$

Using Fubini, we have in one hand

$$\begin{aligned} \int_\delta^{+\infty} \frac{d\alpha}{\alpha^2} \int_{|f| \le \alpha} |f|^2(x)\, dx &= \int_{\mathbb{R}^n} |f|^2(x)\, dx \int_{\max\{\delta, |f|(x)\}}^{+\infty} \frac{d\alpha}{\alpha^2} \\ &\le \int_{\mathbb{R}^n} \frac{|f|^2(x)}{\max\{\delta, |f|(x)\}}\, dx \le \|f\|_{L^1(\mathbb{R}^n)}. \end{aligned} \tag{7.82}$$

In the other hand, we recall that the bad set Ω_α is the union of disjoint cubes $(C_k)_{k\in\mathbb{N}}$ and on each of these cubes the average of $|f|$ is larger than α. Hence we have

$$\mu(\Omega_\alpha)=\sum_{k\in\mathbb{N}}\mu(C_k)\le\alpha^{-1}\sum_{k\in\mathbb{N}}\int_{C_k}|f|(x)\,dx$$
$$=\alpha^{-1}\int_{\Omega_\alpha}|f|(x)\,dx.$$

We write then

$$\begin{aligned}\mu(\Omega_\alpha)\le\alpha^{-1}\int_{\Omega_\alpha}|f|(x)\,dx =&\alpha^{-1}\int_{\Omega_\alpha\cap\{x;|f|(x)>\frac{\alpha}{2}\}}|f|(x)\,dx\\ &+\alpha^{-1}\int_{\Omega_\alpha\cap\{x;|f|(x)<\frac{\alpha}{2}\}}|f|(x)\,dx\\ &\le\frac{\mu(\Omega_\alpha)}{2}+\alpha^{-1}\int_{|f|>\frac{\alpha}{2}}|f|(x)\,dx.\end{aligned}$$

Thus we just proved

$$\frac{\mu(\Omega_\alpha)}{2}\le\alpha^{-1}\int_{|f|>\frac{\alpha}{2}}|f|(x)\,dx. \tag{7.83}$$

Combining (7.81), (7.82) and (7.83), we finally obtain

(7.84)
$$\begin{aligned}\int_\delta^{+\infty}\mu\Big(\Big\{x\in A:|Tg_\alpha(x)|>\frac{\alpha}{2}\Big\}\Big)\,d\alpha\le C_n\Big[\|f\|_{L^1}+\int_\delta^{+\infty}\frac{d\alpha}{\alpha}\int_{|f|>\frac{\alpha}{2}}|f|(x)\,dx\Big]\\ \le C_n\Big[\|f\|_{L^1}+\int|f|(x)\,\mathrm{Log}+\Big(\frac{2|f|(x)}{\delta}\Big)\Big].\end{aligned}$$

Now we bound the contribution of the action of T on the bad part. We have seen in the proof of the primitive formulation of L^p theorem for convolution Calderón-Zygmund kernels that the following inequality holds

$$\int_{\mathbb{R}^n\setminus\widetilde{\Omega}_\alpha}|T\,b_\alpha|\le C(T)\int_{\widetilde{\Omega}_\alpha}|f|(x)\,dx,$$

where $\widetilde{\Omega}_\alpha=\bigcup_{k\in\mathbb{N}}\widetilde{C}_k$ and $\widetilde{C}_k$ are the cubes obtained from the C_k by dilating by the factor $2\sqrt{n}$ leaving the cube centers fixed.

For any $\beta>0$, we bound

$$\begin{aligned}\mu(\{x\in A:\,|Tb_\alpha|(x)\ge\beta\})&\le\mu(\widetilde{\Omega}_\alpha)+\mu(\{x\in\mathbb{R}^n\setminus\widetilde{\Omega}_\alpha;|T\,b_\alpha(x)|>\beta\}\\ &\le(2\sqrt{n})^n\,\mu(\Omega_\alpha)+\frac{C_n}{\beta}\int_{\mathbb{R}^n\setminus\widetilde{\Omega}_\alpha}|T\,b_\alpha|(x)\,dx\\ &\le(2\sqrt{n})^n\,\mu(\Omega_\alpha)+\frac{C_n}{\beta}\int_{\widetilde{\Omega}_\alpha}|f|(x)\,dx.\end{aligned}$$

We apply this inequality to $\beta = \frac{\alpha}{2}$ and we integrate between δ and $+\infty$. We obtain

$$\int_\delta^{+\infty} \mu\Big(\Big\{x \in A:\ |Tb_\alpha|(x) \geq \frac{\alpha}{2}\Big\}\Big)\, d\alpha \leq C_n \int_\delta^{+\infty} \mu(\Omega_\alpha)\, d\alpha$$
$$+ C_n \int_\delta^{+\infty} \frac{d\alpha}{\alpha} \int_{\widetilde{\Omega}_\alpha} |f|(x)\, dx.$$

We decompose again

$$\frac{1}{\alpha} \int_{\widetilde{\Omega}_\alpha} |f|(x)\, dx \leq \frac{1}{\alpha} \int_{\{x\in\widetilde{\Omega}_\alpha; |f|(x)<\frac{\alpha}{2}\}} |f|(x)\, dx + \frac{1}{\alpha} \int_{|f|>\frac{\alpha}{2}} |f|(x)\, dx$$
$$\leq C_n\, \mu(\Omega_\alpha) + \frac{1}{\alpha} \int_{|f|>\frac{\alpha}{2}} |f|(x)\, dx.$$

Hence we have proved

$$\int_\delta^{+\infty} \mu\Big(\Big\{x \in A: |Tg_\alpha(x)| \geq \frac{\alpha}{2}\Big\}\Big)\, d\alpha \leq c \int_\delta^{+\infty} \mu(\Omega_\alpha)\, d\alpha$$
$$+ c \int_\delta^{+\infty} \frac{d\alpha}{\alpha} \int_{|f|>\frac{d\alpha}{\alpha}} |f|(x)\, dx.$$

Using (7.83) again, we then have

$$\int_\delta^{+\infty} \mu\Big(\Big\{x \in A:\ |Tb_\alpha|(x) \geq \frac{\alpha}{2}\Big\}\Big) \leq c \int_{\mathbb{R}^n} |f|(x)\, \mathrm{Log}+\Big(\frac{2|f|(x)}{\delta}\Big)\, dx, \tag{7.85}$$

where c depends on T. Combining (7.84) and (7.85) together with (7.85), we obtain

$$\int_A |T\, f|(x) \leq \delta\, \mu(A) + c \int_{\mathbb{R}^n} |f|(x)\, \mathrm{Log}\Big[e + \frac{2|f|(x)}{\delta}\Big].$$

The inequality (**??**) follows by taking $\delta = 2\, \|f\|_{L^1/\mu(A)}$. This concludes the proof of theorem 7.18. $\square$

8 The L^p−Theorem for Littlewood Paley decompositions

8.1 Bernstein and Nikolsky inequalities

Theorem 8.1. *(Bernstein inequality)*

Let $p \in [1, +\infty]$. There exists a constant $C_{n,p} > 1$ such that, for any $k \in \mathbb{N}^*$ and any $f \in L^p(\mathbb{R}^n)$ satisfying

$$\text{supp}\, \widehat{f} \subset B_{2^k}(0) \backslash B_{2^{k-1}}(0),$$

then $\nabla f \in L^p(\mathbb{R}^n)$ and we have

$$C_{n,p}^{-1} \, \|f\|_{L^p(\mathbb{R}^n)} \leq 2^{-k} \, \|\nabla f\|_{L^p(\mathbb{R}^n)} \leq C_{n,p} \, \|f\|_{L^p(\mathbb{R}^n)}. \tag{8.1}$$

□

Proof of Theorem 8.1. Let χ be a cut of function in $C_c^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} \chi \equiv 0 & \text{in } \; B_{\frac{1}{4}}(0) \cup \big(\mathbb{R}^n \backslash B_4(0)\big) \\ \chi \equiv 1 & \text{in } \; B_1(0) \backslash B_{\frac{1}{2}}(0). \end{cases}$$

By assumption we have

$$\widehat{f}(\xi) = \chi(2^{-k}\xi) \; \widehat{f}(\xi) \;\; \text{in} \;\; \mathcal{S}'(\mathbb{R}^n).$$

Using Proposition **??**, we deduce

$$f(x) = (2\pi)^{-\frac{n}{2}} \; \check{\chi}(2^k x) \, 2^{kn} * f(x).$$

This implies for any $j = 1, \ldots, n$ (using Proposition **??**)

$$\partial_{x_j} f = (2\pi)^{-\frac{n}{2}} \; 2^{k(n+1)} \, \partial_{x_j} \check{\chi}(2^k x) * f \;\; \text{in} \;\; \mathcal{S}'(\mathbb{R}^n).$$

Since $\chi \in C_c^\infty(\mathbb{R}^n)$, $\partial_{x_j} \check{\chi} \in \mathcal{S}(\mathbb{R}^n)$ and then in particular $\partial_{x_j} \check{\chi} \in L^1(\mathbb{R}^n)$. Using Young inequality, we deduce that

$$\|\nabla f\|_{L^p(\mathbb{R}^n)} \leq C_n \; 2^k \, \|\nabla \check{\chi}\|_{L^1(\mathbb{R}^n)} \, \|f\|_{L^p(\mathbb{R}^n)}.$$

This implies the second inequality in (8.1).

We shall now present the proof of the first inequality in (8.1) in the particular case where $p \in (1, +\infty)$.

For the limiting cases respectively $p = 1$ and $p = +\infty$, we shall need a multiplier theorem that takes into account the support of the Fourier transform and that we shall prove in Chapter 7 only. Recall from Chapter 1 that for any $j \in \{1, \ldots, n\}$

$$\widehat{\partial_{x_j} f} = -i \, \xi_j \, \widehat{f}.$$

Multiplying the identity by $i\xi_j$ and summing out j gives[10]

$$\chi\big(2^{-k}\xi\big)\ \sum_{j=1}^n \frac{i\,\xi_j}{|\xi|^2}\ \widehat{\partial_{x_j} f} = \widehat{f}\ \text{ in }\ \mathcal{S}'(\mathbb{R}^n).$$

Denote

$$\begin{aligned} m_{j,k(\xi)} &:= i\,2^k\ \frac{\chi(2^{-k}\xi)}{|\xi|^2}\ \xi_j \\ &= i\,\chi(2^{-k}\xi)\ \frac{2^{-k}\,\xi_j}{|2^{-k}\xi|^2}\ . \end{aligned}$$

We have $m_{j,k(\xi)} = m_j(2^{-k}\xi)$ where

$$m_j(\eta) := i\ \frac{\chi(\eta)\,\eta_j}{|\eta|^2} \in C^\infty_c(\mathbb{R}^n).$$

Hence, it is straightforward to prove that

$$\forall \ell\in\mathbb{N}^n\ \ \exists\, C_\ell>0\ \text{ s.t. }\ \sup_j\,|\partial^\ell m_{j,k}(\xi)| \le \frac{C_\ell}{|\xi|^{|\ell|}}.$$

We can use the multiplyer Theorem 7.10 to deduce

$$2^k\,\|f\|_{L^p(\mathbb{R}^n)} \le C_{p,n}\,\|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

This is the first inequality in (8.1) and this concludes the proof of Theorem 8.1 in the case $p\in(1,+\infty)$. The general case is postponed to Chapter 7.

While the second inequality in (8.1) looks a bit like a "reverse Poincaré inequality", the following theorem could be interpreted as some sort of "reverse Hölder inequality".

Theorem 8.2. *There exists $C_n>0$ such that for any $1\le p\le q\le+\infty$, for any $k\in\mathbb{N}$ and any $f\in L^p(\mathbb{R}^n)$ satisfying*

$$\operatorname{supp}\widehat{f}\subset B_{2^k}(0),$$

then $f\in L^q(\mathbb{R}^n)$ and the following inequality holds

$$\|f\|_{L^q(\mathbb{R}^n)} \le C_n\ (2^k)^{\frac{n}{p}-\frac{n}{q}}\ \|f\|_{L^p(\mathbb{R}^n)}. \tag{8.2}$$

□

Proof of Theorem 8.2. Let $f_k(x) := 2^{kn}\,f(2^k x)$. We have then

$$\widehat{f_k}(\xi) = \widehat{f}(2^{-k}\xi)$$

which gives $\operatorname{supp}\widehat{f_k}\subset B_1(0)$.

[10]We are using here the fact that $\widehat{f}$ is supported away from the origin.

Let χ now be a function in $C_c^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} \chi \equiv 1 & \text{on } B_1(0) \\ \chi \equiv 0 & \text{on } R^n \backslash B_2(0). \end{cases}$$

Because of the choice of χ we have

$$\widehat{f_k}(\xi) = \chi(\xi)\, \widehat{f_k}(\xi).$$

Using Proposition **??**, we deduce

$$f_k = (2\pi)^{-\frac{n}{2}}\, \check{\chi} * f_k \quad .$$

Since we only consider the case $p < q$ and since $p \geq 1$, we have

$$0 < \frac{1}{p} - \frac{1}{q} < 1\ .$$

Hence there exists $r \in (1, \infty)$ such that

$$1 - \frac{1}{r} = \frac{1}{p} - \frac{1}{q}\ .$$

Since $\check{\chi} \in \mathcal{S}(\mathbb{R}^n)$, we have in particular $\check{\chi} \in L^1(\mathbb{R}^n)$ and Young inequality gives then

$$\|f_k\|_{L^q(\mathbb{R}^n)} \leq (2\pi)^{-\frac{n}{2}}\, \|\check{\chi}\|_{L^1(\mathbb{R}^n)}\, \|f_k\|_{L^p(\mathbb{R}^n)}.$$

Hölder inequality gives

$$\|\check{\chi}\|_{L^1(\mathbb{R}^n)} \leq \|\check{\chi}\|_{L^1(\mathbb{R}^n)}^{1-\beta}\, \|\check{\chi}\|_{L^\infty(\mathbb{R}^n)}^{\theta},$$

where $\theta = 1 - \frac{1}{r}$. Choose $c_n = \max\{\|\check{\chi}\|_{L^1}, \|\check{\chi}\|_{L^\infty}\}$ and we have proved

$$\|f_k\|_{L^q(\mathbb{R}^n)} \leq C_n\, \|f_k\|_{L^p(\mathbb{R}^n)}.$$

(8.2) follows by substituting $f_k(x) = 2^{-nk} f(2^{-k}x)$. This concludes the proof of Theorem 8.2. $\square$

8.2 Littlewood Paley projections

In the proof of Theorem 7.10 we introduced a partition of unity over the phase space with each function $\varphi_k = \varphi(2^{-k}\xi)$ being supported in the dyadic annuli $B_{2^{k+1}}(0)\backslash B_{2^{k-1}}(0)$. We shall consider the same partition of unity of the phase space but truncated at 0. Precisely, let $\psi \in C_c^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} \psi(\xi) \equiv 1 & \text{in } B_1(0) \\ \psi(\xi) \equiv 0 & \text{in } \mathbb{R}^n \backslash B_2(0), \end{cases}$$

and denote $\varphi(\xi) := \psi(\xi) - \psi(2\xi)$. We have clearly

$$\operatorname{supp} \varphi \subset B_2(0) \backslash B_{\frac{1}{2}}(0).$$

For $k > 0$ we take $\varphi_k(\xi) := \varphi(2^{-k}\xi)$ while for $k = 0$ we take $\varphi_0(\xi) = \psi(\xi)$. This gives

$$\sum_{k=0}^{N} \varphi_k(\xi) = \psi(2^{-N}\xi).$$

This implies that

$$\sum_{k\in\mathbb{N}} \varphi_k(\xi) = \lim_{N\to+\infty} \sum_{k=0}^{N} \varphi_k(\xi) \equiv 1 \text{ in } \mathbb{R}^n.$$

Definition 8.3. *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $k \in \mathbb{N}$. We define the k-th Littlewood-Paley projection of f associated to the partition of unity $(\varphi_k)_{k\in\mathbb{N}}$ to be $f_k := \mathcal{F}^{-1}(\varphi_k \hat{f})$.*

Because of Bernstein theorem 8.1, we have in particular, by iterating (8.1):

$$\forall p \in [1,\infty] \quad \forall k \in \mathbb{N} \quad \forall q \in \mathbb{N} \quad \sup_{|\ell|=q} \|\partial^\ell f_k\|_{L^p(\mathbb{R}^n)} \sim 2^{kq} \, \|f_k\|_{L^p(\mathbb{R}^n)}.$$

We have for $k > 0$ (using Proposition **??**)

$$f_k = 2^{kn} \, \check{\varphi}\,(2^k x) * f \, (2\pi)^{-\frac{n}{2}}. \tag{8.3}$$

Hence we deduce that for any $p \in [1,\infty]$

$$\sup_{k\in\mathbb{N}} \|f_k\|_{L^p(\mathbb{R}^n)} \le C_{n,\varphi} \, \|f\|_{L^p(\mathbb{R}^n)}. \tag{8.4}$$

By the triangular inequality we also have trivially

$$\|f\|_{L^p(\mathbb{R}^n)} \le \sum_{k\in\mathbb{N}} \|f_k\|_{L^p(\mathbb{R}^n)}. \tag{8.5}$$

The goal of the present chapter is to prove that for any $p \in (1,+\infty)$

$$\|f\|_{L^p(\mathbb{R}^n)} \sim \left\| \Big(\sum_{k\in\mathbb{N}} |f_k|^2 \Big)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}.$$

To that aim we have to present briefly the L^p-spaces for families.

8.3 The spaces $L^p(\mathbb{R}^n, \ell_q)$

We recall the classical notation for any sequence $(a_k)_{k\in\mathbb{N}}$ and any $q \in [1,\infty)$

$$\|a_k\|_{\ell^q} := \Big(\sum_{k\in\mathbb{N}} |a_k|^q \Big)^{\frac{1}{q}}$$

and

$$\|a_k\|_{\ell^\infty} := \sup_{k\in\mathbb{N}} |a_k|\,.$$

It is a well-known fact that $\mathbb{R}^{\mathbb{N}}$ or $\mathbb{C}^{\mathbb{N}}$ equipped with lack of these norms is complete and then define a Banach space.

We now define

$$L^p(\mathbb{R}^n,\ell^q) := \Big\{(f_k)_{k\in\mathbb{N}} \text{ s.t. } f_k\in L^p(\mathbb{R}^n) \Big(\sum_{k\in\mathbb{N}}|f_k|^q\Big)^{\frac{1}{q}}(x)<+\infty$$

$$\text{for almost every } x\in\mathbb{R}^n \text{ and } \Big\|\Big(\sum_{k\in\mathbb{N}}|f_k|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}<+\infty\Big\}.$$

We have the following proposition:

Proposition 8.4. *For any $p\in[1,\infty]$ and $q\in[1,\infty]$ the space $L^p(\mathbb{R}^n,\ell^q)$ defines a Banach space. Moreover for $p\in(1,\infty)$ and $q\in(1,\infty)$*

$$\big(L^p(\mathbb{R}^n,\ell^q)\big)'=L^{p'}(\mathbb{R}^n,\ell^{q'}).$$

Proof of Proposition 8.4. We first prove that $L^p(\mathbb{R}^n,\ell^q)$ is complete. Let $(f_k^j)_{k\in\mathbb{N}}$ be a Cauchy sequence in $L^p(\mathbb{R}^n,\ell^q)$. Then for each $k\in\mathbb{N}$ $(f_k^j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mathbb{R}^n)$. Since $L^p(\mathbb{R}^n)$ defines a Banach space, there exists $(f_k^\infty)_{k\in\mathbb{N}}$ such that

$$\forall k\in\mathbb{N}\quad f_k^j\longrightarrow f_k^\infty \ \text{ strongly in } L^p(\mathbb{R}^n).$$

This implies in particular that for any $N\in\mathbb{N}$

$$\Big(\sum_{k=0}^N|f_k^j|^q\Big)^{\frac{1}{q}}\longrightarrow\Big(\sum_{k=0}^N|f_k^\infty|^q\Big)^{\frac{1}{q}} \ \text{ strongly in } L^p(\mathbb{R}^n).$$

Let $F_N(x):=(\sum_{k=0}^N|f_k^\infty|^q)^{\frac{1}{q}}$. Because of the previous strong convergence we have

$$\begin{aligned}\|F_N\|_{L^p(\mathbb{R}^n)}&\le\limsup_{j\to+\infty}\Big\|\Big(\sum_{k=0}^N|f_k^j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}\\&\le\limsup_{j\to+\infty}\|(f_k^j)\|_{L^p(\mathbb{R}^n,\ell^q)}<+\infty.\end{aligned}$$

$(F_N)_{N\in\mathbb{N}}$ is a monotone sequence of L^p functions whose L^p norm is uniformly bounded. By using Beppo Levi monotone convergence theorem, we deduce that F_N strongly converges in L^p to a limit which is obvioulsly equal to $(\sum_{k=0}|f_k^\infty|^q)^{\frac{1}{q}}$. It implies that $(f_k^\infty)_{k\in\mathbb{N}}\in L^p(\mathbb{R}^n,\ell^q)$. It remains to prove

$$(f_k^j)_{k\in\mathbb{N}}\longrightarrow(f_k^\infty)_{k\in\mathbb{N}} \ \text{ strongly in } L^p(\mathbb{R}^n,\ell^q).$$

Let $\varepsilon>0$ and let respectively $j_0\in\mathbb{N}$ and $N_0\in\mathbb{N}$ such that

$$\text{i)}\qquad \sup_{j,\ell\ge j_0}\Big\|\Big(\sum_{k\in\mathbb{N}}|f_k^j-f_k^\ell|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}<\frac{\varepsilon}{3}\,.$$

$$\text{ii)}\qquad \Big\|\Big(\sum_{k>N_0}|f_k^\infty|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}<\frac{\varepsilon}{3}\,.$$

$$\text{iii)}\qquad \Big\|\Big(\sum_{k>N_0}|f_k^{j_0}|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}<\frac{\varepsilon}{3}\,.$$

We then deduce that
$$\sup_{j\geq j_0}\Big\|\Big(\sum_{k>N_0}|f_k^j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}<\frac{2\varepsilon}{3}.$$
Hence we have
$$\limsup_{j\to+\infty}\Big\|\Big(\sum_{k>\mathbb{N}_0}|f_k^j-f_k^\infty|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}<\varepsilon.$$
Since for each $k\in\mathbb{N}$ $f_k^j\to f_k^\infty$ strongly in $L^p(\mathbb{R}^n)$, we have
$$\lim_{j\to+\infty}\Big\|\Big(\sum_{k=0}^{N_0}|f_k^j-f_k^\infty|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}=0.$$

Hence for j large enough, we have
$$\Big\|\Big(\sum_{k=0}^{+\infty}|f_k^j-f_k^\infty|^q\Big)^{\frac{1}{q}}\Big\|<\varepsilon$$
which implies the convergence of $(f_k^j)_{k\in\mathbb{N}}$ towards $(f_k^\infty)_{k\in\mathbb{N}}$ in $L^p(\mathbb{R}^n,\ell^q)$.

We prove now the second part of Proposition 8.4. Let p and q in $(1,+\infty]$. Let $T\in(L^p(\mathbb{R}^n,\ell^q))'$. Let $k_0\in\mathbb{N}$ and denote $L^p_{k_0}(\mathbb{R}^n,\ell^q)$ the subspace of $(f_k)_k\in L^p(\mathbb{R}^n,\ell^q)$ such that $f_k\equiv 0$ for $k\neq k_0=0$. $L^p_{k_0}(\mathbb{R}^n,\ell^q)$ is obviously isomorphic to $L^p(\mathbb{R}^n)$ and, using Riesz representation theorem, we define the existence of $g_{k_0}\in L^{p'}(\mathbb{R}^n)$ such that
$$T_{|_{L^p_{k_0}(\mathbb{R}^n,\ell^q)}}\big((f_k)\big)=\int_{\mathbb{R}^n}f_{k_0}(x)\,g_{k_0}(x)\,dx.$$
Let
$$L^p_{\leq k_0}(\mathbb{R}^n,\ell^q):=\big\{(f_k)_{k\in\mathbb{N}}\in L^p(\mathbb{R}^n,\ell^q)\ \text{ such that } f_k\equiv 0 \text{ for } k>k_0\big\}.$$
By linearity we have
$$T_{|_{L^p_{\leq k_0}(\mathbb{R}^n,\ell^q)}}\big((f_k)\big)=\sum_{k\leq k_0}\int_{\mathbb{R}^n}f_k(x)\,g_k(x)\,dx.$$
Let
$$\begin{array}{rccc}\Pi_{k_0}: & L^p(\mathbb{R}^n,\ell^q) & \longrightarrow & L^p_{\leq k_0}(\mathbb{R}^n,\ell^q)\\ & (f_k)_{k\in\mathbb{N}} & \longrightarrow & (f_k)_{k\leq k_0}.\end{array}$$
It is not difficult to prove that for any $(f_k)_{k\in\mathbb{N}}$
$$\lim_{k_0\to+\infty}\Pi_{k_0}\big((f_k)_{k\in\mathbb{N}}\big)=(f_k)_{k\in\mathbb{N}}\ \text{ in }\ L^p(\mathbb{R}^n,\ell^q).$$
Hence, by continuity of T, we deduce that
$$\forall(f_k)_{k\in\mathbb{N}}\in L^p(\mathbb{R}^n,\ell^q)\quad T\big((f_k)_{k\in\mathbb{N}}\big)=\sum_{k\in\mathbb{N}}\int_{\mathbb{R}^n}f_k(x)\,g_k(x)\,dx.$$

It remains to prove that $(g_k)_{k\in\mathbb{N}} \in L^{p'}(\mathbb{R}^n, \ell^{q'})$.

Let $k_0 \in \mathbb{N}$ and denote $f_k^{k_0} \in L^p_{\le k_0}(\mathbb{R}^n, \ell^q)$, the element defined by

$$\forall k \le k_0 \quad f_k^{k_0} := \frac{\dfrac{g_k}{|g_k|^{2-q'}}}{\Big(\sum\limits_{k=0}^{k_0} |g_k|^{q'}\Big)^{1-\frac{p'}{q'}}} \,.$$

We have that $\forall k \le k_0$

$$|f_k^{k_0}|(x) \le \frac{|g_k|^{q'-1}}{|g_k|^{q'-p'}} = |g_k|^{\frac{p'}{p}} \in L^p(\mathbb{R}^n).$$

Because of the continuity of T we have in one hand

$$\big|T\big((f_k^{k_0})\big)\big| \le C_T \, \|(f_k^{k_0})\|_{L^p(\mathbb{R}^n,\ell^q)} = C_T \Bigg[\int_{\mathbb{R}^n} \Big(\sum_{k=0}^{k_0} |g_k|^{q'}\Big)^{\frac{p'}{q'}}\Bigg]^{\frac{1}{p}}.$$

In the other hand, a direct computation gives

$$T\big((f_k^{k_0})\big) = \int_{\mathbb{R}^n} \Big(\sum_{k=0}^{k_0} |g_k|^{q'}(x')\Big)^{\frac{p'}{q'}} \, dx.$$

Since $p > 1$, we have proved

$$\int_{\mathbb{R}^n} \Big(\sum_{k=0}^{k_0} |g_k|^{q'}(x)\Big)^{\frac{p'}{q'}} \, dx \le C_{T,p} \,.$$

The constant in the right-hand side of the inequality is independent of k_0. Hence $(g_k) \in L^{p'}(\mathbb{R}^n, \ell^{q'})$ and this concludes the proof of Proposition 8.4. □

8.4 The L^p-theorem for Littlewood-Paley decompositions

The goal of the present subsection is to give a proof of the following theorem which is the main achievement of the course.

Theorem 8.5. *Let $(\varphi_k)_{k\in\mathbb{N}}$ be a dyadic partition of unity of the phase space and let $p \in (1, \infty)$, there exists $1 < C$ such that for any $f \in L^p(\mathbb{R}^n)$:*

$$C^{-1} \, \|f\|_{L^p(\mathbb{R}^n)} \le \|(f_k)_{k\in\mathbb{N}}\|_{L^p(\mathbb{R}^n,\ell^2)} \le C \, \|f\|_{L^p}, \tag{8.6}$$

where $(f_k)_{k\in\mathbb{N}}$ is the Littlewood-Paley decomposition of f relative to the partition of unity $(\varphi_k)_{k\in\mathbb{N}}$. □

Proof of Theorem 8.5. For any $f \in \mathcal{S}(\mathbb{R}^n)$, we denote $\forall x \in \mathbb{R}^n$

$$S(f)(x) := \Big(\sum_{k\in\mathbb{N}} |f_k|^2(x)\Big)^{\frac{1}{2}}.$$

By Minkowski inequality, we deduce that S is a sub-additive map.

We first prove that S is strong $(2,2)$. Indeed, using Plancherel theorem, we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} |S(f)|^2(x)\,dx &= \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^n} |f_k|^2(x)\,dx \\
&= \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^n} \widehat{f}_k(\xi)\,\overline{\widehat{f}}_k(\xi)\,d\xi \qquad (8.7)\\
&= \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^n} \varphi_k^2(\xi)\,|\widehat{f}|^2(\xi)\,d\xi\;.
\end{aligned}
$$

Since $\operatorname{supp}\varphi_k \subset B_{2^{k+1}}(0)\backslash B_{2^{k-1}}$, each $\xi\in\mathbb{R}^n$ is contained in the support of at most 3 different φ_k. Hence we have the bound

$$
\forall \xi\in\mathbb{R}^n \quad \sum_{k\in\mathbb{N}} \varphi_k^2(\xi) \le 3\,\|\varphi\|^2_{L^\infty(\mathbb{R}^n)}. \tag{8.8}
$$

Combining (8.7) and (8.8), we obtain that S is indeed strong $(2,2)$.

We claim now that S is weak $(1,1)$. Let $K_k(x) := \check{\varphi}_k(x)$. In particular for $k>1$, we have $K_k(x) = 2^{kn}\,\check{\varphi}(2^k x)$ and

$$
f_k = (2\pi)^{-\frac{n}{2}}\,2^{kn}\,\check{\varphi}(2^k x) * f.
$$

In order to prove the claim, we shall be using the following lemma which is the Hörmander condition for families:

Lemma 8.6. *(Hörmander condition for families)*

Under the notations above, we have the existence of $B>0$ such that

$$
\forall y\in\mathbb{R}^n \quad \int_{|x|>2|y|} \|K_k(x-y)-K_k(x)\|_{\ell^2}\,dx \le B < +\infty. \tag{8.9}
$$

□

Proof of Lemma 8.6. Let $y\neq 0$ and denote $v := \frac{y}{|y|}$. For any $x\in\mathbb{R}^n$, one has

$$
|K_k(x-y)-K_k(x)| \le \int_0^{|y|} \left|\frac{\partial K_k}{\partial v}\right|(x-tv)\,dt.
$$

Using *Minkowski integral inequality*, one has

$$
\begin{aligned}
\int_{|x|>2|y|} \|K_k(x-y)-K_k(x)\|_{\ell^2}\,dx &\le \int_{|x|>2|y|} \left(\sum_{k\in\mathbb{N}} \left|\int_0^{|y|} \left|\frac{\partial K_k}{\partial v}\right|(x-tv)\,dt\right|^2\right)^{\frac12} dx \\
&\le \int_{|x|>2|y|} \int_0^{|y|} \|\nabla K_k\|_{\ell^2}(x-tv)\,dt\,dx.
\end{aligned}
$$

We interchange the order of integration and we proceed to the change of variable $z := x - tv$. This gives

$$\int_{|x|>2|y|} \|K_k(x-y) - K_k(x)\|_{\ell^2}\, dx \le |y| \int_{|z|>|y|} \|\nabla K_k\|_{\ell^2}(z)\, dz. \tag{8.10a}$$

We have for each $k \in \mathbb{N}^*$

$$|\nabla K_k|(z) = 2^{k(n+1)} |\nabla \check{\varphi}|(2^k z).$$

Since $\varphi \in C_c^\infty(\mathbb{R}^n)$, we have that $\check{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ and hence, obviously, we have in particular

$$|\nabla \check{\varphi}|(x) \le C\ \min\{1; |x|^{-n-2}\}.$$

This implies then

$$|\nabla K_k|(z) \le C\ \min\{2^{k(n+1)}; 2^{-k}\, |z|^{-n-2}\}.$$

For each z we denote by k_z the integer part of

$$\log_2 |z|^{-1} \quad \big(\text{i.e. } k_z := [\log_2\ |z|^{-1}]\big).$$

We write

$$\begin{aligned}
\Big(\sum_{k\in\mathbb{N}} |\nabla K_k|^2(z)\Big)^{\frac12} &\le \Big(\sum_{k\le k_z} |\nabla K_k|^2(z)\Big)^{\frac12} + \Big(\sum_{k=k_z+1}^{\infty} |\nabla K_k|^2(z)\Big)^{\frac12}\\
&\le C\, \Big(\sum_{k\le k_z} 2^{2k(n+1)}\Big)^{\frac12} + C\, |z|^{-n-2} \Big(\sum_{k=k_z+1}^{\infty} 2^{-2k}\Big)^{\frac12}\\
&\le \sqrt{2}\, C\ 2^{k_z(n+1)} + \frac{C}{\sqrt{2}}\, |z|^{-n-2}\, 2^{-k_z}.
\end{aligned}$$

Using the fact that $2^{k_z} \sim \frac{1}{|z|}$, we deduce

$$\Big(\sum_{k\in\mathbb{N}} |\nabla K_k|^2(z)\Big)^{\frac12} \le \frac{C'}{|z|^{n+1}}.$$

Inserting this last inequality in (8.10) gives then

$$\begin{aligned}
\int_{|x|>2|y|} \|K_k(x-y) - K_k(x)\|_{\ell^2}\, dx &\le C'\, |y| \int_{|z|>|y|} \frac{dz}{|z|^{n+1}}\\
&\le B\ |y| \int_{|y|}^{+\infty} \frac{d\rho}{\rho^2}\\
&\le B < +\infty.
\end{aligned}$$

This concludes the proof of Lemma 8.6. $\square$

Continuation of the proof of Theorem 8.5. Let $\alpha > 0$, we proceed to a Calderón-Zygmund decomposition of f for the threshold α. We write $f = g_\alpha + b_\alpha$ where g_α and b_α are respectively the good and bad parts of the decomposition. Using the subadditivity of S, we have

$$\mu\big(\{x; S(f)(x) > \alpha\}\big) \leq \mu\Big(\Big\{x; S(g_\alpha)(x) > \frac{\alpha}{2}\Big\}\Big) + \mu\Big(\Big\{x; S(b_\alpha)(x) > \frac{\alpha}{2}\Big\}\Big). \tag{8.10b}$$

Using the fact that S is strong $(2,2)$, we deduce

$$\begin{aligned} \frac{\alpha^2}{4}\,\mu\Big(\Big\{x\in\mathbb{R}^n; S(g_\alpha)(x) > \frac{\alpha}{2}\Big\}\Big) &\leq C\int_{\mathbb{R}^n} |g_\alpha|^2(x)\,dx \\ &\leq C\,2^{n+1}\alpha\int_{\mathbb{R}^n}|f|(x)\,dx. \end{aligned} \tag{8.11}$$

We recall the notations from Chapter 4:
The bad part of $\mathbb{R}^n$ for the decomposition is a union of disjoint cubes with faces parallel to the canonical hyperplanes: $\Omega = \bigcup_{\ell\in\mathbb{N}} C_\ell$ and $\widetilde{C}_\ell$ are the dilations of these cubes by the factor $2\sqrt{n}$ leaving each center c_ℓ fixed. This dilation factor is chosen in such a way that

$$\forall x\in\mathbb{R}^n\backslash\widetilde{C}_\ell \quad \forall y\in C_\ell \quad |x-c_\ell| \geq 2|y-c_\ell|.$$

Denote as usual $\widetilde{\Omega} = \bigcup_{\ell\in\mathbb{N}}\widetilde{C}_\ell$.

We estimate

$$\int_{\mathbb{R}^n\backslash\widetilde{\Omega}} |S(b_\alpha)|(x)\,dx = \int_{\mathbb{R}^n\backslash\widetilde{\Omega}} \Big|\sum_{k\in\mathbb{N}}|K_k * b_\alpha|^2(x)\Big|^{\frac{1}{2}} dx.$$

We write $b_\alpha = \sum_{\ell\in\mathbb{N}} b_\ell$ where $b_\ell = b\,\mathbf{1}_{c_\ell}$ and we use Minkowski inequality to obtain

$$\int_{\mathbb{R}^n\backslash\widetilde{\Omega}} |S(b_\alpha)|(x)\,dx \leq \sum_{\ell\in\mathbb{N}}\int_{\mathbb{R}^n\backslash\widetilde{\Omega}} \|K_k * b_\ell\|_{\ell^2}(x)\,dx. \tag{8.12}$$

Using the fact that $\int_{C_\ell} b_\ell(y)\,dy = 0$, we write

$$\begin{aligned} \|K_k * b_\ell\|_{\ell^2}(x) &= \Big\|\int_{y\in C_\ell} K_k(x-y)\;b_\ell(y)\,dy\Big\|_{\ell^2} \\ &= \Big\|\int_{y\in c_\ell}\big[K_k\big(x-c_\ell-(y-c_\ell)\big) - K_k(x-c_\ell)\big]\;b_\ell(y)\,dy\Big\|_{\ell^2}. \end{aligned} \tag{8.13}$$

Using again *Minkowski integral inequality* and continuing (8.12) and (8.13), we obtain by the mean of Lemma 8.6

$$\begin{aligned} \int_{\mathbb{R}^n\backslash\widetilde{\Omega}} |S(b_\alpha)|(x)\,dx &\leq \sum_{\ell\in\mathbb{N}}\int_{\mathbb{R}^n\backslash\widetilde{\Omega}} dx\int_{C_\ell}|b_\ell(y)|\;\big\|K_k\big(x-c_\ell-(y-c_\ell)-K_k(x-c_\ell)\big\|_{\ell^2}\,dy \\ &\leq \sum_{\ell\in\mathbb{N}}\int_{C_\ell}|b_\ell(y)|\,dy\int_{|x-c_\ell|>2|y-c_\ell|} \big\|K_k(x-c_\ell-(y-c_\ell))- K_k(x-c_\ell)\big\|_{\ell^2}\,dx \\ &\leq B\sum_{\ell\in\mathbb{N}}\int_{C_\ell}|b_\ell(y)|\,dy \leq B\int_{\mathbb{R}^n}|b_\alpha(y)|\,dz \leq 2B\int_{\mathbb{R}^n}|f(y)|\,dy. \end{aligned}$$

This implies that

$$\sup_{\beta>0} \beta\, \mu\big(\big\{x \in \mathbb{R}^n \backslash \widetilde{\Omega};\ S(b_\alpha)(x) > \beta\big\}\big) \leq 2B \int_{\mathbb{R}^n} |f(y)|\, dy.$$

Applying this inequality to $\beta := \frac{\alpha}{2}$ and recalling that $|\widetilde{\Omega}| \leq C_n\, \alpha^{-1} \int_{\mathbb{R}^n} |f(x)|\, dx$, we deduce

$$\alpha\, \mu\Big(\Big\{x \in \mathbb{R}^n;\ S(b_\alpha)(x) > \frac{\alpha}{2}\Big\}\Big) \leq C \int_{\mathbb{R}^n} |f(x)|\, dx.$$

Combining this inequality with (8.10b) and (8.11) gives that S is weak $(1,1)$ and the claim is proved. Using now Marcinkiewicz interpolation theorem 4.2, we deduce that S is strong (p,p) for $p \in (1,2]$.

We claim now that S is strong (p,p) for $p \in (2,+\infty)$. We shall use a duality argument. Thanks to Proposition 8.4, using Hahn Banach theorem, we have

$$\begin{aligned}
\Big[\int |S(f)|^p\,(x)\, dx\Big]^{\frac{1}{p}} &= \|(f_k)\|_{L^p(\mathbb{R}^n,\ell^2)} = \sup_{\|(h_k)\|_{L^{p'}(\mathbb{R}^n,\ell^2)}\leq 1} \sum_{k\in\mathbb{N}} \int f_k(x)\, h_k(x)\, dx \\
&= \sup_{\|(h_k)\|_{L^{p'}(\mathbb{R}^n,\ell^2)}\leq 1} \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^n} K_k * f(x)\, h_k(x)\, dx \\
&= \sup_{\|(h_k)\|_{L^{p'}(\mathbb{R}^n,\ell^2)}\leq 1} \int_{\mathbb{R}^n} f(x) \sum_{k\in\mathbb{N}} K_k^{\#} * h_k(x)\, dx.
\end{aligned}$$

Therefore, in order to prove that S is strong (p,p) for $p > 2$, it suffices to prove that the operator S^* defined by

$$S^*(h_k)_{k\in\mathbb{N}} := \sum_{k\in\mathbb{N}} K_k^{\#} * h_k$$

maps continuously $L^{p'}(\mathbb{R}^n,\ell^2)$ into $L^{p'}(\mathbb{R}^n)$. Precisely, we are proving the following lemma:

Lemma 8.7. *Under the above notations, for any $p' \in (1,2]$, there exists $C > 0$ such that $\forall (h_k) \in L^{p'}(\mathbb{R}^n,\ell^2)$, we have*

$$\Big\| \sum_{k\in\mathbb{N}} K_k^{\#} * h_k \Big\|_{L^{p'}(\mathbb{R}^n)} \leq C\, \|(h_k)\|_{L^{p'}(\mathbb{R}^n,\ell^2)} \tag{8.14}$$

Proof of Lemma 8.7. We use a natural extension of Marcinkiewicz interpolation theorem 4.2 to the framework of mappings from $L^{p'}(\mathbb{R}^n,\ell^2)$ into $L^{p'}(\mathbb{R}^n)$ whose proof is left to the reader in order to infer that the lemma is proved if it holds for $p' = 2$ and if there exists $C > 0$ such that

$$|S^*(h_k)_{k\in\mathbb{N}}|_{L^{1,\infty}(\mathbb{R}^n)} \leq C\, \|(h_k)\|_{L^1(\mathbb{R}^n,\ell^2)}. \tag{8.15}$$

We then first consider the case $p' = 2$.

To justify all steps in the computations below, we can of course restrict to elements $(h_k)_{k\in\mathbb{N}} \in L^2(\mathbb{R}^n, \ell^2)$ such that $h_k \in \mathcal{S}(\mathbb{R}^n)$ and $h_k \equiv 0$ for k large enough. It is not difficult to prove that this class is dense in $L^2(\mathbb{R}^n, \ell^2)$.

We have by using Plancherel theorem:

$$\begin{aligned}
\int_{\mathbb{R}^n} |S^*(h_k)|^2 &= (2\pi)^{-n} \int_{\mathbb{R}^n} \Big(\sum_{k\in\mathbb{N}} \widehat{K_k^\#}\, \widehat{h}_k\Big)\overline{\Big(\sum_{\ell\in\mathbb{N}} \widehat{K_\ell^\#}\, \widehat{h}_\ell\Big)} \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \sum_{k,\ell\in\mathbb{N}} \varphi_k^\#(\xi)\, \overline{\varphi_\ell^\#(\xi)}\, \widehat{h}_k(\xi)\, \overline{\widehat{h}_\ell}(\xi)\, d\xi.
\end{aligned}$$

Recall that $\operatorname{supp}\varphi_k(\xi) \subset B_{2^{k+1}}(0)\backslash B_{2^{k-1}}(0)$, hence

$$\varphi_k^\#(\xi)\, \varphi_\ell^\#(\xi) \not\equiv 0 \Longrightarrow |k-\ell| \leq 3.$$

This implies that

$$\begin{aligned}
\int_{\mathbb{R}^n} |S^*(h_k)|^2(x)\, dx &= (2\pi)^{-n} \int_{\mathbb{R}^n} \sum_{|k-\ell|<4} \varphi_k^\#(\xi)\, \varphi_\ell^\#(\xi)\, \widehat{h}_k(\xi)\, \overline{\widehat{h}_\ell}(\xi)\, d\xi \\
&\leq C\,(2\pi)^{-n} \int_{\mathbb{R}^n} 7 \sum_{k\in\mathbb{N}} |\varphi_k^\#(\xi)|^2\, |\widehat{h}_k|\,(\xi)\, d\xi \\
&\leq 7\,(2\pi)^{-n}\, \|\varphi\|^2_{L^\infty(\mathbb{R}^n)} \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^n} |\widehat{h}_k|^2(\xi)\, d\xi \\
&\leq C\, \|(h_k)\|_{L^2(\mathbb{R}^n,\ell^2)}.
\end{aligned}$$

Hence we have proved (8.14) for $p' = 2$.

We establish now (8.15). Let

$$H(x) := \Big(\sum_{k\in\mathbb{N}} |h_k|^2(x)\Big)^{\frac{1}{2}}.$$

We fix $\alpha > 0$ and we proceed to a Calderón-Zygmund decomposiiton for H. As usual, we denote by $\Omega = \bigcup_{\ell\in\mathbb{N}} C_\ell$ the union of the bad cubes relative to this decomposition. For each $k \in \mathbb{N}$, we write $h_k = g_k + b_k$, where

$$g_k(x) = \begin{cases} h_k(x) & \text{for } x \in \mathbb{R}^n\backslash\Omega \\ \displaystyle\fint_{C_\ell} h_k(y)\, dy & \text{for } x \in C_\ell\ (\ell \in \mathbb{N}). \end{cases}$$

Since $H(x) \leq \alpha$ on $\mathbb{R}^n\backslash\Omega$ and $\fint_{C_\ell}(H(y)\, dy \leq 2^n\alpha$ for any ℓ, we deduce, using Minkowski inequality, that

$$\|(g_k)\|_{L^\infty(\mathbb{R}^n,\ell^2)} \leq 2^n\alpha. \tag{8.16}$$

For any $k \in \mathbb{N}$ and $\ell \in \mathbb{N}$, we denote

$$b_{k,\ell} := b_k\, \mathbf{1}_{C_\ell},$$

where $\mathbf{1}_{C_\ell}$ denotes the characteristic function of the bad cube C_ℓ. Observe that we have fixed

$$\forall k, \ell \in \mathbb{N} \quad \fint_{C_\ell} b_{k,\ell}(y)\, dy = 0. \tag{8.17}$$

Moreover, using Minkowski inequality, we have also for any $\ell \in \mathbb{N}$

$$\begin{aligned} \fint_{C_\ell} \Big(\sum_{k\in\mathbb{N}} |b_{k,\ell}|^2(y) \Big)^{\frac{1}{2}} dy &\le \fint_{C_\ell} \Big(\sum_{k\in\mathbb{N}} |h_k - \fint_{C_\ell} h_k|^2(y) \Big)^{\frac{1}{2}} dy \\ &\le \fint_{C_\ell} \|h_k\|_{\ell^2}(y)\, dy + \Big\| \fint_{C_\ell} h_k \Big\|_{\ell^2}. \end{aligned}$$

Using Minkowski integral inequality, we then deduce

$$\forall \ell \in \mathbb{N} \quad \fint_{C_\ell} \|b_{k,\ell}\|_{\ell^2}(y)\, dy \le 2 \fint_{C_\ell} \|h_k\|_{\ell^2}(y)\, dy. \tag{8.18}$$

Finally, recall that from the fundamental properties of the Calderón-Zygmund decomposition one has

$$\mu(\Omega) = \sum_{\ell\in\mathbb{N}} \mu(C_\ell) \le \frac{\displaystyle\int_{\mathbb{R}^n} \|h_k\|_{\ell^2}(y)\, dy}{\alpha}. \tag{8.19}$$

Using the strong $(2,2)$ property, we have

$$\alpha^2 \mu\Big(\Big\{ x \in \mathbb{R}^n;\ |S^*(g_k)|(x) > \frac{\alpha}{2} \Big\} \Big) \le C\, \|(g_k)\|^2_{L^2(\mathbb{R}^n,\ell^2)}.$$

Combining this inequality with (8.16) gives then

$$\begin{aligned} \alpha\, \mu\Big(\Big\{ x \in \mathbb{R}^n;\ |S^*(g_k)|(x) > \frac{\alpha}{2} \Big\} \Big) &\le C \int_{\mathbb{R}^n} \|(g_k)\|_{\ell^2}(y)\, dy \\ &\le C \int_{\mathbb{R}^n} \|(h_k)\|_{\ell^2}(y)\, dy\,, \end{aligned} \tag{8.20}$$

where we used again Minkowski integral inequality.

Denote as usual $\widetilde{C}_\ell$ the dilated cubes by the factor $2\sqrt{n}$ and $\widetilde{\Omega} = \bigcup_{\ell\in\mathbb{N}} \widetilde{C}_\ell$ with respect to the center c_l of C_l. We estimate now

$$\int_{\mathbb{R}^n\setminus\widetilde{\Omega}} |S^*(b_k)|(x)\, dx \le \int_{\mathbb{R}^n\setminus\widetilde{\Omega}} \sum_{\ell\in\mathbb{N}} \Big(\sum_{k\in\mathbb{N}} |K_k^* * b_{k,\ell}|(x) \Big)\, dx.$$

As usual we write

$$\begin{aligned} |K_k^{\#} * b_{k,\ell}|(x) &= \Big| \int_{C_\ell} K_k^{\#}\big(x - c_\ell - (y - c_\ell)\big)\, b_{k,\ell}(y)\, dy \Big| \\ &= \Big| \int_{C_\ell} \big[K_k^{\#}\big(x - c_\ell - (y - c_\ell)\big) - K_k^{\#}(x - c_\ell) \big]\, b_{k,\ell}(y)\, dy \Big|. \end{aligned}$$

We then bound using Cauchy-Schwarz inequality

$$\sum_{k\in\mathbb{N}} |K_k^{\#} * b_{k,\ell}|(x) \leq \int_{c_\ell} \sum_{k\in\mathbb{N}} \left|K_k^{\#}(x-c_\ell-(y-c_\ell)-K_k^{*}(x-c_\ell)\right| |b_{k,\ell}(x)|\, dy$$
$$\leq \int_{c_\ell} \|K_k^{\#}\big(x-c_\ell-(y-c_\ell)\big)-K_k^{\#}(x-c_\ell)\|_{\ell_2}\, \|b_{k,\ell}(y)\|_{\ell^2}\, dy.$$

This gives

$$\int_{\mathbb{R}^n\setminus\widetilde{\Omega}} |S^*(b_k)|(x)\, dx \leq$$
$$\sum_{\ell\in\mathbb{N}} \int_{C_\ell} \|b_{k,\ell}\|_{\ell^2}(y)\, dy \int_{|x-c_\ell|>2|y-c_\ell|} \|K_k^{\#}\big(x-c_\ell-(y-c_\ell)\big)-K_k(x-c_\ell)\|_{\ell^2}\, dy.$$

Using Lemma 8.6 (i.e. Hörmander property for families), we then deduce

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\widetilde{\Omega}} |S^*(b_k)|(x)\, dx &\leq C \sum_{\ell\in\mathbb{N}} \int_{C_\ell} \|b_{k,\ell}\|_{\ell^2}(y)\, dy\\ &= C \int_{\mathbb{R}^n} \|b_k\|_{\ell^2}(y)\, dy\\ &\leq 2\, C \int_{\mathbb{R}^n} \|h_k\|_{\ell^2}(y)\, dy.\end{aligned} \tag{8.21}$$

Combining (8.19), (8.20) and (8.21) gives

$$\alpha\, \mu\big(\{x\in\mathbb{R}^n;\ |S^*(h_k)|(x)>\alpha\}\big) \leq C \int_{\mathbb{R}^n} \|h_k\|_{\ell^2}(y)\, dy,$$

which is the weak $(1,1)$ property for S^* (8.15). We then deduce Lemma 8.7. $\square$

End of the proof of Theorem 8.5. We recall the identity

$$\left[\int_{\mathbb{R}^n} |S(f)|^p(x)\, dx\right]^{\frac{1}{p}} = \sup_{\|(h_k)\|_{L^{p'}(\mathbb{R}^n,\ell^2)}\leq 1} \int_{\mathbb{R}^n} f(x)\ S^*(h_k)(x)\, dx.$$

Since by Lemma 8.7 S^* is continuously mapping $L^{p'}(\mathbb{R}^n,\ell^2)$ into $L^{p'}(\mathbb{R}^n)$ for any $p'\in(1,2]$, we deduce then $\forall p\in[2,\infty)$ $\exists C_p>0$ such that

$$\left[\int_{\mathbb{R}^n} |S(f)|^p(x)\, dx\right]^{\frac{1}{p}} \leq C_p\, \|f\|_{L^p(\mathbb{R}^n)}.$$

Hence we have proved the second inequality in (8.6). It remains to prove the first one in order to conclude the proof of the theorem.

We use the following duality argument

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}^n)} &= \sup_{\|g\|_{L^{p'}(\mathbb{R}^n)}\leq 1} \int_{\mathbb{R}^n} f(x)\, g(x)\, dx\\ &= \sup_{\|g\|_{L^{p'}(\mathbb{R}^n)}\leq 1} \int_{\mathbb{R}^n} \sum_{k,\ell\in\mathbb{N}} f_k(x)\ g_\ell(x)\, dx.\end{aligned}$$

Since $\operatorname{supp} \widehat{f_k} \cap \operatorname{supp} \widehat{g_\ell} = \emptyset$ for $|k-\ell| \geq 4$, we deduce

$$\begin{aligned}
\|f\|_{L^p(\mathbb{R}^n)} &= \sup_{\|g\|_{L^{p'}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \sum_{|k-\ell|<4} f_k(x)\, g_\ell(x)\, dx \\
&\leq \sup_{\|g\|_{L^{p'}(\mathbb{R}^n)} \leq 1} 7 \int_{\mathbb{R}^n} \|f_k\|_{\ell^2}(x)\, \|g_k\|_{\ell^2}(x)\, dx \\
&\leq \sup_{\|g\|_{L^{p'}(\mathbb{R}^n)} \leq 1} 7\, \|(f_k)\|_{L^p(\mathbb{R}^n,\ell^2)} \|(g_k)\|_{L^{p'}(\mathbb{R}^n,\ell^2)} \\
&\leq C\, \|(f_k)\|_{L^p(\mathbb{R}^n,\ell^2)},
\end{aligned}$$

where we used

$$\|(g_k)\|_{L^p(\mathbb{R}^n,\ell^2)} \leq C\, \|g\|_{L^{p'}(\mathbb{R}^n)}.$$

This concludes the proof of the Theorem 8.5. □

9 Some important Function Spaces and their Littlewood Paley characterizations

9.1 Besov and Triebel Lizorkin Spaces

Under Construction

9.2 The Hardy Space $\mathcal{H}^1(\mathbb{R}^n)$

9.2.1 Historical origins of the Hardy spaces $\mathcal{H}^p$

The Hardy spaces in one variable have their original setting in complex analysis. They first appeared as spaces of holomorphic functions and were introduced with the aim of characterizing boundary values of holomorphic functions on the unit disk $\mathbb{D} := \{|z| < 1\}$. Namely, let us look at the following problem: what are the possible functions $S^1 \to \mathbb{C}$ arising as boundary values of some holomorphic function $F : \mathbb{D} \to \mathbb{C}$?

This question, as just stated, is too vague: due to the lack of compactness of $\mathbb{D}$, holomorphic functions defined on $\mathbb{D}$ could exhibit a wild behaviour as we approach the boundary (for instance, we can prescribe arbitrary values of F on any discrete subset of $\mathbb{D}$). In order to obtain a meaningful notion of boundary value, it is natural to impose integrability conditions on our functions F.

As a motivation of the forthcoming definitions, let us make a *heuristic* remark: if the trace of F on $\partial\mathbb{D} = S^1$ is some complex function $f \in L^p(S^1)$, for some $1 \leq p \leq \infty$, then F (which is holomorphic and thus harmonic) is given by the Poisson integral of f. In polar coordinates we have the formula

$$F(re^{i\theta}) = \int_{S^1} P_r(e^{i(\theta-\eta)}) f(e^{i\eta})\, d\eta.$$

The Poisson kernel is everywhere positive and satisfies $\int_{S^1} P_r(e^{i\eta})\, d\eta = 1$ for any r, so (by Young's inequality on the group S^1)

$$\|F(r\cdot)\|_{L^p(S^1)} \le \|f\|_{L^p(S^1)}$$

and in particular all the norms in the left-hand side remain bounded as $r \uparrow 1$.

In 1915 Hardy observed that, for any holomorphic function $F : \mathbb{D} \to \mathbb{C}$, the map $r \mapsto \|F(r\cdot)\|_{L^p}$ is nondecreasing (for an arbitrary $0 < p \le \infty$). These observations lead us to define the space

$$H^p(\mathbb{D}) := \left\{ F : \mathbb{D} \to \mathbb{C} \text{ holomorphic with } \lim_{r \uparrow 1} \|F(r\cdot)\|_{L^p(S^1)} < +\infty \right\}.$$

When $p > 1$, using the weak* compactness of $L^p(S^1)$, it is not difficult to show that any $F \in H^p(\mathbb{D})$ is given by the Poisson integral of a complex-valued function $f \in L^p(S^1)$ satisfying $\widehat{f}(k) = 0$ for all $k < 0$. Conversely, given any such f, its Poisson integral lies in $H^p(\mathbb{D})$. Moreover, one can show that

$$F(r\cdot) \to f \text{ in } L^p(S^1) \quad \text{and} \quad \lim_{r \uparrow 1} F(re^{i\theta}) = f(e^{i\theta}) \text{ for a.e. } \theta,$$

so that f deserves to be regarded as the set of boundary values of F (we mention that for a.e. θ one has even a *nontangential* convergence of F to $f(e^{i\theta})$). This settles the problem for $1 < p \le \infty$. Let us also remark that the condition $\widehat{f}(k) = 0$ (for all $k < 0$) amounts to saying that $\Im(f)$, up to constants, equals the Hilbert-Riesz transform of $-\Re(f)$. When $1 < p < \infty$ the Hilbert-Riesz transform maps $L^p(S^1)$ into itself, so any function in $L^p(S^1)$ arises as the real part of the trace of some element of $H^p(\mathbb{D})$.

The case of $0 < p \le 1$ is more difficult. F. Riesz, in a paper published in 1923, introduced the notation $H^p(\mathbb{D})$ for these spaces of holomorphic function (the letter H stands of course for Hardy) and proved many interesting properties, such as the following factorization theorem.

Theorem 9.1. *Any $F \in H^p(\mathbb{D})$ can be written as $F = BG$, for suitable holomorphic functions $B, G : \mathbb{D} \to \mathbb{C}$ such that $|B| \le 1$, $G \neq 0$ everywhere and $G \in H^p(\mathbb{D})$ (B is the so-called* Blaschke product *associated to the zeros of F).*

This theorem enabled him to prove the existence of a trace $f \in L^p(S^1)$ such that we have again all the convergence results mentioned before for the case $p > 1$: the trick is that, as $G \neq 0$ everywhere, one can take a k-th root of G (for an arbitrary $k > \frac{1}{p}$) and, observing that $G^{1/k} \in H^{kp}(\mathbb{D})$ and $B \in H^\infty(\mathbb{D})$, we get back to the previous case.

Again, when $p = 1$, it can be shown that the possible traces are precisely the complex-valued functions $f \in L^1(S^1)$ satisfying $\widehat{f}(k) = 0$ for all $k < 0$. As before, the possible real values of traces of functions form the set

$$\mathcal{H}^1(S^1) = \left\{ f \in L^1(S^1) : \mathcal{R}f \in L^1(S^1) \right\}$$

(here $\mathcal{R}$ denotes the Hilbert-Riesz transform). But $\mathcal{R}$ does not map $L^1(S^1)$ into itself any longer, so this set is a proper subspace of $L^1(S^1)$.

These results still hold in the upper half-plane $\mathbb{R}^2_+ := \{z = x + iy : y > 0\}$, replacing S^1 with $\mathbb{R}$ (see [**?**]), leading to the definition of $\mathcal{H}^1(\mathbb{R})$.

Later, in 1960, Stein and Weiss introduced the systems of conjugate harmonic functions in several variables, inspiring the correct definition of Hardy spaces in higher dimension.

The first characterization avoiding conjugate functions was provided by Burkholder, Gundy and Silverstein in 1971: they proved that a holomorphic function belongs to $\mathcal{H}^p$ if and only if the nontangential maximal function of its real part lies in L^p. The importance of this result lies in the fact that it allows to decide the membership of a function f to $\mathcal{H}^p$ by looking just at f itself.

In 1972 Fefferman and Stein, in a single pioneering paper, provided new real characterizations of the Hardy spaces, introducing different useful maximal functions and showing that the Poisson kernel can be replaced essentially by any other kernel. In this paper Fefferman and Stein also proved that singular integrals map Hardy spaces to themselves (and in particular $\mathcal{H}^1$ to L^1), as well as the duality $(\mathcal{H}^1)^* = BMO$.

The Littlewood Paley characterization of these spaces was first given by Peetre, while the atomic decomposition was obtained by Coifman (in one dimension) and Latter (in arbitrary dimension).

9.2.2 Equivalent definitions and basic properties of $\mathcal{H}^1(\mathbb{R}^n)$

We now introduce the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$, with a strong emphasis on the modern real-variable point of view outlined in the last part of [**?**, FS16]

This important space can be characterized in many useful ways: indeed, $\mathcal{H}^1(\mathbb{R}^n)$ is the space of all functions in $L^1(\mathbb{R}^n)$ satisfying one of the equivalent definitions provided by Theorem 9.2 ((8) being the closest to the historical one).

Before stating the theorem, let us recall that the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space, with the following increasing sequence of (semi)norms:

$$\|\psi\|_k := \sup_{x\in\mathbb{R}^n} (1+|x|^2)^{k/2} \sum_{|\alpha|\le k} \left| \frac{\partial^{|\alpha|}\psi}{\partial x^\alpha} \right| (x), \quad k \ge 0.$$

Theorem 9.2. *Fix any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \varphi(x)\,dx \neq 0$. There exists an $N \ge 0$ (independent of φ) such that the following are equivalent, for a function $f \in L^1(\mathbb{R}^n)$:*

1. *the* vertical maximal function $\mathcal{M}^v_\varphi f(x) := \sup_{t>0} |\varphi_t * f|(x)$ *lies in* $L^1(\mathbb{R}^n)$*;*

2. *the* conical maximal function

$$\mathcal{M}^c_\varphi f(x) := \sup_{\substack{t>0,\\ y\in B_t(x)}} |\varphi_t * f|(y)$$

lies in $L^1(\mathbb{R}^n)$*;*

3. *the* tangential maximal function

$$\mathcal{M}^t_\varphi f(x) := \sup_{\substack{t>0,\\ y\in\mathbb{R}^n}} |\varphi_t * f|(y) \left(1 + \frac{|y-x|}{t}\right)^{-n-1}$$

lies in $L^1(\mathbb{R}^n)$*;*

4. *the* grand maximal function

$$\mathcal{GM}f(x) := \sup\left\{ |\psi_t * f|(x) \mid t > 0, \psi \in \mathcal{S}(\mathbb{R}^n), \|\psi\|_N \le 1 \right\}$$

lies in $L^1(\mathbb{R}^n)$*;*

5. *the similar grand maximal function*

$$\mathcal{GM}'f(x) := \sup\left\{ |\varphi_t * f|(x) \mid t > 0, \psi \in C_c^\infty(B_1(0)), \|\nabla\psi\|_{L^\infty} \le 1 \right\}$$

lies in $L^1(\mathbb{R}^n)$*;*

6. *there exists an* atomic decomposition*, namely there exist* $\lambda_k \ge 0$ *and* ∞*-atoms* a_k *(see Definition 9.4) such that*

$$f = \sum_{k=0}^{\infty} \lambda_k a_k \ \textit{in}\ L^1(\mathbb{R}^n), \quad \sum_{k=0}^{\infty} \lambda_k < +\infty;$$

7. *the vertical maximal function with the Poisson kernel, i.e.* $\mathcal{M}_P^v f$*, lies in* $L^1(\mathbb{R}^n)$ *(notice that* $P(x) = \mathcal{F}^{-1}(e^{-t|\xi|}) = \frac{c_n}{(1+|x|^2)^{(n+1)/2}} \notin \mathcal{S}(\mathbb{R}^n)$*);*

8. *the tempered distributions* $\mathcal{R}_j f := \mathcal{F}^{-1}\left(-i\frac{\xi_j}{|\xi|}\widehat{f}(\xi)\right)$ *belong to* $L^1(\mathbb{R}^n)$*;*

9. f *belongs to the homogeneous Triebel-Lizorkin space* $\dot{F}^0_{1,2}(\mathbb{R}^n)$*.*

Each of the preceding statements defines also a norm on $\mathcal{H}^1(\mathbb{R}^n)$: (1) defines the norm $\left\|\mathcal{M}_\varphi^v f\right\|_{L^1}$ (and similarly for (2), (3), (4), (5), (7)), (6) induces the norm $\inf \sum_k \lambda_k$ (the infimum ranging among all the possible decompositions), (8) provides the norm $\|f\|_{L^1} + \sum_{j=1}^n \|\mathcal{R}_j f\|_{L^1}$ and (9) defines $\|f\|_{\dot{F}^0_{1,2}}$ (which is a norm on $L^1(\mathbb{R}^n) \cap \dot{F}^0_{1,2}(\mathbb{R}^n)$).

The proof of this theorem is scattered across the next sections. We will show the following diagram of implications (with the corresponding bounds on the induced norms):

$$\begin{tikzcd}
 & & (7) \arrow[ddll, Rightarrow, "\text{for a suitable } \varphi"'] & (8) \arrow[l, Rightarrow] \\
 & & (4) \arrow[r, Rightarrow] \arrow[dll, Rightarrow] & (6) \arrow[ul, Rightarrow] \arrow[u, Rightarrow] \arrow[ddl, Rightarrow] \\
(1) \arrow[r, Rightarrow] & (2) \arrow[r, Rightarrow] & (3) \arrow[u, Rightarrow] & \\
 & & (5) \arrow[ull, Rightarrow, "\text{for a suitable } \varphi"] &
\end{tikzcd}$$

We left out (9) in the diagram, since its equivalence with the other definitions is slightly involved and invokes the vector-valued space $\mathcal{H}^1(\ell^2)$, which will be introduced in Section 9.2.8.

Similarly, as we will see in Section 9.2.4, the proof of (1) $\Rightarrow$ (2) is quite circuitous and uses a particular refinement of the proofs of (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Let us rely on definition (1) as for now, i.e. let $\mathcal{H}^1(\mathbb{R}^n)$ denote the space of functions $f \in L^1(\mathbb{R}^n)$ satisfying (1) and let $\|f\|_{\mathcal{H}^1} := \left\|\mathcal{M}_\varphi^v f\right\|_{L^1}$. A first basic question is whether $C_c^\infty(\mathbb{R}^n)$ is included in $\mathcal{H}^1(\mathbb{R}^n)$. Surprisingly, this property (which holds for most of the common functional spaces) fails for $\mathcal{H}^1(\mathbb{R}^n)$, as the next proposition shows.

Proposition 9.3. *If $f \in \mathcal{H}^1(\mathbb{R}^n)$, then $\int f(x)\,dx = 0$.*

Proof. Assume by contradiction that $m := \left|\int f(x)\,dx\right| \neq 0$. Choose any $x_0 \neq 0$ such that $c := |\varphi|(x_0) \neq 0$. Then we can find $R > 0$ such that

$$\left|\int_{B_R(0)} f(x)\,dx\right| \geq \frac{m}{2}, \quad \|\varphi\|_{L^\infty} \int_{\mathbb{R}^n \setminus B_R(0)} |f|(x)\,dx \leq \frac{cm}{4}.$$

For any $z \in \mathbb{R}^n$ close to 0 and any large $r > 0$ we have

$$\begin{aligned}
&r^n |\varphi_r * f|(r(x_0+z)) \geq \left|\int_{B_R(0)} \varphi(x_0+z-r^{-1}y)f(y)\,dy\right| - \frac{cm}{4} \\
&\geq c\left|\int_{B_R(0)} f(y)\,dy\right| - \int_{B_R(0)} \left|\varphi(x_0+z-r^{-1}y)-\varphi(x_0)\right| |f|(y)\,dy - \frac{cm}{4} \\
&\geq \frac{cm}{4} - \|f\|_{L^1} \max_{y\in B_R(0)} \left|\varphi(x_0+z-r^{-1}y)-\varphi(x_0)\right| - \frac{cm}{8} \geq \frac{cm}{8}
\end{aligned}$$

provided that $\|f\|_{L^1} \max_{y\in B_R(0)} |\varphi(x_0+z-r^{-1}y)-\varphi(x_0)| \leq \frac{cm}{8}$, which holds if $|z| < \epsilon$ and $r > \epsilon^{-1}$ for some small ϵ. We can assume that $\epsilon < \frac{|x_0|}{2}$. For such z, r it holds

$$\mathcal{M}_\varphi^v f(r(x_0+z)) \geq \frac{cm}{8} r^{-n} \gtrsim |r(x_0+z)|^{-n}.$$

But $E := \{r(x_0+z) \mid |z| < \epsilon, r > \epsilon^{-1}\}$ is an open cone minus a bounded set, so

$$\int \mathcal{M}_\varphi^v f(x)\,dx \geq \int_E \mathcal{M}_\varphi^v f(x)\,dx \gtrsim \int_E |x|^{-n}\,dx = +\infty,$$

contradicting the fact that $f \in \mathcal{H}^1(\mathbb{R}^n)$. $\square$

As shown by the next proposition, the mean-zero property is the only requirement which a function in $C_c^\infty(\mathbb{R}^n)$ needs to satisfy in order to be in $\mathcal{H}^1(\mathbb{R}^n)$.

Definition 9.4. *For any $1 < p \leq \infty$, a p-atom is a function $a \in L^p$ supported in some ball B, with zero mean and*

$$\|a\|_{L^p} |B|^{1/p'} \leq 1.$$

We remark that (for $1 < p < \infty$) the last condition can be rewritten as

$$|B| \left(\fint_B |a|^p \right)^{1/p} \leq 1.$$

By Hölder's inequality, any q-atom a is also a p-atom for every $1 < p \leq q$ and $\|a\|_{L^1} \leq 1$. We now show that the $\mathcal{H}^1$-norm is bounded, as well.

Remark 9.5. *$\mathcal{M}_\varphi^v f \lesssim Mf$ pointwise, since letting $h(x) := \max_{|x'| \geq |x|} |\varphi|(x')$ we have (noticing that h is radial and that the superlevel sets $\{h > s\}$ are either open balls or empty, for all $s > 0$)*

$$\begin{aligned} |\varphi_t * f|(x) \leq & \int t^{-n} h(t^{-1}y) |f|(x-y)\, dy \\ = & \int h(y) |f|(x - ty)\, dy \\ = & \int_0^\infty \int_{\{h>s\}} |f|(x-ty)\, dy\, ds \\ \leq & Mf(x) \int_0^\infty |\{h > s\}| \\ = & Mf(x) \int h(y)\, dy \lesssim Mf(x), \end{aligned}$$

as $\int h(y)\, dy$ is finite. The same proof with P in place of f shows that $\mathcal{M}_P^v f \leq Mf$.

Proposition 9.6. *If a is a p-atom supported in B, then $a \in \mathcal{H}^1(\mathbb{R}^n)$ and*

$$\|a\|_{\mathcal{H}^1(\mathbb{R}^n)} \lesssim \|a\|_p |B|^{1/p'} \leq 1.$$

The implied constant depends on n, p and φ.

Proof. Let $B = B_r(x_0)$. For $x \in B_{2r}(x_0)$ we use the last remark to estimate

$$\mathcal{M}_\varphi^v a(x) \lesssim Ma(x),$$

which gives (by Hölder's inequality and Hardy-Littlewood maximal inequality)

$$\int_{B_{2r}(x_0)} \mathcal{M}_\varphi^v a(x)\, dx \lesssim \int_{B_{2r}(x_0)} Ma(x)\, dx \leq |B_{2r}(x_0)|^{1/p'} \|Ma\|_{L^p} \lesssim |B|^{1/p'} \|a\|_{L^p}.$$

For $x \notin B_{2r}(x_0)$ we use instead the mean-zero property:

$$\varphi_t * a(x) = \int (\varphi_t(x-y) - \varphi_t(x-x_0)) a(y)\, dy.$$

By the mean value theorem, $|\varphi_t(x-y) - \varphi_t(x-x_0)| \leq r |\nabla \varphi_t|(x-z)$ for some z on the segment joining x_0 to y. So $|z - x_0| \leq r$, thus

$$|\nabla \varphi_t|(x-z) = t^{-n-1} |\nabla \varphi| \left(\frac{x-z}{t} \right) \lesssim t^{-n-1} \left(\frac{x-z}{t} \right)^{-n-1} \lesssim |x - x_0|^{-n-1}$$

since $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Hence,

$$\int_{\mathbb{R}^n \setminus B_{2r}(x_0)} \mathcal{M}_\varphi^v a(x)\, dx \lesssim r \|a\|_{L^1} \int_{\mathbb{R}^n \setminus B_{2r}(x_0)} |x - x_0|^{-n-1}\, dx \lesssim |B|^{1/p'} \|a\|_{L^p}.$$

□

Proposition 9.7. *For any $f \in \mathcal{H}^1(\mathbb{R}^n)$ we have $\|f\|_{L^1} \lesssim \|f\|_{\mathcal{H}^1}$.*

Proof. We assume (without loss of generality) that $\int \varphi(x)\,dx = 1$. Recall that $\lim_{h\to 0} \|f(\cdot + h) - f\|_{L^1} = 0$ for all functions $f \in L^1(\mathbb{R}^n)$. Thus,

$$\begin{aligned}\|\varphi_t * f - f\|_{L^1} &= \int \left| \int \varphi(y)\,(f(x-ty) - f(x))\,dy \right| dx \\ &\le \int\int |\varphi|\,(y)\,|f(x-ty) - f(x)|\,dx\,dy \to 0\end{aligned}$$

as $t \to 0$, by the dominated convergence theorem: indeed, the inner integral is bounded by $2\|f\|_{L^1} |\varphi|\,(y)$ and tends to 0 for all y, by the aforementioned property of functions in $L^1(\mathbb{R}^n)$. So $\|f\|_{L^1} = \lim_{t\to 0} \|\varphi_t * f\|_{L^1} \le \left\|\mathcal{M}_\varphi^v f\right\|_{\mathcal{H}^1}$. □

Proposition 9.8. *The space $\mathcal{H}^1(\mathbb{R}^n)$ is a Banach space.*

Proof. If (f_j) is a Cauchy sequence in $\mathcal{H}^1(\mathbb{R}^n)$, then we have

$$\|f_j - f_k\|_{L^1} \lesssim \|f_j - f_k\|_{\mathcal{H}^1} \to 0 \text{ as } j,k \to \infty$$

by Proposition 9.7, so (f_j) is a Cauchy sequence in $L^1(\mathbb{R}^n)$. Hence, $f_j \to f$ for some $f \in L^1(\mathbb{R}^n)$. For any $x \in \mathbb{R}^n$ we have

$$|\varphi_t * f|\,(x) = \lim_{j\to\infty} |\varphi_t * f_j|\,(x) \le \liminf_{j\to\infty} \mathcal{M}_\varphi^v f_j(x),$$

so $\mathcal{M}_\varphi^v f(x) \le \liminf_{j\to\infty} \mathcal{M}_\varphi^v f_j(x)$ and, by Fatou's lemma, we deduce

$$\|f\|_{\mathcal{H}^1} = \left\|\mathcal{M}_\varphi^v f\right\|_{L^1} \le \liminf_{j\to\infty} \left\|\mathcal{M}_\varphi^v f_j\right\|_{L^1} < +\infty.$$

So $f \in \mathcal{H}^1(\mathbb{R}^n)$. Moreover, since $f - f_j = \lim_{k\to\infty}(f_k - f_j)$ in $L^1(\mathbb{R}^n)$, the same argument shows that

$$\|f - f_j\|_{\mathcal{H}^1} \le \liminf_{k\to\infty} \|f_k - f_j\|_{\mathcal{H}^1}.$$

But the right-hand side can be made small at will, by taking j large enough (since (f_j) is a Cauchy sequence in $\mathcal{H}^1(\mathbb{R}^n)$). This proves that $f_j \to f$ in $\mathcal{H}^1(\mathbb{R}^n)$. □

Proposition 9.9. *If $(f_j)_{j\in\mathbb{N}}$ is a bounded sequence in $\mathcal{H}^1(\mathbb{R}^n)$ then, up to extracting a subsequence, there exists $f \in \mathcal{H}^1(\mathbb{R}^n)$ such that $f_j \overset{*}{\rightharpoonup} f$ in $\mathcal{S}'(\mathbb{R}^n)$.*

This result is related to the fact that $\mathcal{H}^1(\mathbb{R}^n)$ is a dual space. Notice that the same statement is false in $L^1(\mathbb{R}^n)$: for instance, it is easy to see that $\varphi_t \overset{*}{\rightharpoonup} (\int \varphi(x)\,dx)\delta$ weak star. In general, a distributional limit of functions in $L^1(\mathbb{R}^n)$ is a finite measure which can possess a singular part.

Proof. We assume (without loss of generality) that $\int \varphi(x)\,dx = 1$. Recall that the dual space of $C_0(\mathbb{R}^n)$, the space of continuous functions which are infinitesimal at infinity, is $C_0(\mathbb{R}^n)^* = \mathcal{M}(\mathbb{R}^n)$, the space of finite (signed) measures. $L^1(\mathbb{R}^n)$ is isometrically embedded into $\mathcal{M}(\mathbb{R}^n)$: a function $g \in L^1(\mathbb{R}^n)$ can be regarded as the finite measure $g\,dx$.

By Proposition 9.7, (f_j) is bounded in $L^1(\mathbb{R}^n)$ as well. Since $C_0(\mathbb{R}^n)$ is separable, by Banach-Alaoglu any closed ball in its dual is weakly* sequentially compact, so there exists a subsequence, which we still denote (f_j), and a measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ such that

$$f_j\, dx \overset{*}{\rightharpoonup} \mu \text{ in } C_0(\mathbb{R}^n)^*.$$

We claim that μ is absolutely continuous with respect to the Lebesgue measure. Indeed, for any $x \in \mathbb{R}^n$ and any $t > 0$ we have

$$\varphi_t * f_j(x) = \int \varphi_t(x-y)\, f_j(y)\, dy \to \int \varphi_t(x-y)\, d\mu(y) =: \varphi_t * \mu(x).$$

Arguing as in the previous proof, we deduce

$$\sup_{t>0} |\varphi_t * \mu|(x) \leq \liminf_{j\to\infty} \mathcal{M}_\varphi^v f_j =: g.$$

It is easy to check that $\varphi_t(-\cdot) * \rho \to \rho$ in $C_0(\mathbb{R}^n)$, for any $\rho \in C_0(\mathbb{R}^n)$. This implies, using Fubini's theorem, that $\varphi_t * \mu\, dx \overset{*}{\rightharpoonup} \mu$. Let now E be a Borel set with $|E| = 0$. We can find a decreasing sequence of open sets (V_k) such that $E \subseteq \cap_k V_k$ and $|V_k| = 0$. By weak* convergence we have

$$|\mu|(E) = |\mu|(V_k) \leq \liminf_{t\to 0} \int_{V_k} |\varphi_t * \mu|(x)\, dx \leq \int_{V_k} g(x)\, dx.$$

But $g \in L^1(\mathbb{R}^n)$ (by Fatou's lemma, since $\liminf_{j\to\infty} \left\|\mathcal{M}_\varphi^v f_j\right\|_{L^1} < +\infty$), so taking the limit as $k \to \infty$ we deduce

$$|\mu|(E) \leq \lim_{k\to\infty} \int_{V_k} g(x)\, dx = 0.$$

Hence the claim is proved, i.e. $\mu = f\, dx$ for some $f \in L^1(\mathbb{R}^n)$. We deduce that $f \in \mathcal{H}^1(\mathbb{R}^n)$ as in the previous proof. The convergence $f_j \overset{*}{\rightharpoonup} f$ in $\mathcal{S}'(\mathbb{R}^n)$ follows from the fact that $\mathcal{S}(\mathbb{R}^n)$ injects continously into $C_0(\mathbb{R}^n)$. $\square$

9.2.3 $\mathcal{H}^1 \to \mathcal{H}^1$ boundedness of Calderón-Zygmund operators

In this section we will take for granted Theorem 9.2 (with an abuse of notation, we will denote by $\|\cdot\|_{\mathcal{H}^1}$ any of the equivalent norms introduced above) and we will show why $\mathcal{H}^1(\mathbb{R}^n)$ is the good replacement of $L^1(\mathbb{R}^n)$ from the point of view of harmonic analysis.

Namely, its norm has the same behaviour as the L^1-norm: for any $\lambda > 0$,

$$\|f_\lambda\|_{\mathcal{H}^1} = \|f\|_{\mathcal{H}^1}$$

(to be precise, this identity becomes $\|f\|_{\mathcal{H}^1} \|f_\lambda\|_{\mathcal{H}^1} \lesssim \|f\|_{\mathcal{H}^1}$ if we use the norm given by (9), with an implied constant independent of f and λ). Furthermore, Calderón-Zygmund operators map $\mathcal{H}^1(\mathbb{R}^n)$ into itself: this property holds also for $L^p(\mathbb{R}^n)$ with $1 < p < \infty$, but it dramatically fails for $L^1(\mathbb{R}^n)$.

Let us mention that, as another confirmation of the appropriateness of Hardy spaces, if one carries over the theory into the general case of $\mathcal{H}^p(\mathbb{R}^n)$ spaces then, for $1 < p < \infty$, they collapse to $L^p(\mathbb{R}^n)$ (for which many important results in harmonic analysis already hold).

Theorem 9.10. *Let $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a Calderón-Zygmund kernel, i.e. a measurable function satisfying (for some finite constants $A, B > 0$)*

- $|K|(x) \le A|x|^{-n}$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$
- $\int_{|x|>2|y|} |K(x-y) - K(x)|\, dx^n \le B$ *for all* $y \in \mathbb{R}^n$
- $\int_{r<|x|<R} K(x)\, dx^n = 0$ *for any* $0 < r < R < +\infty$.

Let $K_\epsilon := K\mathbf{1}_{\mathbb{R}^n \setminus B_\epsilon(0)}$. Then, for any $f \in \mathcal{H}^1(\mathbb{R}^n)$, $K_\epsilon * f \in L^1(\mathbb{R}^n)$ and the limit

$$K * f := \lim_{\epsilon \to 0} K_\epsilon * f$$

exists in $L^1(\mathbb{R}^n)$. We have the estimate

$$\|K * f\|_{L^1} \le C(n, A, B)\, \|f\|_{\mathcal{H}^1}.$$

Proof. Recall that $K_\epsilon \in L^2$ still satisfies the above conditions (with B possibly replaced by $C(n)B$) and that $\left\|\widehat{K}_\epsilon\right\|_{L^\infty} \le C(n, A, B)$. Fix any $f \in \mathcal{H}^1(\mathbb{R}^n)$: by the characterization involving the atomic decomposition, we can find $\lambda_k \ge 0$ and ∞-atoms a_k with $f = \sum_k \lambda_k a_k$ and $\sum_k \lambda_k \lesssim \|f\|_{\mathcal{H}^1}$.

It suffices to prove the thesis for ∞-atoms: once this is done, for any $\epsilon > 0$

$$\|K_\epsilon * f\|_{L^1} \le \sum_k \lambda_k \|K_\epsilon * a_k\|_{L^1} \lesssim \sum_k \lambda_k \lesssim \|f\|_{\mathcal{H}^1}.$$

Moreover, $(K_\epsilon * f)$ is Cauchy in $\mathcal{H}^1(\mathbb{R}^n)$ as $\epsilon \to 0$: indeed, for an arbitrary N,

$$\begin{aligned}
&\limsup_{\epsilon, \epsilon' \to 0} \|K_\epsilon * f - K_{\epsilon'} * f\|_{L^1} \\
&\le \limsup_{\epsilon, \epsilon' \to 0} \sum_{k \le N} \lambda_k \|K_\epsilon * a_k - K_{\epsilon'} * a_k\|_{L^1} + \limsup_{\epsilon, \epsilon' \to 0} \sum_{k > N} \lambda_k \|K_\epsilon * a_k - K_{\epsilon'} * a_k\|_{L^1} \\
&\le 0 + \limsup_{\epsilon, \epsilon' \to 0} \sum_{k > N} \lambda_k \left(\|K_\epsilon * a_k\|_{L^1} + \|K_{\epsilon'} * a_k\|_{L^1}\right) \lesssim \sum_{k > N} \lambda_k,
\end{aligned}$$

which can be made arbitrarily small by letting $N \to +\infty$. Thus $K_\epsilon * f$ converges in $L^1(\mathbb{R}^n)$ and the limit satisfies the same estimate.

Let now a be an ∞-atom supported in $\overline{B}_R(x_0)$. Recall that

$$\|K_\epsilon * a\|_{L^2} \le C(n, A, B)\, \|a\|_{L^2} \le C(n, A, B)$$

and that $\lim_{\epsilon \to 0} K_\epsilon * a$ exists in L^2. Using Hölder's inequality we infer that $(K_\epsilon * a)\mathbf{1}_{\overline{B}_{2R}(x_0)}$ satisfies

$$\left\|(K_\epsilon * a)\mathbf{1}_{\overline{B}_{2R}(x_0)}\right\|_{L^1} \le C(n, A, B)$$

and converges in L^1 as $\epsilon \to 0$. Moreover, using the mean-zero property of a, for any $x \in \mathbb{R}^n \setminus \overline{B}_{2R}(x_0)$ we can write

$$K_\epsilon * a(x) = \int_{\mathbb{R}^n} \left(K_\epsilon(x-y) - K_\epsilon(x-x_0)\right)\, a(y)\, dy^n,$$

so that

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_{2R}(x_0)} |K_\epsilon * a|(x)\,dx^n \le & \int_{\mathbb{R}^n\setminus B_{2R}(x_0)}\int_{\mathbb{R}^n} |K_\epsilon(x-y)-K_\epsilon(x-x_0)|\,|a(y)|\,dy^n\,dx^n \\ = & \int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B_{2R}(x_0)} |K_\epsilon(x-y)-K_\epsilon(x-x_0)|\,|a(y)|\,dx^n\,dy^n \\ \le & B\,\|a\|_{L^1} \le B.\end{aligned}$$

Adding this to the preceding inequality we deduce that $\|K_\epsilon * a\|_{L^1} \le C(n,A,B)$. Finally, $(K_\epsilon * a)\mathbf{1}_{\mathbb{R}^n\setminus\overline{B}_{2R}(x_0)}$ is Cauchy in L^1 as well, since

$$(K_\epsilon * a)\mathbf{1}_{\mathbb{R}^n\setminus\overline{B}_{2R}(x_0)} = (K_{\epsilon'} * a)\mathbf{1}_{\mathbb{R}^n\setminus\overline{B}_{2R}(x_0)}$$

whenever $\epsilon,\epsilon' \le R$. □

The multiplier version of Calderón-Zygmund theorem holds as well, with the following statement.

Theorem 9.11. *Assume $m \in C^\infty(\mathbb{R}^n\setminus\{0\})$ satisfies*

$$\sup_{\xi\in\mathbb{R}^n\setminus\{0\}} |\xi|^{|\alpha|}\left|\frac{\partial^{|\alpha|}m}{\partial\xi^\alpha}(\xi)\right| < +\infty$$

for any $\alpha\in\mathbb{N}^n$. Then, for any $f\in\mathcal{H}^1(\mathbb{R}^n)$, the distribution $m\widehat{f}\in L^\infty(\mathbb{R}^n)$ lies in $\mathcal{F}(L^1(\mathbb{R}^n))$ and

$$\left\|\mathcal{F}^{-1}(m\widehat{f})\right\|_{L^1} \lesssim \|f\|_{\mathcal{H}^1}.$$

Proof. Take an atomic decomposition $f=\sum\lambda_k a_k$ as in the preceding proof and fix a dyadic partition of unity $(\psi_\ell)_{\ell\in\mathbb{Z}}$ in $\mathbb{R}^n\setminus\{0\}$. Recall that the kernels $K_N := \mathcal{F}^{-1}\left(\sum_{\ell=-N}^N \psi_\ell m\right)\in\mathcal{S}(\mathbb{R}^n)$ satisfy the Hörmander condition for some constant B independent of N and have equibounded Fourier transforms. Thus we can argue as in the previous proof (without the need of truncating the kernel K_N, since it is a Schwartz function) and we get

$$\|K_N * a_k\|_{L^1} \lesssim 1.$$

But, by Plancherel's theorem, $K_N * a_k \to \mathcal{F}^{-1}(m\widehat{a}_k)$ in $L^2(\mathbb{R}^n)$ as $N\to\infty$, thus

$$\mathcal{F}^{-1}(m\widehat{a}_k)\in L^1(\mathbb{R}^n) \text{ and } \left\|\mathcal{F}^{-1}(m\widehat{a}_k)\right\|_{L^1}\lesssim 1$$

(by Fatou's lemma, since a subsequence $K_{N_j} * a_k$ converges a.e. to $\mathcal{F}^{-1}(m\widehat{a}_k)$). Thus the limit

$$g := \sum_k \lambda_k\mathcal{F}^{-1}(m\widehat{a}_k)$$

exists in $L^1(\mathbb{R}^n)$ and satisfies $\|g\|_{L^1}\lesssim\|f\|_{\mathcal{H}^1}$, as well as $\widehat{g}=\sum_k\lambda_k(m\widehat{a}_k)=m\widehat{f}$. □

Actually, in the preceding theorems we can easily upgrade the $\mathcal{H}^1\to L^1$ boundedness to $\mathcal{H}^1\to\mathcal{H}^1$.

Corollary 9.12. *Under the hypotheses of Theorem 9.11, for any $f \in \mathcal{H}^1(\mathbb{R}^n)$ we have*

$$\mathcal{F}^{-1}m\widehat{f} \in \mathcal{H}^1(\mathbb{R}^n), \qquad \left\|\mathcal{F}^{-1}m\widehat{f}\right\|_{\mathcal{H}^1} \lesssim \|f|_{\mathcal{H}^1}.$$

Proof. By the characterization of $\mathcal{H}^1(\mathbb{R}^n)$ using Riesz transforms, it suffices to show that $\mathcal{R}_j\mathcal{F}^{-1}(m\widehat{f}) \in L^1(\mathbb{R}^n)$ with an estimate on its L^1-norm (for any $1 \leq j \leq n$). But

$$\mathcal{R}_j\mathcal{F}^{-1}(m\widehat{f}) = \mathcal{F}^{-1}\left(-i\frac{\xi_j}{|\xi|}m(\xi)\widehat{f}(\xi)\right)$$

and the multiplier still satisfies the hypotheses of Theorem 9.11. □

Corollary 9.13. *Under the hypotheses of Theorem 9.10, for any $f \in \mathcal{H}^1(\mathbb{R}^n)$ we have $K_\epsilon * f \in \mathcal{H}^1(\mathbb{R}^n)$ and the limit*

$$K * f := \lim_{\epsilon\to 0} K_\epsilon * f$$

exists in $\mathcal{H}^1(\mathbb{R}^n)$, with the estimate

$$\|K * f\|_{\mathcal{H}^1} \leq C(n, A, B)\,\|f\|_{\mathcal{H}^1}.$$

Proof. From Corollary 9.12 we know that, for any $1 \leq j \leq n$, $\mathcal{R}_jf \in \mathcal{H}^1(\mathbb{R}^n)$ with $\|\mathcal{R}_jf\|_{\mathcal{H}^1} \lesssim \|f\|_{\mathcal{H}^1}$. Moreover,

$$\mathcal{R}_j(K_\epsilon * f) = \mathcal{F}^{-1}\left(-i\frac{\xi_j}{|\xi|}\widehat{K}_\epsilon(\xi)\widehat{f}(\xi)\right) = K_\epsilon * (\mathcal{R}_jf),$$

so, by the conclusion of Theorem 9.10, $(\mathcal{R}_j(K_\epsilon * f))$ is Cauchy as $\epsilon \to 0$. As a consequence, $(K_\epsilon * f)$ is Cauchy in $\mathcal{H}^1(\mathbb{R}^n)$. □

9.2.4 Equivalence of some maximal functions

The goal of this section is to prove the equivalence among the norms defined by (1), (2), (3) and (4).

Trivially, we have

$$\mathcal{M}^v_\varphi f \lesssim \mathcal{G}\mathcal{M}f$$

pointwise (with the implied constant depending only on φ), so $\left\|\mathcal{M}^v_\varphi f \lesssim \mathcal{G}\mathcal{M}f\right\|_{L^1} \lesssim \|\mathcal{G}\mathcal{M}f\|_{L^1}$ and (4) $\Rightarrow$ (1) hold as well.

We also remark the following pointwise inequalities:

$$\mathcal{M}^v_\varphi f \leq \mathcal{M}^c_\varphi f \leq 2^{n+1}\mathcal{M}^t_\varphi f$$

pointwise (the second inequality follows from the fact that $2^{n+1}\left(1 + \frac{|y-x|}{t}\right)^{-n-1} \geq 1$ whenever $y \in B_t(x)$).

Let us now turn to the first nontrivial inequality, namely the fact that $\left\|\mathcal{M}^t_\varphi f\right\|_{L^1} \lesssim \left\|\mathcal{M}^c_\varphi f\right\|_{L^1}$, which will give (2) $\Rightarrow$ (3).

Lemma 9.14. *For any $x \in \mathbb{R}^n$ we have*

$$\mathcal{M}^t_\varphi f(x) \leq \left(M\left|\mathcal{M}^c_\varphi f\right|^{n/(n+1)}\right)^{(n+1)/n}(x).$$

Proof. The key observation is the fact that $|\varphi_t * f|(y) \leq \mathcal{M}_\varphi^c f(z)$ whenever $z \in B_t(y)$ (since $z \in B_t(y)$ is equivalent to $y \in B_t(z)$). From this it follows that

$$\begin{aligned}|\varphi_t * f|^{n/(n+1)}(y) \leq & \frac{1}{|B_t(y)|}\int_{B_t(y)} (\mathcal{M}_\varphi^c f)^{n/(n+1)}(z)\,dz \\ \leq & \frac{|B_{t+|y-x|}(x)|}{|B_t(y)|} \fint_{B_{t+|y-x|}(x)} (\mathcal{M}_\varphi^c f)^{n/(n+1)}(z)\,dz,\end{aligned}$$

which gives

$$|\varphi_t * f(y)|^{n/(n+1)}\left(1+\frac{|y-x|}{t}\right)^{-n} \leq M\left|\mathcal{M}_\varphi^c f\right|^{n/(n+1)}(x).$$

Raising both sides to the power $\frac{n+1}{n}$ we obtain the thesis. $\square$

Corollary 9.15. *Using the $L^{(n+1)/n}$-boundedness of the Hardy-Littlewood maximal function, we deduce*

$$\left\|\mathcal{M}_\varphi^t f\right\|_{L^1} \leq \left\|M\left|\mathcal{M}_\varphi^c f\right|^{n/(n+1)}\right\|_{L^{(n+1)/n}}^{(n+1)/n} \lesssim \left\|\mathcal{M}_\varphi^c f\right\|_{L^1}.$$

Now we prove that the grand maximal function $\mathcal{GM}f$ is controlled pointwise by $\mathcal{M}_\varphi^t f$, which will trivially give (3) $\Rightarrow$ (4) and $\|\mathcal{GM}f\|_{L^1} \lesssim \left\|\mathcal{M}_\varphi^t f\right\|_{L^1}$. The choice of the seminorm $\|\cdot\|_N$ will be specified by the proof of the next lemma, which roughly says that every $\psi \in \mathcal{S}(\mathbb{R}^n)$ is a superposition of dilations of φ.

Lemma 9.16. *Any $\psi \in \mathcal{S}(\mathbb{R}^n)$ can be written as a series*

$$\psi = \sum_{k=0}^{\infty} \eta^{(k)} * \varphi_{2^{-k}}$$

converging in $\mathcal{S}(\mathbb{R}^n)$, where the functions $\eta^{(k)} \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$\int (1+|y|)^{2(n+1)}\left|\eta^{(k)}\right|(y)\,dy \lesssim 2^{-k(n+2)}\|\psi\|_N$$

for a suitable seminorm $\|\cdot\|_N$ depending only on n (while the implied constant depends also on φ).

Proof. Let $(\rho_k)_{k\in\mathbb{N}}$ be a (inhomogeneous) dyadic partition of unity in $\mathbb{R}^n$, which can be obtained by taking $\rho_0 \in C_c^\infty(B_2)$, $\rho_0 \equiv 1$ in a neighbourhood of $\overline{B}_1$ and letting $\rho_k := \rho_0(2^{-k}\cdot) - \rho_0(2^{-(k-1)}\cdot)$ for $k > 0$ (so that, for $k > 0$, ρ_k is supported in the open annulus $B_{2^{k+1}} \setminus \overline{B}_{2^{k-1}}$).

Since $\int \varphi(x)\,dx \neq 0$, we have $\widehat{\varphi}(0) \neq 0$. By continuity we can find $k_0 \geq 0$ such that $\widehat{\varphi}(\xi) \neq 0$ for all $\xi \in \overline{B}_{2^{1-k_0}}$. For $k \geq k_0$ let $\eta^{(k)} \in \mathcal{S}(\mathbb{R}^n)$ be defined by

$$\widehat{\eta^{(k)}} := \frac{\rho_{k-k_0}\widehat{\psi}}{\widehat{\varphi}(2^{-k}\cdot)}$$

(notice that the right-hand side makes sense on $\overline{B}_{2^{k-k_0+1}}$ and vanishes near the boundary of this ball, so it can be smoothly extended by 0 on the complement). Let $\eta^{(k)} := 0$ for $k < k_0$. The series

$$\sum_{k=0}^{\infty} \widehat{\eta^{(k)}}\widehat{\varphi}(2^{-k}) = \sum_{k=k_0}^{\infty} \rho_{k-k_0}\widehat{\psi}$$

converges to $\widehat{\psi}$ in $\mathcal{S}(\mathbb{R}^n)$, so (by the continuity of $\mathcal{F}^{-1}$) we also have

$$\sum_{k=0}^{\infty} \eta^{(k)} * \varphi_{2^{-k}} = \psi$$

in $\mathcal{S}(\mathbb{R}^n)$. We can find a seminorm $\|\cdot\|_{N''}$ such that $\int (1+|y|)^{2(n+1)} |\eta| (y)\, dy \lesssim \|\widehat{\eta}\|_{N''}$, so that for $k \geq k_0$

$$\int (1 + |y|)^{2(n+1)} \left|\eta^{(k)}\right| (y)\, dy \lesssim \left\| \frac{\rho_{k-k_0}\widehat{\psi}}{\widehat{\varphi}(2^{-k}\cdot)} \right\|_{N''}.$$

Using the Leibniz rule it is easy to see that, for a suitable bigger seminorm $\|\cdot\|_{N'}$ independent of φ,

$$\left\| \frac{\rho_{k-k_0}\widehat{\psi}}{\widehat{\varphi}(2^{-k}\cdot)} \right\|_{N''} \lesssim 2^{-k(n+2)} \left\|\widehat{\psi}\right\|_{N'}$$

(the implied constant, however, will depend on φ and k_0, i.e. on φ). We can finally find $\|\cdot\|_N$ such that $\left\|\widehat{\psi}\right\|_{N'} \lesssim \|\psi\|_N$. □

Corollary 9.17. *For any $x \in \mathbb{R}^n$ we have $\mathcal{GM}f(x) \lesssim \mathcal{M}^t_\varphi f(x)$.*

Proof. Fix $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\|\psi\|_N \leq 1$. Since $\psi_t = \sum_{k=0}^{\infty} \varphi_{2^{-k}t} * (\eta^{(k)})_t$,

$$\begin{aligned}
|\psi_t * f| (x) \leq & \sum_{k=0}^{\infty} \left|\varphi_{2^{-k}t} * (\eta^{(k)})_t * f\right| (x) \\
\leq & \sum_{k=0}^{\infty} \int |\varphi_{2^{-k}t} * f| (x-y)\ t^{-n} \left|\eta^{(k)}\right| (t^{-1}y)\, dy \\
\leq & \mathcal{M}^t_\varphi f(x) \sum_{k=0}^{\infty} \int \left(1 + \frac{|y|}{2^{-k}t}\right)^{n+1} t^{-n} \left|\eta^{(k)}\right| (t^{-1}y)\, dy.
\end{aligned}$$

But the last integral is bounded by

$$2^{k(n+1)}\int \left(1 + \frac{|y|}{t}\right)^{n+1} t^{-n} \left|\eta^{(k)}\right| (t^{-1}y)\, dy = 2^{k(n+1)}\int (1 + |y|)^{n+1} \left|\eta^{(k)}\right| (y)\, dy \lesssim 2^{-k}$$

for all $k \geq 0$. So we obtain $|\psi_t * f| (x) \lesssim \mathcal{M}^t_\varphi f(x)$ and the thesis follows by taking the supremum over $t > 0$ and over $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\|\psi\|_N \leq 1$. □

In order to prove the implication (1) ⇒ (2), we need some technical preliminaries. Fix $0 < \epsilon < 1$ and define the following modified maximal functions:

$$\widetilde{\mathcal{M}}^c_\varphi f(x) := \sup_{0<t<\epsilon^{-1}, y \in B_t(x)} |\varphi_t * f|(y) \left(\frac{t}{t+\epsilon+\epsilon|y|}\right)^{n+1},$$

$$\widetilde{\mathcal{M}}^t_\varphi f(x) := \sup_{0<t<\epsilon^{-1}, y \in \mathbb{R}^n} |\varphi_t * f|(y) \left(1+\frac{|y-x|}{t}\right)^{-n-1} \left(\frac{t}{t+\epsilon+\epsilon|y|}\right)^{n+1},$$

$$\widetilde{\mathcal{GM}} f(x) := \sup\left\{ |\psi_t * f|(x) \left(\frac{t}{t+\epsilon+\epsilon|x|}\right)^{n+1} \,\middle|\, 0<t<\epsilon^{-1}, \psi \in \mathcal{S}(\mathbb{R}^n), \|\psi\|_N \le 1 \right\}.$$

Clearly $\widetilde{\mathcal{M}}^c_\varphi f$ converges to $\mathcal{M}^c_\varphi f$ pointwise from below, as $\epsilon \to 0$, and most importantly it always lies in $L^1(\mathbb{R}^n)$: from $t+\epsilon|y| \ge \epsilon t + \epsilon|y| \ge \epsilon|x|$ we infer

$$\begin{aligned} |\varphi_t * f|(y) \left(\frac{t}{t+\epsilon+\epsilon|y|}\right)^{n+1} &\le \|\varphi_t\|_{L^\infty} \|f\|_{L^1} \frac{t^{n+1}}{(\epsilon+\epsilon|x|)^{n+1}} \\ &\lesssim t^{-n} \|f\|_{L^1} t^{n+1} \epsilon^{-n-1} (1+|x|)^{-n-1} \\ &\le \epsilon^{-n-2} (1+|x|)^{-n-1} \in L^1(\mathbb{R}^n). \end{aligned}$$

Lemma 9.18. *We still have*

$$\left\|\widetilde{\mathcal{GM}} f\right\|_{L^1} \lesssim \left\|\widetilde{\mathcal{M}}^t_\varphi f\right\|_{L^1} \lesssim \left\|\widetilde{\mathcal{M}}^c_\varphi f\right\|_{L^1},$$

the implied constants being independent of ϵ and f.

Proof. The second inequality is proved exactly as we did for the original maximal functions: again we have, whenever $0 < t < \epsilon^{-1}$ and $z \in B_t(y)$,

$$|\varphi_t * f|(y) \left(\frac{t}{t+\epsilon+\epsilon|y|}\right)^{n+1} \le \widetilde{\mathcal{M}}^c_\varphi f(z)$$

and, raising this inequality to the power $\frac{n}{n+1}$, averaging as z varies in $B_t(y)$ and then raising to the power $\frac{n+1}{n}$, we get again

$$\widetilde{\mathcal{M}}^t_\varphi f(x) \le \left(M \left|\widetilde{\mathcal{M}}^c_\varphi f\right|^{n/(n+1)}\right)^{(n+1)/n} (x)$$

for any $x \in \mathbb{R}^n$, from which the second inequality follows (using the $L^{(n+1)/n}$-boundedness of the Hardy-Littlewood maximal function).

Let us turn to the first inequality. Using the decomposition $\psi_t = \sum_{k=0}^\infty \varphi_{2^{-k}t} * \left(\eta^{(k)}\right)_t$ we obtain again (for any $x \in \mathbb{R}^n$ and any $0 < t < \epsilon^{-1}$)

$$|\psi_t * f|(x) \le \sum_{k=0}^\infty \int |\varphi_{2^{-k}t} * f|(x-y)\, t^{-n} \left|\eta^{(k)}\right| (t^{-1}y)\, dy,$$

but now we estimate (using $0 < 2^{-k}t < \epsilon^{-1}$)

$$|\varphi_{2^{-k}t} * f|(x-y) \le \widetilde{\mathcal{M}}^t_\varphi f(x) \left(1+\frac{|y|}{2^{-k}t}\right)^{n+1} \left(\frac{2^{-k}t+\epsilon+\epsilon|x-y|}{t}\right)^{n+1}.$$

Inserting this into the preceding inequality and multiplying both sides by $\left(\frac{t}{t+\epsilon+\epsilon|x|}\right)^{n+1}$ we arrive at

$$|\psi_t * f|(x)\left(\frac{t}{t+\epsilon+\epsilon|x|}\right)^{n+1} \leq \widetilde{\mathcal{M}}^t_\varphi f(x)\sum_{k=0}^{\infty} I_k,$$

where I_k denotes the following integral:

$$I_k := \int \left(1+\frac{|y|}{2^{-k}t}\right)^{n+1}\left(\frac{2^{-k}t+\epsilon+\epsilon|x-y|}{t+\epsilon+\epsilon|x|}\right)^{n+1} t^{-n}\left|\eta^{(k)}\right|(t^{-1}y)\,dy.$$

The second factor in the definition of I_k is bounded by

$$\left(\frac{t+\epsilon+\epsilon|x|+\epsilon|y|}{t+\epsilon+\epsilon|x|}\right)^{n+1} = \left(1+\frac{\epsilon|y|}{t+\epsilon+\epsilon|x|}\right)^{n+1} \leq \left(1+\frac{|y|}{t}\right)^{n+1},$$

where we used the assumption $\epsilon < 1$, while the first factor is again bounded by $2^{k(n+1)}\left(1+\frac{|y|}{t}\right)^{n+1}$. Thus,

$$\begin{aligned} I_k \leq & 2^{k(n+1)}\int\left(1+\frac{|y|}{t}\right)^{2(n+1)} t^{-n}\left|\eta^{(k)}\right|(t^{-1}y)\,dy \\ = & 2^{k(n+1)}\int(1+|y|)^{2(n+1)}\left|\eta^{(k)}\right|(y)\,dy \lesssim 2^{-k}, \end{aligned}$$

in view of the statement of Lemma 9.16. So we get

$$|\psi_t * f|(x)\left(\frac{t}{t+\epsilon+\epsilon|x|}\right)^{n+1} \lesssim \widetilde{\mathcal{M}}^t_\varphi f(x)$$

and taking the supremum over $0 < t < \epsilon^{-1}$ we obtain the pointwise bound $\widetilde{\mathcal{GM}}f(x) \lesssim \widetilde{\mathcal{M}}^t_\varphi f(x)$, from which we infer the first inequality of the thesis. □

Theorem 9.19. *For any $0 < \epsilon < 1$ we have $\left\|\widetilde{\mathcal{M}}^c_\varphi f\right\|_{L^1} \lesssim \left\|\mathcal{M}^v_\varphi f\right\|_{L^1}$ (the implied constant is independent of ϵ).*

Proof. We claim that it suffices to bound the integral $\int_E \widetilde{\mathcal{M}}^c_\varphi f(x)\,dx$ on the 'bad' set

$$E := \left\{\widetilde{\mathcal{GM}}f \leq \lambda\widetilde{\mathcal{M}}^c_\varphi f\right\},$$

for some large enough λ. Indeed,

$$\int_{\mathbb{R}^n\setminus E}\widetilde{\mathcal{M}}^c_\varphi f(x)\,dx \leq \lambda^{-1}\int_{\mathbb{R}^n\setminus E}\widetilde{\mathcal{GM}}f(x)\,dx \leq C\lambda^{-1}\int\widetilde{\mathcal{M}}^c_\varphi f(x)\,dx,$$

since by the preceding lemma $\left\|\widetilde{\mathcal{GM}}f\right\|_{L^1} \leq C\left\|\widetilde{\mathcal{M}}^c_\varphi f\right\|_{L^1}$ (for some C depending only on n and φ). Choosing $\lambda := 2C$ we get

$$\int_{\mathbb{R}^n\setminus E}\widetilde{\mathcal{M}}^c_\varphi f(x)\,dx \leq \frac{1}{2}\int\widetilde{\mathcal{M}}^c_\varphi f(x)\,dx.$$

We can now subtract the finite quantity $\frac{1}{2}\int_{\mathbb{R}^n\setminus E}\widetilde{\mathcal{M}}^c_\varphi f(x)\,dx$ from both sides (this step is the reason why we needed to introduce these modified maximal functions: the same integral with $\mathcal{M}^c_\varphi f$ could a priori be infinite) and obtain

$$\frac{1}{2}\int_{\mathbb{R}^n\setminus E}\widetilde{\mathcal{M}}^c_\varphi f(x)\,dx \leq \frac{1}{2}\int_E \widetilde{\mathcal{M}}^c_\varphi f(x)\,dx,$$

so that

$$\int \widetilde{\mathcal{M}}^c_\varphi f(x)\,dx = \int_{\mathbb{R}^n\setminus E}\widetilde{\mathcal{M}}^c_\varphi f(x)\,dx + \int_E \widetilde{\mathcal{M}}^c_\varphi f(x)\,dx \leq 2\int_E \widetilde{\mathcal{M}}^c_\varphi f(x)\,dx.$$

Fix now $x \in E$ and let (y,t) such that $0 < t < \epsilon^{-1}$, $y \in B_t(x)$ and

$$|\varphi_t * f|(y)\left(\frac{t}{t+\epsilon+\epsilon|y|}\right)^{n+1} \geq \frac{1}{2}\widetilde{\mathcal{M}}^c_\varphi f(x).$$

We aim at showing that the same inequality holds, with $\frac{1}{4}$ in place of $\frac{1}{2}$, for all y' in a small ball $B_{\eta t}(y)$ ($0 < \eta < 1$ will be specified later). Once this is achieved, we will have

$$\begin{aligned}\frac{1}{4}\widetilde{\mathcal{M}}^c_\varphi f(x) &\leq \left(\fint_{B_{\eta t}}(\mathcal{M}^v_\varphi f)^{1/2}(y')\,dy'\right)^2\\ &\leq \left(\left(\frac{t+\eta t}{\eta t}\right)^n \fint_{B_{t+\eta t}(x)}(\mathcal{M}^v_\varphi f)^{1/2}(y')\,dy'\right)^2\\ &\leq \left(\frac{1+\eta}{\eta}\right)^{2n}\left(M(\mathcal{M}^v_\varphi f)^{1/2}\right)^2(x),\end{aligned}$$

from which the thesis follows as usual (integrating over E and using the L^2-boundedness of the Hardy-Littlewood maximal function).

Let $g(y') := \varphi_t * f(y')\left(\frac{t}{t+\epsilon+\epsilon|y|}\right)^{n+1}$. The function g is locally Lipschitz and is smooth on $\mathbb{R}^n \setminus \{0\}$, so for $y' \in B_{\eta t}(y)$

$$|g(y') - g(y)| \leq \eta t \sup_{z\in B_{\eta t}(y)\setminus\{0\}} |\nabla g|(z).$$

We compute

$$\nabla g(z) = t^{-1}(\nabla\varphi)_t * f(z)\left(\frac{t}{t+\epsilon+\epsilon|y|}\right)^{n+1} - \varphi_t * f(z)\frac{(n+1)t^{n+1}}{(t+\epsilon+\epsilon|y|)^{n+2}}\cdot\epsilon\frac{z}{|z|}.$$

But, writing $z = x + th$ (with $|h| < 1+\eta < 2$),

$$\varphi_t*f(z) = \int t^{-n}\varphi\left(\frac{x+th-u}{t}\right)f(u)\,du = \int t^{-n}\varphi\left(\frac{x-u}{t}+h\right)f(u)\,du = (\varphi(\cdot+h))_t*f(x)$$

and similarly $(\nabla\varphi)*f(z) = (\nabla\varphi(\cdot+h))_t*f(x)$. Assuming without loss of generality $\epsilon < \frac{1}{4}$, we also have

$$t+\epsilon+\epsilon|z| \geq t+\epsilon+\epsilon(|x|-(1+\eta)t) \geq \frac{1}{2}(t+\epsilon+\epsilon|x|)$$

(as $t - \epsilon(1+\eta)t \geq \frac{t}{2}$). Putting everything together,

$$\begin{aligned}
|\nabla g|(z) \lesssim & t^{-1}\left(|\nabla\varphi * f|(x) + |\varphi * f|(z)\frac{\epsilon t}{t+\epsilon+\epsilon|z|}\right)\left(\frac{t}{t+\epsilon+\epsilon|z|}\right)^{n+1} \\
\lesssim & t^{-1}\big(|(\nabla\varphi(\cdot+h))_t * f|(x) + |(\varphi(\cdot+h))_t * f(x)|\big)\left(\frac{t}{t+\epsilon+\epsilon|x|}\right)^{n+1} \\
\lesssim & t^{-1}\widetilde{\mathcal{GM}}f(x),
\end{aligned}$$

thanks to the fact that the quantities $\sup_{|h|<2}\|\varphi(\cdot+h)\|_N$ and $\sup_{|h|<2}\left\|\frac{\partial\varphi}{\partial x_i}(\cdot+h)\right\|_N$ are finite (for $i=1,\dots,n$). Hence,

$$|g(y') - g(y)| \leq \eta t \cdot C' t^{-1}\widetilde{\mathcal{GM}}f(x) \leq \eta C'\lambda\widetilde{\mathcal{M}}^c_\varphi f(x)$$

(for some C' depending only on n and φ), as $x \in E$. Choosing $\eta := \min\left(\frac{1}{2}, \frac{1}{4C'\lambda}\right)$ we arrive at

$$g(y') \geq g(y) - |g(y) - g(y')| \geq \frac{1}{2}\widetilde{\mathcal{M}}^c_\varphi f(x) - \frac{1}{4}\widetilde{\mathcal{M}}^c_\varphi f(x) = \frac{1}{4}\widetilde{\mathcal{M}}^c_\varphi f(x),$$

which is what we wanted to obtain. □

9.2.5 Further remarks

We collect in this section the proofs of some easier parts of Theorem 9.2. By what we proved in the previous section, given $\varphi' \in \mathcal{S}(\mathbb{R}^n)(\mathbb{R}^n)$ with $\int \varphi'(x)\,dx \neq 0$, we have

$$\left\|\mathcal{M}^v_{\varphi'} f\right\|_{L^1} \lesssim \|\mathcal{GM}f\|_{L^1} \lesssim \|\mathcal{M}_\varphi f\|_{L^1}$$

and similarly $\left\|\mathcal{M}^v_\varphi f\right\|_{L^1} \lesssim \left\|\mathcal{M}^v_{\varphi'} f\right\|_{L^1}$. So $\mathcal{M}^v_\varphi f$ and $\mathcal{M}^v_{\varphi'} f$ have comparable L^1-norms. This shows that, in order to prove (5) ⇒ (1) and (7) ⇒ (1), we are free to choose φ at will (provided it satisfies $\int \varphi(x)\,dx \neq 0$).

Proof of (5) ⇒ (1). As just remarked, we can assume $\varphi \in C^\infty_c(B_1)$ and $\|\nabla\varphi\|_{L^\infty} \leq 1$. The thesis follows from the trivial pointwise inequality $\mathcal{M}^v_\varphi f \leq \mathcal{GM}'f$. □

Proof of (7) ⇒ (1). First of all, we claim that there exists a continuous function $\rho : [1,+\infty) \to \mathbb{R}$ such that ρ is rapidly decreasing at infinity (i.e. $\sup_t t^k|\rho|(t) < +\infty$ for every $k \geq 0$) and

$$\int_1^\infty \rho(t)\,dt = 1, \quad \int_1^{+\infty} t^k\rho(t)\,dt = 0 \text{ for } k = 1,2,\dots$$

(these integrals make sense by the rapid decrease assumption on ρ).

An explicit example is the following:

$$\rho(t) := \frac{e}{\pi t}\Im\left(\exp\left(e^{3\pi i/4}(t-1)^{1/4}\right)\right).$$

The rapid decrease at infinity is clear since $|\rho|(t) \leq \frac{e}{\pi t} \exp\left(\Re\left(e^{3\pi i/4}(t-1)^{1/4}\right)\right) = \frac{e}{\pi t} \exp\left(-\frac{1}{\sqrt{2}}(t-1)^{1/4}\right)$. Let

$$g : \Omega := \mathbb{C} \setminus \{t \in \mathbb{R}, t \geq 1\} \to \mathbb{C}, \quad g(z) := \frac{e}{\pi} \exp\left(e^{3\pi i/4}(t-1)^{1/4}\right),$$

where $(z-1)^{1/4}$ means the unique holomorphic function $h : \Omega \to \mathbb{C}$ such that $h^4(z) = z - 1$ and $\lim_{\epsilon \to 0^+} h(t + \epsilon i) = (t-1)^{1/4}$ for every real $t > 1$. We remark that $z \mapsto e^{3\pi i/4}(t-1)^{1/4}$ maps Ω into $\left\{re^{i\theta} \mid r > 0, \theta \in \left(\frac{3}{4}\pi, \frac{5}{4}\pi\right\}\right)$ and so

$$|g|(z) \leq \frac{e}{\pi} \exp\left(\Re\left(e^{3\pi i/4}(z-1)^{1/4}\right)\right) \leq \frac{e}{\pi} \exp\left(-\frac{1}{\sqrt{2}}|z-1|^{1/4}\right) \lesssim |z|^{-k}.$$

Let γ be the loop (in Ω) obtained by concatenating the parametrized paths

$$t + \epsilon i \ (t \in [1, R]), \quad \sqrt{R^2 + \epsilon^2} e^{i\alpha} \ (\alpha \in [\alpha_0, 2\pi - \alpha_0]),$$
$$R - t - \epsilon i \ (t \in [0, R-1]), \quad 1 + \epsilon e^{-i\alpha} \ (\alpha \in [\frac{\pi}{2}, \frac{3\pi}{2}]),$$

for arbitrary $\epsilon, R > 0$. By Cauchy's theorem we have $\int_\gamma z^{k-1} g(z)\, dz = 0$ for $k = 1, 2, \ldots$ and

$$\int_\gamma z^{-1} g(z)\, dz = 2\pi i g(0) = 2ei \exp\left(e^{3\pi i/4} e^{\pi i/4}\right) = 2i.$$

Taking the imaginary part of both identities, sending $\epsilon \to 0$ and then $R \to \infty$ (and noticing that the contributions of the two circular arcs are infinitesimal), we get precisely

$$2\int_1^\infty t^k \rho(t)\, dt = 0 \text{ for } k = 1, 2, \ldots, \quad 2\int_1^\infty \rho(t)\, dt = 2,$$

which is the claim.

We now build a Schwartz function out of the Poisson kernel: let

$$\varphi(x) := \int_1^\infty \rho(t) P_t(x)\, dt.$$

This integral converges (as $|P_t|(x) = t^{-n} P(t^{-1}x) \leq t^{-n} P(0) \leq P(0)$) and defines a function in $L^1(\mathbb{R}^n)$, since

$$\int_{\mathbb{R}^n} |\varphi|(x)\, dx \leq \int_1^\infty \int_{\mathbb{R}^n} |\rho|(t) P_t(x)\, dx\, dt = \int_1^\infty |\rho|(t)\, dt < +\infty.$$

Moreover, using Fubini's theorem, $\widehat{\varphi}(\xi) = \int_1^\infty \rho(t) e^{-t|\xi|}\, dt$. It is easy to show inductively that, for $\xi \neq 0$ and any multiindex $\alpha \neq 0$,

$$\frac{\partial^{|\alpha|} \widehat{\varphi}(\xi)}{\partial \xi^\alpha} = \int_1^\infty \rho(t) \cdot t\, Q_\alpha(t, \xi, |\xi|^{-1}) e^{-t|\xi|}\, dt$$

for a suitable polynomial Q_α. In particular, $\widehat{\varphi}$ is smooth on $\mathbb{R}^n \setminus \{0\}$ and all its derivatives are rapidly decreasing at infinity. Moreover, $\widehat{\varphi}$ is clearly continuous. Given any $\alpha \neq 0$, we write

$$e^{-s} = \sum_{k<K} \frac{(-s)^k}{k!} + R_K(s)$$

and notice that $s^{-K}\left|R_K\right|(s)$ is bounded for $s \in \mathbb{R} \setminus \{0\}$ close to the origin, while it is infinitesimal as $|s| \to \infty$; thus $\left|R_K\right|(s) \lesssim |s|^K$. This implies

$$\frac{\partial^{|\alpha|}\widehat{\varphi}(\xi)}{\partial \xi^\alpha} = \int_1^\infty \rho(t) \cdot t\, Q_\alpha(t, \xi, |\xi|^{-1}) \left(\sum_{k<K} \frac{(-t\,|\xi|)^k}{k!} + R_K(t\,|\xi|) \right) dt.$$

Calling d and d' the degrees of Q_α with respect to its first and last argument, respectively, we obtain that for every $K > d'$

$$\int_1^\infty \left|\rho(t) \cdot t\, Q_\alpha(t, \xi, |\xi|^{-1})\right| \left|R_K\right|(t\,|\xi|)\, dt \lesssim \int_1^\infty t^{1+d+K}\, |\xi|\, dt \lesssim |\xi|$$

(whenever $0 < |\xi| \leq 1$), while

$$\int_1^\infty \rho(t) \cdot t\, Q_\alpha(t, \xi, |\xi|^{-1}) \sum_{k<K} \frac{(-t\,|\xi|)^k}{k!}\, dt = 0$$

by the special properties satisfied by η. This shows that all the derivatives of $\widehat{\varphi}$ extend continuously up to the origin, hence $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ and we deduce that $\varphi \in \mathcal{S}(\mathbb{R}^n)$, as well. Finally, $\int_{\mathbb{R}^n} \varphi(x)\, dx = \int_1^\infty \int_{\mathbb{R}^n} \rho(t) P_t(x)\, dx = \int_1^\infty \rho(t)\, dt = 1$ and

$$\mathcal{M}^v_\varphi f(x) \leq \int_1^\infty |\rho|\,(t) \mathcal{M}^v_P f(x)\, dt \lesssim \mathcal{M}^v_P f(x)$$

for any $f \in L^1(\mathbb{R}^n)$, showing that (7) $\Rightarrow$ (1) for this particular φ. $\square$

Proof of (6) $\Rightarrow$ (5). Let $\psi \in C_c^\infty(B_1(0))$ with $\|\nabla\psi\|_{L^\infty} \leq 1$. Given an ∞-atom supported in $B_r(x_0)$, we have

$$|\psi_t * a|\,(x) \leq \|\psi_t\|_{L^1} \|a\|_{L^\infty} = \|\psi\|_{L^1} \|a\|_{L^\infty} \lesssim \|a\|_{L^\infty}$$

for any $x \in B_{2r}(x_0)$. Fix now $x \in \mathbb{R}^n \setminus B_{2r}(x_0)$ and notice that $\psi_t * a(x) = 0$ if $t < |x - x_0| - r$ (since in this case $\psi_t(x - \cdot)$ and a are supported in the disjoint balls $\overline{B}_t(x)$ and $\overline{B}_r(x_0)$). Assume instead that $t \geq |x - x_0| - r$: in this case we get $t \geq \frac{|x - x_0|}{2}$, so

$$\begin{aligned} |\psi_t * a|\,(x) &\leq \int |\psi_t(x-y) - \psi_t(x - x_0)|\, |a|\,(y)\, dy \\ &\leq r\, \|\nabla\psi_t\|_{L^\infty} \|a\|_{L^1} \\ &\lesssim r t^{-n-1} \lesssim \frac{r}{|x - x_0|^{n+1}} \end{aligned}$$

(as $\nabla\psi_t(x) = t^{-n-1}\nabla\psi(t^{-1}x)$). Thus,

$$\begin{aligned}\|\mathcal{GM}'f\|_{L^1} &\lesssim \int_{B_{2r}(x_0)} \|a\|_{L^\infty}\, dx + \int_{\mathbb{R}^n\setminus B_{2r}(x_0)} \frac{r}{|x-x_0|^{n+1}}\, dx \\ &\lesssim \|a|_{L^\infty} |B_r(x_0)| + r\int_{2r}^{\infty} \frac{\rho^{n-1}\, d\rho}{\rho^{n+1}} \lesssim 1.\end{aligned}$$

Hence, if $f = \sum_k \lambda_k a_k$ is an atomic decomposition,

$$\|\mathcal{GM}'f\|_{L^1} \le \left\|\sum_k \lambda_k \mathcal{GM}' a_k\right\|_{L^1} \lesssim \sum_k \lambda_k.$$

□

Proof of (6) ⇒ (7). The proof of Proposition 9.6 can be repeated verbatim, with φ replaced by P, to show that

$$\|\mathcal{M}_P^v a\|_{L^1} \lesssim 1$$

for any ∞-atom a. Hence, if $f = \sum_k \lambda_k a_k$ is an atomic decomposition,

$$\|\mathcal{M}_P^v f\|_{L^1} \le \left\|\sum_k \lambda_k \mathcal{M}_P^v a_k\right\|_{L^1} \lesssim \sum_k \lambda_k.$$

□

Proof of (6) ⇒ (8). It suffices to notice that the proof of Theorem 9.11 used only the atomic decomposition of f. So, choosing $m(\xi) := -i\frac{\xi_j}{|\xi|}$, we deduce

$$\|\mathcal{R}_j f\|_{L^1} \lesssim \inf \sum \lambda_k$$

(the infimum ranging over all the possible atomic decompositions). Moreover, for any decomposition

$$\|f\|_{L^1} \le \sum_k \lambda_k \|a_k\|_{L^1} \le \sum_k \lambda_k.$$

Thus, $\|f\|_{L^1} + \sum_{j=1}^n \|\mathcal{R}_j f\|_{L^1} \lesssim \inf \sum_k \lambda_k$. □

9.2.6 Characterization with the Riesz transforms

We now show the implication (8) ⇒ (7) among the equivalent definitions of $\mathcal{H}^1(\mathbb{R}^n)$. The proof will implicitly show the corresponding inequality on the norms, namely

$$\|\mathcal{M}_P^v f\|_{L^1} \lesssim \|f\|_{L^1} + \|\mathcal{R}_1 f\|_{L^1} + \cdots + \|\mathcal{R}_n f\|_{L^1}.$$

Assume that f and all its Riesz transforms are in $L^1(\mathbb{R}^n)$. So far we have tacitly allowed any function to be either real or complex valued, but now it is convenient to assume f real valued (without loss of generality, as $\mathcal{R}_j$ maps real functions to real distributions). The functions

$$u_j(x,t) := P_t * \mathcal{R}_j f(x) \text{ for } 1 \le j \le n, \quad u_{n+1}(x,t) := P_t * f(x)$$

form a *system of conjugate harmonic functions* on $\mathbb{H}^{n+1} := \{(x,t) \in \mathbb{R}^{n+1} : t > 0\}$, i.e. they satisfy the following system of *generalized Cauchy-Riemann equations*:

$$\sum_{j=1}^{n+1} \frac{\partial u_j}{\partial x_j} = 0, \quad \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j} \text{ for any } 1 \le j,k \le n+1,$$

where x_{n+1} is an alias for the auxiliary variable t. This can be checked using the formulas

$$\mathcal{F}(P_t * f)(\xi) = (2\pi)^{-n/2} e^{-t|\xi|} \widehat{f}(\xi), \quad \mathcal{F}(P_t * \mathcal{R}_j f)(\xi) = -(2\pi)^{-n/2} i \frac{\xi_j}{|\xi|} e^{-t|\xi|} \widehat{f}(\xi).$$

Clearly it suffices to prove that

$$\sup_{t>0} |u|(\cdot,t) \in L^1(\mathbb{R}^n),$$

where $u := (u_1, \dots, u_{n+1})$. We could bound $|u|(x,t)$ by the Hardy-Littlewood maximal function of $(f, \mathcal{R}_1 f, \dots, \mathcal{R}_n f)$ at x, but this would be useless (as the Hardy-Littlewood maximal function satisfies only a weak (1,1) bound). Rather, we aim at showing that $|u|^q(x,t) \lesssim Mg(x)$ for some $q < 1$ and some $g \in L^{1/q}(\mathbb{R}^n)$ with

$$\|g\|_{L^{1/q}}^{1/q} \lesssim \|f\|_{L^1} + \|\mathcal{R}_1 f\|_{L^1} + \dots + \|\mathcal{R}_n f\|_{L^1},$$

from which the thesis will follow since

$$\left\| \sup_{t>0} |u|(\cdot,t) \right\|_{L^1} \lesssim \|Mg\|_{L^{1/q}}^{1/q} \lesssim \|g\|_{L^{1/q}}^{1/q}.$$

Pick now $\frac{n-1}{n} < q < 1$ (so that in particular $q > 0$). From Lemma 9.20 below, we know that $(|u|^2 + \epsilon^2)^{q/2}$ is subharmonic (for any $\epsilon > 0$). Thus it satisfies the following version of the maximum principle: for any $\Omega \Subset \mathbb{H}^{n+1}$ and any $h \in C^0(\overline{\Omega})$ harmonic in Ω, the implication

$$(|u|^2 + \epsilon^2)^{q/2} \le h \text{ on } \partial\Omega \;\Rightarrow\; (|u|^2 + \epsilon^2)^{q/2} \le h \text{ on } \overline{\Omega}$$

holds. Sending $\epsilon \to 0$, it is easy to check that $|u|^q$ satisfies the same property. Lemma 9.21 below tells us that this property applies also with the harmonic function $h(x,t) := P_{t-\delta} * |u|^q(x,\delta)$ on the unbounded domain $\{(x,t) : t > \delta\} \subseteq \mathbb{H}^{n+1}$, for any $\delta > 0$.

Notice that

$$\sup_{\delta>0} \||u|^q(\cdot,\delta)\|_{L^{1/q}}^{1/q} = \sup_{\delta>0} \|u(\cdot,\delta)\|_{L^1} \le \|f\|_{L^1} + \|\mathcal{R}_1 f\|_{L^1} + \dots + \|\mathcal{R}_n f\|_{L^1}.$$

Since any closed ball in $L^{1/q}$ is weakly sequentially compact, we can find a sequence $\delta_k \to 0$ and a function $g \in L^{1/q}(\mathbb{R}^n)$ (whose $L^{1/q}$-norm satisfies the same bound) such that $|u|^q(\cdot,\delta_k) \rightharpoonup g$. Since $P_{t-\delta_k} \to P_t$ in $L^{(1/q)'}$, we deduce that

$$|u|^q(x,t) \le \lim_{k\to\infty} (P_{t-\delta_k} * |u|^q(\cdot,\delta_k))(x) = P_t * g(x).$$

Finally, by Remark 9.5, we have $P_t * g \le Mg$, which was our goal. It remains to prove the two lemmas.

Lemma 9.20. *For any $q \geq \frac{n-1}{n}$ the function $(|u|^2 + \epsilon^2)^{q/2}$ is subharmonic, i.e.*

$$\Delta\left((|u|^2 + \epsilon^2)^{q/2}\right) \geq 0.$$

Proof. We will use the shorthand notation $\partial_j := \frac{\partial}{\partial x_j}$. We compute

$$\partial_j (|u|^2 + \epsilon^2)^{q/2} = q(|u|^2 + \epsilon^2)^{(q/2)-1} u \cdot \partial_j u,$$

$$\sum_j \partial^2_{jj}(|u|^2+\epsilon^2)^{q/2} = \sum_j q(q-2)(|u|^2+\epsilon^2)^{(q/2)-2}(u\cdot\partial_j u)^2 + \sum_j q(|u|^2+\epsilon^2)^{(q/2)-1} |\partial_j u|^2$$

(using $\Delta u = 0$). The thesis follows immediately if $q \geq 2$, so we can assume $\frac{n-1}{n} \leq q < 2$, i.e. $0 < 2 - q \leq \frac{n+1}{n}$. It suffices to prove that

$$\frac{n+1}{n} \sum_j (u \cdot \partial_j u)^2 \leq |u|^2 \sum_j |\partial_j u|^2 .$$

This inequality is a consequence of the generalized Cauchy-Riemann equations: indeed, the matrix $A := (\partial_j u_k(x))_{jk}$ is symmetric, so (by the spectral theorem) we can find $P \in \mathbb{O}(n+1)$ and a diagonal matrix D such that $A = P^t D P$. The coefficients on the diagonal of D are the eigenvalues $\lambda_1, \ldots, \lambda_{n+1}$ of A. We remark that

$$\sum_j \lambda_j = \mathrm{tr}(D) = \mathrm{tr}(A) = 0.$$

We pick j_0 such that $|\lambda_{j_0}| = \max_j |\lambda_j|$. By Cauchy-Schwarz we have

$$(n+1)\,|\lambda_{j_0}|^2 = n\,|\lambda_{j_0}| + \left|\sum_{j\neq j_0} \lambda_j\right|^2 \leq n \sum_j |\lambda_j|^2 ,$$

so, letting $v := P \begin{pmatrix} u_1(x) \\ \vdots \end{pmatrix}$, we can estimate

$$\begin{aligned} \frac{n+1}{n} \sum_j |u \cdot \partial_j u|^2 =& \frac{n+1}{n} \sum_j \left| A \begin{pmatrix} u_1(x) \\ \vdots \end{pmatrix} \right|^2 = \frac{n+1}{n} |Dv|^2 \leq \frac{n+1}{n} |\lambda_{j_0}|^2 |v|^2 \\ \leq& \sum_j |\lambda_j|^2 |u|^2 (x). \end{aligned}$$

We finally observe that

$$\sum_j |\lambda_j|^2 = \mathrm{tr}(D^t D) = \mathrm{tr}(P A^t P^t P A P^t) = \mathrm{tr}(A^t A) = \sum_j |\partial_j u|^2 .$$

□

Lemma 9.21. *For $t > \delta$ we have $|u|^q (x,t) \leq (P_{t-\delta} * |u|^q (\cdot, \delta))(x)$.*

Proof. Let us first prove that, for every $\eta > 0$, there exists an arbitrarily large radius $R > 0$ such that $|u| \le \eta$ on the set $\{(x,t) : t \ge \delta, |(x,t)| \ge R\}$. From the mean-value property of harmonic functions, for any $(x,t))$ in this set we have

$$|u|(x,t) \le \frac{1}{\left|B_{t/2}(x,t)\right|} \int_{B_{t/2}(x,t)} |u|(y,s)\,dy\,ds.$$

If $|x| \le t$ then $t \ge \frac{R}{\sqrt{2}}$ and we can estimate

$$|u|(x,t) \le \frac{1}{\left|B_{t/2}(x,t)\right|} \int_{\mathbb{R}^n \times \left(\frac{t}{2}, \frac{3}{2}t\right)} |u|(y,s)\,dy\,ds \lesssim At^{-n} \lesssim AR^{-n}$$

(where $A := \sup_{s>0} \|u(\cdot,s)\|_{L^1} < +\infty$), which becomes small at will taking R large enough. Otherwise, if $|x| > t$, then $|x| \ge \frac{R}{\sqrt{2}}$ and any point $(y,s) \in B_{t/2}(x,t)$ satisfies $|y| > \frac{|x|}{2}$, so

$$|u|(x,t) \lesssim t^{-n-1} \int_{t/2}^{3t/2} \int_{|y|>|x|/2} |u|(y,s)\,dy\,ds \lesssim \int_{t/2}^{3t/2} s^{-n-1} \int_{|y|>R/\sqrt{8}} |u|(y,s)\,dy\,ds.$$

But the latter quantity can be uniformly estimated by

$$\int_{\delta/2}^{\infty} s^{-n-1} \int_{|y|>R/\sqrt{8}} |u|(y,s)\,dy\,ds,$$

which can be made arbitrarily small taking R large enough, thanks to the dominated convergence theorem (since the inner integral is bounded by A and tends to 0 as $R \to +\infty$).

Now $h(x,t) := (P_{t-\delta} * |u|^q(\cdot,\delta))(x)$ is harmonic on $\{(x,t) : t > \delta\}$ and extends continuously to the boundary $\mathbb{R}^n \times \{\delta\}$, where it coincides with $|u|^q$. So we have proved that

$$|u|^q(x,t) \le (P_{t-\delta} * |u|^q(\cdot,\delta))(x) + \eta^q$$

on the boundary of $S_R := \{(x,t) : t > \delta, |(x,t)| < R\}$ for any R large enough. We deduce that this inequality is also true on S_R itself. Thus, letting $R \to +\infty$, we infer that it holds on $\{(x,t) : t > \delta\}$. The thesis follows as we let $\eta \to 0$. $\square$

9.2.7 Existence of the atomic decomposition

In this section we show that any function $f \in L^1(\mathbb{R}^n)$ with $\mathcal{GM}f \in L^1(\mathbb{R}^n)$ admits an atomic decomposition

$$f = \sum_{k=0} \lambda_k a_k$$

with $\lambda_k \ge 0$, (a_k) a collection of ∞-atoms and $\sum_k \lambda_k \lesssim \|\mathcal{GM}f\|_{L^1}$, thereby proving the implication (4) $\Rightarrow$ (6) and the bound on the corresponding norms.

[$\cdots$ work in progress $\cdots$]

9.2.8 Littlewood-Paley characterization

In this section we are going to prove that $\mathcal{H}^1(\mathbb{R}^n) = \dot{F}^0_{1,2}(\mathbb{R}^n)$, in the sense specified by Theorem 9.23, denoting by $\mathcal{H}^1(\mathbb{R}^n)$ the space of functions satisfying (any of) the definitions $(1)-(8)$, whose equivalence has been established in the previous sections.

We fix a function $\psi \in C^\infty_c(B_2 \setminus \overline{B}_{1/2})$ such that $\sum_{j\in\mathbb{Z}} \psi(2^{-j}\xi) = 1$ for any $\xi \in \mathbb{R}^n \setminus \{0\}$. Recall that such a ψ can be produced by taking any $\phi \in C^\infty_c(B_2)$ such that $\phi \equiv 1$ in a neighbourhood of $\overline{B}_1$ and letting $\psi := \phi - \phi(2\cdot)$.

We let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be any function such that $\int_{\mathbb{R}^n} \varphi \, dx^n \neq 0$ and $\operatorname{supp}\varphi \subseteq B_2$. For instance we can take $\varphi := \mathcal{F}^{-1}(\phi)$ for any ϕ as above (using $\int_{\mathbb{R}^n} \varphi \, dx^n = (2\pi)^{-n/2}\phi(0) \neq 0$).

Lemma 9.22. *For any $f \in \mathcal{S}'(\mathbb{R}^n)$ and any $r \in (0,\infty)$ we have*

$$\sup_{t>0} |\varphi_t * P_j f|(x) \leq C(n,r) M |P_j f|^r (x)^{1/r}.$$

Proof. Recall that, whenever $v \in \mathcal{S}'(\mathbb{R}^n)$ has its Fourier transform supported in B_1, we have the inequality

$$\sup_z \frac{|v(x-z)|}{(1+|z|)^{n/r}} \lesssim (M|v|^r)^{1/r}(x).$$

More generally, if $\widehat{u}$ is supported in B_s, letting $v := u(s^{-1}\cdot)$ we obtain

$$\sup_z \frac{|u(x-z)|}{(1+s|z|)^{n/r}} = \sup_z \frac{|v(sx-z)|}{(1+|z|)^{n/r}} \lesssim (M|v|^r)^{1/r}(sx) = (M|u|^r)^{1/r}(x).$$

If $\frac{2}{t} \leq 2^{j-1}$ (i.e. if $t \geq 2^{2-j}$) we have $\varphi_t * P_j f \equiv 0$, since the supports of $\widehat{\varphi}_t$ and $\psi(2^{-j}\cdot)$ are disjoint in this case. Assume now that $t \leq 2^{2-j}$: in this case $\psi(2^{-j}\cdot)$ is supported in $B_{8/t}$, so choosing any $N > \frac{n}{r} + n$ and estimating $|\varphi(z)| \lesssim (1+|z|)^{-N}$ we get

$$|\varphi_t * P_j f|(x) \lesssim \int t^{-n} \frac{|P_j f|(x-z)}{(1+t^{-1}|z|)^N}\, dz \leq \sup_z \frac{|P_j f|(x-z)}{(1+t^{-1}|z|)^{n/r}} \int \frac{t^{-n}}{(1+t^{-1}|z|)^{N-n/r}}\, dz.$$

The last integral is a finite constant independent of t, while

$$\sup_z \frac{|P_j f|(x-z)}{(1+t^{-1}|z|)^{n/r}} \lesssim \sup_z \frac{|P_j f|(x-z)}{(1+\frac{8}{t}|z|)^{n/r}} \lesssim (M|P_j f|^r)^{1/r}(x).$$

$\square$

Before stating and proving the next theorem, we introduce the vector-valued Hardy space $\mathcal{H}^1(\mathbb{R}^n, \ell^2)$: it is the subspace of

$$L^1(\mathbb{R}^n, \ell^2) := \left\{ (f_j)_{j\in\mathbb{Z}} \subseteq L^1(\mathbb{R}^n) : \int \left(\sum_j |f_j|^2 \right)^{1/2} < +\infty \right\}$$

made of elements (f_j) satisfying one of the equivalent definitions (1) – (7) in vectorized form. For instance, (1) amounts to ask that

$$\sup_{t>0} \|\varphi_t * (f_j)\|_{\ell^2} = \sup_{t>0} \|(\varphi_t * f_j)\|_{\ell^2} \in L^1(\mathbb{R}^n).$$

Their equivalence comes from the fact that the proofs for the scalar case can be repeated verbatim for the vectorial case (we exclude definition (8) since its equivalence with the other definitions uses real numbers in an essential way, due to the appeal to the spectral theorem).

Theorem 9.23. *For any $f \in \mathcal{H}^1(\mathbb{R}^n)$ we have*

$$\|(P_j f)_{j\in\mathbb{Z}}\|_{L^1(\ell^2)} \lesssim \|f\|_{\mathcal{H}^1}.$$

Conversely, if for some $f \in \mathcal{S}'(\mathbb{R}^n)$ we have $\|(P_j f)_{j\in\mathbb{Z}}\|_{L^1(\ell^2)} < \infty$, then there exists a unique polynomial Q such that $f - Q \in \mathcal{H}^1(\mathbb{R}^n)$; moreover

$$\|f - Q\|_{\mathcal{H}^1(\mathbb{R}^n)} \lesssim \|(P_j f)_{j\in\mathbb{Z}}\|_{L^1(\ell^2)}.$$

Proof. The first statement follows immediately from the $\mathcal{H}^1 \to L^1(\ell^2)$ version of Theorem 9.10, applied with $K_j := \mathcal{F}^{-1}(\eta(2^{-j}\cdot))$, with assumptions (1) and (3) replaced by the validity of the $L^2 \to L^2(\ell^2)$ bound (which holds as a consequence of Plancherel's theorem). This variant of Theorem 9.10 is simply obtained by vectorizing the same proof (and in this case there is no need of truncating the kernel). Recall that this ℓ^2-valued kernel satisfies the Hörmander condition

$$\int_{|x|>2|y|} \|(K_j(x-y) - K_j(x))\|_{\ell^2}\, dx^n \lesssim 1.$$

We now turn to the converse. Pick $\eta := \psi(2\cdot) + \psi + \psi\left(\frac{\cdot}{2}\right)$ and notice that $\eta \equiv 1$ near the support of ψ. Let

$$\tilde{P}_j : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n), \quad \tilde{P}_j(g) := \mathcal{F}^{-1}(\eta(2^{-j}\cdot)\mathcal{F}g)$$

and remark that $\tilde{P}_j P_j = P_j$. Applying the $\mathcal{H}^1(\ell^2) \to L^1$ version of Theorem 9.10 with $f := (P_j f)_{j\in\mathbb{Z}}$ and $K_j := \mathcal{F}^{-1}(\eta(2^{-j}\cdot))$, we can estimate

$$\left\|\sum_{j=-N}^{N} P_j f\right\|_{L^1} = \left\|\sum_{j=-N}^{N} \tilde{P}_j P_j f\right\|_{L^1} \lesssim \left\|\sum_{j=-N}^{N} P_j f\right\|_{\mathcal{H}^1(\ell^2)}.$$

We can similarly estimate the L^1-norm of $\mathcal{R}_k \sum_{j=-N}^{N} P_j f$, using the ℓ^2-valued kernel $K_j := \mathcal{F}^{-1}\left(-i\frac{\xi_k}{|\xi|}\eta(2^{-j}\cdot)\right)$. Thus,

$$\left\|\sum_{j=-N}^{N} P_j f\right\|_{\mathcal{H}^1} \lesssim \left\|\sum_{j=-N}^{N} P_j f\right\|_{\mathcal{H}^1(\ell^2)} = \left\|\sup_{t>0}\left(\sum_{j=-N}^{N} |\varphi_t * P_j f|^2\right)^{1/2}\right\|_{L^1}.$$

Using Lemma 9.22 with any $0 < r < 1$, as well as the Hardy-Littlewood maximal estimate for $L^{1/r}(\ell^{2/r})$, the last quantity is bounded up to constants by

$$\begin{aligned}\left\| \left(\sum_{j=-N}^{N} (M\,|P_j f|^r)^{2/r} \right)^{1/2} \right\|_{L^1} &= \|(M\,|P_j f|^r)\|_{L^{1/r}\left(\ell_N^{2/r}\right)}^{1/r} \lesssim \|(|P_j f|^r)\|_{L^{1/r}\left(\ell_N^{2/r}\right)}^{1/r} \\ &= \|(P_j f)\|_{L^1(\ell^2)}\,,\end{aligned}$$

where ℓ_N^p denotes the truncated space of sequences $a = (a_{-N}, \ldots, a_N)$ with the norm $\|a\|_{\ell_N^p} := \left(\sum_{j=-N}^{N} |a_j|^p\right)^{1/p}$. The same argument shows that the partial sums $\sum_{j=-N}^{N} P_j f$ form a Cauchy sequence in $\mathcal{H}^1(\mathbb{R}^n)$ and thus, by Proposition 9.8, converge to some $g \in \mathcal{H}^1(\mathbb{R}^n)$.

But $\mathcal{F}\left(\sum_{j=-N}^{N} P_j f\right) \to \widehat{f}$ in $\mathcal{D}'(\mathbb{R}^n \setminus \{0\})$, so the tempered distribution $\widehat{f} - \widehat{g}$ is supported in $\{0\}$. This means that

$$Q := f - g = \mathcal{F}^{-1}\left(\widehat{f} - \widehat{g}\right)$$

is a polynomial. So $f - Q = g \in \mathcal{H}^1(\mathbb{R}^n)$ and, letting $N \to \infty$ in the above estimate, we also have

$$\|f - Q\|_{\mathcal{H}^1} = \|g\|_{\mathcal{H}^1} \lesssim \|(P_j f)\|_{L^1(\ell^2)}\,.$$

$\square$

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