Study of topological singularities in function spaces between manifolds and applications to the Calculus of Variations

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presented by

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Introduction

Our intention in this thesis is to study some important problems related to the "toplogical singular set" of the maps in a given function space. This object, which is to be defined without ambiguity for some categories of these mappings, is the obstruction which characterizes the non-approximability of a mapping in this space by the smooth mappings. These topological singularities and their properties are the base of some interesting results on the weakly harmonic maps into the sphere or on the weak or strong density of smooth maps in function spaces. They have become an independent subject of study with important questions to solve, related to different domains such as Functional Analysis, Geometric Measure Theory, Topology and Geometry.

Here we limit ourselves to the Sobolev spaces between manifolds. But remark that the same problems are worthy to ask for any other function space. Consider two compact riemannien manifolds M and N of respective dimensions n and k, such that N is closed and isometrically embedded in some euclidien space $\mathbb{R}^N$. for $p \geq 1$, the Sobolev space $W^{1,p}(M,N)$ is defined by

$$W^{1,p}(M,N) := \{u \in W^{1,p}(M,\mathbb{R}^N);\ u(x) \in N \text{ p.p. dans } M\}.$$

This space is equipped with the induced weak and strong topologies of $W^{1,p}(M,\mathbb{R}^n)$ and is closed under the convergence in these topologies. The p-energy functional is defined by $E_p(u) := \int_M |\nabla u|^p$ and is called the Dirichlet energy $E(u) := \int_M |\nabla u|^2$ for $p = 2$. Also, for a map $\varphi \in C^\infty(\partial M, N)$ we set

$$W^{1,p}_\varphi(M,N) := \{u \in W^{1,p}(M,N);\ u|_{\partial M} = \varphi\}.$$

For definitions concerning the Geometric Measure Theory the reader can refer to [16] or [28]. Meanwhile, we will refer to integer multiplicity rectifiable currents (respectively real multiplicity currents) with finite mass by the term i.m. rectifiable (respectively normal) currents.

Harmonic mappings into the sphere

Let us begin with a variational problem which leads us, in a natural way, to problems related to topological singularities.

Consider the Sobolev space $H^1(\Omega, S^2)$ where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a bounded open set and S^2 is the 2-dimensional unit sphere in $\mathbb{R}^3$. We call u a weakly harmonic map if it is a critical point for the functional E, i.e. if and only if we have

$$\frac{d}{dt} E\left(\frac{u+tv}{|u+tv|}\right)_{|t=0} = 0 \quad \text{for all} \quad v \in C_c^\infty(\Omega, \mathbb{R}^3)\,.$$

In other words, u is weakly harmonic in the Sobolev space $H^1(\Omega, S^2)$ if it satisfies the following equation in the sense of distributions :

$$\begin{cases} -\Delta u = u|\nabla u|^2 & \text{in} \quad \Omega \\ \\ u(x) \in S^2 & \text{a.e.} \end{cases}$$

Let $\varphi : \partial\Omega \to S^2$ be a smooth map which has a regular extension into Ω. The existence of a weakly harmonic map equal to φ on the boundary can be easily proved by a straightforward minimizing argument. By the way, the uniqueness and regularity questions for weakly harmonic maps in $H^1_\varphi(\Omega, S^2)$ have not the same answers as in the classic cases, i.e. when the target manifold is an euclidean space.

The smoothness of harmonic extensions into S^2

One of the important problems which is still open is if smooth harmonic extensions of φ into Ω exist. In the first step one may want to minimize the Dirichlet energy in $H^1_\varphi(\Omega, S^2)$ and prove the regularity of the solution. But in fact if we define

$$\mu_\varphi := \inf_{H^1_\varphi(\Omega, S^2)} E(u) \leq \inf_{C^\infty_\varphi(\overline{\Omega}, S^2)} E(u) =: \bar{\mu}_\varphi\,,$$

the strict inequality

$$\mu_\varphi < \bar{\mu}_\varphi$$

happens sometimes (See [22]). Thus minimizers of E are not necessarily smooth and we should find other harmonic maps which could be a suitable candidate for a smooth solution. Meanwhile R.Schoen and K.Uhlenbeck ([35]) proved that these minimizers are smooth in Ω except on a finite set of points.

In trying to attack this problem, another functional on $H^1_\varphi(\Omega, S^2)$ has been studied which is called the "relaxed energy". In fact, the relaxed energy is the largest sequentially lower semi-continuous functional on $H^1_\varphi(\Omega, S^2)$ which is less than E on $C^\infty_\varphi(\Omega, S^2)$:

Definition 1 *The relaxed energy $\mathcal{F}$ of E on $H^1_\varphi(\Omega, S^2)$ is defined to be*

$$\mathcal{F}(u) := \inf\left\{ \liminf_{n\to\infty} E(u_n)\,;\; u_n \in C^\infty_\varphi(\Omega, S^2)\,,\; u_n \rightharpoonup u \right\}\,. \tag{0.1}$$

Since the smooth maps which take φ as their boundary value are weakly sequentially dense in $H^1_\varphi(\Omega, S^2)$ (See [2]), we observe that $\mathcal{F}$ is well defined. Moreover $\mathcal{F}$ is sequentially lower semi-continuous with respect to the weak topology in $H^1_\varphi(\Omega, S^2)$ and we have

$$\inf_{H^1_\varphi(\Omega,S^2)} \mathcal{F} = \inf_{C^\infty_\varphi(\Omega,S^2)} E\,. \tag{0.2}$$

This equation shows the importance of study of $\mathcal{F}$. Since the infimum of $\mathcal{F}$ in $H^1_\varphi(\Omega, S^2)$ is achieved, the question which should be considered then is whether a minimizer of $\mathcal{F}$ is weakly harmonic and to what extent it is regular.

In this line, F.Bethuel, H.Brezis, J.M. Coron and E.Lieb (See [5] and [10]) showed the striking fact that, for $n = 3$, the relaxed energy achieves the following elegant algebric formula :

$$\mathcal{F}(u) = F(u) := E(u) + 8\pi L(u) \tag{0.3}$$

where

$$L(u) := \frac{1}{4\pi} \sup_{\substack{\psi\,:\,\Omega \to \mathbb{R} \\ |d\psi|_\infty \leq 1}} \left\{ \int_\Omega u^*\omega_V \wedge d\psi - \int_{\partial\Omega} \varphi^*\omega_V \wedge \psi \right\} \tag{0.4}$$

where ω_V is the volume form on S^2 (or can be replaced by any 2-form ω, $\int_{S^2} \omega = 4\pi$). In particular this yields that the critirical points of $\mathcal{F}$ are weakly harmonic. F.Bethuel and H.Brezis showed also that the minimizers of $\mathcal{F}$ are smooth in Ω except on a finite set of points (See[4]).

The intuitive approach for $L(u)$ is that if $u \in H^1_\varphi(\Omega, S^2)$ is smooth in Ω except on a set of finite points $\{p_1, ..., p_m\}$, taking the degree d_i on the point p_i, then $L(u)$ is the minimum length of the segments connecting these singularities with respect to the multiplicities (See [10]). In other words

$$L(u) = m_i \left(\sum_{i=1}^m d_i\, [[p_i]],\, \Omega \right)$$

where $m_i(\Omega, S^2)$, for the i.m. rectifiable 0-current $\mathbf{S}$, is defined by

$$m_i(\mathbf{S}, \Omega) := \inf \left\{ \mathbf{M}(\mathbf{T})\,;\, T \in \mathcal{R}_1(\mathbb{R}^3),\, \operatorname{spt}\mathbf{T} \subset \overline{\Omega}\,,\, \partial\mathbf{T} = \mathbf{S} \right\}.$$

In the first chapter, we study the same approach for $n > 3$ but this generalisation meets obstacles. One may introduce for ω, any 2-form on S^2 which satisfies $\int_{S^2} \omega = 1$:

$$L(u) := \sup_{\substack{\psi \in \Omega^{n-3}(\overline{\Omega}) \\ |d\psi|_\infty \leq 1}} \left\{ \int_\Omega u^*\omega \wedge d\psi - \int_{\partial\Omega} \varphi^*\omega \wedge \psi \right\} \tag{0.5}$$

as a generalization of $L(u)$ in the 3-dimensional case. Observe that L is independant of the choice of ω and is continuous on $H^1_\varphi(\Omega, S^2)$ for $\Omega \subset \mathbb{R}^n$ and the functional

$$F(u) := E(u) + 8\pi L(u) \tag{0.6}$$

would still be weakly lower semi-continuous. But we have this theorem :

Theorem 1 (I.1) *For every $\Omega \subset \mathbb{R}^4$ and every map $\varphi \in C^\infty(\partial\Omega, S^2)$, smoothly extendable onto Ω, there exists $u \in H^1_\varphi(\Omega, S^2)$ such that*

$$F(u) < \mathcal{F}(u)\,.$$

Moreover there exists a domain $\Omega \subset \mathbb{R}^4$ and $\varphi \in C^\infty(\partial\Omega, S^2)$, smoothly extendable onto Ω, for which this gap phenomenon exists :

$$\inf_{H^1_\varphi(\Omega,S^2)} E < \inf_{H^1_\varphi(\Omega,S^2)} F < \inf_{C^\infty_\varphi(\Omega,S^2)} E\,.$$

■

The difference with the case $n = 3$ lies in the value which $L(u)$ represents. We shall consider a map $u \in H^1_\varphi(\Omega, S^2)$, which is smooth except on a finite union of $(n-3)$-dimensional submanifolds of Ω : $\{\sigma_1, ...\sigma_m\}$ (We say that $u \in R^\infty(\Omega, S^2)$). The degree d_i of u on each σ_i is well defined and we set the topological singularity of u, $\mathbf{S}_u$, to be

$$\mathbf{S}_u := \sum_{i=1}^{m} d_i\,[[\sigma_i]]. \tag{0.7}$$

Calculating $L(u)$, we see that

$$L(u) = \sup_{|d\psi|_\infty \le 1} \int_{\mathbf{S}_u} \psi \le \sup_{\|d\psi\|^*_\infty \le 1} \int_{\mathbf{S}_u} \psi = m_r(\mathbf{S}_u, \Omega) \tag{0.8}$$

where $\|.\|^*$ is the co-mass norm on the space of forms and

$$m_r(\mathbf{S}_u, \Omega) := \inf\left\{\mathbf{M}(\mathbf{T})\,;\, \mathbf{T} \in \mathcal{D}_{n-2}(\mathbb{R}^n)\,, \partial\mathbf{T} = \mathbf{S}_u,\, \text{spt}\,\mathbf{T} \subset \overline{\Omega}\right\}$$

is the mass of the minimal normal (real) current in Ω with boundary $\mathbf{S}_u$.

Meanwhile, $m_i(\mathbf{S}_u, \Omega)$, the minimal mass of i.m. rectifiable currents in $\overline{\Omega}$ which are bounded by $\mathbf{S}_u$, is still proportional to the energy needed for removing the singularities of u and estimating it weaky by smooth maps (See the further proposition 1). Here arises the main question which should be answered if we want to continue as above, that is if

$$m_r(\mathbf{S}, \Omega) = m_i(\mathbf{S}, \Omega) \quad \forall \mathbf{S} \in \mathcal{R}_{n-3}(\Omega).$$

But contrary to the case $n = 3$, the answer is no for $n > 3$. Specially, for $n = 4$, there exists a curve $[[\Gamma]]$ in $\mathbb{R}^4$ for which

$$m_r([[\Gamma]]) \le \frac{1}{2} m_i(2[[\Gamma]]) < m_i([[\Gamma]]).$$

This gap phenomenon was firstly proved by L.C.Young in [42]. F.Morgan in [27] and B.White in [37] have given other examples of such curves in $\mathbb{R}^4$. This is then the origin of the facts proved in theorem 1.

Remark 1 *We have always the relation*

$$\sup_{\substack{\|d\psi\|^*_\infty \leq 1 \\ spt\psi \subset \overline{\Omega}}} \int_{\mathbf{S}} \psi = m_r(\mathbf{S}, \Omega)$$

which is due to the fact that there exists always a calibration for minimizing normal currents (See Chapter I for the references).

The topological singularities and the relaxed energy

The question which arises then is to find the equivalent formula for (0.3) for the relaxed energy when $n > 3$. Regarding (0.8), we can consider L, for $n = 3$ as a continuous extension of $m_i(\mathbf{S}_u, \Omega)$ $(= m_r(\mathbf{S}_u, \Omega))$ into all $H^1(\Omega, S^2)$. Therefore, for generalizing the result to higher dimensions, one should extend the definition of the topological singularities over $H^1(\Omega, S^2)$:

Definition 2 *Let $u \in H^1_\varphi(\Omega, S^2)$. We define the topological singularity of u to be the current $\mathbf{S}_u \in \mathcal{D}_{n-3}(\Omega)$ defined by*

$$\mathbf{S}_u(\alpha) := \int_\Omega u^*\omega \wedge d\alpha \qquad \forall \alpha \in \mathcal{D}^{n-3}(\Omega).$$

Here $\mathcal{D}^k(\Omega)$ is the set of smooth k-forms on Ω with compact support (See[16], 2.2.3) and ω is some 2-form on S^2 for which $\int_{S^2} \omega = 1$.

Remark 2 *F.Béthuel, J.M.Coron, F.Demengel et F.Hélein ([6]) proved that "$\mathbf{S}_u = 0$" is the necessary and sufficient condition for $u \in H^1(\Omega, S^2)$ to be approximable by smooth maps in the strong topology. This is the reason behind the choice of "topological singularity" as the name for $\mathbf{S}_u$.*

This definition coincides with the one given for R^∞ maps in (0.7) (See [16], vol II section 5.4.2. The reader can also find the detailed proof of this fact in Chapter II). Observe that the expression $m_i(\mathbf{S}_u, \Omega)$ has a meaning for any $u \in H^1_\varphi(\Omega, S^2)$ only if S_u is a boundary for an i.m. rectifiable current. Although this necessary condition is satisfied for $n = 3$, the proof for $n > 3$ is not the same and we are forced to use the methods developed in [16] for the cartesian currents to prove it. The difficulty lies on the fact that the question of the strong continuity of m_i for $n > 3$, even over $R^\infty(\Omega, S^2)$, is still open. This is also the obstacle to identify the functional

$$\widetilde{F}(u) := E(u) + 8\pi m_i(\mathbf{S}_u, \Omega)$$

with the relaxed energy.

Open Question 1 *Is $\mathcal{F} = \widetilde{F}$?*

■

Precisely we have this proposition proved in Chapter II :

Proposition 1 (II.3.1) *Let $u \in H^1_\varphi(\Omega, S^2)$, then $\mathbf{S}_u$ is the boundary of some i.m.rectifiable current. Set*

$$\widetilde{F}(u) := E(u) + 8\pi m_i(\mathbf{S}_u, \Omega).$$

$\widetilde{F}$ is lower semi-continuous with respect to the weak topology in $H^1_\varphi(\Omega, S^2)$ and its critical points are weakly harmonic. Moreover

$$\widetilde{F}(u) \leq \mathcal{F}(u)\,, \quad \forall u \in H^1_\varphi(\Omega, S^2).$$

■

We will talk about the problem of topological singularities for maps into spheres in a more general context.

The multiplicity of S^2-valued harmonic extenstions

In the second chapter, we will answer to the question of multiplicity of harmonic extensions into S^2 for a smooth mapping $\varphi : \Omega \to S^2$, $n = \dim\Omega > 3$. Here is the theorem we prove in this chapter.

Theorem 2 (Theorem II.1) *Let Ω be a regular bounded domain in $\mathbb{R}^n$, $n > 3$, and φ a non-constant smooth map from $\partial\Omega$ into S^2. Then φ admits infinitely many weakly harmonic extensions.*

■

Remark 3 *This result is independent of the choice of the metric on S^2 .*

In [21], R. Hardt, D.Kinderlehrer and F.H.Lin had proved the existence of infinitely many weakly harmonic extensions to an axially symmetric boundary condition in $H^1(\mathbf{B}^3, S^2)$ where $\mathbf{B}^3$ is the unit ball in $\mathbb{R}^3$. The method consists in constructing a non-axially symmetric harmonic extension and then one obtains infinitely many different harmonic maps with the same boundary data by rotating this extension around the symmetry axis.

Another method consists in finding new weakly harmonic maps minimizing the variants of the relaxed energy already presented in this Introduction. This has been done by F.Bethuel, H.Brezis and J.-M.Coron in [5] where they introduced such functionals which they called "relaxed energies". Using these functionals they proved for $n = 3$ that if φ is not homotopic to a constant or if we have this gap condition

$$\min_{H^1_\varphi(\Omega,S^2)} E(u) < \inf_{C^\infty_\varphi(\Omega,S^2)} E(u)$$

then φ admits infinitely many weakly harmonic extensions inside Ω. Using the same gap condition, T.Isobe proved the corresponding result for the case $n \geq 4$ in [26], still using the relaxed energies whose definition was extended to higher dimensions.

At last, using his *strict dipole insertion lemma*, (the 3-dimensional version of the furthur lemma 1) proved in [32], T.Rivière showed that if Ω is a regular bounded domain of $\mathbb{R}^3$, a non constant smooth boundary data $\varphi : \partial\Omega \to S^2$ admits always infinitely many weakly harmonic extensions (Appeared in [33]).

Let us consider the same method for $n \geq 4$. Although the F-energy presented in (0.6) is not the relaxed energy, its minimizers are still weakly harmonic. Proving this fact in Chapter II, we will produce new weakly harmonic maps using this energy. But the difficult step consists in finding some equivalent construction in any dimensions of the insertion of 2 singular points with the strict inequality like in [32] for $n = 3$. In the first sight it seems that we should insert this time a couple of singularities of dimesnion $n-3$ (e.g. two circle-singularities for when $n = 4$). But the dipole for $n = 3$ is nothing else than the sphere $S^0 = S^{n-3}$ in 3 dimensions. So it appears that ([32], lemma A.1) can be generalized by inserting this time an $(n-3)$-dimensional singular sphere. This lemma, technically more involved than the 3 dimensional case, is the main step to prove theorem 2.

Lemma 1 (II.4.1) *Let Ω be a bounded regular domain in $\mathbb{R}^n$ and u a regular non-constant map from Ω to S^2. Let x_0 be a point of Ω for which $\nabla u(x_0) \neq 0$. Then for every $\rho > 0$ there exists a map $v \in H^1(\Omega, S^2)$ and $0 < \delta < \rho$ such that*

(i) $v = u$ *on* $\Omega \backslash B_\rho(x_0)$

(ii) $\mathbf{S}_v = [[\sigma]]$

(iii) $E(v) < E(u) + 8\pi\omega_{n-2}\delta^{n-2} = E(u) + 8\pi L(v, u)$

where σ is an $(n-3)$-dimensional sphere of center x_0 and radius δ and ω_k is the volume of the unit k-dimensional disk.

∎

Topological singularities in $W^{1,p}(M, S^p)$

Considering the characteristics of $\mathbf{S}_u$, the topological singular set defined in the previous section for any map $u \in H^1_\varphi(\Omega, S^2)$, it is interesting and natural to consider the problem of the topological singular set $\mathbf{S}_u$ for the maps $u \in W^{1,p}(M, S^p)$ when p is any integer and M a compact manifold.

Any map $u \in W^{1,p}(\mathbf{B}^n, S^p)$, $p < n$, is the strong limit of smooth maps if and only if $d(u^*\omega_{S^p}) = 0$ in the sense of distributions (See [6]). By this, we can generalize definition 2 for this space. :

Definition 3 *Let $u \in W^{1,p}(M, S^p)$. We define the "local" topological singularity of u, $\mathbf{S}_u \in \mathcal{D}_{n-p-1}(M)$, to be the current defined by*

$$\mathbf{S}_u(\alpha) := \int_M u^*\omega \wedge d\alpha \qquad \forall \alpha \in \mathcal{D}^{n-p-1}(M).$$

Here $\mathcal{D}^k(M)$ is the set of smooth k-forms on M with compact support (*See*[*16*], 2.2.3) *and ω is some p-form on S^p for which $\int_{S^p} \omega = 1$.*

We recall that $m_i(\mathbf{S})$ (resp. $m_r(\mathbf{S})$) is the minimal mass of i.m. rectifiable (resp. normal) currents supported in M and bounded by $\mathbf{S}$. Two questions regarding the topological singularities in $W^{1,p}(M, S^p)$ are still open for <u>almost</u> all values for p :

Open Question 2 *Is $\mathbf{S}_u$ the boundary of some i.m. rectifiable current, when M is a closed manifold?*

$\blacksquare$

Open Question 3 *Assume that the answer to the previous question is positive. Then, do $m_i(\mathbf{S}_{u_m})$ tend to $m_i(\mathbf{S}_u)$ if $u_m \to u$ strongly in $W^{1,p}$?*

$\blacksquare$

Minimal normal and i.m. rectifiable currents

$\mathbf{S}_u$ is effectively the boundary of some normal current in $W^{1,p}(M, S^p)$ and

$$m_r(\mathbf{S}_{u_m} - \mathbf{S}_u) \to 0$$

for any convergent sequence in $W^{1,p}(M, S^p)$. As a consequence, if for any i.m.rectifiable current $\mathbf{S}$ of dimension $n-p-1$, $m_i(\mathbf{S}) \le C m_r(\mathbf{S})$ for some constant $C > 0$, the two above questions will have positive answers. But, except for $p = 1$ or $p = n-1$, we do not know if this constant exist.

Open Question 4 *Assume that $k \neq 0, n-2$. Is the quantity*

$$\frac{m_i(\mathbf{S})}{m_r(\mathbf{S})} = \sup_{l \in \mathbb{N}} \frac{l m_i(\mathbf{S})}{m_i(l\mathbf{S})}\,,$$

equi-bounded uniformly over all i.m. rectifiable k-currents supported in a compact subset of $\mathbb{R}^n$?

$\blacksquare$

Remark 4 *As we already mentioned, $m_i(\mathbf{S}) = m_r(\mathbf{S})$ is always true if $k = 0, n-2$ (For the references see the discussion about the calibrations in the first chapter).*

A geometric interpretation for $\mathbf{S}_u$

M.Giaquinta, G.Modica and J.Soucek gave another definition for $\mathbf{S}_u$, which is equivalent to the ours (See [16], vol II, section 5.4.2). $\mathbf{S}_u$ is defined to be the horizontal part of ∂G_u, when G_u is the rectifiable graph of u, considered as a cartesian current in $M \times S^p$ (See [16], vol I). Considering this fact and using the characteristics of the cartesian currents and the polyconvex envelopes of the Dirichlet energy discussed in [16], we proved that the Question 2 has a positive answer for $p = 2$ (See the above proposition 1).

And if S^p is an H-space?

In Chapter III we answer to the two above questions regarding the topological singularities of $u \in W^{1,p}(M, S^P)$ for $p = 3$ and 7. The particularity of these two cases reside in the fact that S^3 and S^7 (alongside with S^1 and S^0) are the only spheres which are H-spaces, i.e. there is a smooth multiplication

$$\kappa : S^p \times S^p \to S^p$$

such that the induced homotopic homeomorphism

$$\kappa_* : \pi_p(S^p) \oplus \pi_p(S^p) \to \pi_p(S^p)$$

is the sum of elements in $\pi_p(S^p)$ ([8], section VI.15). As a result, the method we use does not work for other values of p. Here is our main result

Theorem 3 (III.1) *Let $p = 3$ or 7, $p < n = dim\, M$ and $u \in W^{1,p}(M, S^p)$, $\partial M = \emptyset$. Then $\mathbf{S}_u$ is the boundary of an i.m. rectifiable current in M. Moreover, $m_i(\mathbf{S}_{u_m} - \mathbf{S}_u) \to 0$ if the u_m converge strongly to u in $W^{1,p}(M, S^p)$.*

■

A new perspective for the topological singularities

In the last chapters of this thesis we will try to generalize the notion of the topological singular set for certain categories of Sobolev spaces $W^{1,p}(\mathbf{B}^n, N)$. We will explain how these efforts let us to prove some theorems about the sequentially weak density of smooth maps in these spaces. We will use locally lipschitz projections of N over its $[p]$-skeletons, the results of F.J.Almgren, W.Browder and .Lieb about the inverse images for the Sobolev maps into spheres ([1]) and the singularity removing propositions adapted to our situation. We recall that the topological singularities should be defined to identify the obstruction to the non-approximability of a Sobolev map between M and N by the smooth maps from M into N. The singularities we consider detect the local obstructions of the approximability, therefore we will emphasize in the Chapter IV on the case $M = \mathbf{B}^n$, where $\mathbf{B}^n$ is the n-dimensional unit disk. F.Hang and F.H.Lin [20] have recently showed the possible existence of "global" obstructions when the topology of the domain M is not trivial. So one should be careful when considering the Sobolev spaces $W^{1,p}(M, N)$ for generic compact smooth manifold M.

Flat chains with coefficients in normed groups

In [7], F.Bethuel and X.Zheng proved that smooth maps are not dense in $W^{1,p}(\mathbf{B}^n, N)$, if $p < n$ and $\pi_{[p]}(N) \neq 0$. In this case, the best one can do is to approximate the maps in $W^{1,p}(\mathbf{B}^n, N)$ by maps which are smooth away from a finite union $\Sigma = \bigcup_{i=1}^r \Sigma_i$ of smooth $(n-p-1)$-dimensional submanifolds of $\mathbf{B}^n$. This set of maps is called $R^{\infty,p}(\mathbf{B}^n, N)$. A map $v \in R^{\infty,p}(\mathbf{B}^n, N)$ realizes elements σ_x of $\pi_{[p]}(N)$ on the $[p]$-spheres centered at any point $x \in \Sigma(v)$ and contained in the normal $[p]+1$ plane to $T_x\Sigma(v)$. If for some $x \in \Sigma(v)$, σ_x is non trivial, then v can not be approximated by smooth maps in the strong topology (See [2]).

As an example, the smooth maps are not dense in $W^{1,1}(\mathbf{B}^2, \mathbb{RP}^2)$ since $\pi_1(\mathbb{RP}^2) \neq 0$. Then, $v \in R^{\infty,1}(\mathbf{B}^2, \mathbb{RP}^2)$ is smooth except on a finite number of points in $\mathbf{B}^2 : \{p_1, \dots, p_r\}$. If v has the non-zero homotopy type of $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$ around one of these points, it can not be approximated by the smooth maps in $W^{1,1}(\mathbf{B}^2, \mathbb{RP}^2)$ (We can construct such v). The idea is then to identify and define properly the "topological singular set" of such v, which allows us to extend the definition to any map $u \in W^{1,1}(\mathbf{B}^2, \mathbb{RP}^2)$.

The usual method, using the differential forms and proposed by F.Bethuel, J.M.Coron, F.Demengel and F.Hélein in [6] is not helpful since $\pi_1(\mathbb{RP}^2)$ is not torsion free and the homotopy cycles in $\mathbb{RP}^2$ are not detected by the 1-forms. For the same reasons, the approach of ([16], vol II, section 4.4.2) by M.Giaquinta, G.Modica and J.Soucek, using the graph of Sobolev maps is not satisfactory.

The idea would be to use the flat chains with coefficients in a normed abelian group G, which are the generalizations of normal ($G = \mathbb{R}$) and rectifiable ($G = \mathbb{Z}$) currents. This theory was first introduced by H.Federer [13] and W.Fleming [15]. Recently there have been remarkable advances by B.White ([38] and [39]). In fact, we can imagine the topological singular set of v, $\mathbf{S}_v$, as a 0-chain with coefficients in $\mathbb{Z}_2$:

$$\mathbf{S}_v := \sum_{i=1}^r \sigma_{p_i}[[p_i]] \quad (\sigma_{p_i} := [v(\partial B_\delta(p_i))]_{\pi_1(\mathbb{RP}^2)} \in \mathbb{Z}_2).$$

The question would be to understand the behaviour of $\mathbf{S}_{v_m}$ for a convergent sequence $v_m \to u \in W^{1,1}(\mathbf{B}^2, \mathbb{RP}^2)$ and possibly to prove a convergence of the chains $\mathbf{S}_{v_m}$ in the flat norm to some $\mathbb{Z}_2$-chain we would call the topological singularity of u.

Naturally, regarding what we mentioned about the realizations of elements of $\pi_{[p]}(N)$ by $v \in R^{\infty,p}(\mathbf{B}^n, N)$ around its singularities, we can proceed in the same way for maps in $W^{1,p}(\mathbf{B}^n, N)$, i.e. to define the topological singularities of $v \in R^{\infty,p}(\mathbf{B}^n, N)$ as a $\pi_{[p]}(N)$ -chain and to study the behaviour of these chains for the convergent sequences $v_m \to u \in W^{1,p}(\mathbf{B}^n, N)$. Nevertheless, this program is not suitable for all Sobolev spaces, as shows the example $W^{1,3}(\mathbf{B}^4, S^2)$ treated by R.Hardt et T.Rivière (See [24]).

In Chapter IV, we will prove this theorem :

Theorem 4 (IV.1 and IV.1 *bis*) *Assume that* $\mathbf{B}^n$ *is the unit disk in* $\mathbb{R}^n$ *and that* N *is a compact riemannian manifold of dimension* $k \geq [p]$, $\partial N = \emptyset$. *We assume also that either* $[p] = 1$ *and* $\pi_1(N)$ *is abelian, or* $[p] = 3, 7, n-1$ *and* N *is* $([p]-1)$*-connected, i.e.*

$$\pi_1(N) = \cdots = \pi_{[p]-1}(N) = 0.$$

Then $\mathbf{S}_u$, *the topological singularity of* $u \in W^{1,p}(\mathbf{B}^n, N)$ *is well defined as a flat* $\pi_{[p]}(N)$*-chain and the flat norm of* $\mathbf{S}_{u_m} - \mathbf{S}_u$ *converge to zero if* $u_m \to u$ *in* $W^{1,p}(\mathbf{B}^n, N)$. *Moreover,* u *is the strong limit of maps in* $C^\infty(\mathbf{B}^n, N)$ *if and only if* $\mathbf{S}_u = 0$. *Also, if* $u|_{\partial \mathbf{B}^n} = \varphi$ *is smooth and smoothly extendable over* $\mathbf{B}^n$, $\mathbf{S}_u$ *will be the boundary of some flat* $\pi_{[p]}(N)$*-chain of finite mass (and as a result of rectifiable support) and "*$\mathbf{S}_u = 0$*" would be the necessary and sufficient condition for* u *to be the strong limit of maps in* $C^\infty_\varphi(\mathbf{B}^n, N)$.

■

Remark 5 *Regarding* $W^{1,1}(\mathbf{B}^2, \mathbb{RP}^2)$, *we have this remarkable fact that we can identify* $\mathbf{S}_u$ *for any map* u *in this space to a* $\mathbb{Z}_2$*-valued Borel measure of total finite variation. The reader can refer to [38] where B.White give the conditions on* G *for which a finite mass flat* G*-chain has a rectifiable support.*

We should add some other remarks. First, the reason we can not state the same results for all values of $[p]$ is what we explained in the previous section, i.e. $[p] = 1, 3, 7$ and $n-1$ are the only values for which there is a proof for the integral flat convergence of the topological singularities of a convergent sequence in $W^{1,[p]}(\mathbf{B}^n, S^{[p]})$. Second, we can extend these results for $[p] = 3, 7$ and $n-1$, even if $\pi_1(N) \neq 0$, under certain conditions (See the proof of theorem 4 in Chapter IV). At last, we should recall that there are examples of $([p]-1)$-connected manifolds whose $[p]$-th homotopy group is not torsion free otherwise the cases we consider would reduce to those already studied in [6]. As an exmaple, the Stiefel manifolds $V_k(\mathbb{R}^n)$, when $n-k$ is odd, are $(n-k-1)$-connected and $\pi_{n-k}(V_k(\mathbb{R}^n)) = \mathbb{Z}_2$ is not torsion free (See [25])

F.Hang and F.H.Lin [20] have found examples where the absence of the local obstructions in not sufficient for that a map $u \in W^{1,p}(M, N)$ be strongly approximable by smooth maps. Precisely, there is a map in $H^1(\mathbb{CP}^2, S^2)$ for which $d(u^*\omega) = 0$ but u is not in the strong closure of smooth maps in this space. Also there are maps in $W^{1,3}(\mathbb{CP}^3, \mathbb{CP}^2)$ which are not strongly approximable by smooth maps though $\pi_3(\mathbb{CP}^2) = 0$. The necessary and sufficient conditions for that a Sobolev map between two manifolds be approximable by the smooth maps are still unknown for the general case.

Finally we ask this question for which we have no definite answer :

Open Question 5 *How should one define the topological singular set of maps in* $W^{1,1}(\mathbf{B}^n, N)$ *when* $\pi_1(N)$ *is not abelian? The same question can also be asked about the functional spaces* $H^{\frac{1}{2}}(M, N)$.

■

In Chapter V, when we consider the problem of weak density of smooth maps in $W^{1,1}(\mathbf{B}^n, N)$ for non-abelain fundamental group, we will try to explain the obstacles regarding this situation and to put the bases for a future response to this question.

Weak density of smooth maps and the connections

While the question of flat convergence for the singularity chains of a sequence of convergent maps $v_m \in R^{\infty,p}(\mathbf{B}^n, N)$ remain to be answered (See theorem 4), one can ask also a weaker question : Does the flat norm of $\mathbf{S}_{v_m}$ remain bounded as $v_m \to u$? This is another problem we address in Chapter IV about the uniform boundedness of the mass $\mathbf{M}(\mathbf{T}_m)$ of a minimal connection $\mathbf{T}_m$ ($\partial \mathbf{T}_m = \mathbf{S}_m$) as $v_m \to u$.

Related to this question is the problem of the weak density of smooth maps in $W^{1,p}(M, N)$. Although the density of smooth maps for the weak topology can be easily handled from the one for the strong topology (See [2]), the question of the density of smooth maps in $W^{1,p}(M, N)$ for the sequentially weak topology , where $p \in \mathbb{N}$, is more involved : For $p \in \mathbb{N}$, $\pi_p(N) \neq 0$, does there exist for any $u \in W^{1,p}(M, N)$ a sequence $u_m \in C^\infty(M, N)$ such that $u_m \rightharpoonup u$ in $W^{1,p}$? The case $M = \mathbf{B}^3$, $N = S^2$, $p = 2$ was treated by F.Bethuel, H.Brezis, J.M.Coron and E.Lieb in [10], and [3]. F.Bethuel mentioned that the answer is yes for $M = \mathbf{B}^n$, $N = S^p$, $p \geq 2$ in [2]. In [19], P.Hajlasz proved that if N is $(p-1)$-connected, any map in $W^{1,p}(M, N)$ is the weak limit of a sequence of smooth maps in this space. Observe that this result can be also deduced from the work of F.Bethuel, J.-M.Coron, F.Demengel and F.Hélein in [6] for when $M = \mathbf{B}^n$ and $\pi_p(N)$ is torsionless.

As we will explain in Chapters IV and V, the control of the mass of the minimal chain connecting $\mathbf{S}_{v_m}$ for $v_m \in R^{\infty,p}(\mathbf{B}^n, N)$ converging strongly to u permits to give a positive answer to the sequentially weak density of smooth maps. This appraoch is different from the one used by P.Hajlasz and can be used for proving his theorem and some other partial results regarding the weak sequential denstiy of maps in $W^{1,p}(\mathbf{B}^n, N)$. Specially, Hajlasz's method is not adapted when we wish to approach $u \in W^{1,p}_\varphi(\mathbf{B}^n, N)$ in the weak topology by a sequence of maps in $C^\infty_\varphi(\mathbf{B}^n, N)$ (He does not mention this question in [19]). The case $p = 1$ is more involved when $\pi_1(N)$ is non-abelian and we will discuss it in an independent chapter (Chapter V). The reason is that in this case we can not identify an element of $\pi_1(N)$ without fixing its base point, so defining the topological singularities as the flat chains with coefficeints in $\pi_1(N)$ meets obstacles. There are some other technical complications which we will mention in Chapter V.

In theorems 2 *bis*, 3 *bis* in Chapter IV and in the theorem 1 *bis* in Chapter V we will prove :

Theorem 5 *Assume that $\mathbf{B}^n$ is the unit disk in $\mathbb{R}^n$ and that N is a compact riemannien* *manifold of dimension $k \geq [p]$, $\partial N = \emptyset$, and either $[p] = 1$ or N is $([p]-1)$-connected, i.e.*

$$\pi_1(N) = \cdots = \pi_{[p]-1}(N) = 0.$$

Also assume that $\varphi : \partial \mathbf{B}^n \to N$ admits a smooth extension over $\mathbf{B}^n$. Then, for any map $u \in W^{1,p}_{\varphi}(\mathbf{B}^n, N)$, there exists a sequence of smooth maps $u_m : \mathbf{B}^n \to N$, $u_m|_{\partial \mathbf{B}^n} = \varphi$, such that $\|u_m - u\|_{L^p} \to 0$ and that $\|u_m\|_{W^{1,p}}$ is bounded by a constant.

■

Remark 6 *Naturally if $p \geq 2$, we can always find a subsequence of such a sequence, converging weakly to u. But the question of sequentially weak density of smooth maps in $W^{1,1}(\mathbf{B}^n, N)$ is still open.*

We can extend the results of theorem 5 for $p \geq 2$ when $\pi_2(N)$ is finitely generated. Specially for $p = 2$

Theorem 6 (IV.4 et IV.4 *bis*) *If $\pi_2(N)$ is finitely generated, we have the sequentially weak density of $C^{\infty}(\mathbf{B}^n, N)$ (resp. $C^{\infty}_{\varphi}(\mathbf{B}^n, N)$) in $H^1(\mathbf{B}^n, N)$ (resp. $H^1_{\varphi}(\mathbf{B}^n, N)$).*

■

The recent developments by F.Hang and F.H.Lin [20] have shown that one should consider the global topology of M for extending these results to any smooth compact manifold M as the domain using the same methods. We hope to expose in near future how our proofs for the sequentially weak density of smooth maps in the Sobolev spaces can be adapted to any domain.

Remark 7 *We do not have always the equi-boundedness of the mass of minimal connections for $\mathbf{S}_{v_m}$ when $v_m \to u$ in $W^{1,p}(\mathbf{B}^n, N)$: For instance, there exist $v_m \in R^{3,\infty}(\mathbf{B}^4, S^2)$ such that*

$$\inf\{\mathbf{M}(\mathbf{T}_m)\,;\ \mathbf{T}_m \textit{ is a } \mathbb{Z}-\textit{chain such that } \partial \mathbf{T}_m = \mathbf{S}_{v_m}\} \longrightarrow +\infty$$

as $v_m \to u$ in $W^{1,3}(\mathbf{B}^4, S^2)$ (See [24]). However it is not excluded that the smooth maps be sequentially weakly dense in $W^{1,3}(\mathbf{B}^4, S^2)$.

Open Question 6 *Are the smooth maps are sequentially weakly dense in $W^{1,3}(\mathbf{B}^4, S^2)$? Also, regarding the results obout the sequentially weak density of smooth maps in $H^{\frac{1}{2}}(S^2, S^1)$ by T.Rivière ([34]), we ask the same questions about $H^{\frac{1}{2}}(\mathbf{B}^n, N)$.*

■

Remark 8 *Meanwhile, using a global obstruction, F.Hang and F.H.Lin proved that the smooth maps are not sequentially dense in $W^{1,3}(\mathbb{CP}^2, S^2)$ (See[20]) .*

This question remains open too for some other cases which are not put forward in this thesis.

Chapter I

Relaxing the Dirichlet energy

Relaxing the Dirichlet energy for maps into S^2 in high dimensions

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We consider various ways of relaxing the Dirichlet energy of maps into sphere.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with regular boundary and let

$$H^1(\Omega, S^2) = \{u \in H^1(\Omega, \mathbb{R}^3)\,;\, u(x) \in S^2 \quad \text{a.e. on}\, \Omega\}$$

and

$$H^1_\varphi(\Omega, S^2) = \{u \in H^1(\Omega, S^2)\,;\, u = \varphi \quad \text{on}\, \partial\Omega\}$$

where φ is a given boundary data. For $u \in H^1_\varphi(\Omega, S^2)$ the Dirichlet energy is given by $E(u) = \int_\Omega |\nabla u|^2$. We assume that φ is in $C^\infty(\partial\Omega, S^2)$ and can be extended into Ω by a smooth map.

We call u to be a weakly harmonic map if it is a critical point for the functional E, i.e. if and only if we have

$$\frac{d}{dt}E\left(\frac{u+tv}{|u+tv|}\right)_{|t=0} = 0 \quad \text{for all} \quad v \in C_c^\infty(\Omega, \mathbb{R}^3)\,.$$

In other words, u is weakly harmonic in the Sobolev space $H^1(\Omega, S^2)$ if it satisfies the following equation in the sense of distributions :

$$\begin{cases} -\Delta u = u|\nabla u|^2 \quad \text{in} \quad \Omega \\ \\ u(x) \in S^2 \quad \text{a.e.} \end{cases} \tag{1.1}$$

Assuming $\varphi : \partial\Omega \to S^2$ is as above, one of the important problems which is still open is if smooth harmonic extensions of φ into Ω exist. In the first step one may want to minimize the Dirichlet energy in $H^1_\varphi(\Omega, S^2)$ and prove the regularity of the solution. But in fact if we define

$$\mu_\varphi := \inf_{H^1_\varphi(\Omega,S^2)} E(u) \leq \inf_{C^\infty_\varphi(\overline{\Omega},S^2)} E(u) =: \bar{\mu}_\varphi\,,$$

the strict inequality

$$\mu_\varphi < \bar{\mu}_\varphi$$

happens sometimes (See [22]). Thus minimizers of E are not necessarily smooth and we should find other harmonic maps which could be a suitable candidate for a smooth solution. Meanwhile R.Schoen and K.Uhlenbeck ([35]) proved that these minimizers are smooth in Ω except on a finite set of points.

In trying to attack this problem, another functional on $H^1_\varphi(\Omega, S^2)$ has been studied which is called the “relaxed energy”. In fact, the relaxed energy is the largest sequentially lower semi-continuous functional on $H^1_\varphi(\Omega, S^2)$ which is less than E on $C^\infty_\varphi(\Omega, S^2)$:

Definition 1.1 *The relaxed energy $\mathcal{F}$ of E on $H^1_\varphi(\Omega, S^2)$ is defined to be*

$$\mathcal{F}(u) := \inf\left\{ \liminf_{n\to\infty} E(u_n)\,;\; u_n \in C^\infty_\varphi(\Omega, S^2)\,,\; u_n \rightharpoonup u \right\}\,. \tag{1.2}$$

Since the smooth maps which take φ as their boundary value are weakly sequentially dense in $H^1_\varphi(\Omega, S^2)$ (See [2]), we observe that $\mathcal{F}$ is well defined. Moreover $\mathcal{F}$ is sequentially lower semi-continuous with respect to the weak topology in $H^1_\varphi(\Omega, S^2)$ and we have

$$\inf_{H^1_\varphi(\Omega,S^2)} \mathcal{F} = \inf_{C^\infty_\varphi(\Omega,S^2)} E\,. \tag{1.3}$$

This equation shows the importance of study of $\mathcal{F}$. Since the infimum of $\mathcal{F}$ in $H^1_\varphi(\Omega, S^2)$ is achieved, the question which should be considered then is whether a minimizer of $\mathcal{F}$ is weakly harmonic and to what extent it is regular.

In this line, F.Bethuel, H.Brezis, J.M. Coron and E.Lieb (See [5] and [10]) showed the striking fact that, for $n = 3$, the relaxed energy achieves the following elegant algebric formula :

$$\mathcal{F}(u) = F(u) := E(u) + 8\pi L(u)$$

where

$$L(u) := \frac{1}{4\pi} \sup_{\substack{\psi : \Omega \to \mathbb{R} \\ |d\psi|_\infty \leq 1}} \left\{ \int_\Omega u^* \omega_V \wedge d\psi - \int_{\partial\Omega} \varphi^* \omega_V \wedge \psi \right\} \tag{1.4}$$

where ω_V is the volume form on S^2 (or can be replaced by any 2-form ω, $\int_{S^2} \omega = 4\pi$). In particular this yields that the critrical points of $\mathcal{F}$ are weakly harmonic. F.Bethuel and H.Brezis showed also that the minimizers of $\mathcal{F}$ are smooth in Ω except on a finite set of points (See[4]).

The intuitive approach for $L(u)$ is that if $u \in H^1_\varphi(\Omega, S^2)$ is smooth in Ω except on a set of finite points $\{p_1, ..., p_m\}$, taking the degree d_i on the point p_i, then $L(u)$ is the minimum length of the segments connecting these singularities with respect to the multiplicities (See [10]). In other words

$$L(u) = m_i \left(\sum_{i=1}^m d_i\, [[p_i]],\, \Omega \right)$$

where we define for the i.m. rectifiable 0-current $\mathbf{S}_u = \sum_{i=1}^m d_i\, [[p_i]]$:

$$m_i(\mathbf{S}_u, \Omega) := \inf \left\{ \mathbf{M}(\mathbf{T})\, ;\, T \in \mathcal{R}_1(\mathbb{R}^3),\, \text{spt}\, \mathbf{T} \subset \overline{\Omega}\, ,\, \partial\mathbf{T} = \mathbf{S}_u \right\} .$$

Here we study the same approach for $n > 3$ but this generalisation meets obstacles. One may introduce for ω, any 2-form on S^2 which satisfies $\int_{S^2} \omega = 1$:

$$L(u) := \sup_{\substack{\psi \in \Omega^{n-3}(\overline{\Omega}) \\ |d\psi|_\infty \leq 1}} \left\{ \int_\Omega u^* \omega \wedge d\psi \, - \int_{\partial\Omega} \varphi^* \omega \wedge \psi \right\} \tag{1.5}$$

as a generalization of $L(u)$ in the 3-dimensional case. Observe that L is independant of the choice of ω and is continuous on $H^1_\varphi(\Omega, S^2)$ for $\Omega \subset \mathbb{R}^n$ and the functional

$$F(u) := E(u) + 8\pi L(u) \tag{1.6}$$

would still be weakly lower semi-continuous. But we have this theorem :

Theorem 1 *For every $\Omega \subset \mathbb{R}^4$ and every map $\varphi \in C^\infty(\partial\Omega, S^2)$, smoothly extendable onto Ω, there exists $u \in H^1_\varphi(\Omega, S^2)$ such that*

$$F(u) < \mathcal{F}(u)\, . \tag{1.7}$$

Moreover there exists a domain $\Omega \subset \mathbb{R}^4$ and $\varphi \in C^\infty(\partial\Omega, S^2)$, smoothly extendable onto Ω, for which this gap phenomenon exists :

$$\inf_{H^1_\varphi(\Omega,S^2)} E < \inf_{H^1_\varphi(\Omega,S^2)} F < \inf_{C^\infty_\varphi(\Omega,S^2)} E\,. \tag{1.8}$$

■

The difference with the case $n = 3$ lies in the value which $L(u)$ represents. We shall consider a map $u \in H^1_\varphi(\Omega, S^2)$, which is smooth except on a finite union of $(n-3)$-dimensional submanifolds of Ω : $\{\sigma_1, ...\sigma_m\}$. The degree d_i of u on each σ_i is well defined and we define $\mathbf{S}_u := \sum_{i=1}^m d_i\,[[\sigma_i]]$. Calculating $L(u)$, we see that

$$L(u) = \sup_{|d\psi|_\infty \leq 1} \int_{\mathbf{S}_u} \psi \leq \sup_{\|d\psi\|^*_\infty \leq 1} \int_{\mathbf{S}_u} \psi = m_r(\mathbf{S}_u, \Omega) \tag{1.9}$$

where $\|.\|^*$ is the co-mass norm on the space of forms and

$$m_r(\mathbf{S}_u, \Omega) := \inf\left\{\mathbf{M}(\mathbf{T})\,;\, \mathbf{T} \in \mathcal{D}_{n-2}(\mathbb{R}^n)\,, \partial\mathbf{T} = \mathbf{S}_u,\, \operatorname{spt}\mathbf{T} \subset \overline{\Omega}\right\}$$

is the mass of the minimal normal (real) current in Ω with boundary $\mathbf{S}_u$. The last equation in (1.9) is due to the fact that there exists always a calibration for minimizing normal currents, which we shall discuss later in this paper (See proposition 2.3). Meanwhile, $m_i(\mathbf{S}_u, \Omega)$, the minimal mass of i.m. rectifiable currents in $\overline{\Omega}$ which are bounded by $\mathbf{S}_u$, is still proportional to the energy needed for removing the singularities of u. Here arises the main question which should be answered if we want to continue as above, that is if

$$m_r(\mathbf{S}, \Omega) = m_i(\mathbf{S}, \Omega) \quad \forall \mathbf{S} \in \mathcal{R}_{n-3}(\Omega).$$

But contrary to the case $n = 3$, the answer is no for $n > 3$. Specially, for $n = 4$, there exists a curve $[[\Gamma]]$ in $\mathbb{R}^4$ for which

$$m_r([[\Gamma]]) < m_i([[\Gamma]]).$$

This gap phenomenon was firstly proved by L.C.Young in [42]. F.Morgan in [27] and B.White in [37] have given other examples of such curves in $\mathbb{R}^4$.

Remark 1.1 *Meanwhile, in [29], we observed that the critical points of F are still weakly harmonic in $H^1_\varphi(\Omega, S^2)$ and we used this to prove the existence of infinitely many weakly harmonic extensions of φ onto Ω.*

Finally we may search for the amount of energy needed to relax the Dirichlet energy. In section 3 we prove that the topological singular set $\mathbf{S}_u$ of any $u \in H^1_\varphi(\Omega, S^2)$ is the boundary of some i.m. rectifiable current. Then the evidences we discuss in this paper propose that $\mathcal{F}$ coincides with

$$\widetilde{F}(u) := E(u) + 8\pi m_i(\mathbf{S}_u, \Omega).$$

We can only prove that $\widetilde{F} \leq \mathcal{F}$, the inequality in the other direction being still an open problem (See proposition 3.1 and the remark following). Meanwhile, we can prove $\widetilde{F} \geq \mathcal{F}$ when we consider the problem of relaxing the 3-energy of maps into S^3. We will present this example in a forthcoming paper.

2 Preliminaries

2.1 The subspace $R^\infty_\varphi(\Omega, S^2)$

Definition 2.1 *We say that $u \in H^1_\varphi(\Omega, S^2)$ is in $R^\infty_\varphi(\Omega, S^2)$ if and only if u is smooth except on $B = \bigcup_{i=1}^m \sigma_i \cup B_0$, a compact subset of Ω, where $\mathcal{H}^{n-3}(B_0) = 0$ and the σ_i , $i = 1, \cdots, m$ are disjoint smooth embeddings of the open $(n-3)$-dimensional unit disk. Moreover we assume that any two σ_i and σ_j can meet only on their boundaries.*

Remark 2.1 *According to ([2], theorem 2 bis), $R^\infty_\varphi(\Omega, S^2)$ is dense in $H^1_\varphi(\Omega, S^2)$.*

Definition 2.2 *Let $u \in H^1_\varphi(\Omega, S^2)$. We define the current $\mathbf{S}_u \in \mathcal{D}_{n-3}(\Omega)$ to be the current defined by*

$$\mathbf{S}_u(\alpha) := \int_\Omega u^*\omega \wedge d\alpha \qquad \forall \alpha \in \mathcal{D}^{n-3}(\Omega). \tag{2.1}$$

Here $\mathcal{D}^k(\Omega)$ is the set of smooth k-forms on Ω with compact support (*See*[*16*], 2.2.3) *and ω is some 2-form on S^2 for which $\int_{S^2} \omega = 1$.*

A simple observation shows that the definition of $\mathbf{S}_u$ is independent of the choice of ω due to the fact that the difference of two closed forms on S^2 is exact. The existence of the integral (2.1) is a direct consequence of the following inequality :

$$|u^*\omega| \leq \frac{1}{8\pi}|\nabla u|^2 \qquad \text{a.e. on}\, \Omega \tag{2.2}$$

where $4\pi\omega = \omega_V$ is the standard volume form of S^2.

Definition 2.3 *Let $u \in R^\infty_\varphi(\Omega, S^2)$ and let $B = \bigcup \sigma_i \cup B_0$ be the singular set of u. Suppose that each σ_i is oriented by a smooth $(n-3)$-vectorfield $\vec{\sigma}_i$. For $a \in \sigma_i$ and N_a the 3-dimensional plane orthogonal to σ_i at a. Consider the 3-disk $M_{a,\delta} = B_\delta(a) \cap N_a$ oriented by the 3-vector $\vec{M}_a$ such that $\vec{\sigma}_i(a) \wedge \vec{M}_a = (-1)^n \vec{\xi}_{\mathbb{R}^n}$. Then the topological degree of u on the 2 dimensional sphere $\Sigma_{a,\delta} = \partial M_{a,\delta}$ is well defined and is independent of the choice of a for δ small enough. We should call this integer the degree of u on σ_i and denote it by*

$$deg_{\sigma_i} u\,.$$

We shall mention here some useful facts which we have already proved in [29]. Recall that any k-dimensional rectifiable subset $\mathcal{M}$ of $\mathbb{R}^n$ considered with a multiplicity θ and oriented by a unit k-vector field $\vec{\xi}$ defines a rectifiable current as follows

$$\tau(\mathcal{M}, \theta, \vec{\xi})(\alpha) := \int_{\mathcal{M}} < \vec{\xi}, \alpha > \theta \, d\mathcal{H}^k \qquad \forall \alpha \in \mathcal{D}^k(\mathbb{R}^n).$$

Lemma 2.1 *Let $\omega = \frac{1}{4\pi}\omega_V$ and $u \in R^\infty_\varphi(\Omega, S^2)$. Then the $(n-2)$-vectorfield $\vec{D}(u)$ defined on $\Omega \backslash B$ by the equation*

$$< \vec{D}(u)(x), \Psi > \omega_{\mathbb{R}^n} := u^*\omega(x) \wedge \Psi \quad \forall \Psi \in \Lambda^{n-2}(\mathbb{R}^n) \tag{2.3}$$

is a simple $(n-2)$-vectorfield tangent to the smooth manifold $u^{-1}(y)$ for all regular value $y = u(x) \in S^2$. Meanwhile

$$|\vec{D}(u)| = \frac{1}{4\pi}|J_2 u| \quad a.e. \ on \ \Omega. \tag{2.4}$$

■

An element of $\Lambda_k(\mathbb{R}^n)$ is called *simple* if and only if it equals the exterior product of k vectors of $\mathbb{R}^n$ ([13], 1.6.1).

In the view of lemma 2.1, for any $y \in S^2$ a regular value of $u \in R^\infty_\varphi(\Omega, S^2)$, the current

$$\mathbf{T}^u_y := \tau\left(u^{-1}(y), 1, \frac{\vec{D}(u)}{|\vec{D}(u)|}\right) \tag{2.5}$$

is well defined. Moreover

Proposition 2.1 *Consider $u \in R^\infty_\varphi(\Omega, S^2)$ and $\mathbf{T}^u_y$ as in (2.5), then for almost all $y \in S^2$, $\mathbf{T}^u_y$ is a rectifiable current in $\mathbb{R}^n$ with support in $\overline{\Omega}$ and*

$$\partial \mathbf{T}^u_y = \mathbf{S}_u + \tau\left(\varphi^{-1}(y), 1, \frac{\vec{D}(\varphi)}{|\vec{D}(\varphi)|}\right) \tag{2.6}$$

where the $(n-3)$-vectorfield $\vec{D}(\varphi)$ on $\partial\Omega$ is defined by the equation

$$< \vec{D}(\varphi)(x), \Psi > \omega_{E_x} := \varphi^*\omega(x) \wedge \Psi \quad \forall \Psi \in \Lambda_{n-3}(E_x)$$

where $E_x = T_x(\partial\Omega)$ is the tangent space to $\partial\Omega$ at x and ω_{E_x} is its unit volume form.

■

Proposition 2.2 *Let $u \in R^\infty_\varphi(\Omega, S^2)$ and $B = \bigcup_i \sigma_i \cup B_o$ its singular set. Then*

$$\mathbf{S}_u = \sum_i (deg_{\sigma_i} u)\tau(\sigma_i, 1, \vec{\sigma}_i).$$

■

2.2 Calibrations and minimizing real currents

Let $\mathbf{T}$ be a normal current in $\mathcal{D}_m(\mathbb{R}^n)$ with support in a compact set : K.

Definition 2.4 *The measurable form α in $\Omega^m(\mathbb{R}^n)$ is called to be a calibration for $\mathbf{T}$ in K if*

$$\begin{cases} (i)\ \alpha \text{ is exact}, \\ \\ (ii)\ \|\alpha_{|K}\|_\infty^* \leq 1\,, \\ \\ (iii)\ \mathbf{T}(\alpha) = \mathbf{M}(\mathbf{T})\,. \end{cases} \tag{2.7}$$

We say then that $\mathbf{T}$ is calibrated in K.

We have this interesting proposition which shows the importance of calibrations in the study of minimal currents :

Proposition 2.3 *The real current $\mathbf{T}$ is calibrated in K if and only if it has the minimal mass among all the real currents supported in K and taking the same boundary. Specially for any open bounded set Ω in $\mathbb{R}^n$ and any real flat chain $\mathbf{S}$ in Ω we have*

$$m_r(\mathbf{S},\Omega) = \sup_{\|d\psi\|_\infty^* \leq 1} \mathbf{S}(\psi). \tag{2.8}$$

■

We omit the proof since it is the same as the proof for ([18], proposition 4.35, p. 59). The interesting fact is that, as a result, a minimal i.m. rectifiable current is calibrated if and only if it is also a minimal real current for the same boundary. The only cases where this always happens are when the minimal current is of dimension or codimension 1 in Ω. In other words if $dim\, \mathbf{S} = 0$ or $n-2$, then

$$m_r(\mathbf{S},\Omega) = m_i(\mathbf{S},\Omega). \tag{2.9}$$

For the proof and some counterexamples when the conditions are not satisfied see ([14], section 5). The readers can refer to ([16], vol II, section 1.3.4) for more details. In [1], the authors present an interesting proof of (2.9) for $dim\, \mathbf{S} = n-2$. Also different proofs for the zero dimensional case can be found in [10] and [12]. For other counterexamples see [27], [37] and [42].

2.3 The F-energies

For any 2-form ω on S^2 satisfying $\int_{S^2} = 1$ and $u \in H^1_\varphi(\Omega, S^2)$ we define

$$L(u) := \sup_{\substack{\psi\, \in\, \Lambda^{n-3}(\overline{\Omega}) \\ \|d\psi\|_\infty \leq 1}} \left\{ \int_\Omega u^*\omega \wedge d\psi \; - \int_{\partial\Omega} \varphi^*\omega \wedge \psi \right\} \tag{2.10}$$

and

$$L^*(u) := \sup_{\substack{\psi \in \Lambda^{n-3}(\overline{\Omega}) \\ \|d\psi\|_\infty^* \leq 1}} \left\{ \int_\Omega u^*\omega \wedge d\psi - \int_{\partial\Omega} \varphi^*\omega \wedge \psi \right\} \tag{2.11}$$

where $|.|$ and $\|.\|^*$ are respectively the euclidean and the co-mass norms on the space of forms. The definitions are independent of the choice of ω (See [29]), so from now on we put $\omega = (1/4\pi)\omega_V$.

Remark 2.2 *L and L^* are both continuous with respect to the H^1 norm in $H^1_\varphi(\Omega, S^2)$. The proof is the same as for the case $n = 3$ in [5].*

We have

Lemma 2.2 *For any $u \in H^1_\varphi(\Omega, S^2)$, $\mathbf{S}_u$ is a real flat chain. Moreover we have*

$$L(u) \leq L^*(u) = m_r(\mathbf{S}_u, \Omega)\,. \tag{2.12}$$

■

Proof : Set

$$\mathbf{D}_u(\alpha) := \int_\Omega u^*\omega \wedge \alpha \qquad \forall \alpha \in \mathcal{D}^{n-2}(\Omega).$$

Since by (2.2) we have $8\pi\mathbf{M}(\mathbf{D}_u) \leq E(u)$, $\mathbf{D}_u$ is a normal current. Moreover, by definition, $\mathbf{S}_u = \partial\mathbf{D}_u$, so $\mathbf{S}_u$ is a real flat chain. We have, using (2.8),

$$\begin{aligned} m_r(\mathbf{S}_u - \mathbf{S}_v, \Omega) &= \sup_{\substack{\psi \in \Lambda^{n-3}(\overline{\Omega}) \\ \|d\psi\|_\infty^* \leq 1}} (\mathbf{S}_u - \mathbf{S}_v)(\psi) \\ &\leq C\|\nabla u\|_2 \|\nabla v\|_2 (\|\nabla u - \nabla v\|_2), \end{aligned}$$

where the last inequality is obtained by the same method as in ([5], theorem 1). As a result $m_r(\mathbf{S}_u, \Omega)$ is continuous with respect to the strong topology in $H^1_\varphi(\Omega, S^2)$.

Meanwhile, if $u \in R^\infty_\varphi(\Omega, S^2)$, using the co-area formula and proposition 2.1 successively we obtain

$$\begin{aligned} \int_\Omega u^*\omega \wedge d\psi - \int_{\partial\Omega} \varphi^*\omega \wedge \psi &= \int_\Omega u^*\omega \wedge d\psi - \int_\Omega \phi^*\omega \wedge d\psi \\ &= \int_{S^2} (\mathbf{T}^u_w(d\psi) - \mathbf{T}^\phi_w(d\psi))\, dw \\ &= \mathbf{S}_u(\psi). \end{aligned}$$

This implies

$$L^*(u) = \sup_{\substack{\psi \in \Lambda^{n-3}(\overline{\Omega}) \\ \|d\psi\|_\infty^* \le 1}} \left\{ \int_\Omega u^*\omega \wedge d\psi - \int_{\partial\Omega} \varphi^*\omega \wedge \psi \right\}$$

$$= \sup_{\|d\psi\|_\infty^* \le 1} \mathbf{S}_u(\psi) = m_r(\mathbf{S}_u, \Omega).$$

Since $L^*(u)$ and $m_r(\mathbf{S}_u, \Omega)$ are continuous in H^1-norm and considering the fact that $R^\infty_\varphi(\Omega, S^2)$ is dense in $H^1_\varphi(\Omega, S^2)$ for the strong topology, we deduce the equality in (2.12). Moreover, $L \le L^*$ as $\|\psi\|_\infty^* \le \|\psi\|_\infty$ for all differential forms. ■

Definition 2.5 *We define the F-energies to be*

$$F(u) := E(u) + 8\pi L(u)$$

and

$$F^*(u) := E(u) + 8\pi L^*(u).$$

2.4 Sequentially weak density of smooth maps in $H^1_\varphi(\Omega, S^2)$

Let us recall some facts about maps in $R^\infty_\varphi(\Omega, S^2)$:

Proposition 2.4 *There exists $C > 0$ such that for all $u \in R^\infty_\varphi(\Omega, S^2)$ we have*

$$8\pi m_i(\mathbf{S}_u) \le E(u) + C. \tag{2.13}$$

Moreover there exists a sequence $u_m \in R^\infty_\varphi(\Omega, S^2)$ such that

$$\begin{cases} \mathbf{S}_{u_m} = 0 \\ u_m = u \text{ on } K_m \\ \mu(K_m) \to 0 \text{ as } m \to \infty \\ E(u_m) \le E(u) + 8\pi m_i(\mathbf{S}_u) + \dfrac{1}{m} \\ u_m \rightharpoonup u \text{ in } H^1 \end{cases} \tag{2.14}$$

■

(2.13) is proved in [1]. In ([2], section VI), the author, suggesting (2.14) and considering (2.13), remarked that smooth maps are sequentially dense in $H^1_\varphi(\Omega, S^2)$ for the weak topology, as in the case $n = 3$ (See [3]). Recent developments by F.Hang and F.H.Lin showed that this argument should be modified for when the domain is not contractible. They remarked that "$\mathbf{S}_u = 0$" is not always the sufficient condition for the strong approximability of $u \in H^1(M, S^2)$ by smooth maps and we should consider global topological obstructions too (See [20]). But the arguments used in [6] work locally and therefore if $\mathbf{B}^n$ is the n-dimensional unit disk in $\mathbb{R}^n$, for any map $u \in H^1_\varphi(\mathbf{B}^n, S^2)$, there exists a sequence of smooth maps $u_m \in C^\infty_\varphi(\mathbf{B}^n, S^2)$ such that

$$\begin{cases} (i)\, u_m \rightharpoonup u \text{ in } H^1 \\ \\ (ii)\, E(u_m) \leq 2E(u) + C + O(\dfrac{1}{m}) \end{cases} \tag{2.15}$$

We will present our method for proving (2.14) in a forthcoming paper where we will treat the question of sequentially weak density of smooth maps in Sobolev spaces between manifolds ([31]).

3 A lower bound for the relaxed energy

Proposition 3.1 *Let $u \in H^1_\varphi(\Omega, S^2)$, then $\mathbf{S}_u$ is the boundary of some i.m.rectifiable current. Set*

$$\widetilde{F}(u) := E(u) + 8\pi m_i(\mathbf{S}_u, \Omega). \tag{3.1}$$

$\widetilde{F}$ is lower semi-continuous with respect to the weak topology in $H^1_\varphi(\Omega, S^2)$ and

$$\widetilde{F}(u) \leq \mathcal{F}(u)\,, \quad \forall u \in H^1_\varphi(\Omega, S^2). \tag{3.2}$$

■

Remark 3.1 *We do not prove that $\widetilde{F}$ is the relaxed energy. A stronger result for the case $\Omega = \mathbf{B}^n$ would be to show that $m_i(\mathbf{S}_u, \mathbf{B}^n)$ is continuous in $H^1_\varphi(\mathbf{B}^n, S^2)$, which is still an open problem (Compare with Remark 2.2 and lemma 2.2).*

Proof : For the sake of simplicity we prove the proposition for $\Omega = \mathbf{B}^n$, the n-dimensional unit disk. For the general case we can replace smooth maps by maps satisfying the condition $\mathbf{S}_{u_m} = 0$.

Let $u \in H^1_\varphi(\Omega, S^2)$ and consider a sequence of smooth maps converging weakly to u as in (2.15). Since u_m is smooth, $\partial G_{u_m} = 0$, where G_{u_m} is the graph of u_m. Also since the Dirichlet energy is regular (See [16], vol II, section 5.2.1), the G_{u_m} are equi-bounded in mass. By the Compactness theorem, there is an i.m. rectifiable n-current T supported in $\Omega \times S^2$ such that $G_{u_m} \rightharpoonup T$ up to some sub-sequence. By ([16], vol I, section 5.5.2, proposition 3), $G_{u_m} \in cart^{2,1}(\Omega \times S^2)$ for all m. So by the closure theorem ([16], vol I, section 5.5.2, theorem 6) and the Structure theorem ([16], vol II, section 5.2.1) we have

$$T = G_u + L_T \times [[S^2]] \in cart^{2,1}(\Omega \times S^2)$$

while L_T is an $(n-2)$-dimensional i.m. rectifiable current in Ω. From ([16], vol II, section 1.2.4, proposition 15) and (2.15) we deduce that

$$8\pi\mathbf{M}(L_T) \leq E(u) + C. \tag{3.3}$$

Now let π and $\hat{\pi}$ be the respective projections of $\Omega \times S^2$ on Ω and S^2. Since $\partial T = 0$, for any 2-form ω on S^2 and any compactly supported $(n-3)$-form α in Ω we have

$$\int_\Omega u^*\omega \wedge d\alpha = G_u(\pi^*(d\alpha) \wedge \hat{\pi}^*\omega) = \partial G_u(\pi^*\alpha \wedge \hat{\pi}^*\omega) = -\partial L_T(\alpha).$$

So $\mathbf{S}_u = \partial(-L_T)$, which proves the first claim of the proposition. Moreover, as a consequence, $\widetilde{F}$ is well defined for the maps in $H^1_\varphi(\Omega, S^2)$.

Let u_m be a sequence of maps in $H^1_\varphi(\Omega, S^2)$ converging weakly to u. We will prove that

$$\widetilde{F} \leq \liminf_{m\to\infty} \widetilde{F}(u_m). \tag{3.4}$$

Put

$$\beta := \liminf_{m\to\infty} \widetilde{F}(u_m).$$

Passing to some subsequence of u_m if necessary, we have $\widetilde{F}(u_m) \to \beta < +\infty$. Let $-L_m$ be the mass minimizing i.m.rectifiable current taking $\mathbf{S}_{u_m}$ as its boundary. The u_m are equi-bounded in energy while the L_m are equi-bounded in mass. So, using the same argumets as above, we see that the cartesian currents

$$T_m := G_{u_m} + L_m \times [[S^2]]$$

converge to some current $T := G_u + L \times [[S^2]]$ in $cart^{2,1}(\Omega \times S^2)$, up to a subsequence. By ([16], vol II, section 1.2.4, proposition 15) we get

$$\begin{aligned}
\widetilde{F}(u) &= E(u) + 8\pi m_i(\mathbf{S}_u) \leq E(u) + 8\pi\mathbf{M}(L) \\
&\leq \liminf_{m\to\infty}(E(u_m) + 8\pi\mathbf{M}(L_m)) \\
&= \liminf_{m\to\infty}(E(u_m) + 8\pi m_i(\mathbf{S}_{u_m})) \\
&= \beta.
\end{aligned}$$

This proves (3.4). Thus $\widetilde{F}$ is lower semi-continuous with respect to the weak topology. (3.2) follows immediately as $\widetilde{F}$ coinsides with E on smooth maps. ∎

4 Proof of theorem 1

4.1 Proof of (1.7)

We observe that regarding lemma 2.2 and proposition 3.1, it suffices to prove the existence of a map $u \in R^\infty_\varphi(\Omega, S^2)$ which satisfies

$$m_r(\mathbf{S}_u, \Omega) < m_i(\mathbf{S}_u, \Omega). \tag{4.1}$$

This happens for $n = 4$. Specially there is a curve Γ in $\mathbb{R}^4$ for which $m_r([[\Gamma]]) < m_i([[\Gamma]])$ (See [42], [14] and [16], vol II for more details). For any $\Omega \subset \mathbb{R}^4$ and boundary data φ, we can construct a map $u \in R^\infty_\varphi(\Omega, S^2)$ which is smooth except on such a curve, supported in a small ball in Ω. The method is almost the same as the one used by the authors in [1] for constructing a map with prescribed singularities and constant boundary value, so we will not expose the details in this paper. This map will satisfy (4.1).

4.2 Sketch of the proof for (1.8)

a) For $0 < \delta' < \delta$, we construct a domain $\Omega_{\delta,\delta'} \subset \mathbb{R}^4$ and a map $\varphi_{\delta,\delta'} \in C^\infty(\partial\Omega_{\delta,\delta'}, S^2)$ which is extendable onto $\Omega_{\delta,\delta'}$. We put

$$H^1_{\delta,\delta'} := H^1_{\varphi_{\delta,\delta'}}(\Omega_{\delta,\delta'}, S^2)$$

and

$$C^\infty_{\delta,\delta'} := H^1_{\delta,\delta'} \cap C^\infty(\Omega_{\delta,\delta'}, S^2).$$

b) We prove that

$$\inf_{H^1_{\delta,\delta'}} F^* - \inf_{C^\infty_{\delta,\delta'}} E = O(\delta) - k$$

when $k > 0$.

c) Regarding the fact that $F \leq F^*$ the theorem is proved by choosing δ small enough.

4.3 Construction of $\Omega_{\delta,\delta'}$

Let $\mathbf{B}$ be an integer multiplicity m-rectifiable current in $\mathbb{R}^n$, without boundary. Put

$$m_i(\mathbf{B}) := \min\{\mathbf{M}(\mathbf{T})\,;\, \partial\mathbf{T} = \mathbf{B}\,,\ \mathbf{T} \in \mathcal{R}_{m+1}(\mathbb{R}^n)\} \tag{4.2}$$

where $\mathbf{M}(\mathbf{T})$ is the mass of $\mathbf{T}$. By [42] there exists Γ, a closed curve on $K \subset \mathbb{R}^4$, a surface homeomorph to the Klein bottle and $\mathbf{A}$, integer multiplicity rectifiable surface in $\mathbb{R}^4$ such that :

$$\begin{cases} (i)\ \partial\mathbf{A} = 2[[\Gamma]] \\[2ex] (ii)\ \mathbf{M}(\mathbf{A}) < 2m_i([[\Gamma]]) \\[2ex] (iii)\ \mathrm{spt}\mathbf{A} = K \end{cases} \tag{4.3}$$

where

$$[[\Gamma]] := \tau(\Gamma, 1, \vec{v})$$

is the integer multiplicity rectifiable current based on Γ and oriented by the unit tangent vectorfield $\vec{v}$. By slight modifications of Γ and K around their singular subsets, we may consider them to be smooth. Let $\vec{n}$ be a smooth normal vectorfield on $K \subset \mathbb{R}^4$. We recall that $\Gamma \subset K$ and we put :

$$\Sigma_\delta := \{x + t\vec{n}(x)\, ;\, 0 \leq t \leq \delta\, ,\, x \in \Gamma\}$$

and

$$\Gamma_\delta := \{x + \delta\vec{n}(x)\, ;\, x \in \Gamma\}$$

We observe that for Σ_δ and Γ_δ suitably oriented and δ sufficiently small we have :

$$\partial[[\Sigma_\delta]] = [[\Gamma_\delta]] - [[\Gamma]]. \tag{4.4}$$

Let V_δ be the tubular neighborhood of Γ_δ :

$$V_\delta := \{y \in \mathbb{R}^4\, ;\, d(y, \Gamma) \leq \delta\}.$$

For each $y \in \Gamma$ and $0 < \delta' < \delta$ let $B(\delta, \delta', y)$ be the 2-dimensional disk in $\mathbb{R}^4$ centered at y and with radius δ' which is orthogonal to Σ_δ and observe that

$$B(\delta, \delta') := \bigcup_{y \in \Gamma} B(\delta, \delta', y)$$

is a 3-dimensional submanifold of $\mathbb{R}^4$. We shall construct $\Omega_{\delta,\delta'}$ such that $B(\delta, \delta') \subset \Omega_{\delta,\delta'}$.

Let T be a smooth surface such that

$$\begin{cases} (i)\, \partial T = [[\Gamma_\delta]] \\ (ii)\, T \cap B(\delta, \delta') = \emptyset \\ (iii)\, \vec{n}(x) \text{ is the outward tangent to } T \text{ at } x + \delta\vec{n}(x) \in \partial T\, ,\ \forall x \in k. \end{cases} \tag{4.5}$$

Such T exists : As $\pi_1(\mathbb{R}^4 \backslash V_\delta) = 0$, there exists some smooth $T_0 \subset \mathbb{R}^4 \backslash V_\delta$ such that $\partial T_0 = [[\Gamma]]$. So if $T_1 = \Sigma_\delta \cup T_0$ we get $\partial T_1 = [[\Gamma_\delta]]$. T is obtained by smoothing T_1 in a neighborhood of Γ. Let $\vec{e_1}, \vec{e_2}$ be 2 smooth orthonormal vectorfields on T such that for each $y \in \Gamma_\delta$, $\vec{e_1}(y)$ and $\vec{e_2}(y)$ be tangent to $B(\delta, \delta', y)$. We put

$$U_\delta := \{x + t_1\vec{e_1}(x) + t_2\vec{e_2}(x)\, ;\ (t_1^2 + t_2^2)^{\frac{1}{2}} \leq \delta\}.$$

We choose δ small enough and some $\delta' < \delta$ such that

$$B(\delta, \delta') \cap W_{\delta'} = \emptyset$$

where $W_{\delta'} := \{x \in \mathbb{R}^4 \,;\; d(x,K) \le \delta'\}$. This is possible since Γ_δ has no intersection with K.

For every $x \in \Gamma$, $y = x + \delta\vec{n}(x)$, let $C(\delta,\delta',y)$ be the cone with the vertex x and the base $B(\delta,\delta',y)$ and put

$$C(\delta,\delta') := \bigcup_{y\in\Gamma_\delta} C(\delta,\delta',y).$$

We define the map $\pi : C(\delta,\delta') \to B(\delta,\delta')$ as follows : For every $z \in C(\delta,\delta',y)$, $\pi(z)$ is the intersection of the line $x - z$ and the disk $B(\delta,\delta',y)$, where x is the vortex of the cone $C(\delta,\delta',y)$. Then we put

$$\Omega_{\delta,\delta'} := C(\delta,\delta') \cup U_{\delta'} \cup W_{\delta'}. \tag{4.6}$$

$\Omega_{\delta,\delta'}$ is a domain in $\mathbb{R}^4$ which contains a tubular neighborhood of K and a one of T while $\partial\Omega_{\delta,\delta'}$ contains the set $B(\delta,\delta')$.

4.4 Construction of $\varphi_{\delta,\delta'}$

Let B be the unit disk in $\mathbb{R}^2$ and $\nu : B \to S^2$ be the smooth covering map as defined in [4], which satisfies these conditions :

$$\begin{cases} (i)\; \nu|_{\partial B} = const. = e \in S^2\,,\;\; \nu(0) = -e \\[2ex] (ii)\; \displaystyle\int_B |\nabla\nu|^2 = 4\pi \\[2ex] (iii)\; \text{For } z \ne e \text{ in } S^2,\;\; \#\,\omega^{-1}(z) = 1 \text{ and } \deg(\nu, B, 0) = 1. \end{cases} \tag{4.7}$$

We define the map $\phi_{\delta,\delta'} \in C^\infty(\Omega_{\delta,\delta'}, S^2)$ as follows :

$$\phi_{\delta,\delta'}(z) := \begin{cases} \nu\left(\left(\dfrac{t_1}{\delta'}, \dfrac{t_2}{\delta'}\right)\right) & \text{if } z = x + t_1\vec{e_1} + t_2\vec{e_2} \in U_{\delta'} \\[3ex] e & \text{if } z \notin U_{\delta'} \end{cases}$$

And we put

$$\varphi_{\delta,\delta'} := \phi_{\delta,\delta'|_{\partial\Omega}}\,.$$

4.5 Estimation for $\inf E$ on $C^\infty_{\delta,\delta'}$

Let $u \in C^\infty_{\delta,\delta'}$. By (2.2), (2.4) and the co-area formula we get :

$$\begin{aligned} \int_{\Omega_{\delta,\delta'}} |\nabla u|^2 \;&\ge 8\pi \int_{\Omega_{\delta,\delta'}} |u^*\omega| = 2\int_{\Omega_{\delta,\delta'}} |J_2 u| \\ &= 2\int_{S^2} dw \int_{u^{-1}(w)} 1 \;= 2\int_{S^2} \mathbf{M}(\mathbf{T}^u_w)\, dw. \end{aligned} \tag{4.8}$$

while considering the propositions 2.1 and 2.2 we have

$$\partial \mathbf{T}^u_w = [[\varphi^{-1}_{\delta,\delta'}(w)]]. \tag{4.9}$$

Meanwhile, for each $w \neq e \in S^2$, there exists some surface $S_{w,\delta} \subset B(\delta,\delta')$ such that

$$\partial[[S_{w,\delta}]] = [[\varphi^{-1}_{\delta,\delta'}(w)]] - [[\varphi^{-1}_{\delta,\delta'}(-e)]] = [[\varphi^{-1}_{\delta,\delta'}(w)]] - [[\Gamma_\delta]].$$

for suitable orientations. Using this and regarding (4.4) we get

$$\begin{aligned} |\, m_i\left([[\Gamma]]\right) - m_i\left([[\varphi^{-1}_{\delta,\delta'}(w)]]\right) | \quad &\le |\, m_i\left([[\Gamma]] - [[\varphi^{-1}_{\delta,\delta'}(w)]]\right) | \\ &\le |\Sigma_\delta| + |B(\delta,\delta')| = O(\delta). \end{aligned}$$

This estimation, combined with (4.8) and (4.9) gives :

$$E(u) \ge 2\int_{S^2} m_i\left([[\varphi^{-1}_{\delta,\delta'}(w)]]\right) dw = 8\pi m_i([[\Gamma]]) + O(\delta) \quad \forall u \in C^\infty_{\delta,\delta'}$$

and as a result

$$\inf_{C^\infty_{\delta,\delta'}} E \ge 8\pi m_i\left([[\Gamma]]\right) + O(\delta)\,. \tag{4.10}$$

4.6 Estimation for $\inf F^*$ on $H^1_{\delta,\delta'}$

We put for $z \in \Omega_{\delta,\delta'}$:

$$u_{\delta,\delta'} := \begin{cases} \varphi_{\delta,\delta'}(\pi(z)) & \text{if } z \in C(\delta,\delta') \\ e & \text{if } z \notin C(\delta,\delta') \end{cases}$$

We have for $K > 0$ independent of δ and δ' :

$$\begin{cases} |\nabla u_{\delta,\delta'}| = 0 & \text{on } \Omega_{\delta,\delta'}\backslash C(\delta,\delta') \\ |\nabla u_{\delta,\delta'}| \le |\nabla \varphi_{\delta,\delta'}|\,|\nabla \pi| \le \dfrac{K}{\delta'} & \text{on } C(\delta,\delta') \\ u_{\delta,\delta'}|_{\partial\Omega_{\delta,\delta'}} = \varphi_{\delta,\delta'} \end{cases} \tag{4.11}$$

Therefore

$$E(u_{\delta,\delta'}) \le \int_{C(\delta,\delta')} \frac{K^2}{\delta'^2} \le \frac{K^2}{\delta'^2}|C(\delta,\delta')| = O(\delta). \tag{4.12}$$

As a result $u_{\delta,\delta'} \in H^1_{\varphi_{\delta,\delta'}}(\Omega_{\delta,\delta'}, S^2)$. We should estimate $L^*(u_{\delta,\delta'})$: Pay attention that $u_{\delta,\delta'} \in R^\infty_{\varphi_{\delta,\delta'}}(\Omega_{\delta,\delta'}, S^2)$ as it is smooth on $\Omega_{\delta,\delta'}\backslash\Gamma$. Proposition 2.2 and a simple topological observation show that if Γ is suitably oriented we have

$$\mathbf{S}_{u_{\delta,\delta'}} = [[\Gamma]].$$

Recall that $\partial\mathbf{A} = 2[[\Gamma]]$ and that spt$\mathbf{A} \subset W_{\delta'} \subset \Omega_{\delta,\delta'}$ (See (4.6)) . So referring to lemma 2.2 and using (4.12) we have :

$$F^*(u_{\delta,\delta'}) = E(u_{\delta,\delta'}) + 8\pi m_r(\mathbf{S}_{u_{\delta,\delta'}}, \Omega_{\delta,\delta'}) \leq 4\pi\mathbf{M}(\mathbf{A}) + O(\delta)$$

and as a result

$$\inf_{H^1_{\delta,\delta'}} F^* \leq F^*(u_{\delta,\delta'}) \leq 4\pi\mathbf{M}(\mathbf{A}) + O(\delta)\,. \tag{4.13}$$

4.7 End of the proof

Combining (4.10) and (4.13) we get

$$\inf_{H^1_{\delta,\delta'}} F^* - \inf_{C^\infty_{\delta,\delta'}} E = O(\delta) - 4\pi(2m_i\,([[\Gamma]]) - \mathbf{M}(\mathbf{A}))\,.$$

But regarding (4.3) we know that

$$2m_i\,([[\Gamma]]) - \mathbf{M}(\mathbf{A}) > 0\,.$$

Therefore by choosing δ small enough, for $\Omega = \Omega_{\delta,\delta'}$ and $\varphi = \varphi_{\delta,\delta'}$ we get :

$$\inf_{H^1_\varphi(\Omega,S^2)} F^* - \inf_{C^\infty_\varphi(\Omega,S^2)} E < 0\,.$$

This shows that

$$\inf_{H^1_\varphi(\Omega,S^2)} F < \inf_{C^\infty_\varphi(\Omega,S^2)} E \tag{4.14}$$

as $F \leq F^*$.

Now F is coercive and weakly lower semi-continuous (As we mentionned in [29], the proof is as in [5] for $n = 3$). So its minimum is achieved by some $v \in H^1_\varphi(\Omega, S^2)$. We claim that $\mathbf{S}_v \neq 0$. If not,

$$F(v) = \widetilde{F}(v) \geq \inf_{H^1_\varphi(\Omega,S^2)} \widetilde{F}\,,$$

where $\widetilde{F}(u) = E(u) + 8\pi m_i(\mathbf{S}_u)$. Meanwhile using the same arguments as above we can prove that

$$\inf_{H^1_\varphi(\Omega,S^2)} F < \inf_{H^1_\varphi(\Omega,S^2)} \widetilde{F}.$$

This leads to a contradiction, so $\mathbf{S}_v$ can not be zero. As a result,

$$L(v) = \sup_{\substack{\psi \in \Omega^{n-3}(\overline{\Omega}) \\ |d\psi|_\infty \leq 1}} \int_\Omega \mathbf{S}_v > 0\,,$$

which implies :

$$\inf_{H^1_\varphi(\Omega,S^2)} E \leq E(v) < E(v) + 8\pi L(v) = F(v) = \inf_{H^1_\varphi(\Omega,S^2)} F.$$

This completes the proof of theorem 1. ∎

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Chapter II

Multiplicity of S^2-valued harmonic maps

Existence of infinitely many weakly harmonic maps from a domain in $\mathbb{R}^n$ into S^2 for non-constant boundary data

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We prove existence of infinitely many weakly harmonic maps from a domain of $\mathbb{R}^n$ into S^2 for non-constant smooth boundary data.

1 Introduction

Consider the Sobolev space :

$$H^1(\Omega, S^2) = \{u \in H^1(\Omega, \mathbb{R}^3)\,;\, u(x) \in S^2 \quad \text{a.e. on}\,\Omega\}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set and S^2 is the 2-dimensional unit sphere in $\mathbb{R}^3$. For $u \in H^1(\Omega, S^2)$ the energy $E(u) = \int_\Omega |\nabla u|^2$ is well defined. We call u a weakly harmonic map if it is a critical point for the functional E, i.e. if and only if we have

$$\frac{d}{dt} E\left(\frac{u+tv}{|u+tv|}\right)_{|t=0} = 0 \quad \text{for all} \quad v \in C_c^\infty(\Omega, \mathbb{R}^3)\,.$$

In other words, u is weakly harmonic in the Sobolev space $H^1(\Omega, S^2)$ if it satisfies the following equation in the sense of distributions :

$$\begin{cases} -\Delta u = u|\nabla u|^2 \quad \text{in} \quad \Omega \\ \\ u(x) \in S^2 \quad \text{a.e.} \end{cases}$$

Let $\varphi : \partial\Omega \to S^2$ be a smooth map which has a regular extension into Ω. The existence of a weakly harmonic map equal to φ on the boundary can be easily proved by a straightforward minimizing argument. By the way, the uniqueness and regularity questions for weakly harmonic maps in $H^1_\varphi(\Omega, S^2)$ have not the same answers as in the classic cases, i.e. when the target manifold is an euclidean space.

In this paper we consider the question of uniqueness of such extensions. In [21], R. Hardt, D.Kinderlehrer and F.H.Lin had proved the existence of infinitely many weakly harmonic extensions to an axially symmetric boundary condition in $H^1(B^3, S^2)$ where B^3 is the unit ball in $\mathbb{R}^3$. The method consists in constructing a non-axially symmetric harmonic extension and then one obtains infinitely many different harmonic maps with the same boundary data by rotating this extension around the symmetry axis.

Another method consists in finding new harmonic maps by defining new functionals whose critical points are still weakly harmonic. This has been done by F.Bethuel, H.Brezis and J.-M.Coron in [5] where they introduced such functionals which they called "relaxed energies". Using these functionals they proved for $n = 3$ that if φ is not homotopic to a constant or if

$$\min_{H^1_\varphi(\Omega,S^2)} E(u) < \inf_{C^\infty_\varphi(\Omega,S^2)} E(u)$$

then φ admits infinitely many weakly harmonic extensions inside Ω. Using the same gap condition, T.Isobe proved the corresponding result for the case $n \geq 4$ in [26], still using the relaxed energies whose definition was extended to higher dimensions.

At last, using his *strict dipole insertion lemma*, proved in [32], T.Rivière showed that if Ω is a regular bounded domain of $\mathbb{R}^3$, a non constant smooth boundary data $\varphi : \partial\Omega \to S^2$ admits always infinitely many weakly harmonic extensions (Appeared in [33]). The method, first proposed by F.Bethuel, H.Brezis and J.-M.Coron, consists in producing infinitely many distinct weakly harmonic maps in an inductive process by minimizing the relaxed energies.

The main difficulty in adapting the approach in [33] to higher dimensions is first generalizing the concept of relaxed energies as appeared in [5] to what we will call the F-energies in a suitable way and proving the desired properties for these new energies. Another difficult step consists in finding some equivalent construction in any dimensions of the insertion of 2 singular points with the strict inequality like in [32] for $n = 3$. It appears that ([32], lemma A.1) can be generalized (via some technical difficulties) by inserting this time $(n-3)$-dimensional singular spheres. Our main result is the following

Theorem 1 *Let Ω be a regular bounded domain in $\mathbb{R}^n$, $n \geq 3$, and φ a non-constant smooth map from $\partial\Omega$ into S^2. Then φ admits infinitely many weakly harmonic extensions.*■

Remark 1.1 *This result is independent of the choice of the metric on S^2 . For the details compare with [33].*

Remark 1.2 *It seems that the main difficulty to overcome in order to extend the result for p-harmonic maps into S^p, using the same method, is to prove the lower semi-continuity of the generalized relaxed energies which can be defined also in these cases in a natural way.*

The paper is organized as follows. In section 2 we recall some elementary facts needed for our work using concepts of Geometric Measure Theory. In section 3 we introduce the F-energies and discuss their characteristics. The readers can refer to [16] for more elaborated discussion of these subjects. Then in section 4 we prove our main result using the strict insertion lemma which we shall prove in the last part of the paper.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set and let

$$H^1(\Omega, S^2) = \{u \in H^1(\Omega, \mathbb{R}^3)\,;\, u(x) \in S^2 \quad \text{a.e. on}\,\Omega\}$$

and

$$H^1_\varphi(\Omega, S^2) = \{u \in H^1(\Omega, S^2)\,;\, u = \varphi \quad \text{on}\,\partial\Omega\}$$

where φ is a given boundary data. For $u \in H^1_\varphi(\Omega, S^2)$ the Dirichlet energy is given by $E(u) = \int_\Omega |\nabla u|^2$. We assume that φ is in $C^\infty(\partial\Omega, S^2)$ and can be extended into Ω by a smooth map.

2.1 The subspace $R^\infty_\varphi(\Omega, S^2)$

Definition 2.1 *We say that $u \in H^1_\varphi(\Omega, S^2)$ is in $R^\infty_\varphi(\Omega, S^2)$ if u is smooth except on $B = \bigcup_{i=1}^m \sigma_i \cup B_0$, a compact subset of Ω, where $\mathcal{H}^{n-3}(B_0) = 0$ and the σ_i , $i = 1, \cdots, m$ are smooth embeddings of the unit disk of dimension $n-3$. Moreover we assume that any two different faces of B, σ_i and σ_j, may meet only on their boundaries.*

Remark 2.1 *In ([2], theorem 2bis), F. Bethuel has proved that $R^\infty_\varphi(\Omega, S^2)$ is dense in $H^1_\varphi(\Omega, S^2)$ for the strong topology.*

Definition 2.2 *Let $u \in H^1_\varphi(\Omega, S^2)$. We define the current $\mathbf{S}_u \in \mathcal{D}_{n-3}(\Omega)$ to be the current defined by*

$$\mathbf{S}_u(\alpha) := \int_\Omega u^*\omega \wedge d\alpha \qquad \forall \alpha \in \mathcal{D}^{n-3}(\Omega). \tag{2.1}$$

Here $\mathcal{D}^k(\Omega)$ is the set of smooth k-forms on Ω with compact support (See[16], 2.2.3) and ω is some 2-form on S^2 for which $\int_{S^2}\omega = 1$.

Let ω_1 and ω_2 be two such forms on S^2. We have $\omega_1 - \omega_2 = d\beta$ where β is some smooth 1-form on S^2 extendable to $\mathbb{R}^3$. Let $u \in H^1_\varphi(\Omega, S^2)$ and consider a sequence $u_m \in C^\infty(\Omega, \mathbb{R}^3)$ converging to u in H^1. We have

$$u_m^*(d\beta) = d\,(u_m^*\beta)$$

and by passing to the limit, we observe that this holds true for u in the sense of distributions. This proves the independence of $\mathbf{S}_u$ from the choice of ω as we have :

$$d(u^*\omega_1) - d(u^*\omega_2) = du^*(d\beta) = 0$$

in the sense of distributions. Now the existence of the integral (2.1) is a direct consequence of the following inequality :

$$|u^*\omega| \leq \frac{1}{8\pi}|\nabla u|^2 \qquad \text{a.e. on}\,\Omega \tag{2.2}$$

where $4\pi\omega = \omega_V$ is the standard volume form of S^2.

We shall give a description of $\mathbf{S}_u$ for $u \in R^\infty_\varphi(\Omega, S^2)$. Clearly if u is smooth a standard operation on pull-back yields

$$d(u^*\omega) = u^*(d\omega) = 0$$

and as a consequence we deduce for $u \in R^\infty_\varphi(\Omega, S^2)$ that

$$spt\mathbf{S}_u \subseteq B.$$

Definition 2.3 *Let $u \in R^\infty_\varphi(\Omega, S^2)$ and let $B = \bigcup \sigma_i \cup B_0$ be the singular set of u. Suppose that each σ_i is oriented by a smooth $(n-3)$-vectorfield $\vec{\sigma}_i$. For $a \in \sigma_i$ let N_a be the 3-dimensional plane orthogonal to σ_i at a. Consider the 3-disk $M_{a,\delta} = B_\delta(a) \cap N_a$ oriented by the 3-vector $\vec{M}_a$ such that $\vec{\sigma}_i(a) \wedge \vec{M}_a = (-1)^n \vec{\xi}_{\mathbb{R}^n}$. Then the topological degree of u on the 2 dimensional sphere $\Sigma_{a,\delta} = \partial M_{a,\delta}$ is well defined and is independent of the choice of a for δ small enough. We call this integer the degree of u on σ_i and denote it by*

$$deg_{\sigma_i} u\,.$$

Our first goal is to show that for $u \in R^\infty_\varphi(\Omega, S^2)$, $\mathbf{S}_u$ is the integer multiplicity rectifiable current $\sum_{i=1}^m (deg_{\sigma_i} u)\,\tau(\sigma_i, 1, \vec{\sigma}_i)$. Recall that any k-dimensional rectifiable subset $\mathcal{M}$ of $\mathbb{R}^n$ considered with a multiplicity θ and oriented by a unit k-vector field ξ defines a rectifiable current as follows

$$\tau(\mathcal{M}, \theta, \xi)(\alpha) := \int_{\mathcal{M}} < \xi, \alpha > \theta\, d\mathcal{H}^k \qquad \forall \alpha \in \mathcal{D}^k(\mathbb{R}^n).$$

Lemma 2.1 *Let* $\omega = \frac{1}{4\pi}\omega_V$ *and* $u \in R^\infty_\varphi(\Omega, S^2)$. *Then the* $(n-2)$*-vectorfield* $\vec{D}(u)$ *defined on* $\Omega\backslash B$ *by the equation*

$$< \vec{D}(u)(x), \Psi > \omega_{\mathbb{R}^n} := u^*\omega(x) \wedge \Psi \quad \forall \Psi \in \Lambda^{n-2}(\mathbb{R}^n) \tag{2.3}$$

is a simple $(n-2)$*-vectorfield tangent to the smooth manifold* $u^{-1}(y)$ *for all regular value* $y = u(x) \in S^2$. *Meanwhile*

$$|\vec{D}(u)| = \frac{1}{4\pi}|J_2 u| \quad a.e. \ on \ \Omega. \tag{2.4}$$

■

Remember that an element of $\Lambda_k(\mathbb{R}^n)$ is called *simple* if and only if it equals the exterior product of k vectors of $\mathbb{R}^n$ ([13], 1.6.1).

Proof : Write

$$u^*\omega = \sum_{i<j} u_{ij}\, dx^i \wedge dx^j \quad \text{a.e. on } \Omega$$

and $u_{ij} = 0$ for $i \geq j$. For almost all $x \in \Omega$, $u^*\omega(x)$ is in $\Lambda^2(\mathbb{R}^n)$. Using (2.3) a short calculation shows that

$$\vec{D}(u)(x) = \sum_{\sigma \in S_n} \frac{1}{(n-2)!} u_{\sigma(1),\sigma(2)} \frac{\partial}{\partial x^{\sigma(3)}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\sigma(n)}}$$

and we get

$$\vec{D}(u)(x) \wedge \vec{\eta} =< \vec{\eta}, u^*\omega(x) > \vec{\xi}_{\mathbb{R}^n} \quad \forall \vec{\eta} \in \Lambda_2(\mathbb{R}^n). \tag{2.5}$$

So if $y \in S^2$ is a regular point for u we have $\vec{D}(u)(x) \neq 0$ and if $\vec{v}$ is any vector tangent to $u^{-1}(y)$ at x by (2.5) we obtain

$$\vec{D}(u)(x) \wedge \vec{v} \wedge \vec{w} =< \vec{v} \wedge \vec{w}, u^*\omega(x) > \vec{\xi}_{\mathbb{R}^n} \quad \forall \vec{w} \in \Lambda_1(\mathbb{R}^n)$$

and since $Du(x) \cdot \vec{v} = 0$

$$< \vec{v} \wedge \vec{w}, u^*\omega(x) >= \frac{1}{4\pi} < \Lambda_2(Du)(x) \cdot (\vec{v} \wedge \vec{w}), \omega(y) >$$

$$= \frac{1}{4\pi} < Du(x) \cdot \vec{v} \wedge Du(x) \cdot \vec{w}, \omega(y) >= 0$$

Therefore $\vec{D}(u)(x) \wedge \vec{v} = 0$ for any $\vec{v}$ tangent to $u^{-1}(y)$ at x and as a result $\vec{D}(u)(x)$ is a simple $(n-2)$-vector associated to tangent space of $u^{-1}(y)$ at x (See [13], 1.6.1). Now using (2.5) and by duality we get (2.4) as $\omega = \frac{1}{4\pi}\omega_V$ and so

$$|\vec{D}(u)(x)| = |u^*\omega(x)| = \frac{1}{4\pi}|\Lambda_2(Du)(x)| = \frac{1}{4\pi}|J_2 u(x)| \quad \text{a.e. on } \Omega.$$

■

For any $y \in S^2$, regular value of $u \in R^\infty_\varphi(\Omega, S^2)$, we define the current

$$\mathbf{T}^u_y := \tau\left(u^{-1}(y), 1, \frac{\vec{D}(u)}{|\vec{D}(u)|}\right). \tag{2.6}$$

Proposition 2.1 *Consider $u \in R^\infty_\varphi(\Omega, S^2)$ and $\mathbf{T}^u_y$ as in (2.6), then for almost all $y \in S^2$, $\mathbf{T}^u_y$ is a rectifiable current in $\mathbb{R}^n$ with support in $\overline{\Omega}$ and*

$$\partial \mathbf{T}^u_y = \mathbf{S}_u + \tau\left(\varphi^{-1}(y), 1, \frac{\vec{D}(\varphi)}{|\vec{D}(\varphi)|}\right) \tag{2.7}$$

where the $(n-3)$-vectorfield $\vec{D}(\varphi)$ on $\partial\Omega$ is defined by the equation

$$< \vec{D}(\varphi)(x), \Psi > \omega_{E_x} := \varphi^*\omega(x) \wedge \Psi \quad \forall \Psi \in \Lambda_{n-3}(E_x)$$

where $E_x = T_x(\partial\Omega)$ is the tangent space to $\partial\Omega$ at x and ω_{E_x} is its unit volume form.

■

Proof : First observe that by Sard's theorem, for almost all $y \in S^2$, $u^{-1}(y)$ is a countable union of smooth submanifolds supported in $\overline{\Omega}$. Moreover by lemma 1, $\frac{\vec{D}(u)}{|\vec{D}(u)|}$ is associated to the tangent space of $u^{-1}(y)$. So by co-area formula we have

$$\int_{S^2} \mathbf{M}(\mathbf{T}^u_y)\, dy = \int_\Omega |J_2 u| \leq \frac{1}{2}\int_\Omega |\nabla u|^2$$

and we deduce that $\mathbf{M}(\mathbf{T}^u_y) < +\infty$ for almost all y, i.e. $\mathbf{T}^u_y$ is rectifiable. The claim about $\partial\mathbf{T}^u_y$ is proved in 4 steps :

(i) We prove that $\partial\mathbf{T}^u_y$ is a flat chain.

(ii) We give an expression for $\partial\mathbf{T}^u_y$ of the form

$$\sum_i r^i_y\, \tau(\sigma_i, 1, \vec{\sigma}_i) + \tau\left(\varphi^{-1}(y), 1, \frac{\vec{D}(\varphi)}{|\vec{D}(\varphi)|}\right)$$

using the constancy theorem.

(iii) We prove that $r^i_y = deg_{\sigma_i} u$.

(iv) At last (2.7) would be proved using the definition of $\mathbf{S}_u$ and the co-area formula.

Step (i) : Since u is smooth on $\Omega \backslash B$ we observe that

$$\operatorname{spt}(\partial\mathbf{T}^u_y) \subseteq \partial\Omega \cup B \tag{2.8}$$

if y is a regular value for u. We know that u is smooth near $\partial\Omega$ and we have $u^{-1}(y)\cap\partial\Omega = \varphi^{-1}(y)$, $\vec{\xi}_{\mathbb{R}^n} = (-1)^{n-1}\vec{\xi}_{E_x} \wedge \vec{n}$ for all $x \in \partial\Omega$. Using (2.5) for $\vec{D}(u)$ and $\vec{D}(\varphi)$ we get that

$$\vec{D}(u) = (-1)^{n-1}\vec{D}(\varphi) \wedge \vec{n}_{ext} \quad \text{for regular points} \quad x \in \partial\Omega$$

when $\vec{n}_{ext}$ is the outward unit tangent vector to $u^{-1}(y)$ at x. So considering the rules of orientation of manifolds we get

$$\partial \mathbf{T}^u_y \llcorner \partial\Omega = \tau\left(\varphi^{-1}(y), 1, \frac{\vec{D}(\varphi)}{|\vec{D}(\varphi)|}\right) \tag{2.9}$$

which is a rectifiable current for the regular values of u and φ.

For proving the claim we put $\mathbf{S}_y = \partial\mathbf{T}^u_y - \tau\left(\varphi^{-1}(y), 1, \frac{\vec{D}(\varphi)}{|\vec{D}(\varphi)|}\right)$ and consider the set

$$B_\varepsilon = \{x|\, d(x,B) < \varepsilon\}$$

the ε-neighborhood of B in Ω. By (2.8) and (2.9) we get

$$\partial(\mathbf{T}^u_y \llcorner B_\varepsilon) = \mathbf{T}^u_y \llcorner \partial B_\varepsilon + \mathbf{S}_y \quad , \quad \mathrm{spt}\mathbf{S}_y \subseteq B. \tag{2.10}$$

Since u is smooth on ∂B_ε , $\mathbf{T}^u_y \llcorner \partial B_\varepsilon$ is an $(n-3)$-dimensional normal current. Now using the co-area formula we get

$$\int_{S^2} \mathbf{M}(\mathbf{T}^u_y \llcorner B_\varepsilon)\, dy = \int_{B_\varepsilon} |J_2 u| \leq \frac{1}{2}\int_{B_\varepsilon} |\nabla u|^2 \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

So for almost all $y \in S^2$, $\mathbf{M}(\mathbf{T}^u_y \llcorner B_\varepsilon) \to 0$. By (2.10) we deduce that $\mathbf{S}_y$ is a flat chain as it is a flat-norm limit of normal currents $\mathbf{T}^u_y \llcorner \partial B_\varepsilon$.

Step (ii) : $\mathbf{S}_y$ is a flat chain in Ω without boundary. By the Constancy Theorem ([16], 5.3.1, theorem 3) applied successively to the σ_i, there exist real numbers r^i_y such that

$$\mathrm{spt}(\mathbf{S}_y - r^i_y \tau(\sigma_i, 1, \vec{\sigma}_i)) \subseteq \Omega\backslash\sigma_i \quad i = 1, \cdots, m$$

and as a result

$$\mathrm{spt}(\mathbf{S}_y - \sum_{i=1}^m r^i_y \tau(\sigma_i, 1, \vec{\sigma}_i)) \subseteq \Omega\backslash\bigcup_{i=1}^m \sigma_i\,.$$

Meanwhile $B = \bigcup_i \sigma_i \cup B_0$ where $\mathcal{H}^{n-3}(B_0) = 0$. So since the support of $\mathbf{S}_y$ lies in B, $\mathbf{S}_y - \sum_{i=1}^m r^i_y \tau(\sigma_i, 1, \vec{\sigma}_i)$ is an $(n-3)$-dimensional flat chain supported in B_0, therefore

$$\mathbf{S}_y = \sum_{i=1}^m r^i_y \tau(\sigma_i, 1, \vec{\sigma}_i)$$

and so

$$\partial\mathbf{T}^u_y = \sum_i r^i_y\, \tau(\sigma_i, 1, \vec{\sigma}_i) + \tau\left(\varphi^{-1}(y), 1, \frac{\vec{D}(\varphi)}{|\vec{D}(\varphi)|}\right). \tag{2.11}$$

Step (iii) : We begin this part by proving the following lemma.

Lemma 2.2 *Let M be a 3-dimensional smooth manifold supported in Ω oriented by $\vec{M}$ a smooth 3-vectorfield. Let $\mathbf{M} = \tau(M, 1, \vec{M})$ and $\Sigma = \partial\mathbf{M}$. Then for almost all $y \in S^2$,*

$$\mathbf{k}(\partial\mathbf{T}^u_y, \mathbf{M}) = (-1)^n \int_\Sigma u^*\omega$$

where $\mathbf{k}(\mathbf{S}, \mathbf{T})$ is the kronecker index of $\mathbf{S}$ and $\mathbf{T}$ as defined in $([16], vol. 1, 5.3.4)$.

■

Proof : For almost all $y \in S^2$ regular value for $(u|_\Sigma)$ (2.11) is valid and Σ transversally intersects $u^{-1}(y)$ at each point of their intersection. So we have :

$$\int_\Sigma u^*\omega = \sum_{x\in\Sigma\cap u^{-1}(y)} < \vec{\Sigma}(x), \frac{u^*\omega(x)}{|u^*\omega(x)|} >= \mathbf{k}(\mathbf{T}^u_y, \Sigma) \tag{2.12}$$

Consider the translation $\tau^a : \mathbb{R}^n \to \mathbb{R}^n,\ \tau^a(x) = x + a$. Considering the definition of the kronecker index and ([16], 5.3.4, theorem 2) we observe that there exists a small enough such that

(i) spt $\tau^a_\#\Sigma \subset \Omega\backslash B$, spt $\tau^a_\#\mathbf{M} \subset \Omega$,

(ii) $\mathbf{T}^u_y \cap \tau^a_\#\Sigma$, $\mathbf{T}^u_y \cap \tau^a_\#\mathbf{M}$ and $\partial\mathbf{T}^u_y \cap \tau^a_\#\mathbf{M}$ exist,

(iii) $\mathbf{k}(\mathbf{T}^u_y, \Sigma) = \mathbf{k}(\mathbf{T}^u_y, \tau^a_\#\Sigma)$, $\mathbf{k}(\partial\mathbf{T}^u_y, \mathbf{M}) = \mathbf{k}(\partial\mathbf{T}^u_y, \tau^a_\#\mathbf{M})$ and

(iv) $\partial(\mathbf{T}^u_y \cap \tau^a_\#\mathbf{M}) = \partial\mathbf{T}^u_y \cap \tau^a_\#\mathbf{M} + (-1)^{n-3}\mathbf{T}^u_y \cap \tau^a_\#\Sigma$.

Therefore by (2.12)

$$\begin{aligned}
&\mathbf{k}(\partial\mathbf{T}^u_y, \mathbf{M}) = \mathbf{k}(\partial\mathbf{T}^u_y, \tau^a_\#\mathbf{M}) = (\partial\mathbf{T}^u_y \cap \tau^a_\#\mathbf{M})(1) \\
&= (-1)^n(\mathbf{T}^u_y \cap \tau^a_\#\Sigma)(1) = (-1)^n\mathbf{k}(\mathbf{T}_y, \tau^a_\#\Sigma) = (-1)^n\mathbf{k}(\mathbf{T}^u_y, \Sigma) \\
&= (-1)^n \int_\Sigma u^*\omega.
\end{aligned}$$

which proves the lemma. ■

Now take $\mathbf{M}_{a,\delta} = \tau(M_{a,\delta}, 1, \vec{M}_{a,\delta})$ as in the definition 3. Applying lemma 2 and (2.12) to $\mathbf{M}_{a,\delta}$ we get :

$$deg_{\sigma_i} u = \int_{\Sigma_{a,\delta}} u^*\omega = (-1)^n\mathbf{k}(\partial\mathbf{T}^u_y, \mathbf{M}_{a,\delta}) = r^i_y\,. \tag{2.13}$$

Step (iv) : Let $\alpha \in \mathcal{D}^{n-3}(\Omega)$. By the co-area formula and (2.4) we get :

$$\int_\Omega u^*\omega \wedge d\alpha = \frac{1}{4\pi}\int_{S^2} dy \int_{u^{-1}(y)} \frac{u^*\omega\wedge d\alpha}{|u^*\omega|}$$

$$= \frac{1}{4\pi}\int_{S^2} dy \int_{u^{-1}(y)} < \frac{\vec{D}(u)}{|\vec{D}(u)|}, d\alpha > \, d\mathcal{H}^{n-2}$$

$$= \frac{1}{4\pi}\int_{S^2} \mathbf{T}^u_y(d\alpha)\, dy = \frac{1}{4\pi}\int_{S^2} \partial\mathbf{T}^u_y(\alpha)\, dy$$

and since $\alpha|_{\partial\Omega} = 0$ using (2.11) and (2.13) we obtain :

$$\begin{aligned} \int_\Omega u^*\omega \wedge d\alpha &= \frac{1}{4\pi}\int_{S^2} dy \sum_{i=1}^m (deg_{\sigma_i} u)\tau(\sigma_i, 1, \vec{\sigma}_i)(\alpha) \\ &= \sum_{i=1}^m (deg_{\sigma_i} u)\, \tau(\sigma_i, 1, \vec{\sigma}_i)(\alpha) \end{aligned} \tag{2.14}$$

which completes the proof of proposition 1 regarding the definition of $\mathbf{S}_u$ and the formula for $\partial\mathbf{T}^u_y$ in (2.11). ■

Corollary 2.1 *Let $u \in R^\infty_\varphi(\Omega, S^2)$ and $B = \bigcup_i \sigma_i \cup B_o$ its singular set. Then*

$$\mathbf{S}_u = \sum_i (deg_{\sigma_i} u)\tau(\sigma_i, 1, \vec{\sigma}_i).$$

■

Proof : Refer to the relation (2.14) in the proof of proposition 1. ■

3 The F-energy on $H^1_\varphi(\Omega, S^2)$

In this section we define for any $v \in H^1_\varphi(\Omega, S^2)$ a functional F_v on $H^1_\varphi(\Omega, S^2)$ which has two interesting properties. First, it is lower semi-continuous and second, its critical points are also the critical points of the energy E, i.e. the critical points of F, in particular its minimizers, would be weakly harmonic maps. In fact this "F-energy" is a natural generalization of the "relaxed energy" in dimension 3 introduced in [5], except that in higher dimensions the functional F may not be a relaxed energy for $H^1_\varphi(\Omega, S^2)$: i.e. there exist cases where

$$\inf_{H^1_\varphi(\Omega,S^2)} F < \min_{C^\infty_\varphi(\Omega,S^2)} E\,,$$

(See [30]).

Definition 3.1 *Let u, v be two maps in $H^1_\varphi(\Omega, S^2)$. We define the connection between u and v to be*

$$L(u,v) = \sup_{\substack{\psi \in \Omega^\infty_{n-3}(\overline{\Omega}) \\ |d\psi|_\infty \le 1}} \left\{ \int_\Omega u^*\omega \wedge d\psi - \int_\Omega v^*\omega \wedge d\psi \right\} \tag{3.1}$$

where ω is any 2-form on S^2 with $\int_{S^2} \omega = 1$. We will often take $\omega = \frac{1}{4\pi}\omega_V$ which is more suitable for computations.

Remark 3.1 *We recall that the mass of currents is in fact the dual of the comass norm of differential forms (See[13], 4.1.7). So, from Geometric Measure Theory point of view, it would be more natural to use the comass norm of $d\psi$ instead of its euclidean norm in the definition of L. Meanwhile the euclidean norm is preferred for the relative simplicity of the proof of lower semi-continuity of F_v .*

Proposition 3.1 *We have the following inequality :*

$$L(u,v) \le C\|\nabla u - \nabla v\|_2(\|\nabla u\|_2 + \|\nabla v\|_2) \quad \forall u, v \in H^1_\varphi(\Omega, S^2). \tag{3.2}$$

∎

Proof : We write

$$d\psi = \sum_{1 \le i_3 < i_4 < \cdots < i_n \le n} \psi_{i_3 i_4 \cdots i_n} dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_n}$$

and we have

$$\sum_{i_3 < i_4 < \cdots < i_n} |\psi_{i_3 i_4 \cdots i_n}|^2 = |d\psi|^2 \le 1.$$

Now by simple calculations we obtain :

$$< \vec{\xi}_{\mathbb{R}^n}, u^*\omega \wedge d\psi >= \frac{1}{8\pi} \sum_{\substack{i_3 < i_4 < \cdots < i_n \\ \{i_1, \cdots, i_n\} = \{1, \cdots, n\}}} u \cdot (u_{x^{i_1}} \wedge u_{x^{i_2}})\, \psi_{i_3 i_4 \cdots i_n}$$

and the proposition is proved using the same method used in [5], Theorem 3. ∎

Now let $u \in H^1_\varphi(\Omega, S^2)$ and for $u_0 \in C^\infty_c(\Omega, S^2)$ consider the variation $u(t) = \frac{u + tu_0}{|u + tu_0|}$. As a consequence for t small enough $u(t) \in H^1_\varphi(\Omega, S^2)$ and we have :

Lemma 3.1 *For all $u, v \in H^1_\varphi(\Omega, S^2)$ and for t small enough $L(u(t), v) = L(u, v)$.*

∎

Proof : Pay attention that if $u_n \to u$ in H^1 then for t small enough we have $u_n(t) \to u(t)$ in H^1. So in the view of the proposition 2 and by using the fact that $R^\infty_\varphi(\Omega, S^2)$ is dense in $H^1_\varphi(\Omega, S^2)$ (See Remark 3), it suffices for us to prove this lemma for $u, v \in R^\infty_\varphi(\Omega, S^2)$. For such u and v we get by the co-area formula and proposition 1 :

$$\int_\Omega u^*\omega \wedge d\psi - \int_\Omega v^*\omega \wedge d\psi = \frac{1}{4\pi}\int_{S^2} (\mathbf{T}^u_y - \mathbf{T}^v_y)(d\psi)\, dy = (\mathbf{S}_u - \mathbf{S}_v)(\psi). \tag{3.3}$$

Meanwhile for $u \in R^\infty_\varphi(\Omega, S^2)$, using the corollary 1, we have $\mathbf{S}_u = \mathbf{S}_{u(t)}$ as u and $u(t)$ have the same singular set and the same degrees on its components. By (3.3) we get :

$$L(u(t), v) = \sup_{|d\psi|_\infty \le 1} \left(\mathbf{S}_{u(t)} - \mathbf{S}_v\right)(\psi) = \sup_{|d\psi|_\infty \le 1} \left(\mathbf{S}_u - \mathbf{S}_v\right)(\psi) = L(u, v)$$

and the lemma is proved. ■

Proposition 3.2 *For fixed $v \in H^1_\varphi(\Omega, S^2)$ let*

$$F_v(u) := E(u) + 8\pi L(u, v).$$

Then F_v is a lower semi-continous functional on $H^1_\varphi(\Omega, S^2)$ and its critical points are weakly harmonic maps.

■

Remark 3.2 *T.Isobe has proved the lower semi-continuity of the functionals*

$$F_{\psi,\lambda}(u) = E(u) + 8\pi\lambda\left\{\int_\Omega u^*\omega \wedge d\psi - \int_{\partial\Omega} \varphi^*\omega \wedge \psi\right\}$$

for $\lambda < C(n)$, $n \ge 4$ (See [26]). But what we need here is the same result for $\lambda = 1$ for which we have to prefer another argument.

Proof : Again as in the proposition 2, the proof of lower semi-continuity of F_v is the same as the proof of lower semi-continuity of the relaxed energy in [5]. Using lemma 3 we obtain

$$\frac{d}{dt}F_v(u(t))_{|t=0} = \frac{d}{dt}E(u(t))_{|t=0} + 8\pi\frac{d}{dt}L(u(t), v)_{|t=0} = \frac{d}{dt}E(u(t))_{|t=0}$$

so as a result the critical points of F_v are those of E. ■

4 Proof of the main theorem

We shall state here the main result of the paper.

Theorem 1 *Let Ω be a regular bounded domain in $\mathbb{R}^n$, $n \ge 3$, and φ a non-constant smooth map from $\partial\Omega$ to S^2. Then φ admits infinitely many weakly harmonic extensions.*

■

For proving this theorem we apply a method proposed by F. Bethuel, H. Brezis and J.-M. Coron which uses the F-energy as an efficient tool for finding the new weakly harmonic maps and a technical lemma which we shall prove in the following section.

Lemma 4.1 *Let Ω be a bounded regular domain in $\mathbb{R}^n$ and u a regular non-constant map from Ω to S^2. Let x_0 be a point of Ω for which $\nabla u(x_0) \neq 0$. Then for every $\rho > 0$ there exists a map $v \in H^1(\Omega, S^2)$ and $0 < \delta < \rho$ such that*

(i) $v{=}u$ *on* $\Omega \backslash B_\rho(x_0)$

(ii) $\mathbf{S}_v = \tau(\sigma, 1, \vec{\sigma})$

(iii) $E(v) < E(u) + 8\pi\omega_{n-2}\delta^{n-2} = E(u) + 8\pi L(v, u)$

where σ is an $(n-3)$-dimensional sphere of center x_0 and radius δ and ω_k is the volume of the unit k-dimensional disk.

■

This lemma, called the strict insertion of singularities, was firstly proved for the case $n = 3$ by T. Rivière in [32]. The computations used rely on the previous computations for inserting coverings of S^2 in dimension 2 (See [9]). The axially symmetric version of it was proved in [23].

Proof of theorem 1 : Two situations may take place :

(1) There are infinitely many distinct minimizers for E in $H^1_\varphi(\Omega, S^2)$ and so the problem is solved.

(2) There are only a finite number of minimizers for E on $H^1_\varphi(\Omega, S^2)$.

In this case let $w_1, \cdots, w_m$ be the minimizing maps. By the partial regularity theory of [35] and considering the fact that φ is not constant we deduce the existence of Ω_1, an open subset of Ω, on which w_1 is smooth and some $x_0 \in \Omega_1$ for which $\nabla w_1(x_0) \neq 0$. For some $\rho > 0$ which will be fixed later we apply the lemma 4 to w_1 on Ω_1 and name the transformed map v_1. So we have

$$E(v_1) < E(w_1) + 8\pi L(v_1, w_1). \tag{4.1}$$

Now suppose that u_1 is a minimizing map for F_{v_1} on $H^1_\varphi(\Omega, S^2)$. By proposition 4 such maps exist and are weakly harmonic. We shall prove that for ρ sufficiently small u_1 is different from all the w_i. We distinguish two cases :

(a) $L(w_k, w_1) = 0$: By (4.1) we obtain

$$F_{v_1}(u_1) \leq F_{v_1}(v_1) = E(v_1) < E(w_1) + 8\pi L(v_1, w_1). \tag{4.2}$$

Moreover subadditionality of L gives

$$|L(v_1, w_1) - L(v_1, w_k)| \leq L(w_k, w_1) = 0, \tag{4.3}$$

so $L(v_1, w_1) = L(v_1, w_k)$ and using the fact that $E(w_1) = E(w_k)$, (4.2) implies

$$F_{v_1}(u_1) < F_{v_1}(w_k).$$

This strict inequality proves naturally that $u_1 \neq w_k$ when $L(w_1, w_k) = 0$.

(b) $L(w_k, w_1) > 0$: We have

$$L(w_k, v_1) + L(v_1, w_1) \geq L(w_k, w_1), \tag{4.4}$$

meanwhile by the lemma 4

$$L(v_1, w_1) = \omega_{n-2}\delta^{n-2} < \omega_{n-2}\rho^{n-2}, \tag{4.5}$$

thus

$$\begin{aligned} F_{v_1}(w_k) &= E(w_k) + 8\pi L(w_k, v_1) \\ &\geq E(w_1) + 8\pi(L(w_k, w_1) - \omega_{n-2}\rho^{n-2}). \end{aligned} \tag{4.6}$$

Now it is sufficient to choose $\rho > 0$ such that for all w_k verifying $L(w_k, w_1) > 0$ we have the inequality

$$0 < \omega_{n-2}\rho^{n-2} < \frac{L(w_k, w_1)}{2}, \tag{4.7}$$

then by (4.5) we have $L(w_k, w_1) - \omega_{n-2}\rho^{n-2} > \omega_{n-2}\rho^{n-2} > L(v_1, w_1)$ and this, added to (4.6) implies :

$$F_{v_1}(w_k) > F_{v_1}(w_1) \geq F_{v_1}(u_1),$$

which combined with part (a) proves that u_1 is different from all the w_k .

We construct by induction a sequence u_j of distinct weakly harmonic maps in $H^1_\varphi(\Omega, S^2)$ which are also different from the w_i, using the same method. Choose ρ_{j+1} such that

$$\begin{cases} 0 < \omega_{n-2}\rho_{j+1}^{n-2} < Min\left\{\dfrac{L(w_k, w_1)}{2}; \quad \text{for } k \text{ verifying} \quad L(w_k, w_1) > 0\right\} \\ and \\ 0 < \omega_{n-2}\rho_{j+1}^{n-2} < Min\left\{\dfrac{E(u_i) - E(w_1)}{8\pi}; \quad i = 1, \cdots, j\right\} \end{cases} \tag{4.8}$$

Let u_{j+1} be a minimizer of $F_{v_{j+1}}$ when v_{j+1} is the transformed map of w_1 on $B_{\rho_{j+1}}(x_0)$ as in lemma 4. Again the first inequality in (4.8) assures that u_{j+1} is distinct from the w_i. For seeing that $u_{j+1} \neq u_i$ for $i \leq j$, using the strict inequality of lemma 4 we observe that

$$F_{v_{j+1}}(u_{j+1}) \leq F_{v_{j+1}}(v_{j+1}) = E(v_{j+1}) < E(w_1) + 8\pi\omega_{n-2}\rho_{j+1}^{n-2}. \tag{4.9}$$

Moreover from (4.8) we have

$$8\pi\omega_{n-2}\rho_{j+1}^{n-2} < E(u_i) - E(w_1). \tag{4.10}$$

Thus combining (4.9) and (4.10) imply that $E(u_{j+1}) \leq F_{v_{j+1}}(u_{j+1}) < E(u_i)$. This yields that $u_{j+1} \neq u_i$ for $i \leq j$ and completes the proof of the theorem. ∎

5 The strict insertion of a singular sphere

We would follow the method used by T. Rivière in [32] for the case $n = 3$.

5.1 Notations

We replace x_0 by 0 using a suitable translation in $\mathbb{R}^n$. We choose also an orthonormal basis $(\vec{i}, \vec{j}, \vec{k}_1, \cdots, \vec{k}_{n-2})$ for $\mathbb{R}^n$ such that

$$u_x(0) \neq 0, \quad u_x(0) \cdot u_y(0) = 0. \tag{5.1}$$

(See [9]). Let $(x, y, z_1, \cdots, z_{n-2})$ be the coordinates in the new basis. We introduce also the polar coordinates (r, θ) , $(R, \theta_1, \cdots, \theta_{n-4}, \varphi)$ as follows

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z_1 = R\cos\theta_1 \\ z_2 = R\sin\theta_1\cos\theta_2 \\ . \\ . \\ . \\ z_{n-3} = R\sin\theta_1\cdots\sin\theta_{n-4}\cos\varphi \\ z_{n-2} = R\sin\theta_1\cdots\sin\theta_{n-4}\sin\varphi \end{cases} \tag{5.2}$$

where $0 \leq \theta_i \leq \pi$, $0 \leq \varphi \leq 2\pi$ and $|\mathbf{z}| = R$ for $\mathbf{z} = (z_1, \cdots, z_{n-2})$.

Now for δ sufficiently small and $R \in [0, \delta + \delta^2]$ we define two unit vector fields

$$I(\mathbf{z}) = \frac{u_x(0, 0, \mathbf{z})}{|u_x(0, 0, \mathbf{z})|}, \quad K(\mathbf{z}) = u(0, 0, \mathbf{z}). \tag{5.3}$$

Since u takes its values in S^2, I and K are orthogonal. Let $a = |u_x(0)|$ and $b = |u_y(0)|$. We define $J(\mathbf{z})$ to be a smooth vectorfield such that (I, J, K) form an orthonormal basis. We verify then

$$\begin{aligned} u_x(0, 0, \mathbf{z}) &= (a + O(R))I(\mathbf{z}) \\ u_y(0, 0, \mathbf{z}) &= O(R)I(\mathbf{z}) + (b + O(R))J(\mathbf{z}). \end{aligned} \tag{5.4}$$

5.2 Sketch of the proof

We shall transform u in the region

$$C^\delta = \{(x, y, \mathbf{z}) \in \Omega \,|\quad 0 \leq R \leq \delta + \delta^2, \quad 0 \leq r \leq 2\delta^2\}.$$

For δ sufficiently small , the transformed map v would be singular exactly on the $(n-3)$-dimensional sphere $\sigma = \{(0, 0, \mathbf{z});\, R = \delta\}$ and will satisfy

$$deg_\sigma v = 1, \quad E(v) < E(u) + 8\pi\omega_{n-2}\delta^{n-2}. \tag{5.5}$$

For this aim we define the map u^δ as follows

(a) $u = u^\delta$ outside C^δ

(b) In the region

$$c^\delta = \{(x, y, \mathbf{z}) \,|\, R < \delta - \delta^2, 0 \leq r \leq 2\delta^2\}$$

u^δ would be an interpolation between u ouside c^δ and a conformal map on each disk centered at $(0, 0, \mathbf{z})$ and of radius δ^2 in the region

$$c_1^\delta = \{(x, y, \mathbf{z}) \,|\, R < \delta - \delta^2\,, 0 \leq r \leq \delta^2\}$$

exactly as it is described by T.Rivière in [32], following the method of H.Brezis and J.-M. Coron in [9] .

(c) For the region $\tilde{c}^\delta = C^\delta \backslash c^\delta$, u^δ will be the conjugation of the value of u^δ on $\partial\tilde{c}^\delta$ with the projection $\Pi : \tilde{c}^\delta \to \partial\tilde{c}^\delta$ which is defined as follows : For $p \in \tilde{c}^\delta$, $\Pi(p)$ is the intersection with $\partial\tilde{c}^\delta$ of the line orthogonal to σ which passes through p.

It will be showed that $v = u^\delta$ for δ small enough is a desired map. In the last step we will prove that $L(u, v) = \omega_{n-2}\delta^{n-2}$, the volume of the $(n-2)$-disk of the boundary σ.

5.3 The construction of u^δ in c^δ

For $(x, y, \mathbf{z}) \in c^\delta$ we define

(i) If $r < \delta^2$:

$$u^\delta = \frac{2\lambda}{\lambda^2 + r^2}(xI(\mathbf{z}) + yJ(\mathbf{z}) - \lambda K(\mathbf{z})) + K(\mathbf{z}) \tag{5.6}$$

where $\lambda = c\delta^4$ and c will be fixed later.

(ii) If $\delta^2 \leq r \leq 2\delta^2$:

$$\begin{aligned} u^\delta &= (A_1 r + B_1)I(\mathbf{z}) + (A_2 r + B_2)J(\mathbf{z}) \\ &\quad + \sqrt{1 - (A_1 r + B_1)^2 - (A_2 r + B_2)^2}\, K(\mathbf{z}) \end{aligned} \tag{5.7}$$

where A_i and B_i depend only on $\mathbf{z}, \theta, r$ as follows :

$$\begin{cases} 2\delta^2 A_i + B_i = u_i(2\delta^2\cos\theta, 2\delta^2\sin\theta, \mathbf{z}) \\ \text{for } i = 1,2\,(u_i \text{ is the } i\text{-th coordinate of } u \text{ in } (I(\mathbf{z}), J(\mathbf{z}), K(\mathbf{z})) \\ \delta^2 A_1 + B_1 = \dfrac{2\lambda\delta^2}{\lambda^2+\delta^4}\cos\theta \\ \delta^2 A_2 + B_2 = \dfrac{2\lambda\delta^2}{\lambda^2+\delta^4}\sin\theta. \end{cases} \tag{5.8}$$

The estimates for $E(u^\delta)$ in $c_2^\delta = c^\delta \backslash c_1^\delta$

Following the same computations as in [9] or [32] we have the following estimates on c_2^δ for fixed $\mathbf{z}$:

$$\begin{cases} \displaystyle\int_{\delta^2 \le r \le 2\delta^2} |\nabla_{xy} u^\delta(x,y,\mathbf{z})|^2\,dxdy \\ = 4\pi\delta^4(a^2+b^2-2c^2+(a^2+b^2+8c^2-4ac-4bc)\ln 2) + O(\delta^5). \\ \left|\dfrac{\partial u^\delta}{\partial z_i}(x,y,\mathbf{z})\right| = \left|\dfrac{\partial u}{\partial z_i}(0,0,\mathbf{z})\right| + O(\delta^2) \quad \text{for } i = 1,\cdots,n-2. \\ |\nabla u^\delta| \le C \quad \text{for } C>0 \text{ independent of } \delta. \end{cases} \tag{5.9}$$

Note that by $\nabla_{xy} u$ we mean the matrix of first partial derivatives of u in x and in y. As a result we have the following estimate for the energy on c_2^δ :

$$\begin{aligned} &\int_{c_2^\delta} |\nabla u^\delta|^2 \\ &= 4\pi\omega_{n-2}\delta^{n+2}(a^2+b^2-2c^2+(a^2+b^2+8c^2-4ac-4bc)\ln 2) \\ &+\pi((2\delta)^2-(\delta^2))\int_{0\le R\le \delta-\delta^2} |\nabla_{\mathbf{z}} u(0,0,\mathbf{z})|^2\,d\mathbf{z} + O(\delta^{n+3}). \end{aligned} \tag{5.10}$$

The estimates for $E(u^\delta)$ in c_1^δ.

Firstly for a fixed $\mathbf{z}$, u^δ is a conformal diffeomorphism from the disk $B^2((0,0,\mathbf{z}),\delta^2)$ into S^2 and we get :

$$\begin{aligned} \int_{r\le\delta^2} |\nabla_{xy} u^\delta(x,y,\mathbf{z})|^2\,dxdy &= 2\,\mathrm{Area}(u^\delta(B^2((0,0,\mathbf{z}),\delta^2),\mathbf{z})) \\ &= 8\pi - 8\pi c^2\delta^4 + O(\delta^5). \end{aligned} \tag{5.11}$$

and by integration on $\mathbf{z}$ we obtain :

$$\begin{aligned}
&\int_{c_1^\delta} |\nabla_{xy} u^\delta(x,y,\mathbf{z})|^2 \, dxdydz_1 \cdots dz_{n-2} \\
&= \frac{\omega_{n-2}}{(n-2)} \int_0^{\delta-\delta^2} R^{n-3} dR \int_{r\leq\delta^2} |\nabla_{xy} u^\delta(x,y,\mathbf{z})|^2 \, dxdy \\
&= 8\pi\omega_{n-2}(\delta-\delta^2)^{n-2} - 8\pi\omega_{n-2} c^2 \delta^{n+2} + O(\delta^{n+3}).
\end{aligned} \tag{5.12}$$

Meanwhile we estimate the $\mathbf{z}$-derivatives of u^δ in c_1^δ. Firstly we have

$$\frac{\partial u^\delta}{\partial z_i}(x,y,\mathbf{z}) = \frac{2\lambda}{\lambda^2+r^2}\left(x\frac{dI}{dz_i} + y\frac{dJ}{dz_i} - \lambda\frac{dK}{dz_i}\right) + \frac{dK}{dz_i} \quad \text{for } i = 1,\cdots,n-2. \tag{5.13}$$

We estimate $\frac{\partial u^\delta}{\partial z_i}(x,y,\mathbf{z})$ in two regions :

(a) $r \leq \delta^3$: Using (5.13) we observe that for $0 \leq r \leq \delta^2$:

$$|\nabla_{z_i} u^\delta| \leq \left|\frac{dK}{dz_i}\right| + \frac{2\lambda}{(\lambda^2+r^2)^{\frac{1}{2}}} \leq C \quad \text{independent of } \delta \tag{5.14}$$

and as a result

$$\int_{r\leq\delta^3} |\nabla_{\mathbf{z}} u^\delta|^2 dxdy = O(\delta^6), \tag{5.15}$$

which implies

$$\int_{c_1^\delta} |\nabla_{\mathbf{z}} u^\delta|^2 = O(\delta^{n+4}). \tag{5.16}$$

(b) $\delta^3 \leq r \leq \delta^2$: We have

$$\left|\frac{2\lambda}{\lambda^2+r^2}\left(x\frac{dI}{dz_i} + y\frac{dJ}{dz_i}\right)\right| \leq \frac{2\lambda r}{\lambda^2+r^2} \leq C\frac{\lambda}{r} = O(\delta). \tag{5.17}$$

So using (5.13)

$$\frac{\partial u^\delta}{\partial z_i}(x,y,\mathbf{z}) = \left(\frac{r^2-\lambda^2}{r^2+\lambda^2}\right)\frac{\partial u}{\partial z_i}(0,0,\mathbf{z}) + O(\delta) \quad \text{for} \quad \delta^3 \leq r \leq \delta^2,$$

and we get

$$\begin{aligned}
\int_{\delta^3\leq r\leq\delta^2} |\nabla_{\mathbf{z}} u^\delta|^2 \, dxdy &= \left(2\pi \int_{\delta^3}^{\delta^2} \left(\frac{r^2-\lambda^2}{r^2+\lambda^2}\right)^2 r\, dr\right) |\nabla_{\mathbf{z}} u(0,0,\mathbf{z})|^2 + O(\delta^5) \\
&= \pi\delta^4 |\nabla_{\mathbf{z}} u(0,0,\mathbf{z})|^2 + O(\delta^5).
\end{aligned}$$

This last estimate combined with (5.16) yields

$$\int_{c_1^\delta} |\nabla_{\mathbf{z}} u^\delta|^2 = \pi\delta^4 \int_{0\le R\le \delta-\delta^2} |\nabla_{\mathbf{z}} u(0,0,\mathbf{z})|^2 \, dz_1 \cdots dz_{n-2} + O(\delta^{n+3}) \,. \tag{5.18}$$

At last combining (5.12) and (5.18) we obtain :

$$\begin{aligned} \int_{c_1^\delta} |\nabla u^\delta|^2 &= 8\pi\omega_{n-2}(\delta-\delta^2)^{n-2} - 8\pi\omega_{n-2}c^2\delta^{n+2} \\ &\quad + \pi\delta^4 \int_{0\le R\le \delta-\delta^2} |\nabla_{\mathbf{z}} u(0,0,\mathbf{z})|^2 \, d\mathbf{z} + O(\delta^{n+3}) \,. \end{aligned} \tag{5.19}$$

The evaluation of $E(u^\delta)$ on $\tilde{c}^\delta$

As briefly mentioned above in the the sketch of the proof, u^δ in the region $\tilde{c}^\delta$ is defined as follows : We define the projection $h : \tilde{c}^\delta \to \sigma$ by

$$h(x,y,z_1,z_2,\cdots,z_{n-2}) = (0,0,\frac{\delta z_1}{R},\cdots,\frac{\delta z_{n-2}}{R}) \tag{5.20}$$

Then the projection Π , defined on

$$\tilde{c}^\delta = \{(x,y,\mathbf{z}) \,|\, \delta-\delta^2 \le R \le \delta+\delta^2 \,, \quad 0 \le r \le \delta^2\}$$

sends each point p to the intersection between $\partial\tilde{c}^\delta$ and the line passing through p and $h(p)$. We take

$$u^\delta = \left(u^\delta|_{\partial\tilde{c}^\delta}\right) \circ \Pi.$$

Pay attention that the points p and $\Pi(p)$ lie in the 3-plane orthogonal to σ at $h(p)$.

Using the co-area formula we have

$$\int_{\tilde{c}^\delta} |\nabla u^\delta|^2 = \int_\sigma d\mathcal{H}^{n-3} \int_{h^{-1}(w)} \frac{|\nabla u^\delta|^2}{|J_{n-3}h|} \, d\mathcal{H}^3 \tag{5.21}$$

Moreover $|J_{n-3}h| = \left(\frac{\delta}{R}\right)^{n-3}$ and

$$\begin{aligned} h^{-1}(w) = \{(x,y,R,\theta_1,\cdots,\theta_{n-4},\varphi) \in \tilde{c}^\delta \,|\, \delta-\delta^2 \le R \le \delta+\delta^2 \,, 0 \le r \le 2\delta^2 \,, \\ \theta_i = const. \text{ for } i=1,\cdots,n-4, \text{ and } \varphi = const.\} \end{aligned}$$

is a cylinder of the height $2\delta^2$, of radius $2\delta^2$ and of center $w \in \sigma$. We now estimate the value of $\int_{h^{-1}(w)} R^{n-3} |\nabla u^\delta|^2 \, dx\, dy\, dR$.

We write $h^{-1}(w)$ as the union of two separate regions G_w and H_w :

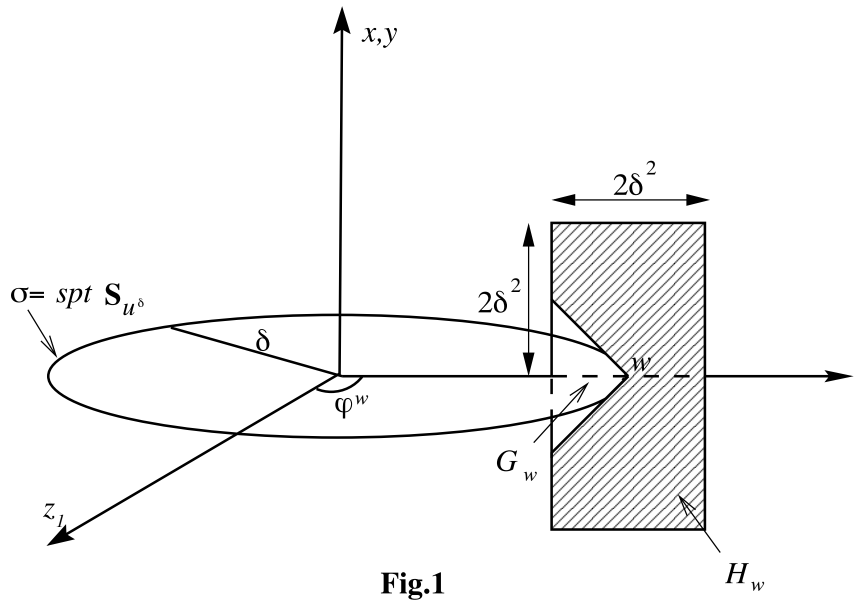
x,y
2δ²
2δ²
σ= spt $\mathbf{S}_{u^\delta}$
δ
w
φ^w
G_w
z_1
H_w

Fig.1

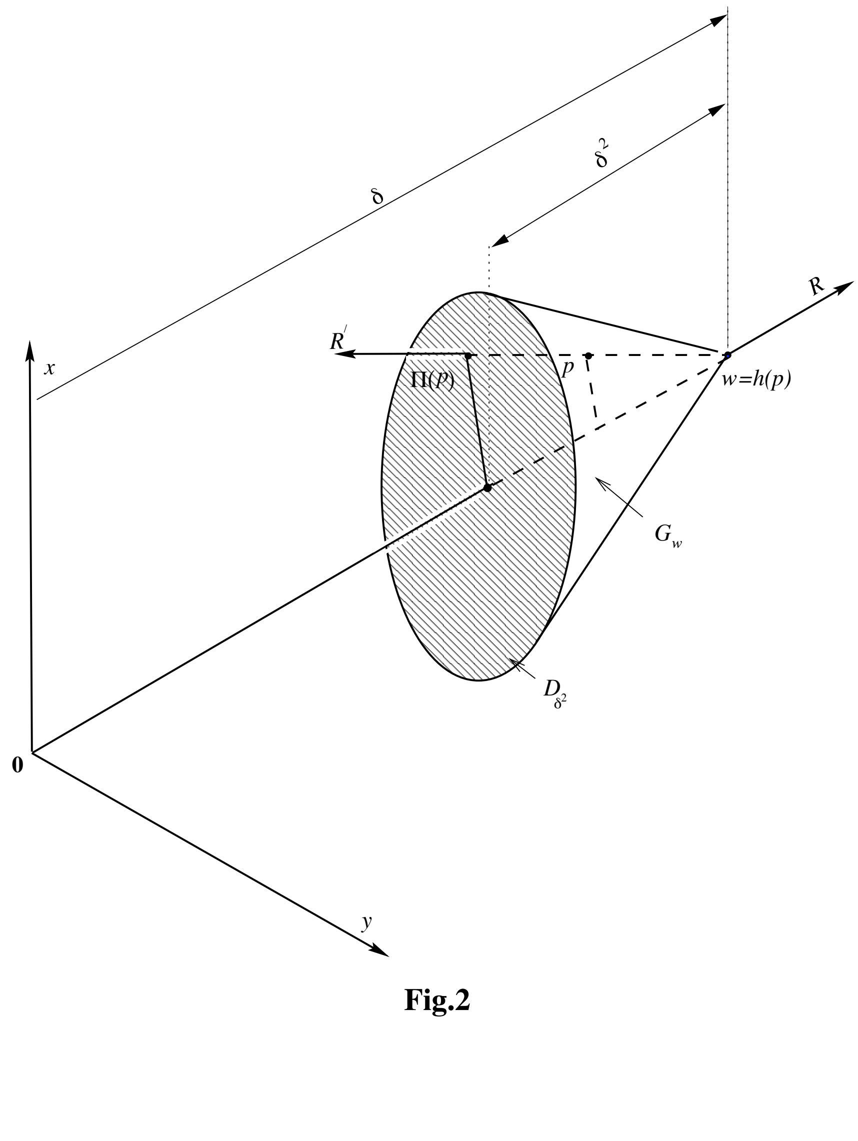
δ
δ²
R
R′
Π(p)
p
w=h(p)
x
G_w
D_δ²
0
y

Fig.2

(1) $G_w = \Pi^{-1}(\partial c_1^\delta \cap h^{-1}(w))$ is the little 3-cone of vertex w, lying in the plane orthogonal to σ at w, whose end is the disk D_{δ^2} of center $(0, 0, \delta - \delta^2, \theta_1^w, \cdots, \theta_{n-4}^w, \varphi^w)$ and of radius δ^2. Pay attention that on this disk u^δ is the conformal map defined in (5.6).

(2) H_w is the complementar of G_w in $h^{-1}(w)$: i.e. $H_w = \Pi^{-1}(\partial\tilde{c}^\delta \backslash \partial c_1^\delta \cap h^{-1}(w))$.

See Fig.1 and Fig.2 to visualize these regions for $n = 4$. For estimating $|\nabla u^\delta|$ on G_w we proceed by changing the coordinates. Let R' be the distance of the point $p = (x, y, z_1, \cdots, z_{n-2}) \in G_w$ from w, the vertex of the cone, and let x' and y' be the two first coordinates of $\Pi(p)$ in D_{δ^2} (See Fig.2). We have

$$\begin{cases} x' = \dfrac{\delta^2 x}{\delta - R} \\[2ex] y' = \dfrac{\delta^2 y}{\delta - R} \\[2ex] R' = \sqrt{r^2 + (\delta - R)^2} \\[2ex] \theta_i = \theta_i \ , \ \varphi = \varphi \end{cases} \quad \text{and} \quad \begin{cases} x = \dfrac{x'R'}{\sqrt{\delta^4 + r'^2}} \\[2ex] y = \dfrac{y'R'}{\sqrt{\delta^4 + r'^2}} \\[2ex] \delta - R = \dfrac{\delta^2 R'}{\sqrt{\delta^4 + r'^2}} \end{cases} \tag{5.22}$$

Now u^δ is constant on the rays passing by w, so we get

$$u^\delta(x', y', R', \theta_1, \cdots, \theta_{n-4}, \varphi) = u^\delta(x', y', \sqrt{\delta^4 + r'^2}, \theta_1, \cdots, \theta_{n-4}, \varphi) \tag{5.23}$$

i.e. $\dfrac{\partial u^\delta}{\partial R'} = 0$. Also by a simple calculation of the derivatives using (5.22) we have for the point $(x, y, \mathbf{z}) \in G_w$:

$$\begin{cases} \dfrac{\partial u^\delta}{\partial x} = \left(\dfrac{\sqrt{\delta^4 + r'^2}}{R'}\right) \dfrac{\partial u^\delta}{\partial x'}(x', y', \sqrt{\delta^4 + r'^2}) \\[3ex] \dfrac{\partial u^\delta}{\partial y} = \left(\dfrac{\sqrt{\delta^4 + r'^2}}{R'}\right) \dfrac{\partial u^\delta}{\partial y'}(x', y', \sqrt{\delta^4 + r'^2}) \\[3ex] \dfrac{\partial u^\delta}{\partial R} = \left(\dfrac{x'\sqrt{\delta^4 + r'^2}}{\delta^2 R'}\right) \dfrac{\partial u^\delta}{\partial x'}(x', y', \sqrt{\delta^4 + r'^2}) \\[3ex] \qquad + \left(\dfrac{y'\sqrt{\delta^4 + r'^2}}{\delta^2 R'}\right) \dfrac{\partial u^\delta}{\partial y'}(x', y', \sqrt{\delta^4 + r'^2}). \end{cases} \tag{5.24}$$

and in the same line by calculating the Jacobian of the new coordinates we have :

$$dx\,dy\,dR = \frac{\delta^2 R'^2}{(\delta^4 + r'^2)^{\frac{3}{2}}}\,dx'dy'dR'. \tag{5.25}$$

Using (5.2) and doing the same work, we get :

$$|\nabla u^\delta|^2 = \left|\frac{\partial u^\delta}{\partial x}\right|^2 + \left|\frac{\partial u^\delta}{\partial y}\right|^2 + \left|\frac{\partial u^\delta}{\partial R}\right|^2 + I \tag{5.26}$$

where

$$I = \frac{1}{R^2}\left(\left|\frac{\partial u^\delta}{\partial \theta_1}\right|^2 + \frac{1}{\sin^2\theta_1}\left|\frac{\partial u^\delta}{\partial \theta_2}\right|^2 + \cdots + \frac{1}{\sin^2\theta_1 \sin^2\theta_2 \cdots sin^2\theta_{n-4}}\left|\frac{\partial u^\delta}{\partial \varphi}\right|^2\right).$$

Using (5.23) and applying (5.14) and (5.26) to the points of D_{δ^2} we obtain

$$\begin{aligned} I(x', y', R') &= \frac{(\delta - \delta^2)^2}{R^2} I(x', y', \sqrt{\delta^4 + r'^2}) \\ &\leq \frac{(\delta - \delta^2)^2}{R^2}\left|\nabla_{\mathbf{z}} u^\delta(x', y', \sqrt{\delta^4 + r'^2})\right|^2 \leq C\frac{(\delta - \delta^2)^2}{R^2} \end{aligned} \tag{5.27}$$

Therefore by integrating directly over the cone G_w we deduce from (5.27) :

$$\int_{G_w} R^{n-3} I\,dx\,dy\,dR = O(\delta^{n+3}). \tag{5.28}$$

Furthermore considering (5.22), (5.24), (**??**) and (5.26) we estimate the integral

$$J = \int_{G_w} R^{n-3}(|\nabla u^\delta|^2 - I)$$

as follows

$$
\begin{aligned}
J &= \int_{G_w} R^{n-3}\left(\left|\frac{\partial u^\delta}{\partial x}\right|^2+\left|\frac{\partial u^\delta}{\partial y}\right|^2+\left|\frac{\partial u^\delta}{\partial R}\right|^2\right) dx\,dy\,dR \\
&= \int_{D_{\delta^2}} dx'\,dy' \int_0^{\sqrt{\delta^4+r'^2}} R^{n-3}\frac{\delta^2 R'^2}{(\delta^4+r'^2)^{\frac{3}{2}}}\left[\frac{\delta^4+r'^2}{R'^2}\left|\nabla_{x'y'}u^\delta(x',y',\sqrt{\delta^4+r'^2})\right|^2\right. \\
&\qquad\qquad\left.+\left(\frac{\delta^4+r'^2}{\delta^4 R'^2}\right)\left(x'\left|\frac{\partial u^\delta}{\partial x'}\right|+y'\left|\frac{\partial u^\delta}{\partial y'}\right|\right)^2\right] dR' \\
&= \int_{D_{\delta^2}} \frac{\delta^2}{\sqrt{\delta^4+r'^2}}\left|\nabla_{x'y'}u^\delta\right|^2\int_{-\delta^2}^{\delta}\frac{\sqrt{\delta^4+r'^2}}{\delta^2}R^{n-3}\,dR \\
&\quad+\int_{D_{\delta^2}} dx'\,dy'\frac{1}{\delta^2\sqrt{\delta^4+r'^2}}\left(x'\left|\frac{\partial u^\delta}{\partial x'}\right|+y'\left|\frac{\partial u^\delta}{\partial y'}\right|\right)^2\int_{\delta-\delta^2}^{\delta}\frac{\sqrt{\delta^4+r'^2}}{\delta^2}R^{n-3}\,dR.
\end{aligned}
\tag{5.29}
$$

Using the inequality

$$
|\nabla_{xy}u^\delta|^2 \le C\frac{\delta^8}{(\delta^8+r^2)^2} \quad \text{on} \quad D_{\delta^2} \tag{5.30}
$$

which is established in [9] we obtain

$$
\begin{aligned}
&\int_{D_{\delta^2}} dx'\,dy'\frac{1}{\delta^2\sqrt{\delta^4+r'^2}}\left(x'\left|\frac{\partial u^\delta}{\partial x'}\right|+y'\left|\frac{\partial u^\delta}{\partial y'}\right|\right)^2\int_{\delta-\delta^2}^{\delta}\frac{\sqrt{\delta^4+r'^2}}{\delta^2}R^{n-3}\,dR \\
&\le C\int_0^{\delta^2}\delta^{n+3}\frac{r^3}{(\delta^8+r^2)^2}\,dr = O(\delta^{n+3}\ln(1/\delta)).
\end{aligned}
\tag{5.31}
$$

And combining (5.11), (5.26), (5.28), (5.29) and (5.31), finally we get :

$$
\begin{aligned}
\int_{G_w} R^{n-3}|\nabla u^\delta|^2 &= \frac{8\pi}{n-2}(\delta^{n-2}-(\delta-\delta^2)^{n-2}) \\
&\quad -\frac{8\pi}{n-2}c^2\delta^{n+2}+O(\delta^{n+3}\ln(1/\delta)).
\end{aligned}
\tag{5.32}
$$

Now, using the estimates in (5.9) and the fact that $u^\delta = u$ on $\partial\tilde{c}^\delta\backslash\partial c^\delta$ we observe that $|\nabla u^\delta|$ is bounded on ∂H_w and therefore following the same method as the one used for G_w we get

$$
\int_{H_w} R^{n-3}|\nabla u^\delta|^2 = O(\delta^{n+3}). \tag{5.33}
$$

which conjugated with (5.21) and (5.32) yields

$$\int_{\tilde{\mathcal{C}}^\delta} |\nabla u^\delta|^2 = 8\pi\omega_{n-2}(\delta^{n-2} - (\delta - \delta^2)^{n-2}) \\ -8\pi\omega_{n-2}c^2\delta^{n+2} + O(\delta^{n+3}\ln(1/\delta)). \tag{5.34}$$

The estimate for the energy of u in C^δ

Similarly as in [32] we have the following estimate :

$$\int_{C^\delta} |\nabla u|^2 = 4\pi\omega_{n-2}\delta^{n+2}(a^2 + b^2) \\ +4\pi\delta^4 \int_{0\le R\le\delta-\delta^2} |\nabla_{\mathbf{z}} u(0,0,\mathbf{z})|^2 d\mathbf{z} + O(\delta^{n+3}). \tag{5.35}$$

5.4 The end of proof of lemma 4

Conjugating (5.10), (5.19) ,(5.34) and (5.35) we obtain :

$$\int_\Omega |\nabla u^\delta|^2 = 8\pi\omega_{n-2}\delta^{n-2} \\ -4\pi\omega_{n-2}\delta^{n+2}(4c^2 - (a^2 + b^2 + 8c^2 - 4ac - 4bc)\ln 2) \\ +O(\delta^{n+3}\ln(1/\delta)) \tag{5.36}$$

and by choosing a suitable c such that

$$4c^2 - (a^2 + b^2 + 8c^2 - 4ac - 4bc)\ln 2 > 0$$

we can be sure that for δ small enough $v = u^\delta$ would satisfy the strict inequality (5.5). For example put $c = \max\left\{\frac{a}{2}, \frac{b}{2}\right\}$. It is easy to verify that the degree of v on its only singular set, i.e. $\sigma = \{(0,0,\mathbf{z})\,|R = \delta\}$ is one. By the way as in (**??**) :

$$L(v,u) = \sup_{\substack{\psi \in \Omega^\infty_{n-3}(\overline{\Omega}) \\ |d\psi|_\infty \le 1}} \left\{ \int_\Omega v^*\omega \wedge d\psi - \int_\Omega u^*\omega \wedge d\psi \right\} \\ = \sup_{\substack{\psi \in \Omega^\infty_{n-3}(\overline{\Omega}) \\ |d\psi|_\infty \le 1}} \mathbf{S}_v(\psi) \tag{5.37}$$

as $\mathbf{S}_u = 0$. Meanwhile using the corollary 1 :

$$|\mathbf{S}_v(\psi)| = |\tau(\sigma, 1, \vec{\sigma})(\psi)| = |\mathbf{T}(d\psi)| \le \mathbf{M}(\mathbf{T}) \tag{5.38}$$

for every current $\mathbf{T}$ which takes σ as its boundary, using the fact that $|d\psi|_\infty \leq 1$. Putting $\mathbf{T} = \mathbf{T}_0 = \tau(B_\delta, 1, \vec{B_\delta})$ where B_δ is the $(n-2)$-ball of the center 0 and of radius δ , we obtain combining (5.37) and (5.38) :

$$L(v, u) \leq \omega_{n-2}\delta^{n-2}. \tag{5.39}$$

Now take $\psi_0 = z_1 \wedge dz_2 \wedge \cdots \wedge dz_{n-2}$. A simple observation shows that $\mathbf{T}_0(d\psi_0) = \mathbf{M}(\mathbf{T}_0) = \omega_{n-2}\delta^{n-2}$, so again using (5.37) and (5.38) we obtain easily that

$$L(v, u) \geq \omega_{n-2}\delta^{n-2}$$

which completes the proof regarding (5.39). ■

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Chapter III

Topological singularities in $W^{1,3}(M, S^3)$

On topological singular set of maps with finite 3-energy into S^3

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We prove that the topological singular set of a map in $W^{1,3}(M, S^3)$ is the boundary of an integer multiplicity rectifiable current in M, where M is a closed smooth manifold of dimension greater than 3. Also we prove that the mass of the minimal i.m. rectifiable current taking this set as the boundary is a strongly continuous functional on $W^{1,3}(M, S^3)$.

1 Introduction

Let M be an oriented smooth closed riemannien manifold of dimension n, and N any closed riemannien manifold isometrically embedded in $\mathbb{R}^N$. Let

$$W^{1,p}(M, N) := \{u \in W^{1,p}(M, \mathbb{R}^N)\,;\, u(x) \in N \quad \text{a.e. on}\, M\}.$$

For $u \in W^{1,p}(M, N)$ the p-energy is given by $E(u) = \int_M |\nabla u|^p dvol_M$.

In [7], F.Bethuel and X.Zheng proved that smooth maps are not strongly dense in $W^{1,p}(M,N)$ if $p<n$ and $\pi_{[p]}(N)\neq 0$, [p] being the integer part of p. In this case, one may want to characterize the maps in $W^{1,p}(M,N)$ which are approximable by smooth maps and identify the obstruction for maps which are not. Precisely, we would like to associate to any map $u\in W^{1,p}(M,N)$ a topological singualr set, $\mathbf{S}_u$, which characterizes the approximability of u by smooth maps, i.e. u would be the strong limit of smooth maps if and only if $\mathbf{S}_u=0$.

In this line, F.Bethuel proved in [3] that $u\in W^{1,2}(\mathbf{B}^n,S^2)$ is strongly approximable by maps in $C^\infty(\mathbf{B}^n,S^2)$, if and only if $d(u^*\omega_{S^2})=0$ in the sense of distributions. Here $\mathbf{B}^n$ is the n-dimensional unit disk. The same result holds for the space $W^{1,p}(\mathbf{B}^n,S^p)$ for any other integer p (See [6]). Thus, the "local" topological obstruction for maps in $W^{1,p}(M,S^{[p]})$ can be defined as a current :

Definition *Let $p<n$ and $u\in W^{1,p}(M,S^{[p]})$. The topological singular set of u, $\mathbf{S}_u\in\mathcal{D}_{n-[p]-1}(M)$, is the current defined by*

$$\mathbf{S}_u(\alpha):=\int_M u^*\omega\wedge d\alpha \qquad \forall\alpha\in\mathcal{D}^{n-[p]-1}(M).$$

Here $\mathcal{D}^k(M)$ is the set of smooth k-forms on M with compact support (See[16], 2.2.3) and ω is any $[p]$-form on $S^{[p]}$ for which $\int_{S^{[p]}}\omega=1$.

Remark 1.1 *Recent developments by F.Hang and F.H.Lin [20] showed that the condition "$\mathbf{S}_u=0$", though being necessary for the strong approximability of a map $u\in W^{1,p}(M,S^p)$ by smooth maps in this space, is not always sufficient due to some obstructions lying in the "global" topological structure of certain domains. Precisely, there is a map $u\in H^1(\mathbb{CP}^2,S^2)$ for which $d(u^*\omega)=0$ while u is not in the strong closure of smooth maps in $H^1(\mathbb{CP}^2,S^2)$.*

Two important problems about $\mathbf{S}_u$, $u\in W^{1,p}(M,S^p)$, are still open for almost every integer p. First, we do not know whether $\mathbf{S}_u$ is always the boundary of an i.m. rectifiable current, i.e. if it is an integral flat chain. This has been proved for $p=1$ or $n-1$ (See [16], vol II, section 5.4.3) or $p=2$ (See [30]). The second problem arises if the answer to the first one is positive. Set for $\mathbf{S}$, any integral flat chain in M of dimension k,

$$m_i(\mathbf{S}):=\inf\left\{\mathbf{M}(\mathbf{T})\,;\,T\in\mathcal{R}_{k+1}(M),\ \partial\mathbf{T}=\mathbf{S}_u\right\},$$

the minimal mass of i.m. rectifiable currents taking $\mathbf{S}$ as the boundary. Then the question would be to determine whether $m_i(\mathbf{S}_{u_m}-\mathbf{S}_u)\to 0$ if u_m converges strongly to u in $W^{1,p}(M,S^p)$. The answer is yes for $p=1$ or $n-1$, (See [5] and [16], vol II, section 5.4.2), while we do not know whether this is the case for the maps in $H^1(\mathbf{B}^4,S^2)$. We encounter this case when considering the problem of relaxing the Dirichlet energy for maps into S^2.

As we saw in [30], generalizing to higher dimensions the algebraic formula given in [5] for the relaxed Dirichlet energy from a 3 dimensional domain into S^2 is possible if we prove that $m_i(\mathbf{S}_u)$ is strongly continuous on $H^1(\mathbf{B}^n, S^2)$.

Another case where the second problem shows its importance is when we try to define a topological singular set for maps in $W^{1,p}(\mathbf{B}^n, N)$. In [6], F.Bethuel, J.M.Coron, F.Demengel and F.Helein gave a discription of this set for when N is $([p]-1)$-connected and $\pi_{[p]}(N)$ is torsion free. Considering the problem for when $\pi_{[p]}(N)$ has torsions, the author and T.Rivière ramarked that we can define this set as a flat $\pi_{[p]}(N)$-chain if these two questions come to have a positive answer for $[p]$. As an example, the topological singular set of any map in $u \in W^{1,1}(\mathbf{B}^n, \mathbb{RP}^2)$ is a flat $\mathbb{Z}_2$-chain, and is equal to zero if and only if u is a strong limit of smooth maps in $W^{1,1}(\mathbf{B}^n, \mathbb{RP}^2)$ (See [31]).

In this paper we solve these problems for $p = 3$ and 7. The particularity of these two cases reside in the fact that S^3 and S^7 (alongside with S^1) are the only spheres which have this property : There is a smooth multiplication

$$\kappa : S^k \times S^k \to S^k$$

such that the induced homotopic homeomorphism

$$\kappa_* : \pi_k(S^k) \oplus \pi_k(S^k) \to \pi_k(S^k)$$

is the sum of elements in $\pi_k(S^k)$. As a result, the method we use does not work for other values of p. Here is our main result

Theorem 1 *Let $p = 3$ or 7, $p < n = dim\, M$ and $u \in W^{1,p}(M, S^p)$. Then $\mathbf{S}_u$ is the boundary of an i.m. rectifiable current in M. Moreover, $m_i(\mathbf{S}_{u_m} - \mathbf{S}_u) \to 0$ if u_m converges strongly to u in $W^{1,p}(M, S^p)$.*

■

If M is not closed we set

$$W^{1,p}_\varphi(M, N) := \{u \in W^{1,p}(M, N)\,;\, u = \varphi \quad \text{on } \partial M\}$$

where φ is a given boundary data. We assume that φ is in $C^\infty(\partial M, N)$ and can be extended into M by a smooth map. Then we have

Theorem 1 bis *Let $p = 3$ or 7, $p < n = dim\, M$ and $u \in W^{1,p}_\varphi(M, S^p)$. Then $\mathbf{S}_u$ is the boundary of an i.m. rectifiable current in M. Moreover, $m_i(\mathbf{S}_{u_m} - \mathbf{S}_u) \to 0$ if u_m converges strongly to u in $W^{1,p}_\varphi(M, S^p)$.*

■

Considering the question of topological singular sets, using the methods of [31], we have these corollaries. The readers may refer to [15], [38] and [31] respectively for definitions and more details.

Corollary 1.1 *Let $\mathbf{B}^n$ be the n-dimensional unit disk, $n > [p] = 3$ or 7, and assume that N is a closed $([p]-1)$-connected riemannien manifold of dimension equal or greater than $[p]$. Then $\mathbf{S}_u$, the topological singular set of any $u \in W^{1,p}(\mathbf{B}^n, N)$, is well defined as a flat $\pi_{[p]}(N)$-chain and the flat norm of $\mathbf{S}_{u_m} - \mathbf{S}_u$ converges to 0 if $u_m \to u$ in $W^{1,p}(\mathbf{B}^n, N)$. Moreover u is a strong limit of smooth maps in $W^{1,p}(\mathbf{B}^n, N)$ if and only if $\mathbf{S}_u = 0$.*

■

Remark 1.2 *The cases where N is not $([p]-1)$-connected are more involved. The readers can refer to [24], where T. Rivière and R. Hardt have treated the relatively difficult case of $W^{1,3}(\mathbf{B}^4, S^2)$.*

Corollary 1.1 bis *Let $\mathbf{B}^n$ be the n-dimensional unit disk, $n > [p] = 3$ or 7, and assume that N is a closed $([p]-1)$-connected riemannien manifold of dimension equal or greater than $[p]$. We assume also that $\varphi \in C^\infty(\partial \mathbf{B}^n, N)$ is smoothly extendable into $\mathbf{B}^n$. Then u is a strong limit of smooth maps in $W^{1,p}_\varphi(\mathbf{B}^n, N)$ if and only if $\mathbf{S}_u = 0$.*

■

2 Some known facts

Definition 2.1 *We say that $u \in W^{1,p}(M, S^p)$ is in $R^{\infty,p}(M, S^p)$ if u is smooth except on $B = \bigcup_{i=1}^m \sigma_i \cup B_0$, a compact subset of M, where $\mathcal{H}^{n-p-1}(B_0) = 0$ and the σ_i, $i = 1, \cdots, m$ are smooth embeddings of the unit disk of dimension $n-p-1$. Moreover we assume that any two different faces of B, σ_i and σ_j, may meet only on their boundaries.*

Theorem 2 (Bethuel,[2]) *$R^{\infty,p}(M, S^p)$ is dense in $W^{1,p}(M, S^p)$ for the strong topology.*

■

We recall the definition of $\mathbf{S}_u$, the topological singular set of u :

Definition 2.2 *Let $u \in W^{1,p}(M, S^p)$. We define the current $\mathbf{S}_u \in \mathcal{D}_{n-p-1}(M)$ to be the current defined by*

$$\mathbf{S}_u(\alpha) := \int_M u^*\omega \wedge d\alpha \qquad \forall \alpha \in \mathcal{D}^{n-p-1}(M). \tag{2.1}$$

Here $\mathcal{D}^k(M)$ is the set of smooth k-forms on M with compact support (See[16], 2.2.3) and ω is some p-form on S^p for which $\int_{S^p} \omega = 1$.

Let ω_1 and ω_2 be two such forms on S^p. We have $\omega_1 - \omega_2 = d\beta$ where β is some smooth 1-form on S^p extendable to $\mathbb{R}^{p+1}$. Let $u \in W^{1,p}(M, S^p)$ and consider a sequence $u_m \in C^\infty(M, \mathbb{R}^{p+1})$ converging to u in $W^{1,p}$. We have

$$u_m^*(d\beta) = d\,(u_m^*\beta)$$

and by passing to the limit, we observe that this holds true for u in the sense of distributions. This proves the independence of $\mathbf{S}_u$ from the choice of ω as we have :

$$d(u^*\omega_1) - d(u^*\omega_2) = du^*(d\beta) = 0$$

in the sense of distributions. Now the existence of the integral (2.1) is a direct consequence of the following inequality :

$$|u^*\omega| \leq \frac{1}{p^{p/2}\alpha_p}|\nabla u|^p \qquad \text{a.e. on } M \tag{2.2}$$

where $\alpha_p := |S^p|$ and $\alpha_p\omega = \omega_V$, is the standard volume form of S^p.

We shall give a description of $\mathbf{S}_u$ for $u \in R^{\infty,p}(M, S^p)$. Clearly if u is smooth a standard operation on pull-back yields

$$d(u^*\omega) = u^*(d\omega) = 0$$

and as a consequence we deduce for $u \in R^{\infty,p}(M, S^p)$ that

$$spt\mathbf{S}_u \subseteq B.$$

Definition 2.3 *Let $u \in R^{\infty,p}(M, S^p)$ and let $B = \bigcup \sigma_i \cup B_0$ be the singular set of u. Suppose that each σ_i is oriented by a smooth $(n-p-1)$-vectorfield $\vec{\sigma}_i$. For $a \in \sigma_i$ let N_a be any $(p+1)$-dimensional smooth submanifold of M, orthogonal to σ_i at a. Consider the embedded $(p+1)$-disk $M_{a,\delta} = B_\delta(a) \cap N_a$ oriented by the $(p+1)$-vectorfield $\vec{M}_a$ such that $(-1)^{n-p}\vec{\sigma}_i(a) \wedge \vec{M}_a$ is the fixed orientation of M. Then the topological degree of u on the p-dimensional topological sphere $\Sigma_{a,\delta} = \partial M_{a,\delta}$ is well defined and is independent of the choice of a and N_a for δ small enough. We call this integer the degree of u on σ_i and denote it by*

$$deg_{\sigma_i} u\,.$$

Remember that any k-dimensional rectifiable subset $\mathcal{M}$ of M considered with a multiplicity θ and oriented by a unit k-vector field ξ defines a rectifiable current as follows

$$\tau(\mathcal{M}, \theta, \xi)(\alpha) := \int_{\mathcal{M}} < \xi, \alpha > \theta\, d\mathcal{H}^k \qquad \forall \alpha \in \mathcal{D}^k(M).$$

We should recall some useful results.

Lemma 2.1 *If u_m is a sequence of maps in $W^{1,p}(M,S^p)$ converging to u, $\mathbf{S}_{u_m}$ tends to $\mathbf{S}_u$ in the sense of currents. That is, for any α, smooth $(n-p-1)$-form in M, we have*

$$\mathbf{S}_u = \lim_{m\to\infty} \mathbf{S}_{u_m}(\alpha).$$

Equivalently

$$m_r(\mathbf{S}_{u_m} - \mathbf{S}_u) \to 0 \quad \textit{if} \quad u_m \to u \quad \textit{in} \quad W^{1,p}(M,S^p),$$

where $m_r(\mathbf{S})$ is the minimal mass of normal currents taking $\mathbf{S}$ as their boundary.

$\blacksquare$

Lemma 2.2 *Let M be a compact riemannien manifold. Then for any $u \in R^{\infty,p}(M,S^p)$, $\mathbf{S}_u$ is the integer multiplicity rectifiable current $\sum_{i=1}^{m}(deg_{\sigma_i} u)\,\tau(\sigma_i,1,\vec{\sigma}_i)$. Meanwhile, if ∂M is empty, or if $u|_{\partial M}$ is homotopic to a constant, then $\mathbf{S}_u$ is the boundary of some i.m. rectifiable current of finite mass.*

$\blacksquare$

The reader can find the proofs of these statements for the case $p=2$ in [29] and [30], M being a domain in $\mathbb{R}^n$. The proofs are essentially the same for other values of p and any smooth compact manifold.

Remark 2.1 *By lamma 2.1, theorem 1 would come true for any p if $\frac{m_i(\mathbf{S})}{m_r(\mathbf{S})} < C$, for any integral flat $(n-p-1)$-chain $\mathbf{S}$ in M. The existence of such a constant is an open problem except for when dim $\mathbf{S} = 0, n-2$, where we have the equality $m_i(\mathbf{S}) = m_r(\mathbf{S})$ for any integral flat chain. Refer to [1], [10], [12], [14] and [16], vol II, section 1.3.4 for proofs and different aspects of the problem.*

Theorem 3 (Almgren, Browder and Lieb, [1]) *Let M be as above, $u \in R^{\infty,p}(M,S^p)$, such that either ∂M is empty or $u|_{\partial M}$ is constant, then*

$$m_i(\mathbf{S}_u) \le \frac{1}{p^{p/2}\alpha_p}\int_M |\nabla u|^p dvol_M$$

$\blacksquare$

3 Proof of theorem 1

We identify S^3 (respectively S^7) with the unit spheres in quaternions (respectively Cayley numbers) and observe that they inherit the product structure on these spaces. If we show the quaternion product (respectively Cayley product) by $\kappa(x,y) := x \bullet y$, κ will be a smooth map from $S^k \times S^k \to S^k$, k=3,7, and will satisfy this condition : The induced homotopic homeomorphism

$$\kappa_* : \pi_k(S^k) \oplus \pi_k(S^k) \to \pi_k(S^k)$$

is the sum of elements in $\pi_k(S^k)$. The spheres of dimensions $0, 1, 3$ and 7 are the only spheres for which such κ exist (See [8], section VI.15, p. 412). By $x^{-1} \in S^k$ we mean the right inverse of $x \in S^k$. Set for $u, v \in W^{1,p}(M, S^p)$ and $x \in M$

$$u \bullet v^{-1}(x) := u(x) \bullet v(x)^{-1}.$$

Lemma 3.1 *Let $u, v \in W^{1,p}(M, S^p)$, p=3,7, then $u \bullet v^{-1} \in W^{1,p}(M, S^p)$. Moreover if $\{u_m\}$ is a strongly convergent sequence in $W^{1,p}(M, S^p)$, then $E(u_m \bullet u_k^{-1}) \to 0$ if $m, k \to +\infty$.*

■

Proof : Straight computations show that

$$\nabla(u \bullet v^{-1}) = \nabla u \bullet v^{-1} - u \bullet (v^{-1} \bullet (\nabla v \bullet v^{-1}))$$

which yields

$$|\nabla(u \bullet v^{-1})| \leq |\nabla u| + |\nabla v|$$

as $|u| = |v| = 1$. Thus $u \bullet v^{-1} \in W^{1,p}(M, S^p)$. The smoothness of operations and the Lebesgue dominant convergece yields the second part of lemma. ■

Lemma 3.2 *If $u, v \in R^{\infty,p}(M, S^p)$, p=3,7, then $u \bullet v^{-1} \in R^{\infty,p}(M, S^p)$ and we have*

$$\mathbf{S}_{u \bullet v^{-1}} = \mathbf{S}_u - \mathbf{S}_v \tag{3.1}$$

.

■

Proof : That $u \bullet v^{-1} \in R^{\infty,p}(M, S^p)$ is a direct result of smoothness of the product. The relation (3.1) can be deduced from lemma 2.2 and the fact that for any $(n-p-1)$-dimensional face of $B(u \bullet v^{-1})$ we have :

$$deg_\sigma(u \bullet v^{-1}) = deg_\sigma u - deg_\sigma v.$$

■

Now we present the proof of theorem 1. Let $u \in W^{1,p}(M, S^p)$, p=3,7. By theorem 2 there exists a sequence of maps $u_m \in R^{\infty,p}(M, S^p)$ such that $u_m \to u$ in $W^{1,p}(M, S^p)$. By lemma 3.1, there exist a subsequence u_{m_k} of u_m such that

$$E(u_{m_k} \bullet u_{m_{k+1}}^{-1}) \leq \frac{p^{p/2}\alpha_p}{2^{k+1}}.$$

Meanwhile, using theorem 3 and (3.1), we observe that there is an i.m. rectifiable current $\mathbf{L}_k$ such that

$$\begin{cases} \partial \mathbf{L}_k = \mathbf{S}_{u_{m_k} \bullet u_{m_{k+1}}^{-1}} = \mathbf{S}_{u_{m_k}} - \mathbf{S}_{u_{m_{k+1}}} \\ \\ \mathbf{M}(\mathbf{L}_k) \leq \dfrac{1}{2^k} \end{cases}$$

Choose a finite mass i.m. rectifiable current $\mathbf{L}_0$ such that $\partial\mathbf{L}_0 = \mathbf{S}_{u_{m_1}}$ and put

$$\mathbf{L} := \mathbf{L}_0 - \sum_{i=1}^{+\infty} \mathbf{L}_i.$$

So $\mathbf{M}(\mathbf{L}) < +\infty$ and $\mathbf{L}$ is also an i.m. rectifiable current. Observe that if

$$\mathbf{I}_k := \mathbf{L}_0 - \sum_{i=1}^{k} \mathbf{L}_i,$$

then

$$\partial\mathbf{I}_k = \mathbf{S}_{u_{m_{k+1}}}.$$

Meanwhile $\mathbf{M}(\mathbf{I}_k - \mathbf{L}) \to 0$. This, using lemma 2.1, yields

$$\partial\mathbf{L} = \mathbf{S}_u.$$

(So far we have proved that $\mathbf{S}_u$ is the boundary of some i.m. rectifiable current in M). Moreover,

$$m_i(\mathbf{S}_{u_{m_{k+1}}} - \mathbf{S}_u) \leq \mathbf{M}(\mathbf{I}_k - \mathbf{L}) \to 0 \quad \text{as} \quad k \to +\infty.$$

Consequently, for any convergent sequence $u_m \in R^{\infty,p}(M,S^p)$,

$$m_i(\mathbf{S}_{u_m} - \mathbf{S}_u) \to 0 \tag{3.2}$$

As a result, for any $u \in W^{1,p}(M,S^p)$, $m_i(\mathbf{S}_u) \leq CE(u)$ for $C > 0$ independent of u. Meanwhile, by the strong density of $R^{\infty,p}(M,S^p)$ in $W^{1,p}(M,S^p)$ and lemma 2.1, lemma 3.2 is true for maps in $W^{1,p}(M,S^p)$ too. Using the same method and the proved facts about $\mathbf{S}_u$, we can prove (3.2) for any convergent sequence $u_m \in W^{1,p}(M,S^p)$, i.e.

$$m_i(\mathbf{S}_{u_m} - \mathbf{S}_u) \to 0 \quad \text{if} \quad u_m \to u \quad \text{in} \quad W^{1,p}(M,S^p).$$

■

Theorem 1 *bis* is proved following the same ideas.

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Chapter IV

Topological singularities and connections

Weak density of smooth maps into manifolds for the Dirichlet energy

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We consider the problem of topological singularities for Sobolev maps into closed manifolds. Using this approach, we prove that smooth maps are weakly sequentially dense in the Sobolev space $W^{1,2}(\mathbf{B}^n, N)$ for any closed manifold N whose second homotopy group is of finite type.

1 Introduction

1.1 Aspects of the problem

Questions regarding the density of smooth maps in a given function space between manifolds arrised in calculus of variations. It is becoming a field on its own with widely open problems.

The most studied function spaces are the Sobolev spaces $W^{1,p}(M,N)$ of maps from a compact n-dimensional manifold M into a closed riemannien manifold isometrically embedded in some $\mathbb{R}^N$:

$$W^{1,p}(M,N) := \left\{u \in W^{1,p}(M,\mathbb{R}^N)\,; \quad u(x) \in N \text{ a.e. } x \in M\right\}.$$

In [36], [7], and [2], respectively R.Shoen, K.Uhlenbeck, X.Zheng and F.Bethuel shed light on the approximability or non approximability by smooth maps of maps in $W^{1,p}(\mathbf{B}^n,N)$, where $\mathbf{B}^n$ is the n-dimensional unit disk. They showed that the lack of approximability is due to the existence of "topological singular set" for u which is characterized by local realizations by u of non-zero elements of $\pi_{[p]}(N)$ around points in $\mathbf{B}^n$, where $[p]$ is the integer part of p. (The notion of topological singular set is still vague and remains to be precisely defined). In particular they proved that if $\pi_{[p]}(N) = 0$ then any map in $W^{1,p}(\mathbf{B}^n,N)$ can be approximated by smooth maps for the strong topology.

In the case $\pi_{[p]}(N) \neq 0$, the best one can do is to approximate the maps in $W^{1,p}(\mathbf{B}^n,N)$ by maps which are smooth away from a finite union $\Sigma = \bigcup_{i=1}^r \Sigma_i$ of smooth $(n-p-1)$ dimensional submanifolds of $\mathbf{B}^n$. This set of maps is called $R^{\infty,p}(\mathbf{B}^n,N)$. A map $v \in R^{\infty,p}(\mathbf{B}^n,N)$ realizes elements σ_x of $\pi_{[p]}(N)$ on the $[p]$-spheres centered at any point $x \in \Sigma(v)$ and contained in the normal $[p]+1$ plane to $T_x\Sigma(v)$. If for some $x \in \Sigma(v)$, σ_x is non trivial, then v can not be approximated by smooth maps in the strong topology (See [2]). Furthermore one can assign to v a $\pi_{[p]}(N)$-chain which is carried by $\Sigma(v)$ with "multiplicity" σ_x at each point x of $\Sigma(v)$. This $\pi_{[p]}(N)$-chain can be called the topological singular set $\mathbf{S}_v$ of v in $R^{\infty,p}(\mathbf{B}^n,N)$. One of the major questions would be to understand the behavior of $\mathbf{S}_{v_m}$ for a sequence of maps $v_m \in R^{\infty,p}(\mathbf{B}^n,N)$ converging to any $u \in W^{1,p}(\mathbf{B}^n,N)$ and eventually to prove a "flat-norm" convergence of $\mathbf{S}_{v_m}$ to a unique flat $\pi_{[p]}(N)$-chain $\mathbf{S}_u$ we could call the topological singular set of u.

In this paper, we prove the convergence of the $\pi_{[p]}(N)$-chains $\mathbf{S}_{v_m}$ for any convergent sequence of maps in $W^{1,p}(\mathbf{B}^n,N)$ when $[p] = n-1$ if N is $([p]-1)$-connected, i.e.

$$\pi_1(N) = \cdots = \pi_{[p]-1}(N) = 0$$

or when $[p] = 1$ if $\pi_1(N)$ is abelian. The problem is still open for almost every other value for $[p]$. In fact, if we set for $\mathbf{S}$, any integral flat chain in $\mathbf{B}^n$ of dimension k,

$$m_i(\mathbf{S}) := \inf\left\{\mathbf{M}(\mathbf{T})\,;\, T \in \mathcal{R}_{k+1}(\mathbf{B}^n),\ \partial\mathbf{T} = \mathbf{S}_u\right\},$$

the minimal mass of i.m. rectifiable currents taking $\mathbf{S}$ as the boundary, the question would be to determine whether $m_i(\mathbf{S}_{u_m} - \mathbf{S}_{u_k}) \to 0$ if u_m converges strongly to u in $W^{1,p}(\mathbf{B}^n,S^p)$. The answer is yes for $p = 1$ or $n-1$, (See [5] and [16], vol II, section 5.4.2), while we do not know whether this is the case even for the maps in $H^1(\mathbf{B}^4,S^2)$.

Meanwhile, the above program should not work in the described picture for any p and N (See [24]). But one can ask also a weaker question : Does the flat norm of $\mathbf{S}_{v_n}$ remain bounded as $v_m \to u$? This is another problem we address in this paper about the uniform boundedness of the mass $\mathbf{M}(\mathbf{T}_m)$ of a minimal connection $\mathbf{T}_m$ ($\partial \mathbf{T}_m = \mathbf{S}_m$) as $v_m \to u$. In this paper we will restrict to the cases where $p \in \mathbb{N}$, $\pi_p(N) \neq 0$, $\pi_q(N) = 0$ for $1 < q < p$ and when $p = 1$ we assume that $\pi_1(N)$ is abelian.

Related to this question is the problem of the weak density of smooth maps in $W^{1,p}(\mathbf{B}^n, N)$. Although the density of smooth maps for the weak topology can be easily handled from the one for the strong topology (See [2] : Smooth maps are dense for the weak topology if and only if $p \in \mathbb{N}$), the question of the density of smooth maps in $W^{1,p}(\mathbf{B}^n, N)$ for the <u>sequentially</u> weak topology , where $p \in \mathbb{N}$, is more involved : For $p \in \mathbb{N}$, $\pi_p(N) \neq 0$, does there exist for any $u \in W^{1,p}(\mathbf{B}^n, N)$ a sequence $u_m \in C^\infty(\mathbf{B}^n, N)$ such that $u_m \rightharpoonup u$ in $W^{1,p}$? The case $N = S^2$, $p = 2$ was treated by F.Bethuel, H.Brezis, J.M.Coron and E.Lieb in [10], and [3]. F.Bethuel mentioned that the answer is yes for $N = S^p$, $p \geq 2$ in [2]. In [19], P.Hajlasz has proved that the answer is yes when N is $(p-1)$-connected. No counter example to the above stated question is known.

As we will explain below the control of the mass of the minimal chain connecting $\mathbf{S}_{v_m}$ for $v_m \in R^{\infty,p}(\mathbf{B}^n, N)$ converging strongly to u permits to give a positive answer to the sequentially weak density of smooth maps. This appraoch is different from the one used by P.Hajlasz and can be used for proving his theorem and some other partial results regarding the weak sequential denstiy of maps in $W^{1,p}(\mathbf{B}^n, N)$.

Remark 1.1 *We do not have always the equi-boundedness of the mass of minimal connections for $\mathbf{S}_{v_m}$ when $v_m \to u$ in $W^{1,p}(\mathbf{B}^n, N)$: For instance, there exist $v_m \in R^{3,\infty}(\mathbb{B}^4, S^2)$ such that*

$$\inf \{\mathbf{M}(\mathbf{T}_m) \,;\ \mathbf{T}_m \textit{ is a } \mathbb{Z} - \textit{chain such that } \partial \mathbf{T}_m = \mathbf{S}_{v_m}\} \longrightarrow +\infty$$

as $v_m \to u$ in $W^{1,3}(\mathbf{B}^4, S^2)$ (See [24]). However it is not excluded that the smooth maps be sequentially weakly dense in $W^{1,3}(\mathbf{B}^4, S^2)$.

Recent developments by F.Hang and F.H.Lin in [20] showed that one should be careful while considering a generic smooth compact manifold M as the domain. Specially there are cases when the condition "$\mathbf{S}_u = 0$" is not sufficient to guarantee the strong approximability of u by the smooth maps in $W^{1,p}(M, N)$, even when $N = S^p$. This happens because the condition $\mathbf{S}_u = 0$ is a local one and can not "detect" probable "global" topological obstructions in a topologically non-trivial domain.

1.2 Main results

Our first main result is the following :

Theorem 1 *Let $\mathbf{B}^n$ be the unit disk in $\mathbb{R}^n$. Assume that $[p] = 1$ and $\pi_1(N)$ is abelian or $[p] = n - 1$ and N is a closed $([p] - 1)$-connected riemannien manifold of dimension equal* *or greater than $[p]$. Then $\mathbf{S}_u$, the topological singular set of any $u \in W^{1,p}(\mathbf{B}^n, N)$, is well defined as a flat $\pi_{[p]}(N)$-chain and the flat norm of $\mathbf{S}_{u_m} - \mathbf{S}_u$ converges to 0 if $u_m \to u$ in $W^{1,p}(\mathbf{B}^n, N)$. Moreover u is a strong limit of smooth maps in $W^{1,p}(\mathbf{B}^n, N)$ if and only if $\mathbf{S}_u = 0$.*

∎

Remark 1.2 *The approach used in ([16], vol II, section 5.4.2) for defining a topological singularity for Sobolev maps considers only the real homological singularities. This is not adapted when the homotopy type singularities are not seen by the real homology, as in the case $W^{1,1}(\mathbf{B}^n, \mathbb{RP}^2)$ discussed below.*

Remark 1.3 *We can extend these results to $[p] = 3$ or 7. This will be treated in a forthcoming paper.*

We may also ask the same questions about the spaces of maps with fixed boundary value : For $\varphi \in C^\infty(\partial\mathbf{B}^n, N)$, admitting a smooth extension $\phi : \mathbf{B}^n \to N$, we define

$$C^\infty_\varphi(\mathbf{B}^n, N) := \{u \in C^\infty(\mathbf{B}^n, N)\,; \quad u = \varphi \text{ on } \partial\mathbf{B}^n\}$$

and

$$W^{1,p}_\varphi(\mathbf{B}^n, N) := \left\{u \in W^{1,p}(\mathbf{B}^n, N)\,; \quad u = \varphi \text{ a.e. on } \partial\mathbf{B}^n\right\}.$$

Theorem 1 bis *Let $\mathbf{B}^n$ be the unit disk in $\mathbb{R}^n$. Assume that $[p] = 1$ and $\pi_1(N)$ is abelian or $[p] = n - 1$ and N is a closed $([p] - 1)$-connected riemannien manifold of dimension equal or greater than $[p]$. We assume also that φ is smoothly extendable into $\mathbf{B}^n$. Then $u \in W^{1,p}_\varphi(\mathbf{B}^n, N)$ is a strong limit of smooth maps in $C^\infty_\varphi(\mathbf{B}^n, N)$ if and only if $\mathbf{S}_u$, its topological singular $\pi_{[p]}(N)$-chain, is zero.*

∎

In this paper, we give a new proof of this theorem :

Theorem 2 (Hajlasz, [19]) *Let $\mathbf{B}^n$ be the unit disk in $\mathbb{R}^n$. and N be any k-dimensional closed manifold. Assume that for some integer $2 \le p \le k$, N is $(p-1)$-connected, i.e.*

$$\pi_q(N) = 0 \text{ for } q < p.$$

Then for every $u \in W^{1,p}(\mathbf{B}^n, N)$ there is a sequence of maps $u_m \in C^\infty(\mathbf{B}^n, N)$ such that u_m converge weakly to u in $W^{1,p}(\mathbf{B}^n, N)$.

∎

Remark 1.4 *The result can also be deduced from [6] when $\pi_p(N)$ is torsion free, which is not always the case. As an example, the Stiefel manifolds $V_k(\mathbb{R}^n)$, when $n-k$ is odd, are $(n-k-1)$-connected and $\pi_{n-k}(V_k(\mathbb{R}^n)) = \mathbb{Z}_2$ is not torsion free (See [25]).*

Meanwhile, our method has this privilege that it can be used also for the fixed boundary case. This result is not mentionned by P.Hajlasz and can not be deduced directly from his proof.

Theorem 2 bis *Let N be a closed smooth manifold. Assume that for some integer $2 \leq p \leq k$, N is $(p-1)$-connected. Also assume that $\varphi : \partial\mathbf{B}^n \to N$ is a smooth map, smoothly extendable to $\mathbf{B}^n$. Then for every $u \in W^{1,p}_\varphi(\mathbf{B}^n, N)$ there is a sequence of maps $u_m \in C^\infty_\varphi(\mathbf{B}^n, N)$ such that u_m converge weakly to u in $W^{1,p}_\varphi(\mathbf{B}^n, N)$.*

$\blacksquare$

We can extend the above results on the sequentially weak density of smooth maps to the Sobolev spaces $W^{1,p}(M, N)$ when M is a smooth compact manifold of dimension greater than p and N satisfies the above conditions, using the same methods. But the proofs should be modified to surmount the obstacles related to the "global" topological structure of M.

If $p = 1$, we do not prove that smooth maps are sequentially dense in $W^{1,1}(\mathbf{B}^n, N)$. Meanwhile, assuming that $\pi_1(N)$ is abelian, by controling the mass of connections for a convergent sequence in $W^{1,1}(\mathbf{B}^n, N)$, a weaker result is obtained. The non-abelian case is more involved and will be treated in a forthcoming paper.

Definition 1.1 *Let Ω be a domain in $\mathbb{R}^n$ and let u_m be a bounded sequence in $L^1(\Omega)$. u_m is said to converge in the biting sense to $u \in L^1(\Omega)$ if for every $\varepsilon > 0$ there exists a measurable set $E \subset \Omega$ such that $\mu(E) < \varepsilon$ and $u_m \rightharpoonup u$ weakly in $L^1(\Omega \backslash E)$.*

Theorem 3 *Let $\mathbf{B}^n$ be the unit disk in $\mathbb{R}^n$ and N be any closed manifold. Assume that $\pi_1(N)$ is abelian. then for every $u \in W^{1,1}(\mathbf{B}^n, N)$ there is a sequence of maps $u_m \in C^\infty(\mathbf{B}^n, N)$ such that ∇u_m tend to ∇u in the biting sense.*

$\blacksquare$

Theorem 3 bis *Let $\mathbf{B}^n$ be the unit disk in $\mathbb{R}^n$ and N be any k-dimensional closed manifold. Assume that $\varphi \in C^\infty(\partial\mathbf{B}^n, N)$ is smoothly extendable to $\mathbf{B}^n$. If $\pi_1(N)$ is abelian, for every $u \in W^{1,1}_\varphi(\mathbf{B}^n, N)$ there is a sequence of maps $u_m \in C^\infty_\varphi(\mathbf{B}^n, N)$ such that ∇u_m tend to ∇u in the biting sense.*

$\blacksquare$

Further observations showed that we can solve the problem for any closed manifold N when $p = 2$ if $\pi_2(N)$ is of finite type.

Theorem 4 *Let $\mathbf{B}^n$ be the unit disk in $\mathbb{R}^n$ and N be any closed manifold for which $\pi_2(N)$ is finitely generated. Then for every $u \in W^{1,2}(\mathbf{B}^n, N)$, there is a sequence of smooth maps $u_m : \mathbf{B}^n \to N$ converging weakly to u in $W^{1,2}$.*

∎

Theorem 4 bis *Let N be a s above. Assume that $\varphi \in C^\infty(\partial\mathbf{B}^n, N)$ is smoothly extendable to $\mathbf{B}^n$. Then for every $u \in W^{1,2}_\varphi(\mathbf{B}^n, N)$, there is a sequence of smooth maps $u_m \in C^\infty_\varphi(\mathbf{B}^n, N)$ converging weakly to u in $W^{1,2}$.*

∎

For some technical reasons, we will prefer to replace in the proofs the domain $\mathbf{B}^n$ by the n-dimensional cube $\mathcal{C}^n$. Naturally this does not affect the results as these two domains are diffeomorph to each other.

2 Preliminaries

2.1 Flat chains over a coefficient group

Let G be an abelian group. $|\,.\,| : G \to \mathbb{R}^+$ is called a norm on G if

$$(i)\, \forall g \in G, \quad |-g| = |g|,$$

$$(ii)\, \forall g, h \in G, \quad |g+h| \leq |g| + |h|\,,$$

$$(iii)\, |g| = 0 \text{ if and only if } g = 0.$$

We assume that G is a complete metric space with respect to the metric $d(g,h) := |g-h|$.

Let K be any compact convex subset of $\mathbb{R}^n$. We introduce the spaces of polyhedral k-chains, flat k-chains and finite mass flat k-chains in K, with coefficients in G. The readers can refer to [15] and [37] for more details.

Definition 2.1 *$\mathcal{P}_k(K,G)$ is the space of all G-linear sums of oriented k-dimensional polyhedras in K. For $P = \sum_{i=1}^m g_i[[\sigma_i]] \in \mathcal{P}_k(K,G)$, where $g_i \in G$ and σ_i, $i = 1, \ldots, m$, are non-overlapping k-dimensional polyhedras, we define the mass and the boundary of P respectively to be :*

$$\mathbf{M}(P) := \sum_{i=1}^m |g_i|\, vol(\sigma_i),$$

$$\partial P := \sum_{i=1}^m g_i\, \partial[[\sigma_i]] \in \mathcal{P}_{k-1}(K,G).$$

Definition 2.2 *Let $P \in \mathcal{P}_k$ be a polyhedral G-chain. The flat norm of P is :*

$$\mathcal{F}(P) := \inf\left\{\mathbf{M}(P - \partial B) + \mathbf{M}(B); \quad B \in \mathcal{P}_{k+1}\right\}.$$

Definition 2.3 *The space of flat k-chains, $\mathcal{F}_k(K,G)$, is the $\mathcal{F}$-completion of $\mathcal{P}_k(K,G)$. For $A \in \mathcal{F}_k(K,G)$, we define the mass of A to be :*

$$\mathbf{M}(A) := \inf\left\{\liminf_{n\to\infty} \mathbf{M}(P_n)\,; \quad P_n \xrightarrow{\mathcal{F}} A,\ P_n \in \mathcal{P}_k(K,G)\right\}.$$

$\mathcal{M}_k(K,G)$ is the set of flat k-chains in $\mathcal{F}_k(K,G)$ with finite mass and is a complete metric space with respect to the flat norm. Finally, for Ω being any open set in $\mathbb{R}^n$, we define $\mathcal{F}_k(\Omega,G)$ to be the union of all the $\mathcal{F}_k(K,G)$ among convex compact sets $K \subset \Omega$.

We recall some usefull results :

Lemma 2.1 *The boundary map $\partial : \mathcal{P}_k \to \mathcal{P}_{k-1}$ is continuous with respect to the $\mathcal{F}$-norm and so it can be extended to a unique $\mathcal{F}$-continuous map $\partial : \mathcal{F}_k \to \mathcal{F}_{k-1}$.*

■

Lemma 2.2 *Any homomorphism $\chi : G \to H$ between groups, which is continuous with respect to their norms, induces a $\mathcal{F}$-continuous group homomorphism*

$$\chi_* : \mathcal{F}_k(K,G) \to \mathcal{F}_k(K,H).$$

Moreover, χ_* commutes with ∂, i.e. :

$$\chi_*(\partial A) = \partial\chi_*(A), \quad \forall A \in \mathcal{F}_k(K,G) \tag{2.1}$$

and

$$\mathbf{M}(\chi_*(A)) \leq C\mathbf{M}(A), \quad \forall A \in \mathcal{M}_k(K,G)$$

if $|\chi(g)| \leq C|g|$ for all $g \in G$.

■

2.2 The subspaces $\mathcal{R}^{\infty,p}(\mathcal{C}^n, N)$ and $R^{\infty,p}(\mathcal{C}^n, N)$

Definition 2.4 *Let $\mathcal{C}^n := [-\frac{1}{2}, \frac{1}{2}]^n$ be the unit cube in $\mathbb{R}^n$. $u \in W^{1,p}(\mathcal{C}^n, N)$ is in $\mathcal{R}^{\infty,p}(\mathcal{C}^n, N)$ if u is smooth except on $\Sigma(u) = \sum_{i=1}^r \Sigma_i$, where for $i = 1, \ldots, r$, Σ_i is a subset of a linear subspace of $\mathbb{R}^n$ of dimension $n-p-1$ and $\partial\Sigma_i$ is a subset of a inear subspace of dimension $n-p-2$.*

Theorem (Bethuel, [2]) *$\mathcal{R}^{\infty,p}(\mathcal{C}^n, N)$ (respectively $\mathcal{R}^{\infty,p}_\varphi(\mathcal{C}^n, N)$) is dense in $W^{1,p}(\mathcal{C}^n, N)$ (respectively $W^{1,p}_\varphi(\mathcal{C}^n, N)$) for the strong topology.* ■

Let $u \in \mathcal{R}^{\infty,p}(\mathcal{C}^n, N)$. There is some compact subset of $\mathcal{C}^n$, $B = \bigcup_{i=1}^{\mu} \sigma_i$, where the σ_i, $i = 1, \ldots, \mu$ are non-overlapping $(n-p-1)$-dimensional polyhedras, such that $\Sigma(u) \subset B$ and that every $n-p-2$ dimensional face of B belongs to at least two σ_i. Moreover we can assume that any two different faces of B intersect only on their boundaries. Let

$$\|x\| := \max_{i=1,\ldots,n} |x_i| \quad \text{for}\, x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

and for $\delta > 0$ put

$$V^\delta := \{y \in \mathcal{C}^n;\ \|y - B\| \leq \delta\}$$

where

$$\|y - B\| := \inf\{\|y - x\|;\ x \in B\}.$$

Also for $\delta > 0$ and some orthonormal base $\{e_1^i, \ldots, e_{p+1}^i\}$ orthogonal to σ_i, set

$$\sigma_i^\delta := \left\{ x + \sum_{j=1}^{p+1} t_j e_j^i\,;\ x \in \sigma_i\,,\ \max_{j=1,\ldots,p+1} |t_j| \leq \delta \right\}$$

and define $\pi_i : \sigma_i^\delta \to \sigma_i$ to be the smooth projection

$$\pi_i \left(x + \sum_{j=1}^{p+1} t_j e_j^i \right) := x.$$

For δ_0 small enough, we consider a lipschitz projection $\pi : V^{\delta_0} \to B$ with the following properties :

(i) $V^\delta = \bigcup_{i=1}^{\mu} V_i^\delta$, where the $V_i^\delta := \pi^{-1}(\sigma_i) \cap V^\delta$ are non-overlapping n-polyhedras in $\mathbb{R}^n$ which intersect only on lower dimensional faces.

(ii) There are lipschitz diffeomorphisms

$$f_i : V_i^{\delta_0} \to \sigma_i^{\delta_0},$$

such that

$$\begin{cases} f_i(V_i^\delta) = \sigma_i^\delta & \forall \delta < \delta_0 \\ \pi|_{V_i^\delta} = \pi_i \circ f_i|_{V_i^\delta} & \\ f_i([x, \pi(x)]) = [f_i(x), \pi(x)] & \forall x \in V_i^\delta \end{cases}$$

where by $[p, q]$ we mean the segment joining the two points in $\mathcal{C}^n$.

Definition 2.5 *For $y \in V^\delta \backslash B$, let $h_\delta(y)$ be the unique point on ∂V^δ which is on the ray from $\pi(y)$ to y. Then naturally $\pi(h_\delta(y)) = \pi(y)$ and h_δ is locally lipschitz on $V^\delta \backslash B$. We set*

$$u_\delta(y) := \begin{cases} u(h_\delta(y)) & \text{if } y \in V^\delta \\ \\ u(y) & \text{otherwise} \end{cases} \tag{2.2}$$

Definition 2.6 *We set*

$$R^{\infty,p}(\mathcal{C}^n, N) := \{u_\delta\,;\, u \in \mathcal{R}^{\infty,p}(\mathcal{C}^n, N)\}$$

and we say u is radial if $u \in R^{\infty,p}(\mathcal{C}^n, N)$.

By computing the integral of u_δ on V_i^δ by the mean of f_i as new coordinates we observe that for $\delta_1 > 0$ sufficiently small, there is some constant K, depending only on B, for which :

$$\begin{cases} \displaystyle\int_{\partial V^\delta} |\nabla u|^p \leq \frac{K}{\delta_1} \int_{V^{\delta_1}} |\nabla u|^p \\ \\ \displaystyle\int_{V^\delta} |\nabla u_\delta|^p \leq \delta K \int_{\partial V^\delta} |\nabla u|^p \end{cases} \tag{2.3}$$

for $\delta \in I_0$, a positive measure subset of $[0, \delta_1]$.

Remark 2.1 *As a result, $R^{\infty,p}(\mathcal{C}^n, N)$ is also dense in $W^{1,p}(\mathcal{C}^n, N)$ for the strong topology.*

We recall that there are canonical isomorphismps between $\pi_p(N, x)$ and $\pi_p(N, y)$ for $x, y \in N$ if and only if $\pi_1(N)$ is abelian for $p = 1$ and $\pi_1(N) = 0$ for $p > 1$. We assume that these conditions are satisfied so that we can talk about the homotopy classes of maps from S^p into N as elements of $\pi_p(N)$.

Definition 2.7 *Let $u \in \mathcal{R}^{\infty,p}(\mathcal{C}^n, N)$ and $\Sigma(u) \subset B = \bigcup_{i=1}^{\mu} \sigma_i$ be its singular set. Assume that each σ_i is oriented by a smooth $(n-p-1)$-vectorfield $\vec{\sigma}_i$. For $a \in \sigma_i$, let N_a be the $(p+1)$-dimensional plane orthogonal to σ at a. Consider the $(p+1)$-disk $M_{a,\delta} = B_\delta(a) \cap N_a$ oriented by the $(p+1)$-vector $\vec{M}_a$ such that $\vec{\sigma}_i(a) \wedge \vec{M}_a = \xi_{\mathbb{R}^n}$. u is continuous on the p-dimensional oriented sphere $\Sigma_{a,\delta} = \partial M_{a,\delta}$. The homotopic singularity of u at σ_i is*

$$[u, \sigma_i] := [u|_{\Sigma_{a,\delta}}]_{\pi_p(N)}\,, \tag{2.4}$$

i.e. the homotopy class of $u|_{\Sigma_{a,\delta}}$ in $\pi_p(N)$, which is independent of the choices of a and δ.

Definition 2.8 *We define the topological singularity of $u \in \mathcal{R}^{\infty,p}(\mathcal{C}^n, N)$ to be the $\pi_p(N)$-polyhedral chain*

$$\mathbf{S}_u := \sum_{i=1}^{\mu} [u, \sigma_i]\, [[\sigma_i]] \in \mathcal{P}_{n-p-1}(\mathcal{C}^n, \pi_p(N)),$$

where $\Sigma(u) \subset B = \bigcup_{i=1}^{\mu} \sigma_i$ is its singular set.

Remark 2.2 *u suffices to be continuous on $\mathcal{C}^n \backslash B$ for $\mathbf{S}_u$ to be well defined.*

2.3 A useful lemma

Let $\mathbf{B}^l$ be the unit disk in $\mathbb{R}^l$. We denote

$$U^l := \left\{ (x,y) \in \mathbf{B}^l \times \mathbf{B}^l \,;\, x \neq y \right\}$$

and

$$U^l_\delta := \left\{ (x,y) \in U^l \,;\, y \notin B(x,\delta) \right\}.$$

Definition 2.9 *For $(x,y) \in U^l$, we define $p(x,y)$ to be the unique point on $\partial\mathbf{B}^l$ which is on the ray from x to y.*

Clearly p is well-defined and smooth on U^l. As U^l_δ is compact, we have for some constant $C(l,\delta) > 0$:

$$\sup_{(x,y)\in U^l_\delta} |\nabla p(x,y)| \leq C(l,\delta) < +\infty.$$

We have

Lemma 2.3 *Let $1 \leq p < l$ be an integer. Then*

$$\int_{B(0,1-\delta)} |\nabla_y \, p(x,y)|^p \; dx \leq C(l,p,\delta) \tag{2.5}$$

when $C(l,p,\delta)$ depends on l, p and δ and not on y. ■

Proof : Let $x \in B(0,1-\delta)$. We distinguish two cases :

(i) $y \notin B(x,\delta)$. Then $(x,y) \in U^l_\delta$ and we get

$$|\nabla_y p(x,y)| \leq C(l,\delta) \leq \frac{2C(l,\delta)}{|y-x|} \,. \tag{2.6}$$

(ii) Otherwise $y \in B(x,\delta) \subset \mathbf{B}^l$. Then

$$p(x,y) = p\left(\frac{y-x}{|y-x|}\delta + x, x\right)$$

and so

$$|\nabla_y p(x,y)| \leq \left|\nabla_y p\left(\frac{y-x}{|y-x|}\delta + x, x\right)\right| \frac{\delta l}{|y-x|} \leq \frac{\delta l C(l,\delta)}{|y-x|} \,, \tag{2.7}$$

as

$$\left(\frac{y-x}{|y-x|}\delta + x, x\right) \in U^l_\delta.$$

Using the inequalities (2.6) and (2.7), the lemma is proved. ■

3 An example : $W^{1,1}_{\varphi}(\mathcal{C}^2, \mathbb{RP}^2)$

3.1 Notations

Let $f : S^2 \to \mathbb{R}^6$ be the map :

$$f(x,y,z) := (\frac{\sqrt{2}}{2}x^2\,,\frac{\sqrt{2}}{2}y^2\,,\frac{\sqrt{2}}{2}z^2\,,xy\,,yz\,,zx). \tag{3.1}$$

f induces an embedding of the 2-dimensional Real Projective Space, $\mathbb{RP}^2$, into $\mathbb{R}^6$. A property of this embedding is that the minimum length of the cycle homotopic to the non-zero element of $\pi_1(\mathbb{RP}^2) \simeq \mathbb{Z}_2$ is π, independent of the choice of the base point. We define a norm on the 2-group $\pi_1(\mathbb{RP}^2)$:

$$|a| := 1 \quad \text{if} \quad a \neq 0, \quad := 0 \quad \text{otherwise}. \tag{3.2}$$

Also we define the map $g : \mathbf{B}^2 \to \mathbb{RP}^2$ as follows :

$$g(x_1,x_2) := f\left(x_1, x_2, \sqrt{1-(x_1^2+x_2^2)}\right). \tag{3.3}$$

Now let $w_0 = f(1,0,0) \in \mathbb{RP}^2$ and put

$$\mathcal{G} = f\left(\left\{(x,y,z) \in S^2\,;\, z=0\right\}\right).$$

$\mathcal{G}$ is a length minimizing generator of $\pi_1(\mathbb{RP}^2)$ passing through w_0. For $w \in \mathbb{RP}^2 \backslash \mathcal{G}$ we define the projection

$$p_w : \mathbb{RP}^2 \backslash \{w\} \to \mathcal{G}$$

as follows :

$$p_w(w') := g((p(g^{-1}(w), g^{-1}(w'))) \quad \forall w' \in \mathbb{RP}^2 \backslash \{w\} \tag{3.4}$$

where p is the map given in definition 2.9. Observe that p_w is well defined for $w' \in \mathcal{G}$ as in this case we would have $p_w(w') = w'$ independent of the choice of $g^{-1}(w')$. Let us fix $\varepsilon > 0$ such that

$$Vol(\mathbb{RP}^2 \backslash \mathcal{G}_\varepsilon) > 2\pi\,,$$

where

$$\mathcal{G}_\varepsilon := \left\{y \in \mathbb{RP}^2\,;\, d(y,\mathcal{G}) < \varepsilon\right\}$$

is the ε-neighbourhood of $\mathcal{G}$ in $\mathbb{RP}^2$.

Lemma 3.1 *Let $\mathcal{G}$ and p_w be as above. Then :*

(i) $p_w : \mathbb{RP}^2 \backslash \{w\} \to \mathcal{G}$ is well defined and smooth.

(ii) For any cycle $\mathcal{G}' \subset \mathbb{RP}^2 \backslash \{w\}$ we have :

$$[\mathcal{G}']_{\pi_1(\mathbb{RP}^2)} = \chi([p_w(\mathcal{G}')]_{\pi_1(\mathcal{G})}) \tag{3.5}$$

where $\chi : \pi_1(\mathcal{G}) \simeq \mathbb{Z} \to \pi_1(\mathbb{RP}^2) \simeq \mathbb{Z}_2$ *is an onto homomorphism.*

(iii) For any $w' \in \mathbb{RP}^2$ *we have :*

$$\int_{\mathbb{RP}^2 \setminus \mathcal{G}_\varepsilon} |\nabla p_w(w')|\, dw = C_0 = C_0(\varepsilon) < +\infty. \tag{3.6}$$

■

Proof : We observe that g^{-1} is well defined and smooth on $\mathbb{RP}^2 \setminus \mathcal{G}$, while in a neighbourhood of $\mathcal{G}$, p_w is a projection along smooth curves orthogonal to $\mathcal{G}$. This proves the first part of the lemma. Now observe that the injection map $i : \mathcal{G} \to \mathbb{RP}^2$ induces a homomorphism

$$\chi : \pi_1(\mathcal{G}) \to \pi_1(\mathbb{RP}^2)$$

which is onto as $[\mathcal{G}]$ is the generator of $\pi_1(\mathbb{RP}^2)$. So, since p_w is smooth on $\mathbb{RP}^2 \setminus \{w\}$, we get

$$[\mathcal{G}']_{\pi_1(\mathbb{RP}^2)} = [p_w(\mathcal{G}')]_{\pi_1(\mathbb{RP}^2)} = \chi([p_w(\mathcal{G}')]_{\pi_1(\mathcal{G})})$$

which proves (3.5).

Now let

$$N_\varepsilon := \mathbb{RP}^2 \setminus \mathcal{G}_\varepsilon$$

and observe that for $g : \mathbf{B}^2 \to \mathbb{RP}^2$ as in (3.3) :

(i) $N_{\varepsilon/2} = g(B(0, 1-\delta))$, for some $0 < \delta < 1$,

(ii) $g|_{B(0,1-\delta)}$ is an embedding.

We prove (3.6) : Let $w \in N_\varepsilon \subset N_{\varepsilon/2}$. If $w' \notin \mathcal{G}_{\varepsilon/2}$ then since g^{-1} is smooth on $N_{\varepsilon/2}$, using (2.5) and (3.4), we get for some $C_0(\delta) > 0$:

$$\int_{N_\varepsilon} |\nabla p_w(w')|\, dw \le \int_{N_{\varepsilon/2}} |\nabla p_w(w')|\, dw \le C_0(\delta).$$

If not, the map $\tilde{p} : N_\varepsilon \times \overline{\mathcal{G}}_{\varepsilon/2} \to \mathcal{G}$:

$$\tilde{p}(w, w') := p_w(w')$$

is smooth on its compact domain because $N_\varepsilon \cap \overline{\mathcal{G}}_{\varepsilon/2} = \emptyset$. So there exists $K > 0$, independent of w, w' for which

$$|\nabla p_w(w')| \le K$$

if $w' \in \mathcal{G}_{\varepsilon/2}$, $w \in N_\varepsilon$. This completes the proof of (3.6). ■

3.2 Study of $R^{\infty,1}_{\varphi}(\mathcal{C}^2, \mathbb{RP}^2)$

Let $u \in R^{\infty,1}_{\varphi}(\mathcal{C}^2, \mathbb{RP}^2)$. We observe that $\mathbf{S}_u \in \mathcal{P}_0(\mathcal{C}^2, \pi_1(\mathbb{RP}^2))$ is is in fact the sum $\sum_{i=1}^{\mu}[u,p_i]\,[[p_i]]$ where $\{p_1, ..., p_\mu\}$ are the singularities of u and $[u,p_i]$ is the class of $u(\partial B(p_i,\delta))$ in $\pi_1(\mathbb{RP}^2)$ for δ small enough.

Definition 3.1 *$I \in \mathcal{F}_1(\mathcal{C}^2, \pi_1(\mathbb{RP}^2))$ is a connection for u if $\partial I = \mathbf{S}_u$.*

Proposition 3.1 *For $u \in R^{\infty,1}_{\varphi}(\mathcal{C}^2, \mathbb{RP}^2)$, there exists $I \in \mathcal{P}_1(2, \pi_1(\mathbb{RP}^2))$ such that*

$$\begin{cases} \partial I = \mathbf{S}_u \\ \\ \mathbf{M}(I) \leq C \displaystyle\int |\nabla u| + C \end{cases} \tag{3.7}$$

for some constant $C > 0$ depending only on φ.

■

Remark 3.1 *Any $I \in \mathcal{P}_1(\mathcal{C}^2, \pi_1(\mathbb{RP}^2))$ is a set of non oriented segments while $\mathbf{M}(I)$ is simply the total length of these segments.*

Corollary 3.1 *For any $u \in R^{\infty,1}_{\varphi}(\mathcal{C}^2, \mathbb{RP}^2)$, there exists a connection $I_u \in \mathcal{F}_1(2, \pi_1(\mathbb{RP}^2))$ of minimal mass which satisfies*

$$\mathbf{M}(I_u) \leq C \int |\nabla u| + C.$$

(Use the compactness result of [13], section 4.2.26, p. 432.) ■

Proof of proposition 3.1 : First we assume that $\varphi \equiv w_0$ is constant. Let u be a map in $R^{\infty,1}_{\varphi}(\mathcal{C}^2, \mathbb{RP}^2)$ for which $\mathbf{S}_u = \sum_{i=1}^{\mu}[u,p_i]\,[[p_i]]$. Let A be the set of regular values of u in $\mathbb{RP}^2$. By Sard's theorem, $\mathcal{H}^2(A) = vol(\mathbb{RP}^2) = 4\pi$. We estimate the integral

$$J := \int_{\mathbb{RP}^2 \backslash \mathcal{G}_\varepsilon} \int_{\mathcal{C}^2} |\nabla(p_w \circ u)(x)|\; dx\, dw. \tag{3.8}$$

We have by (3.6) :

$$J \leq \int_{\mathcal{C}^2} \int_{\mathbb{RP}^2 \backslash \mathcal{G}_\varepsilon} |\nabla p_w(u(x))|\, |\nabla u(x)|\, dw\, dx \leq C_0 \int_{\mathcal{C}^2} |\nabla u|.$$

As a result, considering (3.8), there exists some positive measure set $W \subset \mathbb{RP}^2 \backslash \mathcal{G}_\varepsilon$ such that :

$$\int_{\mathcal{C}^2} |\nabla(p_w \circ u)| \leq \frac{C_0}{2\pi} \int_{\mathcal{C}^2} |\nabla u| \quad \text{for all } w \in W. \tag{3.9}$$

Since u is *radial*, for some regular $w \in A \cap W$, $u^{-1}(w)$ is a finite subset of $\mathcal{C}^2$. We have :

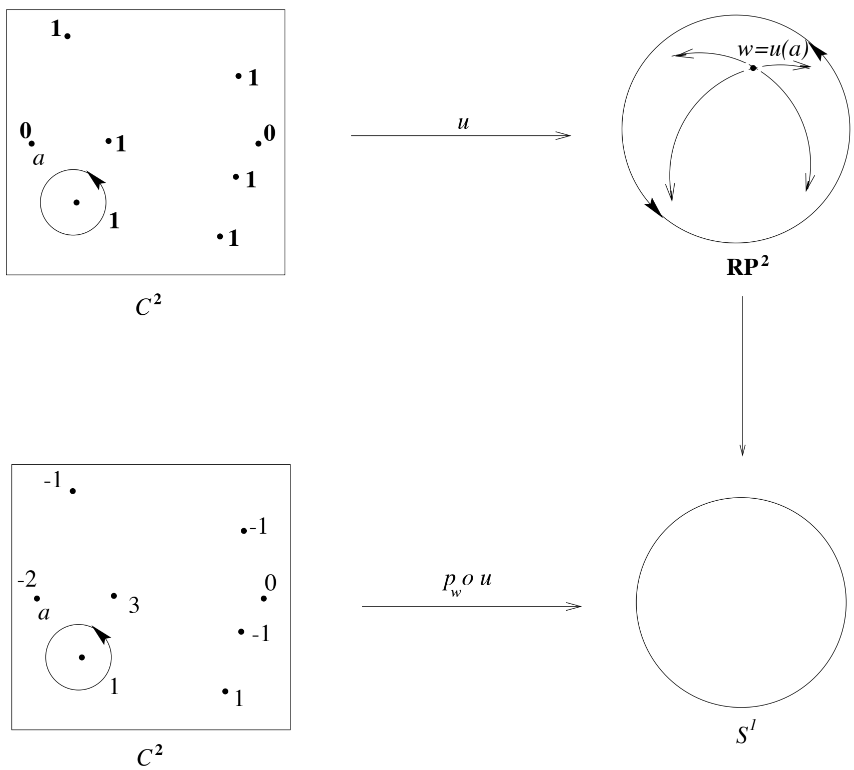


Fig.1
Projection of u into S^1

Lemma 3.2 *There exists $w \in W$ such that the map*

$$\tilde{u} := p_w \circ u : \mathcal{C}^2 \to \mathcal{G}$$

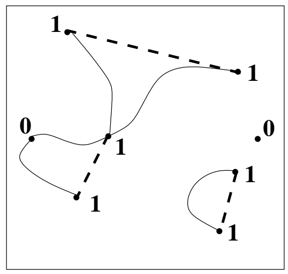
is in $\mathcal{R}^{\infty,1}_{w_0}(\mathcal{C}^2, \mathcal{G})$. Moreover if we consider the additive group $\mathbb{Z}$ with its usual norm, for some $\tilde{I} \in \mathcal{P}_1(\mathcal{C}^2, \mathbb{Z})$, for which $\partial \tilde{I} = \mathbf{S}_{\tilde{u}}$, the following properties hold :

$$\begin{cases} L(\tilde{I}) = \inf \left\{ L(\tilde{I}') \, ; \quad \tilde{I}' \in \mathcal{F}_1(\mathcal{C}^2, \mathbb{Z}) \, , \quad \partial \tilde{I}' = \mathbf{S}_{\tilde{u}} \right\} \\ \\ L(\tilde{I}) \leq \dfrac{1}{\pi} \displaystyle\int_{\mathcal{C}^2} |\nabla \tilde{u}|. \end{cases} \tag{3.10}$$

where $L(\tilde{I})$ is the $\mathbb{Z}$-mass of $\tilde{I}$.

■

Remark 3.2 *Observe that $\pi_1(\mathcal{G}) \simeq \mathbb{Z}$. Moreover $L(\tilde{I})$ is the length of minimal connections connecting the singularities of $\tilde{u}$, introduced in [10].*



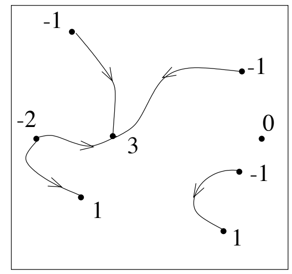


Fig.2
Connections for u and for $p_w \circ u$

For a proof of this lemma, see [11], propositions 1 and 2. Observe that the best constant in inequality (3.10) is achieved by the mean of co-area formula as in [1].

Using lemma 3.2, we finish the proof of the proposition : Consider the homomorphism χ in (3.5). By lemma 2.2, χ induces a group homomorphism

$$\chi_* : \mathcal{P}_k(\mathcal{C}^2, \mathbb{Z}) \to \mathcal{P}_k(\mathcal{C}^2, \pi_1(\mathbb{RP}^2)).$$

We consider $\tilde{I}$ as in lemma 3.2 and we set $I := \chi_*(\tilde{I})$. We deduce that

$$\partial I = \chi_*(\mathbf{S}_{\tilde{u}}). \tag{3.11}$$

Meanwhile, by lemma 3.1, part (ii), we observe that, for all points $p \in \mathcal{C}^2$, there exists δ small enough for which :

$$[u, p] = [u(\partial B_\delta(p))]_{\pi_1(\mathbb{RP}^2)} = \chi([\tilde{u}(\partial B_\delta(p))]_{\pi_1(\mathcal{G})}) = \chi([\tilde{u}, p])$$

and as a result :

$$\mathbf{S}_u = \chi_*(\mathbf{S}_{\tilde{u}}). \tag{3.12}$$

For visualizing this phenomenon see Fig.1 where we compare the singualrities of u and $p_w \circ u$. Comparing this with (3.11) we obtain

$$\partial I = \mathbf{S}_u \,.$$

Observe that $|\chi(z)| \leq |z|$ for all $z \in \mathbb{Z}$, thus we have by lemma 3.2 :

$$\mathbf{M}(I) = \mathbf{M}(\chi_*(\tilde{I})) \leq L(\tilde{I}) \leq \frac{1}{\pi} \int_{\mathcal{C}^2} |\nabla(p_w \circ u)| \,.$$

So using the inequality (3.9), we get

$$\mathbf{M}(I) \leq \frac{C_0}{2\pi^2} \int_{\mathcal{C}^2} |\nabla u|.$$

This completes the proof for constant boundary datas. In Fig.2 we have illustrated two connections for u and one for $p_w \circ u$. We show how the minimal polyhedral connection for u (the thin dashed segments) comes to be lesser in mass from the image of any connection of $p_w \circ u$ under χ_* (the thick curves).

Now consider the case of non-constant φ. We extend u over the cube

$$\widetilde{\mathcal{C}}^2 := \{x \in \mathbb{R}^2 ; \quad \|x\| \leq \frac{1}{2} + \varepsilon\}$$

for some $\varepsilon > 0$ as follows :

$$u(x) := \phi\left(\frac{1/2 + \varepsilon - \|x\|}{\varepsilon}\, x\right) \quad \forall x \in \widetilde{\mathcal{C}}^2 \backslash \mathcal{C}^2 ,$$

while ϕ is the smooth extention of φ onto $\mathcal{C}^2$. Now u is constant on the boundary of $\widetilde{\mathcal{C}}^2$ and we have clearly

$$\int_{\widetilde{\mathcal{C}}^2} |\nabla u| \leq \int_{\mathcal{C}^2} |\nabla u| + C_1$$

where C_1 depends only on φ. Applying the proposition to u on $\widetilde{\mathcal{C}}^2$ as above, we obtain some $I' \in \mathcal{P}_1(\widetilde{\mathcal{C}}^2, \mathbb{Z}_2)$ for which $\partial I' = \mathbf{S}_u$ and $M(I') \leq CE(u) + C$. Now since spt $\mathbf{S}_u$ is a compact set in $\mathcal{C}^2$, we observe that there is an open $U \subset \mathcal{C}^2$ such that spt $\mathbf{S}_u \subset U$ and ∂U is a convex polygone. Let Π denote the lipschitz map which leaves U fixed and radially projects points outside U onto its boundary. This map induces a map

$$\Pi_\# : \mathcal{P}_k(\widetilde{\mathcal{C}}^2, \mathbb{Z}_2) \to \mathcal{P}_k(\mathcal{C}^2, \mathbb{Z}_2)$$

which commutes with the boundary map. Moreover

$$\mathbf{M}(\Pi_\#(I')) \leq \mathrm{lip}\Pi\, \mathbf{M}(I').$$

So as spt $\mathbf{S}_u \subset U$, it is easy to see that $I := \Pi_\#(I')$ satisfies the conditions of proposition 3.1. ■

Now we present another important result concerning the maps in $R^{\infty,1}_\varphi(\mathcal{C}^2, \mathbb{RP}^2)$. The same singularity removing proposition was proved in [3] for $H^1(B^3, S^2)$.

Proposition 3.2 *Let $I \in \mathcal{P}_1(\mathcal{C}^2, \pi_1(\mathbb{RP}^2))$ be a connection for $u \in R^{\infty,1}_\varphi(\mathcal{C}^2, \mathbb{RP}^2)$. Then there are maps $v_m \in C^\infty_\varphi(\mathcal{C}^2, \mathbb{RP}^2)$ such that*

$$\begin{cases} v_m = u \text{ on } \mathcal{C}^2 \backslash K_m , \\ |K_m| \leq \dfrac{1}{m} , \\ \displaystyle\int_{\mathcal{C}^2} |\nabla v_m| \leq \int_{\mathcal{C}^2} |\nabla u| + C\mathbf{M}(I) + \frac{1}{m} \cdot \end{cases} \tag{3.13}$$

for some constant $C > 0$ independent of u.

■

This proposition is a special case of proposition 5.1 which is proved in the next section.

3.3 Topological singularities for maps in $W^{1,1+s}(\mathcal{C}^2, \mathbb{RP}^2)$

We give a proof for theorem 1 for $M = \mathcal{C}^2$, $N = \mathbb{RP}^2$ and $[p] = 1$. Let u be a map in $W^{1,p}(\mathcal{C}^2, \mathbb{RP}^2)$ such that $[p] = 1$. We intend to define $\mathbf{S}_u$, the topological singular chain of u as a flat $\mathbb{Z}_2$-chain. In fact we are to prove that for any sequence of maps $u_m \in R^{\infty,p}(\mathcal{C}^2, \mathbb{RP}^2) \subset R^{\infty,1}(\mathcal{C}^2, \mathbb{RP}^2)$, $\mathbf{S}_{u_m}$ is a convergent sequence in $\mathcal{F}_0(\mathcal{C}^2, \mathbb{Z}_2)$ and that the limit is independent of the choice of the sequence u_m.

Let u_m be such a sequence. Set as in (3.8)

$$J_m := \int_{\mathbb{RP}^2 \setminus \mathcal{G}_\varepsilon} \int_{\mathcal{C}^2} |\nabla(p_w \circ u)(x) - \nabla(p_w \circ u_m)(x)| \, dx \, dw.$$

We are to prove that $J_m \to 0$. First observe that for fixed $x \in \mathcal{C}^2$

$$|\nabla(p_w \circ u)(x) - \nabla(p_w \circ u_m)(x)| \leq C(|\nabla p_w(u(x))| + |\nabla p_w(u_m(x))|) \in L^1(\mathbb{RP}^2 \setminus \mathcal{G}_\varepsilon)$$

(See 3.6). Now, since $\nabla(p_w \circ u_m)$ converge for almost every $w \in \mathbb{RP}^2 \setminus \mathcal{G}_\varepsilon$ to $\nabla(p_w \circ u)$, by Lebesgue dominant convergence we get

$$\int_{\mathbb{RP}^2 \setminus \mathcal{G}_\varepsilon} |\nabla(p_w \circ u)(x) - \nabla(p_w \circ u_m)(x)| \, dw \to 0$$

for almost every $x \in \mathcal{C}^2$. Also we have

$$\int_{\mathbb{RP}^2 \setminus \mathcal{G}_\varepsilon} |\nabla(p_w \circ u)(x) - \nabla(p_w \circ u_m)(x)| \, dw \leq C_0(\varepsilon)(|\nabla u(x)| + |\nabla u_m|) \in L^1(\mathcal{C}^2).$$

Thus, again using the Lebesgue dominant convergence, we obtain that J_m tends to 0 for $m \to +\infty$. As a result, there exists $w \in \mathbb{RP}^2 \setminus \mathcal{G}_\varepsilon$ such that

$$p_w \circ u_m \to p_w \circ u \quad \text{in} \quad W^{1,1}(\mathcal{C}^2, S^1)$$

and that w is a regular value for all u_m, i.e.

$$p_w \circ u_m \in \mathcal{R}^{\infty,1}(\mathcal{C}^2, S^1).$$

Meanwhile, any flat chain with multiplicity in $\mathbb{Z}$ is also a real current, defining a dual functional on the space of compactly supported smooth differential forms. Now if we set $\mathbf{S}_{p_w \circ u}$ to be the real 0-current (distribution) defined as follows :

$$\mathbf{S}_{p_w \circ u}(\alpha) := \frac{1}{2\pi} \int_{\mathcal{C}^2} (p_w \circ u)^*(d\theta) \wedge d\alpha \quad \forall \alpha \in C_c^\infty(\mathcal{C}^2, \mathbb{R}),$$

we get

$$m_r(\mathbf{S}_{p_w \circ u_m} - \mathbf{S}_{p_w \circ u}) \to 0$$

where by $m_r(\mathbf{S})$ we mean the minimal mass of normal currents getting $\mathbf{S}$ as their boundary (See [16], vol II, section 5.4.2, theorem 2). Moreover, for a 0-dimensional integral flat chain $\mathbf{S}$ in $\mathbb{R}^n$ the minimal i.m. rectifiable current taking $\mathbf{S}$ as the boundary is also the minimal real current, i.e. we have

$$m_r(\mathbf{S}) = m_i(\mathbf{S}) := \inf\{\mathbf{M}(\mathbf{T}); \mathbf{T} \in \mathcal{R}_1(\mathbb{R}^n), \quad \partial\mathbf{T} = \mathbf{S}\}$$

(See [14]). As a result, $\mathbf{S}_{p_w \circ u}$ is the boundary of some i.m. rectifiable current ($\mathbf{S}_{p_w \circ u} \in \mathcal{F}_0(\mathcal{C}^2, \mathbb{Z})$) and we get

$$\mathcal{F}(\mathbf{S}_{p_w \circ u_m} - \mathbf{S}_{p_w \circ u}) \leq m_i(\mathbf{S}_{p_w \circ u_m} - \mathbf{S}_{p_w \circ u}) \to 0\,.$$

Using lemma 2.2 and (3.12) we obtain that the flat $\mathbb{Z}_2$-chain

$$\mathbf{S}_u := \chi_*(\mathbf{S}_{p_w \circ u}) = \lim_{m\to\infty} \chi_*(\mathbf{S}_{p_w \circ u_m}) = \lim_{m\to\infty} \mathbf{S}_{u_m}$$

is independent of the choice of w and that $\mathcal{F}(\mathbf{S}_{u_m} - \mathbf{S}_u) \to 0$. Since any two sequences converging to u can be restructured to a single converging sequence, $\mathbf{S}_u$ is independent of the converging sequence u_m too.

Now suppose that $\mathbf{S}_u = 0$. Consequently for any sequence of maps u_m converging to u in $W^{1,p}(\mathcal{C}^2, S^1)$, there is polyhedral $\mathbb{Z}_2$-chains I_m such that

$$\mathbf{M}(I_m) \to 0$$

and that spt $(\partial I_m - \mathbf{S}_{u_m}) \subset \partial\mathcal{C}^2$ (This is what we call a connection when we do not fix a boundary data). Using the same method as for the singularity removing proposition 3.2, we prove the existence of a sequence of smooth maps $v_m : \mathcal{C}^2 \to \mathbb{RP}^2$ which converge to u in $W^{1,1}$ (Here we use the fact that $\mathbf{M}(I_m) \to 0$). Consequently, u is homotopical to constant on any generic 1-skeleton of $\mathcal{C}^2$. Using this and referring to [2], the proof of theorem 1, we can approximate strongly u by smooth maps in $W^{1,p}(\mathcal{C}^2, \mathbb{RP}^2)$. This completes the proof of theorem 1 for this special case. ∎

3.4 Study of sequential weak density in $W^{1,1}_\varphi(\mathcal{C}^2, \mathbb{RP}^2)$

We prove theorem 3 bis for $n = 2$ and $N = \mathbb{RP}^2$: For every $u \in W^{1,1}_\varphi(\mathcal{C}^2, \mathbb{RP}^2)$, there are $u_m \in C^\infty_\varphi(\mathcal{C}^2, \mathbb{RP}^2)$ such that $u_m \to u$ in $L^1(\mathcal{C}^2)$ and ∇u_m converge in the biting sense to ∇u.

Proof : First we approximate u by a sequense $u_k \in R^{\infty,1}_\varphi(\mathcal{C}^2, \mathbb{RP}^2)$ (See remark 2.1). Passing to a subsequence if necessary, we can assume that energies of u_k are bounded by the same constant. So, by proposition 3.1, there are polyhedral connections I_k for u_k such that their masses are equi-bounded. Using proposition 3.2, we construct maps $u_{k,m}$, which converge almost everywhere to u_k and have equi-bounded energies too. As a result, $u_{m,m}$ tend in L^1 to u and their gradients are equi-bounded in L^1 norm. By ([16], Vol I, section 1.2.7), $\nabla u_{m,m}$ converge in L^1 in the biting sense. Furthermore the limit can not be other than ∇u, since $u_{m,m}$ converge strongly to u in L^1. ∎

4 Controling the mass of connections

We assume that $p > 1$ and that N is a $(p-1)$-connected smooth compact manifold of dimension $k \geq p$, i.e.

$$\pi_q(N) = 0 \text{ for } q < p.$$

Using the fact that N is $(p-1)$-connected, we generalize the result of proposition 3.1 to maps in $\mathcal{R}_\varphi^{\infty,p}(\mathcal{C}^n, N)$. This is what we prove in proposition 4.1. As before, the main idea is to conjugate u with a projection of N on the generators of its p-homotpy group.

Consider some triangulation of N and for $1 \leq l \leq k$, let N^l be the l-skeleton of N. So $N = N^k$. Observe that by ([40], theorem (1.6), p. 215), N^p is $(p-1)$-connected and the homomorphisms

$$\chi^{p,l} : \pi_p(N^p) \longrightarrow \pi_p(N^l),$$

induced by the injection maps $i_{p,l} : N^p \to N^l$, are onto. As a result, using ([17], Corollary 3.5, p. 38) , N^p is of the homotopy type of a bouquet of p-spheres and we obtain that $\pi_p(N^p)$ is finitely generated. Let $g_1, \dots, g_\beta$ be its generators. As a result, $\pi_p(N^l)$ is finitely generated too. We choose its generators among $\{\chi^{p,l}(g_1), \dots, \chi^{p,l}(g_\beta)\}$ and we define a norm on $\pi_p(N^l)$, $p \leq l \leq k$, as follows : For $a \in \pi_p(N^l)$, $|a|$ is the smallest length of a product of generators of $\pi_p(N^l)$ representing a. Observe that there is some constant $C > 0$ such that

$$|\chi^{p,l}(g)| \leq C|g|, \quad \forall g \in \pi_p(N^p). \tag{4.1}$$

Since $\pi_1(N) = 0$, $\mathbf{S}_u \in \mathcal{P}_{n-p-1}(\mathcal{C}^n, \pi_p(N))$ is well defined for any $u \in \mathcal{R}_\varphi^{\infty,p}(\mathcal{C}^n, N)$ (See definition 2.8). We proceed as before by generalizing the concept of connections :

Definition 4.1 *We say that* $\mathbf{T} \in \mathcal{F}_{n-p}(\mathcal{C}^n, \pi_p(N))$ *is a connection for* $u \in \mathcal{R}_\varphi^{\infty,p}(\mathcal{C}^n, N)$ *if* $\partial \mathbf{T} = \mathbf{S}_u$.

We write

$$N^l = \bigcup_{i=1}^{s_l} \xi_i^l(\mathbf{B}^l),$$

where

$$\xi_i^l : \mathbf{B}^l \to N_i^l := \xi_i^l(\mathbf{B}^l), \; i = 1, \dots, s_l$$

are diffeomorphisms and each two N_i^l are rather disjoint or intersecting on a lower dimensional face in N^{l-1}.

Now let $w \in N_1^l \times \cdots \times N_{s_l}^l$, $w = (w_1, ..., w_{s_l})$ be such that $w_i \notin N^{l-1}$. Define

$$p_w^l : N^l \backslash \{w_1, \dots, w_{s_l}\} \to N^{l-1}$$

as follows :

$$p_w^l(y) := \begin{cases} \xi_i^l(p((\xi_i^l)^{-1}(w_i), (\xi_i^l)^{-1}(y))) & \text{if } y \in N_i^l \backslash N^{l-1} \\ \\ y & \text{otherwise} \end{cases}$$

where p is the projection defined in definition 2.9.

Lemma 4.1 *Let $p+1 \leq l \leq k$, then*

(i) p_w^l is well defined and locally Liptchitz on $N \backslash \{w_1, \ldots, w_{s_l}\}$.

(ii) For any p-dimensional cycle $\mathcal{G}' \subset N \backslash \{w_1, \ldots, w_{s_l}\}$ we have :

$$[\mathcal{G}']_{\pi_p(N^l)} = \chi^l([p_w^l(\mathcal{G}')]_{\pi_p(N^{l-1})} \tag{4.2}$$

where

$$\chi^l : \pi_p(N^{l-1}) \to \pi_p(N^l)$$

is the homomorphism induced by the injection map $i_l : N^{l-1} \to N^l$.

(iii) For any $w' \in N^l$:

$$\int_{N^l_{1,\varepsilon} \times \cdots \times N^l_{s_l,\varepsilon}} |\nabla p_w(w')|^p \, dw \leq C(p,l,\varepsilon) < +\infty \,, \tag{4.3}$$

where for $1 \leq i \leq s_l$ and $0 < \varepsilon < 1$:

$$N^l_{i,\varepsilon} := \xi_i^l \left(B^l(0, 1-\varepsilon) \right).$$

■

Remark 4.1 *Since N is $(p-1)$-connected, $\pi_p(N) \equiv H_p(N, \mathbb{Z})$ (Hurewicz theorem). So the homotopy class of p-cycles in N is well defined.*

Proof : Using (2.5), the lemma is proved as for lemma 3.1. ■

Now let us estimate the integral

$$J := \int_{N^l_{1,\varepsilon} \times \cdots \times N^l_{s_l,\varepsilon}} \int_{\mathcal{C}^n} |\nabla (p_w \circ u)(x)|^p \, dx dw. \tag{4.4}$$

for $u \in W^{1,p}(\mathcal{C}^n, N^l)$, for $p < l$. By (4.3) we have

$$\begin{aligned} J &\leq \int_{\mathcal{C}^n} \int_{N^l_{1,\varepsilon} \times \cdots \times N^l_{s_l,\varepsilon}} |\nabla p_w(u(x))|^p |\nabla u(x)|^p \, dw dx \\ &\leq C(p,l,\varepsilon) \int_{\mathcal{C}^n} |\nabla u|^p. \end{aligned}$$

As a result, by considering (4.4), there is some positif measure set $W \subset N^l_\varepsilon := N^l_{1,\varepsilon} \times \cdots \times N^l_{s_l,\varepsilon} \subset \mathbb{R}^{ls_l}$ for which :

$$\int_{\mathcal{C}^n} |\nabla (p_w \circ u)|^p \leq \frac{C(p,l,\varepsilon)}{\mathcal{H}^{ls_l}(N^l_\varepsilon)} \int_{\mathcal{C}^n} |\nabla u|^p \quad \forall w \in W. \tag{4.5}$$

Lemma 4.2 *Let $l > p$ and $u^l \in \mathcal{R}^{p,\infty}_{w_0}(\mathcal{C}^n, N^l)$ for some $w_0 \in N^{l-1}$. Then there is a map $u^{l-1} : \mathcal{C}^n \to N^{l-1}$ and $C > 0$, independent of u^l, such that*

(i) $u^{l-1} \in \mathcal{R}^{\infty,p}_{w_0}(\mathcal{C}^n, N^{l-1})$,

(ii) $\displaystyle\int_{\mathcal{C}^n} |\nabla u^{l-1}|^p \leq C \int_{\mathcal{C}^n} |\nabla u^l|^p \,,$

(iii) $\mathbf{S}_{u^l} = \chi^l_*(\mathbf{S}_{u^{l-1}})$
where $\chi^l : \pi_p(N^{l-1}) \to \pi_p(N^l)$ is the homomorphism induced by the injection map $i_l : N^{l-1} \to N^l$.

■

Proof : Let us fix $0 < \varepsilon < 1$ and consider the set $W \subset N^l_\varepsilon$ as in (4.5). Also we fix ε_1, ε_2, $\varepsilon_3 > 0$ and $0 < \delta < \delta_1$ such that

$$\frac{C(l,p,\varepsilon)}{\mathcal{H}^{ls_l}(N^l_\varepsilon)} \left(K^2 \int_{V^{\delta_1}} |\nabla u^l|^p + \delta K \varepsilon_2 + \varepsilon_1 \right) + \varepsilon_3 \leq \int_{\mathcal{C}^n} |\nabla u^l|^p \tag{4.6}$$

where K, δ and δ_1 satisfy (2.3). For almost all $w = (w_1, \ldots, w_{s_l}) \in W$, w_i's are regular values for $u^l|_{\mathcal{C}^n \setminus V^\delta}$ and $u^l|_{\partial V^\delta}$, which are smooth on their domains. Using (2.3) and by the co-area formula we obtain that for almost all $w \in W$, $(u^l)^{-1}(w_i) \cap (\mathcal{C}^n \setminus V^\delta)$ is a finite mass smooth submanifold of $\mathcal{C}^n \setminus V^\delta$, of dimension $n - l$, while its boundary is also a finite mass submanifold of ∂V^δ, of dimension $n - l - 1$. We fix such w and we observe that for all $\varepsilon' > 0$, there is $f_{\varepsilon'}$, some lipschitz diffeomorphism of $\mathcal{C}^n$, such that $f_{\varepsilon'}$ is the identity map except on a small neighbourhood of $\bigcup_{i=1}^{s_l} (u^l)^{-1}(w_i)$, and we have :

$$\begin{cases} f_{\varepsilon'}(V^\delta) = V^\delta \ , f_{\varepsilon'}(\partial V^\delta) = \partial V^\delta \\[2mm] (u^l \circ f_{\varepsilon'})^{-1}(w_i) \cap (\mathcal{C}^n \setminus V^\delta) \text{ is a polyhedral } (n-l)\text{-submanifold of } \mathcal{C}^n \setminus V^\delta \\[2mm] (u^l \circ f_{\varepsilon'})^{-1}(w_i) \cap (\partial V^\delta) \text{ is a polyhedral } (n-l-1)\text{-submanifold of } \partial V^\delta. \\[2mm] \displaystyle\int_{\mathcal{C}^n} \left|\nabla (u^l \circ f_{\varepsilon'}) - \nabla u^l\right|^p < \varepsilon' \\[2mm] \displaystyle\int_{\partial V^\delta} \left|\nabla (u^l \circ f_{\varepsilon'}) - \nabla u^l\right|^p < \varepsilon' \end{cases} \tag{4.7}$$

Let $\varepsilon' = \min\{\varepsilon_1, \varepsilon_2\}$ and denote $v^l := (u^l \circ f_{\varepsilon'})_\delta$. Using (2.3) and (4.7) we get :

$$\begin{aligned}
\int_{\mathcal{C}^n} |\nabla v^l|^p &= \int_{V^\delta} |\nabla (u^l \circ f_{\varepsilon'})_\delta|^p + \int_{\mathcal{C}^n \setminus V^\delta} |\nabla (u^l \circ f_{\varepsilon'})|^p \\
&\leq \delta K \int_{\partial V^\delta} |\nabla (u^l \circ f_{\varepsilon'})|^p + \int_{\mathcal{C}^n \setminus V^\delta} |\nabla u^l|^p + \varepsilon_1 \\
&\leq \delta K \varepsilon_2 + \delta K \int_{\partial V^\delta} |\nabla u^l|^p + \int_{\mathcal{C}^n \setminus V^\delta} |\nabla u^l|^p + \varepsilon_1 \\
&\leq \int_{\mathcal{C}^n} |\nabla u^l|^p + (\delta K \varepsilon_2 + \varepsilon_1 + K^2 \int_{V^{\delta_1}} |\nabla u^l|^p).
\end{aligned} \tag{4.8}$$

We observe that v^l is continuous on $\mathcal{C}^n \setminus B$ and since $f_{\varepsilon'}$ is a diffeomorphism, it has the same homotopic singularity as u^l on components of B. Now by (4.5) we have :

$$\int_{\mathcal{C}^n} |\nabla (p_w \circ v^l)|^p \leq \frac{C(l,p,\varepsilon)}{\mathcal{H}^{ls_l}(N^l_\varepsilon)} \int_{\mathcal{C}^n} |\nabla v^l|^p \,. \tag{4.9}$$

So as a result $v^{l-1} := p_w \circ v^l \in W^{1,p}_{w_0}(\mathcal{C}^n, N^{l-1})$. Observe that by construction v^{l-1} is locally lipschitz away from

$$\Sigma(v^{l-1}) = \bigcup_{i=1}^{s_l} (u^l \circ f_{\varepsilon'})^{-1}_\delta (w_i) \cup B.$$

Moreover by (4.7), $(u^l \circ f_{\varepsilon'})^{-1}_\delta (w_i)$ is a finite union of $(n-l)$-dimensional polyhedrals supported in $\mathcal{C}^n$. Thus, since $n - l \leq n - p - 1$, we can find some $u^{l-1} \in \mathcal{R}^{\infty,p}_{w_0}(\mathcal{C}^n, N^{l-1})$ such that u^{l-1} has the same topologic singularities as v^{l-1}, and

$$\int_{\mathcal{C}^n} |\nabla u^{l-1} - \nabla v^{l-1}|^p \leq \varepsilon_3 \,.$$

This fact, combined with (4.6), (4.8) and (4.9) yields :

$$\int_{\mathcal{C}^n} |\nabla u^{l-1}|^p \leq \left(\frac{C(l,p,\varepsilon)}{\mathcal{H}^{ls_l}(N^l_\varepsilon)} + 1 \right) \int_{\mathcal{C}^n} |\nabla u^l|^p \,.$$

We have proved so far parts (i) and (ii) of lemma 4.2. Part (iii) is a direct consequence of (4.2) and the construction of u^{l-1}, using the same argument as in proof of proposition 3.1 (See (3.12)). ∎

Lemma 4.3 *Let N be a $(p-1)$-connected smooth compact manifold. Let $u \in \mathcal{R}^{\infty,p}_\varphi(\mathcal{C}^n, N^p)$ such that φ is constant. Then there exists polyhedral chain $\mathbf{T} \in \mathcal{P}_{n-p}(\mathcal{C}^n, \pi_p(N^p))$ such that*

$$\begin{cases} \partial T = \mathbf{S}_u \\ \\ \mathbf{M}(T) \leq C \displaystyle\int_{\mathcal{C}^n} |\nabla u|^p \end{cases} \tag{4.10}$$

for some constant $C > 0$ independent of u.

∎

Proof : As we observed above, N^p is $(p-1)$-connected too and it is finitely generated. Let $g_1, \dots, g_\beta$ be its generators. By ([17], Corollary 3.5, P.38), we observe that there are smooth maps $p_i : N^p \to S^p$, $i = 1, \dots, \beta$, such that

$$[p_i(\mathcal{G})]_{\pi_p(S^p)} = \alpha_i([\mathcal{G}]_{\pi_p(N^p)}) \quad \text{for any } \; p-\text{cycle } \mathcal{G} \subset N^p, \tag{4.11}$$

where, for every $a \in \pi_p(N^p)$,

$$a = \sum_{i=1}^{\beta} \alpha_i(a) g_i$$

is its unique decomposition. Meanwhile, for every $u \in \mathcal{R}_\varphi^{\infty,p}(\mathcal{C}^n, N^p)$, $p_i \circ u$ is in $\mathcal{R}_\varphi^{\infty,p}(\mathcal{C}^n, S^p)$. Since φ is constant, by [1] and the approximation theorem (5.6) in [15], there is $\mathbf{T}_i \in \mathcal{P}_{n-p}(\mathcal{C}^n, \mathbb{Z})$ such that

$$\begin{cases} \partial \mathbf{T}_i = \mathbf{S}_{p_i \circ u} \\ \\ \mathbf{M}(\mathbf{T}_i) \leq C_i \displaystyle\int_{\mathcal{C}^n} |\nabla u(p_i \circ u)|^p \end{cases} \tag{4.12}$$

where $C_i > 0$ is independent of u. (See also [29] for detailed discussion for S^2).

Now consider the injectif group homomorphism $\kappa^i : \mathbb{Z} \to \pi_p(N)$, $i = 1, \dots, \beta$, defined by $\kappa^i(n) = n g_i$. Observe that we have

$$\sum_{i=1}^{\beta} \kappa^i(\alpha_i(a)) = a \quad \forall a \in \pi_p(N^p),$$

which combined with (4.11) gives :

$$\kappa_*^i(\mathbf{S}_{p_i \circ u}) = \mathbf{S}_u \,.$$

Moreover, κ_*^i satisfies

$$\mathbf{M}(\kappa_*^i(T)) \leq C_i' \, \mathbf{M}(T),$$

for some constant C_i' independent of T. We set

$$\mathbf{T} := \sum_{i=1}^{\beta} \kappa_*^i(\mathbf{T}_i).$$

So $\mathbf{T}$ is a polyhedral $\pi_p(N^p)$-chain, of dimension $n-p$ and supported in $\mathcal{C}^n$. Using lemma 2.2 and (4.12) we obtain

$$\partial \mathbf{T} = \sum_{i=1}^{\beta} \kappa_*^i(\mathbf{S}_{p_i \circ u}) = \mathbf{S}_u$$

and

$$\mathbf{M}(T) \leq \sum_{i=1}^{\beta} C_i' \mathbf{M}(\mathbf{T}_i) \leq \sum_{i=1}^{\beta} C_i' C_i \int_{\mathcal{C}^n} |\nabla (p_i \circ u)|^p \,.$$

This completes the proof since the p_i are smooth. ∎

Using the above stated lemmas, we prove the following important result :

Proposition 4.1 *For any integer p, $2 \leq p \leq k$, let N be a k-dimensional $(p-1)$-connected compact smooth manifold. Let $\mathcal{C}^n$ be the unit cube in $\mathbb{R}^n$. Then for $u \in R^{\infty,p}_{\varphi}(\mathcal{C}^n, N)$, there is $\mathbf{T} \in \mathcal{P}_{n-p}(\mathcal{C}^n, \pi_p(N))$ such that*

$$\begin{cases} \partial \mathbf{T} = \mathbf{S}_u \\ \\ \mathbf{M}(\mathbf{T}) \leq C \displaystyle\int |\nabla u|^p + C \end{cases} \tag{4.13}$$

for some constant $C > 0$ independent of u.

■

Corollary 4.1 *For any $u \in R^{\infty,p}_{\varphi}(\mathcal{C}^n, N)$, there is a minimal connection $\mathbf{T}_u \in \mathcal{F}_{n-p}(\mathcal{C}^n, \pi_p(N))$ which satisfies*

$$\mathbf{M}(\mathbf{T}_u) \leq C \int |\nabla u|^p + C.$$

(See corollary 3.1). ■

Proof of proposition 4.1 : It is sufficient to prove the proposition for $\varphi = w_0 \in N^p$, constant. Using the same method as in the proof of proposition 3.1, combined with the approximation theorem (5.6) in [15], the proof is generalized for any smooth boundary data.

Write $N^k = N$ and $u^k = u$. Using lemma 4.2 successively we obtain a map $u^p \in \mathcal{R}^{\infty,p}_{w_0}(\mathcal{C}^n, N^p)$, which satisfies

$$\begin{cases} \displaystyle\int_{\mathcal{C}^n} |\nabla u^p|^p \leq C_1 \int_{\mathcal{C}^n} |\nabla u|^p \\ \\ \chi_*(\mathbf{S}_{u^p}) = \mathbf{S}_u \end{cases} \tag{4.14}$$

where $\chi : \pi_p(N^p) \to \pi_p(N)$ is the natural homomorphism and C_1 is independent of u. We apply lemma 5.1 to u^p and get some $\mathbf{T}_p \in \mathcal{P}_{n-p}(\mathcal{C}^n, \pi_p(N^p))$ such that

$$\mathbf{M}(\mathbf{T}_p) \leq C_2 \int_{\mathcal{C}^n} |\nabla u^p|^p$$

and

$$\partial \mathbf{T}_p = \mathbf{S}_{u^p}.$$

Combining with (4.14) and applying lemma 2.2, using (4.1), we observe that $\mathbf{T} := \chi_*(\mathbf{T}_p)$ satifies (4.13). ■

5 Removing the singularities using finite energy

In the section, we prove that we can remove the singularities of a map $u \in \mathcal{R}_\varphi^{\infty,p}(\mathcal{C}^n, N)$ by modifying it along one of its polyhedral connections and using an energy almost proportional to the mass of the connection. The idea first appeared in [3] for $H^1(B^3, S^2)$. Our proof uses a different approach since the situation is technically more involved. Note that we use the same norm defined for $\pi_p(N)$ as in section 4 and the method may not work for non-equivalent norms. This is the exact statement of what we prove in this section :

Proposition 5.1 *Let $p > 1$ be an integer and let N be a k-dimensional simply connected closed manifold. Assume that $\pi_p(N)$ is finitely generated. If $\mathbf{T} \in \mathcal{P}_{n-p}(\mathcal{C}^n, \pi_p(N))$ is a connection for $u \in \mathcal{R}_\varphi^{\infty,p}(\mathcal{C}^n, N)$, there are maps $u_m \in C_\varphi^\infty(\mathcal{C}^n, N)$ such that*

$$\begin{cases} u_m \xrightarrow{L^p} u \quad \text{as } m \to \infty \\ \displaystyle\int_{\mathcal{C}^n} |\nabla u_m|^p \leq \int_{\mathcal{C}^n} |\nabla u|^p + C\mathbf{M}(\mathbf{T}) + O(\frac{1}{m}) \end{cases} \tag{5.1}$$

for $C > 0$ independent of u. The same result holds when $p = 1$ if $\pi_1(N)$ is abelian.

■

First we prove two lemmas necessary for the proof of this proposition.

Lemma 5.1 *For every $g \in \pi_p(N)$, there exists an open covering of N, $\{U_1^g, \ldots, U_{\nu_g}^g\}$, and smooth maps*

$$\omega_{g,j} : \mathbf{B}^p \times U_j^g \to N, \quad j = 1, \ldots, \nu_g$$

such that

$$\begin{cases} \omega_{g,j}(.|_{\partial \mathbf{B}^p}, y) \equiv y & \forall y \in N \\ [\omega_{g,j}(., y)]_{\pi_p(N)} = g & \forall y \in N \\ \displaystyle\int_{\mathbf{B}^p} |\nabla_x \omega_{g,j}(., y)|^p dx \leq C|g| & \forall y \in N \\ |\nabla \omega_{g,j}|_\infty \leq C_g \end{cases} \tag{5.2}$$

where $C > 0$ is independent of g and j.

■

Proof : Let $h_1, \ldots, h_\gamma$ be the generators of $\pi_p(N)$. Since N is compact we can find a finite open covering of N, $\{U_1, \ldots, U_\nu\}$, and smooth maps

$$\omega_{i,j} : \mathbf{B}^p \times U_j \to N$$

such that for all i, j and all $y \in N$ we have

$$\begin{cases} \omega_{i,j}(.|_{\partial \mathbf{B}^p}, y) \equiv y \\ \\ [\omega_{i,j}(., y)]_{\pi_p(N)} = h_i \,. \end{cases} \tag{5.3}$$

Now we write $g \in \pi_p(N)$ in its minimal length decomposition

$$g = h_{i_1} + \cdots + h_{i_s},$$

where $s = |g|$. For $y \in N$, $x \in \mathbf{B}^p$ and $\rho = 1, \ldots, s$, we set

$$\omega_{g,x}(y) := \omega_{i_\rho, j_\rho}\left(sx - (\rho - 1)\frac{x}{|x|}, y_\rho\right) \quad \text{if} \quad \frac{\rho - 1}{s} \leq |x| \leq \frac{\rho}{s}\,,$$

where $y_s := y \in U_{j_s}$ and for $\rho = 1, \ldots, s-1$,

$$y_\rho := \omega_{i_{\rho+1}, j_{\rho+1}}(0, y_{\rho+1}) \in U_{j_\rho}.$$

Observe that by slightly modifying $\omega_{g,x} : \mathbf{B}^p \to N$, we can assume that it is smooth on its domain. Moreover it will satisfy

$$\begin{cases} \omega_{g,y}|_{\partial \mathbf{B}^p} \equiv y \\ \\ [\omega_{g,y}]_{\pi_p(N)} = g \\ \\ \displaystyle\int_{\mathbf{B}^p} |\nabla \omega_{g,y}|^p \leq Cs = C|g| \end{cases}$$

for $C > 0$ independent of g and y. Another observation shows that $\omega_{g,y}$ depends smoothly on y in small neighbourhoods. Since N is compact, we can find a finite open covering for it, $\{U_1^g, \ldots, U_{\nu_g}^g\}$, such that for $j = 1, \ldots, \nu_g$

$$\omega_{g,j}(x, y) := \omega_{g,y}(x), \quad \text{if} \quad y \in U_j^g$$

satisfy (5.2). ∎

Lemma 5.2 *Let $u \in \mathcal{R}_\varphi^{\infty,p}(\mathcal{C}^n, N)$ and $\Sigma \subset \mathcal{C}^n$ be an oriented polyhedral of dimension $n - p$ such that u is continuous on Σ except probably on its boundary. Then for every $g \in \pi_p(N)$, there is a sequence $u_m \in W_\varphi^{1,p}(\mathcal{C}^n, N)$ and $C > 0$ independent of g and u such that*

$$\begin{cases} u_m = u \quad on \quad \mathcal{C}^n \backslash K_m \\ \\ |K_m| \to 0 \quad as \quad m \to \infty \\ \\ \displaystyle\int_{\mathcal{C}^n} |\nabla u_m|^p \leq \int_{\mathcal{C}^n} |\nabla u|^p + C|g||\Sigma| + \frac{1}{m} \end{cases} \tag{5.4}$$

and

$$\mathbf{S}_{u_m} = \mathbf{S}_u - g[[\partial \Sigma]].$$

■

Proof : We identify $\mathbb{R}^n$ with $\mathbb{R}^{n-p} \times \mathbb{R}^p$ with variables $X \in \mathbb{R}^{n-p}, Y \in \mathbb{R}^p$. Without loss of generality we can assume that Σ lies in the plane $\mathbb{R}^{n-p} \times \{0\}$. We divide Σ in polyhedrals of equal dimension

$$\Sigma := \bigcup_{j=1}^{\nu_g} \bigcup_{i=1}^{i_j} \Sigma_j^i$$

such that $u(\Sigma_j^i) \subset U_j^g$ for all i, j. We choose B as in section 2.2 such that

$$\bigcup_{j=1}^{\nu_g} \bigcup_{i=1}^{i_j} \partial \Sigma_j^i \subset B$$

and we replace u by u_{δ_1} for δ_1 small enough (See definition 2.5). This doesn't change much the energy of u and $\mathbf{S}_{u_{\delta_1}} = \mathbf{S}_u$, so it is sufficient to prove the lemma for $u = u_{\delta_1}$. Since u is radial, we have for some constant $C_1 > 0$

$$|\nabla u(x)| \leq C_1 \text{ if } x \in \mathcal{C}^n \backslash V_{\delta_1}, \quad |\nabla u(x)| \leq \frac{C_1}{\|x - B\|} \text{ if } x \in V_{\delta_1}. \tag{5.5}$$

We set for $\eta << \delta < \delta_1$ and $(X, Y) \in \mathcal{C}^n \backslash V_\delta$

$$v^{\delta,\eta}(X,Y) := \begin{cases} u(X,Y) & \text{if } (X,0) \notin \Sigma \text{ or if } |Y| \geq \eta \\ u\left(X, 2Y - \eta \dfrac{Y}{|Y|}\right) & \text{if } (X,0) \in \Sigma \text{ and } \frac{\eta}{2} \leq |Y| \leq \eta \\ \omega_{g,j}\left(\dfrac{2}{\eta} Y, u(X,0)\right) & \text{if } (X,0) \in \bigcup_{i=1}^{i_j} \Sigma_j^i \text{ and } |Y| \leq \frac{\eta}{2} \end{cases} \tag{5.6}$$

We set

$$\Sigma^\eta := \{(X,Y);\ (X,0) \in \Sigma,\ |Y| \leq \eta\}.$$

and we observe that $vol(\partial V^\delta \cap \Sigma^\eta) = O(\eta^p)$. Using (2.2), (2.3) and (5.5) we get

$$\int_{V_\delta} |\nabla(v^{\delta,\eta} \circ h_\delta) - \nabla u|^p \leq \delta K \int_{\partial V_\delta \cap \Sigma^\eta} \left(\frac{C_g C_2}{\eta} + C_1\right)^p \leq O(\delta)$$

for $C_2 > 0$ independent of δ. Moreover for fixed δ we have

$$\begin{aligned} \int_{\mathcal{C}^n \backslash V_\delta} |\nabla v^{\delta,\eta} - \nabla u|^p &\leq C|g||\Sigma| + \int_{\Sigma^\eta} \left(\frac{C_g C_2}{\delta} + C_1\right)^p \\ &\leq C|g||\Sigma| + O(\eta). \end{aligned}$$

As a result, by choosing successively suitable δ and η, $u_m := (v^{\delta,\eta})_\delta$ will satisfy (5.4). Moreover we have

$$\mathbf{S}_{u_m} = \mathbf{S}_u \pm g[[\partial \Sigma]].$$

If necessary, we get the good sign by replacing g by $-g$ above. ■

Proof of proposition 5.1 **:** We write

$$\mathbf{T} = \sum_{i=1}^{\theta} g_i[[\Sigma^i]].$$

Put $u_m^0 := u$ and for $i = 1, \ldots, \theta$, let u_m^i be the m-th element of the sequence obtained by applying lemma 5.2 to u_m^{i-1} for Σ^i, g_i. We get

$$\begin{aligned} \mathbf{S}_{u_m^\theta} &= \mathbf{S}_u - \sum_{i=1}^{\theta} g_i[[\partial\Sigma^i]] \\ &= \mathbf{S}_u - \partial\mathbf{T} = 0. \end{aligned}$$

and we observe that u_m^θ satisfy (5.1). Pay attention that $\mathbf{S}_{u_m^\theta} = 0$ means that u_m^θ, restricted to almost every small enough p-cycle in $\mathcal{C}^n$, is homotopic to constant in N. Using this and referring to [2], the proof of theorem 1, we can approximate strongly u_m^θ by smooth maps in $C_\varphi^\infty(\mathcal{C}^n, N)$. This completes the proof. ∎

6 Proof of theorems 1,2 and 3

Theorems 1 and 1 *bis* are proved using the same arguments as for $W^{1,1+\varepsilon}(\mathcal{C}^2, \mathbb{RP}^2)$, regarding the fact that we have developped the necessary tools above. Observe that the equality

$$m_i(\mathbf{S}_u) = m_r(\mathbf{S}_u)$$

holds true for any integral flat chain $\mathbf{S}$ in $\mathbb{R}^n$ if and only if $\mathbf{S}$ is of dimension 0 or codimension 2 in $\mathbb{R}^n$ (See [14]). Thus our method can not be used for $[p]$ taking a value other than 1 or $n-1$.

Considering propositions 4.1 and 5.1, theorem 2 bis is proved the same as in section 3.4. The only difference is that since $p > 1$, a bounded sequence in $W_\varphi^{1,p}(\mathcal{C}^n, N)$ has a weakly convergent subsequence. Theorem 2 is proved following the same ideas. The only important difference is that a chain $\mathbf{T}$ is said to be a connection for $u \in \mathcal{R}^{\infty,p}(\mathcal{C}^n, N)$ if $\operatorname{spt}(\partial T - \mathbf{S}_u) \subset \partial\mathcal{C}^n$ (Compare with definition 4.1).

Propositions 4.1 and 5.1 hold for $p = 1$, abelian $\pi_1(N)$, thus theorems 3 and 3 bis are proved with the same method.

7 Proof of theorem 4

Let N be any closed manifold. We prove that the smooth maps are sequentially weakly dense dense in $W^{1,2}(\mathcal{C}^n, N)$. Regarding what we proved above, we should prove the theorem for $\pi_1(N) \neq 0$. Trying to adapt the method used for proving theorem 2 , the first problem we confront is that in this case there are not canonical isomorphisms between the homotopy groups $\pi_p(N,x)$ with different base points. Thus, we can not talk about $[u,\sigma_i]$ as in definition 2.7 without fixing a base point in N. Another difficulty is that N^2 may not be of the same homotopy type as a bouquet of spheres.

For surmounting these problems we consider the smooth riemannien manifold $\widetilde{N}$, the universal covering of N, and the corresponding fibration $F:\widetilde{N}\to N$. We assume that $\widetilde{N}$ is embedded isometrically in some $\mathbb{R}^{N'}$ and that F is a local isometry. We consider N^2 as defined in section 4 and again using ([40], theorem (1.6), p. 215) we observe that $\pi_1(N)=\pi_1(N^2)$ and for $2\le l\le k$, the homomorphisms

$$\chi^{2,l}:\pi_2(N^2)\longrightarrow\pi_2(N^l),$$

induced by the injection maps $i_{2,l}:N^2\to N^l$, are onto. Meanwhile, since N is compact, $\pi_2(N)$ and $\pi_2(N^2)$ are finitely generated. Set

$$\widetilde{N^2}:=F^{-1}(N^2).$$

Since $\pi_1(N^2)=\pi_1(N)$ and using the homotopy theory, we deduce that $\widetilde{N^2}$ is the universal covering of N^2 as a CW-complex and that $F|_{\widetilde{N^2}}$ is the corresponding fibration. Observe that this diagram is commutative :

$$\begin{array}{ccc}\pi_2(\widetilde{N^2}) & \stackrel{\widetilde{\chi}^{2,k}}{\longrightarrow} & \pi_2(N)\\ \downarrow(F|_{\widetilde{N^2}})_* & & \downarrow F_*\\ \pi_2(N^2) & \stackrel{\chi^{2,k}}{\longrightarrow} & \pi_2(N)\end{array}\tag{7.1}$$

where $\widetilde{\chi}^{2,k}:\widetilde{N^2}\to\widetilde{N}$ is induced by the injection map $\tilde{i}_{2,k}:\widetilde{N^2}\to\widetilde{N}$ and is onto. Also

$$F_*:\pi_2(\widetilde{N})\to\pi_2(N)\quad\text{and}\quad(F|_{\widetilde{N^2}})_*:\pi_2(\widetilde{N^2})\to\pi_2(N^2)$$

are isomorphisms. Thus, since $\pi_1(\widetilde{N^2})=\pi_1(\widetilde{N})=0$, using ([17], Corollary 3.5, P. 38) and the fact that $\pi_2(N^2)$ is finitely generated, we obtain that $\widetilde{N^2}$ is of the homotopy type of a finite bouquet of spheres.

Any $u\in\mathcal{R}^{2,\infty}(\mathcal{C}^n,N)$ can be lifted to a map $\tilde{u}:\mathcal{C}^n\to\widetilde{N}$ as $\pi_1(\mathcal{C}^n\backslash\Sigma(u))=0$. (Remember that $\pi_1(\mathcal{C}^n)=0$ and that $\Sigma(u)$ is of codimension 3 in $\mathcal{C}^n$). Since F is a local isometry, we get that $\tilde{u}\in\mathcal{R}^{2,\infty}(\mathcal{C}^n,N)$ and that

$$\int_{\mathcal{C}^n}|\nabla\tilde{u}|^2=\int_{\mathcal{C}^n}|\nabla u|^2.\tag{7.2}$$

Since $\pi_1(\widetilde{N})=0$, $\mathbf{S}_{\tilde{u}}$ is well defined as in definition 2.8.

Let $u \in W^{1,2}(\mathcal{C}^n, N)$ and $u_m \in \mathcal{R}^{2,\infty}(\mathcal{C}^n, N)$ a sequence converging strongly to u. Using the same method as in proposition 4.1 we can prove the existence of some constant $C > 0$ independent of u_m, and maps $u_m^2 \in \mathcal{R}^{2,\infty}(\mathcal{C}^n, N^2)$ such that

$$\int_{\mathcal{C}^n} |\nabla u_m^2|^2 \leq C \int_{\mathcal{C}^n} |\nabla u_m|^2. \tag{7.3}$$

Meanwhile, if we consider the liftings $\tilde{u}_m^2 \in \mathcal{R}^{2,\infty}(\mathcal{C}^n, \widetilde{N^2})$, we get

$$\widetilde{\chi}_*^{2,k}(\mathbf{S}_{\tilde{u}_m^2}) = \mathbf{S}_{\tilde{u}_m}. \tag{7.4}$$

This is a result of the commutativity of diagram (7.1) and the construction of u_m^2, using the same method as in lemma 4.2. Since $\widetilde{N^2}$ is of homotopy type of a bouquet of spheres, using the arguments of lemma 5.1, we observe that for any map $\tilde{v}^2 \in \mathcal{R}^{2,\infty}(\mathcal{C}^n, \widetilde{N^2})$ we can find $\mathbf{T}_2$, a $\pi_2(\widetilde{N^2})$-chain, supported in a finite union of smooth submanifolds of M of dimension $n-2$ and connecting $\mathbf{S}_{\tilde{v}^2}$, such that

$$\operatorname{spt}(\partial \mathbf{T}_2 - \mathbf{S}_{\tilde{v}^2}) \subset \partial \mathcal{C}^n \quad \text{and} \quad \mathbf{M}(\mathbf{T}_2) \leq C \int_{\mathcal{C}^n} |\nabla \tilde{v}^2|^2, \tag{7.5}$$

where C is independent of v^2. Regarding (7.2), (7.3), (7.4) and (7.5), we prove the same result for any map $\tilde{u} \in \mathcal{R}^{2,\infty}(\mathcal{C}^n, \widetilde{N})$ which is a lifting of a map $u \in \mathcal{R}^{2,\infty}(\mathcal{C}^n, N)$, this time using $\pi_2(\widetilde{N})$-chains.

Also, the equivalent statement of proposition 5.1 is proved for maps from $\mathcal{C}^n$ into $\widetilde{N}$, though $\widetilde{N}$ may not be compact. This is possible as $\pi_2(\widetilde{N})$ is still finitely generated. (Here we use countable proper open covers of $\widetilde{N}$ in place of finite covers and remove the singularities by modifying the maps in neighbourhoods of smooth $(n-2)$-dimesnional polyhedrals in $\mathcal{C}^n$. Applying the singularity removing proposition to $\tilde{u}_m$ and its connection, we deduce the existence of some maps $\tilde{v}_m^k \in \mathcal{R}^{2,\infty}(\mathcal{C}^2, \widetilde{N})$ such that $\mathbf{S}_{\tilde{v}_m^k} = 0$, $\tilde{v}_m^k \to \tilde{u}_m$ in L^2-norm and

$$\int_{\mathcal{C}^n} |\nabla \tilde{v}_m^k|^2 \leq C \int_{\mathcal{C}^n} |\nabla \tilde{u}_m|^2 + O(\frac{1}{k}) \tag{7.6}$$

are equi-bounded. Set

$$v_m^k := F \circ \tilde{v}_m^k \in \mathcal{R}^{2,\infty}(\mathcal{C}^n, N).$$

Since $\mathbf{S}_{\tilde{v}_m^k} = 0$, the v_m^k do not realize any non trivial homotopy class of $\pi_2(N)$ around their singularities. So we can apprximate them strongly by maps $u_m^k \in C^\infty(\mathcal{C}^n, N)$. By (7.6), the u_m^k are equi-bounded in Dirichlet energy and for a suitable subsequence $u_m^{k(m)}$, they converge strongly to u in L^2. So there is a subsequence of $u_m^{k(m)} \in C^\infty(\mathcal{C}^n, N)$ which converges weakly to u in $W^{1,2}$. ∎

Theorem 4 *bis* is proved using the same method. Also, using the same arguments we can prove that smooth maps are sequentially dense in $W^{1,p}(\mathbf{B}^n, N)$, if $\pi_p(\widetilde{N})$, being the first non-trivial homotopy group of $\widetilde{N}$, is of finite type.

Chapter V

Study of $W^{1,1}(\mathbf{B}^n, N)$ for non-abelian $\pi_1(N)$

Weak density of smooth maps in $W^{1,1}(\mathbf{B}^n, N)$

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We prove that smooth maps are dense in the sense of biting convergence in $W^{1,1}(\mathbf{B}^n, N)$ when N is a closed riemannien manifold.

1 Introduction

Let $\mathbf{B}^n$ be the unit disk in $\mathbb{R}^n$ and N a closed riemannien manifold isometrically embedded in $\mathbb{R}^N$. Set

$$W^{1,1}(\mathbf{B}^n, N) := \{u \in W^{1,1}(\mathbf{B}^n, \mathbb{R}^N); u(x) \in N \text{ for a.e.} x \in \mathbf{B}^n\}$$

This space inherits the strong and the weak topology of $W^{1,1}(\mathbf{B}^n, \mathbb{R}^N)$ and is closed under the weak convergence of maps in $W^{1,1}$. The energy of a map $u \in W^{1,1}(\mathbf{B}^n, N)$ is defined to be $\int_{\mathbf{B}^n} |\nabla u|$.

Based on the work of R.Shoen, K.Uhlenbeck, X.Zheng and F.Bethuel in [36], [7], and [2], we know that smooth maps from $\mathbf{B}^n$ into N are not dense in $W^{1,1}(\mathbf{B}^n, N)$ if

$\pi_1(N) \neq 0$. In fact, they showed that the lack of approximability is due to local realizations by $u \in W^{1,1}(\mathbf{B}^n, N)$ of non-zero elements of $\pi_1(N)$ around points in $\mathbf{B}^n$. In particular they proved that if $\pi_1(N) = 0$ then any map in $W^{1,1}(\mathbf{B}^n, N)$ can be approximated by smooth maps for the strong topology. A major question would be to determine a criteria for a map to be approximable by smooth maps in $W^{1,1}(\mathbf{B}^n, N)$, i.e. we try to define $\mathbf{S}_u$, "the topological singular set " of u, which would be equal to zero if and only if u is a strong limit of smooth maps in $W^{1,1}(\mathbf{B}^n, N)$.

In the case $\pi_1(N) \neq 0$, one can approximate the maps in $W^{1,1}(\mathbf{B}^n, N)$ by maps which are smooth away from a finite union $\Sigma = \bigcup_{i=1}^r \Sigma_i$ of smooth $(n-2)$-dimensional submanifolds of $\mathbf{B}^n$. This set of maps is called $R^\infty(\mathbf{B}^n, N)$. A map $v \in R^\infty(\mathbf{B}^n, N)$ realizes elements σ_x of $\pi_1(N, y)$ on the circles centered at any point $x \in \Sigma(v)$ and contained in the normal bidimensional plane to $T_x\Sigma(v)$. If for some $x \in \Sigma(v)$, σ_x is non trivial, then v can not be approximated by smooth maps in the strong topology (See [2]). In [31], the author and T.Rivière observed that if $\pi_1(N)$ is abelian, one can assign to v a $\pi_1(N)$-chain which is carried by $\Sigma(v)$ with "multiplicity" σ_x at each point x of $\Sigma(v)$. This $\pi_1(N)$-chain is called the topological singular set $\mathbf{S}_v$ of v in $R^\infty(\mathbf{B}^n, N)$. Moreover, for a sequence of maps $v_m \in R^\infty(\mathbf{B}^n, N)$ converging strongly to any $u \in W^{1,1}(\mathbf{B}^n, N)$, $\mathbf{S}_{v_m}$ converges in the flat norm to a unique flat $\pi_1(N)$-chain $\mathbf{S}_u$ we called the topological singular set of u.

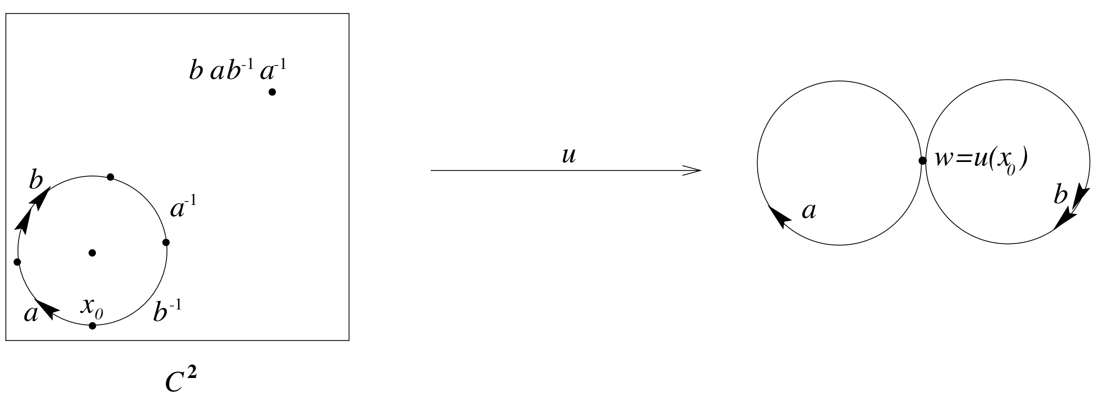

Fig.1
An $(aba^{-1}b^{-1})$-type singularity dipole

This approach confronts important obstacles when $\pi_1(N)$ is not abelian. The major problem is the following : If $\pi_1(N)$ is abelian, its elements are well defined independent of the choice of the base point in N, i.e. we can define isomorphisms $\gamma_\#$ between $\pi_1(N, y)$ and $\pi_1(N, y')$ with the aide of smooth curves γ joining y and y' in N. These isomorphisms do not depend of the choice of γ and so we can identify $\pi_1(N, y)$ and $\pi_1(N, y')$ in a natural manner. In this way, e.g. we can compare the topologic singularity of $u \in R^\infty(\mathbf{B}^2, \mathbb{RP}^2)$ around different points in the square $\mathbf{B}^2$ without ambiguity, though the values of u in $\mathbb{RP}^2$ near these points might differ. But, if $\pi_1(N)$ is not abelian, there is no canonical isomorphism between $\pi_1(N, y)$ and $\pi_1(N, y')$ for two different points $y, y' \in N$. The isomorphisms $\gamma_\#$ would depend on the homotopy class of γ and even a closed curve γ joining y to itself may produce a non-trivial isomorphism of $\pi_1(N, y)$ onto itself. So, talking about the topologic type of a singularity without fixing the base points in $\mathbf{B}^n$ and in N is impossible and we can neither compare the topological type of different singularities nor talk about connecting them by chains with coefficients in $\pi_1(N)$ as before.

Another problem we encounter in the study of this case is that $u \in R^\infty(\mathbf{B}^n, N)$ may have singularities of the type $aba^{-1}b^{-1}$ which are not removable by strong convergence of smooth maps. Meanwhile, following the method used in [31], the conjugation of u with p^a (or p^b), the projections of N on the generating cycles of a (or b), will not "see" these singularities in the first instance, since $p^a \circ u$ (or $p^b \circ u$) would realise the cycles aa^{-1} (or bb^{-1}) in their respectable circle-type targets.

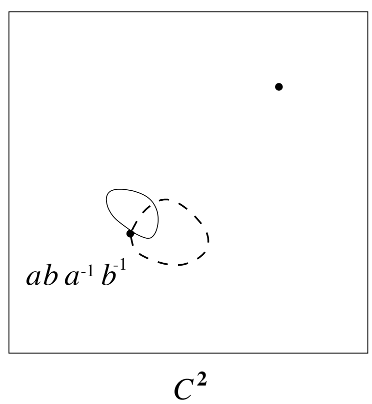

Fig.2
A bad connecting set for the dipole
(Not suitable for removing the singularities)

In this way, the question of defining a topological singular set for maps in $W^{1,1}(\mathbf{B}^n, N)$ is still open for non-abelian $\pi_1(N)$. In this paper, we try to pave the way for understanding the situation by answering another related question. If $\pi_1(N)$ is abelain, we can prove that for any map $u \in W^{1,1}(\mathbf{B}^n, N)$, there is a sequence of smooth maps, $v_m \in C^\infty(\mathbf{B}^n, N)$, such that u is the $W^{1,1}$-weak limit of v_m outside arbitrary small positive measure subsets of $\mathbf{B}^n$ (See definition 1.1 below). The method consists in controling the mass of chains which connect the singular chain of a map $u \in R^\infty(\mathbf{B}^n, N)$ to the boundary of $\mathbf{B}^n$ and then removing the singularities, spending en energy proportional to the mass of these connections (See [31]). The question is then whether this method can be modified to prove the same result for the non-abelian $\pi_1(N)$ case.

For surmounting the above described problems for non-abelian $\pi_1(N)$, we should introduce new elements into the proof. In fact, we search a kind of connecting set $A_u \subset \mathbf{B}^n$ of dimension $n-1$ for the singularities of a map $u \in R^\infty(\mathbf{B}^n, N)$ so that for any point $x \in A_u$ we can identify $a(x)$: the elements of $\pi_1(N, u(x))$ which should be introduced into u (transversally to A_u at x) such that the singularities of u are removed. These connecting sets should also take into account the problems provoked by $aba^{-1}b^{-1}$-type singularities described above. And, last but not least, the one-energy of inserted curves producing $a(x)$ at $x \in A_u$ should be controled uniformly (independent of the choice of x and u) so that the total energy of the modification be uniformly proportional to the volume of A_u, which in its turn is controled by the energy of u. All this is possible for a converging sequence $u_m \to u \in W^{1,1}(\mathbf{B}^n, N)$. So here is the main results of this paper :

Definition 1.1 *Let Ω be a domain in $\mathbb{R}^n$ and let u_m be a bounded sequence in $L^1(\Omega)$. u_m is said to converge in the biting sense to $u \in L^1(\Omega)$ if for every $\varepsilon > 0$ there exists a measurable set $E \subset \Omega$ such that $\mu(E) < \varepsilon$ and $u_m \rightharpoonup u$ weakly in $L^1(\Omega \backslash E)$.*

Theorem 1 *Let $\mathbf{B}^n$ be the unit disk in $\mathbb{R}^n$ and N be any k-dimensional closed manifold. Then for every $u \in W^{1,1}(\mathbf{B}^n, N)$ there is a sequence of maps $u_m \in C^\infty(\mathbf{B}^n, N)$ such that ∇u_m tend to ∇u in the biting sense.*

■

Assume that $\partial \mathbf{B}^n$ is not empty. We may also ask the same questions about the spaces of maps with fixed boundary value : For $\varphi \in C^\infty(\partial \mathbf{B}^n, N)$, admitting a smooth extension $\phi : \mathbf{B}^n \to N$, we define

$$C^\infty_\varphi(\mathbf{B}^n, N) := \{u \in C^\infty(\mathbf{B}^n, N)\,; \quad u = \varphi \text{ on } \partial \mathbf{B}^n\}$$

and

$$W^{1,1}_\varphi(\mathbf{B}^n, N) := \left\{u \in W^{1,1}(\mathbf{B}^n, N)\,; \quad u = \varphi \text{ a.e. on } \partial \mathbf{B}^n\right\}.$$

Theorem 1 bis *Let $\mathbf{B}^n$ be the n-dimensional unit disk and N be any k-dimensional closed manifold. Assume that $\varphi \in C^\infty(\partial \mathbf{B}^n, N)$ is smoothly extendable into $\mathbf{B}^n$. Then for every $u \in W^{1,1}_\varphi(\mathbf{B}^n, N)$ there is a sequence of maps $u_m \in C^\infty_\varphi(\mathbf{B}^n, N)$ such that ∇u_m tend to ∇u in the biting sense.*

■

As a simplified example, consider the space $W^{1,1}(\mathbf{B}^n, \mathcal{S}_2)$, where $\mathcal{S}_2 := S^1_a \vee S^1_b$ is the bouquet of two circles based on the point $w \in \mathbb{R}^2$. $\pi_1(\mathcal{S}_2, w)$ is the free (thus non-abelian) group generated by two generators a and b. Let p^a and p^b be the projection of $\mathcal{S}_2$ onto S^1_a and S^1_b. The idea is to associate to any sequence $u_m \in R^\infty(\mathbf{B}^n, \mathcal{S}_2)$, converging strongly to $u \in W^{1,1}(\mathbf{B}^n, \mathcal{S}_2)$, two points $y_a \in S^1_a$ and $y_b \in S^1_b$ such that

$$A_{u_m} := A^a_{u_m} \cup A^b_{u_m} := (p^a \circ u_m)^{-1}(y_a) \cup (p^b \circ u_m)^{-1}(y_b)$$

is a finite union of smooth submanifolds of $\mathbf{B}^n$ and that for a uniform constant $C > 0$

$$vol(A_{u_m}) \le C \int_{\mathbf{B}^n} |\nabla u| + C.$$

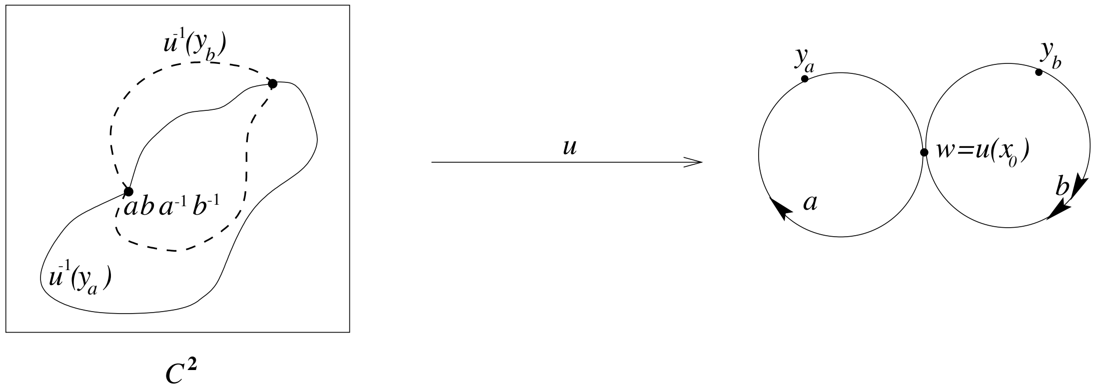

Fig.3
Inverse Images are good connecting sets for the dipole

Then the topological considerations detailed in the paper show that A_{u_m} satisfy the above necessary conditions for suitable connecting sets. Observe that as the image of these "connections" are constant in $\mathcal{S}_2$, the homotopy groups $\pi_1(\mathcal{S}_2, u_m(x))$ for $x \in A_{u_m}$ would have a fixed base point. For a visulaisation of this problem compare Figures 1,2 and 3.

For generalizing these results to any smooth compact manifold M as the domain one should be careful as there may be some global topological obstructions we did not consider in this paper. Refer to the recent work of F.Hang and F.H.Lin [20] where they show that the absence of "local" topological obstructions does not mean the approximability by smooth maps in the strong topology. We hope to extend these results to any domain by adapting our proofs to the new cases.

Finally we mention that the same questions about the density of smooth maps and the topological singularities can be asked about the functional spaces $H^{\frac{1}{2}}(\mathbf{B}^n, N)$, which is also an interesting case.

2 Preliminaries

2.1 The non-abelian fundamental group

Let N be a closed smooth manifold and $y, y' \in N$ two base points. Any curve

$$\gamma : [0,1] \to N$$

for which $\gamma(0) = y$ and $\gamma(1) = y'$, induces a natural isomorphism

$$\gamma_\# : \pi_1(N, y') \to \pi_1(N, y)$$

which depends only on the homotopy class of γ. If $\pi_1(N, y)$ is abelian, these isomorphisms are canonic, that is they do not depend on the choice of the curve γ. In this case we can talk about $\pi_1(N)$ without ambiguity. Otherwise, for referring to a specific element of $\pi_1(N)$, we are obliged to fix a base point for $\pi_1(N)$. Now let us assume that $y = y'$ and consider a curve γ as above. We have

$$\gamma_{\#}(a) = [\gamma]a[\gamma]^{-1}, \quad \forall a \in \pi_1(N, y) \tag{2.1}$$

where $[\gamma]$ is the homotopy class of γ in $\pi_1(N, y)$. Naturally if $\pi_1(N, y)$ is not abelian, these isomorphisms may not be trivial for $[\gamma] \neq 0$. See ([8], section VII.7) for more details.

2.2 The subspace $R^\infty(\mathbf{B}^n, N)$

Definition 2.1 *We say that $u \in W^{1,1}(\mathbf{B}^n, N)$ is in $R^\infty(\mathbf{B}^n, N)$ if u is smooth except on $B = \bigcup_{i=1}^m \sigma_i \cup B_0$, a compact subset of $\mathbf{B}^n$, where $\mathcal{H}^{n-2}(B_0) = 0$ and the σ_i , $i = 1, \cdots, m$ are smooth embeddings of the unit disk of dimension $n-2$. Moreover we assume that any two different faces of B, σ_i and σ_j, may meet only on their boundaries.*

Theorem 2 (Bethuel,[2]) *$R^\infty(\mathbf{B}^n, N)$ is dense in $W^{1,1}(\mathbf{B}^n, N)$ for the strong topology.*

$\blacksquare$

Definition 2.2 *Let $u \in R^\infty(\mathbf{B}^n, S^1)$ and let $B = \bigcup \sigma_i \cup B_0$ be the singular set of u. Suppose that each σ_i is oriented by a smooth $(n-2)$-vectorfield $\vec{\sigma}_i$. For $a \in \sigma_i$ let N_a be any 2-dimensional smooth submanifold of $\mathbf{B}^n$, orthogonal to σ_i at a. Consider the embedded disk $M_{a,\delta} = B_\delta(a) \cap N_a$ oriented by the 2-vectorfield $\vec{M}_a$ such that $(-1)^{n-1}\vec{\sigma}_i(a) \wedge \vec{M}_a$ is the fixed orientation of $\mathbf{B}^n$. Then the topological degree of u on the closed curve $\Sigma_{a,\delta} = \partial M_{a,\delta}$ is well defined and is independent of the choice of a and N_a for δ small enough. We call this integer the degree of u on σ_i and denote it by*

$$deg_{\sigma_i} u\,.$$

Theorem 3 (Almgren, Browder and Lieb, [1]) *Let $u \in R^\infty(\mathbf{B}^n, S^1)$, then for any regular value $y \in S^1$,*

$$\partial[[u^{-1}(y)]] - [[u^{-1}(y)]] \llcorner \partial\mathbf{B}^n = \sum_{i=1}^m (deg_{\sigma_i} u)\ [[\sigma_i]]$$

and

$$\int_{S^1} \mathcal{H}^{n-1}(u^{-1}(y))\, dy \le \int_{\mathbf{B}^n} |\nabla u|\,.$$

$\blacksquare$

3 Proof of theorem 1

As in the case where $\pi_1(N)$ is abelian, we should prove the existence of sets with bounded volume, connecting the singularities of a map in $R^\infty(\mathbf{B}^n, N)$, along which we can modify the map for removing its singularities. Meanwhile, for some technical reasons, we should use the same process for the elements of any strongly convergent sequence $u_m \in R^\infty(\mathbf{B}^n, N)$ when defining these sets.

Let us consider any map u$\in W^{1,1}(\mathbf{B}^n, N)$ and a sequence of maps $u_m \in R^\infty(\mathbf{B}^n, N)$ converging strongly to u. As we mentionned above, such a sequence always exist. We should show the existence of smooth maps $v_m : \mathbf{B}^n \to N$, such that ∇v_m tend in the biting sense to ∇u.

Step 1 : Projection of maps into some one skeleton of N

Consider some triangulation of N and for $1 \leq l \leq k$, let N^l be the l-skeleton of N. So $N = N^k$. Observe that by ([40], theorem (1.6), p. 215), the homomorphism

$$\chi : \pi_1(N^1, y) \to \pi_1(N, y), \tag{3.1}$$

induced by the injection map $i : N^1 \to N$, is onto. Also using ([17], Corollary 3.5, p. 38), N^1 is of the homotopy type of a bouquet of circles and we obtain that $\pi_1(N^1)$ is finitely generated. Let $f : N^1 \to \mathcal{S}_\beta := \bigvee_{i=1}^{\beta} S_i^1$ be a homotopy equivalence between N^1 and the bouquet of β circles, $S_1^1, \ldots, S_\beta^1$, embedded in some euclidean space and based on the fixed point w.

Definition 3.1 *We set*

$$U^l := \left\{ (x, y) \in \mathbf{B}^l \times \mathbf{B}^l \,;\, x \neq y \right\}.$$

For $(x, y) \in U^l$, we define $p(x, y)$ to be the unique point on $\partial \mathbf{B}^l$ which is on the ray from x to y.

Let us write

$$N^l = \bigcup_{i=1}^{s_l} \xi_i^l(\mathbf{B}^l),$$

where

$$\xi_i^l : \mathbf{B}^l \to N_i^l := \xi_i^l(\mathbf{B}^l),\ i = 1, \ldots, s_l$$

are diffeomorphisms and each two N_i^l are rather disjoint or intersecting on a lower dimensional face in N^{l-1}. Let $w \in N_1^l \times \cdots \times N_{s_l}^l$, $w = (w_1, ..., w_{s_l})$ be such that $w_i \notin N^{l-1}$. Define

$$p_w^l : N^l \backslash \{w_1, \ldots, w_{s_l}\} \to N^{l-1}$$

as follows :

$$p^l_w(y) := \begin{cases} \xi^l_i(p((\xi^l_i)^{-1}(w_i), (\xi^l_i)^{-1}(y))) & \text{if } y \in N^l_i \backslash N^{l-1} \\ \\ y & \text{otherwise} \end{cases}$$

where p is the projection defined in definition 3.1. Set for $1 \le i \le s_l$ and $0 < \varepsilon < 1$

$$N^l_{i,\varepsilon} := \xi^l_i \left(\mathbf{B}^l(0, 1-\varepsilon)\right)$$

and

$$N^l_\varepsilon := N^l_{1,\varepsilon} \times \cdots \times N^l_{s_l,\varepsilon}\,.$$

We proved in [31] that

$$\int_{N^l_\varepsilon} \int_{\mathbf{B}^n} |\nabla(p^l_w \circ u)(x)| dx\, dw \le C(l,\varepsilon) \int_{\mathbf{B}^n} |\nabla u|,$$

where $C(l,\varepsilon)$ is independent of u. Moreover, for any sequence of maps $u_m \in R^\infty(\mathbf{B}^n, N)$ converging to u we have

$$J_m := \int_{N^l_\varepsilon} \int_{\mathbf{B}^n} |\nabla(p^l_w \circ u_m)(x) - \nabla(p^l_w \circ u)(x)|\, dx\, dw \to 0 \quad \text{as} \quad m \to +\infty\,.$$

The proof is the same as the one given for $W^{1,1}(\mathbf{B}^2, \mathbb{RP}^2)$ in [31]. Meanwhile, observe that for fixed $w \in N^l$, the isomorphisms

$$\kappa_y := \gamma_\# : \pi_1(N, p^l_w(y)) \to \pi_1(N, y), \tag{3.2}$$

where $\gamma : [0,1] \to N$, $\gamma(0) = y$, $\gamma(1) = p^l_w(y)$ is any smooth curve, are independent of the choice of γ if its trajectory lies entirely in $(p^l_w)^{-1}(p^l_w(y))$. This is because any connected component of $(p^l_w)^{-1}(p^l_w(y))$ is simply-connected. Moreover, for any curve $\alpha : [0,1] \to N$, $\alpha(0) = \alpha(1) = y$, we have

$$\kappa_y \circ \chi \left([p^l_w \circ \alpha]\right) = [\alpha], \tag{3.3}$$

where χ is as in (3.1).

Proposition 3.1 *Let u and $u_m \in R^\infty(\mathbf{B}^n, N)$ be as above. Then, there are $w_l \in N^l_\varepsilon$, $1 < l \le k$, such that for all m*

(i) $u^{l-1} := p^l_{w_l} \circ u^l \in W^{1,1}(\mathbf{B}^n, N^l)$ *and* $u^{l-1}_m := p^l_{w_l} \circ u^l_m \in R^\infty(\mathbf{B}^n, N^{l-1})$

(ii) $u^{l-1}_m \to u^l$ *in* $W^{1,1}$

(iii) $\int_{\mathbf{B}^n} |\nabla u^{l-1}_m| \le K(l,\varepsilon) \int_{\mathbf{B}^n} |\nabla u| + K$

(iv) We have

$$\kappa_{u_m(x)} \circ \chi \left([u^1_m \circ \alpha]\right) = [u_m \circ \alpha],$$

where $\alpha : [0,1] \to \mathbf{B}^n$, $\alpha(0) = \alpha(1)$, is any smooth curve avoiding the singularities of u^1_m.

∎

Regarding the above statements, the proof of this proposition is straightforward.

Step 2 : Defining the inverse images which connect the singularities of u_m

Fix suitable $\varepsilon > 0$ and consider the sequence u_m^1 according to proposition 3.1. Observe that $u_m^1 = \mathcal{P} \circ u_m$ where

$$\mathcal{P} := p_{w_2}^2 \circ \ldots \circ p_{w_k}^k .$$

Set

$$\tilde{u}_m := f \circ u_m^1 : \mathbf{B}^n \to \mathcal{S}_\beta \, .$$

f can be assumed to be smooth, so $\tilde{u}_m \in R^\infty(\mathbf{B}^n, \mathcal{S}_\beta)$. Also, again by propositon 3.1, for some constant $C > 0$ independent of m

$$\int_{\mathbf{B}^n} |\nabla \tilde{u}_m| \leq C \int_{\mathbf{B}^n} |\nabla u| + C. \tag{3.4}$$

We have then

Proposition 3.2 *For $i = 1, \ldots, \beta$, there is $y_i \in S_i^1$, $y_i \neq w$, a regular value of $f \circ \mathcal{P}$, such that y_i is a regular value for any $\tilde{u}_m$ and that for a subsequence of u_m we have*

$$\mathcal{H}^{n-1}(\tilde{u}_{m_k}^{-1}(y_i)) \leq C' \int_{\mathbf{B}^n} |\nabla u| + C',$$

for $C' > 0$ independent of m.

■

Proof : Observe that we can project smoothly $\mathcal{S}_\beta$ on each of the circles $S_1^1, \ldots, S_\beta^1$. Composing $\tilde{u}_m$ with these projections we obtain maps $u_{m,i} : \mathbf{B}^n \to S_i^1$ of energies lesser than that of $\tilde{u}_m$. Also for $y \in S_i^1$, different from w, $\tilde{u}_m^{-1}(y) = u_{m,i}^{-1}(y)$. So by theorem 3 and (3.4) we obtain

$$\int_{S_i^1} \mathcal{H}^{n-1}(\tilde{u}_m^{-1}(y))\, dy \leq C \int_{\mathbf{B}^n} |\nabla u| + C.$$

Thus, by Fatou's lemma

$$\int_{S_i^1} \liminf_{m \to +\infty} \mathcal{H}^{n-1}(\tilde{u}_m^{-1}(y))\, dy \leq C \int_{\mathbf{B}^n} |\nabla u| + C.$$

As a result, the subset

$$\left\{ y \in S_i^1 \, ; \liminf_{m \to +\infty} \mathcal{H}^{n-1}(\tilde{u}_m^{-1}(y)) \leq \frac{1}{2\pi} (C \int_{\mathbf{B}^n} |\nabla u| + C) \right\}$$

is of positive measure in S_i^1. This, combined with Sard's theorem, completes the proof. ■

Now observe that we can write

$$\tilde{u}_m^{-1}(y_i) = \bigcup_{j=1}^{\mu_i} A_m^{i,j} \subset \mathbf{B}^n$$

and

$$(f \circ \mathcal{P})^{-1}(y_i) = \bigcup_{k=1}^{\nu_i} B^{i,k} \subset N$$

where $A_m^{i,j}$ and $B^{i,k}$, respectively the connected components of $\tilde{u}_m^{-1}(y_i)$ and $(\mathcal{P} \circ f)^{-1}(y_i)$, are smooth submanifold of $\mathbf{B}^n$ and N. Moreover, it is obvious that $u_m(A_m^{i,j}) \subset B^{i,k}$ for some $1 \leq k \leq \nu_i$.

Using the isomorphisms κ_y defined above, we want to associate a unique, well defined element of $\pi_1(N, y)$, $a_y^{i,k}$, to any $y \in B^{i,k}$. Since f is a homotopy equivalence, the $f^{-1}(y_i)$ are simply-connected. As a result, since $\mathcal{P}(B^{i,k}) \subset f^{-1}(y_i)$, the $B^{i,k}$ are simply-connected too (See (3.3)). Let $a^i \in \pi_1(\mathcal{S}_\beta, y_i)$ be the homotopy class representing the curves which make only one turn over S_i^1 in one fixed direction. Let $y' \in f^{-1}(y_i)$. Since f is a homotopy equivalence,

$$a_{y'}^i := (f_\#)^{-1}(a) \in \pi_1(N^1, y')$$

is well defined. We set for $y \in B^{i,k}$

$$a_y^{i,k} := k_y \circ \chi(a_{\mathcal{P}(y)}^i) \in \pi_1(N, y)$$

which is well defined by (3.2). Observe that by ([8], section VII, theorem 7.2), for any $\gamma : [0,1] \to B^{i,k}$ we have

$$\gamma_\#(a_{\gamma(1)}^{i,k}) = a_{\gamma(0)}^{i,k} \,. \tag{3.5}$$

Step 3 : Modifying a map along the connecting sets

Here is the main result of this step :

Proposition 3.3 *Let u_m and $A_m^{i,j}$ as above. Then there are maps $v_{m,m'} \in C^\infty(\mathbf{B}^n, N)$ such that*

$$\begin{cases} v_{m,m'} \xrightarrow{L^1} u_m \quad \text{as } m' \to \infty \\ \displaystyle\int_{\mathbf{B}^n} |\nabla v_{m,m'}| \leq \int_{\mathbf{B}^n} |\nabla u_m| + C \sum_{i=1}^{\beta} \sum_{j=1}^{\mu_i} \mathcal{H}^{n-1}(A_m^{i,j}) + O(\frac{1}{m'}) \end{cases}$$

for $C > 0$ independent of u.

■

This singularity removing proposition is proved using the same methods as in ([31], proposition 2.4) with slight modifications. The only major difference is that we need a new version of ([31], lemma 2.9) :

Lemma 3.1 *For every $1 \leq i \leq \beta$, and avery $1 \leq k \leq \nu_i$, there exists an open covering of $B^{i,k}$, $\{U_1^{i,k}, \dots, U_{r_{i,k}}^{i,k}\}$, and smooth maps*

$$\omega_r^{i,k} : [0,1] \times U_r^{i,k} \to B^{i,k}, \quad r = 1, \dots, r_{i,k}$$

such that

$$\begin{cases} \omega_r^{i,k}(0,y) = \omega_r^{i,k}(1,y) = y & \forall y \in B^{i,k} \\ [\omega_r^{i,k}(.,y)]_{\pi_p(N,y)} = a_y^{i,k} & \forall y \in B^{i,k} \\ \displaystyle\int_0^1 |\nabla_x \omega_r^{i,k}(.,y)| dx \leq C & \forall y \in B^{i,k} \\ |\nabla \omega_r^{i,k}|_\infty \leq C & \end{cases}$$

where $C > 0$ is independent of i and k.

■

Using the compatibility condition (3.5) the proof of this lemma is straightforward.

Step 4 : End of proof for theorem 1

Remember that $\tilde{u}_m^{-1}(y_i)$ is the distinct union of the $A_m^{i,j}$. So, by propositions 3.2 and 3.3, $v_{m,m}$ tend in L^1 to u and their gradients are equi-bounded in L^1 norm. By ([16], Vol I, section 1.2.7), $\nabla v_{m,m}$ converge in L^1 in the biting sense. Furthermore the limit can not be other than ∇u, since $v_{m,m}$ converge strongly to u in L^1. ■

Theorem 1 *bis* is proved following the same method.

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