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Weak bundle structures and a variational approach to Yang-Mills Lagrangians in supercritical dimensions

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Abstract

In this thesis we study the Yang-Mills energy of connections over singular G -bundles in dimensions higher than the celebrated dimension four. This functional has a very large invariance group, analogously to the classical parametric Plateau problem. The main goal of the thesis is to give a functional analytic framework in which the Yang-Mills functional becomes coercive and in which the Yang-Mills Plateau problem can be solved.

In the supercritical dimension 5 the tools introduced by K. Uhlenbeck to ensure coercivity do not work anymore and the control of minimizers must be done in suitable spaces of “singular bundles”.

We consider a space of weak connections on singular bundles \mathcal{A}_G , defined by requiring that on codimension-1 slices along spheres the connection forms can be identified with $W^{1,2}$ connections. By a new approximation result we characterize this space as the closure of the space \mathcal{R}^∞ consisting of locally smooth connections on bundles with finitely many topological defects. We then prove the sequential weak closure result which ensures the existence of local minimizers of the Yang-Mills energy in dimension 5. This implies that the space \mathcal{A}_G is the correct setting for the variational study of the Yang-Mills Plateau problem in dimension 5. Our methods are related to the proofs of the closure of rectifiable currents by slicing methods and of the closure theorem for rectifiable scans.

For the case of abelian connections, i.e. when the structure group is $U(1)$, we prove the sequential weak closure of the class of weak curvatures $\mathcal{F}_\mathbb{Z}^p$ defined by requiring a non-local integrality condition on slices in 3 dimensions. In an equivalent formulation, we are required to prove the sequential weak- L^p closure for L^p -vector fields in \mathbb{R}^3 such that their fluxes through “almost all” spheres are integers.

In the case of weak $U(1)$ -curvatures we provide a definition of boundary trace which is preserved under sequential weak convergence. We then prove the optimal interior regularity of minimizers of the Yang-Mills Plateau problem, i.e. we obtain that they are Hölder outside a set of isolated points. Our proof is essentially new, since for the main step we utilize a combinatorial method based on Smirnov's decomposition of 1-currents, instead of the classical energy estimates.

Finally, we provide an optimal new result concerning the existence of energy-controlled global gauges for $SU(2)$ -bundles in dimension 4. Uhlenbeck's coercivity estimate cited above states that under a smallness condition on the L^2 -energy of the curvature, a gauge is found in which the connection's $W^{1,2}$ norm is controlled by such energy. Therefore this result can be applied locally and determines the locations of "bubbling sets" for G -bundles. We provide here an optimal analogue in the case where no bound on the curvature is assumed. In this case we find a global gauge in which we can bound the $L^{4,\infty}$ -norm of the connection form in terms of the energy of the curvature. Such global controlled gauges exist even in the case of "bubbling", while Uhlenbeck's gauges exist just locally outside the bubbling points.

The existence of controlled global gauges is based on a new controlled extension result for maps $u \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$. For such a map we construct an extension to $\tilde{u} : B^4 \rightarrow \mathbb{S}^3$ with a norm control on \tilde{u} in the (optimal) Lorentz-Sobolev space $W^{1,(4,\infty)}(B^4, \mathbb{S}^3)$ in terms of the $W^{1,3}$ norm of u . We also prove analogous optimal controlled extension results for the cases of \mathbb{S}^1 and \mathbb{S}^2 .

We include several appendices in which we review some related topics, giving links between the main topics of this thesis and other fields of research.

Riassunto

Questa tesi verte sullo studio del funzionale di Yang-Mills su fibrati singolari in dimensione maggiore di 4. Tale funzionale ha un gruppo di invarianza molto vasto, e ciò crea un interessante parallelo con il problema di Plateau parametrico. Lo scopo principale di questa tesi è quello di stabilire un quadro generale in cui la minimizzazione analoga al problema di Plateau per il funzionale di Yang-Mills possa essere affrontato utilizzando il metodo diretto del Calcolo delle Variazioni.

Ci concentreremo sulla dimensione 5, nella quale gli strumenti introdotti da K. Uhlenbeck per assicurare la coercività del funzionale di Yang-Mills cessano di essere applicabili, costringendoci in particolare a lavorare in classi di “fibrati singolari” nuovi rispetto alla teoria in dimensione inferiore.

Utilizzeremo lo spazio \mathcal{A}_G , definito richiedendo che lungo le slice di codimensione 1 le nostre forme di connessione siano gauge-equivalenti a connessioni $W^{1,2}$ nel senso di Uhlenbeck. Dimostreremo quindi un nuovo teorema di approssimazione che ci consente di identificare lo spazio sopra definito con la chiusura, rispetto un’opportuna distanza legata alla topologia forte L^2 , dello spazio \mathcal{R}^∞ costituito da connessioni localmente lisce su fibrati aventi un numero finito di difetti topologici. Inoltre dimostrando la chiusura per convergenza debole dello spazio \mathcal{A}_G . Conseguentemente il funzionale di Yang-Mills raggiunge il suo minimo nello spazio \mathcal{A}_G , dimostrando la buona positura del problema di Plateau per il funzionale di Yang-Mills in dimensione 5. I metodi della nostra dimostrazione sono collegati a quelli per la dimostrazione per slicing della chiusura flat delle correnti rettificabili e a quelli per la dimostrazione della chiusura dello spazio degli scan rettificabili.

Nel caso di curvature abeliane, ossia quello in cui il gruppo di gauge è $U(1)$, dimostreremo in dimensione 3 la chiusura debole sequenziale di uno spazio $\mathcal{F}_{\mathbb{Z}}^p$ di curvature definite richiedendo che sia verificata una condizione non locale

di interesse sulle slices. In una formulazione equivalente, il nostro risultato dimostra la chiusura per convergenza debole in L^p della classe composta dai campi vettoriali L^p su \mathbb{R}^3 il cui flusso attraverso “quasi ogni” sfera è un intero.

Nel caso abeliano su descritto, definiremo una nozione di traccia sul bordo, tale da essere preservata per convergenza debole. Inoltre dimostreremo il risultato di regolarità ottimale, valido i minimi del problema di Plateau relativo al funzionale di Yang-Mills: i minimi sono localmente Hölderiani al di fuori di un insieme di punti singolari isolati. La dimostrazione di questo risultato è essenzialmente nuova, in quanto nel passo principale utilizziamo un metodo combinatorio basato sulla decomposizione di Smirnov per 1-correnti normali, invece dei metodi classici.

Infine, dimostreremo un nuovo risultato concernente l’esistenza di gauge controllate e globali per fibrati con gruppo di gauge $SU(2)$ in dimensione 4. Ricordiamo che il risultato di coercività di Uhlenbeck garantisce l’esistenza di gauge in cui si può maggiorare la norma $W^{1,2}$ della connessione in funzione della norma L^2 della curvatura, sotto l’ipotesi però che quest’ultima norma non superi un certo valore. Dunque questo risultato definisce degli insiemi di “bubbling” topologico per fibrati. Qui dimostreremo invece l’esistenza di una gauge globale, in cui la connessione è maggiorata in norma $L^{4,\infty}$ in funzione della norma L^2 della curvatura senza ipotesi di piccolezza. Tali gauge sono chiamate “globali” in quanto, anche quando avviene il fenomeno di “bubbling”, esse continuano a esistere globalmente e non soltanto localmente fuori dai punti di bubbling come le gauge di Uhlenbeck.

Il teorema di esistenza di gauge globali controllate è basato su un nuovo risultato ottimale di estensione per mappe di Sobolev $u \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$. Per tali funzioni costruiremo estensioni $\tilde{u} : B^4 \rightarrow \mathbb{S}^3$ maggiorate in norma $W^{1,(4,\infty)}$ in funzione della norma $W^{1,3}$ di u . Questo risultato è ottimale. Dimostreremo risultati analoghi anche per spazi di mappe di Sobolev fra sfere di dimensioni inferiori.

In alcune appendici descriveremo argomenti collegati, cercando di stabilire legami con altri ambiti.

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Chapter 1

Introduction

Let G be a compact connected Lie group with Lie algebra \mathfrak{g} . Let (M, h) be a compact Riemannian manifold. Over M we consider a principal G -bundle $P \rightarrow M$ and the associated vector bundle $E \rightarrow M$ issued from the adjoint representation $\text{Ad} : G \rightarrow \mathfrak{g}$. A connection on E determines a covariant derivative ∇ assigning sections of $E \otimes T^*M$ to sections of E . ∇ in turn determines an exterior derivative d_∇ which sends E -valued k -forms into E -valued $k + 1$ -forms, i.e. $d_\nabla : \Omega^k(E) \rightarrow \Omega^{k+1}(E)$. We denote $d_\nabla d_\nabla := F$, the curvature form of ∇ . Thus F is an $\text{ad}(E)$ -valued 2-form on M , which can be identified with a \mathfrak{g} -valued 2-form. For notations and conventions regarding G -bundles we refer to [47]. The *Yang-Mills functional* is then defined as

$$\mathcal{YM}(\nabla) = \int_M |F|^2,$$

where $|\cdot|$ is the norm on \mathfrak{g} -valued 2-forms obtained naturally from the Killing form on \mathfrak{g} and from the metric h on M . A connection ∇ is called *Yang-Mills* if it satisfies the Euler-Lagrange equation of critical points of \mathcal{YM} , i.e.

$$d_\nabla^* F_\nabla = 0. \tag{1.1}$$

This equation forms a nonlinear elliptic system if combined with the Bianchi identity

$$d_\nabla F_\nabla = 0.$$

Since Donaldson's work [46] on the invariants of differentiable structures over 4-manifolds an increasing interest has been directed towards the study of Yang-Mills connections, see e.g. [48] and [126] and the references therein. There are several methods available for constructing Yang-Mills connections in 4 dimensions, besides the variational point of view just described. These include the gluing techniques of C. H. Taubes [125] and algebro-geometric methods [11].

In this thesis we pursue a *variational study* of the functional \mathcal{YM} . Our results will be expected to help create examples of Yang-Mills connections in 5 dimensions, a case where far less constructions and examples are available.

Note that for a measurable gauge change $g : M \rightarrow G$, the curvature F_A transforms into $g^{-1}F_Ag$ and we have $|g^{-1}Fg| = |F|$ as a consequence of the invariance of the norm $|\cdot|$. Therefore \mathcal{YM} is also invariant under the group of measurable gauge transformations

$$\mathcal{G} := \{g : M \rightarrow G, g \text{ measurable}\}.$$

This fact establishes a strong analogy between the variational study of \mathcal{YM} and the study of minimal surfaces. Keeping this model problem in mind gives, we think, a good idea of our overall philosophy, thus we pass to describe it.

1.1 The Plateau problem

We recall the parametric formulation of the classical Plateau problem:

Problem 1 (Plateau problem). *Fix a simple closed Jordan curve $\gamma \subset \mathbb{R}^3$, i.e. an injective continuous image of \mathbb{S}^1 into \mathbb{R}^3 . Study the following variational problem:*

$$\inf \{A(u) : u : D^2 \rightarrow \mathbb{R}^3, u \text{ is an immersion}, u|_{\partial D^2} \in \text{Diff}^+(\partial D, \gamma)\}.$$

where $A(u) := \int_{D^2} |\partial_x u \wedge \partial_y u| dx \wedge dy$ is the area of the image of such u and $\text{Diff}^+(\partial D, \gamma)$ is the space of orientation-preserving diffeomorphisms.

The area functional, like \mathcal{YM} , also has a very large invariance group, i.e. the group of orientation-preserving diffeomorphisms $\text{Diff}^+(D^2)$. There are two celebrated strategies to avoid the lack of compactness which this entails:

- Reduce to the case where u is conformal, via reparameterization. For conformal maps u the area functional is equal to the more coercive energy functional $E(u) = \frac{1}{2} \int_{D^2} |du|^2$. The approach of Douglas and Radò to the Plateau problem follows this lead, proving the equivalence of the minimization of area and energy.
- Study the area functional directly on the class of oriented 2-dimensional submanifolds with fixed boundary γ . This is the approach by Federer and Fleming. They introduced a distributional notion of submanifolds,

called currents, for which coercivity of mass (i.e. the natural extension of the area of submanifolds) holds with respect to the weak topology. The more restrictive class of integral currents contains usual submanifolds and satisfies a closure theorem under the above weak convergence. Therefore a minimizer exists in such class. A regularity theory for minimizing currents then completes the study of the above Plateau problem.

In the case of \mathcal{YM} one needs to exploit both kinds of strategies: the first one corresponds in our case to finding Coulomb gauges, in which the connection is controlled by the curvature. The second strategy corresponds to our definitions of weak curvatures and bundles on which sequential weak closure results are true at the same time as the coercivity of \mathcal{YM} . We also proved the optimal regularity theorem for abelian G .

1.1.1 Plateau problem for the Yang-Mills functional

By analogy to Problem 1, we study the Yang-Mills functional by looking at a variational problem defined up to a global gauge.

We first recall some notation. If ∇_0 represents a smooth connection on E then any other smooth connection can be written as $\nabla = \nabla_0 + A$ where A is a section of $\wedge^1 TM \otimes ad(E) = \wedge^1 TM \otimes \mathfrak{g}$. Such A changes into $A^g := g^{-1}dg + g^{-1}Ag$ under a gauge transformation $g : M \rightarrow G$.

Problem 2 (Yang-Mills Plateau problem). *Let (M, h) be a compact manifold with nonempty boundary ∂M . Fix a G -bundle $E \rightarrow M$ as above and consider a fixed connection ∇_0 on E . Study the following minimization problem:*

$$\inf \{ \mathcal{YM}(A) : \exists g \in \mathcal{G} \text{ such that } i_{\partial M}^* A = (i_{\partial M}^* A_0)^g \}. \quad (1.2)$$

We start with some heuristic computations which help to understand which are the right function spaces that we have to consider.

1.1.2 Natural function spaces for the Yang-Mills Plateau problem

In order to have coercivity of the functional \mathcal{YM} the natural choice of a function space will be one in which the curvature form F is in L^2 . We desire to find the natural space for the \mathfrak{g} -valued connection 1-forms A . If \wedge denotes the combination of the usual wedge product of differential forms with the Lie bracket $[\cdot, \cdot]$ on \mathfrak{g} then F_A can be locally expressed by the \mathfrak{g} -valued 2-form

$$F = dA + A \wedge A.$$

Thus we see that it is natural to consider $A \in W^{1,2} \cap L^4$: then $|dA|$ will belong to L^2 , as will do the nonlinear term $A \wedge A$, which is bounded by $|A|^2$.

By representation theory we may assume that G is a group of matrices $G < SO(n)$, in particular G has an isometric immersion $G \rightarrow \mathbb{R}^{n \times n}$ which respects the group operation. This allows to define spaces of Sobolev maps with values in G as usual:

$$W^{k,p}(M, G) := W^{k,p}(M, \mathbb{R}^{n \times n}) \cap \{u \text{ s. t. } u(x) \in G \text{ a.e. } x \in M\}.$$

Therefore we automatically have $W^{k,p}(M, G) \subset L^\infty(M, \mathbb{R}^{n \times n})$.

Since after a change of trivialization g we have $A' = g^{-1}dg + g^{-1}Ag$, it is natural to assume the regularity $g \in W^{2,2}$, which has the consequence that

$$A \in W^{1,2} \cap L^4 \Leftrightarrow A' \in W^{1,2} \cap L^4.$$

We are led to state the following definition:

Definition 1.1 ($W^{2,2}$ -bundle). *Let (M, h) be a 4-dimensional compact Riemannian manifold, possibly with boundary. Fix an atlas (U_i, ϕ_i) on M . A $W^{2,2}$ -bundle over M is identified by a collection of changes of trivialization $g_{ij} \in W^{2,2}(U_i \cap U_j, G)$ satisfying the following cocycle condition for all i, j, k :*

$$g_{ij}g_{jk} = g_{ik} \text{ on } U_i \cap U_j \cap U_k.$$

If the data g_{ij} are smooth functions, then we recover the definition of a smooth G -bundle. The embedding $W^{2,p} \rightarrow C^0$ is true only in dimension $n < p/2$ thus for $p = 2$ we have that $W^{2,2} \rightarrow C^0$ for $n \leq 3$ and $W^{2,2} \rightarrow \text{VMO}$, which ensures the preservation of topology [33], for $n = 4$. Therefore in dimension $n \leq 3$ the $W^{2,2}$ -bundles are homeomorphic to smooth bundles and in dimension $n = 4$ we still have a control on the topology [131]. In dimension $n \geq 5$ we will be instead forced to work on *singular bundles* with some control on the topology possibly only on 4-dimensional slices.

We may define $W^{1,2}$ -connections on $W^{2,2}$ -bundles as follows:

Definition 1.2 (The space $\mathcal{A}^{1,2}$ of $W^{1,2}$ -connections). *Let (M, h) be a 4-dimensional compact Riemannian manifold, possibly with boundary. Fix an atlas (U_i, ϕ_i) on M . Let a $W^{2,2}$ G -bundle $E = E_{W^{2,2}}$ over M be given by data (g_{ij}) as in the previous definition. A $W^{1,2}$ -connection over E is given by a collection of 1-forms $A_i \in W^{1,2} \cap L^4(U_i, T^*U_i \otimes \mathfrak{g})$ such that for all i, j*

$$A_i = g_{ij}^{-1}dg_{ij} + g_{ij}^{-1}A_jg_{ij} \text{ on } U_i \cap U_j.$$

The space of such collections $A = (A_i)_i$ for fixed E will be denoted $\mathcal{A}^{1,2}(E)$.

The union of such spaces $\mathcal{A}^{1,2}(E)$ over all $W^{2,2}$ -bundles E over M is denoted $\mathcal{A}^{1,2}(M)$.

A global connection form $A \in L^2(M, T^*M \otimes \mathfrak{g})$ represents a $W^{1,2}$ -connection over a fixed $W^{2,2}$ -bundle $E \rightarrow M$ if corresponding to some given atlas as above there exist A_i and g_{ij} as above such that for all i we can find functions $g_i \in W^{1,2}(U_i, G)$ such that $A_i = g_i^{-1}dg_i + g_i^{-1}Ag_i$ on U_i .

We denote the spaces of such connections for fixed E (or fixed M) by $\mathcal{A}^2(E)$ (respectively $\mathcal{A}^2(M)$).

We next define the curvature in this case:

Definition 1.3 (Curvature of a $\mathcal{A}^{1,2}$ -connection). *The curvature form $F_A \in L^2(M, \wedge^2 TM \otimes \mathfrak{g})$ corresponding the connection data $A = (A_i)_i \in \mathcal{A}^{1,2}(E)$ to be the collection of the local data $F_i := dA_i + A_i \wedge A_i$ on U_i . Then we automatically have for all i, j*

$$F_j = g_{ij}^{-1}F_i g_{ij} \text{ on } U_i \cap U_j.$$

Observe that $|F_i| = |F_j|$ on $U_i \cap U_j$ by the *ad*-invariance of the norm, thus also in this case $\mathcal{YM}(A)$ can be computed on $A \in \mathcal{A}^{1,2}(E)$ with no ambiguity. We can examine the minimization problem (1.2) on the space $\mathcal{A}^{1,2}(E)$ on a smooth bundle E .

Good behavior in subcritical dimension $n \leq 3$

In small dimension there exists a minimizer on a fixed bundle E :

Theorem 1.4. *If $\dim(M) \leq 3$ then the problem (1.2) has a minimizer in the class $\mathcal{A}^{1,2}(E)$ for each fixed smooth bundle E .*

The proof is a consequence of the embedding $W^{2,2} \subset C^0$ and of the fact that for each i the L^2 -norm of F_i controls $\|A_i\|_{W^{1,2}}$ in regions where no energy concentrates. This phenomenon, proved by Uhlenbeck, will resurface later on, since it is an important source of *coercivity*. **This source of coercivity fails in dimensions $n \geq 5$** , as we will see.

Theorem 1.5 ([132]). *Let $n \leq 4$ and consider a trivial bundle $E \rightarrow K$ over a n -manifold K . There exists ϵ_0 depending only on n with the following property. Assume that $A \in W^{1,2}(K, T^*K \otimes \mathfrak{g})$ and that in some trivialization the curvature form $F := dA + A \wedge A$ satisfies*

$$\|F\|_{L^2(K)} \leq \epsilon_0. \quad (1.3)$$

Then there exists a local gauge transformation $g \in W^{2,2}(K, G)$ such that on K the new expression $A_{\text{coul}} := g^{-1}dg = g^{-1}Ag$ of the connection satisfies

$$d^*A_{\text{coul}} = 0 \quad (1.4)$$

and for C depending only on n ,

$$\|A_{\text{coul}}\|_{W^{1,2}(K)} \leq C\|F\|_{L^2(K)}. \quad (1.5)$$

Remark 1.6. *Because of the Sobolev embedding $W^{1,2} \rightarrow L^{\frac{2n}{n-2}}$ in dimension n , we see that the control (1.5) implies a control in L^4 precisely in dimensions $n \leq 4$.*

Once we know this result, to prove Theorem 1.4 we may decompose the domain into regions K_i (covering all M but finitely many points) on which the curvature satisfies the smallness condition (1.3) for all large k . Then up to subsequence the A_i^k converge weakly in $W^{1,2}$ in these regions. The A_i^k control the g_{ij}^k via the relations from Definition 1.2. By the embedding $W^{2,2} \rightarrow C^0$ valid for $n \leq 3$ we deduce that the g_{ij}^k converge uniformly, thus the limit bundle is still E .

Bubbling in critical dimension $n = 4$

The problem with the above reasoning in critical dimension $n = 4$ is that in this case the g_{ij}^k are not controlled in C^0 anymore. We have a *bubbling phenomenon*, i.e. the bundle changes topology in the limit. We still have the control (1.3) uniformly in k outside a finite set of (quantized) energy concentration points.

Due to Uhlenbeck [132] a classical bundle can be recovered if a $W^{1,2}$ -connection exists, under the assumption that (1.1) holds. From Rivière's Lorentz space techniques [107] it follows that the point removability also holds in general:

Theorem 1.7 (Point removability, cfr. [107]). *Let ∇ be a $W^{1,2}$ -connection on a smooth bundle E over $B^4 \setminus \{0\}$. If the L^2 norm of the curvature F of ∇ is finite, then there exists a gauge in which the bundle P extends to a smooth bundle \tilde{P} over B^4 and the connection ∇ extends to a $W^{1,2}$ connection $\tilde{\nabla}$ over B^4 .*

We present the proof in Theorem 6.2, since it is not proved in the literature. The consequence of this result for the problem (1.2) is the following:

Theorem 1.8. *Let $\dim(M) = 4$, M be a compact Riemannian n -manifold with nonempty boundary ∂M . Fix a G -bundle $E \rightarrow M$. In general, a sequence (A_i^k, g_{ij}^k) of data as in the definition of $\mathcal{A}^{1,2}(E)$ which minimizes $\mathcal{YM}(A)$ under the trace constraint as in Problem (1.2) will have a weakly convergent subsequence, i.e. up to subsequence*

$$g_{ij}^k \xrightarrow{W^{2,2}} g_{ij}^\infty, \quad A_i^k \xrightarrow{W^{1,2}} A_i^\infty$$

and $(A_i^\infty, g_{ij}^\infty)$ are the data of an element of $\mathcal{A}^{1,2}(\tilde{E})$ where \tilde{E} is obtained by modifying E over small neighborhood of a finite number of points $p_1, \dots, p_k \in M$.

This result is analogous to the ‘‘topological bubbling’’ phenomenon first discovered by Sacks and Uhlenbeck [112] in the case of minimal immersions of surfaces. We describe some new results on the control of Coulomb gauges in 4-dimensions in Section 1.6.

Loss of regularity in supercritical dimension 5

The proof of existence of minimizers for Problem (1.2) in dimension $n = 5$ is one of the main results of this thesis. We describe here the kinds of singularities which play a central role. We will see that not only a loss of control on the topology of the underlying bundles happens, but a more drastic loss of control on the regularity of our connections takes place.

Example 1.9. *Fix a topologically nontrivial $SU(2)$ -bundle E over \mathbb{S}^4 , e.g. the simplest $SU(2)$ -instanton having $c_2(E) = 1 \in \mathbb{Z} \sim H^4(\mathbb{S}^4)$. See [58], Ch. 6 for notations and details. Recall in particular that we may use quaternion notation due to the isomorphisms $SU(2) \sim Sp(1)$ and $su(2) \sim Im\mathbb{H}$, under which Pauli matrices correspond to quaternion imaginary units. With these notations, E becomes isomorphic to the tautological \mathbb{H} -line bundle. Consider the connection corresponding to the \mathbb{R} -orthogonal projection on the vertical direction, i.e.*

$$\omega = Im(q_1 d\bar{q}_1 + q_2 d\bar{q}_2).$$

Then on a small ball B_r^5 we construct the curvature F via the projection $\pi : B_1^5 \setminus \{0\} \rightarrow \partial B_1^5$, by defining

$$F = (\pi^* F)|_{B_r^5}.$$

By examining the above formula and using the scale invariance of the Yang-Mills functional in dimension 4 we then see that the energy can be arbitrarily small:

$$\int_{B_r} |F|^2 = Cr.$$

Moreover the bundle has by construction a topological singularity at the origin.

The above singularities are analogous to the function $\frac{x}{|x|} : B^r \rightarrow \mathbb{S}^2$ which witnesses the fact that smooth maps are not dense in $W^{1,2}(B^3, \mathbb{S}^2)$ (see e.g. [18]).

We recall that the second Chern class $c_2(E) \in H^4(M, \mathbb{Z})$ for smooth $SU(2)$ -bundles is represented by the 4-form $\text{tr}(F \wedge F)$. By the theory of characteristic classes [74], for a smooth 4-dimensional submanifold $S \subset M$ we the topology of a $SU(2)$ -bundle $E|_S$ is represented by

$$c_2(E)[S] := \frac{1}{8\pi^2} \int_S \text{tr}(F \wedge F) \in \mathbb{Z}.$$

The bundle of above Example 1.9 satisfies $d(\text{tr}(F \wedge F)) = 8\pi^2 \delta_0$, i.e. a topological singularity is present at the origin.

Since singularities cost very little energy on small balls in dimension $n \geq 5$, Problem (1.2) cannot be studied in the above setting anymore, since the curvatures do not control the connections anymore. More rigorously, assume that we have a sequence of $SU(2)$ -curvatures F_k with the following properties:

- for each k , F_k is a *smooth* curvature on a *fixed* smooth bundle over the 5-ball, $E \rightarrow \mathbb{B}^5$, with corresponding smooth connection data (A_i^k, g_{ij}^k) ;
- $\|F_k\|_{L^2(\mathbb{B}^5)} \leq C$, uniformly in k ;
- for all k there holds $d(\text{tr} F_k \wedge F_k) = 0$.

Then in general the following bad behavior could take place:

- The convergence of the connection data (A_i^k, g_{ij}^k) can be controlled only in a very weak sense:

$$A_i^k \rightharpoonup A_i^\infty \text{ weakly in } L^2, \quad g_{ij}^k \rightharpoonup g_{ij}^\infty \text{ weakly in } W^{1,2}.$$

- The curvature defined distributionally by $F_i^\infty = dA_i^\infty + A_i^\infty \wedge A_i^\infty$ satisfies

$$\text{supp} (d\text{tr}(F_i^\infty \wedge F_k^\infty)) = \overline{\mathbb{B}^5}.$$

The reason for this is that the Sobolev embeddings behind Uhlenbeck's theorem 6.4 and behind the control of the g_{ij} via the formula $A_i = g_i^{-1}dg_i + g_i^{-1}A_i$ fail. We now introduce the setting in which we manage to recover coercivity by different methods.

1.2 Weak closure result in dimension 5

To study the problem (1.2) in 5 dimensions we pursue the direction which led Federer and Fleming to introduce integral currents to solve the Plateau problem. We will define a weaker class of connections and we will prove that such class is closed under L^2 weak convergence, therefore a minimizer of the Yang-Mills Plateau analogue (1.2) will exist in this class.

Definition 1.10. *Let M be a compact Riemannian 5-manifold. We define the class of weak connections of L^2 -curvatures on singular bundles over M as follows:*

$$\mathcal{A}_G(M) := \left\{ \begin{array}{l} F \in L^2(M, \wedge^2 TM \otimes \mathfrak{g}) \text{ such that} \\ \exists A \in L^2(M, T^*M \otimes \mathfrak{g}), dA + A \wedge A \stackrel{\mathcal{D}}{=} F \text{ and} \\ \forall p \in \mathbb{R}^5, \text{ for a.e. } \rho_{inj}(M) > r > 0, i_{\partial B_r(p)}^* A \in \mathcal{A}^{1,2}(\partial B_r(p)). \end{array} \right\}.$$

The number $\rho_{inj}(M)$ above is the injectivity radius of M and the symbol $\stackrel{\mathcal{D}}{=}$ means equality in the distributional sense.

We will often restrict to the case where M is the closed unit ball $\mathbb{B}^5 \subset \mathbb{R}^5$.

This class is suitable for posing the above Yang-Mills Plateau-type problem. Indeed we have:

Theorem 1.11 (Weak closure of \mathcal{A}_G). *Assume that we have a sequence of L^2 curvature forms F_n corresponding to*

$$[A_n] \in \mathcal{A}_G(\mathbb{B}^5)$$

such that

$$\sup_n \|F_n\|_{L^2(\mathbb{B}^5)} < \infty$$

and

$$F_n \rightharpoonup F \text{ in } L^2(\mathbb{B}^5, \wedge^2 \mathbb{R}^5 \otimes \mathfrak{g}).$$

Then F is the curvature form corresponding to some $[A] \in \mathcal{A}_G(\mathbb{B}^5)$ as well.

This theorem, obtained in collaboration with my advisor T. Rivière, is one of the main results of this thesis. For the proof see Chapter 8.

Ideas of the proof: controlling the oscillation of slices

To prove the above result we utilize a control on control on the slices given by Definition 1.10. This control is analogous to the *MBV* control needed in for the closure of integral currents and of rectifiable scans [8, 72] and we use the same abstract setting as those results.

We introduce a geometric distance on gauge classes of sliced connections $A, B \in \mathcal{A}^2(\mathbb{S}^4)$ as follows (see also [47]). We say that A is equivalent to B if for a measurable gauge $g : \mathbb{S}^4 \rightarrow G$ there holds $A = g^{-1}dg + g^{-1}Bg$. Let $[A], [B]$ be the so-obtained equivalence classes. Our distance is defined as follows:

$$\text{dist}([A], [B]) = \inf \{ \|A' - B'\|_{L^2(\mathbb{S}^4)} : A' \in [A], B' \in [B] \}.$$

We then consider the identification $i(t) : \mathbb{S}^4 \rightarrow \partial B_t(x_0) \subset \mathbb{R}^5$ and define $A(t) = i(t)^*A$ for $A \in L^2$. By Definition 1.10, for a.e. t the form $A(t)$ is L^2 and it belongs to $\mathcal{A}^2(\mathbb{S}^4)$. We have the following control, valid on any interval $I \Subset]0, \infty[$:

$$\text{dist}([A(t)], [A'(t)]) \leq C_I \|F\|_{L^2(\mathbb{B}^5)} |t - t'|^{\frac{1}{2}}, \quad \text{for } t, t' \in I.$$

We then use the following abstract theorem:

Theorem 1.12. *Consider a metric space $(\mathcal{Y}, \text{dist})$ on which a function $\mathcal{N} : \mathcal{Y} \rightarrow \mathbb{R}^+$ is defined. Suppose that the following hypothesis is met:*

$$\forall C > 0 \text{ the sublevels } \{\mathcal{N} \leq C\} \text{ are seq. compact.}$$

Suppose $A_n : [0, 1] \rightarrow \mathcal{Y}$ are measurable maps such that

$$\text{dist}(f_n(t), f_n(t')) \leq C |t - t'|^{1/2}$$

and

$$\sup_n \int_0^1 \mathcal{N}(f_n(t)) dt < C.$$

Then f_n have a subsequence which converges pointwise almost everywhere. The limiting function f also satisfies

$$\text{dist}(f(t), f(t')) \leq C|t - t'|^{\frac{1}{2}}, \quad \int_0^1 \mathcal{N}(f(t))dt < C.$$

We will prove in Section 8.4 that if we choose

$$\begin{aligned} \mathcal{Y} &= \{[A] \in \mathcal{A}^2(\mathbb{S}^4)/\sim\}, \\ f_n(t) &= [A_n(t)] \text{ slices as above and } \mathcal{N}([A]) := \int_{\mathbb{S}^4} |F_A|^2, \end{aligned}$$

then the hypotheses of Theorem 1.12 are verified.

The fact that the sublevels of \mathcal{N} are sequentially dist-compact is a consequence of Theorem 1.8 thus uses the point removability result of Theorem 1.7 from dimension 4. We then show that the distance dist on \mathcal{Y} provides enough control to extract a limit of the pointwise dist-converging sequence of slices $A_n(t)$. This gives $W^{1,2}$ -representatives for the slices of the limit connection A , concluding the proof.

1.3 Approximability by regular connections

In order to connect our Definition 1.10 of the class \mathcal{A}_G of weak L^2 -connection forms to classical connections and to show that it is the correct extension of the class of smooth connections on smooth bundles, we also prove a strong density result. The class of smooth connections on finitely-puncture bundles \mathcal{R}^∞ will have \mathcal{A}_G as strong closure. The precise definition is as follows:

Definition 1.13. *Let M^5 be a compact 5-manifold. We will denote by $\mathcal{R}^\infty(M)$ the following space*

$$\mathcal{R}^\infty(M) := \left\{ \begin{array}{l} F \in L^2(M, \wedge^2 TM \otimes \mathfrak{g}) \text{ s. t. } \exists p_1, \dots, p_N \in M \\ \exists E \rightarrow M \setminus \{p_1, \dots, p_N\} \text{ smooth } G\text{-bundle} \\ F \stackrel{\sigma, \text{loc}}{=} dA + A \wedge A, \text{ loc. smooth outside } \{p_i\}_{i=1}^N \end{array} \right\}.$$

The above notation $F \stackrel{\sigma, \text{loc}}{=} dA + A \wedge A$ signifies that we may find a local trivialization σ of the G -bundle $E \rightarrow M \setminus \{p_1, \dots, p_N\}$ in which F represents a smooth connection with connection form A .

In general for a $SU(2)$ -connection form belonging to $\mathcal{R}^\infty(M)$ we have the following control on the topology of the underlying bundle:

$$d(\operatorname{tr} F \wedge F) = 8\pi^2 \sum_{i=1}^N d_i \delta_{p_i}$$

where p_i are the singular points as in the above definition.

We then consider a geometric distance to compare gauge-equivalence classes of \mathfrak{g} -valued 2-forms on M^5 :

Definition 1.14 (Distance on L^2 -curvature classes). *On 2-forms*

$$F, F' \in L^2(M, \wedge^2 TM \otimes \mathfrak{g})$$

we impose the equivalence relation \sim defined as follows:

$$F \sim F' \text{ if there exists a measurable } g : M \rightarrow G \text{ such that } F = g^{-1} F' g.$$

We then define the following distance on equivalence classes of curvatures:

$$\operatorname{dist}([F], [F']) := \inf \{ \|F - g^{-1} F' g\|_{L^2(M)} : g : M \rightarrow G \text{ measurable} \}.$$

With these definitions we have the following approximation result:

Theorem 1.15. *Let M be a compact 5-manifold. Any L^2 curvature form F corresponding to a connection form A with $[A] \in \mathcal{A}_G(M)$ can be approximated by 2-forms F_n corresponding to $[A_n] \in \mathcal{R}^\infty(M)$ with respect to the pseudodistance $\operatorname{dist}([F], [F_n])$ of Definition 1.14.*

This result, proved in collaboration with my advisor T. Rivière, is one of the main results of this thesis. For the proof see Chapter 7. To illustrate the new difficulties, we make a small parenthesis describing previously known results.

Previous results and new difficulties

Results of the same kind as Theorem 1.15 were proved by Bethuel [19] for Sobolev maps between manifolds $u \in W^{1,p}(B^n, \mathbb{S}^{n-1})$ in supercritical dimension $n > p$ and by Kessel and Rivière [83, 84] for weak curvatures in supercritical dimension in the case of an *abelian* structure group G (cfr. also [5] and the references therein).

We cite such result in the case of $W^{1,2}(B^3, \mathbb{S}^2)$ -maps:

Theorem 1.16 ([19]). *For a given map $u \in W^{1,2}(B^3, \mathbb{S}^2)$ we can find maps*

$$u_n \in \mathcal{R}^{\infty,2}(B^3, \mathbb{S}^2) := \left\{ \begin{array}{l} u \in W^{1,2}(B^3, \mathbb{S}^2) : \exists \Sigma \subset B^3 \text{ finite, s.t.} \\ u \in C_{loc}^\infty(B^3 \setminus \Sigma, \mathbb{S}^2) \end{array} \right\}$$

such that $u_n \rightarrow u$ strongly in $W^{1,2}$.

The idea behind this result is to use finer and finer subdivisions (e.g. cubeulations) of B^3 to define the approximants. One then first approximates the restriction on the boundaries of the subdividing sets, where one can apply the classical results for subcritical dimensions. Then a Calderon-Zygmund type procedure is applied:

- On “good” small cubes where not much energy (i.e. not much L^2 -norm of $|\nabla u|$) concentrates it will be possible to extend the boundary approximation in a controlled way, by extending harmonically u into the ambient space $\mathbb{R}^3 \supset \mathbb{S}^2$ and then projecting on \mathbb{S}^2 . Since we control ∇u , the map to be approximated does not oscillate much thus it is well approximated by such extension.
- On the remaining “bad” cubes we extend the boundary approximation radially, creating one single singularity. We will have just a bound on the approximant (in terms of the average of ∇u) and not a bound on the approximation error. However the total volume of bad cubes is doomed to become negligible in the limit, since a quantized amount of energy is concentrated on each cube. This provides the basis for a suitable dominated convergence result, showing that as the subdivisions refine, the approximants converge strongly.

Since each so-constructed approximant is continuous except at finitely many centers of bad cubes, we can then apply classical smoothing methods to improve the local regularity to C^∞ .

The new difficulty in the case of non-abelian structure groups is that with each local approximation on a boundary of a cube comes a particular gauge g in which the approximation is valid. Such gauges interact wildly and an important new difficulty is to control all their impacts at the same time, while getting a control on the connection forms.

Ideas of our proof

We also consider cubical grids of smaller and smaller sizes for the approximation. By using a partition of unity and the fact that by definition $[A] \in \mathcal{A}_G$ is

equivalent to a $\mathcal{A}^{1,2}$ -connection on 4-dimensional slices, we can perform the approximation on the skeletons of critical dimension 4 by classical methods.

We define “good” cubes the cubes C of the grid for which the following quantities defined in terms of A, F and the average \bar{F} of F on C

$$\int_{\partial C} |F - \bar{F}|^2, \quad \int_{\partial C} |F|^2 \quad \text{and} \quad \int_{\partial C} |A|^2$$

satisfy suitable scale-invariant smallness conditions. On each boundary of good cube we then apply Uhlenbeck’s [132] result cited in Theorem 1.5, which gives locally a gauge change g_i such that

$$d^*(A^{g_i}) = 0, \quad \|A^{g_i}\|_{W^{1,2}} \leq C \|F\|_{L^2},$$

$$i^* F^{g_i} \stackrel{D}{=} dA^{g_i} + A^{g_i} \wedge A^{g_i}.$$

We desire to use these gauges to control the approximation process, because of the $W^{1,2}$ -control that they give on the local connection forms. Then we perform harmonic extensions $\tilde{g}_i, \tilde{A}^{g_i}$ of g_i, A^{g_i} and we show that $F_{\tilde{A}^{g_i}}$ approximates $(\bar{F})^{g_i}$.

We thus end up with $i^*(\tilde{A}^{g_i})^{\tilde{g}_i^{-1}} = i^*A$ on the boundary. On the other hand the curvature $\tilde{g}_i(F_{\tilde{A}^{g_i}})^{\tilde{g}_i^{-1}}$ of $(\tilde{A}^{g_i})^{\tilde{g}_i^{-1}}$ still approximates \bar{F} since in general $|g^{-1}F_1g - g^{-1}F_2g| = |g^{-1}(F_1 - F_2)g| = |F_1 - F_2|$ by the invariance of the norm.

If we perform a smoothing on the 4-skeleton before applying this procedure we will also have that the approximating connections are continuous up to gauge, thus we can apply a classical mollification to conclude.

1.4 Abelian curvatures and vector fields with integer fluxes

In this section we consider the case of the abelian group $G = U(1) = \{e^{i\theta} : \theta \in \mathbb{S}^1\}$ in dimension $n = 3$. In this case $\mathfrak{g} \sim \mathbb{R}$ and A, F are respectively 1- and 2-forms in the usual sense.

The motivation for studying the Yang-Mills functional in this case is that it provides a *simplified model* for the behavior of general curvatures, due to

the *absence of nonlinearities*: In this case the local expression of connection and curvature are expected to satisfy

$$F = dA.$$

This simplification allows to investigate connections to several other fields of research, furnishing several wide open directions in which generalizations involving the (nonlinear) nonabelian connections could point.

A second simplification is the fact that not only the expression of the curvature F is gauge invariant, but *the local expression of F is completely independent of the gauge*.

For the abelian case we will introduce the Yang-Mills functional with exponent p allowed to be different than 2:

$$\mathcal{YM}_p(F) := \int_M |F|^p.$$

We start by giving a definition of weak curvatures which looks different than Definition 1.10:

Definition 1.17 (Abelian weak curvatures). *An L^p -curvature of a singular $U(1)$ -bundle over B^3 is a measurable real-valued 2-form F satisfying*

•

$$\int_{B^3} |F|^p dx^3 < \infty,$$

• For all $x \in B^3$ and for almost all $0 < r < \text{dist}(x, \partial B^3)$ we have

$$\frac{1}{2\pi} \int_{\partial B_r(x)} i_{\partial B_r(x)}^* F \in \mathbb{Z},$$

where $i_{\partial B_r(x)}^*$ is the inclusion map of $\partial B_r(x)$ in B^3 .

We call $\mathcal{F}_{\mathbb{Z}}^p(B^3)$ the class of all such 2-forms F .

We now explain why such weak curvatures in dimension 3 and with exponents $p < 3/2$ give the closest analogy to nonabelian L^2 -curvatures corresponding to connection classes $[A] \in \mathcal{A}_G$ in 5-dimensions.

Topological singularities and dimension 3

In the nonabelian case topological singularities were naturally appearing in 5 dimensions due to the scaling behavior of the functional \mathcal{YM} . For the most commonly appearing group $SU(2)$ we have that the second Chern class $c_2 \in H^4(M, \mathbb{Z})$ characterizes $SU(2)$ -bundles (i.e. complex Hermitian bundles E of rank 2) over closed manifolds M up to isomorphism (see [74]). By Chern-Weil theory (see [85]) the value of c_2 on a 4-cycle Σ^4 can be expressed via the curvature F of the bundle restricted to Σ^4 as follows:

$$c_2(E)[\Sigma^4] = \frac{1}{8\pi^2} \int_{\Sigma^4} \text{tr}(F \wedge F) \in \mathbb{Z}.$$

In order to reproduce a similar example for $U(1)$ -bundles we look after the *characteristic class which helps distinguishing $U(1)$ -vector bundles* (i.e. complex Hermitian line bundles). Such class is the first Chern class $c_1 \in H^2(M, \mathbb{Z})$. By Chern-Weil theory the value of c_1 on a 2-cycle Σ^2 is represented by the curvature F as

$$c_1(E)[\Sigma^2] = \frac{1}{2\pi} \int_{\Sigma^2} F \in \mathbb{Z}.$$

Since c_1 is 2-dimensional it is therefore possible to imitate the construction of Example 1.9 and *obtain topological point-singularities* precisely when the complement of a point has nontrivial 2-dimensional homology. This is why we work in *dimension 3*.

Equivalence of different definitions of $\mathcal{F}_{\mathbb{Z}}^p(B^3)$

Our definition of L^p -weak curvatures just requires the integrality of the first Chern class to hold on spherical slices. We note that passing from Definition 1.17 to an analogue of the 5-dimensional Definition 1.10 does not change the space $\mathcal{F}_{\mathbb{Z}}^p(B^3)$:

Proposition 1.18. *For each L^p -integrable curvature 2-form F on a Riemannian 3-manifold M the following two properties are equivalent:*

- $F \in \mathcal{F}_{\mathbb{Z}}^p(M)$ according to Definition 1.17.
- There exists an L^p -connection form A on B^3 such that for all centers $x_0 \in B^3$ and a.e. $\rho_{inj} > r > 0$ the slice $i_{\partial B_r(x_0)}^* A$ is gauge-equivalent to a locally $W^{1,p}$ connection form.

This result is a straightforward consequence of the next theorem:

Theorem 1.19 (T. Kessel, T. Rivière [83, 84]). *Let $p \geq 1$ and let M be a Riemannian 3-manifold. Any weak L^p -curvature over a singular $U(1)$ -bundle as defined above can be approximated in the strong L^p -norm by elements of $\mathcal{R}_\infty(M)$, i.e. by smooth curvatures on bundles over the finitely punctured manifold M .*

Proof of Proposition 1.18: The proof of our non-abelian analogue stated in Theorem 1.15 shows that $\mathcal{R}_\infty(M)$ is dense also in the space of 2-forms satisfying the second definition. Both conditions stated in the proposition are closed under strong L^p -convergence. Theorem 1.19 then concludes the proof. \square

Choice of the interesting exponent p

For $p \geq 3/2$ any finite energy curvature will automatically have no topological singularity. In particular we have:

Theorem 1.20 ($n = 3$ is subcritical for $p > 3/2$). *Let M be a Riemannian 3-manifold with $\partial M \neq \emptyset$ and let $p > 3/2$ and let $E \rightarrow M$ be a smooth Hermitian line bundle. Then for each smooth curvature F_0 over ∂M the Yang-Mills Plateau problem*

$$\inf \{ \mathcal{YM}_p(A) : i_{\partial M}^* F = F_0 \} .$$

has a minimizer which is the curvature of a $W^{1,p}$ -connection on the same bundle E .

Thus dimension 3 is supercritical just for $p < 3/2$. From now on we restrict to this case.

1.4.1 Weak closure in the abelian case

One of the main results of this thesis, obtained in collaboration with my advisor Tristan Rivière is the following:

Theorem 1.21 (Weak closure of the space $\mathcal{F}_\mathbb{Z}^p(B^3)$). *Let $p > 1$ and assume that*

$$F_n \in \mathcal{F}_\mathbb{Z}^p(B^3), \quad F_n \rightharpoonup F_\infty \text{ weakly in } L^p, \quad \sup_n \|F_n\|_{L^p} < \infty .$$

Then $\mathcal{F}_\mathbb{Z}^p(B^3)$. The same is not true for $p = 1$.

This theorem is discussed in Chapter 2, see Theorem 2.2 there. A consequence is the existence of minimizers for the functional \mathcal{YM}_p in the class $\mathcal{F}_{\mathbb{Z}}^p(B^3)$.

Slice distance and 2-dimensional integrability result

In order to prove the sequential weak closure of $\mathcal{F}_{\mathbb{Z}}^p(M)$ we use slices by spheres as in the nonabelian case, but with some distinctions. Because of the linearity of the formula $F = dA$ we can easily recover local expressions of A from F , therefore *we slice directly the curvature instead of slicing the connection*. Moreover the expression of the curvature is independent on the gauge. By looking just at slices of F we obtain an explicit geometric distance between gauge classes of connections on \mathbb{S}^2 :

Definition 1.22 (Abelian slice distance). *Let F_1, F_2 be two L^p -integrable 2-forms on \mathbb{S}^2 . We define*

$$\text{dist}(F_1, F_2) := \inf \left\{ \|\alpha\|_{L^p(\mathbb{S}^2)} : d\alpha = F_1 - F_2 - \sum_{i=1}^n \delta_{x_i} - \partial I \right\}$$

where the infimum is done on all α , all finite sums of Dirac masses δ_{x_i} and all finite mass integer multiplicity integral 1-currents I .

The fact that this defines a distance is nontrivial. We need the following representation result for vector fields in 2-dimensions, which is itself an important result of this thesis:

Theorem 1.23 (Integrability Theorem). *Let $p > 1$, let Ω be a compact 2-dimensional manifold, possibly with boundary and θ be the volume form of \mathbb{S}^1 . Then the following equality holds*

$$\begin{aligned} & \{u^*\theta : u \in W^{1,p}(\Omega, \mathbb{S}^1), \text{deg}(u|_{\partial\Omega}) = 0\} \\ & \quad = \\ & \{\alpha : \alpha \in L^p(\Omega, \wedge^1 \mathbb{R}^2), \exists I \in \mathcal{I}_1(\Omega), [d\alpha] = \partial I\}, \end{aligned}$$

where $\mathcal{I}_1(\Omega)$ represents the finite mass integral rectifiable 1-currents on Ω and $[d\alpha]$ is the distribution associated to $d\alpha$ by imposing

$$\langle [d\alpha], \varphi \rangle = \int_{\Omega} d\alpha \wedge \varphi \quad \forall \varphi \in \mathcal{D}_0(\Omega).$$

The proof of this result is based on a density result for a class similar to \mathcal{R}^∞ and will be presented in Chapter 3. This result is related to the study of the distributional Jacobian of \mathbb{S}^1 -valued maps (which would correspond to the case $p = 1$, not treated here). See [5] and the references therein.

Control of the slices

For the distance dist defined above we obtain the same control as for the nonabelian case, i.e. we control the slice oscillations in terms of the curvature. This allows the use of a Hölder slice oscillation control:

Theorem 1.24. *The slice-function of a weak $U(1)$ -curvature $F \in \mathcal{F}_{\mathbb{Z}}^p(B^3)$ satisfies the following kind of bounds:*

$$\text{dist}(F(t), F(t')) \leq C \|F\|_{L^p(B^3)} |t - t'|^{1-1/p}.$$

A more refined statement is present in Theorem 4.11. The proof of this result is more challenging than in the nonabelian case due to the complication of our distance, which is formulated in terms of curvatures rather than in terms of connection forms.

1.4.2 The definition of the boundary trace

Let M be a Riemannian 3-manifold with smooth boundary. In order for the Plateau analogue

$$\inf \{ \mathcal{YM}_p(F) : F \in \mathcal{F}_{\mathbb{Z}}^p(M), i_{\partial M}^* F = \phi \}$$

to be well posed, we have to give a precise meaning to the notation $i_{\partial M}^* F = \phi$. A priori this notation does not have a meaning for general L^p -forms F (see the discussion of Section 4.6 for more details). Before giving the corresponding definition we note down important features which a boundary trace definition should have in order for the above minimization problem to be meaningful.

For a general 2-form $\phi \in C^\infty(\partial M, \wedge^2 T\partial M)$ we are looking for a definition of the class $\mathcal{F}_{\mathbb{Z},\phi}^p(M)$ of “weak $U(1)$ -curvatures with boundary restriction ϕ ” satisfying the following requirements:

- **(closure)** For any L^p -regular 2-form ϕ on ∂M , the class $\mathcal{F}_{\mathbb{Z},\phi}^p(M)$ is closed by sequential weak L^p -convergence.
- **(nontriviality)** If $\phi \neq \psi$ are two L^p -regular 2-forms on $\partial\Omega$, then $\mathcal{F}_{\mathbb{Z},\phi}^p(M) \cap \mathcal{F}_{\mathbb{Z},\psi}^p(M) = \emptyset$.
- **(compatibility)** For any smooth 2-form ϕ , $\mathcal{F}_{\mathbb{Z},\phi}^p \cap \mathcal{R}^\infty(M)$ are exactly the 2-forms $F \in \mathcal{R}^\infty(M)$ such that $i_{\partial M}^* F = \phi$, where $i_{\partial M}$ is the inclusion map.

In the case of $M = B^3$ we give the following definition:

Definition 1.25. *If $\phi \in C^\infty(\partial B^3, \wedge^2 T\partial B^3)$ then we say that a weak $U(1)$ -curvature $F \in \mathcal{F}_{\mathbb{Z}}^p(B^3)$ has ϕ as a boundary trace and we write $i_{\partial B^3}^* F = \phi$ if*

$$\text{dist}(F(t), \phi) \rightarrow 0 \quad \text{as } t \rightarrow 1^-,$$

where $F(t)$ is the slice of F along the sphere ∂B_t

A similar definition can be adopted for general compact 3-manifolds with boundary, by using appropriate foliations near ∂M .

One of the main results of this thesis is the following type of result:

Theorem 1.26 (Good definition of the boundary trace). *For $M = B^3$ the classes $\mathcal{F}_{\mathbb{Z}, \varphi}^p(M)$ defined by using Definition 1.25 satisfy the above three properties.*

A similar result holds for more general 3-manifolds M , as described in Chapter 4.

Idea of the proof

The proof of Theorem 1.26 is based on a careful study of the distance d performed in Chapter 4. We show that using the distance d metrizes the weak convergence of L^p -equibounded slices. As we saw above the slice distance is also Hölder-continuous for well-behaved slicing functions. Thus we automatically have local bounds on how much sequences of slice functions of weakly convergent weak $U(1)$ -curvatures F_n can oscillate near a fixed slice. This provides the main tool for the proof.

1.5 Regularity of minimizers in the abelian case

One of the main results of this thesis, proved in Chapter 5, is the regularity of minimizers for the functional \mathcal{YM}_p . The precise result is the following one (see Theorem 5.2):

Theorem 1.27. *Let $p \in]1, 3/2[$, and let $F \in \mathcal{F}_{\mathbb{Z}}^p(B^3)$ be a minimizer for the problem*

$$\inf \{ \mathcal{YM}_p(F) : F \in \mathcal{F}_{\mathbb{Z}}^p(B^3), i_{\partial B^3}^* F = \phi \}.$$

Then F is locally Hölder-continuous away from a locally finite set $\Sigma \subset B^3$.

The minimizer exists by the weak closure of $\mathcal{F}_{\mathbb{Z}}^p(B^3)$ described in the preceding section. A similar procedure works when the closed ball B^3 is replaced by a compact Riemannian manifold M^3 .

The proof of regularity proceeds roughly along the same steps as the proof of the regularity of harmonic maps [115], [69] and [70]. The main difference and novelty of our result with respect to previous regularity results is the approach to the following ϵ -regularity Theorem (cfr. Theorem 5.3):

Theorem 1.28 (ϵ -regularity). *Let $M = B^3$ as above. There exists $\epsilon_p > 0$ such that for any local \mathcal{YM}_p -minimizer $F \in \mathcal{F}_{\mathbb{Z}}^p$, if $B_r^3(x_0) \subset B^3$ and*

$$r^{2p-3} \int_{B_r(x_0)} |F|^p d\mathcal{H}^3 < \epsilon_p,$$

then

$$dF = 0 \quad \text{on } B_{r/2}(x_0).$$

This result is in fact the crucial point of the regularity theory, because it allows to pass from knowing that a curvature F has small energy to the fact that it satisfies an elliptic system of equations in the weak sense. Once we know this, the fact that F is Hölder (at least on balls where energy does not concentrate) follows from the regularity of elliptic systems by K. Uhlenbeck [130] and P. Tolksdorf [127], which we present in Appendix D.

The proof of the ϵ -regularity theorem is done via a procedure of approximation and reduction to a combinatorial problem, instead of using elliptic estimates. Indeed suppose for a moment that we had the information that $F \in \mathcal{R}^\infty$. Then by smoothness and Chern-Weil theory we have

$$dF = * \sum_{i=1}^N d_i \delta_{p_i}, \quad d_i \in \mathbb{Z},$$

where p_i are the singular points of the bundle corresponding to F . We wish to have a procedure which allows to *remove singularities* from F while (1) decreasing $\int_{B^3} |F|^p$ and (2) maintaining the boundary value of F .

Equivalent formulation in terms of vector fields

The idea for the construction of competitors in the ϵ -regularity proof is best explained in the equivalent formulation in terms of vector fields. We use the

identification of k -covectors β with simple $(n - k)$ -vectors $*\beta$ valid in \mathbb{R}^n , which for oriented manifolds translates into Poincaré duality. In our case a 2-form F is identified with a vector field X by requiring

$$F(W \wedge V) = X \cdot (W \times V)$$

pointwise for all couples of vectors V, W .

The equivalent of $\mathcal{F}_{\mathbb{Z}}^p(M)$ is the following space

Definition 1.29 (vector fields with integer fluxes). *Let M be a compact oriented Riemannian 3-manifold. We define $L_{\mathbb{Z}}^p(M, TM)$ to be the class of vector fields $X \in L^p(M, TM)$ such that*

$$\int_{\partial B_r^3(a)} X \cdot \nu \in \mathbb{Z}, \quad \forall a \in M, \text{ a.e. } r < \rho_{inj}(M),$$

where $\nu : \partial B_r^3(a)$ is the outward unit normal vector to a geodesic r -ball.

The integrality condition above and the one in the definition of $\mathcal{F}_{\mathbb{Z}}^p$ differ just by a normalization factor.

Main construction used for the ϵ -regularity

Let now $X \in L_{\mathbb{Z}}^p(B^3, \mathbb{R}^3)$ correspond to $F \in \mathcal{R}^\infty(B^3)$ as above. Then X is smooth outside a finite set of points $\{p_i, \dots, p_k\}$ and

$$\operatorname{div} X = \sum_i d_i \delta_{p_i}, \quad d_i \in \mathbb{Z}.$$

The flow of this vector field conserves mass outside the points p_i , therefore knowing (1) how this flow behaves near the singular set (2) the set of its trajectories, gives us global information on X itself. We wish to decompose X using its flow in a geometric manner, and then just inverse the directions of some flow trajectories in order to “annihilate some sources with some sinks”. In other words we would like to insert some dipoles tailored on $(X, \{p_i\})$ using the structure of the flow trajectories of X for this tailoring.

It is not immediately clear that this strategy can be formalized and that we can find a criterion for the “source/sink annihilation” to work. In the next section we describe our solution to these problems.

1.5.1 Smirnov decomposition and combinatorial flows

The first result which we use is Smirnov's theorem on the decomposition of 1-currents. This theorem in the special case of 1-currents representable as L^1 -vector fields can be summarized as follows: given a L^1 -current X we can represent X as a superposition of "topologically simple" rectifiable currents, i.e. currents supported either (in the boundaryless case) on so-called *solenoids*, of which typical examples are strange attractors of dynamical systems, or (in the case with boundary) on non-self-intersecting *Lipschitz curves*. Moreover such decomposition can be done *without cancellations* and such that also the *boundaries* of the currents represented by the above curves superpose without cancellations. The precise statement is as follows:

Theorem 1.30 ([120]). *Assume T is a finite mass 1-current on \mathbb{B}^n with finite mass boundary ∂T . Then there exists a total decomposition $T = A + C$ such that $\partial C = 0$ and A is acyclic.*

A can be further decomposed into a superposition of arcs as follows. There exists a finite positive Borel measure μ on the space of arcs such that

$$\langle A, \omega \rangle = \int \langle [\gamma], \omega \rangle d\mu(\gamma), \quad (1.6)$$

$$\langle \|A\|, \phi \rangle = \int \langle \|\gamma\|, \phi \rangle d\mu(\gamma), \quad (1.7)$$

$$\langle \partial A, f \rangle = \int \langle \partial[\gamma], f \rangle d\mu(\gamma), \quad (1.8)$$

$$\langle \|\partial A\|, \phi \rangle = \int \langle \|\partial[\gamma]\|, \phi \rangle d\mu(\gamma), \quad (1.9)$$

for all $\omega \in C^\infty(B^n, \wedge^1 \mathbb{R}^n)$, $f \in C^\infty(B^n)$, $\phi \in C^0(B^n)$. C can be decomposed into a superposition of elementary solenoids, i.e. there exists a finite Borel positive measure ν on solenoids Sol such that

$$\langle C, \omega \rangle = \int \langle S, \omega \rangle d\nu(S), \quad (1.10)$$

$$\langle \|C\|, \phi \rangle = \int \langle \|S\|, \phi \rangle d\nu(S). \quad (1.11)$$

We give precise definitions and discuss some proofs and generalizations in Appendix A.

Note that if we are interested in preserving the boundary value while decreasing the energy, then removing the cyclic part of our X is very much in our interest. In other words our minimizers X will be acyclic, i.e. no "strange

attractor"-looking currents will take part into their decomposition. Even if we are not acyclic (indeed we just suppose that our locally smooth X approximates a minimizer, not that it is one), since curves corresponding to the decompositions of A, C have μ -a.e. disjoint supports, we can again just modify the acyclic part A , without having to manipulate cycles.

For acyclic X we associate a discretized structure to X by "grouping together" curves in the support of μ based on where their endpoints are. This corresponds to what we mentioned above, i.e. to doing a decomposition of X tailored on its flow structure. This discretized structure is well-represented by a weighted directed graph, i.e. to a *combinatorial flow*, such that however Kirchhoff's law of preservation of the flow at nodes of the graph) is satisfied only up to the integer errors d_i corresponding to the degrees of our singularities.

How and under which hypotheses can we decrease the energy of this discretized flow while preserving its boundary value? A simple answer which is enough for our purposes is the following:

Proposition 1.31. *Assume that X_d is a combinatorial flow as above, given by a directed weighted graph $G = (E, V)$ with weight function $c : E \rightarrow \mathbb{R}^+$, such that except for a set of edges $S \subset V$ called "boundary of X_d ", Kirchhoff's law relative to the given directions and to the weights c is valid with "errors" belonging to \mathbb{Z} . If*

$$\sum_{s \in S} c(s) < 1$$

then we can find another flow X'_d corresponding to a graph on the same vertices $G' = (E', V)$ and another function $f : E' \rightarrow \mathbb{R}^+$ such that:

- *The edge sets E, E' agree up to orientation.*
- *$f \leq c$, i.e. the new flow has **lower energy** than the old one.*
- *On the subset S the orientations of the edges E and E' coincide and $f = c$, i.e. the new flow has the **same boundary value**.*
- *The new flow satisfies Kirchhoff's law with no error, i.e. **we removed the charges**.*

The proof of the above result is done using a maxflow-mincut result. The main idea of the proof is that since we assume that the energy on the boundary is smaller than the quantization gap, the proof reduces to a kind of remainder result.

This allows to construct competitors satisfying $\operatorname{div}X = 0$ and with lower energy and thus settles the ϵ -regularity proof.

1.5.2 Some open problems

Regularity in the abelian case and Smirnov decomposition

The proof of the ϵ -regularity for weak $U(1)$ -curvatures uses a technique which is very unusual for regularity results, i.e. it uses a combinatorial construction related to the Maxflow-Mincut theorem. It would be interesting to obtain the same result using traditional tools:

Open Problem 1. *Prove Theorem 5.3 without using combinatorial tools.*

Possibly such a proof of ϵ -regularity would allow to tackle the regularity problem for stationary $U(1)$ -curvatures, which is surely very interesting:

Open Problem 2. *The stationarity equation for the functional \mathcal{YM}_p is given in Section 5.4.4, equation (5.15). Is it true that stationary $U(1)$ -curvatures have isolated singular points?*

Another possible approach for answering Open Problem 2 would be to extend a combinatorial characterization by weighted directed graphs of general curvatures in $\mathcal{F}_{\mathbb{Z}}^p(B^3)$. At this level we can already state this question:

Open Problem 3. *Prove Theorem 5.3 without using Theorem 1.19.*

Indeed, before applying our combinatorial method to the ϵ -regularity proof in Chapter 5 we have to reduce to the case in which our weak curvature has finitely many charges, i.e. we need to first use the approximation theorem 1.19.

Infinite graphs and extended Smirnov theorems

A possible approach to Open Problems 2 and 3 could be to reason as in the proof of the existence of minimal connections (see Theorem 2.10), but obtain a limit of the weighted graphs described in Section 5.2.2. The existence of such graph for general $F \in \mathcal{F}_{\mathbb{Z}}^p(B^3)$ is strongly related to the question treated in Section B.4.1, i.e. whether there can be a notion of Smirnov decomposition for currents in a class which is wider than just normal currents. In Example B.22 we provide a flat 1-current which does not have a Smirnov decomposition due to the fact that its support is totally disconnected. We were not able to provide such an example within the class $\mathcal{F}_{\mathbb{Z}}^p(B^3)$. We state the question in terms of vector fields with integer fluxes.

Open Problem 4. *Is an analogue of Smirnov's decomposition theorem A.6 valid for L^p -vector fields with integer fluxes $L^p_{\mathbb{Z}}(B^3)$ with $p > 1$?*

Besides Smirnov's theorem, if we desire to associate a weighted graph to $F \in \mathcal{F}_{\mathbb{Z}}^p(B^3)$ directly and use it to prove regularity results, it could be necessary to have an effective analogue of the maxflow-mincut theorem C.3 for infinite networks. Results of this type are already available in the case of locally finite graphs (see Section C.2) however we expect stronger results to be necessary:

Open Problem 5. *Is a maxflow-mincut theorem like Theorem C.6 valid for the wider class of infinite networks G such that every subgraph of G has finite boundary capacity?*

1.6 Results on Coulomb gauges and Nonlinear Sobolev Spaces

1.6.1 The search for a global gauge in dimension 4 and Lorentz spaces

Sometimes, e.g. for the control of the 4-dimensional Yang-Mills flow $\partial_t d_{\nabla} = -d_{\nabla}^* F_{\nabla}$, a quantitative control of the singularities of the connection ∇ is required (see [122]). This control is achieved in the L^2 -small curvature regime (i.e. when no singularity is present) via Uhlenbeck's Theorem 1.5. This theorem provides gauges $g(t)$ in which $A(t)$ is $W^{1,2}$ -controlled by $F(t)$ and the flow provides weak equations controlling the behavior of $A(t)$ for short time. This control breaks down at energy level ϵ_0 because of the possible formation of singularities, or "topological bubbles" of the bundle. If p is such a bubbling point, the connection form will satisfy

$$|A| \sim \frac{1}{\text{dist}(\cdot, p)}$$

near p . We achieved a quantitative control on such bubbling by working in function spaces X over \mathbb{R}^4 in which a function like $f(x) = \frac{1}{|x|}$ has finite norm. Optimal candidate for such a space X is the Lorentz space

$$L^{4,\infty} := \left\{ f \in L^1_{loc}(\mathbb{R}^4) : \sup_{\alpha > 0} |\{x : |f|(x) > \alpha\}| < \infty \right\}.$$

The following theorem, obtained in collaboration with my advisor Tristan Rivière, is one of the main results of this thesis:

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Theorem 1.32 (Globally controlled gauges). *Let M^4 be a Riemannian 4-manifold. There exists a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with the following properties. Let ∇ be a $W^{1,2}$ connection over an $SU(2)$ -bundle over M . Then there exists a global $W^{1,(4,\infty)}$ section of the bundle (possibly allowing singularities) over the whole M^4 such that in the corresponding trivialization ∇ is given by $d + A$ with the following bound.*

$$\|A\|_{L^{(4,\infty)}} \leq f(\|F\|_{L^2(M)}),$$

where F is the curvature form of ∇ .

For the proof see Chapter 9 and Section 9.5. The technique of proof uses a new Lorentz-Sobolev extension for functions in $W^{1,3}(\mathbb{S}^3, SU(2))$ (see Section 1.6.2), together with a discussion of the energy concentration possibilities for the curvature and Uhlenbeck's Theorem 6.4. We provide a more extended summary of the proof in Section 9.5.1.

It would be interesting to obtain also the Coulomb condition besides the above control. This is however an open question:

Open Problem 6. *Prove that it is possible to find $L^{4,\infty}$ -controlled global Coulomb gauges as in Theorem 1.32. In other words, prove that it is possible to find a gauge as in Theorem 1.32, but with the further requirement that $d^*A = 0$ in such gauge.*

1.6.2 Controlled extension of Sobolev maps into manifolds

The main ingredient of Theorem is the following optimal extension result, also obtained in collaboration with my advisor Tristan Rivière:

Theorem 1.33. *There exists a function $f_1 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with the following property. Suppose $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$. then there exists an extension $u \in W^{1,(4,\infty)}(B^4, \mathbb{S}^3)$ of ϕ such that the following estimate holds:*

$$\|\nabla u\|_{L^{4,\infty}(B^4)} \leq f_1(\|\nabla \phi\|_{L^3}).$$

The originality of this result with respect to the previous ones [22] or [93] is that whereas the previous works were concerned with the existence of an extension, in our case a control is provided in term of the boundary value. We show in Section 9.2.7 that even under the hypothesis $\deg(\phi) = 0$ such that a

$W^{1,4}$ -extension surely exists, no energy control will be available.

We discuss the relevance of our theorem, several possible extensions and related phenomena in Section 9.2.

Here we point out the main open questions in the area of controlled nonlinear extensions and some analogues of Theorem 1.33. An useful tool to control the topology of a manifold N are the fundamental groups $\pi_m(N)$ which is a quotient of $C^0(\mathbb{S}^m, \mathbb{N})$. To say that any map in this space is continuously extendable to B^{m+1} amounts to asserting that $\pi_m(N) = 0$.

We consider here the *controlled* extension problem for maps $\mathbb{S}^m \rightarrow \mathbb{S}^n$. As is usually the case the interesting new features appear when smooth maps are not dense in $W^{1,p}(S^m, S^n)$, in which case we expect topological obstructions to gradually disappear as p decreases. The first facts to note are the following:

- For extensions of maps from $W^{1,p}(S^m, S^n)$ to B^{m+1} the natural space given by continuous Sobolev and trace embeddings is $W^{1, \frac{m+1}{m}p}(B^{m+1}, S^n)$ (see Sec. 9.2.1 and 9.2.2).
- For $p < \frac{n+1}{m+1}m$ the controlled extensions exist (see Sec. 9.2.1).
- $p > m$ the extension question reduces to a purely topological problem (see Sec. 9.2.2).

The open cases when $p < m$ are the thus among the following ones:

Open Problem 7. *Assume that $\frac{n+1}{m+1}m \leq p < m$ and $m > n$. For which such choices of m, n, p does there exist a finite function $f_{m,n,p} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for every $\phi \in W^{1,p}(S^m, S^n)$ there exists an extension $u \in W^{1, \frac{m+1}{m}p}(B^{m+1}, S^n)$ for which the estimate*

$$\|u\|_{W^{1, \frac{m+1}{m}p}(B^{m+1}, S^n)} \leq f_{m,n,p} (\|\phi\|_{W^{1,p}(S^m, S^n)})$$

holds? Does the estimate hold for $p = m$ for the norm $W^{1, (m+1, \infty)}(B^{m+1}, S^n)$?

The above problem is partially understood or solved just in some cases:

- Due to a relation of extension problems to lifting problems we answer the above problem for $n = 2 < m$ and $\frac{3m}{m+1} \leq p < \frac{4m}{m+1}$, see Prop. 1.36 and Section 9.2.4.

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- In particular we cover all p for the dimensions $m = 3, n = 2$.
- For $n = 1, m \geq 3$ and $\frac{3m}{m+1} \leq p < m$ [22] prove that no extension exists.

It will be interesting in the future to look at the link of extension and lifting problems in detail. It is possible to do this also in the case of \mathbb{S}^1 -valued maps and in nonlocal Sobolev spaces, e.g. using the results of [28].

In the critical case $p = m$ left aside in the above Open Problem we have the following results:

- Using the Hopf lifts as in [72, 73] we prove Theorem 1.34 which is the solution to case $p = m = n = 2$ (see Sec. 9.3).
- The extension exists but cannot be controlled in the above Sobolev norm, making the Lorentz-Sobolev weakening of Theorem 1.33 and of Theorem 1.34 below optimal (see Sec. 9.2.5). This is analogous to the case of global gauges in 4-dimensions pointed out in the introduction.
- We also prove an analogous result for $p = m = n = 1$ (see Theorem 9.13) however this is not the natural space to look at, unlike higher dimensions. In this case indeed the trace space $H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$ is the natural space to look at, because $W^{1,1}(\mathbb{S}^1, \mathbb{S}^1)$ does not continuously embed in it (we recall a counterexample in 9.2.3).

These theorems leave open higher dimensional cases:

Open Problem 8. *Assume $n \geq 4$. Does there exist a finite function $f_n : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for each $\phi \in W^{1,n}(\mathbb{S}^n, \mathbb{S}^n)$ we can find an extension $u \in W^{1,(n+1,\infty)}(B^{n+1}, \mathbb{S}^n)$ for which the estimate*

$$\|u\|_{W^{1,(n+1,\infty)}(B^{n+1}, \mathbb{S}^n)} \leq f_n(\|\phi\|_{W^{1,n}(\mathbb{S}^n, \mathbb{S}^n)})$$

holds?

Unlike linear Sobolev spaces, not only the topology of the domain must be compared to the Sobolev exponent p , but also the dimension and structure of the constraint (i.e. the target manifold) plays a critical role. This is also related to the topological global obstructions to density results for smooth functions between manifolds in F. Hang-F. Lin [64, 65] (see also T. Isobe [77]).

A general tool allowing extensions the projection trick of Section 9.2.1, which works well for Sobolev exponents smaller than the target dimension

plus one. Lifting theorems allow to increase this dimension thus to apply the projection trick with higher exponents.

Using the Hopf fibration $H : \mathbb{S}^3 \rightarrow \mathbb{S}^2$ we construct controlled lifts and apply a version of the projection trick obtaining the following theorem with much less effort than for the 3-dimensional case of Theorem 1.33:

Theorem 1.34 (see Section 9.3). *Suppose $\phi \in W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$ is given. Then there exists $u \in W^{1,(3,\infty)}(B^3, \mathbb{S}^2)$ such that in the sense of traces $u|_{\partial B^3} = \phi$ and such that the following estimate holds, for a constant independent of ϕ .*

$$\|u\|_{W^{1,(3,\infty)}(B^3)} \leq C \|\phi\|_{W^{1,2}(\mathbb{S}^2)} (1 + \|\phi\|_{W^{1,2}(\mathbb{S}^2)}).$$

The Hopf fibration has a natural structure of $U(1)$ -bundle with nontrivial characteristic class, $P \rightarrow S^2$. Lifting a map $\phi : X \rightarrow S^2$ to a $\tilde{\phi} : X \rightarrow S^3$ for which $H \circ \tilde{\phi} = \phi$ corresponds to giving the trivialization of the pullback bundle ϕ^*P . Analogous lifts are interesting to study for general principal G -bundles, using universal connections. The next case after the one with target S^2 is the $SU(2)$ -bundle of the introduction, which corresponds to the Hopf fibration $S^7 \rightarrow S^4$.

The Hopf lift seems to be much more difficult to extend the case where the target is S^3 . We cannot use principal bundles because $\pi_2(G) = 0$ for all compact Lie groups G . For other fibrations the following question is open:

Open Problem 9. *Is it possible to find a fibration $\pi : E \rightarrow \mathbb{S}^3$ with compact fiber M and a constant $C > 0$ such that for each $\phi \in W^{1,3}(\mathbb{R}^3, \mathbb{S}^3)$ there exists a lift $\tilde{\phi} : \mathbb{R}^3 \rightarrow E$ satisfying the estimate $\|\nabla \tilde{\phi}\|_{L^{(3,\infty)}} \leq C f(\|\nabla \phi\|_{L^3})$ for some finite function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$?*

The controlled Hopf lift result for S^2 yields also an answer to Open Problem 7 for dimensions $m = 3, n = 2$:

Theorem 1.35. *Assume $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^2)$. Then there exists a controlled extension $u \in W^{1,(4,\infty)}(B^4, \mathbb{S}^2)$ with the control*

$$\|u\|_{W^{1,(4,\infty)}(B^4, \mathbb{S}^2)} \leq C \|\phi\|_{W^{1,3}(\mathbb{S}^3, \mathbb{S}^2)} (1 + \|\phi\|_{W^{1,3}(\mathbb{S}^3, \mathbb{S}^2)}).$$

If instead we have $\phi \in W^{1,p}(\mathbb{S}^3, \mathbb{S}^2)$ for $9/4 \leq p < 3$ then there exists an extension $u \in W^{1, \frac{4}{3}p}(B^4, \mathbb{S}^2)$ with

$$\|u\|_{W^{1, \frac{4}{3}p}(B^4, \mathbb{S}^2)} \leq C \|\phi\|_{W^{1,p}(\mathbb{S}^3, \mathbb{S}^2)} (1 + \|\phi\|_{W^{1,p}(\mathbb{S}^3, \mathbb{S}^2)}).$$

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The same proof allows to also answer Open Problem 9 for $n = 2 < m$ for some exponents p :

Proposition 1.36. *Assume $n = 2, m \geq 3$ and $\frac{3m}{m+1} \leq p < \frac{4m}{m+1}$ and consider a $\phi \in W^{1,p}(\mathbb{S}^m, \mathbb{S}^2)$. Then there exists a controlled extension $u \in W^{1, \frac{m+1}{m}p}(B^{m+1}, \mathbb{S}^2)$ with*

$$\|u\|_{W^{1, \frac{m+1}{m}p}(B^4, \mathbb{S}^2)} \leq C \|\phi\|_{W^{1,p}(\mathbb{S}^3, \mathbb{S}^2)} (1 + \|\phi\|_{W^{1,p}(\mathbb{S}^3, \mathbb{S}^2)}).$$

Chapter 2

Weak closure for $U(1)$ -curvatures in 3 dimensions

2.1 Introduction

In this chapter we prove the weak closure theorem for the class $\mathcal{F}_{\mathbb{Z}}^p$ of weak L^p -curvatures on singular $U(1)$ -bundles, in 3 dimension. This chapter is based on joint work with my advisor Tristan Rivière [PR1]. This result parallels the one of Chapter 8 done for the nonabelian case in 5 dimensions.

The main difference is that here we prove closure based on the definition of $\mathcal{F}_{\mathbb{Z}}^p$ in terms of an integrality condition rather than based on the existence of local $W^{1,p}$ -representatives of the curvature. This is equivalent to the general definition of $\mathcal{F}_{\mathbb{Z}}$ precisely by the higher linearity available in the abelian case, as discussed in Section 1.4.1. We will discuss the relation of the slice definitions and slice distances in Chapter 4. For a motivation of our setting see the discussion in Section 1.4.

2.1.1 Definitions and results

We recall the definition of the class $\mathcal{F}_{\mathbb{Z}}^p(B^3)$:

$$\mathcal{F}_{\mathbb{Z}}^p(B^3) := \left\{ \begin{array}{l} F \in L^p(B^3, \wedge^2 \mathbb{R}^3) \text{ s. t.} \\ \forall x \in B^3, \text{ a.e. } 0 < r < \text{dist}(x, \partial B^3) \\ \frac{1}{2\pi} \int_{\partial B_r(x)} i_{\partial B_r(x)}^* F \in \mathbb{Z} \end{array} \right\},$$

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where $i_{\partial B_r(x)}^*$ is the pullback via the inclusion of $\partial B_r(x)$ into B^3 . We will often use a different normalization in which the above factor $\frac{1}{2\pi}$ disappears.

As discussed in Section 1.4 studying $\mathcal{F}_{\mathbb{Z}}^p(B^3)$ is equivalent to studying the class $L_{\mathbb{Z}}^p(B^3, \mathbb{R}^3)$ of vector fields $X \in L^p(B^3, \mathbb{R}^3)$ such that

$$\int_{\partial B_r^3(a)} X \cdot \nu \in \mathbb{Z}, \quad \forall a \in B^3, \text{ a.e. } r < \text{dist}(a, \partial B^3),$$

where $\nu : \partial B_r^3(a) \rightarrow \mathbb{S}^2$ is the outward unit normal vector.

For $p \geq 3/2$ the above integrality condition implies $\text{div} X = 0$, while for $1 \leq p < 3/2$ it is possible for X to have a dense set of singularities. We are interested in the variational problem

$$\inf\{\|X\|_{L^p} : X \cdot \nu = \phi \text{ on } \partial\Omega, X \text{ has integer fluxes}\}. \quad (2.1)$$

The study of this problem proceeds as follows. Any minimizing sequence X_k has a subsequence converging weakly in L^p to some vector field X_∞ . We should consider the question of whether X_∞ has still integer fluxes or not.

To have a positive answer we have to exclude wild oscillations of the X_k , which might “average out” their fluxes. Such oscillations take place if $p = 1$ but in the other cases we have a *closure result*:

Theorem 2.1. *For $1 < p < 3/2$ the class $L_{\mathbb{Z}}^p(B^3, \mathbb{R}^3)$ is weakly sequentially precompact. More rigorously, if*

$$X_k \in L_{\mathbb{Z}}^p(B^3, \mathbb{R}^3), \quad X_k \xrightarrow{\text{weak-}L^p} X_\infty, \quad \forall k \quad \|X_k\|_{L^p} \leq C,$$

then $X_\infty \in L_{\mathbb{Z}}^p(B^3, \mathbb{R}^3)$.

For $p = 1$ given any vector-valued Radon measure $X \in \mathcal{M}^3(B^3)$ where

$$\mathcal{M}^3(B^3) := \{(\mu_1, \mu_2, \mu_3) \mid \mu_i \text{ signed Radon measure on } B^3\},$$

we can find a sequence $X_k \in L_{\mathbb{Z}}^1(B^3, \mathbb{R}^3)$ such that $X_k \rightharpoonup X$ weakly in the sense of measures.

The version of the above theorem involving weak curvatures is the following:

Theorem 2.2 (Theorem 2.1, curvature version). *Let $p > 1$ and assume that*

$$F_n \in \mathcal{F}_{\mathbb{Z}}^p(B^3), \quad F_n \rightharpoonup F_\infty \text{ weakly in } L^p, \quad \sup_n \|F_n\|_{L^p} < \infty.$$

Then $\mathcal{F}_{\mathbb{Z}}^p(B^3)$. The same is not true for $p = 1$.

Recall from the discussion of Section 1.4 that the integers obtained by integrating a curvature along a 2-cycle corresponds in the smooth case to the first Chern class of the bundle on that cycle. The above theorem can be reformulated by saying that this interpretation “survives” under weak convergence.

We will prove below the following result:

Proposition 2.3 (Interesting exponents are $p < 3/2$). *If $F \in \mathcal{F}_{\mathbb{Z}}^p(B^3)$ and $p \geq 3/2$ then $F = dA$ for some $A \in W^{1,p}(B^3, \wedge^1 \mathbb{R}^3)$.*

We can prove directly that the class $\mathcal{F}_{\mathbb{Z}}^p$ is also closed in the *strong* L^p -topology:

Lemma 2.4. *The class $\mathcal{F}_{\mathbb{Z}}^p(B^3)$ is closed for the L^p topology.*

Proof. We take a sequence $F_k \in \mathcal{F}_{\mathbb{Z}}^p(B^3)$ such that $F_k \xrightarrow{L^p} F_{\infty}$. If we take $x \in B^3, R < \text{dist}(x, \partial B^3)$, then there holds

$$\|F_k - F_{\infty}\|_{L^p}^p \geq \int_{B_R(x)} |F_k - F_{\infty}|^p dx \geq \int_0^R \left| \int_{\partial B_r(x)} i_{\partial B_r(x)}^*(F_k - F_{\infty}) d\mathcal{H}^2 \right|^p dr.$$

Therefore the above L^p -functions

$$f_k : [0, R] \rightarrow \mathbb{Z}, \quad f_k(r) := \int_{\partial B_r(x)} i_{\partial B_r(x)}^* F_k d\mathcal{H}^2$$

converge to the analogously defined function f_{∞} in L^p , therefore also pointwise almost everywhere, thus proving that F_{∞} also belongs to $\mathcal{F}_{\mathbb{Z}}^p$. \square

This also follows from the fact that the class \mathcal{R}^{∞} of smooth curvatures on smooth bundles over a finitely punctured B^3 is strongly dense in $\mathcal{F}_{\mathbb{Z}}^p(B^3)$ (see [83, 84] and Theorem 2.5 below).

2.1.2 How we obtain the weak closure

We prove the Weak Closure Theorem 2.2 by introducing a slice distance between the slices of fixed center appearing in the definition of $\mathcal{F}_{\mathbb{Z}}^p(B^3)$. Since our slices are by spheres and each slice has integer total F -area, we consider the following space of slices:

$$Y := L^p(\mathbb{S}^2, \wedge^2 \mathbb{R}^2) \cap \left\{ h : \int_{\mathbb{S}^2} h \in \mathbb{Z} \right\}.$$

To prove that the integrality required by Theorem 2.2 for the weak limit F_∞ holds as required by the definition of $\mathcal{F}_\mathbb{Z}^p$ we may restrict to the slices along $\partial B_r(x)$ for fixed x and $r \in [\rho, 2\rho]$. In this range of radii we can rescale our slices to obtain elements of Y without inducing large distortions because the corresponding function $f(x + r\omega) = (r, \omega) \in [\rho, 2\rho] \times \mathbb{S}^2$ is bilipschitz.

We therefore reduce to studying a sequence of slice functions

$$h_n : [\rho, 2\rho] \rightarrow Y.$$

Suppose that we are able to obtain boundedness of the L^p -norms of $h_n(s)$ for a.e. s . If we use Banach-Alaoglu's theorem then for each s such that $h_n(s) \in Y$ for all n we will find a convergent subsequence $n(s)$ depending on s . We have to identify the limit $h_\infty(s)$ with the s -slice of F_∞ .

the problem which stops us at this point is that the set of s is uncountable: we have no guarantee yet that all the subsequences $n(s)$ have a common subsequence. To pass from an uncountable set of "relevant" s to a countable subset which still controls the behavior of $h_n(s)$ for all s we need a *control on the oscillations* of the h_n . For this we introduce a new distance d on Y which is suited for using the behavior of the F_n for controlling the oscillations of the slice functions $h_n(\cdot)$.

For $h_1, h_2 \in Y$ we define

$$d(h_1, h_2) := \inf \left\{ \|\alpha\|_{L^p} : h_1 - h_2 = d\alpha + \partial I + \sum_{i=1}^N d_i \delta_{a_i} \right\},$$

where the infimum is taken over all triples given by an L^p -integrable 1-form α , an integer 1-current I of finite mass and an N -ple of couples (a_i, d_i) , where $a_i \in \mathbb{S}^2$ and $d_i \in \mathbb{Z}$.

By the identifications given by Poincaré duality we can equivalently consider h_i to be functions and we can replace α and $d\alpha$ respectively by a vector field V of the same regularity (i.e. belonging to $L^p(\mathbb{S}^2)$) and by its distributional divergence $\operatorname{div}V$ on \mathbb{S}^2 .

The fact that d is a metric is not immediate (we prove it in Section 4.2): in particular the implication

$$d(h_1, h_2) = 0 \Rightarrow h_1 = h_2$$

depends upon the result of Chapter 3 which says that flow lines of a L^p -vector field on \mathbb{S}^2 with $\operatorname{div}V = \partial I$ where I is an integer multiplicity rectifiable 1-current of finite mass can be represented as preimages $u^{-1}(y)$, $y \in \mathbb{S}^1$, for some $u \in W^{1,p}(\mathbb{S}^2, \mathbb{S}^1)$.

The estimate connecting the distance d above to the ideas [139, 8, 72] is (see Proposition 2.11) a bound on the Lipschitz constant of the slice function

$$h : [\rho, 2\rho] \rightarrow (Y, d), \quad x \mapsto h(x) := T_x^* i_{\partial B_x(a)}^* F,$$

where $T_x(\theta) := a + x\theta$ maps \mathbb{S}^2 to $\partial B_x(a)$. We estimate the Lipschitz constant of h in terms of the maximal function of the L^1 -function

$$f : [\rho, 2\rho] \rightarrow \mathbb{R}^+, \quad f(x) := \|h(x)\|_{L^p}^p = \|i_{\partial B_x(a)}^* F\|_{L^p}^p,$$

an estimate in the same spirit of the one used in [72], which was a generalization of the approach of [8] (see Appendix E).

The oscillation control of the $h_n(s)$ (for h_n coming from F_n as in Theorem 2.2) comes from the abstract theorem 2.13.

We note that a stronger control on the oscillations of the slices will follow in Section 4.4, where we prove that slices are Hölder with respect to the metric on spheres coming from the parameterization by center and radius. In particular the existence of a limit will follow from the usual compactness result for Hölder functions, see Section 4.4.1.

2.1.3 Overview of the chapter

In *Section 2.3* we prove a modified version of Theorem 9.1 of [72], which from the uniform $L^{p,\infty}$ -bound on a sequence of maximal functions Mf_n defined as above (which is a direct consequence of the uniform L^p -bound on the sequence of curvatures F_n considered initially), allows us to deduce a kind of *locally uniform* pointwise convergence of the slices $h_n(x)$ for a.e. x , up to the extraction of a subsequence. This uniformity is the main advantage of our whole construction, and this is why we have to introduce the above distance and maximal estimate. The seed from which our technique grew was planted by [8], and first developed in [72].

Section 2.5 is devoted to the verification of the hypotheses of the abstract Theorem 2.13, and *Section 2.6* concludes that we can extract a subsequence

as requested by Theorem 2.2.

The last *Section 2.7* is devoted to the proving the “ $p = 1$ ” part of Theorem 2.1, thereby also justifying the assumption “ $p > 1$ ” of Theorem 2.2.

2.2 Ideas for the definition of the distance

2.2.1 Strong approximation result

We recall some results of [83, 84] which justify the above definition of d . They all depend on the following theorem:

Theorem 2.5 (Strong approximation via curvatures in \mathcal{R}_∞ , [83, 84]). *Let $p \geq 1$ and assume $F \in \mathcal{F}_\mathbb{Z}^p(B^3)$. Then F is approximable by classical curvatures belonging to $\mathcal{R}_\infty \cap L^p$. In other words, for $k \in \mathbb{N}$ there exists*

- a finite set $\Sigma_k \subset B^3$,
- a smooth Hermitian line bundle E_k over $B^3 \setminus \Sigma_k$,
- a smooth curvature F^k on E_k such that $F^k \in L^p(B^3 \setminus \Sigma_k, \wedge^2 \mathbb{R}^3)$

such that

$$\|F - F^k\|_{L^p} \rightarrow 0.$$

Because of the fact that $c_1(E_k)$ is an integral cohomology class represented by F^k it follows that

$$dF^k = 2\pi \sum_{p \in \Sigma_k} d_i \delta_p, \quad \text{where } d_i \in \mathbb{Z}.$$

2.2.2 Calibrations and minimal connection

We assume for simplicity that $F^k = F$ (in particular F is smooth) on a neighborhood of ∂B^3 . We thus have that

$$\int_{\partial B^3} i^* F^k \in 2\pi\mathbb{Z} \text{ is independent of } k.$$

It follows that for each k, k' the numbers of topological singularities of F^k and of $F^{k'}$ are equal, thus there exists an integer multiplicity 1-current of minimal mass $I_{k,k'}$ connecting them, i.e. we can identify

$$dF^k - dF^{k'} = \partial I_{k,k'}.$$

Definition 2.6 (Minimal connection). *If p_1^+, \dots, p_k^+ and p_1^-, \dots, p_k^- are finite sequences of points inside B^3 , then we call minimal connection of the p^+ 's to the p^- 's the integral 1-current I realizing the minimum in the following problem:*

$$\min \left\{ \mathbb{M}(J) : \partial J = \sum_i \delta_{p_i^+} - \delta_{p_i^-} \right\}.$$

It is not difficult to see that the minimal connection is represented by a finite number of segments $[p_{\sigma(i)}^-, p_i^+]$ such that σ is a permutation on k elements, satisfying

$$\sum_i |p_i^+ - p_{\sigma(i)}^-| = \inf_{\tau \in S_k} \sum_i |p_i^+ - p_{\tau(i)}^-|.$$

Following E. Sandier [113] we can construct a *calibration* and formulate the problem of minimizing mass for a fixed singular set as a dual problem:

Proposition 2.7 (Existence of a calibration following [113]). *Assume that we have two sequences of points $p_i^\pm, i = 1, \dots, k$ inside B^3 . Then there exists a 1-Lipschitz function $f : B^3 \rightarrow \mathbb{R}$ such that*

$$\sum_i f(p_i^+) - \sum_i f(p_i^-) = \text{length of a minimal connection.} \quad (2.2)$$

Proof. It is sufficient to define on $S := \{p_i^\pm, i = 1, \dots, k\}$ a 1-Lipschitz f satisfying (2.2), because thereafter we can use Kirszbraun's theorem. Up to relabeling we may assume that a minimal connection corresponds to the identity permutation. Consider the set of functions g satisfying

$$g(p_i^+) - g(p_i^-) = |p_i^+ - p_i^-|.$$

Let f realize the smallest possible Lipschitz norm on S among functions as above. Assume that this norm is $\lambda > 1$ by contradiction. Consider the directed graph G on k vertices with edges corresponding to the couples (i, j) for which $f(p_j^\pm) - f(p_i^\pm) = \lambda |p_j^\pm - p_i^\pm|$ for some choice of signs. G has at least one edge by hypothesis.

Suppose that G contains a source or a sink, at vertex i . We can then change the two values at $f(p_i^\pm)$ into $f(p_i^\pm) + \alpha$ for small α , such that the graph G for the new function misses the edges touching i and no new edge is created.

We can repeat this procedure and obtain an empty G (which contradicts the minimality of λ) unless G has directed loops.

Now consider the graph \tilde{G} on $2k$ vertices $p_i^\pm, i = 1, \dots, k$ containing edges (p_i^-, p_i^+) and the edge (p_i^\pm, p_j^\pm) for $i \neq j$ whenever $f(p_j^\pm) - f(p_i^\pm) = \lambda|p_j^\pm - p_i^\pm|$. We call the first kind of edges 1-edges and the second type λ -edges, distinguishing between the different Lipschitz constants that f has on them. Since G has a directed loop, \tilde{G} has a loop where all the λ -edges are directed in the same sense and possibly just some 1-edges are not directed that way. Consider the segments $[p_{i_\alpha}^-, p_{i_\alpha}^+]$ of that loop which have this incorrect direction (there must be at least one because the total increase of f along the loop is zero). We see that

$$\sum_{\alpha} f(p_{i_{\alpha+1}}^+) - f(p_{i_\alpha}^-) = \sum_{\alpha} f(p_{i_\alpha}^+) - f(p_{i_\alpha}^-)$$

Since f only increases along the loop between $p_{i_{\alpha+1}}^+$ and $p_{i_\alpha}^-$ we have

$$\begin{aligned} f(p_{i_{\alpha+1}}^+) - f(p_{i_\alpha}^-) &= \lambda \sum (\text{lengths of } \lambda\text{-segments}) + \\ &\quad + \sum (\text{lengths of 1-segments}) \\ &\geq |p_{i_{\alpha+1}}^+ - p_{i_\alpha}^-|, \end{aligned}$$

where the sums are on the segments corresponding to edges of the loop between $p_{i_{\alpha+1}}^+$ and $p_{i_\alpha}^-$. The last inequality is strict unless $\lambda = 1$. Therefore

$$\sum_{\alpha} |p_{i_\alpha}^+ - p_{i_\alpha}^-| \geq \sum_{\alpha} |p_{i_{\alpha+1}}^+ - p_{i_\alpha}^-|$$

with strict inequality unless $\lambda = 1$. But if the inequality is strict then we contradict the fact that we started with a minimal connection. \square

Remark 2.8. *The proof remains valid also if we replace B^3 with an a general metric space E .*

A corollary of the above result is the following:

Proposition 2.9 (Estimate of the connection via the curvature). *Let $F \in R_\infty(B^3)$ be a smooth curvature on a Hermitian line bundle over $B^3 \setminus \{p_i^\pm, i = 1, \dots, k\}$ such that $F = 0$ in a neighborhood of ∂B^3 . Assume that the topological degree of the bundle around each punctured point is ± 1 i.e. that*

$$dF = \sum_i \delta_{p_i^+} - \sum_i \delta_{p_i^-}.$$

Then the length L of the minimal connection connecting the p_i^- to the p_i^+ is estimated by the L^1 -norm of F :

$$L \leq \|F\|_{L^1(B^3)}. \tag{2.3}$$

Proof. We associate a current I_F to F as follows: for $\phi \in C^\infty(\bar{B}^3, \wedge^1 \mathbb{R}^3)$ define

$$\langle I_F, \phi \rangle := \int_{B^3} F \wedge \phi.$$

We can minimize mass among currents supported in B^3 and having fixed boundary equal to ∂I_F and we have that

$$\min \left\{ \mathbb{M}(I) : \partial I = \sum_i \delta_{p_i^+} - \sum_i \delta_{p_i^-} \right\}$$

can be rewritten as follows:

$$\min \left\{ \sup \{ \langle I, \alpha \rangle : \|\alpha\|_{L^\infty} \leq 1 \} : \partial I = \sum_i \delta_{p_i^+} - \sum_i \delta_{p_i^-} \right\}.$$

For I as above and $\alpha = df$ and f as in (2.2) we have

$$\langle I, df \rangle = \sum_i (f(p_i^+) - f(p_i^-)) = L.$$

Note that I_F is a competitor in the above minimization and

$$\mathbb{M}(I_F) = \int_{B^3} |F|.$$

Using Proposition 2.7 we conclude the proof. \square

Now by the density result of Theorem 2.5 we deduce the existence of connecting integral 1-currents for curvatures $F \in \mathcal{F}_{\mathbb{Z}}^p(B^3)$:

Theorem 2.10 (Minimal connections for $\mathcal{F}_{\mathbb{Z}}^p(B^3)$, [83, 84]). *Assume $p \geq 1$ and let $F \in \mathcal{F}_{\mathbb{Z}}^p(B^3)$ be smooth near ∂B^3 . Then there exists a finite set of charges $\pm \delta_{x_i}$ and a finite mass rectifiable integer 1-current I such that for the 1-current associated to F there holds*

$$\partial I_F \llcorner B^3 = \sum_i \pm \delta_{x_i} + \partial I.$$

Proof. Using Theorem 2.5 we may find a sequence $F^k \in \mathcal{R}_\infty(B^3)$ which coincide with F in a neighborhood of ∂B^3 and such that

$$\|F^{k+1} - F^k\|_{L^1(B^3)} \leq 2^{-k}.$$

Since $F^{k+1} - F^k$ is a curvature satisfying the hypotheses of Proposition 2.9, there exists an integral rectifiable 1-current I_k satisfying

$$\partial I_k = \partial F^{k+1} - \partial F^k, \quad \mathbb{M}(I_k) \leq 2^k.$$

We may then write the following formula where the infinite sum converges in L^1 and in the sense of distributions:

$$F = F^1 + \sum_{k=1}^{\infty} (F^{k+1} - F^k).$$

If $\pm\delta_{x_i}$ are the singularities corresponding to F^1 we have

$$\partial I_F = \sum_i \pm\delta_{x_i} + \partial \left(\sum_k I_k \right).$$

By the closure of rectifiable integer 1-currents under mass convergence and since the above infinite sum converges in mass, we obtain that

$$I := \sum_k I_k$$

is an integral rectifiable finite mass 1-current which proves our result. \square

2.2.3 The connecting current I and a controllable slice distance

Now we go back to the question on how to define a distance which allows to control the oscillations of slices done along concentric spheres. We use Proposition 2.10 to understand what the situation is expected to be. The schematic picture of the behavior of $F \in \mathbb{Z}$ in this case is given in Figure 2.1.

Compare now the condition

$$dF = \sum_i n_i \delta_{x_i} + \partial I, \quad \int_x^y \int_{\mathbb{S}^2} |F|^p(a + r\omega) d\omega r^2 dr < C$$

and the definition of the slice distance (call $T_x^* i^*_{\partial B_x(a)} F = sl_x F$)

$$d(sl_x F, sl_y F)^p = \inf \left\{ \int_{\mathbb{S}^2} |\alpha|^p \left| \begin{array}{l} \alpha \in L^p(\mathbb{S}^2, T^*\mathbb{S}^2) \text{ and} \\ d\alpha = sl_y F - sl_x F + \sum_i n_i \delta_{p_i} + \partial I \end{array} \right. \right\}.$$

We see that the oscillation of T^*i^*F along the transverse (radial) direction might give a control on the slice distance. We prove this in the next section.

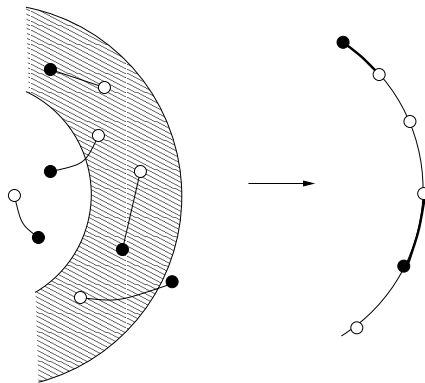


Figure 2.1: We represent schematically (i.e. we forget for a moment that we are in a 3-dimensional setting) the form F and the current I given by Proposition 2.10. On the left the portion between two spherical shells is shown, and the current I is represented by a collection of segments, where the boundary components with opposite signs are represented by small balls. An integration in the radial direction reduces us to the picture on the right, where part of the boundary of I projects to a boundary of a current, while for boundaries of components of I which are only partly inside the spherical shell, we obtain a number of Dirac masses. Such number is finite for almost every couple of slices. Applying Stokes' theorem we can compare the slices with the derivative of our 2-form inside the spherical shell.

2.3 Using the metric d for the weak closure

We hereby consider a 2-form h on $B^3 := B_1^3(0)$ such that $i_{\partial B^3}^* h = 0$ and we suppose that for a fixed point $a \in B^3$ and for $0 < s' < s < \text{dist}(a, \partial B^3)$ there holds

$$\forall x \in [s', s], \quad \int_{\partial B_x(a)} i_{\partial B_x(a)}^* h \in \mathbb{Z}.$$

We also suppose that there exists an integral 1-current I in $[0, 1]^3$ such that ∂I can be represented by $*dh$. In this case we have the following result:

Proposition 2.11. *Under the above hypotheses, for each subinterval $K \subset [s', s]$ there exists a function $M_K \in L^{1,\infty}(K, \mathbb{R})$, such that there holds*

$$[M_K(x)]^{1/p} \geq \text{esssup}_{x \neq \tilde{x} \in K} \frac{d(h(x), h(\tilde{x}))}{|x - \tilde{x}|}, \quad (2.4)$$

Where the 2-form $h(x) := T_x^* i_{\partial B_x(a)}^* h$ on \mathbb{S}^2 corresponds to the restriction $i_{\partial B_x(a)}^* h$ through the affine map $T_x : \mathbb{S}^2 \rightarrow \partial B_x(a)$, $T_x(\theta) := a + x\theta$.

Proof. Without loss of generality, we may suppose that $s = 1$ and that a is the origin. We start by observing that given a subinterval $K' = [t, t + \delta] \subset K$, we may consider (in polar coordinates) a function $\bar{\varphi}(\theta, r) = \varphi(\theta)$ on $B_1(0) \setminus \{0\}$ and identify the 2-form h with the 1-form $*h$. Then for $x \in]0, 1]$, $i_{\partial B_x(0)}^* h$ will be identified with a 1-form tangent to $\partial B_x(0)$, and therefore $h(x)$ is identified with a 1-form (or, after fixing the standard metric, with a 1-vector field) on \mathbb{S}^2 . Observe that

$$\begin{aligned} \left\langle \varphi, *_{\mathbb{S}^2} d_{\mathbb{S}^2} \left(\int_t^{t+\delta} h(x) dx \right) \right\rangle_{\mathbb{S}^2} &= \int_{\mathbb{S}^2} \nabla \varphi(\theta) \cdot \left(\int_t^{t+\delta} h(x)(\theta) dx \right) d\theta \\ &= \int_t^{t+\delta} \int_{\mathbb{S}^2} \langle d\varphi(\theta), h(x)(\theta) \rangle dx d\theta \\ &= \int_{\Omega} \langle d\varphi(\theta), i_{\partial B_x}^* (*h)(\theta) \rangle dV \\ &= \int_{\Omega} \langle d\bar{\varphi}, *h \rangle dV \\ &= \int_{\Omega} \langle \bar{\varphi}, *dh \rangle dV + \int_{\partial B_{t+\delta}} *(i_{\partial B_{t+\delta}}^* h) \bar{\varphi} d\sigma \\ &\quad - \int_{\partial B_t} *(i_{\partial B_t}^* h) \bar{\varphi} d\sigma \\ &= \int_{\Omega} \langle \bar{\varphi}, *dh \rangle dV + \int_{\mathbb{S}^2} h(t+\delta) \varphi d\theta \\ &\quad - \int_{\mathbb{S}^2} h(t) \varphi d\theta, \end{aligned}$$

where $\Omega := B_1 \setminus B_{s'}$. We used above the definition of $h(x)$ and the fact that since $\bar{\varphi}$ depends only on θ we have that for any one-form ω there holds $\langle d\bar{\varphi}, \omega \rangle_{\partial B_x} = \langle d\bar{\varphi}, i_{\partial B_x}^* \omega \rangle_{\partial B_x}$. We now use the property relating the 1-current I to the form h :

$$\int_{\Omega} \langle \bar{\varphi}, *dh \rangle dV = \langle \bar{\varphi}, (\partial I)_{\perp} \Omega \rangle.$$

The following formula holds for C^1 -approximations $\chi_{\epsilon} \in C_c^{\infty}(]0, 1[^3)$ of the characteristic function of Ω :

$$(\partial I)_{\perp} \Omega = \lim_{\epsilon \rightarrow 0} (\partial I)_{\perp} \chi_{\epsilon} = \lim_{\epsilon \rightarrow 0} [\partial(I_{\perp} \chi_{\epsilon}) + I_{\perp}(d\chi_{\epsilon})] = \partial(I_{\perp} \Omega) + \lim_{\epsilon \rightarrow 0} I_{\perp}(d\chi_{\epsilon}),$$

and the last term can be expressed in terms of slices along the proper function

$$\begin{aligned} f : B_1 \setminus B_{s'} &\rightarrow [s', 1] \\ (\theta, r) &\mapsto r, \end{aligned}$$

keeping in mind that $\Omega = f^{-1}([t, t + \delta])$: we have

$$\lim_{\epsilon \rightarrow 0} I_{\perp}(d\chi_{\epsilon}) = \langle I, f, t + \delta \rangle - \langle I, f, t \rangle,$$

and we observe therefore that for almost all values of t and $t + \delta$ the above contribution is an integer 0-current, so from

$$\int_{s'}^1 \mathbb{M}\langle I, f, \tau \rangle d\tau = \mathbb{M}(I_{\perp} f^{\#}(\chi_{[s', 1]} d\tau)) \leq C_{s'} \mathbb{M}(I) < \infty,$$

we obtain that it has also finite mass for almost all choices of t and $t + \delta$, therefore it is a finite sum of Dirac masses with integer coefficients. We now use the following easy lemma:

Lemma 2.12. *With the above notations, if \bar{J} is a finite mass rectifiable integer 1-current in $B_y \setminus B_x$ for $1 > y > x > 0$, then there exists a finite mass rectifiable integer 1-current supported on ∂B_x such that*

- for all functions $\bar{\varphi}(\theta, r) = \varphi(\theta)\chi(r)$ where $\chi \in C_c^{\infty}(]0, 1])$ and $\chi \equiv 1$ on $[x, y]$, there holds $\langle \bar{\varphi}, \partial \bar{J} \rangle = \langle \varphi, \partial J \rangle$,
- $\mathbb{M}(\bar{J}) \leq \mathbb{M}(J)$

Applying the above lemma to $\bar{J} = I_{\perp} \Omega$, we obtain

$$\langle \partial(I_{\perp} \Omega), \bar{\varphi} \rangle = \langle \partial J, \varphi \rangle,$$

where J is a finite mass rectifiable integer 1-current. We can summarize what shown so far by writing (all the objects being defined on \mathbb{S}^2)

$$\begin{aligned} *d \left(\int_t^{t+\delta} h(x) dx \right) &= h(t+\delta) - h(t) + \langle I, f, t+\delta \rangle - \langle I, f, t \rangle + \partial J \\ &= h(t+\delta) - h(t) + \sum_{i=1}^N d_i \delta_{a_i} + \partial J. \end{aligned}$$

Therefore, by definition of the metric $d(\cdot, \cdot)$, it follows that

$$d(h(t), h(t+\delta)) \leq \left\| \int_t^{t+\delta} h(x) dx \right\|_{L^p(\mathbb{S}^2)}.$$

We further compute:

$$\begin{aligned} d(h(t), h(t+\delta)) &\leq \left[\int_{\mathbb{S}^2} \left| \int_t^{t+\delta} h(r)(\theta) dr \right|^p d\theta \right]^{1/p} \\ &\leq \delta^{1-\frac{1}{p}} \left[\int_t^{t+\delta} \int_{\mathbb{S}^2} |h(r)(\theta)|^p dr d\theta \right]^{1/p} \\ &\leq \delta \left[M_K \left(\int_{\mathbb{S}^2} |h(\cdot)|^p \right) (t) \right]^{1/p}, \end{aligned}$$

where $M_K f$ is the uncentered maximal function of f on the interval K , defined as

$$M_K f(x) = \sup \left\{ \frac{1}{|B_\rho(Y)|} \int_{B_\rho(Y)} |f| : x \in B_\rho(y) \subset K \right\}.$$

□

2.4 The almost everywhere pointwise convergence theorem

We next call

$$N_I h(t) := \left[M_I \left(\|h(r)\|_{L^p(D)}^p \right) (t) \right]^{1/p},$$

where $D = [0, 1]^2$ or $D = \mathbb{S}^2$.

Then the following is a restatement of the equation (2.4) in terms of $N_k h$:

$$\text{For all } x, y \in I, \text{ there holds } N_I h(x)|x - y| \geq d(h(x), h(y)). \quad (2.5)$$

Consider now the metric space

$$Y := [L^p(D), d(\cdot, \cdot)] \cap \{h : \int_D h \in \mathbb{Z}\}. \quad (2.6)$$

It is clear that $f := [t \mapsto \|h(t)\|_{L^p(D)}^p] \in L^1([s', s])$ for all $0 < s' < s \leq 1$, therefore, by the usual Vitali covering argument for $M_I f$ we obtain that there exists a dimensional constant C for which

$$\sup_{\lambda > 0} \lambda^p |\{t \in I : N_I h(t) > \lambda\}| \leq C \int_I |f(x)| dx. \quad (2.7)$$

We can now prove the following analogue of [72]’s Theorem 9.1 (a proof is provided just in order to convince the reader that the hypotheses in the original statement can be changed: in fact it is completely analogous to the original one).

Theorem 2.13. *Suppose that for each $n = 1, 2, \dots, h_n : [0, 1] \rightarrow Y$ is a measurable function such that for all subintervals $I \subset [0, 1]$ there holds*

$$\sup_{\lambda > 0} \lambda^p |\{t \in I : N_I h_n(t) > \lambda\}| \leq \mu_n(I) \quad (2.8)$$

for some function $N_I h_n$ satisfying (2.5), where μ_n are positive measures on $[0, 1]$ such that $\sup_n \mu_n([0, 1]) < \infty$. We also suppose that a lower semicontinuous functional $\mathcal{N} : Y \rightarrow \mathbb{R}^+$ is given, and that

- the sublevels of \mathcal{N} are sequentially compact
- there holds

$$\sup_n \int_{[0,1]} \mathcal{N}(h_n(x)) dx < L < \infty \text{ for some } L \in \mathbb{R}. \quad (2.9)$$

Then the sequence h_n has a subsequence that converges pointwise almost everywhere to a limiting function $h : X \rightarrow Y$ satisfying

- $\int_{[0,1]} \mathcal{N}(h(x)) dx \leq L$,
- $\forall I \subset [0, 1], \sup_{\lambda > 0} \lambda^p |\{t \in I : \tilde{N}_I h(t) > \lambda\}| \leq \sup_n \mu_n(I)$, where again $\tilde{N}_I h$ satisfies (2.5).

Remark 2.14. *In Theorem 2.13 we considered the interval $[0, 1]$ instead of $[s', s]$ just for the sake of simplicity; the above results clearly extend also to the general case.*

Proof. Claim 1. It is enough to find a subsequence $f_{n'}$ which is pointwise a.e. Cauchy convergent. Indeed, in such case for a.e. $x \in [0, 1]$ there will exist a unique limit $f(x) := \lim f_{n'}(x) \in \hat{Y}$, the completion of Y . For such x we can then use Fatou's lemma and (2.9), obtaining for a.e. x a further subsequence n'' (which depends on x), along which $\mathcal{N}(f_{n''}(x))$ stays bounded. By compactness of the sublevels of \mathcal{N} we then have that $f(x) \in Y$.

Next, the lower semicontinuity of \mathcal{N} implies that the property (2.9) passes to the limit, while for the other claimed property we may take

$$\tilde{N}_I h(x) := \sup_{I \ni \tilde{x} \neq x} \frac{d(h(x), h(\tilde{x}))}{|x - \tilde{x}|}$$

and then use (2.5) to obtain

$$d(h(x), h(\tilde{x})) = \lim_{n'} d(h_{n'}(x), h_{n'}(\tilde{x})) \leq \liminf_{n'} N_I h_{n'}(x) |x - \tilde{x}|,$$

which gives (2.8) for $\tilde{N}_I h_{n'}$, since it shows that $\tilde{N}_I h(x) \leq \liminf_{n'} N_I h_{n'}(x)$. This proves Claim 1.

Desired properties. We will obtain the desired subsequence (n') by starting with $n_0(j) = j$ and successively extracting a subsequence $n_k(j)$ of $n_{k-1}(j)$ for increasing k . In parallel to this (for each $k \geq 0$)

- we will select countable families \mathcal{I}_k of closed subintervals of $[0, 1]$ which cover $[0, 1]$ up to a nullset Z_k
- for $I \in \mathcal{I}_k$ we will give a point $c_I \in I$ such that $y_{j,I} := h_{n_k(j)}(c_I)$ are Cauchy sequences for all $I \in \mathcal{I}_k$ and

$$\limsup_j N_I h_{n_k(j)}(c_I) \leq \frac{1}{k|I|} \quad (2.10)$$

Claim 2. The above choices guarantee the existence of a pointwise almost everywhere Cauchy subsequence $h_{n'}$. Indeed, we can then take a diagonal subsequence $j' = n_j(j)$, and use the fact that the nullsets Z_k have as union a nullset Z . Then for $I \in \mathcal{I}_k$ with k big enough, we have $d(f_{i'}(c_I), f_{j'}(c_I)) < \epsilon/3$ for i', j' big enough, while for $x \in I$, by (2.10) there exists C close to 1 such that

$$d(h_{i'}(x), h_{i'}(c_I)) \leq N_I h_{i'}(c_I) |I| \leq C \frac{1}{k}.$$

From these two estimates it follows that for all $x \in [0, 1] \setminus Z$ the sequence $h_{j'}$ is Cauchy, as desired.

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Obtaining the desired properties. The subsequence $n_k(j)$ of $n_{k-1}(j)$ will be also obtained by a diagonal extraction applied to a nested family of subsequences $n_{k-1} \prec m_1 \prec m_2 \prec \dots$ (where $a \prec b$ means that $b(j)$ is a subsequence of $a(j)$). We describe now the procedure used to pass from n_{k-1} to m_1 .

We choose an integer q such that

$$q > 2k^p \sup_n \mu_n([0, 1])$$

and we let \mathcal{I} be the decomposition of $[0, 1]$ into $2q$ non-overlapping subintervals of equal length. Then for each n we can find q “good” intervals in \mathcal{I} having μ_n -measure less than $1/(2k^p)$. The possible choices of such subsets of intervals being finite, we can find one such choice of subintervals $\{I_1, \dots, I_q\} \subset \mathcal{I}$ and a subsequence $m_0 \succ n_{k-1}$ such that for any of these fixed “good” intervals and for any $j \in \mathbb{N}$, there holds

$$\mu_{m_0(j)}(I_i) < \frac{1}{2k^p}. \quad (2.11)$$

For a fixed interval I_i , we now give a name to the set of points where (2.10) is falsified at step $m_0(j)$:

$$E_{m_0(j)} := \left\{ x \in I_i : N_{I_i} h_{m_0(j)}(x) > \frac{1}{k|I_i|} \right\}. \quad (2.12)$$

Then by (2.8), (2.11), (2.12) and since $|I_i| \leq |I| = 1$, we obtain

$$|E_{m_0(j)}| \leq k^p |I_i|^p \mu_{m_0(j)}(I_i) < \frac{1}{2} |I_i|^p \leq \frac{1}{2} |I_i|.$$

for j large enough, and therefore by Fatou lemma we get

$$\int_{I_i} \liminf_j \left[\chi_{E_{m_0(j)}}(x) + \frac{|I_i|}{3L} \mathcal{N}(h_{m_0(j)}(x)) \right] dx \leq \frac{1}{2} |I_i| + \frac{|I_i|}{3L} L = \frac{5}{6} |I_i|$$

Therefore we can find $c_{I_i} \in I_i$ and a subsequence $m_1 \succ m_0$ so that along m_1 we have

$$\chi_{E_{m_1(j)}}(c_{I_i}) + \frac{|I_i|}{3L} \mathcal{N}(h_{m_1(j)}(c_{I_i})) < 1,$$

in particular $c_{I_i} \notin E_{m_1(j)}$ for all j , and $\mathcal{N}(f_{m_1(j)}(c_{I_i}))$ is bounded. The latter fact allows us to find a Cauchy subsequence $m_2 \succ m_1$, while the former one gives us the desired property (2.10) for I_i . We can further extract such subsequences in order to obtain the same property for all the “good” intervals I_1, \dots, I_q . These intervals cover $1/2$ of the Lebesgue measure of $[0, 1]$, so we may continue the argument by an easy exhaustion, covering $[0, 1]$ by “good” intervals up to a set of measure zero. \square

2.5 Verification of the properties needed in the Abstract Theorem

We have seen that the functions $N_I h_n$ defined in Section 2.4 satisfy the hypotheses (2.5) and (2.8), as follows from (2.7) if we choose

$$\mu_n(I) := C \int_I \|h_n(x)\|_{L^p(D^2)}^p dx.$$

In order to use the abstract theorem 2.13, we specify the space

$$Y := \{h \in L^p(D, \wedge^2 D) : \int_D h \in \mathbb{Z}\}, \quad (2.13)$$

where D is a 2-dimensional domain (for example $[0, 1]^2$ or \mathbb{S}^2) and we define the functional $\mathcal{N} : Y \rightarrow \mathbb{R}^+$ by

$$\mathcal{N}(h) := \int_D |h|^p dx. \quad (2.14)$$

We show in Chapter 4 that Y is a complete metric space with respect to the distance d :

Proposition 2.15. *The above defined function d is a distance on $L^p(D, \wedge^2 \mathbb{R}^2)$, both in the case when $D = [0, 1]^2$ and in the case $D = \mathbb{S}^2$.*

Proof. See Theorem 4.3 □

We must now show that \mathcal{N} satisfies the properties stated in Theorem 2.13, namely that it is sequentially lower semicontinuous and that it has sequentially compact sublevels. The proofs are given in the following two propositions.

Proposition 2.16. *Under the notations (2.14) and (2.13), the functional $\mathcal{N} : Y \rightarrow \mathbb{R}^+$ is sequentially lower semicontinuous.*

Proof. In other words, we must prove that if $h_n \in Y$ is a sequence such that for some $h_\infty \in Y$ there holds

$$d(h_n, h_\infty) \rightarrow 0, \quad (2.15)$$

then we also have

$$\liminf_{n \rightarrow \infty} \mathcal{N}(h_n) \geq \mathcal{N}(h_\infty). \quad (2.16)$$

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We may suppose that the sequence $\mathcal{N}(h_n)$ is bounded, i.e. the h_n are bounded in L^p . Up to extracting a subsequence we then have

$$h_n \xrightarrow{L^p} k_\infty,$$

for some $k_\infty \in L^p$. By taking as a test function $f \equiv 1$, which is in the dual space L^q since D is bounded, we also obtain that $k_\infty \in Y$. Up to extracting a subsequence we may also assume that for all n we have $\int_D h_n = \int_D k_\infty \in \mathbb{Z}$. By the lower semicontinuity of the norm with respect to weak convergence, we have:

$$\liminf_{n \rightarrow \infty} \mathcal{N}(h_n) \geq \mathcal{N}(k_\infty).$$

This implies (2.16) if we prove

$$h_\infty = k_\infty. \quad (2.17)$$

We now write (2.15) using the definition of d : there must exist finite mass integer 1-currents I_k and vector fields X_k converging to zero in L^p such that

$$h_k - h_\infty = \operatorname{div} X_k + \partial I_k + \delta_0 \int_D (h_k - h_\infty) = \operatorname{div} X_k + \partial I_k.$$

Now we proceed as before, i.e. we define ψ_k and φ_k by

$$\begin{cases} h_k - h_\infty = \Delta \psi_k, \int_D \psi_k = 0 \\ \Delta \varphi_k = \operatorname{div} X_k, \end{cases}$$

so that $\operatorname{div}(\nabla(\psi_k - \varphi_k)) = \partial I_k$. We also have that $\nabla \varphi_k \rightarrow 0$ in L^p and $\nabla \psi_k$ is bounded in $W^{1,p}$, thus up to extracting a subsequence we may assume that

$$\nabla \psi_k \xrightarrow{W^{1,p}} \nabla \psi_\infty.$$

Now by Proposition 4.7 we can write

$$\nabla(\psi_k - \varphi_k) = \nabla^\perp u_k$$

for functions $u_k \in W^{1,p}(D, \mathbb{R}/2\pi\mathbb{Z})$ such that $\|\nabla u_k\|_{L^p} \leq C$. Up to extracting a subsequence we have $\nabla u_k \rightharpoonup \nabla u_\infty$ weakly in L^p , thus also in L^1_{loc} , and in particular

$$\nabla \psi_\infty = \nabla^\perp u_\infty.$$

Since weak- $W^{1,p}$ -convergence implies \mathcal{D}' -convergence, we have as in the proof of Proposition 4.6 that

$$\partial I_k \xrightarrow{\mathcal{D}'} \partial I_\infty + \operatorname{div}(\nabla^\perp u_\infty) = \operatorname{div} \nabla \psi_\infty,$$

where I_∞ is an integer finite mass 1-current. By Lemma 4.8 we have than that $\partial I_\infty = 0$, which implies that

$$h_k - h_\infty \xrightarrow{\mathcal{D}'} 0.$$

Therefore we have (2.17), which concludes the proof. \square

Proposition 2.17. *Under the notations (2.14) and (2.13), and for any $C > 0$, the set $\{h \in Y : \mathcal{N}(h) \leq C\}$ is d -sequentially compact.*

Proof. We must prove that whenever we have a sequence h_n in Y such that $\|h_n\|_{L^p}$ is bounded, then up to extracting a subsequence we have that for some $k_\infty \in Y$ there holds

$$d(h_n, k_\infty) \rightarrow 0. \quad (2.18)$$

We surely have a subsequence of the h_n which is weakly- L^p -convergent to a function $k_\infty \in L^p$. Then, as in the proof of Proposition 2.16 we have $\int_D k_\infty \in \mathbb{Z}$ and up to extracting a subsequence we may assume that $\int_D (h_n - k_\infty) = 0$ for all n . Then we define ψ_n to be the solution of

$$\begin{cases} \Delta \psi_n = h_n - k_\infty \\ \int_D \psi_n = 0, \end{cases}$$

and we claim that

$$\|\nabla \psi_n\|_{L^p} \rightarrow 0. \quad (2.19)$$

This is enough to conclude, since we can then set $X_n = \nabla \psi_n$ which gives an upper bound of $d(h_n, k_\infty)$ which converges to zero, proving (2.18).

In order to prove (2.19) we express

$$\nabla \psi_n(x) = \int_D \nabla G(x, y) [h_n(y) - k_\infty(y)] dy,$$

where G is the Green function of D . We know that $\nabla G \in L^q$ for all $q < 2$ and we also have that the sequence $h_n - k_\infty$ converges to zero weakly in L^p and is bounded in L^p . From the weak convergence we then obtain the pointwise convergence

$$\nabla \psi_n(x) \rightarrow 0 \text{ for all } x. \quad (2.20)$$

We can then use the L^p -boundedness of $h_n - k_\infty$ together with the Young inequality

$$\|\nabla \psi_n\|_{L^r} \leq \|\nabla G\|_{L^q} \|h_n - k_\infty\|_{L^p},$$

for q as above. We then have that $\|\nabla \psi_n\|_{L^r}$ are bounded once the following equivalent relations hold:

$$\frac{1}{r} > \frac{1}{p} + \frac{1}{2} - 1 \Leftrightarrow r < \frac{2p}{2-p},$$

In particular we have the boundedness in L^r for some $r > p$. This together with the pointwise convergence (2.20) and with the L^p -boundedness gives (2.19), as desired. \square

2.6 Proof of the Weak Closure Theorem

Our strategy will be to apply Theorem 2.13 to the sequence h_n arising from the F_n of Theorem 2.2. We start with two relatively elementary lemmas.

Lemma 2.18. *Suppose that $d(h_n(t), h_\infty(t)) \rightarrow 0$ for almost all $t \in I$. Then for all $\alpha, \epsilon > 0$ there exists a subset $E_{\alpha, \epsilon} \subset I$ such that $|E_{\alpha, \epsilon}| < \epsilon$ and that there exists $N_{\alpha, \epsilon}$ such that $n > N_{\alpha, \epsilon}$ and $t \in E_{\alpha, \epsilon}$ imply*

$$d(h_n(t), h_\infty(t)) < \alpha$$

Proof. Call $E_{m, n} := \{x \in I : d(h_i(x), h_\infty(x)) \leq 1/m \text{ for } i \geq n\}$. Then for fixed $m_\alpha > \alpha^{-1}$, the sets $E_{m_\alpha, n}$ form an increasing sequence whose union is I . It follows that $|E_{m_\alpha, n}| \rightarrow |I|$, so we find $N_{\alpha, \epsilon}$ such that $|I \setminus E_{m_\alpha, N_{\alpha, \epsilon}}| \leq \epsilon$. We then choose $E_{\alpha, \epsilon} := E_{m_\alpha, N_{\alpha, \epsilon}}$. It is easy to verify that this set is as desired. \square

Lemma 2.19. *Fix $x \in I$ and a 2-form $h_\infty(x)$. For all $c > 0$ there exists $\epsilon > 0$ such that*

$$\left. \begin{array}{l} d(h(x), h_\infty(x)) < \alpha \\ \int |h(x)|^p \leq A \end{array} \right\} \Rightarrow \int h(x) = \int h_\infty(x).$$

Proof. Suppose by contradiction that there exists a $A > 0$ such that for all $k \in \mathbb{N}$ there exists h_k such that

$$\begin{aligned} d(h_k(x), h_\infty(x)) &\leq \frac{1}{k} \\ \int |h_k(x)|^p &\leq A \\ \int h_k(x) &\neq \int h_\infty(x) \end{aligned}$$

By the second property, we can extract a subsequence $h_{k'}(x)$ of the $h_k(x)$ converging weakly in L^p . In particular we would then have

$$\mathbb{Z} \ni \int h_{k'}(x) \rightarrow \int h'_\infty(x)$$

In particular, for some $N \in \mathbb{N}$ large enough, the subsequence $h_{k''} := h_{k'+N}(x)$ satisfies

$$\int h_{k''}(x) = \int h'_\infty(x).$$

We now prove that $h_\infty(x) = h'_\infty(x)$. It is enough to prove that $h_{k''}(x) \xrightarrow{d} h'_\infty(x)$. and this follows exactly as in the proof of Proposition 2.17. We thus contradicted the assumption $\int h_k(x) \neq \int h_\infty(x)$, as desired. \square

Proof of Theorem 2.2. By the L^p -boundedness of the F_n , it is clear that we may find a weakly converging subsequence $F_{n'} \xrightarrow{L^p} F_\infty$. We suppose by contradiction that there exists a point $x \in B_1(0)$ and two radii $0 < s' < s < \text{dist}(x, \partial B_1(0))$ such that

$$\exists S \subset [s', s] \text{ s.t. } \mathcal{H}^1(S) > 0 \text{ and } \forall t \in S, \quad \int_{\partial B_t(x)} i_{\partial B_t(x)}^* F \notin \mathbb{Z}. \quad (2.21)$$

We then identify the forms given by

$$\tilde{F}_n|_{\partial B_r(x)} := i_{\partial B_r(x)}^* F \text{ for } r \in [s', s],$$

with functions (defined almost everywhere) $h_n : [s', s] \rightarrow Y$ (with the notations of Section 2.4). We suppose without affecting the proof that $[s', s] = I$ (see also Remark 2.14). By Theorem 2.13, we can assume (up to extracting a subsequence) that there exists h_∞ such that for almost all $t \in I$ there holds $d(h_n(t), h_\infty(t)) \rightarrow 0$.

We call

$$\left| \int h_n(x) - \int h_\infty(x) \right| := f_n(t).$$

Since we have $f_n \geq 0$, if we prove that the f_n converge in L^1 -norm, then the almost everywhere pointwise convergence follows, implying the fact that $|S| = 0$ and reaching the desired contradiction. To prove Theorem 2.2 we therefore have to prove that

$$\lim_{n \rightarrow \infty} \int f_n(t) dt = 0. \quad (2.22)$$

We start by calling

$$F_{n,A} := \left\{ t \in I : \int |h_n(t)|^p \geq A \right\}.$$

It clearly follows that (with C as in the statement of the theorem)

$$|F_{n,A}| \leq \frac{1}{A} \int \left(\int |h_n(t)|^p \right) dt = \frac{C}{A}$$

Now take A such that the above quantity is smaller than ϵ , and use Lemma 2.19 to obtain a constant α such that $d(h_n(t), h_\infty(t)) < \alpha$ implies $f_n(t) = 0$ for t such that $\int |h_n(t)|^p < A$, i.e. for $t \notin F_{n,A}$. With such choice of α apply Lemma 2.18 and obtain a set $E_{\alpha,\epsilon}$ so that $|I \setminus E_{\alpha,\epsilon}| < \epsilon$ and an index $N_{\alpha,\epsilon}$ such that for $n \geq N_{\alpha,\epsilon}$ and for $t \in E_{\alpha,\epsilon}$ there holds $d(h_n(t), 0) < \alpha$, and therefore

$f_n(t) = 0$.

For $n > N_{\alpha, \epsilon}$, the function $f_n(t)$ can therefore be nonzero only on $E_{\alpha, \epsilon} \cup F_{n, A}$, and we have

$$\begin{aligned} \int f_n(t) dt &\leq \int_{E_{\alpha, \epsilon} \cup F_{n, A}} f_n(t) dt \\ &\leq |E_{\alpha, \epsilon} \cup F_{n, A}|^{1-1/p} \left[\int |f_n(t)|^p dt \right]^{1/p} \\ &\leq (2\epsilon)^{1-1/p} C, \end{aligned}$$

whence the claim (2.22) follows by the arbitrariness of $\epsilon > 0$, finishing the proof of our result. \square

2.7 The case $p = 1$

We prove here the result stated in the Main Theorem 2.1 for $p = 1$, thereby showing also that the thesis of Theorem 2.2 cannot hold when $p = 1$. We consider the case when the domain is $[0, 1]^3$ for simplicity. The case of general domains is totally analogous.

Proposition 2.20. *Consider a signed Radon measure $X \in \mathcal{M}^3([0, 1]^3)$, with total variation equal to 1. Then there exists a family of vector fields $X_k \in L^1_{\mathbb{Z}}$ such that*

1. *There are two constants $0 < c < C < \infty$ such that*

$$\begin{cases} \forall k & c < \|X_k\|_{L^1([0, 1]^3)} < C \\ & \mathbb{M}(\operatorname{div} X_k) \rightarrow \infty \end{cases}$$

2. *$\operatorname{div} X_k = \partial I_k$ for a sequence of integer rectifiable currents I_k of bounded mass, and finally*

$$X_k \rightharpoonup X$$

From the above, it immediately follows:

Corollary 2.21. *The class $\mathcal{F}_{\mathbb{Z}}^1$ is not closed by weak convergence.*

The following holds for all $p < \frac{n}{n-1}$ in n dimensions:

Lemma 2.22. *Given a segment $[a, b] \subset \mathbb{R}^n$ of length $\epsilon > 0$ and a number $\delta > 0$, if $p < \frac{n}{n-1}$ then it is possible to find a vector field $X \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ with*

$$\begin{aligned} \operatorname{div} X &= \delta_a - \delta_b, \\ \operatorname{spt} X &\subset [a, b] + B_\epsilon(0), \\ \|X\|_{L^p} &\leq C\epsilon^{n-(n-1)p} \end{aligned}$$

where C is a geometric constant, and for two sets A, B , we denote $A + B := \{a + b : a \in A, b \in B\}$.

Proof. We may suppose that $a = (-\epsilon, 0, \dots, 0), b = (\epsilon, 0, \dots, 0)$ first. We then define the piecewise smooth

$$\begin{cases} X(\pm\epsilon(t-1), \epsilon st) &= \left(\frac{1}{\epsilon t^{n-1} |B_1^{n-1}|}, \pm \frac{s}{\epsilon t^{n-1} |B_1^{n-1}|} \right) \text{ for } (t, s) \in [0, 1] \times B_1^{n-1} \\ X(x, y) &= (0, 0) \text{ if } |x| + |y|_{\mathbb{R}^{n-1}} > \epsilon. \end{cases}$$

Then clearly $\operatorname{spt} X \subset [a, b] + B_\epsilon$, and using the divergence theorem it is also easily shown that $\operatorname{div} X = \delta_a - \delta_b$ in the sense of distributions. For the last estimate, we observe that

$$|X(x, y)| \leq \frac{C}{(\epsilon - |x|)^{n-1}} \chi_{\{|x|+|y|\leq\epsilon\}}(x, y),$$

so we can estimate

$$\begin{aligned} \int_{\mathbb{R}^2} |X|^p dx dy &\leq C \int_0^\epsilon \frac{(\epsilon - x)^{n-1}}{(\epsilon - x)^{(n-1)p}} dx \\ &= C\epsilon^{n-(n-1)p} \end{aligned}$$

□

Proof of Proposition 2.20. We will do our construction first in the simpler model case $F = dy \wedge dz \llcorner [0, 1]^3$. The modifications leading to the general case are treated separately.

- *The case of $X \equiv (1, 0, 0)$.* We consider the collections of segments in $[0, 1]^3$ given by

$$\mathcal{S}_k := \left\{ [(-2^{-3k-1}, 0, 0), (2^{-3k-1}, 0, 0)] + (a, b, c) : (a, b, c) \in 2^{-k}\mathbb{Z}^3 \cap [0, 1]^3 \right\}.$$

We then define an integral rectifiable 1-current I_k as the canonical integration from right to left along all the segments of \mathcal{S}_k . There clearly holds

$$\mathbb{M}(I_k) = 2^{-3k}(2^k - 1)^3 \rightarrow 1, \quad (2.23)$$

and it is a standard exercise in geometric measure theory (based on the approximation of $\mathcal{H}^3 \llcorner [0, 1]^3$ by sums of Dirac measures in the points $2^{-k}\mathbb{Z}^3 \cap]0, 1[^3$) to show that there holds:

$$I_k \rightharpoonup \mathcal{H}^3 \llcorner [0, 1]^3 \otimes dx \simeq (1, 0, 0). \quad (2.24)$$

We can then use Lemma 2.22 for each one of the segments in \mathcal{S}_k and with $\delta = \frac{1}{2}\epsilon = 2^{-3k}$ (which produces a set of $(2^k - 1)^3$ vector fields with disjoint supports, which can then be consistently extended to zero outside the set of the supports) each of whose L^1 -norms is equal to $C\epsilon^2 \max\{\delta^{-1}, \epsilon^{-1}\} = 2^{-3k}C$, which is proportional to the mass of the respective segment. Therefore (using (2.24)), property (1) follows.

The last point of the proposition follows by proving that also the vector fields X_k converge as 1-currents to the diffuse current X . The strategy used is as the one usually adopted for the proof of the convergence of the I_k : for a fixed smooth vector field a and for $k \rightarrow \infty$ we may approximate

$$\begin{aligned} \langle X_k, a \rangle &:= \sum_{\sigma \in \mathcal{S}_k} \int_{\text{spt} X^\sigma} X_k \cdot a \\ &= \sum_{P \in \mathbb{Z}^3 \cap]0, 1[^3} \left[\left(\int_{\text{spt} X^\sigma} X_k(x) dx \right) \cdot a(P) + \right. \\ &\quad \left. + \int_{\text{spt} X^\sigma} X_k(x) \cdot Da(P)[x - P] dx \right] + O_a(2^{-3k}) \\ &= \sum_{P \in 2^{-k}\mathbb{Z}^3 \cap]0, 1[^3} 2^{-3k}(1, 0, 0) \cdot a(P) + O_a(2^{-3k}) \\ &\rightarrow \int_{[0, 1]^3} a(x) \cdot (1, 0, 0) dx \end{aligned}$$

where the integral containing the differential Da is zero by the symmetry properties of X^σ and using the fact that

$$\begin{aligned} \frac{|O_a(\epsilon)|}{\epsilon} &\leq \sup \left\{ \left| \frac{a(x + \epsilon u) - a(x)}{\epsilon} - Da(x)[u] \right| : x \in B_1^3, u \in \mathbb{S}^2 \right\} \\ &\rightarrow 0 \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

- *The case of $X = (\rho, 0, 0) \in \mathcal{M}^3([0, 1]^3)$, where ρ is a probability density on $[0, 1]^3$.* In this case we consider the 2^{3k} disjoint cubes \mathcal{C}_k having the same centers as the segments in \mathcal{S}_k and side-length 2^{-k} , and in the above construction we substitute to the segment $\sigma_k \in \mathcal{S}_k$ the segment σ'_k having the same center, but length equal to $\rho(C_k)$, where $C_k \in \mathcal{C}_k$ is the cube with center equal to the one of σ_k and σ'_k . The newly obtained currents I'_k will still satisfy (2.23) and the analogous of (2.24) given by:

$$I'_k \rightharpoonup \rho \otimes dx \simeq (\rho, 0, 0).$$

It is then easy to apply suitable modifications to the above proof showing that also in this case property (2) holds.

- *The general case.* We can write (by Radon-Nikodym decomposition):

$$X = (\rho_1^+ - \rho_1^-, \rho_2^+ - \rho_2^-, \rho_3^+ - \rho_3^-),$$

where ρ_i are positive Radon measures of mass less than 1. Doing separately the construction in the previous point for all the ρ_i we obtain integer rectifiable currents I_k of mass bounded by 6, each of which is supported on finitely many segments. Applying Lemma 2.22 to each of the above segments, we obtain vector fields converging as before to the measure X , and since the supports of the vector fields obtained in this way superpose not more than 6 times, the estimate of the Lemma (used here for $p = 1$) still holds, up to changing the constant.

□

Chapter 3

Integrability of L^p -vector fields in 2 dimensions

3.1 Introduction

In this section we will present a result which helps for a technical step in the study of the slice distance used for the closure theorem of weak curvatures in the abelian case $G = U(1)$. The result consists in realizing L^p -vector fields V on a 2-dimensional Riemannian manifold as gradients of S^1 -valued $W^{1,p}$ -functions. This chapter is based on [P1].

This can be done precisely when the divergence of V can be represented as the boundary of a integral rectifiable 1-current of finite mass, and in fact for $p > 1$ there is a bijective correspondence between the two points of view. Therefore the result has an independent interest.

3.1.1 Presentation of the problem

Consider a vectorfield $V \in L^p(B^2, \mathbb{R}^2)$. If $\operatorname{div} V = 0$ then by the Poincaré Lemma we know that there exists a $W^{1,p}$ -function ψ with

$$V = \nabla^\perp \psi. \quad (3.1)$$

The next case in which the situation is relatively standard, is when (in the sense of distributions)

$$\operatorname{div} V = 2\pi \sum_{i=1}^N n_i \delta_{x_i}, \quad \text{for some } n_i \in \mathbb{Z} \setminus \{0\} \text{ and } x_i \in B^2. \quad (3.2)$$

Note that we cannot have $V \in L^p$ unless $p < 2$ (consider the model case $V(x) = \frac{x}{|x|^2}$, corresponding to $N = 1, x_1 = (0, 0), n_1 = 1$ in (3.2)).

The representation (3.1) holds then just locally outside the points x_i , and the local representations do not lift to a global one. If $p \geq 1$ then we obtain that the function ψ is locally harmonic and V is locally holomorphic. Therefore, it is possible to find a representation of the form (3.1) for a function $\psi \in W^{1,p}(B^2, \mathbb{R}/2\pi\mathbb{Z})$, by taking $\psi = \text{Arg}(V) + C$ for any constant C . Equivalently, one could use the Green function for the Laplacian to obtain a harmonic solution of $\nabla g = V$, and then from the regularity of g the existence of ψ would follow.

If we now consider the preimage $\psi^{-1}(y)$ of any regular value $y \in \mathbb{R}/2\pi\mathbb{Z}$ of ψ , then we see by Sard's theorem that this will be a rectifiable set, and with the orientation corresponding to the vectorfield ∇g , we can also consider this set as an integral current I_ψ on \bar{B}^2 . The boundary of this current is precisely the sum of Dirac masses in (3.2) (without the “ 2π ” factor):

$$\partial I_{\psi \lrcorner B^2} = \sum_{i=1}^N n_i \delta_{x_i} = \frac{1}{2\pi} \text{div} V. \quad (3.3)$$

When passing to the case where we allow $N = \infty$ in (3.2), we have to face the new difficulty that not all the formal infinite sums of Dirac masses can be represented as the distributional divergence of an L^p -vectorfield. The most obvious restriction (depending on the Fubini theorem) is seen as follows. Let Σ be a closed smooth Jordan curve and consider its perturbations $\Sigma(t), t \in [-\epsilon, \epsilon]$ via a family of diffeomorphisms. Then the flux $f(t)$ of V through $\Sigma(t)$ should satisfy again $f \in L^p([-\epsilon, \epsilon])$. In particular, it cannot happen that the algebraic sum of the Dirac masses inside Σ stays infinite for a set of times t of positive measure.

If we assume for a moment that a rectifiable 1-current I as in (3.3) exists, the above condition would translate by saying that the mass of the slice of I along $\Sigma(t)$ is a L^p -function of t .

In this work we prove a *necessary and sufficient condition* for a representability property like (3.1) to hold. Consider a smooth domain $\Omega \subset \mathbb{R}^2$ or $\Omega = \mathbb{S}^2 \simeq \mathbb{C} \cup \{\infty\}$. The main result of this chapter is the following:

Theorem 3.1 (Integrability Theorem, first version). *Suppose we have a vector field $V \in L^p(\Omega, \mathbb{R}^2)$ with $p > 1$, whose divergence can be represented by the*

boundary of an integral 1-current I on Ω , i.e.

$$\frac{1}{2\pi} \int V \cdot \nabla \phi = \langle I, d\phi \rangle \quad \forall \phi \in C_c^\infty(\Omega). \quad (3.4)$$

Then there exists a $W^{1,p}$ -function $u : \Omega \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ such that $V = \nabla^\perp u$ and $u|_{\partial\Omega}$ has zero degree. Viceversa, for any $u \in W^{1,p}(\Omega, \mathbb{R}/2\pi\mathbb{Z})$ with $\deg(u|_{\partial\Omega}) = 0$, the vector field $\nabla^\perp u$ belongs to L^p and has divergence equal to the boundary of a current in $\mathcal{I}_1(\Omega)$, in the sense of (3.4).

The zero degree condition on $\partial\Omega$ in the above theorem can be removed in the following way. Consider a L^p -vectorfield V such that

$$\frac{1}{2\pi} \operatorname{div} V = \partial I + \sum_{i=1}^N n_i \delta_{x_i} \quad \text{for some } n_i \in \mathbb{Z} \setminus \{0\} \text{ and } x_i \in \Omega. \quad (3.5)$$

Then we can find, via the Green function method sketched in the introduction, a vectorfield V' satisfying (3.2) and a function $\psi' \in W^{1,p}(\Omega, \mathbb{R}/2\pi\mathbb{Z})$ satisfying (3.1), with

$$\begin{aligned} \deg(\psi'|_{\partial\Omega}) &= \sum_{i=1}^N n_i \\ \frac{1}{2\pi} \operatorname{div}(V - V') &= \partial I, \end{aligned}$$

and we can apply the integrability Theorem to $V - V'$ obtaining a function $\psi \in W^{1,p}(\Omega, \mathbb{R}/2\pi\mathbb{Z})$ with degree zero on $\partial\Omega$ and which satisfies $\nabla^\perp \psi = V - V'$. Then $\psi + \psi'$ will satisfy

$$\begin{aligned} \nabla^\perp(\psi + \psi') &= V \\ \deg((\psi + \psi')|_{\partial\Omega}) &= \sum_{i=1}^N n_i. \end{aligned}$$

With this construction we obtain the following generalization

Corollary 3.2. *Suppose we have a L^p -vector field V satisfying (3.5). Then there exists a $W^{1,p}$ -function $u : \Omega \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ such that $V = \nabla^\perp u$ and $u|_{\partial\Omega}$ has degree $\sum_{i=1}^N n_i$. Viceversa, for any $u \in W^{1,p}(\Omega, \mathbb{R}/2\pi\mathbb{Z})$ with $\deg(u|_{\partial\Omega}) = d \in \mathbb{Z}$, the vector field $\nabla^\perp u$ belongs to L^p and satisfies (3.5), where $d = \sum n_i$.*

In the case $p = 1$, a result similar to the Integrability Theorem above is a sub-case of the result of [5]. An equivalent statement of such result is (see also Section 3.1.2 where different notations are proposed):

Proposition 3.3 ([5]). *For each integral 1-current I of finite mass on Ω there exists a map $\psi \in W^{1,1}(\Omega, \mathbb{R}/2\pi\mathbb{Z})$ such that (in the sense of distributions)*

$$\partial I = \frac{1}{2\pi} \operatorname{div}(\nabla^\perp \psi).$$

The distribution $\operatorname{div}(\nabla^\perp \psi)$ is called *distributional Jacobian* of ψ .

Remark 3.4. *As seen in Example 3.25, for $p > 1$, unlike the case $p = 1$, a large subclass of the boundaries of integral currents is **not** realized as distributional Jacobian of maps in $W^{1,p}(B^2, \mathbb{S}^1)$, therefore we must ask for a higher integrability condition for the current I : this is why the existence of the L^p -vectorfield V is imposed.*

3.1.2 Different formulations of the Integrability Theorem

We have at least three ways of looking at the manifold \mathbb{S}^1 , namely:

1. as a subset of \mathbb{R}^2 : $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$,
2. via a parameterization: $\mathbb{S}^1 = \{(\cos(t), \sin(t)) : t \in \mathbb{R}\}$,
3. as a group quotient: $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$.

When considering $W^{1,p}$ -maps on B^2 with values in \mathbb{S}^1 , these three points of view lead to three possible spaces:

1. $W_1 = \{u \in W^{1,p}(B^2, \mathbb{R}^2) : u_1^2(x) + u_2^2(x) = 1, \text{ a. e. } x \in B^2\}$, which is just the usual definition of $W^{1,p}(B^2, \mathbb{S}^1)$,
2. $W_2 = \{(\cos(\psi), \sin(\psi)) : \psi \in W^{1,p}(B^2, \mathbb{R})\}$,
3. $W_3 = \{u \in W^{1,p}(B^2, \mathbb{R})\} / \sim$, where $u_1 \sim u_2$ if $u_1 - u_2$ is a measurable map with values on $2\pi\mathbb{Z}$ a.e. We denote this space by $W^{1,p}(B^2, \mathbb{R}/2\pi\mathbb{Z})$.

W_1 is isomorphic as a (topological vector space) to W_3 via the diffeomorphism $\phi : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{S}^1$, $t \mapsto (\cos(t), \sin(t))$. On the other hand, the space W_2 is **different** than W_1, W_3 because of the following result:

Theorem 3.5 ([39]). *If $1 \leq p < 2$ and $u \in W^{1,p}(B^n, \mathbb{S}^1)$ then the following statements are equivalent:*

- u can be strongly approximated by smooth maps $u_k \in C^\infty(B^n, \mathbb{S}^1)$

- $d(u^*\theta) = 0$ in the sense of distributions
- There exists $\tilde{u} \in W^{1,p}(B^n, \mathbb{R})$ such that $u = (\cos(\tilde{u}), \sin(\tilde{u}))$.

In our work, the space W_3 seems notationally lighter, but since W_1 is more common, we would like to reformulate the Integrability Theorem here:

Theorem 3.6 (Integrability Theorem, second version). *Let $V \in L^p(\Omega, \mathbb{R}^2)$ with $p > 1$ be a vectorfield satisfying (3.4) for an integral 1-current I . Then there exist a map $u \in W^{1,p}(\Omega, \mathbb{S}^1)$ with degree zero on $\partial\Omega$ such that $V = u_2 \nabla^\perp u_1 - u_1 \nabla^\perp u_2$. Viceversa, for any map $u \in W^{1,p}(\Omega, \mathbb{S}^1)$ with zero degree on the boundary, the vectorfield $u_2 \nabla^\perp u_1 - u_1 \nabla^\perp u_2$ is in L^p and has divergence equal to the boundary of an integral current.*

We describe how to pass from the first to the Theorem in Section 3.3.1.

Our result can be reformulated in somewhat more geometrical terms by identifying differential forms $\alpha \in L^p(\Omega, \wedge^1 \Omega)$ with vector fields.¹ $V_\alpha \in L^p(\Omega, \mathbb{R}^2)$ by setting $V_\alpha = (\alpha_2, -\alpha_1)$ if $\alpha = \alpha_1 dx + \alpha_2 dy$, so that $d\alpha$ corresponds to $\operatorname{div} V_\alpha$. We also observe that if we consider the tangent space of $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ to be identified with \mathbb{R} in the canonical way, then $V_{u^*\theta}$ can be identified with $\nabla^\perp u$. We obtain therefore the following alternative formulation:

Theorem 3.7 (Integrability Theorem, third version). *Let $p > 1$, let Ω be either a regular open domain in \mathbb{R}^2 or the sphere \mathbb{S}^2 , and let θ be the volume form of \mathbb{S}^1 . Then the following equality holds*

$$\begin{aligned} \{u^*\theta : u \in W^{1,p}(\Omega, \mathbb{S}^1), \operatorname{deg}(u|_{\partial\Omega}) = 0\} \\ = \\ \{\alpha : \alpha \in L^p(\Omega, \wedge^1 \mathbb{R}^2), \exists I \in \mathcal{I}_1(\Omega), [d\alpha] = \partial I\}, \end{aligned}$$

where $\mathcal{I}_1(\Omega)$ represents the finite mass integral rectifiable 1-currents on Ω and $[d\alpha]$ is the distribution associated to $d\alpha$ by imposing

$$\langle [d\alpha], \varphi \rangle = \int_{\Omega} d\alpha \wedge \varphi \quad \forall \varphi \in \mathcal{D}_0(\Omega).$$

¹This is a special instance of the identification of k -covectors α with $(n-k)$ -vectors V in an n -dimensional oriented manifold M given by imposing

$$\langle \beta, V \rangle = \langle \beta \wedge \alpha, \vec{M} \rangle$$

for all $(n-k)$ -covectors β , where \vec{M} is an orienting n -vector field of M .

3.1.3 Ingredients of the proof

The proof of the first part of our theorem follows from a density result: We prove that the class of L^p -vectorfields with finitely many topological singularities is dense in the class of vectorfields satisfying the condition (3.4). This fact is proved in Section 3.2. The proof is in the spirit of the work [19] of Bethuel (see also [18, BCDH91, 30, 65] for related results) and is inspired by the ideas present in [84].

It is easy to prove the first part of the Integrability Theorem for V having finitely many singularities. We can then pass to the limit the $W^{1,p}$ -maps u_k obtained in the simpler case for an approximating sequence $V_k \xrightarrow{L^p} V$, in order to achieve the representation result in the first part of the Integrability Theorem (see Section 3.3).

The second part of the theorem is a direct consequence of a coarea formula (see for example [89]), which is related to the Sard theorem for Sobolev spaces (for which see among others [24, 41, 56]). We state here just the result that we need:

Theorem 3.8. *If $f \in W_{loc}^{1,p}(M^m, N^n)$ for some manifolds M, N , then there exists a Borel representative of f such that $f^{-1}(y)$ is countably $(m - n)$ -rectifiable and has finite \mathcal{H}^{m-n} -measure for almost all $y \in N$ and such that for every measurable function g there holds*

$$\int_M g(x) |J_f(x)| d\mathcal{H}^m(x) = \int_N \left(\int_{f^{-1}(y)} g(x) d\mathcal{H}^{m-n}(x) \right) d\mathcal{H}^n(y), \quad (3.6)$$

where $|J_f(x)| = \sqrt{\det(Df_x \cdot Df_x^T)}$.

3.2 A density result

We consider two classes of vector fields:

$$\mathcal{V}_{\mathbb{Z}} := \{V \in L^p(D, \mathbb{R}^2) : (3.4) \text{ holds}\},$$

and

$$\mathcal{V}_R := \{V \in \mathcal{V}_{\mathbb{Z}} : V \text{ is smooth outside a finite set } S \subset D\}.$$

Since $\mathcal{V}_{\mathbb{Z}}$ is closed in L^p , it is clear that $\overline{\mathcal{V}_R}^{L^p} \subset \mathcal{V}_{\mathbb{Z}}$. We desire to prove the following result:

Proposition 3.9. *With the above notations, $\overline{\mathcal{V}_R}^{-L^p} = \mathcal{V}_\mathbb{Z}$ holds.*

By the remarks about \mathcal{V}_R and $\mathcal{V}_\mathbb{Z}$, we just have to prove that any $V \in \mathcal{V}_\mathbb{Z}$ can be approximated up to an arbitrary small error $\varepsilon > 0$ in L^p -norm, by some $V_\varepsilon \in \mathcal{V}_R$. The strategy of our proof is to choose first a “grid of circles of radius r ”, on which we mollify appropriately V , and then to extend the mollified vector field inside each circle by creating finitely many singularities (note that the number of singularities might become unbounded for $r \rightarrow 0$), and by staying L^p -near the initial V . Finally, we will patch together the extensions on each of the balls bounded by these circles, obtaining the desired approximant V_ε . The way in which we “fill the r -balls” will be by either radial or harmonic extension: we decide the method to apply depending on the degree of V_m on the respective ball (we are guided in this by the result of Demengel [39] cited in Theorem 3.5).

3.2.1 Choice of a good covering

Lemma 3.10. *Given $r > 0$, there exists a natural number N , a set of centers $\{x_1, \dots, x_N\}$ and a positive measure subset $E \subset [3/4r, r]^N$ such that for all $(r_1, \dots, r_N) \in E$*

- *The balls $\{B_1, \dots, B_N\}$, where $B_i = B_{r_i}(x_i)$ cover B^2 .*
- *The smaller balls $B_{\frac{3}{8}r_i}(x_i)$ are disjoint.*
- *For some constant depending only on p and on the dimension, there holds*

$$\sum_{i=1}^N \int_{\partial B_i} |V \cdot n_{B_i}|^p dx \leq C_{2,p} r^{-1} \|V\|_{L^p(B^2)}^p, \quad (3.7)$$

where n_{B_i} is the outer normal to the ball B_i .

Proof. See Section 3.4 □

The next lemma is needed in order to translate properties of the current I to the vector field V .

Lemma 3.11 (Slicing of 1-currents). *Given a piecewise smooth domain $\Omega \subset B^2$, for almost all $t \in [-\varepsilon, \varepsilon]$ the following properties hold:*

- *The slice $\langle I, \text{dist}_{\partial\Omega}, t \rangle$ exists and is a rectifiable 0-current with multiplicity in $2\pi\mathbb{Z}$.*

- The map $\int_{\partial\Omega_t} V(y) \cdot n_t(y) d\mathcal{H}^1(y)$ (where n_t is the unit normal to $\partial\Omega_t$) is well-defined and coincides with the number $\langle I, \text{dist}_{\partial\Omega}, t \rangle(1) \in 2\pi\mathbb{Z}$.

Proof. See Section 3.5 □

Combining the Lemmas 3.10 and 3.11 we obtain:

Lemma 3.12. *Given $r > 0$, there exists a set of balls $\{B_1, \dots, B_N\}$ with radii in $[3/4r, r]$ such that the thesis of Lemma 3.10 holds and that for any Ω which is the closure of a connected component of $B^2 \setminus \cup_{i=1}^N \partial B_i$ the slice $\langle I, \text{dist}_{\partial\Omega}, 0 \rangle$ exists, is a rectifiable 0-current with multiplicity in $2\pi\mathbb{Z}$ and*

$$\langle I, \text{dist}_{\partial\Omega}, 0 \rangle(1) = \int_{\partial\Omega} V(y) \cdot n_\Omega(y) d\mathcal{H}^1(y) \in 2\pi\mathbb{Z}.$$

Proof. We can use Lemma 3.10 first, obtaining a set $E \subset [3/4r, r]^N$. For a cover $\{B'_1, \dots, B'_N\}$ corresponding to a density point of E , we can then apply Lemma 3.11 for all the closures of connected components of $B^2 \setminus \cup \partial B'_i$, and then consider the slices for $t \leq 0$ only. □

3.2.2 Mollification on the boundary and estimates on good and bad balls

Lemma 3.13. *For a choice of balls B_i as in Lemma 3.12, it is possible to find a vector field $V_m \in C^\infty(\cup_i \partial B_i, \mathbb{R}^2)$ such that for all the regions Ω as in Lemma 3.12 there holds*

$$\forall i, \int_{\partial\Omega} V_m \cdot n_\Omega d\mathcal{H}^1 = \int_{\partial\Omega} V \cdot n_\Omega d\mathcal{H}^1 \in 2\pi\mathbb{Z} \quad (3.8)$$

$$\|V_m - V\|_{L^p(\cup_i \partial B_i)} \leq \varepsilon_m. \quad (3.9)$$

Proof. It is enough to find V_m satisfying (3.8) and (3.9), and defined only on $\cup_i \partial B_i \setminus \{x : \exists i \neq j, x \in \partial B_i \cap \partial B_j\} := \cup_i \partial B_i \setminus I$. Indeed, then we can modify it on a neighborhood of I in $\cup_i \partial B_i$, defining a global smooth vector field, without affecting the requirements (3.8) and (3.9). See Figure 3.1.

We now find V_m as above. From Lemma 3.12 it follows that $\sum_i \chi_{\partial B_i} V \cdot n_{B_i} \in L^p(\cup_i \partial B_i)$ and has integral in $2\pi\mathbb{Z}$. Therefore we can take its mollification as a definition of the normal component of V_m , automatically satisfying (3.8) by the properties of the mollification. Then we can mollify the component of V parallel to $\cup \partial B_i$, and take the resulting function as the parallel component of V_m , thereby verifying (3.9) too. □

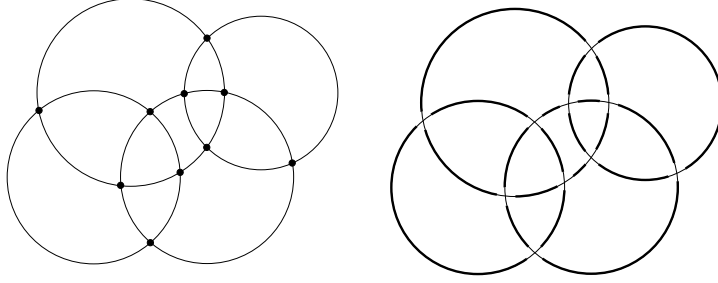


Figure 3.1: We represent schematically the procedure used to construct the vectorfield of Lemma 3.13. V_m is initially defined outside the finite set of points I which is marked thicker in the drawing on the left. Then we keep V_m fixed on the set which is thick on the right, and modify it near the crossings to obtain the final vectorfield.

Lemma 3.14. Suppose \mathcal{B}_n are families of finitely many balls which cover B^2 such that each point is not covered more than C times and

$$\max_{B \in \mathcal{B}_n} (\text{diam} B) \rightarrow 0 \quad (n \rightarrow \infty)$$

Then there holds

$$\sum_{B \in \mathcal{B}_n} \|V - \bar{V}\|_{L^p(B)} \rightarrow 0 \quad (n \rightarrow \infty). \quad (3.10)$$

Proof. We take a smooth approximant $W = W_\varepsilon$ such that

$$\|V - W\|_{L^p(B^2)} \leq \varepsilon/4C.$$

Then, we can use Poincaré's inequality

$$\|W - \bar{W}\|_{L^p(B)} \leq Cr_B^{1/p} \|\nabla W\|_{L^p(B)},$$

and for n big enough there will hold

$$\sum_{B \in \mathcal{B}_n} \|W - \bar{W}\|_{L^p(B)} \leq \varepsilon/2.$$

Putting together the above two estimates, we obtain

$$\begin{aligned} \sum_{B \in \mathcal{B}_n} \|V - \bar{V}\|_{L^p(B)} &\leq \\ &\leq \sum_{B \in \mathcal{B}_n} \|V - W\|_{L^p(B)} + \sum_{B \in \mathcal{B}_n} \|W - \bar{W}\|_{L^p(B)} + \sum_{B \in \mathcal{B}_n} \|\bar{V} - \bar{W}\|_{L^p(B)} \\ &\leq 2 \sum_{B \in \mathcal{B}_n} \|V - W\|_{L^p(B)} + \varepsilon/2 \\ &\leq 2C \|V - W\|_{L^p(B^2)} + \varepsilon/2 \\ &\leq \varepsilon, \end{aligned}$$

as desired. \square

We now distinguish the balls B_i based on the value of the integral $\int_{\partial B_i} V \cdot n_{B_i} d\mathcal{H}^1$: we call B_i a **good** ball in case such integral is zero, and a **bad** ball in case it is in $2\pi\mathbb{Z} \setminus \{0\}$.

Lemma 3.15. *There exists a constant $C > 0$ such that if we have a cover as in Lemma 3.10 with radii not greater than $r := \varepsilon$, then the number of bad balls satisfies the following estimate:*

$$\#(\text{bad balls}) \leq C\varepsilon^{p-2} \|V\|_{L^p}^p.$$

Proof. For a bad ball B we have

$$1 \leq \left| \int_{\partial B} V \cdot n_B d\mathcal{H}^1 \right|,$$

whence we deduce successively

$$1 \leq C\varepsilon^{p-1} \int_{\partial B} |V \cdot n_B|^p d\mathcal{H}^1$$

and (by summing and using Lemma 3.10)

$$\#(\text{bad balls}) \leq C\varepsilon^{p-1} \sum_{B \text{ bad}} \int_{\partial B} |V \cdot n_B|^p d\mathcal{H}^1 \leq C\varepsilon^{p-2} \|V\|_{L^p}^p,$$

as desired. \square

Remark 3.16. *We observe that by Theorem 3.5, on a good ball the normal component*

$$v_m - v : \partial B_i \rightarrow \mathbb{R}^2, \quad v_m - v = n_{B_i} [(V_m - V) \cdot n_{B_i}]$$

satisfies $v_m - v = \nabla^\perp a_m$ for some $W^{1,p}$ -function $a_m : \partial B_i \rightarrow \mathbb{R}$.

Remark 3.17 (explanation of the notation). *If we associate to the form $\alpha = \alpha_1 dx + \alpha_2 dy$ the vectorfield $V_\alpha = (\alpha_2, -\alpha_1)$, then in an orthonormal frame n_{B_i}, t_{B_i} given by the normal and tangential unit vectors on ∂B_i we see that taking the normal projection done on vectorfields, corresponds to restricting the associated form α , obtaining $i_{\partial B_i}^* \alpha$ where $i_{\partial B_i} : \partial B_i \rightarrow \Omega$ is the inclusion. We can explain our notations above by saying that objects arising from restrictions of forms will be denoted by lower case letters.*

The following is a well-known result from the theory of elliptic PDEs.

Lemma 3.18. *Let \tilde{a} be a function on the boundary of the unit 2-ball \mathbb{S}^1 having zero mean. Consider the harmonic extension \tilde{A} of \tilde{a} over B_1 satisfying*

$$\begin{cases} \Delta \tilde{A} = 0 \\ \tilde{A} = \tilde{a} \text{ on } \mathbb{S}^1 \end{cases} \quad (3.11)$$

Then the following estimate holds:

$$\|\nabla \tilde{A}\|_{L^p(B_1)} \leq C \|\nabla \tilde{a}\|_{L^p(\mathbb{S}^1)}. \quad (3.12)$$

We will consider a'_m on the boundary ∂B of a small ball instead of \tilde{a} on ∂B_1 , and obtain a harmonic extended function, denoted by A'_m , satisfying the analogue of (3.11). Taking into account the scaling factors we then obtain the following estimate analogous to (3.12) on a ball B_r of radius r :

$$\|\nabla A'_m\|_{L^p(B_r)} \leq Cr^{1/p} \|v_m - v\|_{L^p(\partial B_r)}. \quad (3.13)$$

We claim that extending $V_m := \nabla^\perp A'_m + \bar{V}$ inside B_r , we obtain the desired approximation:

Lemma 3.19. *If B is a **good ball** of radius ε on whose boundary we have $\|V - V_m\|_{L^p(\partial B)} < \varepsilon$, then the extended smooth vector field V_m defined as above satisfies on B*

$$\|V - V_m\|_{L^p(B)} \leq C\varepsilon^{\frac{p-1}{p}} \|v_m - v\|_{L^p(\partial B)} + \|V - \bar{V}\|_{L^p(B)}.$$

Proof. We can then write

$$\|V - V_m\|_{L^p(B)} \leq \|V - \bar{V}\|_{L^p(B)} + \|\nabla^\perp A'_m\|_{L^p(B)}.$$

The second term above is estimated as in (3.13), by $C\varepsilon^{1/p} \|v_m - v\|_{L^p(\partial B)}$, and the estimate (3.9) gives then $\varepsilon \|v_m - v\|_{L^p(\partial B)}^p \leq C\varepsilon^{p-1}$, finishing the proof. \square

Lemma 3.20. *If $B \subset B_1^2$ is a **bad ball** of radius ε and v_m is the smooth orthogonal vector field on ∂B related to V_m as in Lemma 3.13 and V'_r is the radial extension $V'_r(\theta, \rho) := \frac{\varepsilon}{\rho} v_m(\theta)$ (in polar coordinates centered in the center of B), then with the notation $V_r := V'_r - \bar{V}$, we have the estimate:*

$$\|V - V_r\|_{L^p(B)} \leq \|V - \bar{V}\|_{L^p(B)} + C\varepsilon.$$

Proof. There holds

$$\begin{aligned} \|V - V_r\|_{L^p(B)} &\leq \|V - \bar{V}\|_{L^p(B)} + \|V'_r\|_{L^p(B)}, \\ \|V'_r\|_{L^p(B)}^p &= \int_0^\varepsilon \int_0^{2\pi} \left(\frac{\varepsilon}{\rho}\right)^p |v_m(\theta)|^p d\theta \rho d\rho \\ &= C\varepsilon^2 \|v_m\|_{L^p(\partial B)}^p. \end{aligned}$$

From (3.7), (3.9) and the last equality above we conclude that $\|V'_r\|_{L^p(B)}^p \leq C\varepsilon^p$, as desired. \square

3.2.3 End of proof of Proposition 3.9

Application of Lemmas 3.19 and 3.20

We will use the results of Section 3.2.2 in order to achieve a first global approximation V_1 of V . We again start with the ball B_1 , where we will use Lemma 3.19 or 3.20, respectively when B_1 is a good or a bad ball. The new vector field V_1 obtained by replacing V with the so obtained local approximant on B_1 satisfies the following properties:

- **Good approximation of V on B_1 :** The approximation error in L^p -norm on the ball B_1 is bounded above by $C\varepsilon^{1/p} + \|V - \bar{V}\|_{L^p(B_1)}$.
- **Controlled behavior on the boundary:** The extension inside B_1 is equal to $\nabla^\perp A'_m + \bar{V}$ on the boundaries of the B_i 's, and in particular it has degree equal either to the one of V_m or to zero on any of the boundaries of the domains Ω of Lemma 3.13. Indeed, A'_m is smooth, so $V_m|_{B_1}$ will have divergence either zero (for good balls) or a Dirac mass in the center of B_1 (for bad balls), while on $B_1 \setminus \cup_i \partial B_i$, $V_m = V_1$. Therefore V_1 also has the properties stated in Lemma 3.13.

This allows us to apply iteratively the above construction for the balls $B_j, j = 2, \dots, N$, in order to further modify V_1 . We obtain successively approximants V_2, \dots, V_N according to Lemmas 3.19, 3.20, and we are able to continue ensuring the smallness condition $\|V - V_m\|_{L^p(\partial B_j)}$.

Lemma 3.21. *For each $\bar{\varepsilon} > 0$ there exist a radius bound ε and an approximation error bound ε_m (in Lemma 3.13) such that the approximant V_N constructed above satisfies*

$$\|V - V_N\|_{L^p(B^2)} \leq \bar{\varepsilon}.$$

Proof. Since in Lemma 3.10 the balls $B_{\frac{3}{8}r_i}(x_i)$ are disjoint, we see that no point is covered by more than \bar{C} balls B_i , where \bar{C} is a geometric packing constant depending on our domain Ω . Therefore in our construction we modify our initial V at most \bar{C} times at each point. This induces a factor \bar{C} in our estimates.

By Lemmas 3.19 and 3.20 we can estimate

$$\begin{aligned} \|V - V_N\|_{L^p(B^2)} &\leq \bar{C} \sum_{\text{good } B} \left[C\varepsilon^{\frac{p-1}{p}} \|v_m - v\|_{L^p(\partial B)} + \|V - \bar{V}\|_{L^p(B)} \right] \\ &\quad + \bar{C} \sum_{\text{bad } B} \left[\|V - \bar{V}\|_{L^p(B)} + C\varepsilon \right] \\ &= \bar{C} \sum_{\text{all } B} \|V - \bar{V}\|_{L^p(B)} + C\varepsilon \#(\text{bad balls}) + C\varepsilon^{\frac{p-1}{p}} \varepsilon_m. \end{aligned}$$

Consider now the expression in the last row above: the first term converges to zero by Lemma 3.14, and the last one is small for ε_m small. The middle term can be estimated using Lemma 3.15 and has thus a bound of the form $C\varepsilon^{p-1} \|V\|_{L^p}^p$. Since $p > 1$ and $V \in L^p$, also this term is small for ε small. \square

Smoothing on the boundary

The preceding iteration procedure gives us an L^p -approximant with error $C\varepsilon$ if the radius r of the balls was chosen to be equal to ε . Moreover it is easy to verify that

$$\operatorname{div} V_N = \sum_{i=1}^N \delta_{x_i} \int_{\partial B_i} V_m, \quad \text{locally outside } \cup_i \partial B_i \quad (3.14)$$

where x_i is the center of B_i . The resulting vector field V_N is however not in \mathcal{V}_R : for instance, it is not smooth on all of $\cup_i \partial B_i$. We will thus mollify V_N as follows. We observe that locally near $\cup_i \partial B_i$ on $B^2 \setminus \cup_i \partial B_i$, V_N is represented as $\nabla^\perp A_i := \nabla^\perp A'_i + \bar{V}_i$, where A'_i is smooth and \bar{V}_i is a constant equal to the average of V on a particular B_i . We can take an open cover by small balls of a neighborhood of $\cup_i \partial B_i$ then mollify the functions A_i inside each of these small balls, then use a partition of unity to patch the mollifications into a single smooth function A_ε , introducing an error of less than ε in L^p -norm. Then we can safely define $V_\varepsilon := \nabla^\perp A_\varepsilon$. \square

3.3 Proof of the Integrability Theorem

Proof. We first show how to deduce the second part of Integrability Theorem 3.1 from Proposition 3.9.

The main idea is that, by Proposition 3.9, we can take a sequence $V_n \xrightarrow{L^p} V$ which belongs to \mathcal{V}_R and construct u_n 's such that $V_n = \nabla^\perp u_n$, and they will

be constrained to converge to a u with the desired property $\nabla^\perp u = V$. We remark that if V_n is smooth and divergence-free outside a discrete set Σ , then V_n^\perp is locally holomorphic, and the fact that the divergence around any point of Σ is a Dirac mass with coefficient in $2\pi\mathbb{Z}$ translates into saying that V_n^\perp has degree equal to that coefficient around that point. Consider the divisor D supported on Σ with residue corresponding to the divergence of V_n . In complex notation V_n^\perp becomes a meromorphic function with divisor D , so we can take $u_n := \arg V_n^\perp$, which is well-defined with values in $\mathbb{R}/2\pi\mathbb{Z}$ and satisfies $\nabla u_n = V_n^\perp$.

We have thus functions $u_n \in W^{1,p}(\Omega, \mathbb{S}^1)$ satisfying $V_n = \nabla^\perp u_n$ and therefore $\nabla u_n \xrightarrow{L^p} V$. We can change the u_n by a constant so that $\frac{1}{|\Omega|} \int_\Omega u_n = 0 \in \mathbb{R}/2\pi\mathbb{Z}$. Then by Poincaré's inequality we have that u_n form a L^p -Cauchy sequence, converging therefore to $\bar{u} \in L^p(\Omega, \mathbb{R}/2\pi\mathbb{Z})$. After extracting a subsequence $u_n \xrightarrow{W^{1,p}} u \in W^{1,p}$. Since we have a.e.-convergence too, it must hold $u = \bar{u}$ and $\nabla^\perp u = V$, as desired.

As above, $u_n \xrightarrow{W^{1,p}} u$ and $d(u_n^* \theta)$ are finite sums of Dirac masses with integer coefficients. The fact that for $u \in W^{1,p}(\Omega, \mathbb{R}/2\pi\mathbb{Z})$ the vectorfield $\nabla^\perp u$ has the properties required in the theorem, follows from Theorem 3.8, by taking

$$I = I_z^u = \tau \left(u^{-1}(z), 1, \frac{\nabla^\perp u(x)}{|\nabla^\perp u(x)|} \right),$$

for a common regular value $z \in \mathbb{R}/2\pi\mathbb{Z}$ of all the u_n and of u . With this choice, using the coarea formula (observe that in our case $|J_u| = |\nabla^\perp u|$), we obtain, for all $f \in C_c^\infty(\Omega)$,

$$\begin{aligned} \int_\Omega u^* \theta \wedge df &= \int_\Omega \nabla^\perp u \cdot \nabla f dx = \int_{\mathbb{S}^1} dy \int_{u^{-1}(y)} \left\langle df, \frac{\nabla^\perp u}{|\nabla^\perp u|} \right\rangle d\mathcal{H}^1 \\ &= \int_{\mathbb{S}^1} I_y^u(df) dy = \int_{\mathbb{S}^1} \partial I_y^u(f) dy. \end{aligned}$$

Similarly we obtain for all n :

$$\int_\Omega u_n^* \theta \wedge df = \int_{\mathbb{S}^1} \partial I_y^{u_n}(f) dy = 2\pi \partial I_z^{u_n}(f),$$

since for functions u_n having finitely many singularities, $\partial I_y^{u_n}(f)$ does not depend on y . We have (since $C_c^\infty \subset (W^{1,p})^*$)

$$\int_{\mathbb{S}^1} \partial I_y^{u_n}(f) dy \rightarrow \int_{\mathbb{S}^1} \partial I_y^u(f) dy,$$

Without loss of generality, recalling Theorem 3.8 we may assume that the integrands on the left converge pointwise at z , and that the mass $\mathbb{M}(I_z^u)$ is bounded. This proves the condition (3.4) with $I = I_z^u$, thus finishing the proof. \square

3.3.1 Proof of Theorem 3.1

Proof. We consider the diffeomorphism $\varphi : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{S}^1 \subset \mathbb{R}^2$ given by $t \mapsto (\cos t, \sin t)$, and then instead of the map $u : \Omega \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ obtained in the Integrability Theorem 3.1 we take the map $\bar{u} := \varphi \circ u : \Omega \rightarrow \mathbb{S}^1 \subset \mathbb{R}^2$. We then obtain

$$\begin{aligned} \nabla \bar{u} &= \nabla u \otimes (\nabla \varphi \circ u) \\ &= \begin{pmatrix} -\partial_1 u \sin u & \partial_1 u \cos u \\ -\partial_2 u \sin u & \partial_2 u \cos u \end{pmatrix}, \end{aligned}$$

therefore

$$\bar{u}_1 \nabla^\perp \bar{u}_2 - \bar{u}_2 \nabla^\perp \bar{u}_1 = \cos^2 u \begin{pmatrix} -\partial_2 u \\ \partial_1 u \end{pmatrix} + \sin^2 u \begin{pmatrix} -\partial_2 u \\ \partial_1 u \end{pmatrix} = \nabla^\perp u.$$

This proves the desired identifications, and we only need to prove that if $\bar{u} \in W^{1,p}(\Omega, \mathbb{S}^1)$ then $\bar{u}_1 \nabla^\perp \bar{u}_2 - \bar{u}_2 \nabla^\perp \bar{u}_1 \in L^p(\Omega, \mathbb{R}^2)$. This follows using the relation $\bar{u}_1^2 + \bar{u}_2^2 = 1$ and its consequence $\bar{u}_1 \nabla^\perp \bar{u}_1 = -\bar{u}_2 \nabla^\perp \bar{u}_2$. We have indeed:

$$\begin{aligned} |\bar{u}_1 \nabla^\perp \bar{u}_2 - \bar{u}_2 \nabla^\perp \bar{u}_1|^2 &= \bar{u}_1^2 |\nabla^\perp \bar{u}_2|^2 - 2\bar{u}_1 \bar{u}_2 \nabla^\perp \bar{u}_2 \nabla^\perp \bar{u}_1 + \bar{u}_2^2 |\nabla^\perp \bar{u}_1|^2 \\ &= (\bar{u}_1^2 + \bar{u}_2^2) |\nabla^\perp \bar{u}_2|^2 + (\bar{u}_1^2 + \bar{u}_2^2) |\nabla^\perp \bar{u}_1|^2 \\ &= (\partial_2 \bar{u}_2)^2 + (\partial_1 \bar{u}_2)^2 + (\partial_2 \bar{u}_1)^2 + (\partial_1 \bar{u}_1)^2 \\ &= |\nabla \bar{u}|^2, \end{aligned}$$

and since $u \in W^{1,p}$, this proves the result. \square

3.4 Proof of Proposition 3.10

Our aim here is to prove the following

Proposition 3.22. *Given $r > 0$, there exists a cover of B_1^2 by a finite set of balls $\{B_r(y_1), \dots, B_r(y_N)\}$ such that the balls $B_{r/2}(y_i)$ are disjoint and such that for some constant depending only on p and on the dimension,*

$$\sum_{i=1}^N \int_{\partial B_r(y_i)} |V \cdot n_{B_r(y_i)}|^p dx \leq C_{2,p} r^{-1} \|V\|_{L^p(B^2)}^p, \quad (3.15)$$

where $n_{B_r(y_i)}$ is the outer unit normal vector to the circle $\partial B_r(y_i)$.

Directly from the proof of Proposition 3.22 we can also obtain the more refined result:

Proposition 3.23. *Given $r > 0$, there exists a natural number N , a set of centers $\{x_1, \dots, x_N\}$ and a positive measure subset $E \subset [3/4r, r]^N$ such that for all $(r_1, \dots, r_N) \in E$*

- The balls $\{B_1, \dots, B_N\}$, where $B_i = B_{r_i}(x_i)$ cover B^2 .
- The smaller balls $B_{\frac{3}{8}r_i}(x_i)$ are disjoint.
- For some constant depending only on p and on the dimension, there holds

$$\sum_{i=1}^N \int_{\partial B_i} |V \cdot n_{B_i}|^p dx \leq C_{2,p} r^{-1} \|V\|_{L^p(B^2)}^p.$$

3.4.1 Equivalent definition of the pointwise norm of V

$\langle V, \theta \rangle$ for a vector $\theta \in \mathbb{S}^1 \subset \mathbb{R}^2$, can be expressed as $|V| |\cos \gamma|$ where γ is the angle between θ and V . After noting

$$\int_{\mathbb{S}^1} |\cos \gamma|^p d\theta =: c_p,$$

we can write

$$|V|^p = \frac{1}{c_p} \int_{\mathbb{S}^1} |\langle V, \theta \rangle|^p d\theta. \quad (3.16)$$

We now pass to consider the circle $S_r(x) = \partial B_r(x)$. Then we can write

$$\int_{S_r(x)} V(y) \cdot n_{B_r}(y) dy = \int_{S_r(x)} \left\langle V(y), \left(\frac{y-x}{|y-x|} \right) \right\rangle dy = \int_{\mathbb{S}^1} \langle V(x+r\theta), \theta \rangle r d\theta.$$

Given a positive number r , a point $x \in \mathbb{R}^2$ then belongs to $S_r(y)$ exactly for $y \in S_r(x)$, and we have by (3.16), that

$$\begin{aligned} \int_{S_r(x)} |V(x) \cdot n_{B_r(y)}(x)|^p dy &= \int_{S_r(x)} \left| \left\langle V(x), \left(\frac{x-y}{|x-y|} \right) \right\rangle \right|^p dy \\ &= \int_{\mathbb{S}^1} |\langle V(x), \theta \rangle|^p r d\theta \\ &= c_p r |V(x)|^p. \end{aligned} \quad (3.17)$$

3.4.2 Proposition 3.22 and an extension of it

Proof of Proposition 3.22: We observe that (3.17) can be integrated on \mathbb{R}^2 (after having extended V by zero outside B^2), to give

$$\begin{aligned}
c_p r \int_{B^2} |V(x)|^p dx &= c_p r \int_{\mathbb{R}^2} |V(x)|^p dx \\
&= \int_{\mathbb{R}^2} \int_{S_r(x)} |V(x) \cdot n_{B_r(y)}(x)|^p dy dx \\
&= \int_{\mathbb{R}^2} \int_{S_r(z)} |V(x) \cdot n_{B_r(z)}(x)|^p dx dz \\
&= \int_{B_{1+r}^2} \int_{S_r(z)} |V(x) \cdot n_{B_r(z)}(x)|^p dx dz. \quad (3.18)
\end{aligned}$$

We now define some systems of disjoint balls. We consider a set

$$S = \{x_1, \dots, x_N\} \subset B_{1+r}^2 \text{ s.t. } \begin{cases} \min_{1 \leq i \neq j \leq N} d(x_i, x_j) \geq r \\ S \text{ is maximal} \end{cases} \quad (3.19)$$

and the corresponding set of translates of the ball $B_r(0)$.

$$\mathcal{S} := S + B_r(0) = \{\{x_1 + y, \dots, x_N + y\} : y \in B_r(0)\}$$

Then \mathcal{S} covers B_{1+r} (by maximality in the definition of S) at most C times, where C is a packing number (by the requirement on the mutual distances of elements of S). We can then bound the integral (3.18) from below as follows

$$\begin{aligned}
c_p r \int_{B^2} |V(x)|^p dx &= \int_{B_{1+r}^2} \int_{S_r(z)} |V(x) \cdot n_{B_r(z)}(x)|^p dx dz \\
&\geq \frac{1}{C} \int_{B_r} \left(\sum_{i=1}^N \int_{S_r(x_i+z)} |V \cdot n|^p dy \right) dz
\end{aligned}$$

and it follows that there exists $z \in B_r$ such that

$$\sum_{i=1}^N \int_{S_r(x_i+z)} |V \cdot n|^p dy \leq \frac{C c_p r}{|B_r|} \int_{B^2} |V|^p dx = C_{2,p} r^{-1} \|V\|_{L^p(B^2)}^p.$$

This is enough to prove (3.15). Moreover, again by the maximality of S_0 , the balls $\{B_r(x_i+z)\}_{i=1}^N$ cover B_1^2 , and by the requirement on the distances of the centers in (3.19), the $B_{r/2}(x_i+z)$ are disjoint, proving Proposition 3.22. \square

3.5 Proof of the slicing Lemma

We prove the Lemma 3.11. Suppose that we are given a vector field $V \in L^p(B^2, \mathbb{R}^2)$, for some $p \neq \infty$, such that for some integer multiplicity rectifiable current I we have $\operatorname{div} V = \partial I$. This means more precisely that

$$\int V \cdot \nabla \phi = \langle I, d\phi \rangle, \text{ for all functions } \phi \in C_0^\infty(B^2). \quad (3.20)$$

Here $\langle I, d\phi \rangle$ refers to the action of the current I on the 1-form $d\phi$. If Ω is a piecewise smooth domain, we will also call $\partial\Omega_t$ the set $\{x \text{ s.t. } \operatorname{dist}_{\partial\Omega}(x) = t\}$. By $\operatorname{dist}_{\partial\Omega}$ we here denote the *oriented* distance from $\partial\Omega$, i.e. the function defined on a small neighborhood of $\partial\Omega$ and equal to $\operatorname{dist}_\Omega$ outside Ω and to $-\operatorname{dist}_{\Omega^c}$ inside Ω . Our aim in this section is to prove the following

Proposition 3.24. *Given a piecewise smooth domain $\Omega \subset B^2$, for almost all $t \in [-\varepsilon, \varepsilon]$ the following properties hold:*

- *The slice $\langle I, \operatorname{dist}_{\partial\Omega}, t \rangle$ exists and is a rectifiable 0-current with multiplicity in $2\pi\mathbb{Z}$.*
- *The map $\int_{\partial\Omega_t} V(y) \cdot n_t(y) d\mathcal{H}^1(y)$ (where n_t is the unit normal to $\partial\Omega_t$) is well-defined and coincides with the number $\langle I, \operatorname{dist}_{\partial\Omega}, t \rangle(1) \in 2\pi\mathbb{Z}$.*

Proof. We consider a family of symmetric mollifiers $\varphi_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^+$ supported in $[-\varepsilon, \varepsilon]$, and their primitives $\chi_\varepsilon(x) := \int_{-\infty}^x \varphi_\varepsilon dt$. We will consider a non negative function g which is C_c^∞ -extensions to a neighborhood of $\partial\Omega$ of the constant function equal to 1 on all the Ω_t 's with $t \in [-2\varepsilon, 2\varepsilon]$, and we write the current I as $(\mathcal{M}_I, \theta_I, \tau_I)$, where \mathcal{M}_I is a 1-rectifiable set supporting the current I , τ_I is the orienting vector of I and θ_I is the multiplicity of I . Then the currents approximating the slice $\langle I, f, t \rangle$ (for some Lipschitz function $f : B^2 \rightarrow [-2\varepsilon, 2\varepsilon]$), when it exists, satisfy:

$$\begin{aligned} I \llcorner f^\#(\varphi_\varepsilon(\cdot - t) d\tau)(g) &= \int_{\mathcal{M}_I} \langle \tau_I(x), g(x) \varphi_\varepsilon(f(x) - t) df_x \rangle d\mathcal{H}^1(x) \quad (3.21) \\ &= \int_{\mathcal{M}_I} \langle \tau_I(x), g(x) d(\chi_\varepsilon(\cdot - t) \circ f)_x \rangle d\mathcal{H}^1(x) \\ &= \langle I \llcorner g, dF_\varepsilon \rangle \quad \text{where } F_\varepsilon(x) := \chi_\varepsilon(f_\varepsilon(x) - t), \\ &= \langle I, dF_\varepsilon \rangle \quad \text{since } \operatorname{spt} F_\varepsilon \subset \{g = 1\}, \\ &= \int_{\{x: |f(x)-t| \leq \varepsilon\}} V \cdot \nabla F_\varepsilon dx^2 \quad (\text{by (3.20)}). \end{aligned}$$

Now we take $f(x) := \operatorname{dist}_{\partial\Omega}(x)$, obtaining that a.e. on a tubular neighborhood

$$T(\Omega, 2\varepsilon) := \cup_{-2\varepsilon \leq t \leq 2\varepsilon} \partial\Omega_t,$$

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∇f exists, and on each $\partial\Omega_\tau = \{f = \tau\}$ it is a.e. equal to the unit normal vector n_τ . Therefore we have

$$\begin{aligned}\nabla F_\varepsilon(x) &= \nabla(\chi_\varepsilon(\cdot - t) \circ f_\varepsilon)(x) \\ &= \varphi_\varepsilon(\cdot - t) \circ f(x) \nabla f(x) = [\varphi_\varepsilon(\cdot - t) \circ \text{dist}(x, \partial\Omega)] n_{\text{dist}_{\partial\Omega}(x)}\end{aligned}$$

and

$$\begin{aligned}\int_{\{|f-t|\leq\varepsilon\}} V \cdot \nabla F_\varepsilon dx^2 &= \int_{T(\Omega, 2\varepsilon)} \varphi_\varepsilon \circ \text{dist}_{\partial\Omega_t}(x) V(x) \cdot \nabla(\text{dist}_{\partial\Omega_t})(x) dx^2 \\ &= \int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(t) \left(\int_{\partial\Omega_t} V \cdot n_t d\mathcal{H}^1 \right) dt.\end{aligned}$$

As in the usual theory of slicing, for almost all t 's the currents $I \llcorner f^\#(\varphi_\varepsilon(\cdot - t) d\tau)$ converge weakly to the slice $\langle I, f, t \rangle$ as $\varepsilon \rightarrow 0$. Similarly, V being in L^p , a dominated convergence argument gives also for almost all \bar{t} the convergence

$$\int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(\tau - t) \left(\int_{\partial\Omega_\tau} V \cdot n_{\partial\Omega_\tau} d\mathcal{H}^1 \right) dt \rightarrow \int_{\partial\Omega_t} V \cdot n_{\partial\Omega_t} d\mathcal{H}^1. \quad (3.22)$$

The fact that almost all slices of an integer multiplicity rectifiable current are integer multiplicity rectifiable gives the first point of the Proposition, while the second point follows from (3.21) and (3.22). \square

3.6 Further remarks concerning the Integrability Theorem

We wish first to point out that not all boundaries of rectifiable integral currents ∂I are representable as $u^*\theta$ for $u \in W^{1,p}(\Omega, \mathbb{S}^1)$, if $p > 1$, showing that this case is more subtle than the case $p = 1$ treated in Proposition 3.3. To do this, we use the second formulation of the Integrability Theorem, which says that such $u^*\theta$ would then be equal to V^\perp for some vectorfield $V \in L^p$ satisfying $\text{div}V = \partial I$. We will demonstrate that not all integral currents I have ∂I equal to a divergence of a L^p -vectorfield.

Suppose first that we have a vectorfield V on $B_\varepsilon(p)$ satisfying $\text{div}V = \delta_p$ (where δ_p is the Dirac mass in p). Then for almost all $r \in [0, \varepsilon[$ we have

$$\int_{\partial B_r(p)} V \cdot n_{B_r(p)} d\mathcal{H}^1 = 1, \quad (3.23)$$

and we see that under the constraint (3.23), the minimal L^p -mass is achieved by the radial (in polar coordinates around p) vectorfield

$$V_{min}(\theta, r) = \frac{1}{2\pi r} \hat{r}$$

(by a rearrangement argument and by the convexity of the L^p -norm for $p > 1$). We therefore obtain (for some geometric constant C)

$$\|V\|_{L^p(B_\varepsilon(p))}^p \geq \|V_{min}\|_{L^p(B_\varepsilon(p))}^p = C\varepsilon^{2-p} \quad (3.24)$$

We see that such estimate on the norm of V is only dependent on the fact that $(\operatorname{div}V)_\perp B_\varepsilon(p) = \delta_p$. We can now use a series of inequalities like (3.24) on a series of (disjoint) balls in order to find our counterexample.

Example 3.25. Take a sequence of positive numbers $(a_i)_{i \in \mathbb{N}}$ such that

$$\begin{aligned} \sup_i a_i &= \varepsilon \\ \sum_{i=1}^{\infty} a_i &= 2 \end{aligned} \quad (3.25)$$

$$\sum_{i=1}^{\infty} a_i^{2-p} = +\infty. \quad (3.26)$$

It is possible to achieve this for any $\varepsilon > 0$, since $p > 1$.

Now take a 2-dimensional domain Ω . It is possible to find a series of disjoint balls B_i of radii a_i for any sequence a_i as above, provided that ε is small enough (because $\mathcal{H}^1(\Omega) = \infty$ and for any set C , $\mathcal{H}^1(C) > 0$ implies $H^{2-p}(C) = \infty$). Inside each B_i one can insert two disjoint balls B_i^+, B_i^- of radius $\frac{a_i}{2}$. Call x_i^\pm the center of B_i^\pm , and consider the current

$$I = \sum_{i=1}^{\infty} [x_i^-, x_i^+].$$

Using the estimate (3.26) and the estimates (3.24) on the disjoint balls B_i^\pm , we obtain that any vectorfield satisfying $\operatorname{div}V = \partial I$ must not be in L^p . By our Integrability Theorem (second version) 3.6, we see that none of the currents constructed in this way can possibly have boundary equal to the distributional Jacobian of a map $u \in W^{1,p}(\Omega, \mathbb{S}^1)$.

Chapter 4

Slice distances in the abelian case

4.1 Introduction

In this chapter we study some properties of the slice distance d utilized in Chapters 2 and we define the boundary trace for the Yang-Mills Plateau problem in 3 dimensions. This chapter is based on [P2].

4.1.1 Plateau problem for $U(1)$ -bundles

We recall that for the rigorous treatment of the Plateau problem for $U(1)$ -bundles a suitable setting should consist of the following two ingredients:

- *A class of weak bundles which is closed by sequential weak- L^p convergence of the curvatures:* since as we said the natural energy is the L^p -norm of the curvature, the topology giving precompactness of sublevelsets of the L^p -norm is the weak L^p -topology. In particular any minimizing sequence will have a weakly convergent subsequence. Thus a suitable class of bundles should be closed under this topology.
- *A suitable notion of boundary trace:* if F denotes the curvature of a weak bundle as above, we desire to be able to state the minimization problem which could be formally written as follows

$$\inf \left\{ \int_{\Omega} |F|^p dx : F|_{\partial\Omega} = \phi \right\} \quad (4.1)$$

in a meaningful way. In particular, we would like the weak convergence in the previous point not to disrupt our boundary condition, and to reduce to the usual boundary restriction for locally smooth bundles.

Remark 4.1. *Another possible approach for the creation of nontrivial bundles which are critical for our energy is by minimizing a relaxed energy instead, as suggested in [84] and [78], and in analogy with the case of harmonic maps [20]. In our case a good candidate for such energy would for example be given by*

$$E(F) = \int_{\Omega} |F|^p dx^3 + \sup_{\|d\xi\|_{L^\infty} \leq 1} \int_{\Omega} F \wedge d\xi.$$

The first point above was solved by Theorem 2.2 and is discussed in Chapter 2. The solution of the second point is one of the main results of the present chapter (See Section 4.6).

4.1.2 Definition of the boundary trace

The definition of the boundary trace will be inspired by the good control allowed on slicing functions via the curvature, and the boundary will therefore be regarded as a trace. The same mechanism which ensures a good passage to the limit of slice functions will also ensure the robustness under weak convergence of our boundary traces. We are therefore forced to deepen our study of the slice distance d introduced in Chapter 2.

We recall that the class $\mathcal{F}_{\mathbb{Z}}^p$ of weak $U(1)$ -curvatures was defined (up to a normalization) by requiring that

$$\int_{\partial B_r(x)} i_{\partial B_r(x)}^* F \in \mathbb{Z} \quad \text{for all } x, \text{ a.e. } r > 0.$$

We then identify $\partial B_r(x) \simeq \mathbb{S}^2$ via a homotety and since the above integral condition is scaling-invariant our slicings along concentric spheres give functions

$$h : [r_1, r_2] \rightarrow Y := L^p(\mathbb{S}^2, \wedge^2 T\mathbb{S}^2) \cap \left\{ F : \int_{\mathbb{S}^2} F \in \mathbb{Z} \right\}.$$

The distance d on Y introduced in Chapter 2 and explained in Section 2.1.2 is defined as follows:

$$d(h_1, h_2) := \inf \left\{ \|\alpha\|_{L^p} : h_1 - h_2 = d\alpha + \partial I + \sum_{i=1}^N d_i \delta_{a_i} \right\},$$

where the infimum is taken over all triples given by an L^p -integrable 1-form α , an integer 1-current I of finite mass and an N -ple of couples (a_i, d_i) , where $a_i \in \mathbb{S}^2$ and $d_i \in \mathbb{Z}$.

We then define the class of weak curvatures having boundary trace equal to the smooth 2-form φ as follows:

Definition 4.2. Let $\varphi \in C^\infty(\mathbb{S}^2, \wedge^2 T\mathbb{S}^2)$. We say that a weak $U(1)$ -curvature $F \in \mathcal{F}_{\mathbb{Z}}^p(B^3)$ has boundary trace equal to φ if

$$d(h(r - \epsilon), \varphi) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0^+,$$

where $h : [1/2, 1] \rightarrow Y$ is the slice function for F along spheres with center the origin. We denote the class of such weak curvatures by $\mathcal{F}_{\mathbb{Z}, \varphi}^p(\Omega)$.

For a discussion about the properties ensured by this definition see Section 4.6. It is shown in Section 4.5 that once we have the definition for B^3 the whole setting can be transferred to a domain Ω which is Bilipschitz-equivalent to B^3 . Ensuring the properties of the boundary trace requires a study of the slice distance d , whose results we now briefly describe.

4.1.3 Study of d and properties of the trace

Our first goal will be to prove the following result:

Theorem 4.3. *The above defined function d is a distance on $L^p(\mathbb{S}^2, \wedge^2 T\mathbb{S}^2)$, both in the case when $D = [0, 1]^2$ and in the case $D = \mathbb{S}^2$.*

We will see that the implication $d(h_1, h_2) = 0 \Rightarrow h_1 = h_2$ will be quite involved, requiring the results of Chapter 3. See Section 4.2 for the proof.

The second result will be that the distance d has a good behavior in terms of weak convergence in L^p on Y :

Proposition 4.4. *If $h_n \in Y$ are equibounded in L^p , then*

$$h_n \xrightarrow{d} h_* \Leftrightarrow h_n \xrightarrow{w-L^p} h_*.$$

This result is proven in Section 4.3. The other result which is worth noting is that slices coming from a weak L^p -curvature $F \in \mathcal{F}_{\mathbb{Z}}^p(B^3)$ are d -Hölder continuous:

Proposition 4.5. *Let $F \in \mathcal{F}_{\mathbb{Z}}^p(B^3)$. Let $\mathcal{A} := \{(x, r) : B_r(x) \subset B^3\}$ parameterize the balls contained in B^3 . Consider the slicing function*

$$h : B_{1/2}^3 \times \mathbb{R}^+ \supset \mathcal{A} \rightarrow Y$$

which to (x, r) assigns the form corresponding to restricting F to $\partial B_r(x)$ then pulling the result back via the homothety $T_{x,r} : \mathbb{S}^2 \rightarrow \partial B_r(x)$.

Then h is Hölder-continuous with respect to the product distance on \mathcal{A} and to the distance d on Y and its norm is controlled by F . More precisely if $B, B' \in \mathcal{A}$ then

$$dh(h(B), h(B')) \leq 16 \|F\|_{L^p(B^3)} |B - B'|^{1 - \frac{1}{p}}.$$

For the proof see Section 4.4. The above control is stronger than the one which we used in the Abstract Theorem 2.13, therefore a shorter proof of the closure theorem of Chapter 2 can be provided in Section 4.4.1.

4.1.4 Further results and questions

Other slice distances and optimal transport

We dedicate Section 4.7 to presenting other functions which are closely related to the distance d and a connection to Optimal Transport (see also Appendix B for other results in this direction). More precisely, in [29] the following distance was defined between positive measures μ_1, μ_2 on a domain $\Omega \subset \mathbb{R}^n$:

$$D_p(\mu_1, \mu_2) = \inf \left\{ \left(\int |\sigma|^p \right) : \operatorname{div} \sigma = \mu_2 - \mu_1, \sigma \cdot \nu = 0 \text{ on } \partial\Omega \right\},$$

where ν is the normal to Ω . The interpretation of this minimization is that μ_i represent the initial and final distribution of goods to be transported, σ is a vector field which represents a strategy for transporting the goods, and minimizing the L^p -norm for $p > 1$ will have the effect of penalizing concentration of transport paths.

In our situation the μ_i are replaced in the definition of d by the densities ρ_1, ρ_2 of the 2-forms h_1, h_2 with respect to the volume form of \mathbb{S}^2 (which may change sign, unlike the optimal transport problem). We moreover allow an “error” in the transportation, instead of requiring that ρ_1 is exactly transported to ρ_2 . This is the case because instead of asking that the divergence of σ be equal to $\rho_2 - \rho_1$ we allow the “free” introduction of the error $\sum n_i \delta_{x_i} + \partial I$. This is why the fact that d is still a distance is less obvious. On the other hand the fact that it is actually true leads to the possibility of introducing generalized optimal transport problems. This is still to be investigated.

Reconstructing a weak curvature from information on its slices

We briefly describe in Section 4.8 the relation between studying the slice functions and studying the weak curvatures $F \in \mathcal{F}_{\mathbb{Z}}$. The points which we treat are the following ones:

- If we precisely know all the slices then F is uniquely determined. This is still true if the tangent spaces of the slices which we know span the Grassmannian of 2-planes at each point of B^3 .
- We define a *compatibility condition* as a sufficient condition for an abstract slice function (i.e. a function assigning to each element of a fixed family of 2-cycles a weak curvature form on it) to correspond to a weak curvature F on B^3 . We leave open the question of finding a good candidate for such condition.

4.2 Proof that d is a distance

We prove here Theorem 4.2, i.e. the fact that the distance d on the model space of spherical slices Y is a distance.

Proof of Theorem 4.2: We will prove the three characterizing properties of a metric.

- *Reflexivity:* This is clear since the L^p -norm, the space of integer 1-currents of finite mass and the space of finite sums $\sum_{i=1}^N d_i \delta_{a_i}$ as above, are invariant under sign change.
- *Transitivity:* If we can write

$$\begin{cases} h_1 - h_2 = \operatorname{div} X_\epsilon + \partial I_\epsilon + \sum_{i=1}^N d_i \delta_{a_i} \\ h_2 - h_3 = \operatorname{div} Y_\epsilon + \partial J_\epsilon + \sum_{j=1}^M e_j \delta_{b_j}, \end{cases}$$

where

$$\begin{cases} \|X_\epsilon\|_{L^p} \leq d(h_1, h_2) + \epsilon \\ \|Y_\epsilon\|_{L^p} \leq d(h_2, h_3) + \epsilon, \end{cases}$$

then we put $Z_\epsilon := X_\epsilon + Y_\epsilon$, $K_\epsilon = I_\epsilon + J_\epsilon$ and we consider the singularity set $\{(c_k, f_k)\}$ where

$$\begin{aligned} \{c_k\} &= \{a_i\} \cup \{b_j\} \\ f_k &= \begin{cases} d_i & \text{if } c_k = a_i, c_k \notin \{b_j\} \\ e_j & \text{if } c_k = b_j, c_k \notin \{a_i\} \\ d_i + e_j & \text{if } c_k = a_i = b_j. \end{cases} \end{aligned}$$

We see that K_ϵ is still an integer 1-current of finite mass and that $h_1 - h_3 = \operatorname{div} Z_\epsilon + \partial K_\epsilon + \sum_k f_k \delta_{c_k}$. Then we have:

$$\begin{aligned} d(h_1, h_3) &\leq \|Z_\epsilon\|_{L^p} \leq \|X_\epsilon\|_{L^p} + \|Y_\epsilon\|_{L^p} \\ &\leq d(h_1, h_2) + d(h_2, h_3) + 2\epsilon, \end{aligned}$$

and as $\epsilon \rightarrow 0$ we obtain the transitivity property of $d(\cdot, \cdot)$.

- *Non-degeneracy:* This is the statement of the following proposition.

□

Proposition 4.6. *Under the hypotheses above, $d(h_1, h_2) = 0$ implies $h_1 = h_2$ almost everywhere, for $1 < p < 2$.*

Proof. We may suppose without loss of generality that $\int_D (h_1 - h_2) \in \mathbb{Z}$. We start by taking a sequence of forms X_ϵ such that

$$\begin{cases} \|X_\epsilon\|_{L^p} \rightarrow 0 \\ h_1 - h_2 = \operatorname{div} X_\epsilon + \partial I_\epsilon + \delta_0 \int_D (h_1 - h_2). \end{cases}$$

We would be almost done, if we could control also the convergence of the 1-currents I_ϵ . To do so, we start by expressing the boundaries ∂I_ϵ in divergence form. Therefore, we consider the equations

$$\begin{cases} \Delta \psi = h_1 - h_2 + \delta_0 \int_D (h_2 - h_1) \\ \int_D \psi = 0 \end{cases} \quad (4.2)$$

(by classical results, this equation has a solution whose gradient is in L^q for all q such that $q < 2$ and $q \leq p$) and

$$\begin{cases} \Delta \varphi_\epsilon = \operatorname{div} X_\epsilon \\ \int_D \varphi_\epsilon = 0 \end{cases} \quad (4.3)$$

This second equation can be interpreted in terms of the Hodge decomposition of the 1-form associated to X_ϵ : indeed, for a L^p 1-form α we know by classical results that it can be Hodge-decomposed as

$$\begin{cases} \alpha = df + d^*\omega + h, \text{ where} \\ \int_D f = 0, \int_D *\omega = 0, \Delta h = 0, \text{ and} \\ \|df\|_{L^p} + \|d^*\omega\|_{L^p} + \|h\|_{L^p} \leq C_p \|\alpha\|_{L^p}. \end{cases}$$

Therefore in equation (4.3) we can associate a 1-form α to X_ϵ and take φ_ϵ equal to the function f coming from the above decomposition. Then an easy verification shows that (4.3) is verified.

We have thus, that both (4.2) and (4.3) have a solution, and such solutions satisfy the following estimates:

$$\begin{cases} \|\nabla\varphi_\epsilon\|_{L^p} \leq c_p \|X_\epsilon\|_{L^p} \rightarrow 0 \\ \nabla\psi \in W^{1,p} \subset L^p \text{ since } p^* = \frac{2p}{2-p} > p. \end{cases}$$

Then (supposing $p < 2$) we obtain

$$\begin{cases} \partial I_\epsilon = \operatorname{div}(\nabla(\varphi_\epsilon - \psi)) \\ \|\nabla(\varphi_\epsilon - \psi)\|_{L^p} \text{ is bounded} \end{cases} \quad (4.4)$$

Now we consider the vector field $\nabla(\varphi_\epsilon - \psi) := V_\epsilon \in L^p(D, \mathbb{R}^2)$.

Proposition 4.7. *Suppose that we have a function $V \in L^p(D, \mathbb{R}^2)$ with $p > 1$, for a domain $D \subset \mathbb{R}^2$ or for $D = \mathbb{S}^2$, whose divergence can be represented by the boundary of an integer 1-current I on D , i.e. for all test functions $\gamma \in C_c^\infty(D, \mathbb{R})$ we have*

$$\int_D \nabla\gamma(x) \cdot V(x) dx = \langle I, \nabla\gamma \rangle. \quad (4.5)$$

Then there exists a $W^{1,p}$ -function $u : D \rightarrow \mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ such that $\nabla^\perp u = V$.

Applying Lemma 4.5 to the current I_ϵ of (4.4), we can write

$$\begin{cases} \nabla^\perp u_\epsilon = \nabla(\varphi_\epsilon - \psi) \\ \partial I_\epsilon = \operatorname{div}(\nabla(\varphi_\epsilon - \psi)) \\ \|\nabla u_\epsilon\|_{L^p} \leq C \|\nabla(\varphi_\epsilon - \psi)\|_{L^p} \leq C. \end{cases}$$

Then we have that a subsequence u_k of the u_ϵ converges weakly in $W^{1,p}(D, \mathbb{R}^2)$ to a limit u_0 , and thus it converges in L^1_{loc} , proving that $u_0 \in W^{1,p}(D, \mathbb{R}^2)$.

Now, the u_k converge to u_0 almost everywhere, and thus the limit function u_0 also has almost everywhere values in \mathbb{S}^1 . Since we know now that $u_k \xrightarrow{L^1} u_0$ and that $\|u_k\|_{L^\infty} \leq 1$, we obtain by interpolation $u_k \xrightarrow{L^r} u_0$ for all $r < \infty$. Therefore (by choosing $r = \frac{q}{q-1}$ and by Young's inequality) it follows that

$$\nabla^\perp u_k \xrightarrow{L^1} \nabla^\perp u_0. \quad (4.6)$$

By a generalization of Sard's theorem, the fibers $F_\epsilon(\sigma) := \{x \in D : u_\epsilon(x) = \sigma\}$ for $\sigma \in \mathbb{S}^1$ are rectifiable for almost all σ and can be given a structure of integer 1-currents. Then for almost all $\sigma \in \mathbb{S}^1$ we have

$$\partial[F_\epsilon(\sigma)] = \partial I_\epsilon.$$

By (4.6) we also obtain that the L^p -weak limit $\nabla(\varphi_0 - \psi)$ exists up to extracting a further subsequence, and it is equal to $\nabla^\perp u_0$. Therefore, again by Sard's theorem, its divergence is the boundary of an integer 1-current I_0 , which can be described using a generic fiber $F_0(\sigma)$ of u_0 :

$$\operatorname{div} \nabla(\varphi_0 - \psi) = \partial I_0.$$

Since $u_0 \in W^{1,p}$, by an easy application of the Fubini theorem to the generalized coarea formula, we have that the generic fibers $F(\sigma)$ have finite \mathcal{H}^1 -measure, thus I_0 has finite mass.

Since $\nabla\psi \in L^p$, from

$$\nabla^\perp u_k = \nabla(\psi - \varphi_k) \xrightarrow{L^1} \nabla^\perp u_0$$

we deduce that $\nabla\varphi_k \xrightarrow{L^1} \nabla\varphi_0$. On the other hand, $\nabla\varphi_\epsilon \xrightarrow{L^p} 0$ together with (4.2), implies that there exists an integer 1-current such that

$$h_1 - h_2 = \partial I_0. \quad (4.7)$$

The following lemma concludes the proof. □

Lemma 4.8. *If the boundary of an integer multiplicity finite-mass 1-current I on a domain $D \subset \mathbb{R}^2$ can be represented by a L^p -function for $p \geq 1$, then $\partial I = 0$.*

Proof. Suppose for a moment that $\partial I \neq 0$ and that there exists a function h such that for all $\varphi \in C_c^1(D)$ there holds

$$\langle \varphi, h \rangle = \langle \varphi, \partial I \rangle.$$

If we take a smooth positive radial function $\varphi \in C_c^1(B_1(0))$ which is equal to 1 on $B_{1/2}(0)$ and we consider a point of approximate continuity x_0 of h such that $h(x_0) \neq 0$, then we will also have

$$\begin{aligned} \mathbb{M}(I) \|\nabla \varphi\|_{L^\infty} &\geq \left| \left\langle \nabla_x \left(\frac{1}{\epsilon} \varphi(\epsilon(x - x_0)) \right), I \right\rangle \right| \\ &= \frac{1}{\epsilon} \left| \int \varphi(\epsilon(x - x_0)) h(x) dx \right| \\ &\geq \frac{c |h(x_0)|}{\epsilon}, \end{aligned}$$

which for $\epsilon > 0$ small enough is a contradiction. \square

4.3 d metrizes weak convergence on bounded sequences

Lemma 4.9. *If $h_n, h_*, h_\infty \in Y$, $\|h_n\|_{L^p}$ is bounded, $d(h_*, h_n) \rightarrow 0$ and $h_n \rightharpoonup h_\infty$ weakly in L^p , then $h_* = h_\infty$.*

Proof. From the weak convergence, with no loss of generality $\int h_n = \int h_\infty$ for all n . Define then the potential $\bar{\psi}_n$ by

$$\begin{cases} h_n - h_\infty = \Delta \bar{\psi}_n, \\ \int \bar{\psi}_n = 0 \end{cases}$$

and observe that $\Delta \bar{\psi}_n \rightharpoonup 0$ in L^p thus by elliptic theory and Rellich-Kondrachov's embedding $\|d\bar{\psi}_n\|_{L^{p^*}} \rightarrow 0$.

Now take 1-forms α_n such that (using the definition of d_1 and Proposition 4.21)

$$h_n - h_* = d^* \alpha_n + \Sigma_n, \quad \|\alpha_n\|_p \rightarrow 0$$

where Σ_n is a finite sum of Dirac masses with integer coefficients, and let $\Delta \phi_n = d^* \alpha_n$. Then $\|d\phi_n\|_{L^p} \leq C \|\alpha_n\|_{L^p} \rightarrow 0$ by elliptic estimates. Denote by ψ_n the function satisfying, for some fixed point p in the domain,

$$\Delta \psi_n = h_n - h_* + \delta_p \int (h_n - h_*),$$

and observe that $\|d\psi_n\|_{L^{p^*}}$ is bounded. Then

$$\Delta(\psi_n - \phi_n) = h_n - h_* - d^* \alpha_n = \Sigma'_n,$$

and denoting $v_n = d(\psi_n - \phi_n)$ we obtain for some $q > 1$

$$\begin{cases} d^*v_n = \Sigma'_n \\ \|v_n\|_q \leq C(\|\alpha_n\|_q + \|d\psi\|_q + \|d\bar{\psi}_n\|_q) \leq C \end{cases}$$

therefore (see the Chapter 3) there exist $u_n \in W^{1,q}(\mathbb{S}^2, \mathbb{S}^1)$ such that $u_n^*\theta = v_n$, where θ is the normalized volume 1-form of \mathbb{S}^1 . The end of the proof goes as in Chapter 3, where at the level of the u_n it is possible to find a converging subsequence, and by Sard theorem it is concluded that for a rectifiable 1-current of finite mass I_0 there holds $\partial I_0 = h_\infty - h_*$, thus $h_\infty = h_*$. \square

Proposition 4.10. *If $h_n \in Y$ are equibounded in L^p , then*

$$h_n \xrightarrow{d} h_* \Leftrightarrow h_n \xrightarrow{w-L^p} h_*.$$

Proof. Using the fact that a sequence has a limit h_* if and only if each subsequence has a subsequence converging to h_* and the previous lemma, we obtain immediately the “ \Rightarrow ” implication.

Suppose now $h_n \xrightarrow{w-L^p} h_*$. Then take the potential such that $\Delta\psi_n = h_n - h_*$. By the elliptic estimates and the Rellich-Kondrachov theorem, after extracting a subsequence, $d\psi_n \rightarrow 0$ in L^{p^*} . The limit is zero independent of the subsequence, so $\alpha_n = d\psi_n$ satisfies

$$\begin{cases} h_n - h_* = d^*\alpha_n, \\ \|\alpha_n\|_p \rightarrow 0 \end{cases}$$

which implies $h_n \xrightarrow{d} h_*$. \square

4.4 Regularity on slices

Consider a form $F \in \mathcal{F}_{\mathbb{Z}}^p(\Omega)$. We desire to compare its slices along $\partial B(x, r), \partial B(x', r') \subset \Omega$.

The slices will be given by a function (defined a.e.) $h : \Omega \times \mathbb{R}^+ \rightarrow Y \subset L^p(\mathbb{S}^2)$, where $h(x, r)$ is the function on \mathbb{S}^2 corresponding to the restriction of F to $\partial B(x, r)$, after a homothety and an identification of 2-forms on \mathbb{S}^2 with functions.

Consider the following function $A : \mathbb{S}^2 \times [0, 1] \rightarrow \Omega$:

$$A(\sigma, t) = t(x - x') + x' + [t(r - r') + r']\sigma := x_t + r_t\sigma.$$

Suppose that A is a diffeomorphism onto its image (this is true under the hypothesis (H) formulated below). Then $A^*F \in \mathcal{F}_{\mathbb{Z}}(\mathbb{S}^2 \times [0, 1])$; one can now build a competitor for the infimum in the definition of $d(h(x, r), h(x', r'))$ as follows. Consider

$$\bar{F}(\sigma) = \frac{1}{|r - r'|} \int_0^1 F_{x_t, r_t}^{\parallel}(\sigma) dt, \quad \text{where } F_{x_t, r_t}(\sigma) := r_t^2 F(x_t + \sigma r_t).$$

Here F_{x_t, r_t}^{\parallel} indicates the component of F_{x_t, r_t} parallel to the volume form of the sphere $\partial B_{r_t}(x_t)$. Reasoning along the lines of Proposition 2.11 (See also Proposition 1.19), this gives a competitor for the minimization in the definition of the slice distance $d(h(x, r), h(x', r'))$. To demonstrate this introduce the reparameterization $\rho = r_t$ and compute:

$$\begin{aligned} \int_{\mathbb{S}^2} |\bar{F}|^p(\sigma) d\sigma &= \int_{\mathbb{S}^2} \left(\int_{r'}^r F_{x_{\rho}, \rho}^{\parallel}(\sigma) d\rho \right)^p d\sigma \\ &\leq |r - r'|^{1 - \frac{1}{p}} \int_{r'}^r \int_{\mathbb{S}^2} |F_{x_{\rho}, \rho}|^p d\sigma d\rho. \end{aligned}$$

In order to compare this with the norm of F , note that

$$\begin{aligned} DA(\sigma, t) &= ([t(r - r') + r'] Id_{T\mathbb{S}^2} |x - x' + (r - r')\sigma) \\ &= (r_t Id_{T\mathbb{S}^2} |x - x' + (r - r')\sigma). \end{aligned}$$

Then (assuming $B' \subset B$ for the moment) we pull back the function $|F|^p$:

$$\begin{aligned} \int_{B \setminus B'} |F|^p d\mathcal{H}^3 &= \int_{A^{-1}(B \setminus B')} |F|^p \circ A |DA| \\ &= \int_0^1 \int_{\mathbb{S}^2} |F(x_t + r_t \sigma)|^p r_t^2 |r - r' + \langle \sigma, x - x' \rangle| d\sigma dt. \end{aligned}$$

Our hypothesis on the slices can then be reformulated as follows:

$$(H) \quad |x - x'| \leq \frac{1}{2}(r - r'), \quad 1 \geq r > r'.$$

Under this hypothesis, (since $|\sigma| = 1$) there holds

$$\begin{aligned} \int_{B \setminus B'} |F|^p d\mathcal{H}^3 &\geq \frac{1}{2}(r - r') \int_0^1 \frac{1}{r_t^{2p-2}} \left(\int_{\mathbb{S}^2} |F_{x_t, r_t}|^p d\sigma \right) dt \\ &= \frac{1}{2} \int_{r'}^r \frac{1}{\rho^{2p-2}} \left(\int_{\mathbb{S}^2} |F_{x_{\rho}, \rho}|^p d\sigma \right) d\rho \\ &\geq \frac{1}{2} \int_{r'}^r \left(\int_{\mathbb{S}^2} |F_{x_{\rho}, \rho}|^p d\sigma \right) d\rho \quad \text{if } \rho \leq 1. \end{aligned}$$

Observe that $F_{x_\rho, \rho}$ is the Poincaré dual of $h(x_\rho, \rho)$. Denote:

$$F : \Omega \times \mathbb{R}^+ \xrightarrow{h} Y \xrightarrow{\|\cdot\|_{L^p}^p} \mathbb{R}^+.$$

Then

$$\begin{aligned} d(h(x, r), h(x', r')) &\leq |r - r'|^{1 - \frac{1}{p}} \left(\int_{r'}^r F(x_\rho, \rho) \right)^{1/p} \\ (\text{Under hypotheses (H)}) &\leq 2|r - r'|^{1 - \frac{1}{p}} \left(\int_{B \setminus B'} |F|^p \right)^{1/p}. \end{aligned}$$

Combining the basic estimate above for a couple of segments, we obtain Hölderianity.

Theorem 4.11. *The slice-function $h : \mathcal{A} := B_{\frac{1}{2}} \times \mathbb{R}^+ \cap \{(x, r) : B(x, r) \subset B_1\} \rightarrow L^p(\mathbb{S}^2)$ defined above is Hölder- $(1 - 1/p)$ -continuous with respect to the distance d , and its Hölder constant is bounded by the L^p -norm of F .*

Proof. We desire to see how the above estimates worsen if instead of connecting $B = (x, r)$, $B' = (x', r')$ along a segment, we use a polygonal curve. Consider then γ , consisting in a union of segments $\{S\}$, each of which satisfies (H). For a given segment $S = [\underline{S}, \overline{S}]$ (where $\overline{S} = (x', r')$ is the end with the largest radius) we denote $A_S := B_{\overline{S}} \setminus B_{\underline{S}}$ and $|S_r|$ the difference of the radii of $\underline{S}, \overline{S}$. We then have the following estimate, by the same reasoning as above:

$$\begin{aligned} 2\|F\|_{L^p(A_S)}^p &\geq \int_{\underline{S}}^{\overline{S}} \frac{1}{\rho^{2p-2}} F(s_\rho) d\rho \\ &\geq \overline{S}^{2-2p} |S_r|^{p-1} d(h(\overline{S}), h(\underline{S}))^p. \end{aligned}$$

Summing up and using the triangle inequality,

$$2\#\{S\} \|F\|_{L^p} \sum_{S \in \gamma} |S_r|^{1 - \frac{1}{p}} \geq d(h(B), h(B')).$$

Because of this estimate, the question is how we can join B, B' by some polygonal γ which stays in the allowed set \mathcal{A} and is made of segments verifying (H), such that $\#\{S\}$ is as small as possible and $\sum_{S \in \gamma} |S_r|^{1 - \frac{1}{p}}$ is bounded above.

N can be bounded by 4 because as we sketch below we don't need more than 4 segments, and that $\max_{S \in \gamma} |S_r|$ is bounded by $2|B' - B|$ (also in this

case it's optimal to have a few long segments rather than many short ones). We just briefly describe the kind of γ we use for the estimates.

The worst case that we can face is the one where B, B' are on $\partial\mathcal{A}$, have the same r -coordinate, and are as far from each other as possible. If they are on the part where $x, x' \in \partial B_{\frac{1}{2}}$ with $r < \frac{1}{4}$ then we can take γ to start from B and go “up” in the r -direction with slope 2 until it touches $\partial\mathcal{A}$, then “down” until a very small radius and center $x = 0$, then do the same symmetrically, building up an M -shaped graph. If $r \geq \frac{1}{4}$, then it's better to first go down then up, making a symmetric W -shaped graph. If instead $x, x' \in \partial\mathcal{A} \setminus \partial B_{\frac{1}{2}}$ then again a W -shaped graph is the best option, and if r is large enough a V -shaped graph will be even better.

It is easy (but tedious) to verify that the above constructions verify the estimate on $|S_r|$. We thus end up with the following bound:

$$16\|F\|_{L^p}|B - B'|^{1-\frac{1}{p}} \geq d(h(B), h(B'))$$

□

Remark 4.12. *In general, even though d is Hölder on the slices, Proposition 4.10 does not apply, to give weak continuity on the slices, because the norm boundedness is not verified. This is already clear in the case where the form F is the radial form $F_x(V, W) = \frac{x}{|x|^3} \cdot V \times W$. Denote by $S_{1+\rho}$ the slice along $\partial B(1+\rho, (0, 0, 1))$ for $\rho \in [-\epsilon, +\epsilon]$ (see Figure 4.1). Since these spheres look almost flat near $(0, 0, 0)$ for small ϵ and the integral of F on the portion of a given slice just depends on the solid angle covered by that region, the p -th power of the L^p -norm of the slice $S_{1+\rho}$ on a small ball near the singularity grows like ρ^{2-2p} , i.e. such norm blows up. This fact is indeed to be expected, given Proposition 4.10 and Theorem 4.11, since $\int_{S_{1+\rho}} i^*F = \text{sgn}\rho$, and in particular the slices $S_{1+\rho}$ are not weakly continuous at $\rho = 0$.*

4.4.1 A simplified proof of the closure theorem

Theorem 4.11 and Proposition 4.10 allow a simplification of the proof of the Closure Theorem 2.2. The new proof avoids the Abstract Theorem 2.13. We state and prove here the crucial step from which Theorem 2.2 follows at once.

Lemma 4.13 (Main step of the Closure Theorem). *Let $p \in]1, 3/2[$ as above. Suppose that the 2-forms $F_n \in \mathcal{F}_{\mathbb{Z}}^p(\Omega)$ are weakly convergent to a 2-form $F \in L^p(\Omega)$. Given $B_r(x) \subset \Omega$, consider the slices $S : [r/2, r] \rightarrow L^p(\mathbb{S}^2)$, given by $S(\rho) := i_{x,\rho}^*F$. Then for almost all $\rho \in [r/2, r]$, $S(\rho) \in Y$. In particular the integer flux condition is preserved.*

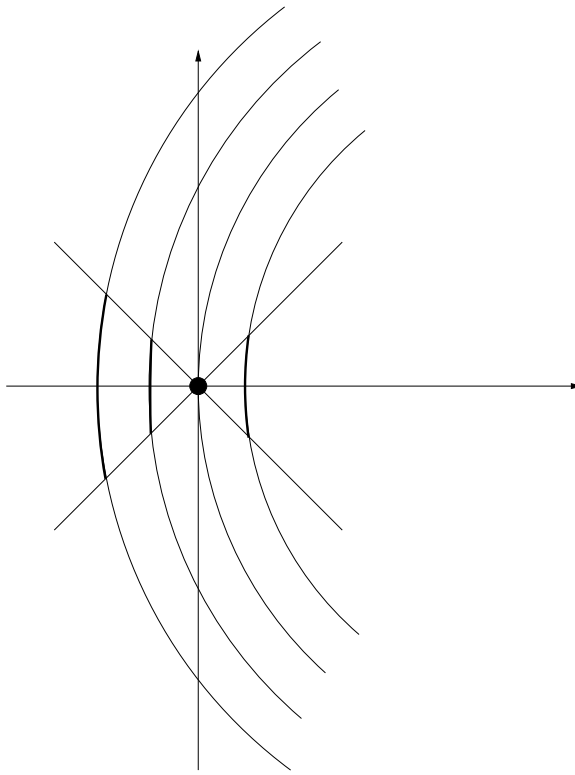


Figure 4.1: We represent schematically the slices passing near the origin. The areas of the thick regions behave like ρ^2 and the integral of F on them is constant and positive, so $|i^*F| \sim \frac{1}{\rho^2}$ and the L^p -norms of the slices is thus $\gtrsim \rho^{2-2p}$.

Proof. W.l.o.g. suppose $x = 0$. By lower semicontinuity of the norm, up to a subsequence $\|F_n\|_{L^p(B_r \setminus B_{r/2})} \leq C$. By Proposition 4.11, the slice functions S_n of F_n are equi-Hölder with respect to our metric d . This means that there exists a pointwise convergent subsequence.

It is evident that the deformation factor of the L^p -norm (coming from the fact that $i_{0,\rho}$ are not isometries) is bounded. Fubini's and Chebychev's theorems imply that we may restrict to a subset of $\rho \in [r, r/2]$ on which the L^p -norms of the $S_n(\rho)$ stay bounded. Then Proposition 4.10 applies. Therefore the slices converge weakly almost everywhere, and testing them on the constant function 1, we see that their degrees also converge. Therefore $S(\rho)$ has integer degree on \mathbb{S}^2 , as desired. \square

4.5 The case of Lipschitz slices

Consider the problem of extending the definition of the distance d to the case of slices different from spheres. The main motivations for this extension are the following:

- A natural question regarding the class $\mathcal{F}_{\mathbb{Z}}^p$ is whether or not the integrality condition is required on spheres can be replaced by a condition on different kinds of surfaces. A particularly interesting case would be one in which the slicing sets tile space, as is the case for the surfaces of cubes.
- The definition of the boundary condition in Section 4.6 is based on slicing. Having more general slice models will allow defining the trace on more general domains.

Given a bilipschitz map $\Psi : \mathbb{S}^2 \rightarrow \Sigma$, define the following distance between L^p -integrable 2-forms on Σ :

$$d_{\Psi}(h_1, h_2) = d_{\mathbb{S}^2}(\Psi^*h_1, \Psi^*h_2).$$

The pullback by bilipschitz functions preserves the integrability class, since

$$|(\Psi^*h)_x| = \sup_{|v| \leq 1, |w| \leq 1} h_{\Psi(x)}(d\Psi_x v, d\Psi_x w) \leq \|d\Psi\|_{\infty}^2 |h_{\Psi(x)}|,$$

and the same holds with Ψ^{-1} instead of Ψ . Analogous estimates imply that different bilipschitz maps induce equivalent distances:

Proposition 4.14. *Suppose $\Psi_1, \Psi_2 : \mathbb{S}^2 \rightarrow \Sigma$ are bilipschitz maps. Then d_{Ψ_i} are distances and they are equivalent:*

$$C^{-1}d_{\Psi_1} \leq d_{\Psi_2} \leq Cd_{\Psi_1}.$$

The constant C depends only on the Lipschitz constants of Ψ, Ψ^{-1} .

Proof. The fact that d_{Ψ_i} satisfy the triangular inequality and the reflexivity follow at once from the analogous properties of d . The non-degeneracy $d_{\Psi_i}(h_2, h_2) = 0 \Leftrightarrow h_1 = h_2$ is a consequence of the inequalities of the thesis, since for $\Psi_1 = id_{\mathbb{S}^2}$ and $\Sigma = \mathbb{S}^2$, d_{Ψ_1} is a distance. If we prove the Proposition for $\Psi_1 = id_{\mathbb{S}^2}, \Sigma = \mathbb{S}^2$, the general case will follow by transitivity of the equivalence between distances. Thus we consider just this case.

We will work with the equivalent definition of d_1 as in Section 4.7.

Fix $h_1, h_2 \in Y$, and consider a competitor α in the definition of $d_1(h_1, h_2)$. In other words, if $h = h_2 - h_1$ and Σ represents a finite sum of Dirac masses, then (interpreting h_i as 2-forms and using the Hodge star on \mathbb{S}^2)

$$d(*\alpha) = h + *\Sigma, \text{ i.e.}$$

$$\forall \phi \in C^\infty(\mathbb{S}^2), \int \phi d(*\alpha) = \int \phi h + \langle \Sigma, h \rangle,$$

The crucial observation is that the above objects extend naturally to the space of Lipschitz functions, and it is equivalent to use $\phi \in \text{Lip}(\mathbb{S}^2)$ instead of $\phi \in C^\infty(\mathbb{S}^2)$ above. If we replace ϕ by $\phi \circ \Psi \circ \Psi^{-1}$ and change variable, we obtain (recall that $\Psi_{\#}\Sigma$ is the image measure):

$$\int d(*(\Psi^*\alpha)) \phi \circ \Psi = \int \Psi^* h \phi \circ \Psi + \langle \Psi_{\#}\Sigma, \phi \circ \Psi \rangle.$$

Since Ψ is bilipschitz, it is a bijection of $\text{Lip}(\mathbb{S}^2)$ into itself, and thus $\Psi^*\alpha$ is a competitor for the distance $d_\Psi(h_1, h_2)$.

Now observe as above that $|\Psi^*\alpha|_x \leq \|d\Psi\|_\infty |\alpha|_{\Psi(x)}$, which leads to the conclusion that

$$\int_{\mathbb{S}^2} |\Psi^*\alpha|_x^p dx \leq \|d\Psi\|_\infty^p \int |\alpha|_{\Psi(x)}^p dx \leq \|d\Psi\|_\infty^p \|d\Psi^{-1}\|_\infty^2 \int |\alpha|_y^p dy.$$

The same holds also with Ψ^{-1} instead of Ψ , so the infimum in the definition of d is comparable with the one in the definition of d_Ψ . \square

4.6 Definition of the boundary value

Let $\Omega \subset \mathbb{R}^3$ be an open bounded smooth domain. Let $\mathcal{F}_{\mathbb{Z}}^p(\Omega)$ be as in Section 4.5. Such class consists of all L^p -integrable 2-forms F such that for generic 2-cycles S bilipschitz-equivalent to \mathbb{S}^2 , there holds

$$\int_S F \in \mathbb{Z}.$$

Given a smooth 2-form φ on $\partial\Omega$, we would like to find a suitable class $\mathcal{F}_{\mathbb{Z},\varphi}^p(\Omega)$ which satisfies the following three conditions:

- **(closure)** for any L^p -regular 2-form φ on $\partial\Omega$, the class $\mathcal{F}_{\mathbb{Z},\varphi}^p(\Omega)$ is closed by sequential weak L^p -convergence.
- **(nontriviality)** if $\varphi \neq \psi$ are two L^p -regular 2-forms on $\partial\Omega$, then $\mathcal{F}_{\mathbb{Z},\varphi}^p(\Omega) \cap \mathcal{F}_{\mathbb{Z},\psi}^p(\Omega) = \emptyset$.
- **(compatibility)** for any smooth 2-form φ , $\mathcal{F}_{\mathbb{Z},\varphi}^p(\Omega) \cap \mathcal{R}^\infty$ are exactly the 2-forms $F \in \mathcal{R}^\infty$ such that $i_{\partial\Omega}^* F = \varphi$, where $i_{\partial\Omega}$ is the inclusion map.

For general L^p -forms (i.e. without the restriction of belonging to $\mathcal{F}_{\mathbb{Z}}^p$) no such class can exist, even if in the closure requirement above we had required *strong* convergence. Indeed, let F, G be different smooth forms, and consider $f_n : [0, \infty[\rightarrow [0, 1]$, $f_n = \chi_{[1/n, \infty[}$. Then $F_n(x) := F(x) + f_n(\text{dist}(x, \partial\Omega))(G(x) - F(x))$ satisfy $F_n \xrightarrow{L^p} G$. Then by compatibility F_n and F should have the same trace, and so by closure G and F should have the same trace, contradicting nontriviality.

At the other extreme, for *locally exact* L^p -forms, using the Poincaré Lemma, we have

$$dF =_{loc} 0 \implies F =_{loc} dA, \quad A \in W_{loc}^{1,p}.$$

Thus one can impose the boundary condition directly on the restrictions to $\partial\Omega$ of $W^{1,p}$ -regular “local primitives” A , using classical trace theorems, and all the above properties follow.

Our new space $\mathcal{F}_{\mathbb{Z}}^p(\Omega)$ is an intermediate space between the two extrema above, escaping both the above reasonings. We therefore use the distance d_S between 2-forms on cycles S , as in Section 4.14. Up to a bilipschitz deformation, we may assume that $\Omega = B^3$, and thus it is enough to define the boundary condition in this case.

The distance d is used to compare the boundary datum with the slices of forms $F \in \mathcal{F}_{\mathbb{Z}}^p$. We abuse notation and denote by $f(x + \rho)$ the form

(with variable $x \in \mathbb{S}^2$) corresponding to the restriction to $\partial B_{1-\rho}$ of the form F . This strange notation is inspired by the analogy to slicing via parallel hyperplanes, instead of spheres. We then define the class $\mathcal{F}_{\mathbb{Z},\varphi}(B^3)$ via the continuity requirement

$$d(f(x + \rho'), \varphi(x)) \rightarrow 0, \text{ as } \rho' \rightarrow 0^+. \quad (4.8)$$

It is clear that the definition (4.8) satisfies the *nontriviality* and *compatibility* conditions above, since $d(\cdot, \cdot)$ is a distance and since for \mathcal{R}^∞ having smooth boundary datum implies that in a neighborhood of ∂B^3 the slices are smooth and converge in the smooth topology to φ . The validity of the *well-posedness* is a bit less trivial, therefore we prove it separately.

Theorem 4.15. *If $F_n \in \mathcal{F}_{\mathbb{Z},\varphi}(B^3)$ are converging weakly in L^p to a form $F \in \mathcal{F}_{\mathbb{Z}}(B^3)$ then also F belongs to $\mathcal{F}_{\mathbb{Z},\varphi}(B^3)$.*

Proof. By weak semicontinuity of the L^p norm we have that F_n are bounded in this norm, $\|F_n\|_{L^p(B_1 \setminus B_{1-h})} \leq C$.

Therefore by Theorem 4.11 the f_n are d -equicontinuous, so a subsequence (which we do not relabel) of the f_n converges to a slice function f_∞ with values in Y a.e.. For all $\rho' \in [0, \rho]$ the forms $f_n(\cdot + \rho')$ are a Cauchy sequence in n , for the distance d . This is enough to imply that f_∞ is equal to the slice of F . Even if F is just defined up to zero measure sets, it still has a d -continuous representative. By uniform convergence it is clear that f still satisfies (4.8). \square

The same proof also gives an apparently stronger result:

Theorem 4.16. *If $F_n \in \mathcal{F}_{\mathbb{Z},\varphi_n}(B^3)$ are converging weakly in L^p to a form $F \in \mathcal{F}_{\mathbb{Z}}(B^3)$ then the forms φ_n converge with respect to the distance d to a form φ and also F belongs to $\mathcal{F}_{\mathbb{Z},\varphi}(B^3)$.*

Remark 4.17. *With a bit more effort one can define the boundary value and prove Theorem 4.16 using just the $L^{p,\infty}$ -bound of the modulus of lipschitzianity of the slices, as given in Theorem 8.2. One can prove the analogous result if one replaces condition (4.8) by the following approximate continuity requirement*

$$\text{for all } \epsilon > 0, \lim_{\rho \rightarrow 0^+} \frac{|[0, \rho] \cap A_\epsilon|}{\rho} = 0, \text{ where } A_\epsilon := \{\rho' : d(f(\cdot + \rho), \varphi) > \epsilon\}.$$

Theorem 4.16 can be reformulated in the formalism of vector fields with integer fluxes:

Proposition 4.18. *Let $X \in L_{\mathbb{Z}}^p(B^3)$. Let $\hat{\rho}(x) = x/|x|$ be the radial vector field defined outside the origin of \mathbb{R}^3 . For $(x, \rho') \in \mathbb{S}^2 \times]0, 1[$, define $\xi(x + \rho') := \hat{\rho} \cdot X(x(1 - \rho'))$. For a given L^p -regular function ϕ defined on ∂B^3 define the class $L_{\mathbb{Z}, \phi}^p(B^3)$ via the following continuity requirement:*

$$d(\xi(x + \rho'), \phi(x)) \rightarrow 0 \text{ as } \rho' \rightarrow 0^+.$$

With this definition we have the following two properties:

1. If $X_n \in L_{\mathbb{Z}, \varphi}^p(B^3)$ converge weakly in L^p to $X \in L_{\mathbb{Z}}^p(B^3)$ then also X belongs to $L_{\mathbb{Z}, \varphi}^p(B^3)$.
2. If $X_n \in L_{\mathbb{Z}, \varphi_n}^p(B^3)$ converge weakly in L^p to $X \in L_{\mathbb{Z}}^p(B^3)$ then φ_n converge with respect to the distance d to some function φ and X belongs to $L_{\mathbb{Z}, \varphi}^p(B^3)$.

Proof. The correspondence between 2-forms and vector fields is used. The restriction operation $F \mapsto i_{\mathbb{S}^2}^* F$ corresponds to the operation $X \mapsto \nu_{\mathbb{S}^2} \cdot X$. The closure of $L_{\mathbb{Z}}^p(B^3)$ under weak convergence being proved in Chapter 2, we have to prove the preservation and convergence of the boundary condition. The needed results are proved in Theorems 4.15 and 4.16 respectively. \square

Remark 4.19. *As noted before, the definition of the distance as in Section 4.5 allows to extend the definition of the boundary value to arbitrary domains*

4.7 Some other definitions of slice distances

We compare here the distance on $\mathcal{Y} := \{h \in L^p(\mathbb{S}^2) : \int_{\mathbb{S}^2} h d\mathcal{H}^2 \in \mathbb{Z}\}$ defined by

$$d(h_1, h_2) := \inf \left\{ \|\alpha\|_{L^p} : h_2 - h_1 = d^* \alpha + \partial I + \sum_{i=1}^N n_i \delta_{a_i} \right\}$$

as in the Introduction, to the following function:

$$d_1(h_1, h_2) := \inf \left\{ \|\alpha\|_{L^p} : h_2 - h_1 = d^* \alpha + \sum_{i=1}^N n_i \delta_{a_i} \right\}$$

We define also another distance d_2 , in the struggle to free our distance d from the presence of an unknown sum of Dirac masses:

$$d_2(h_1, h_2) = \liminf_{\epsilon \rightarrow 0} \inf_{\alpha, A} \{ \|\alpha\|_{L^p} : \text{spt} [(h_2 - h_1) - d^* \alpha] \subset A, A \text{ open}, |A| \leq \epsilon \}.$$

The motivations for introducing these objects are as follows:

1. Sometimes in applications, as for example in Section 4.5, it is easier to deal with definitions in terms of finite, rather than infinite, sets of singularities. This justifies the introduction of d_1 .
2. It is natural to ask whether or not our distance d is induced by a norm defined on the larger space $L^p(\mathbb{S}^2)$. Candidates for such norms are norms which “don’t see small sets”, in particular they should not be sensible to the presence of the singular measures defining d . More importantly, having an underlying norm could perhaps help to define new and more natural notions of critical points for our energy. Investigating the relationship between d and d_2 seems a reasonable first step for that line of research.

We will use the following density result:

Proposition 4.20. *Fix an exponent $p > 1$ and consider the space $\mathcal{V}_{\mathbb{Z}}$ consisting of all L^p -integrable 1-forms α on \mathbb{S}^2 such that*

$$\int_{\mathbb{S}^2} \alpha \wedge d\phi = \langle \partial I, \phi \rangle, \quad \forall \phi \in C^\infty(\mathbb{S}^2),$$

where I is an integer rectifiable 1-current of finite mass on \mathbb{S}^2 . Then its subspace \mathcal{V}_R given by the 1-forms which are smooth outside a discrete (thus finite) set, is dense in the L^p -norm.

4.7.1 The distance d_1

Proposition 4.21. *On \mathcal{Y} there holds $d = d_1$.*

Proof. Clearly $d_1 \geq d$ since the infimum in the definition of d_1 is taken on a smaller class. To prove the opposite inequality, fix $h = h_2 - h_1$ and consider a minimizing sequence α_ϵ as in the definition of d . Then

$$(d^* \alpha_\epsilon) + \left(h - \delta_0 \int_{\mathbb{S}^2} h \right) = \partial I_\epsilon, \quad \|\alpha_\epsilon\|_{L^p} \rightarrow d(h_1, h_2).$$

Consider the function g satisfying the following equation:

$$d^* dg = h - \delta_0 \int_{\mathbb{S}^2} h, \quad \int_{\mathbb{S}^2} g = 0.$$

By standard elliptic theory $\|dg\|_{L^p} \leq C\|h\|_{L^p}$. It follows that

$$d^*(\alpha_\epsilon + dg) = \partial I_\epsilon,$$

and Proposition 4.20 applies then to $\alpha_\epsilon + dg$, giving a decomposition

$$\alpha_\epsilon + dg = f_\epsilon^k + e_\epsilon^k,$$

where $f_\epsilon^k \in \mathcal{V}_R$ and $e_\epsilon^k \xrightarrow{(k \rightarrow \infty)} 0$ in L^p -norm. In particular there exists a measure Σ_ϵ^k of the form $\sum_{i=1}^N n_i \delta_{a_i}$ as in the definition of d_1 , for which

$$d^* f_\epsilon^k = -\Sigma_\epsilon^k = d^*(\alpha_\epsilon + e_\epsilon^k) + h - \delta_0 \int_{\mathbb{S}^2} h.$$

Therefore

$$h = d^*(\alpha_\epsilon + e_\epsilon^k) + \Sigma_\epsilon^k - \delta_0 \int_{\mathbb{S}^2} h.$$

Thus $\alpha_\epsilon + e_\epsilon^k$ are competitors in the infimum defining $d_1(h_1, h_2)$, and as $k \rightarrow \infty, \epsilon \rightarrow 0$, their L^p -norms converge to $d(h_1, h_2)$. This concludes the proof of $d = d_1$. □

4.7.2 The distances d_2 and d_3

A possible choice for the set A in the definition of d_2 (if we interpret A as the set on which $d^*\alpha$ “avoids” as much L^p -norm of $h_2 - h_1$ as possible) could be some neighborhood of a superlevelset of $|h_2 - h_1|$, which gives us a third distance d_3 :

$$d_2(h_1, h_2) \leq \liminf_{k \rightarrow \infty} \{ \|\alpha\|_{L^p} : h_2 - h_1 = d^*\alpha \text{ whenever } |h_2 - h_1| \leq k \} := d_3(h_1, h_2).$$

Lemma 4.22. d_2 is a distance and for $h_1, h_2 \in \mathcal{Y}$ there holds $d(h_1, h_2) \geq d_2(h_1, h_2)$.

Proof. The inequality $d(h_1, h_2) \geq d_2(h_1, h_2)$ follows easily from Proposition 4.21 since we can take as the set A a small neighborhood of the singularities in the definition of d_1 . In particular, it follows that $d_2(h_1, h_2) = 0 \Leftrightarrow h_1 = h_2$. Being the triangular inequality and the symmetry evident for d_2 , and since $d_2(h_1, h_2) = 0 \Rightarrow h_1 = h_2$ follows directly from the Lebesgue continuity property of L^p -forms, d_2 is indeed a distance. □

The other inequalities are still to be investigated:

Open Problem 10. *Is it true that $d = d_2 = d_3$?*

Remark 4.23. We mention here an interesting analogy. A simpler distance similar to d_2 was studied in [29], where for probability measures μ_1, μ_2 on $\Omega \subset \mathbb{R}^n$ bounded open with smooth boundary the following distance was defined:

$$D_{\mathcal{H}}(\mu_1, \mu_2) = \inf_{\sigma \in L^p(\Omega, \mathbb{R}^n)} \left\{ \int_{\Omega} \mathcal{H}(\sigma(x)) dx : d^* \sigma = \mu_1 - \mu_2, \sigma \cdot \nu = 0 \text{ on } \partial\Omega \right\},$$

for a class of functions \mathcal{H} including the case $\mathcal{H}(x) = |x|^p, p > 1$. The connection between our distances and the class of distances $D_{\mathcal{H}}$ would give an interesting connection to the theory of Optimal Transportation, which would strongly echo with the use of basic Optimal Transportation for “minimal connections” connecting singularities of harmonic maps in [30].

4.8 First steps towards a compatibility condition for slices

Since we used slices to define the class of weak curvatures $\mathcal{F}_{\mathbb{Z}}(\Omega)$, it is natural to go one step further and try to construct forms $F \in \mathcal{F}_{\mathbb{Z}}(\Omega)$ by assigning their slices. This kind of problem seems to represent an unexplored area of research, related perhaps to integral geometry. We were not able to find any example of similar problems in the literature. Therefore in the following subsections we attempt to formalize the main questions which have arisen.

Slices on rectifiable cycles and genericity

Consider a 2-form $F \in \mathcal{F}_{\mathbb{Z}}^p(\Omega)$ which is bounded in L^p -norm. Given a Lipschitz 2-cycle $C = \partial K$ on Ω , chosen in a “generic” way such that $i_C^* F$ is in $L^p(C, \mathcal{H}^2)$ and that (in the duality between 2-cycles and 2-forms)

$$\langle C, F \rangle \in \mathbb{Z},$$

we can associate

$$C \mapsto h(C) := i_C^* F \in Y_C,$$

where Y_C is the set of 2-forms h such that

- h is L^p -integrable w.r.t. the surface measure on C ,
- h is \mathcal{H}^2 -a.e. the dual of the unit tangent 2-vector \vec{C} to C ,
- $\langle C, h \rangle$ is an integer.

Open Problem 11. Which such h give rise to an $F \in \mathcal{F}_{\mathbb{Z}}^p$?

We now explain what the requirement that h be defined only for “generic” cycles should mean. For that purpose, denote by \mathcal{C} the *fixed set of Lipschitz cycles* on which a compatibility theory will be defined (useful choices may vary from the set of all spheres to the set of all Lipschitz cycles). The domain of definition of h above should then be given by $\mathcal{C} \setminus R_F$ for some set R_F , possibly depending on F , which belongs to an admissible class of residual sets.

Definition 4.24. Fix a class of cycles \mathcal{C} . We call an **admissible class of residual sets** a class $\mathcal{R} \subset \mathcal{C}$ satisfying the following:

- Suppose that $(C_x)_{x \in [-\epsilon, \epsilon]}$ is a Lipschitz foliation by Lipschitz cycles $C_x \subset \Omega$. Then $C \cap S_N \subset \mathcal{R}$ for all sets S_N of the form

$$S_N = \{C_x : x \in N\} \text{ with } N \subset]-\epsilon, \epsilon[, \quad \mathcal{L}^1(N) = 0.$$

Once we fixed an admissible class of residual sets, we call the complement of a residual set **generic**.

The compatibility question

Consider the question of slice compatibility for a class of slicing cycles \mathcal{C} . Not all applications

$$k : \mathcal{C} \rightarrow \mathcal{Y}_{\mathcal{C}} := \cup_{C \in \mathcal{C}} Y_C \tag{4.9}$$

can be represented as slices h of an underlying form $F \in \mathcal{F}_{\mathbb{Z}}^p$:

Lemma 4.25. Assign to each cycle $C = [\partial B(x, r)]$ the form $h(C) \in Y_C$ equal to $\psi_C^* h(\mathbb{S}^2)$ of a fixed nonzero 2-form $h(\mathbb{S}^2) \in Y_{\mathbb{S}^2}$, where $\psi_C : C \rightarrow \mathbb{S}^2$ is the similitude bijection. The so-obtained function

$$h : \mathcal{C} = \{[\partial B(x, r)] : x \in \Omega, r \in]0, \text{dist}(x, \partial\Omega)\} \rightarrow \mathcal{Y}_{\mathcal{C}}$$

cannot satisfy $h(C) = i_C^* F$ for generic $C \in \mathcal{C}$.

Proof. Assume for a moment that there exists such 2-form $F \in \mathcal{F}_{\mathbb{Z}}^p(\Omega)$. Then for any fixed $M > 0$ we would have $|F| \geq M$ almost everywhere on Ω . Indeed fix $\epsilon > 0$ such that $|E_{\epsilon}| > \epsilon$, where $E_{\epsilon} := \{|h| > \epsilon\}$. Then consider the sets $S(x, r) := \cup_{\rho \leq r} \psi_{B(x, \rho)}^{-1}(E_{\epsilon})$, with the constraint $r < \sqrt{\epsilon/M}$. These sets form a fine covering of Ω , and if $h(\mathbb{S}^2) = (\psi_C^{-1})^* i_C^* F$ for almost all C in the definition of $S(x, r)$, then $|F|$ must be larger than M almost everywhere on $S(x, r)$. By extracting a (not necessarily disjoint) countable cover of Ω by sets $S(x, r)$ up

to zero Lebesgue measure, we obtain that $|F| \geq M$ almost everywhere. By the arbitrariness of M we obtain that F cannot be in L^p , thus contradicting our assumption. \square

Remark 4.26. *Suppose that \mathcal{C} is a family of cycles such that for almost all $x \in \Omega$ the tangent spaces $(T_x C)_{x \in C \in \mathcal{C}}$ span the Grassmannian $G(2,1)$ of 2-planes. Then for any k as in (4.9) there is at most one 2-form F such that $k = k_F$. Indeed, fixing the restrictions $i_C^* F$ at some point x along three linearly independent tangent planes relative to three choices of C , automatically fixes the value of F at x .*

The compatibility requirement between k and F following from Remark 4.26 depends on the pointwise behavior of the single slices. We would like to find a more geometric condition (C) which can be tested by looking only at the function k as in (4.9). See condition (C^*) and Open Problem 12 for an example. The desired condition (C) should also satisfy the following properties.

Definition 4.27. *Suppose that (C) is a property of the function k of (4.9) for a given set of cycles \mathcal{C} . We say that (C) is a **compatibility condition** if the following are true:*

1. *If $F \in \mathcal{F}_{\mathbb{Z}}^p(\Omega)$ for some $p \in]1, 3/2[$ then the function k_F which to a generic $C \in \mathcal{C}$ associates the slice of F along C , satisfies (C) .*
2. *If $F_i \in \mathcal{F}_{\mathbb{Z}}^p(\Omega)$ are a sequence converging L^p -weakly to a form F then k_F (defined as in (1) above) satisfies (C) .*
3. *Whenever k satisfies (C) , there exists a $F \in \mathcal{F}_{\mathbb{Z}}^p(\Omega)$ such that $k = k_F$.*

Since we “know much more” about $\mathcal{F}_{\mathbb{Z}}^p$ than about weak convergence or about slice functions, in general the first point above should prove relatively easier to check.

Example 4.28. *In Chapter 2, a (C) satisfying the first condition was implied. In that article we had a situation where*

$$\mathcal{C} = \{\partial B(x, r) : x \in \mathbb{R}^3, r > 0\}$$

and the generic sets \mathcal{R} were the ones of the form $\{\partial B(x, r) : r \in N\}$ s.t. $\mathcal{L}^1(N) = 0$. Then one had the following condition (C) :

$$(C) : \quad : \text{“the integral of } k(C) \text{ is an integer”}.$$

This is exactly the definition of $\mathcal{F}_{\mathbb{Z}}^p$. As shown by the example from Lemma 4.25, this candidate for condition (C) is too weak to satisfy the second property above.

A simple geometric candidate for compatibility

consider still the case where \mathcal{C} consists of all spheres contained in Ω . If our form $F \in \mathcal{F}_{\mathbb{Z}}^p$ has only finitely many singularities, then the integral $\int_C F$ along each cycle corresponds to an algebraic sum of the degrees of the singularities situated in the interior of C . Now consider two intersecting spheres, C', C'' and suppose that their intersection is a circle D . If we assume that none of the singularities of F is on $C' \cup C''$, we will have then that near D the forms $i_{C'}^* F, i_{C''}^* F$ can be represented respectively as dA', dA'' , for suitable 1-forms A', A'' . It is easy to see (by using Stokes' theorem) that the difference $\int_D A'' - \int_D A'$ must then be an integer, and must equal the algebraic sum of the degrees of all the singularities contained inside $C' \cap C''$. It is thus natural to formulate the following compatibility condition more in general:

$$(C^*) : \quad \forall x \text{ for a.e. circle } D \text{ with center } x, \int_D A' - \int_D A'' \in \mathbb{Z},$$

where $i_{C'}^* k(C') = dA', i_{C''}^* k(C'') = dA''$ locally near D .

It is easy to see that condition (1) of Definition 4.27 is satisfied, while condition (2) is achievable using the techniques leading to the closure theorem 2.2. The third condition is however still to be investigated. We thus formulate the following

Open Problem 12. *Is condition (C^*) a compatibility condition in the sense of Definition 4.27?*

Chapter 5

Interior regularity for abelian curvatures in 3 dimensions

5.1 Introduction

5.1.1 The regularity result

In this chapter we complete the study of the problem, following [P3].

$$\inf \left\{ \mathcal{YM}_p(F) := \int_{B^3} |F|^p : F \in \mathcal{F}_{\mathbb{Z}}^p(B^3), i_{\partial B^3}^* F = \phi \right\}. \quad (5.1)$$

Recall that this problem is equivalent to the minimization of the L^p -norm on $L_{\mathbb{Z}}^p(B^3)$

$$\inf \left\{ \int_{B^3} |X|^p : X \in L_{\mathbb{Z}}^p(B^3), \nu \cdot X = \phi \right\}. \quad (5.2)$$

These problem are interesting just for $p < 3/2$ (see Section 1.4), otherwise they reduce to the case where F is exact, respectively X is a curl. Knowing this stronger fact reduces the problem to a more classical one.

The existence of a minimizer follows via the direct method of the Calculus of Variations, from the weak closure result of Chapter 2.

Without the constraint $X \in L_{\mathbb{Z}}^p(B^3)$ the minimization in (5.2) would yield the minimum $X \equiv 0$ regardless of the choice of ϕ . The fact that with that constraint the minimization is in general nontrivial follows instead from the good behavior of our boundary value as defined in Section 4.6 (in particular

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from the nontriviality condition ensuring that the class of weak curvatures with nontrivial boundary value $\phi \neq 0$ is disjoint from the class of weak curvatures with boundary value zero.

The above results imply the existence of minimizers:

Proposition 5.1. *If ϕ is a 2-form in L^p [respectively, up to Hodge star duality with respect to the standard metric, an L^p -function] on ∂B^3 having integer degree and the Definition 4.2 [respectively, its translation for vector fields] is used for the boundary value $i_{\partial B^3}^* F = \phi$, then the minimum is achieved in Problem (5.1) [resp. (5.2)].*

Proof. We give the proof in the language of forms. Consider a minimizing sequence $F_i \in \mathcal{F}_{\mathbb{Z},\phi}^p(B^3)$ and extract a weakly convergent subsequence, which we label in the same way, abusing notation. We denote by F_∞ the limiting L^p -form. From Theorem 2.2 we know that $F_\infty \in \mathcal{F}_{\mathbb{Z}}^p(B^3)$. Using Theorem 4.15 we deduce that $F_\infty \in \mathcal{F}_{\mathbb{Z},\phi}^p(B^3)$. Thus F_∞ is the desired minimizer. \square

The main result of the present chapter is the following:

Theorem 5.2. *Let $p \in]1, 3/2[$, and let $X \in L^p(B^3, \mathbb{R}^3)$ be a minimizer. Then X is locally Hölder-continuous away from a locally finite set $\Sigma \subset B^3$.*

The new result leading to regularity is the following ϵ -regularity theorem:

Theorem 5.3 (ϵ -regularity). *There exists $\epsilon_p > 0$ such that for any minimizer $X \in L_{\mathbb{Z}}^p$ of the L^p -energy if $B_r^3(x_0) \subset B^3$ and*

$$r^{2p-3} \int_{B_r(x_0)} |X|^p d\mathcal{H}^3 < \epsilon_p, \quad (5.3)$$

then

$$\operatorname{div} X = 0 \quad \text{on } B_{r/2}(x_0). \quad (5.4)$$

In other words, we have that X is a curl in the regions of small energy concentration. Therefore we may apply the regularity theory of [130] (see the review in Appendix D) and obtain the regularity of X in such regions. The rest of the proof of Theorem 5.2 follows the strategy of the regularity theory for harmonic maps. For a discussion of the proof of Theorem 5.3 see the beginning of Section 5.2.

5.1.2 Relation to the regularity theory for harmonic maps

Our regularity result parallels the following result of Schoen and Uhlenbeck [114] (case $p = 2$), later extended by Hardt and Lin [70] (for general $p \in]1, \infty[$) regarding minimizing harmonic maps. The result was proved for more general manifolds, but the special case stated here already presents the main difficulties. The more precise description the singularities is due to Brezis, Coron and Lieb [30].

Theorem 5.4 ([115],[70],[30]). *Suppose $u : B^3 \rightarrow \mathbb{S}^2$ is a map in $W^{1,2}(B^3, \mathbb{S}^2)$ minimizing the L^2 -norm of its differential. Then u has Hölder-continuous derivative outside a locally finite set $\Sigma \subset B^3$. Moreover, u realizes a nontrivial degree around small spheres centered at each point in Σ .*

The analogy of our problem with the one of harmonic maps is also reflected by the fact that in our case the singularities also encode some topology, i.e. they all have a nontrivial degree.

Our approach roughly follows the strategy of the regularity theory for harmonic maps. As in the harmonic map regularity proof, we derive and make use of a monotonicity formula and a stationarity formula (cfr [70] and [103] with our Section 5.5). In Section 5.2 we prove an ϵ -regularity result, in Section 5.3.1 we describe an analogous of the Luckhaus lemma [87], which helps proving the sequential compactness of minimizers. Then we proceed to the study of tangent maps and to the dimension reduction in Section 5.4.

The techniques and results of sections Sections 5.2 and 5.3.1 are quite different from the approaches that we found in the literature, and might shed a different perspective also on the theory of harmonic map regularity. The main new observation is that the ϵ -regularity can be studied on a simple model if we use the fact that the singularities come with an associated integer (the degree, or flux, of our vector field on small spheres surrounding the singularity). The structure that naturally arises is a weighted graph, having vertices that represent the singularities and edges representing the vector field's flow lines. Reducing to this model is allowed by the strong density result of Kessel [83, 84] (the statement is provided in Theorem 1.19).

The approximants to a minimizer (as given by Theorem 2.5) correspond then to normal 1-dimensional currents. We are able to associate a weighted graph to vector fields in \mathcal{R}_∞ , by applying a decomposition result of Smirnov [120] for normal 1-dimensional currents (see Theorem 5.8).

The ϵ -regularity theorem is then obtained by a combinatorial reasoning on these graphs. It relies on an elementary minimax result (the famous “maxflow-mincut” theorem, [57]). See the scheme (9.26) in the next section for a more precise overview of the proof. The same discretization method is the critical step also in the Luckhaus lemma, in Section 5.3.

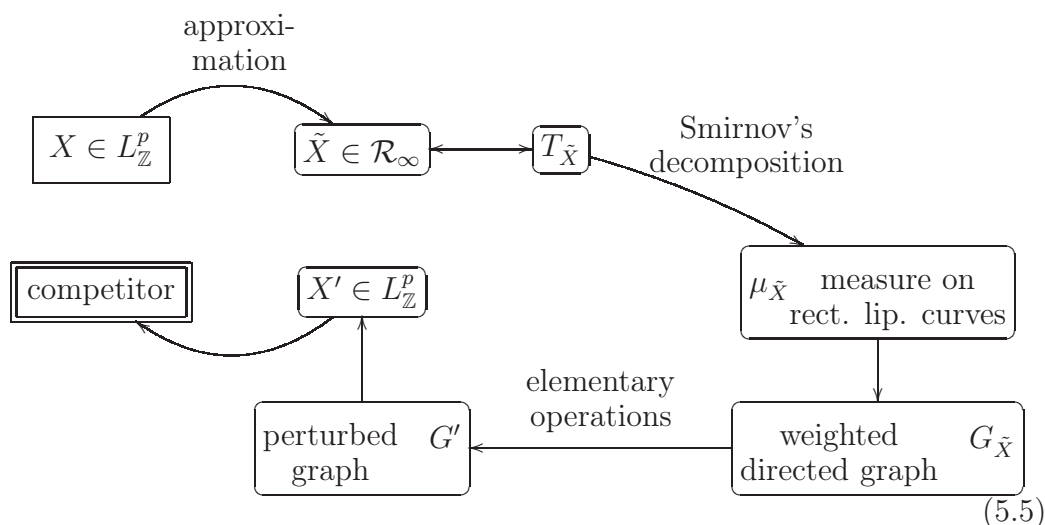
5.2 The ϵ -regularity Theorem

In this section our goal is to prove a so-called ϵ -regularity theorem. This result states that if, for an energy minimizer X on a ball B the energy happens to be small enough, then X has no charges inside a smaller ball. The main steps of the proof can be summarized as follows (see the scheme (9.26) below).

- We first approximate our vector field $X \in L^p_{\mathbb{Z}}$ strongly in L^p -norm by some smoother vector field $\tilde{X} \in \mathcal{R}^\infty(B^3)$ as in Theorem 2.5.
- To \tilde{X} we associate a 1-current $T_{\tilde{X}}$ in a classical way, and we apply to $T_{\tilde{X}}$ a decomposition result due to Smirnov [120] (see also the recent development by Paolini and Stepanov [99, 100]). This result says that a normal current like $T_{\tilde{X}}$ can be decomposed via a measure (on Borel sets for the weak topology) $\mu_{\tilde{X}}$ into a superposition of rectifiable integral currents supported on Lipschitz curves starting and ending on the boundary of $T_{\tilde{X}}$. This result is described in Section 5.2.1.
- Smirnov’s decomposition $\mu_{\tilde{X}}$ in our case (since the boundary $\partial T_{\tilde{X}}$ is supported on a discrete set) gives rise to a weighted directed graph $G_{\tilde{X}}$, by grouping together the curves in the support of $\mu_{\tilde{X}}$ with the same starting and ending point. These constructions are performed in Section 5.2.2.
- We define a way of perturbing $G_{\tilde{X}}$ into another graph G' . For the underlying vector fields, this corresponds to perturbing \tilde{X} into a vector field X' that is (not smooth but) still in $L^p_{\mathbb{Z}}$, and has energy bounded by the energy of \tilde{X} . We call these modifications *elementary operations* (see the definitions at the beginning of Section 5.2.2), and we use the same notation for operations on the graph $G_{\tilde{X}}$ and on the corresponding vector field \tilde{X} .
- If \tilde{X} has little energy on a ball B , then we can perturb it by elementary operations into another vector field X' as above, and which has no

charges inside B . This uses the classical “max flow/min cut” theorem on the graph \tilde{G} (see Section 5.2.3).

- Finally, as the vector fields \tilde{X} approximate better and better the minimizer X , since $p > 1$ we can apply the results of Chapter 2 and extract a subsequence of the perturbed X' that converge weakly to a competitor for X . The comparison of X with the competitor gives a contradiction unless X has no charges in B , proving the result (see Section 5.2.3).



5.2.1 Smirnov’s decomposition of 1-dimensional normal currents

We build our constructions upon Smirnov’s decomposition result for 1-dimensional normal currents [120]. In order to state the results that we use, we need some preliminaries.

Definition 5.5. A 1-current T in \mathbb{R}^3 is called an elementary solenoid if there exists a 1-Lipschitz function $f : \mathbb{R} \rightarrow \mathbb{R}^3$ with $f(\mathbb{R}) \subset \text{spt}(T)$, such that f, T satisfy

$$T = \mathcal{D} - \lim_{T \rightarrow \infty} \frac{1}{2T} f_{\#} \overrightarrow{[-T, T]},$$

$$\mathbb{M}(T) = 1.$$

In the spirit of the above definition, we can identify an oriented Lipschitz curve with a 1-dimensional rectifiable current. We call \mathcal{C}_ℓ the set of all oriented

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curves of length $\leq \ell$, which we endow with the weak topology. All measures on paths described in this section will be positive, σ -finite measures, Borel with respect to the weak topology. The corresponding integrals are understood in the weak sense, i.e.

$S = \int_{\mathcal{C}_\ell} R d\mu(R)$ is the current defined by $S(\phi) = \int_{\mathcal{C}_\ell} R(\phi) d\mu(R)$ for $\phi \in \mathcal{D}^1(\mathbb{R}^3)$.

Definition 5.6. *We say that a 1-current T is decomposed into currents lying in a set $J \subset \mathcal{D}_{1,loc}(\mathbb{R}^3)$ if there is a Borel measure μ supported on J such that*

$$\begin{aligned} T &= \int_J R d\mu(R), \\ \|T\| &= \int_J \|R\| d\mu(R). \end{aligned}$$

$T \in \mathbb{N}_{1,loc}(\mathbb{R}^3)$ is totally decomposed if the same μ also decomposes the boundary:

$$\begin{aligned} \partial T &= \int_J \partial R d\mu(R), \\ \|\partial T\| &= \int_J \|\partial R\| d\mu(R). \end{aligned}$$

Using Birkhoff's theorem (in the appropriate setting), Smirnov proves the following decomposition result.

Theorem 5.7. *$T \in \mathcal{D}_1(\mathbb{R}^3)$, $\partial T = 0$, then T can be decomposed in elementary solenoids.*

For the case $\partial T \neq 0$ there holds instead:

Theorem 5.8. *If $T \in \mathbb{N}_{1,loc}(\mathbb{R}^3)$ then T can be decomposed as follows:*

$$\begin{aligned} T &= P + Q, \\ \|T\| &= \|P\| + \|Q\|, \\ \partial T &= \partial Q, \partial P = 0. \end{aligned}$$

moreover Q can be totally decomposed into simple curves of finite length, i.e. into elements of $\mathcal{C}_\infty := \cup_{\ell>0} \mathcal{C}_\ell$.

For the sketch of the proofs of the above theorem see Appendix A.

Remark 5.9. *We now note some facts that follow easily from the constructions of Smirnov, but are not explicitly stated in his paper:*

1. In the total decomposition of Q above, the paths have in general unbounded (finite) lengths, but almost all of them (w.r.t. the decomposing measure μ) have start point $b(R)$ and end point $e(R)$ on the support of $\partial T = \partial Q$.
2. If T corresponds to a regular vector field (i.e. for all test forms ω , $T(\omega) = \int \omega(X) d\mathcal{L}^3$ and X is regular), then the paths are composed of pieces of trajectories of the flow of X .
3. The functions $b, e : \mathcal{C}_\infty \rightarrow \mathbb{R}^3$ are continuous for the weak topology. In particular, given two Borel sets $A, B \subset \mathbb{R}^3$, the set of paths

$$\{R : \mathbb{M}(R) < \infty, b(R) \in A, e(R) \in B\}$$

is Borel for the weak topology.

4. Suppose that a 1-current T decomposes via a measure μ on the space of 1-currents. If α is a bounded Borel function on $\mathcal{D}_1(\mathbb{R}^3)$, then $\nu = \alpha\mu$ induces by integration a 1-current T_α that is totally decomposed via $|\alpha|\mu$, and satisfies

$$\vec{T}_\alpha = \pm \vec{T} \quad \text{and} \quad \|T_\alpha\| \leq \|\alpha\|_{L^\infty(\mu)} \|T\|.$$

Indeed, this is true for step functions α , and L^1 -convergence at the level of the decomposition induces weak convergence at the level of the decomposed currents.

5. The same result as above holds also in the case of a totally decomposed current T , with the analogous inequality holding also for the boundaries:

$$\|\partial T_\alpha\| \leq \|\alpha\|_{L^\infty(\mu)} \|\partial T\|.$$

5.2.2 Encoding the useful information in a graph

For vector fields $X \in \mathcal{R}_\infty(\Omega)$ the decomposition of Smirnov allows to group the integral trajectories of $X \llcorner \Omega$ according to their start and end points: a generic trajectory could start or end on $\partial\Omega$ or on one of the “charges” (i.e. singularities) of X . We encode this information in a weighted directed graph (i.e. a graph such that to each edge a positive number called “weight” and a direction are assigned). The weights in our encoding graphs keep track of how much of the flux of X is carried by each group of trajectories, and the direction of an edge encodes the direction of the corresponding trajectories. The grouping is done in such a way that there are no flux cancellations within the same group. Thus specifying the flux for a group of trajectories automatically gives a measure of the norm of the restriction of X to those trajectories.

Elementary operations

The following kind of operations will be the ones that we perform on our encoding graphs:

Definition 5.10. *An elementary operation on a directed weighted graph G consists of multiplying by a factor $\alpha \in [-1, 1]$ the weight of an edge, where multiplication of the weight by a negative factor $\alpha < 0$ means inverting the orientation and multiplying by $|\alpha|$.*

We indicate by $G \preceq G'$ the statement that G is achieved from G' , after applying finitely many elementary operations.

We now define the elementary operations on the underlying $X \in \mathcal{R}_\infty$. We use the same name because the two definitions correspond to each other in a natural way, as described in Section 5.2.2.

Definition 5.11. *Consider $X \in \mathcal{R}_\infty$, which we identify with a current $T = T_X$ as in Remark 5.9 (2), and to which we associate P, Q and a measure μ totally decomposing Q as in Theorem 5.8. An elementary operation on X consists in replacing X by the vector field corresponding to $(T_X)_\alpha$ obtained as in Remark 5.9 (5), for some function α that only takes values in $[-1, 1]$ and that is piecewise constant on a family of sets defined via b, e as in Remark 5.9 (3).*

We indicate by $X \preceq X'$ the property of X of being achievable after performing finitely many elementary operations starting from X' .

Remark 5.12. 1. *It is immediate from Remark 5.9 (3) that $X \preceq X'$ implies $\|X\|_{L^p} \leq \|X'\|_{L^p}$ with strict inequality unless $|\alpha| = 1$ in all of our elementary operations.*

2. *\mathcal{R}_∞ is not invariant under elementary operations, since such operations often create jumps in X . In general also the integer divergence condition is not preserved by these modifications.*

3. *From Remark 5.9 (5) it follows however, that for $X \in \mathcal{R}_\infty \cap L^p(\Omega)$, any elementary operation sends X to a vector field $X' \in L^p(\Omega)$ having zero divergence away from the singular set of X .*

Grouping trajectories of $X \in \mathcal{R}_\infty \cap L^1(\Omega)$

Consider $X \in \mathcal{R}_\infty \cap L^1(\Omega)$ and the normal 1-current T_X as in Remark 5.9 (2).

Using Theorem 5.8, we can find a decomposition $T_X = P_X + Q_X$ and a measure μ_X on $\mathcal{C}_\infty := \cup_{\ell > 0} \mathcal{C}_\ell$ that totally decomposes Q_X into finite-length

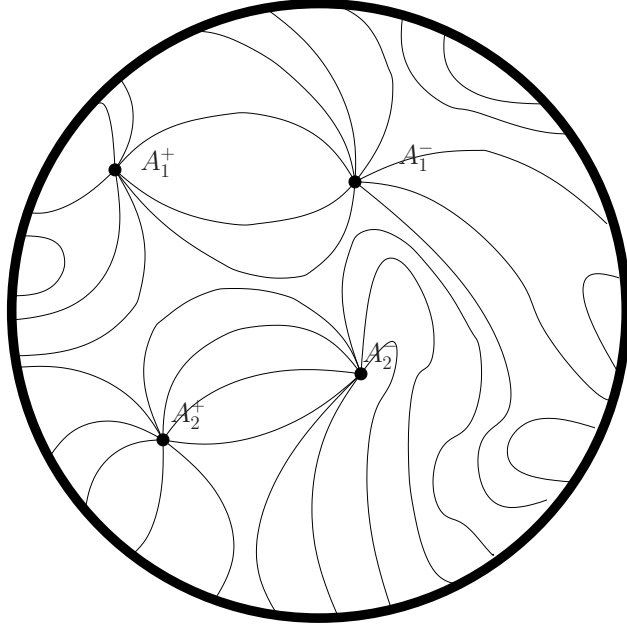


Figure 5.1: We represent schematically (i.e. we forget for a moment that we are in a 3-dimensional setting, and we take Ω to be a ball) the finitely many charges of our vector field $X \in \mathcal{R}_\infty \cap L^1(\Omega)$ as black dots, and some of the supports of the rectifiable currents R of Definition 5.6, as thin lines.

simple paths.

Then note that, due to the special structure of X , $\partial(T_X \llcorner B)$ is supported on $\partial\Omega \cup \{\text{charges of } X\}$. Also, by the total decomposition property of Q_X , there holds

$$\partial(T_X \llcorner B) = \int_{\mathcal{C}_\infty} \partial R d\mu_X(R) = \int_{\mathcal{C}_\infty} (\delta_{e(R)} - \delta_{b(R)}) d\mu_X(R)$$

and $b(R), e(R) \in \text{spt}\partial(T_X \llcorner B)$ for μ_X -a.e. R , so that we can decompose the set of finite length paths into disjoint Borel sets:

$$\mathcal{C}_\infty = C \cup \bigcup_{i,j=0}^n C_{ij},$$

where $\mu_X(C) = 0$ and for all $R \in C_{ij}$ there holds

$$b(R) \in A_i^-, e(R) \in A_j^+,$$

where

$$A_0^\pm := \partial\Omega \cap \{\text{sgn}(X \cdot \nu_\Omega) = \pm 1\}$$

and

$A_i^\pm, i > 0$ enumerate the \pm -charges of X , possibly with repetitions.

By the decomposition theorem 5.8, if

$$C_i^- = \cup_{j=0}^n C_{ij}, C_j^+ = \cup_{i=0}^n C_{ij},$$

then

$$\mu_X(C_i^+) = \sum_{j=0}^n \mu_X(C_{ij}), \quad \mu_X(C_j^-) = \sum_{i=0}^n \mu_X(C_{ij}),$$

and for $i > 0$ it is clear that $\mu_X(C_i^\pm)$ is equal to the charge of A_i^\pm (see also Figure 5.2).

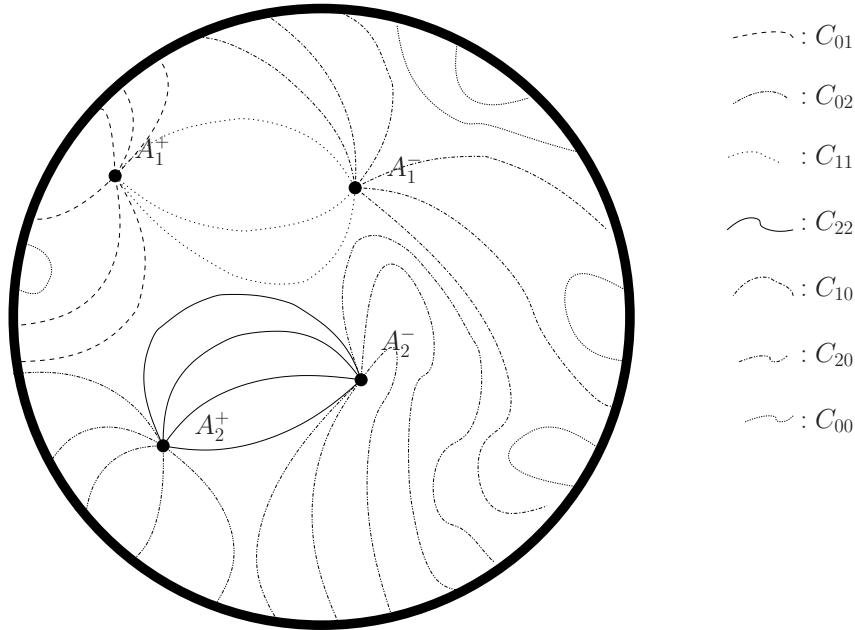


Figure 5.2: In the example of Figure 5.1, we represent with different patterns the supports of curves belonging to different C_{ij} 's. We omit the set C_{ij} if it has $\mu_X(C_{ij}) = 0$.

Associating a graph to a vector field

With the notations of the previous subsection, we associate to X the graph G_X (see Figure 5.3) which has the following features:

- has vertices indexed by $A_i^\pm, i = 0, \dots, n$,
- has a directed edge $A_i^- \rightarrow A_j^+$, for all $0 \leq i, j \leq n$, unless $\mu(C_{ij}) = 0$,
- any edge $A_i^- \rightarrow A_j^+$, it has weight $\mu_X(C_{ij})$ assigned to it.

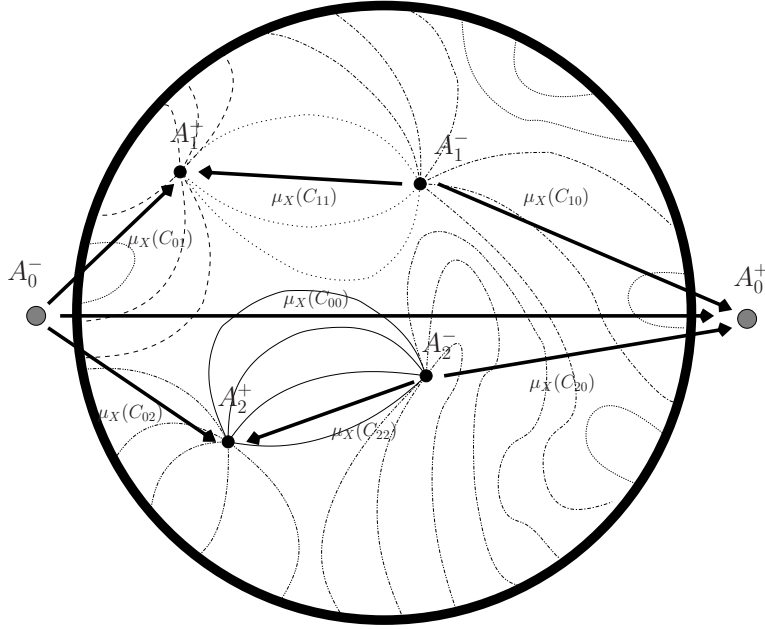


Figure 5.3: We superpose to the picture of Figure 5.2 the associated graph, where on top of each arrow we also describe its weight. The gray vertices A_0^+, A_0^- correspond respectively to start and end points of curves which lie on the boundary.

Further, if $\bar{G} \preceq G_X$ then we associate to \bar{G} a vector field $\bar{X} \preceq X$ such that $\bar{G} = G_{\bar{X}}$, by the following procedure:

- Fix a sequence $G_X = G_0 \preceq G_1 \preceq \dots \preceq G_N = \bar{G}$ such that G_{k+1} is obtained from G_k by an elementary operation. We can still identify the vertices of G_k with those of G_X .
- To each G_k we associate a function $\alpha_k \in L^\infty(\mu_X)$, as follows. We start with $\alpha_0 \equiv 1$. For $k > 0$ if G_{k+1} is obtained from G_k by multiplying the weight on $A_i^- \rightarrow A_j^+$ by $\alpha \in [-1, 1]$ then we define $\alpha_{k+1} := \alpha \chi_{C_{ij}} \alpha_k + \chi_{C_\infty \setminus C_{ij}} \alpha_k$.
- Clearly $\alpha_N \in L^\infty(\mu_X)$ defines an elementary operation on X , and so we call \bar{X} the vector field corresponding to $(T_X)_{\alpha_N}$.

5.2.3 Proof of the ϵ -regularity

Modifications to eliminate charges in the regular case

In this subsection we restrict to vector fields $X \in \mathcal{R}_\infty \cap L^p(\Omega)$ satisfying the conditions of the ϵ -regularity theorem, and we show that we can apply elementary operations decreasing the energy while eliminating the charges of X . The main result is as follows.

Proposition 5.13 (regular case). *Suppose that $X \in \mathcal{R}_\infty \cap L^p(\hat{\Omega})$ and that $\hat{\Omega} \ni \Omega$ is such that $\int_{\partial\Omega} |X| < 1$ and $\int_{\partial\Omega} X \cdot \nu = 0$. Then there exists a second vector field $\bar{X} \in L^p_{\mathbb{Z}}(\hat{\Omega})$ such that $\bar{X} \preceq X$ and*

1. $\bar{X} = X$ on $\overline{\hat{\Omega} \setminus \Omega}$,
2. $\|\bar{X}\|_{L^p(\hat{\Omega})} < \|X\|_{L^p(\hat{\Omega})}$ and
3. $(\text{div} \bar{X}) \llcorner \Omega = 0$.

The inequality of point (2) is strict unless X already satisfies point (3).

Proof. The main idea of the proof is to apply elementary operations to X , so that we cancel out the charges inside Ω . Because of the above constructions, it is enough to do the corresponding operations on the graph G_X that encodes all the information that we need for the proof.

Step 1: structure of the graph G_X . Consider the graph $G := G_X$ defined in Section 5.2.2, and call

- C^+, C^- the sets of vertices of G corresponding to the interior charges of a given sign,
- Σ^\pm the sets of vertices of G corresponding to components of $\partial\Omega$ with local charge \pm , i.e. $\Sigma^\pm = \{A_0^\pm\}$.

The form of our graph is summarized in the following scheme, where we also indicate names for groups of arrows:

$$\Sigma^+ \xrightarrow{\sigma^+} C^- \xleftarrow{\nu} C^+ \xrightarrow{\sigma^-} \Sigma^- .$$

The hypothesis $\int_{\partial\Omega} |X|^p < 1$ implies that the arrows σ^\pm have total weight less than 1. This will be important in the sequel.

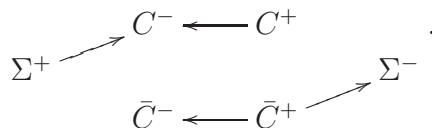
Step 2: elimination of the singularities. We desire to keep the arrows in σ^\pm fixed, and modify the other arrows via elementary operations so that the modified graph satisfies Kirchhoff's law. This can be done as follows:

- We keep (i.e. multiply by +1) all the edges which go directly from Σ^+ to Σ^- . Since these edges are not affected by the elementary operations done in the rest of the proof, we suppose from now on, without loss of generality, that there are no such edges.
- Let's restrict to a connected component of our graph. Suppose first that it has the form drawn above (i.e. it is not degenerate): in this case we can find a maximal Kirchhoff subgraph K connecting Σ^+ to Σ^- , in the undirected graph

$$\Sigma^+ \text{ --- } C^- \text{ --- } C^+ \text{ --- } \Sigma^- .$$

By the “max flow-min cut” theorem, after subtracting such directed subgraph, the remaining edges make a disconnected graph that has 4 possible forms (where we keep the orientations as in the original G):

1. All arrows in ν have been cut, but there are some edges joining Σ to some point charges. These charges correspond to singularities of X , for which at least 1/2-charge flowed from/to Σ . In particular, since the difference $|\sigma^+| - |\sigma^-|$ is constant during our construction, there must be an even number of such charges. This is not possible because the $\int_\Sigma |X \cdot \nu_\Sigma| d\mathcal{H}^2$ was assumed to be smaller than 1.
2. The whole graph has been used, and we end up without leftover edges of the graph. Then again we see that $\int_\Sigma |X \cdot \nu_\Sigma| d\mathcal{H}^2$ is prohibited to be smaller than 1, since in any charge connected to Σ^\pm , the total weight of the arrows from/to the boundary $\partial\Omega$, is $= \frac{1}{2}$, and there are at least 2 such charges.
3. All arrows σ^- have been cut. Then also the arrows in σ^+ have disappeared after eliminating the maximal flow, again because $|\sigma^+| - |\sigma^-|$ is constant (equal to zero) during these modifications. Thus in this case all arrows outside ν are canceled. Then we can multiply by zero the remaining arrows: these arrows are of positive total weight since else we reduce to point (2), which is already excluded. Thus we strictly decrease the L^p -norm of X .
4. The last case is the “generic” one: it could be that after the cut we are left with a graph of the form



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It follows from Lemma 5.15 that in this case it is possible to find another minimal cut that gives a graph as in (3) or in (2) instead, and we conclude the proof.

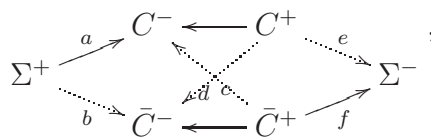
The conclusion of this enumeration is that the only possible cases that allow any singularity at all inside Ω and are compatible with the small boundary energy are the ones corresponding to the case (3) above. Observe that in this case *we are sure to have canceled some edges* i.e. we have decreased the energy of \tilde{X} , as desired.

□

Example 5.14. Consider a regular vector field in $\mathcal{R}_\infty \cap L^p(\Omega)$ that has 5 singularities, one point having charge 1 a second point having charge 2, and the remaining points having charge -1 each (see Figure 5.4). Suppose that the weights of the edges of the associated graph are as in Figure 5.4. We assume from the beginning that μ_X gives no weight to the curves that both start and end point on the boundary (such curves are anyways not affected by our manipulations). The maximal flow pictured on the right corresponds to any of the 3 minimal cuts on the left. In general, no uniqueness of either the maximal flow or the minimal cut is guaranteed. Figure 5.5 exhibits what happens next, in our manipulations. Once we fix the maximal flow of Figure 5.4, we change by elementary operations the flow lines of X , ending up with the graph on the left of Figure 5.5. Since this represents a flow, i.e. obeys Kirchhoff’s law, the curves representing the modified vector field \tilde{X} are concatenated, i.e. that they all start and end on the boundary. This concatenation is “automatically done” by Smirnov’s decomposition, since the associated current $T_{\tilde{X}}$ is totally decomposed (see Definition 5.6). The “canceled flow” on the right of the figure, gives a measure of the amount of L^p -norm of X gained this way.

We must point out that the L^p -energy improvement in passing from X to \tilde{X} depends also from factors not captured by the graph G_X itself, namely on the lengths and concentrations of the curves decomposing the associated current T_X . But for our purposes a subtler analysis along these lines is not needed.

Lemma 5.15. Under the hypotheses of Proposition 5.13 on X , suppose that a connected component of the associated graph G_X has the form



where a minimal cut is given by the arrows in b, c, d, e . Then another minimal cut is given by the arrows in a, b .

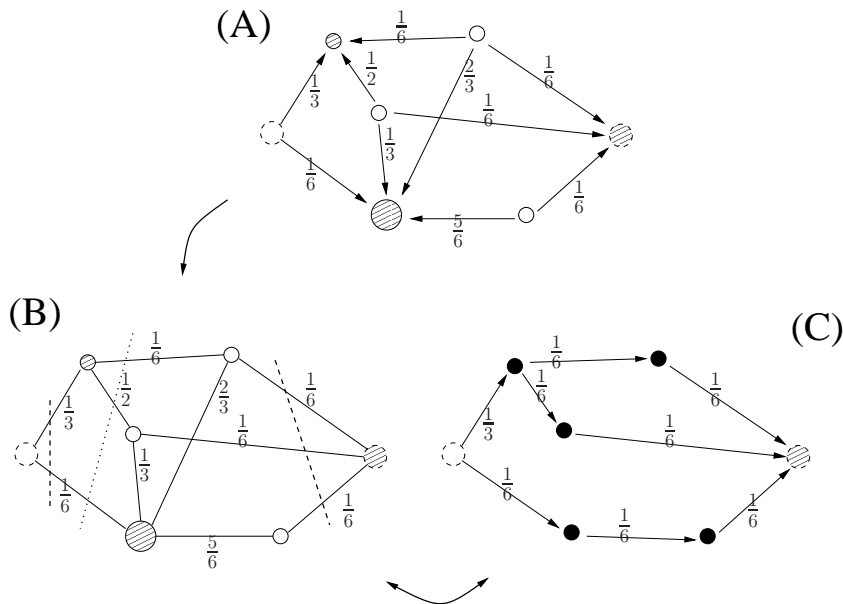


Figure 5.4: (A): a graph corresponding to a possible vector field X having 6 charges (of which the one represented by a larger circle is a double one). In the unoriented graph (B), we represent by dashed lines two minimal cuts separating the vertices with dashed boundaries; a non-minimal cut is corresponds to a dotted line. Observe that the flow through each of the 3 cuts in (A) is the same, but in (B) the sum of edge capacities is larger for the dotted line. In (C) we exhibit the unique maximal flow obtained on (B) between the gray vertices.

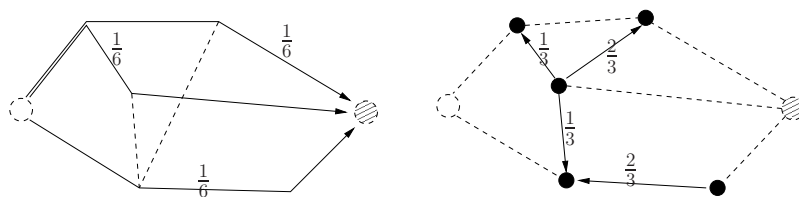


Figure 5.5: Continuing with the example of Figure 5.4, we represent schematically on the left what remains after the cancellation of the charges (in terms of the associated graphs the three arrows of weight $\frac{1}{6}$ actually are substituted by just one arrow of weight $\frac{1}{2}$, but we drew the picture to suggest that a procedure of “concatenating arrows” is actually underlying the operation). On the right we have the flow that results after removing the maximal flow graph out of the initial graph. In our charge removal procedure, we diminish the weights of our graph by the amounts in the right picture, so in this particular \bar{X} has a smaller energy than X .

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Proof. The fact that a, b give a cut is clear from the above diagram. We must prove that such cut is a minimal one.

We indicate by $|x|$ the total flow through the arrows of the group labeled by x . First of all observe that by the zero total flux and small boundary energy hypotheses on X ,

$$|a| + |b| = |e| + |f| < \frac{1}{2},$$

therefore, being b, c, d, e a minimal cut, by comparison with the above cut we obtain

$$|b| + |c| + |d| + |e| < \frac{1}{2}.$$

This implies that *the total number of charges contributing to the vertices C^+ is the same as the number of charges contributing to C^- , and similarly for \bar{C}^+, \bar{C}^-* . Indeed, suppose for contradiction that the numbers of charges contributing to C^+, C^- were not equal. Then the total flow $|a| + |c| + |d| + |e|$ would be ≥ 1 , and this would contradict the fact that $|a|$ and $|c| + |d| + |e|$ are both $< \frac{1}{2}$.

By the consideration in italics above, we obtain that

$$|a| + |d| = |c| + |e|, \quad |b| + |c| = |d| + |f|.$$

Therefore, by definition of a minimal cut

$$|b| + |c| + |d| + |e| \leq |a| + |b|$$

and this gives, using the previous computations,

$$|a| \geq |c| + |d| + |e| = |a| + 2|d|,$$

so $|d| = 0$ and the above inequalities are actually equalities, as desired. \square

The proof of ϵ -regularity

Proof of Theorem 5.3. First of all, we may reduce to the case where the ϵ -regular ball $B(x_0, r)$ of the theorem is the unit ball $B = B(0, 1)$, since the estimates and the function spaces considered are invariant under homotheties and translations of \mathbb{R}^3 . We call $\hat{\Omega}$ the image of the initial B_1 under this transformation.

Step 1: fixing a small energy sphere. We claim that, for any small $\epsilon > 0$, we can find a positive measure set of radii $\rho > 1/2$ such that $\int_{\partial B_\rho} |X|^p < 2\epsilon_p$.

Indeed, if the opposite estimate would hold for a.e. $\rho > 1/2$, then we would obtain

$$\int_B |X|^p \geq \int_{1-\epsilon}^1 \int_{\partial B_\rho} |X|^p > \epsilon_p,$$

therefore

$$\int_B |X|^p d\mathcal{H}^3 \geq (1 - \epsilon)\epsilon_0,$$

and this contradicts our assumption for ϵ small enough. Now from the above boundary energy bound by ϵ_p we get via Hölder's inequality the following bound

$$\int_{\partial B_\rho} |X| \leq (2\epsilon_p)^{\frac{1}{p}} [\mathcal{H}^2(\mathbb{S}^2)]^{\frac{p-1}{p}},$$

and we choose ϵ_p such that the right hand side is equal to 1. This gives the small boundary energy condition as in Proposition 5.13, and the zero flux condition follows from the definition of $L^p_{\mathbb{Z}}(B)$ and from the inequality $|X \cdot \nu_{B_\rho}| \leq |X|$.

Step 2: passing to the approximants. We know that there exist $\tilde{X}_k \in \mathcal{R}_\infty \cap L^p(\hat{\Omega})$ that converge to X in L^p -norm. From the construction leading to this approximation it is clear that we can also further impose the convergence

$$\tilde{X}_k|_{\partial B_\rho} \xrightarrow{L^p} X|_{\partial B_\rho},$$

therefore for k large enough, \tilde{X}_k satisfies the properties required in Proposition 5.13. Applying this proposition, we thus obtain $\bar{X}_k \in L^p_{\mathbb{Z}}(\hat{\Omega})$ which are equal to \tilde{X}_k outside B_ρ and satisfy $\|\bar{X}_k\|_{L^p(\hat{\Omega})} \leq \|\tilde{X}_k\|_{L^p(\hat{\Omega})}$ (with strict inequality if $(\operatorname{div} \tilde{X}_k) \llcorner B_\rho \neq 0$) and $(\operatorname{div} \bar{X}_k) \llcorner B_\rho = 0$.

Step 3: a divergence-free competitor. By weak compactness of $L^p_{\mathbb{Z}}(\hat{\Omega})$ it follows that a subsequence of the $\bar{X}_{k'}$ converges weakly to some $\bar{X} \in L^p_{\mathbb{Z}}(\hat{\Omega})$. The zero divergence condition passes to weak limits, so $\operatorname{div} \bar{X} = 0$ on B_ρ . By sequential weak lowersemicontinuity of the norm, we also deduce

$$\|\bar{X}\|_p \leq \liminf_{k'} \|\bar{X}_{k'}\|_p \leq \liminf_k \|\tilde{X}_k\|_p = \|X\|_p.$$

Since X was a minimizer, all the above inequalities must actually be equalities. We also observe that since the sequence $\bar{X}_{k'}$ converges both weakly and in norm, it must converge also strongly, to \bar{X} . By examining the definition of elementary operations we also observe that the inequality $|\bar{X}_k|(x) \leq |\tilde{X}_k|(x)$ holds almost everywhere for all k , and from it and the a.e. convergence it follows that the same inequality holds also in the limit. Since both \bar{X} and X are minimizers it further follows that $|\bar{X}|(x) = |X|(x)$ almost everywhere.

Step 4: X is also divergence-free. We use the classical regularity theory, namely Lemma 5.17 (which applies since $\operatorname{div} \bar{X} = 0$) and Proposition

5.18 to deduce that \bar{X} is Hölder-continuous in the interior of $B_{r/2}$. It then follows that also $\operatorname{div}X = 0$ on $B_{r/2}$, since in this case $X \in L^\infty_{\mathbb{Z}}(B_{r/2})$. Indeed, using Theorem 2.5 it follows that X can be approximated by vector fields in $\mathcal{R}_{\mathbb{Z}}^\infty(B_{r/2})$ in the strong norm in L^q for $q > 3/2$. But for such exponents the vector fields in $\mathcal{R}_\infty \cap L^q(\Omega)$ are all smooth (and in particular divergence-free, since the divergence is concentrated at their singular points). Thus by approximation also X is divergence-free. \square

5.2.4 A classical consequence: $C^{0,\alpha}$ -regularity

From Theorem 5.3, using an extension by Peter Tolksdorf (and Christoph Hamburger) of the regularity theory first developed by Karen Uhlenbeck, it is relatively straightforward to prove the following extension of it:

Theorem 5.16 (Hölder version of the ϵ -regularity). *If $X \in L^p_{\mathbb{Z}}$ is a minimizer then we can find an $\epsilon_p > 0$ such that if on $B_r^3(x_0) \subset B^3$ the vector field X satisfies (5.3) then on $B_{r/2}(x_0)$ the vector field X is α -Hölder, with α depending only on p and with the Hölder constant of $X|_{B_{r/2}}$ depending only on p and on $\|X\|_{L^p(B_r)}$.*

In order to prove the above theorem, we use the conclusion that $\operatorname{div}X = 0$ of Theorem 5.3 and the Euler equation of the functional $\int_\Omega |X|^p$ to reduce to the by now classical regularity result for systems of equations due to the above cited authors. The main heuristic idea in play here is that roughly “ $\operatorname{div}X = 0$ implies that $X = \nabla f$ for some $W_{loc}^{1,p}$ -function f ”.

In order to use this idea while still keeping rigorous, we use the formulation of our minimization problem in terms of differential 2-forms ω instead of vector fields X .

Lemma 5.17. *The condition that a vector field $X \in L^p_{\mathbb{Z}}(\Omega)$ minimizes the L^p -energy and satisfies $\operatorname{div}X = 0$ implies that the associated 2-form $\omega \in \mathcal{F}_{\mathbb{Z}}^p(\Omega)$ satisfies locally in the sense of the distributions the following equations:*

$$\begin{cases} d\omega = 0 \\ \delta(|\omega|^{p-2}\omega) = 0. \end{cases}$$

Proof. The first equation is a trivial translation of $\operatorname{div}X = 0$ in our new setting. The second one is the Euler equation, and can be directly obtained from the requirement that ω be minimizing, by using the perturbations $\omega \mapsto \omega + \epsilon d\phi$, for $\phi \in C_0^\infty(\wedge^2 T\Omega)$ and taking the derivative in ϵ at $\epsilon = 0$. Since $d\psi$ is exact, it easily follows that the perturbed form is still in $\mathcal{F}_{\mathbb{Z}}^p(\Omega)$. \square

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With the result of the above lemma, we are exactly in the setting of [130], except that that article treats the case $p > 2$, while we are interested in the case $1 < p < 3/2$.

Luckily, the result of [130] was extended in [127], to the case $1 < p < 2$. The article of Tolksdorf considers only the “basic case” where the equations concern a differential of a function instead of the generalization of exact differential forms described by Uhlenbeck, but the setting in which Tolksdorf proves regularity can be translated without much effort into the one of Uhlenbeck, and the techniques present there are not affected by the translation.

Proposition 5.18 ([127]). *If $\omega \in L^p(\wedge^2 T\Omega)$ satisfies the equations of Lemma 5.17 in the weak sense, then ω is α -Hölder, with α depending only on p and with the local Hölder constant of $X|_{B_{r/2}}$ depending only on p and on $\|X\|_{L^p(B_r)}$ for any ball contained in Ω .*

From the above lemma and the proposition, it is straightforward that Theorem 5.16 holds.

5.3 For minimizers X weak convergence implies strong convergence

In this section we prove the following compactness result:

Theorem 5.19. *Suppose $X_k \in L^p_{\mathbb{Z}}(B)$ are minimizers of the L^p -energy, and that $X_k \rightharpoonup X$ weakly in L^p . Then X is also a minimizer and $X_k \rightarrow X$ also L^p -strongly on any ball $B(0, r)$, $r < 1$. In particular, any sequence of minimizers of bounded energy has a strongly convergent subsequence.*

It is a classical result that strong convergence can fail while weak convergence holds, only if some energy is lost in the limit. Thus, it remains to prove that the energy of X on B_r is not lower than the limit of the energies of the X_k on the same ball. The fact that any X obtained as a strong limit of minimizers is a minimizer itself follows from the strong local convergence.

The idea of the proof is to introduce a small parameter $\epsilon > 0$ and to construct an interpolant $\tilde{X}_k \in L^p_{\mathbb{Z}}(B)$ that equals X_k on $B \setminus B_{r+\epsilon}$ and X inside B_r , in such a way that the energy of \tilde{X}_k in the small spherical shell $B \setminus B_{r+\epsilon}$ goes to zero as $\epsilon \rightarrow 0$. This allows us, using the minimization property of X_k , to bound from above the energy of X_k on B_r , by the energy of X on the same

ball.

For the proof of the ϵ -regularity, it is enough to be able to do the constructions for vector fields in $\mathcal{R}_\infty \cap L^p$. The interpolation construction faces again a problem related to possibility that (the approximant of) $X - X_k$ have some singularities in the small shell $B_{r+\epsilon} \setminus B_r$. We deal with this situation again by choosing shells where on the boundaries $X - X_k$ does not have large energy for k large, and by applying the singularity removal operations of Proposition 5.13 from the ϵ -regularity proof. After these elementary operations, we are reduced to an easier situation (see Figure 5.6), where the curves of the Smirnov decomposition of our vector fields all move from one boundary of the shell to the other. In this simpler case, the interpolation can be done via an auxiliary function f satisfying a Neumann boundary value problem in the shell, and the scaling of the classical energy bounds as the thickness of the shell vanishes (see Lemma 5.21), are strong enough for our purposes.

5.3.1 Interpolant construction in the regular case

The result on the existence of the interpolants that we need is the following.

Proposition 5.20. *There exists a constant C depending only on our exponent p from above, such that the following holds. For any numbers R and ϵ such that $R > 1 + \epsilon > 1$, for any $Y \in \mathcal{R}_\infty \cap L^p(B_R)$ having zero flux through $\partial B_{1+\epsilon}$ and through ∂B_1 , having no singularities lying on these two boundaries, and satisfying*

$$\int_{\partial B_r} |Y| d\mathcal{H}^2 < \frac{1}{2},$$

for $r = 1$ and for $r = 1 + \epsilon$, there exists another vector field $\bar{Y} \in \mathcal{R}_\infty \cap L^p(B_R)$, such that

- $\bar{Y} = Y$ on B_1 ,
- $\bar{Y} = 0$ outside $B_{1+\epsilon}$,
- $\|\bar{Y}\|_{L^p(B_{1+\epsilon} \setminus B_1)} \leq \|Y\|_{L^p(B_{1+\epsilon} \setminus B_1)} + C\epsilon^{-\frac{1}{p}} \|Y\|_{L^p(\partial B_1)}$.

Proof of Proposition 5.20. Consider the total decomposition μ of the current T_Y associated to Y . In order to prove Proposition 5.20, we proceed in two steps. In the first one (see Section 5.2.3), we apply some elementary operations $Y|_{B_{1+\epsilon} \setminus B_1}$, obtaining a new vector field Y_1 such that

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- $\bar{Y}_1 := \chi_{B_{1+\epsilon} \setminus B_1} Y_1 + (\chi_{B_R} - \chi_{B_{1+\epsilon} \setminus B_1}) Y$ still belongs to $L^p_{\mathbb{Z}}(B_R)$,
- $\|\bar{Y}_1\|_{L^p(B_R)} \leq \|Y\|_{L^p(B_R)}$,
- $\operatorname{div} Y_1 = 0$ in the interior of $B_{1+\epsilon} \setminus B_1$,
- $\mu_{Y_1} \llcorner S = \mu_{Y|_{B_{1+\epsilon} \setminus B_1}} \llcorner S$ where the Borel (for the weak topology) set S consists of the 1-currents R having boundary on $\partial B_{1+\epsilon} \cup \partial B_1$.

In the second step, we modify the currents $R \in S$ (that up to now were untouched by our construction). We apply an elementary operation in which we cancel (i.e. multiply by 0) the R 's with both boundaries on $\partial B_{1+\epsilon}$, and we let the others unchanged. Then we consider (identifying the current T_2 with a vector field Y_2)

$$T_2 = Y_2 := \int_{S'} R d\mu_{Y_1}(R),$$

where S' are the currents corresponding to Lipschitz curves with one end on $\partial B_{1+\epsilon}$ and the other one on ∂B_1 . It follows $Y_2 \preceq Y_1$ and we see that Y_2 is an L^p -vector field, and since μ_{Y_2} totally decomposes Y_2 , there holds $\operatorname{div} Y_2 = 0$ on $B_{1+\epsilon} \setminus B_1$. The elementary operations decrease the L^p -norms of the boundary values, thus

$$\int_{\partial B_1} |Y_2|^p d\mathcal{H}^2 \leq \int_{\partial B_1} |Y|^p d\mathcal{H}^2. \quad (5.6)$$

We now are in a situation where on one hand

$$\partial T_2 = (\nu \cdot Y_2|_{\partial B_{1+\epsilon}}) \mathcal{H}^2 \llcorner \partial B_{1+\epsilon} - (\nu \cdot Y_2|_{\partial B_1}) \mathcal{H}^2 \llcorner \partial B_1,$$

where ν is the radial vector. On the other hand, by the zero flux condition on Y ,

$$\partial T_2 \llcorner \partial B_{1+\epsilon}(1) = 0 = \partial T_2 \llcorner \partial B_1(1),$$

and by homological reasons this implies that the two boundary parts above are themselves boundaries. So our strategy is to find another L^p -vector field Y_3 whose associated current T_3 has $\partial T_3 = -\partial T_2 \llcorner \partial B_{1+\epsilon}$, and which has good norm estimates. The choice to which we are led is as follows:

$$Y_3 = \nabla f,$$

for f solving

$$\begin{cases} \Delta f = 0 \text{ on } B_{1+\epsilon} \setminus B_1, \\ \partial_\nu f = g \text{ on } \partial B_1, \\ \partial_\nu f = 0 \text{ on } \partial B_{1+\epsilon}, \end{cases} \quad (5.7)$$

for $g := -Y_2 \cdot \nu$. Then we can define \bar{Y} by extending $Y_3 + Y_1 - Y_2$ as zero outside $B_{1+\epsilon}$ and as Y inside B_1 .

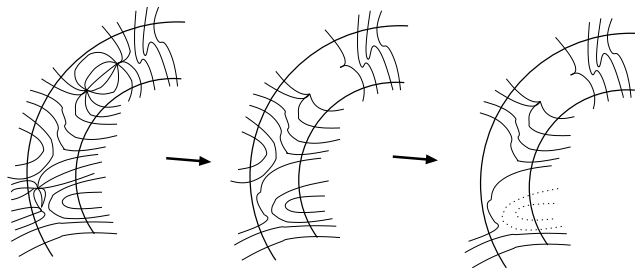


Figure 5.6: We represent schematically, to the left the decomposition of (the current associated to) the vector field Y near $B_{1+\epsilon} \setminus B_1$, in the center the similar decomposition for Y_1 , and to the right the vector field Y_2 , where the part of the decomposition that will stay unmodified (and does not contribute to Y_2) is dotted. The result of subtracting Y_2 and adding Y_3 to Y_1 can be rephrased in a more picturesque way by saying that we are “canceling Y_2 and “replacing it” by Y_3 . We eventually loose a bit in our estimates, since Y_3 “forgets about the support” of Y_2 , and no easy form of a superposition principle holds for our range of exponents p .

The boundary of the associated current $T_{\bar{Y}}$ is equal to $(\partial T_Y) \llcorner \text{int} B_1$, therefore $\bar{Y} \in \mathcal{R}_\infty \cap L^p(B_R)$.

The only fact left to prove in order to obtain Proposition 5.20, is the estimate of the L^p -energy of \bar{Y} , for which we need the following scaling lemma:

Lemma 5.21. *There exists a constant C depending only on the exponent p but not on ϵ , such that the following holds. For any $f \in W^{1,p}(B_{1+\epsilon} \setminus B_1)$ that is a weak solution of the Neumann boundary value equation (5.7), where $g \in L^p(\partial B_1)$, the following estimate holds:*

$$\|\nabla f\|_{L^p(B_{1+\epsilon} \setminus B_1)} \leq C \epsilon^{-\frac{1}{p}} \|g\|_{L^p(\partial B_1)}.$$

Proof. We denote by f_ϵ a solution of (5.7) with parameter ϵ . We observe that the weak formulation of the above Neumann problem states that for all $\phi \in C^\infty(\bar{B}_{1+\epsilon} \setminus B_1)$,

$$\int_{B_{1+\epsilon} \setminus B_1} \nabla f_\epsilon \cdot \nabla \phi = \int_{\partial B_1} \phi g d\mathcal{H}^2.$$

Therefore, for any $\epsilon > 0$ and for any test function ϕ on \mathbb{R}^3 there holds

$$\int_{B_{1+\epsilon} \setminus B_1} \nabla f_\epsilon \cdot \nabla \phi = \int_{B_2 \setminus B_1} \nabla f_1 \cdot \nabla \phi. \tag{5.8}$$

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Now observe that the gradients form a closed subspace of $L^p(B_{1+\epsilon} \setminus B_1, \mathbb{R}^3)$, thus the following equality holds, where $q = \frac{p}{p-1}$:

$$\|\nabla f_\epsilon\|_{L^p(B_{1+\epsilon} \setminus B_1)} = \sup \left\{ \int_{B_{1+\epsilon} \setminus B_1} \nabla f_\epsilon \cdot \nabla \phi_\epsilon : \|\nabla \phi_\epsilon\|_{L^q(B_{1+\epsilon} \setminus B_1)} \leq 1 \right\}. \quad (5.9)$$

Here it is enough to consider functions ϕ_ϵ belonging to $C^\infty(\bar{B}_{1+\epsilon} \setminus B_1)$. For any such test function, there exists another test function ϕ defined via the relation

$$\phi_\epsilon(1+r, \theta) = \phi(1+r/\epsilon, \theta), \quad \forall r \in [0, \epsilon], \forall \theta \in \mathbb{S}^2.$$

The map $\phi \mapsto \phi_\epsilon$ is bijective between $C^\infty(\bar{B}_2 \setminus B_1)$ and $C^\infty(\bar{B}_{1+\epsilon} \setminus B_1)$ and for a geometric constant $C_g \leq 2$, there holds

$$\|\nabla \phi_\epsilon\|_{L^q(B_{1+\epsilon} \setminus B_1)} \leq C_g \epsilon^{\frac{1}{q}-1} \|\nabla \phi\|_{L^q(B_2 \setminus B_1)}.$$

This last fact and (5.8) can be applied to the equivalent definition (5.9), immediately yielding our thesis. Indeed, we can obtain a constant C as in the theorem's formulation, which depends on C_g and on the constant of the classical L^p -regularity estimate for the Neumann problem on the domain $B_2 \setminus B_1$, neither of which depends on ϵ . \square

We thus obtained an estimate of $\|Y_3\|_{L^p(B_{1+\epsilon} \setminus B_1)}$ via $\epsilon^{-\frac{1}{p}} \|Y_2\|_{L^p(\partial B_1)}$, and this suffices because of (5.6). Moreover, $\|Y_1 - Y_2\|_{L^p(B_{1+\epsilon} \setminus B_1)} \leq \|Y\|_{L^p(B_{1+\epsilon} \setminus B_1)}$, because $(Y_1 - Y_2)|_{B_{1+\epsilon} \setminus B_1} \preceq Y|_{B_{1+\epsilon} \setminus B_1}$. This concludes the proof of Proposition 5.20. \square

5.3.2 Proof of Theorem 5.19

In the proof of Theorem 5.19 it is enough to consider the case $r = 1$, and suppose $B = B_R, R > 1$, since the general case follows via the scaling of the energy.

If the X_k and the X would be in $\mathcal{R}_\infty \cap L^p(B)$, then we would apply Proposition 5.20 to $Y_k = X_k - X$ on the shell $B_{1+\epsilon_0} \setminus B_{\epsilon_0}$. In general we cannot rely on this hypothesis, so we use the fact that $\mathcal{R}_\infty \cap L^p(B)$ is dense in $L^p_{\mathbb{Z}}(B)$ and complicate a bit our constructions.

Proof of Theorem 5.19: We proceed in 3 steps.

Step 1: finding a spherical shell of small norm. $X \in L^p(B)$ and the

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X_k converge weakly to it, so by weak lowersemicontinuity, up to forgetting the first terms of the sequence X_k , there holds

$$\|X_k - X\|_{L^p} \leq \|X_k\|_{L^p} + \|X\|_{L^p} \leq 3\|X\|_{L^p}.$$

We fix ϵ_0 and we divide the interval $[1, 1 + \epsilon_0]$ in M smaller intervals I_h of length at least $\epsilon = \epsilon_0/2M$. Then, with the notation

$$A_{I_h} = \{x \in \mathbb{R}^3 : |x| \in I_h\},$$

by pigeonhole principle we can find a subsequence of the X_k and an index h such that

$$\|X_k - X\|_{L^p(A_{I_h})}^p \leq \frac{C}{M} = \epsilon \frac{2C}{\epsilon_0}.$$

From now on we forget about h , and call $I := I_h$. Given any $\delta > 0$, up to choosing another subsequence and changing I slightly, we can also assume

$$\|X_k - X\|_{L^p(\partial B_{\inf I})}^p \leq \delta.$$

Step 2: approximating the interpolant. At this point, with the notation $Y_k := X_k - X$, we use the strong density of $\mathcal{R}_\infty \cap L^p(B)$ in $L^p_{\mathbb{Z}}(B)$ to find an approximant $\tilde{Y}_k \in \mathcal{R}_\infty \cap L^p(B)$ such that the L^p -distance of Y_k and of \tilde{Y}_k on A_I , as well as the L^p -distance of their boundary values, are not larger than ϵ_1 . Similarly we can define approximants \tilde{X}_k, \tilde{X} .

Up to changing I slightly, we can insure that none of the \tilde{Y}_k have any charges on ∂A_I , so that we can apply Proposition 5.20 to them. We obtain $\bar{Y}_k \in \mathcal{R}_\infty \cap L^p(B)$ that is

- L^p -close to Y_k on B_1 ,
- zero on $B \setminus B_{1+\epsilon_0}$.

Up to passing to a subsequence there holds:

$$\begin{cases} \tilde{X}_k - \bar{Y}_k \rightharpoonup \bar{X}_k \in L^p_{\mathbb{Z}}(B), \\ (\tilde{X}_k - \bar{Y}_k)|_{B_{1+\epsilon_0} \setminus B_1} \rightarrow X_k|_{B_{1+\epsilon_0} \setminus B_1}, \\ (\tilde{X}_k - \bar{Y}_k)|_{B_1} \rightarrow X|_{B_1}. \end{cases}$$

The \bar{X}_k defined as above (which depends of the choices of subsequences, on I , and on the parameters $\epsilon_1, \epsilon_0, \epsilon, \delta$), will be our choice of an interpolant between X and X_k .

Step 3: final norm estimates. We can now patch together all our constructions and estimates to obtain the following chain of inequalities. We simplify the notations and write directly $\|\cdot\|_X$ instead of $\|\cdot\|_{L^p(X)}$.

$$\begin{aligned}
\|X_k\|_{B_1} &\leq \|X_k\|_{B_{1+\epsilon_0}} \leq \|\bar{X}_k\|_{B_{1+\epsilon_0}} \quad \text{by minimality of } X_k \\
&\leq \|X\|_{B_1} + \liminf_{\epsilon_1 \rightarrow 0} \|\bar{Y}_k\|_{B_{1+\epsilon_0} \setminus B_1} \quad \text{by lowersemicontinuity} \\
&\leq \|X\|_{B_1} + C \liminf_{\epsilon_1 \rightarrow 0} (\|Y_k\|_{A_I} + \epsilon_1 + C\epsilon^{-1}\|Y_k\|_{\partial B_{\text{inf } I}} + \epsilon_1) \quad \text{using Prop. 5.20} \\
&\leq \|X\|_{B_1} + C \left(\frac{\epsilon}{\epsilon_0} + \frac{\delta}{\epsilon_0} \right),
\end{aligned}$$

and since there is no obstruction to letting ϵ, δ be arbitrarily small, the desired inequality

$$\|X_k\|_{L^p(B_1)} \leq \|X\|_{L^p(B_1)},$$

holds and the thesis follows. \square

5.4 The regularity result

5.4.1 Dimension of the singular set

Definition 5.22. For a vector field $X \in L^p(\Omega)$ defined on some domain Ω , we define the regular set of X , $\text{reg}(X) \subset \Omega$, as the set of those points in a neighborhood of which X is C^1 -regular. The set $\Omega \setminus \text{reg}(X) := \text{sing}(X)$ is called the singular set of X .

Proposition 5.23. If $X \in \mathbb{L}_{\mathbb{Z}}^p(\Omega)$ is a minimizer of the L^p -energy, then for $\Omega' \Subset \Omega$, $\mathcal{H}^{3-2p}(\text{sing}(X) \cap \Omega') = 0$ and $\text{sing}(X)$ is nowhere dense in Ω' .

Proof. Without loss of generality we suppose that X is minimizing with respect to perturbations supported in a neighborhood N of Ω , and we prove the result with Ω instead of Ω' . From Proposition 5.3 we know that $x_0 \in \text{reg}(X)$ if for some $r > 0$ there holds

$$r^{2p-3} \int_{B(x_0, r)} |X|^p \leq \epsilon_0.$$

We can then cover $\text{sing}(X)$ by 2δ -balls $B_1^{2\delta}, \dots, B_l^{2\delta}$ contained in N such that the balls B_k^δ , having the same centers and radius δ , are disjoint. Now, by monotonicity we obtain

$$\epsilon_0 \leq \delta^{2p-3} \int_{B_k^\delta} |X|^p, \quad k = 1, \dots, l.$$

and summing this on k we obtain

$$l\delta^{3-2p} \leq \frac{1}{\epsilon_0} \int_{\cup B_k^\delta} |X|^p \leq \frac{\|X\|_{L^p(\Omega)}^p}{\epsilon_0}. \quad (5.10)$$

After choosing such a family of balls for all δ we obtain the volume estimate

$$\mathcal{H}^3 \left(\bigcup B_k^\delta \right) = l\delta^3 \leq C\delta^{2p} \rightarrow 0,$$

therefore by dominated convergence,

$$\int_{\cup B_k^\delta} |X|^p \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

Inserting this in (5.10) gives, by definition of \mathcal{H}^{3-2p} and by the covering property of our chosen balls, $\mathcal{H}^{3-2p}(\text{sing}(X)) = 0$, as desired.

If we choose a ball $B \subset \Omega$ and we pack it as above with families \mathcal{F}_δ of small disjoint balls of radii $\delta \rightarrow 0$, we see by the scaling reasoning as above that if X has rescaled energy bounded from below by ϵ_0 on all balls for all δ , then X has to have infinite energy on B , which is not the case. Therefore there is a small ball on which the ϵ -regularity theorem 5.3 holds. In particular $\text{sing}(X)$ is nowhere dense. \square

5.4.2 Singular set of weak limits of minimizing vector fields

Proposition 5.24. *Suppose that X_k are minimizers and $X_k \rightharpoonup X_0$. Then $X_k \rightarrow X_0$ locally uniformly on $\Omega' \setminus S_0$, for any $\Omega' \Subset \Omega$. Moreover S_0 is contained in the energy concentration set*

$$\Sigma := \left\{ x \in \Omega : \liminf_{k \rightarrow \infty} \lim_{r \rightarrow 0} r^{2p-3} \int_{B(x,r)} |X_k|^p > 4\epsilon_0 \right\},$$

where ϵ_0 is the constant of the ϵ -regularity theorem 5.3, and $\mathcal{H}^{3-2p}(\Sigma \cap \Omega') = 0$.

Remark 5.25. *It can be proved that $S_0 = \Sigma$, but we don't need this characterization.*

Proof. We can assume up to taking a subsequence that $X_k \rightarrow X_0$ strongly in L^p . We prove that $\mathcal{H}_\infty^{3-2p}(\Sigma) = 0$, and that outside Σ the X_k converge uniformly; this is equivalent to the thesis.

It follows, directly from its definition, that Σ can be covered by finitely many balls B_i , with centers in Σ and radii r_i , and such that for k large enough,

$$(2r_i)^{2p-3} \int_{2B_i} |X_k|^p > 2\epsilon_0, \quad \text{for all } k, i.$$

We fix the choice of this set of balls, such that

$$\sum_i r_i^{3-2p} \leq \mathcal{H}_\infty^{3-2p}(\Sigma) + \epsilon.$$

Then, by the estimates of the ϵ -regularity, it follows that X_k are uniformly Hölder on $\Omega' \setminus \bigcup B_i$, and therefore they have a subsequence converging uniformly on that set. By the reasoning of the proof of Proposition 5.23, as $\delta \rightarrow 0$ the sum $\sum r_i^{3-2p}$ must converge to zero, and by the arbitrariness of ϵ above it follows that $H_\infty^{3-2p}(\Sigma) = 0$. \square

Corollary 5.26. *Let X_k be a minimizer of the L^p -energy, $X_k \rightharpoonup X_0$ and $S_k := \text{sing}(X_k)$ for $i \geq 0$, and $s \geq 0$. Then for any $\Omega' \Subset \Omega$ there holds*

$$\mathcal{H}_\infty^s(S \cap \Omega') \geq \limsup_{k \rightarrow \infty} \mathcal{H}_\infty^s(S_k \cap \Omega').$$

Proof. Consider the balls B_k as in Proposition 5.24, except that this time they are used to approximate $\mathcal{H}_\infty^s(S)$. Then for k large enough there holds

$$S_k \subset \bigcup B_i,$$

and therefore we can obtain

$$\mathcal{H}_\infty^s(S \cap \Omega') + \epsilon \geq \lim_{k \rightarrow \infty} \mathcal{H}_\infty^s(S_k \cap \Omega'),$$

as desired. \square

5.4.3 Monotonicity and tangent maps

We consider now a sequence of blow-ups of a minimizer X around a point x_0 . We call $X_r(x) = \frac{1}{r^2} X(rx + x_0)$, and we observe that

$$X \in L_{\mathbb{Z}}^p(B_r(x_0)) \Leftrightarrow X_r \in L_{\mathbb{Z}}^p(B)$$

Proposition 5.27. *[Monotonicity formula] If $X \in L_{\mathbb{Z}}^p$ is a minimizer of the L^p -energy, then for all $x \in B$ and for almost all $r < \text{dist}(x, \partial B)$ there holds*

$$\frac{d}{dr} \left(r^{2p-3} \int_{B_r(x)} |X|^p d\mathcal{H}^3 \right) = 2p r^{2p-3} \int_{\partial B_r(x)} |X|^{p-2} |X^\parallel|^2 d\mathcal{H}^2 \quad (5.11)$$

where X^\parallel is the component of X orthogonal to $\partial/\partial r$.

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Since the right hand side is positive, the left hand side has a limit $L(x)$ for $r \rightarrow 0^+$, so we can integrate equation (5.11) from 0 to λ , getting

$$\lambda^{2p-3} \int_{B_\lambda} |X|^p d\mathcal{H}^3 - L = 2p \int_{B_\lambda} r^{2p-3} |X|^{p-2} |X^\parallel|^2 d\mathcal{H}^3.$$

As in [70], the function $L(x)$ is actually upper semi-continuous.

The equation (5.11) also implies that

$$\int_{B_1} |X_r|^p = E_r(X) := r^{2p-3} \int_{B_r} |X|^p$$

is increasing in r , therefore the X_λ have a L^p -weakly convergent subsequence $X_{\lambda_i} \rightharpoonup X_0 \in L^p$, $\lambda_i \rightarrow 0$. By a change of variables in the integrated formula we obtain

$$\lambda^{2p-3} \int_{B_\lambda} |X|^p d\mathcal{H}^3 - L = 2p \int_{B_1} r^{2p-3} |X_\lambda|^{p-2} |X_\lambda^\parallel|^2 d\mathcal{H}^3,$$

therefore

$$\lim_{\lambda \rightarrow 0^+} \int_{B_1} r^{2p-3} |X_\lambda|^{p-2} |X_\lambda^\parallel|^2 d\mathcal{H}^3 = 0, \tag{5.12}$$

Since $p' = \frac{p}{p-1}$ and $X_0 \in L^p$, we obtain that $|X_0|^{p-2} X_0 \in L^{p'}$; the weight r^{2p-3} actually worsens the convergence above since it's bounded away from zero, so we obtain that $X_0^\parallel = 0$. This proves more in general the following:

Proposition 5.28. *For any minimizer X , for any $x \in \text{int}(B)$ and for any sequence of rescalings X_{x,λ_i} around x , with $\lambda_i \rightarrow 0$, the weak accumulation points $X_{x,0}$ are radially directed.*

5.4.4 Stationarity and dimension reduction for the singular set

From now on we call s any exponent (smaller than $3 - 2p$, as seen above) for which $\mathcal{H}^s(S \cap \Omega') > 0$, where $S = \text{sing}(X)$ for a minimizer X . Except for x_0 in a set S' such that $\mathcal{H}^s(S') = 0$, there holds

$$\liminf_{\lambda \rightarrow 0} \lambda^{-s} \mathcal{H}^s(S \cap B_{\lambda/2}) > 0, \tag{5.13}$$

where the balls $B_{\lambda/2}$ are all centered at x_0 . As in the previous section, for a subsequence $\lambda_i \rightarrow 0$ our blow-ups converge to a radial tangent map X_0

weakly in L^p , and since they are all minimizers, by Theorem 5.19 they converge strongly, up to taking another subsequence, and also X_0 is a minimizer.

The singular set S_i of X_{λ_i} is the blowup of S , and

$$\lambda_i^{-s} \mathcal{H}^s(S \cap B_{\lambda_i/2}) = \mathcal{H}^s(S_i \cap B_{1/2})$$

and from (5.13) we follow that

$$\mathcal{H}^s(S_0 \cap B_{1/2}) > 0, \quad (5.14)$$

where S_0 is the singular set of X_0 .

Using the radial direction of X_0 and the stationarity (Prop. 5.15) we obtain the following fact.

Lemma 5.29. *For any minimizer X , any tangent map X_0 satisfies*

$$|X_0|(x) = |x|^{-2} |X_0|(x/|x|).$$

Proof. We use the equation (5.15) with respect to a local frame e_1, e_2, e_3 such that the vector e^3 is the radial one and ω associated to X has just the component parallel to $de^1 \wedge de^2$ different from zero (as was proved in Proposition 5.28), and we consider a perturbation field that can be expressed in polar coordinates (ρ, θ) as

$$V(\rho, \theta) = f(\rho)\phi(\theta)\hat{\rho}.$$

We then get from (5.15) that

$$0 = p \int |\omega|^p(\rho, \theta) \frac{1}{\rho} f(\rho) \rho^2 d\rho \phi(\theta) d\theta - \int |\omega|^p(\rho, \theta) \frac{1}{\rho^2} \partial_\rho(\rho^2 f(\rho)) \rho^2 d\rho \phi(\theta) d\theta,$$

By the arbitrariness of $\phi(\theta)$ this translates into the following equation holding for almost all θ

$$\int |\omega|^p(\rho, \theta) [2(p-1)\rho f(\rho) - \rho^2 f'(\rho)] d\rho = 0.$$

This can also be written in terms of $F(\rho) = \rho^{-2p} f(\rho)$ as

$$\int |\omega|^p(\rho, \theta) \rho^{2p} F'(\rho) d\rho = 0$$

and since this holds for all F with support contained in $]0, \infty[$, it must be that

$$|\omega|(\rho, \theta) \rho^2 \text{ is independent of } \rho,$$

as desired. \square

Along the same lines as the above proof (we just have to redefine the orthonormal frames properly), we obtain the following result without difficulty:

Lemma 5.30. *If X_1 is parallel to one coordinate direction e_3 and if the stationarity equation holds, then X_1 is almost everywhere independent of the coordinate x_3 . In particular the thesis is satisfied if X_1 minimizes the energy.*

Remark 5.31. *We note that in Sections 5.4.3 and 5.4.4 until this point just the monotonicity and stationarity formulas were used, without the intervention of any comparison argument. Thus the results proved so far in this subsection are valid not only when X is a minimizer, but also when X is just stationary, i.e. the 2-form F associated to it satisfies*

$$\frac{d}{dt} \Big|_{t=0} \int_{B^3} |\phi_t^* F|^p = 0,$$

for all families of diffeomorphisms $\phi_t : B^3 \rightarrow B^3$ that are differentiably dependent on $t \in [-1, 1]$, equal to the identity in a neighborhood of ∂B^3 , and such that $\phi_0 = id_{B^3}$. This requirement is indeed enough to prove stationarity and monotonicity. On the contrary, the dimension reduction technique that we are about to prove uses the strong convergence result which in turn depends on a comparison argument, thus the following proofs hold only for minimizers X . An intriguing open question is whether or not the uniqueness of tangent maps holds in our case.

We are now ready to apply the dimension reduction technique of Federer to our minimizing vector field X . We start with a radial tangent map X_0 , obtained by blow-up at a point x_0 at which S_0 has positive density with respect to \mathcal{H}^s for some $s < 3 - 2p$ as above, as in (5.14).

As we saw in Section 5.4.3, X_0 is a strong limit of a blowup sequence relative to some $\lambda_i \rightarrow 0^+$. We also know that the singular set S_0 of X_0 has zero \mathcal{H}^{3-2p} -measure and is nowhere dense. It follows from Lemma 5.29, that $|X_0|$ must be (-2) -homogeneous, and $\operatorname{div} X_0 = 0$ locally outside S_0 . Therefore X_0 is itself (-2) -homogeneous outside S_0 , and S_0 is radially invariant, i.e.

$$\lambda S_0 \subset S_0, \quad \forall \lambda > 0.$$

Now we prove that $S_0 = \{0\}$. Indeed, were this not the case, we could find a point $x_1 \in S_0 \cap B_{1/2}$. In this case we could blow up again X_0 with center x_1 , obtaining a tangent map X_1 . By strong convergence we obtain that X_1 would have to be both directed radially and directed along one fixed S_0 direction: this would imply that $X_1 = 0$, contradicting the fact that $x_1 \in S_0$.

The following proposition summarizes the above discussion.

Proposition 5.32. *For a minimizing vector field X , the singular set of any tangent map $\text{sing}(X_0)$ is either empty or contains just the origin.*

After Proposition 5.32 we deduce our main result easily.

Theorem 5.33. *A minimizer X must have finitely many isolated singularities in any open $\Omega' \Subset \Omega$.*

Proof. If X had an accumulating sequence of singular points $\text{sing}(X) \ni x_i \rightarrow x \in \Omega'$, then we can select a small $r > 0$ such that $B(x, r) \subset \Omega'$. Then we can consider the distances

$$\lambda_i = \frac{|x - x_i|}{4},$$

and we observe that for the blowups S_i of ratio λ_i and center x , there holds $\mathcal{H}^0(S_i \cap B_{1/2}) > 2$. This contradicts Proposition 5.32 (where $\mathcal{H}^0(S_0) \leq 1$) and the semicontinuity proved in Corollary 5.26. \square

5.5 Stationarity and monotonicity

5.5.1 Stationarity formula

We consider a smooth diffeomorphism $\varphi_t := id + tV$, where V is a compactly supported vector field and t is small enough. We compute the stationarity formula arising from

$$\frac{d}{dt} \int |\varphi_t^* \omega|^p \Big|_{t=0} = 0.$$

We recall the formula of the norm of the pullback of ω via φ_t , with respect to an orthonormal frame field e_1, e_2, e_3 :

$$\begin{aligned} |(\varphi_t^* \omega)_x|^2 &= \sum_{i,j=1}^n |\omega_{\varphi_t(x)}(d\varphi_t e_i, d\varphi_t e_j)|^2 \\ &= \sum_{i,j=1}^n |\omega_{\varphi_t(x)}(e_i + t dV \cdot e_i, e_j + t dV \cdot e_j)|^2. \end{aligned}$$

To deal better with the above t -derivative, we change variable (we let $y := \varphi_t^{-1}(x)$), so that the point at which we calculate the norm of ω does not depend on t :

$$\int |\varphi_t^* \omega_x|^p dx = \int \left[\left(\sum_{i,j=1}^n |\omega_y(e_i + t dV \cdot e_i, e_j + t dV \cdot e_j)|^2 \right)^{p/2} \det(id + t dV)^{-1} \right] dy$$

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Now we take the derivative of the integrand in $t = 0$, obtaining by easy computations (see for example [103]):

$$p \int |\omega|^{p-2} \sum_{i,j=1}^3 \omega(e_i, e_j) \omega(\nabla_{e_i} V, e_j) - \int |\omega|^p \operatorname{div} V = 0. \quad (5.15)$$

The above formula is justified for minimization problems in L^p , because we are sure that the manipulations done extend to that setting. What ensures that doing the pullback preserves the property of being in $L^p_{\mathbb{Z}}$ as well, is the following:

Proposition 5.34. *Consider a regular foliation*

$$\{\Sigma_\lambda^2 : \lambda \in [-\epsilon, \epsilon]\},$$

i.e. a parameterized set of 2-surfaces in \mathbb{R}^3 such that if $N_\epsilon \Sigma = \cup_\lambda \Sigma_\lambda^2$, then the following (has sense and) holds:

$$\int_{N_\epsilon \Sigma} X \cdot \nu_{\Sigma_\lambda^2} d\mathcal{H}^3 \simeq \int_{-\epsilon}^\epsilon \int_{\Sigma_\lambda^2} X \cdot \nu_{\Sigma_\lambda^2} d\mathcal{H}^2 d\lambda,$$

where $\nu_{\Sigma_\lambda^2}$ is the normal vector of Σ_λ^2 .

The following property is equivalent to the fact that $X \in L^p_{\mathbb{Z}}$: For almost all $\lambda \in [-\epsilon, \epsilon]$ the following holds:

$$\int_{\Sigma_\lambda^2} X \cdot \nu d\mathcal{H}^2 \in \mathbb{Z}. \quad (5.16)$$

Proof. This follows since $\mathcal{R}^\infty \cap L^p$ is dense in $L^p_{\mathbb{Z}}$ in the L^p -norm. Suppose indeed that there exists a closed C^2 -surface Σ such that for a set of $\lambda \in [-\epsilon, \epsilon]$ of measure $\delta > 0$ there holds

$$\int_{\Sigma_\lambda^2} X \cdot \nu d\mathcal{H}^2 \in]a + c, a + 1 - c[, \quad \text{for some } a \in \mathbb{Z}.$$

In particular, whenever $X_i \xrightarrow{L^p} X$, $X_i \in R_\varphi^\infty$ then

$$\begin{aligned} \int_{N_\epsilon \Sigma} |X_i - X|^p d\mathcal{H}^3 &\geq C \int_{-\epsilon}^\epsilon \int_{\Sigma_\lambda^2} |X_i - X|^p d\mathcal{H}^2 d\lambda \\ &\geq C \int_{-\epsilon}^\epsilon |\Sigma_\lambda^2|^{1-p} \left| \int_{\Sigma_\lambda^2} X_i \cdot \nu d\mathcal{H}^2 - \int_{\Sigma_\lambda^2} X \cdot \nu d\mathcal{H}^2 \right|^p d\lambda \\ &\geq C \delta c^p, \end{aligned}$$

contradicting the convergence in L^p -norm stated above. \square

Since for $t < \|X\|_\infty/2$, it follows that ϕ_t is a diffeomorphism and the integral of ω on a sphere S is by definition the same as the integral of $\phi_t^*\omega$ on $\phi_t^{-1}(S)$, we see by the above proposition that $\omega \in \mathcal{F}_\mathbb{Z}^p(\Omega)$ implies that also the perturbations $\phi_t^*\omega$ belong to the same space for t small.

5.5.2 Monotonicity formula

In this section we prove a refinement of the stationarity formula. Since the proof is independent if the dimension n of our domain, we give a formulation in any dimension (the definition of $\mathcal{F}_\mathbb{Z}^p(\Omega)$ now requiring the degree to be an integer on any 2-dimensional sphere). For our applications we will just use the case $n = 3$.

Proposition 5.35 (Monotonicity formula). *If $\omega \in \mathcal{F}_\mathbb{Z}^p$ is stationary, then for all x and almost all $r \in]0, R]$ with the constraint $B_R(x) \subset \Omega$ there holds*

$$\frac{d}{dr} \left(r^{2p-n} \int_{B_r} |\omega|^p dy \right) = 2p r^{2p-n} \int_{\partial B_r} |\omega|^{p-2} |\partial_\rho \omega|^2 d\sigma \quad (5.17)$$

where $\partial_\rho = \frac{\partial}{\partial \rho}$ is the radial derivative.

Proof. We use a strategy similar to [70]. If $F : B_R \rightarrow B_R$ is a weakly differentiable bijective Lipschitz function, and if $\omega \in \mathcal{F}_\mathbb{Z}^p$ then also $F^*\omega \in \mathcal{F}_\mathbb{Z}^p$, so it is a competitor in our minimization. Therefore the stationarity $\left. \frac{d}{dt} \int_{B_R} |F_t^*\omega|^p \right|_{t=1} = 0$, holds provided that $F_0 = id_{B_R}$ and that the family F_t is differentiable in t . Such properties will be clear from our choices of the map F . (5.17) follows from this.

Definition of F

Fix $0 < r < s < R$ such that $0 < t < s/r$. Then we define a function $F = F_{r,s,t} : B_R \rightarrow B_R$ by $F(x) := \eta(|x|)x$, such that

$$\rho := |x| \mapsto |F(x)|$$

is continuous and affine on each of the intervals $[0, r], [r, s], [s, R]$. We define

$$\eta(\rho) = \begin{cases} t & \text{if } \rho \leq r \\ 1 & \text{if } \rho \in [s, r] \end{cases} \quad (5.18)$$

and $\eta|_{]r,s[}$ is defined accordingly:

$$\eta|_{]r,s[}(\rho) := \frac{s - tr}{s - r} + \frac{1}{\rho} \frac{rs(t-1)}{s-r}.$$

Expression of $|F^*\omega|^2$

We do our computation in coordinates. We choose a basis $\{e_0, e_1, \dots, e_{n-1}\}$ with respect to which to write the matrix dF_x , where $e_0 = \partial_\rho$ and the other vectors form an orthogonal basis together with it. Then

$$\frac{\partial F}{\partial x_k} = \eta e_k + \rho\eta' \delta_{0k} e_0. \quad (5.19)$$

Then

$$\begin{aligned} |(F^*\omega)_x|^2 &= \sum_{i,j} [\omega_{F(x)}(dF_x e_i, dF_x e_j)]^2 \\ &= \sum_{i,j=0}^{n-1} \left| \omega \left(\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right) \right|^2 \\ &= \sum_{i,j>0} |\omega(\eta e_i, \eta e_j)|^2 + 2 \sum_{i>0} |\omega((\eta + \rho\eta')e_0, \eta e_i)|^2 \\ &= \eta^4 \sum_{i,j>0} \omega_{ij}^2 + 2\eta^2(\eta + \rho\eta')^2 |\partial_{\rho\lrcorner}\omega|^2. \end{aligned}$$

The derivative in t

We now start the computations for the monotonicity formula.

$$\int_{B_R} |F^*\omega|^p = I + II + III \quad (5.20)$$

where, after a change of variables $y = F^{-1}(x)$,

$$\begin{aligned} I &:= \int_{B_r} |F^*\omega|^p = t^{2p-n} \int_{B_{rt}} |\omega|^p dy, \\ II &:= \int_{B_s \setminus B_r} |F^*\omega|^p \\ III &:= \int_{B_R \setminus B_s} |F^*\omega|^p = \int_{B_R \setminus B_s} |\omega|^p dy \end{aligned}$$

We desire now to change variable also in II and to take $\frac{d}{dt}|_{t=1}$ of the terms above. The easy terms give:

$$\begin{aligned} I' &:= \frac{d}{dt}|_{t=1}(I) = (2p-n) \int_{B_r} |\omega|^p dy + r \int_{\partial B_r} |\omega|^p d\sigma \\ III' &:= \frac{d}{dt}|_{t=1}(III) = 0 \end{aligned}$$

Ingredients for the computations

- We observe that

$$\eta(\rho) + \rho\eta'(\rho) = \frac{s - tr}{s - r},$$

which has t -derivative $\frac{-r}{s-r}$. It is useful to keep in mind that $\eta = 1$ for $t = 1$; this will be used without mention in the calculations.

- If $y = F(x)$ and $\sigma := |y|$, then we can write the expression of η in terms of σ :

$$\sigma = \rho\eta(\rho) = (\rho - r)\frac{s - tr}{s - r} + tr$$

so

$$\rho = f(\sigma) := (s - r)\frac{\sigma - tr}{s - tr} + r$$

and

$$\eta(f(\sigma)) = \frac{s - tr}{s - r} + \left[(s - r)\frac{\sigma - tr}{s - tr} + r \right]^{-1} \frac{rs(t - 1)}{s - r},$$

whence

$$\left. \frac{d}{dt}\eta(f(\sigma)) \right|_{t=1} = -\frac{r}{s - r} + \frac{rs}{\sigma(s - r)}.$$

- From (5.19) it follows that for $\rho := |x| \in [r, s]$,

$$J(dF^{-1}) = [\eta(\rho)^{n-1}(\eta(\rho) + \rho\eta'(\rho))]^{-1},$$

so

$$\begin{aligned} \left. \frac{d}{dt}J(dF^{-1}) \right|_{t=1} &= (1 - n) \left. \frac{d}{dt}\eta \right|_{t=1} + \left. \frac{d}{dt} \left(\frac{s - r}{s - tr} \right) \right|_{t=1} \\ &= (1 - n) \left[-\frac{r}{s - r} + \frac{rs}{\sigma(s - r)} \right] + \frac{r}{s - r}. \end{aligned}$$

The computation of the t -derivative

We call $|i^*\omega_y|^2 = \sum_{i,j>0} \omega_{ij}^2$ and we obtain

$$II = \int_{B_s \setminus B_{rt}} \left(|i^*\omega_y|^2 \eta^4 + 2 \left(\frac{s - tr}{s - r} \right)^2 \eta^2 |\omega_y(\hat{y}, \cdot)|^2 \right)^{p/2} J(dF^{-1}) dy$$

and

$$\begin{aligned}
 II' &:= \left. \frac{d}{dt} II \right|_{t=1} \\
 (\text{derivative of the domain}) &= -r \int_{\partial B_r} |\omega_y|^p d\sigma \\
 (\text{der. of the Jacobian}) &+ \frac{r}{s-r} \int_{B_s \setminus B_r} \left[(n-1) \left(1 - \frac{s}{|y|} \right) |\omega_y|^p + |\omega_y|^p \right] dy \\
 (\text{der. of the main term}) &\frac{p}{2} \left\{ \frac{4r}{s-r} \int_{B_s \setminus B_r} |\omega|^{p-2} \left[\left(\frac{s}{|y|} - 1 \right) - |\partial_{\rho \lrcorner} \omega|^2 \right] dy \right\}.
 \end{aligned}$$

We now take the limit $s \downarrow r$ and we are interested in seeing what the equation $I' + II' + III' = 0$ becomes. The answer is

$$\begin{aligned}
 \lim_{s \downarrow r} II' &= -r \int_{\partial B_r} |\omega|^p d\sigma \\
 &+ 0 + r \int_{\partial B_r} |\omega|^p dy \\
 &+ 0 - 2pr \int_{\partial B_r} |\omega|^{p-2} |\partial_{\rho \lrcorner} \omega|^2 dy \\
 &= -2pr \int_{\partial B_r} |\omega|^{p-2} |\partial_{\rho \lrcorner} \omega|^2 dy,
 \end{aligned}$$

and

$$\lim_{s \downarrow r} I' = (2p-n) \int_{B_r} |\omega|^p dy + r \int_{\partial B_r} |\omega|^p dy$$

Summing up and using the fact that ω is a minimizer of the energy, we get

$$(2p-n) \int_{B_r} |\omega|^p dy + r \int_{\partial B_r} |\omega|^p dy = 2pr \int_{\partial B_r} |\omega|^{p-2} |\partial_{\rho \lrcorner} \omega|^2 dy$$

Multiplying both the r.h.s. and the l.h.s of the above equation by r^{2p-n-1} we get the desired formula

$$\frac{d}{dr} \left(r^{2p-n} \int_{B_r} |\omega|^p dy \right) = 2p r^{2p-n} \int_{\partial B_r} |\omega|^{p-2} |\partial_{\rho \lrcorner} \omega|^2 dy$$

□

In terms of vector fields, we can state the following:

Proposition 5.36 (Monotonicity formula, alternative formulation). *If $X \in L^p_{\mathbb{Z}}$ minimizes the energy, then for almost all $r \in [0, R]$ there holds*

$$\frac{d}{dr} \left(r^{2p-n} \int_{B_r} |X|^p dy \right) = 2p r^{2p-n} \int_{\partial B_r} |X|^{p-2} |X - \langle X, \nu_{B_r} \rangle \nu_{B_r}|^2 d\mathcal{H}^2 \quad (5.21)$$

Chapter 6

Coulomb gauges and point removability in 4 dimensions

In this chapter we prove an improved point removability result based on [107]. This chapter was obtained in collaboration with my advisor Tristan Rivière and is part of [PR3].

6.1 Uhlenbeck Coulomb gauge

In [131] Uhlenbeck proved the following result:

Theorem 6.1 ([131], Thm. 4.6). *Let ∇ be a Yang-Mills connection in a bundle P over $B^4 \setminus \{0\}$. If the L^2 norm of the curvature F of ∇ is finite, then there exists a gauge in which the bundle P extends to a smooth bundle \tilde{P} over B^4 and the connection ∇ extends to a smooth Yang Mills connection $\tilde{\nabla}$ in B^4 .*

We recall that for a connection which in local coordinates is written $\nabla = d + A$, being Yang-Mills means that the curvature $F = F_A$ satisfies in the weak sense

$$d_A^* F_A = 0. \tag{6.1}$$

The regularity theory of Uhlenbeck allows to prove that $W^{1,2}$ Yang-Mills connections $d + A$ on trivial bundles are smooth up to a gauge change in the balls $B_\rho(x)$ such that $\int_{B_\rho(x)} |F|^2 < \epsilon_0$ for a constant ϵ_0 independent of A, F . This uses the regularity theory for the nonlinear (in A) equation (6.1), which when F does not have much energy and A is in Coulomb gauge can be seen

as an elliptic system.

Therefore the main step in the proof of Theorem 6.1 is the proof that we can find a global gauge extending over a neighborhood of the origin, in which the connection is $W^{1,2}$ so that the elliptic regularity can be applied. In Uhlenbeck [131] the elliptic regularity is used however on $B \setminus \{0\}$ in order to provide the needed estimates on concentric annuli. We will describe here how to proceed without this regularity.

Using a result from [107] we obtain that the analogue of Theorem 6.1 holds without the assumption that (6.1) holds. It appears that this result is not present in the literature, although it is hinted at in [12]. We will prove the following

Theorem 6.2 ([131] with no Yang-Mills assumption). *Let ∇ be a $W^{1,2}$ connection in a bundle P over $B^4 \setminus \{0\}$. If the L^2 norm of the curvature F of ∇ is finite, then there exists a gauge in which the bundle P extends to a smooth bundle \tilde{P} over B^4 and the connection ∇ extends to a $W^{1,2}$ connection $\tilde{\nabla}$ in B^4 .*

Theorem 6.2 allows to prove weak compactness for sequences of $W^{1,2}$ -connections with curvatures bounded in L^2 , again removing the assumption that the limit is Yang-Mills present in [117], [47]. The strategy in the paper [117] was to consider minimizing sequences $A_n \in \mathcal{A}^{1,2}$ for the Yang-Mills functional and prove that their connections converge locally weakly in $W^{1,2}$ while the curvatures converge locally weakly in L^2 , outside a finite set of “bad points” where the curvature densities concentrate. This allowed to obtain that the limit (which corresponds to a Yang-Mills minimizer) is Yang-Mills outside those points. The point removability theorem 6.1 which worked under the Yang-Mills assumptions then provided a way for continuing the limit bundle and connection over each bad point. By observing that this last point is the only one where the assumption of having a minimizing sequence was used in [117] we can use our improved Theorem 6.2 to immediately obtain:

Theorem 6.3 ([117] for non-minimizing sequences). *Assume that $A_n \in \mathcal{A}^{1,2}(M)$ on a smooth bundle over a smooth compact 4-manifold M . If $\|F_{A_n}\|_{L^2} \leq C$ for all n then up to extracting a subsequence we have that A_n converge locally weakly in $W^{1,2}$ to a connection $A_\infty \in \mathcal{A}^{1,2}(M)$ over a possibly different bundle.*

6.2 Coulomb gauges and Lorentz-improved regularity

We recall that the connection form A and the curvature form F are related in local coordinates by the distributional equation $F = dA + A \wedge A$. Recall that by Hodge theory the differential DA is controlled via dA and d^*A . It is then heuristically clear that if we desire a control on DA via the curvature we must therefore have some restrictions on d^*A . This was indeed done by Uhlenbeck in [132], where the most difficult part of the result is as follows:

Theorem 6.4 ([132], Thm. 1.3). *There exists a constant ϵ_0 as follows. Assume that $d + A$ is the local expression of a connection in an open set Ω such that $A \in W_{loc}^{1,2}$ and the curvature $F := F_A$ satisfies*

$$\int_{\Omega} |F|^2 \leq \epsilon_0. \quad (6.2)$$

Then there exists a gauge $g \in W_{loc}^{2,2}(\Omega)$ such that the transformed connection form

$$A_g = g^{-1}dg + g^{-1}Ag$$

satisfies

$$d^*A_g = 0 \quad \text{on} \quad \Omega$$

and is controlled by the curvature:

$$\int_{\Omega} |DA_g|^2 + \left(\int_{\Omega} |A_g|^4 \right) \leq C \int_{\Omega} |F|^2. \quad (6.3)$$

This celebrated result allows us to find controlled gauges in concentric dyadic annuli around the origin. To patch together the gauges of two overlapping annuli we use the following result following the techniques of [107] Thm. IV.1.

Theorem 6.5. *Suppose that A and $B = g^{-1}dg + g^{-1}Ag$ are gauge-related connections on a 4-dimensional domain Ω such that*

$$d^*A = d^*B = 0.$$

If $A, B \in W^{1,2}$ then the gauge change g is $W^{2,2} \cap C^0$. Moreover for some $\bar{g} \in G$ we have the bound

$$\|g - \bar{g}\|_{L^\infty \cap W^{2,2}} \lesssim \|A\|_{W^{1,2}}^2 + \|B\|_{W^{1,2}}^2. \quad (6.4)$$

Proof. From

$$dg = gB - Ag$$

because multiplication is continuous from $W^{1,2} \times (W^{1,2} \cap L^\infty)$ to $W^{1,2}$ it follows that $dg \in W^{1,2} \hookrightarrow L^{(4,2)}$ and

$$\|dg\|_{L^{(4,2)}} \lesssim \|A\|_{W^{1,2}} + \|B\|_{W^{1,2}}.$$

from the above equation and using $d^*A = d^*B = 0$ and identifying 1-forms with vector fields we obtain

$$\Delta g = d^*dg = dg \cdot A - B \cdot dg,$$

where both terms are products of elements of $L^{(4,2)}$ therefore belong to $L^{(2,1)}$. We have

$$\|\Delta g\|_{L^{(2,1)}} \lesssim \|dg\|_{L^{(4,2)}} (\|A\|_{L^{(4,2)}} + \|B\|_{L^{(4,2)}}) \lesssim \|A\|_{L^{(4,2)}}^2 + \|B\|_{L^{(4,2)}}^2.$$

By the continuous embeddings $W^{2,(2,1)} \hookrightarrow W^{1,(4,1)} \hookrightarrow L^\infty$ valid in 4 dimensions, we obtain

$$\|g - \tilde{g}\|_{L^\infty \cap W^{2,2}} \lesssim \|A\|_{L^{(4,2)}}^2 + \|B\|_{L^{(4,2)}}^2 := (*),$$

where \tilde{g} is the average of g done in the space \mathbb{R}^N where the group G is isometrically embedded. Since $g \in G$ a.e., we also have

$$\text{dist}_{\mathbb{R}^N}(\tilde{g}, G) \lesssim (*)$$

therefore there exists $\bar{g} \in G$ such that

$$\|g - \bar{g}\|_{L^\infty} \lesssim (*) \lesssim \|A\|_{W^{1,2}}^2 + \|B\|_{W^{1,2}}^2$$

as desired. Note that $W^{1,2}$ curvatures in 4-dimensions can be approximated by smooth curvatures in $W^{1,2}$ -norm (see Lemma 7.12). By applying the above result on balls $B_\rho(x)$ with $\rho \rightarrow 0$ for a.e. x , we obtain that $g \in C^0$ too. \square

Notation: from now on we denote by S_k the spherical shell $B_{2^{-2k}} \setminus B_{2^{-2k-3}}$.

Lemma 6.6. *There exists $\delta > 0$ such that if $\int_{S_k} |F|^2 \leq \delta$ then there exists a global gauge g on S_k in which the connection corresponding to F is represented by a $W^{1,2}$ -form A_k which satisfies*

$$d^*A_k = 0, \quad \|DA_k\|_{L^2(S_k)} + \|A_k\|_{L^4(S_k)} \leq \|F\|_{L^2(S_k)}. \quad (6.5)$$

Proof. Without loss of generality let $k = 0$, because the norms of F , A and DA appearing in (6.5) have the same scaling. We cover S_0 by two charts U_+, U_- which are tubular neighborhoods of opposite half-shells. In U_\pm the connection has the local expression A_\pm . Since the bundle is trivial over U_\pm we can apply Theorem 6.4 and up to a change of gauge A_\pm satisfies (6.5).

On $U_+ \cap U_-$ there exists g such that $A_+ = g^{-1}dg + g^{-1}A_-g$. By Theorem 6.5 we have that $g \in C^0$ and for some $\bar{g} \in G$ there holds

$$\|g - \bar{g}\|_{L^\infty} \lesssim \delta^2. \quad (6.6)$$

in particular it is not possible for g to realize a nontrivial homotopy class $[U_+ \cap U_-, G]$, provided $\delta^2 \leq C_G$ for some C_G depending on the topology of G . Therefore it is possible to extend g in a Lipschitz way over U_- and we find a global trivialization over the whole of S_0 . Applying Theorem 6.4 again we find A_0 as in (6.5). \square

6.3 Proof of Theorem 6.2

Proof. The bundle is non-smooth just at the origin, therefore we may work replacing $B_1(0)$ by a ball $B_\rho(0)$ with $\rho > 0$ on which $\int_{B_\rho} |F|^2 < \delta$. In other words we don't lose any generality if we assume $\int_{B_1} |F|^2 < \delta$. We fix δ later, but it will be smaller than the constant δ of Lemma 6.6 and than the constant ϵ_0 of theorem 6.4.

We apply Lemma 6.6 and we start with the connections A_k defined on S_k and satisfying (6.5). On each $S_{k+1} \cap S_k$ there is a gauge change g_k such that

$$A_{k+1} = g_k^{-1}dg_k + g_k^{-1}A_k g_k. \quad (6.7)$$

By Theorem 6.5 there exist $\bar{g}_k \in G$ such that

$$\|g_k - \bar{g}_k\|_{L^\infty \cap W^{2,2}} \lesssim \|A_k\|_{W^{1,2}}^2 + \|A_{k+1}\|_{W^{1,2}}^2. \quad (6.8)$$

Now we propagate the gauge along the increasing S_k 's. In order to cancel the contributions of the approximating constant gauges \bar{g}_k , we define for example $\bar{A}_1 = \bar{g}_0 A_1 \bar{g}_0^{-1} = \bar{g}_0^{-1}(A_1) = \bar{g}_0^{-1} \circ g_0(A_0)$. This means that \bar{A}_1 differs from A_0 on $S_1 \cap S_0$ just by a small gauge. Similarly define

$$\bar{A}_k := \bar{h}_k(A_k), \quad \bar{h}_k := \prod_{i=0}^{k-1} \bar{g}_i^{-1}.$$

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We use the \bar{A}_k 's as a reference to define a global gauge. Define \tilde{g}_k on $S_{k+1} \cap S_k$ to be such that $\bar{A}_{k+1} = \tilde{g}_k(\bar{A}_k)$, i.e.

$$\tilde{g}_k := \bar{h}_k^{-1} \bar{g}_k^{-1} g_k \bar{h}_k. \quad (6.9)$$

The \tilde{g}_k 's are better than the g_k 's because they don't contain the gauge jumps \bar{g}_k . From (6.8) and (6.5), by multiplying by constants, i.e. by isometries of G , we have

$$\begin{aligned} \|\tilde{g}_k - id\|_{L^\infty \cap W^{2,2}(S_k \cap S_{k+1})} &= \|g_k - \bar{g}_k\|_{L^\infty \cap W^{2,2}(S_k \cap S_{k+1})} \\ &\lesssim \int_{S_k} |F|^2 + \int_{S_{k+1}} |F|^2. \end{aligned} \quad (6.10)$$

Next extend \tilde{g} radially on $S_k^- := B_{2^{-2k-3}} \setminus B_{2^{-2k-4}}$ and on $S_k^+ := B_{2^{-2k+1}} \setminus B_{2^{-2k}}$. Call this extension $\tilde{\tilde{g}}_k$. Note that

$$\sum_{k \geq 1} \int_{S_k} |F|^2 \leq \delta. \quad (6.11)$$

Because of (6.11), (6.11) and because the radial extension is tame enough there holds:

$$\|\tilde{\tilde{g}}_k - id\|_{L^\infty \cap W^{2,2}(S_k^- \cup S_k^+)} \leq \delta.$$

Let δ be small enough so that $\tilde{\tilde{g}}_k = \exp_{id}(\varphi_k)$, $\|\varphi_k\|_{L^\infty \cap W^{2,2}(S_k^- \cup S_k^+ \cup S_k)} \sim \|\tilde{\tilde{g}}_k - id\|_{L^\infty \cap W^{2,2}(S_k^- \cup S_k^+ \cup S_k)}$. This is possible because \exp_{id}^{-1} is well-behaved near the identity.

We create a family of cutoff functions similar to the one used in Littlewood-Paley decompositions. Consider a function $\eta(r)$ which is smooth, decreasing, equal to 0 for $r > 2$ and to 1 for $r < 1$. We can assume $|\eta'| \leq 2$. Then define $\psi_k(x) := \eta(2^{2k}|x|) - \eta(2^{2k+4}|x|)$ and consider $\tilde{\varphi}_k := \psi_k \varphi_k$. We have

$$\begin{aligned} \|\tilde{\varphi}_k\|_{L^\infty} &\leq \|\varphi_k\|_{L^\infty(\mathbb{S}^k)}, \\ \|D^2 \tilde{\varphi}_k\|_{L^2} &\lesssim \|D^2 \varphi_k\|_{L^2(\mathbb{S}^k)} + \|d\psi_k\|_{L^4} \|d\varphi_k\|_{L^4(\mathbb{S}^k)} + \|D^2 \psi_k\|_{L^2} \|\varphi_k\|_{L^\infty(\mathbb{S}^k)} \\ &\lesssim \|\varphi_k\|_{L^\infty \cap W^{2,2}(\mathbb{S}^k)}. \end{aligned}$$

By extending \tilde{g}_k via $\exp \tilde{\varphi}_k$ we obtain a continuous extension of \tilde{g}_k on $S_k \cup S_k^- \cup S_k^+$ which still satisfies the same estimates as \tilde{g} . Use the notation \hat{g}_k . We then define on $B^4 \setminus \{0\}$

$$\lambda := \prod_{i=0}^{\infty} \hat{g}_k.$$

Since \hat{g}_k is nonidentity on at most 5 dyadic rings, this product has locally finitely many factors different than the identity therefore it is well-defined. We

also have that since $W^{2,2} \cap L^\infty$ is an algebra

$$\begin{aligned}
\|\lambda - id\|_{L^\infty \cap W^{2,2}(B_{2^{-2\bar{k}}}\setminus\{0\})} &\lesssim \sum_{k \geq \bar{k}} \|\hat{g}_k - id\|_{L^\infty \cap W^{2,2}(B^4 \setminus \{0\})} \\
&\lesssim \sum_{k \geq \bar{k}} \|\tilde{g}_k - id\|_{L^\infty \cap W^{2,2}(S_k \cup S_k^- \cup S_k^+)} \\
&\lesssim \sum_{k \geq \bar{k}} \|\tilde{g}_k - id\|_{L^\infty \cap W^{2,2}(S_k)} \\
&\lesssim \sum_{k \geq \bar{k}} \int_{S_k} |F|^2.
\end{aligned}$$

In particular we see that $\lambda \rightarrow id$ at zero, therefore the bundle extends, as desired. We must now prove that in this gauge the connection form \tilde{A} is $W^{1,2}$. Recall that if the gauges would be chosen all equal to \tilde{g}_k then the connection would become \tilde{A}_k on S_k , and this is just a constant conjugation of the original A_k as in (6.5). Since the cutoff parts \hat{g}_k on $S_k^- \cup S_k^+$ are controlled in $W^{2,2} \cap L^\infty$ still by the right hand side of (6.9) we obtain using (6.11) and the fact that \hat{g}_k have similar estimates as \tilde{g}_k that

$$\begin{aligned}
\|\tilde{A}\|_{W^{1,2}}^2 &\lesssim \sum_{k \geq 0} \left(\|A_k\|_{W^{1,2}(S_k)}^2 + \|\hat{g}_k(A_{k-1})\|_{W^{1,2}(S_k^-)}^2 + \|\hat{g}_k(A_{k+1})\|_{W^{1,2}(S_k^+)}^2 \right) \\
&\lesssim \sum_{k \geq 0} \left(\|A_k\|_{W^{1,2}(S_k)}^2 + \|\hat{g}_k\|_{W^{2,2}(S_k^-)}^2 + \|\hat{g}_k\|_{W^{2,2}(S_k^+)}^2 \right) \\
&\lesssim \sum_{k \geq 0} \|A_k\|_{W^{1,2}(S_k)}^2 + \sum_{k \geq 0} \|A_k\|_{W^{1,2}(S_k)}^4 \\
&\lesssim \delta + \delta^2.
\end{aligned}$$

In the last passage we used (6.11) and the inequality between ℓ^2 and ℓ^4 . This concludes the proof of Theorem 6.2. \square

Chapter 7

Approximation of nonabelian connections in 5 dimensions

In this chapter we prove the fact that L^2 weak curvature forms F corresponding to connection classes $[A] \in \mathcal{A}_G(\mathbb{B}^5)$ can be strongly approximated up to gauge by curvatures which are locally smooth on bundles with finitely many defects, i.e. correspond to elements of $\mathcal{R}^\infty \cap \mathcal{A}_G(\mathbb{B}^5)$. This Chapter is based on a joint work with my advisor Tristan Rivière [PR3].

7.1 Introduction

For us G will be a nonabelian compact Lie group. By representation theory we may assume that G is a subgroup of $SO(n)$ for n large enough and in particular G is embedded in \mathbb{R}^N for $N = n \times n$ in such a way that the group operations stay continuous. Note that by compactness, any measurable map into G will automatically be L^∞ .

Recall the classical definition of Sobolev connections in 4-dimensions:

Definition 7.1. *Let (M, h) be a 4-dimensional compact Riemannian manifold. Fix an atlas (U_i, ϕ_i) on M . A $W^{1,2}$ connection 1-form A is a collection of local expressions $A_i \in W^{1,2}(U_i, T^*M \otimes \mathfrak{g})$ and measurable gauge changes $g_{ij} : U_i \cap U_j \rightarrow G$ such that for all i, j on $U_i \cap U_j$ there holds*

$$A_i = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1} A_j g_{ij}.$$

We require that g_{ij} satisfy the cocycle condition

$$g_{ij} g_{jk} = g_{ik}$$

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whenever the three terms in expression are all defined. We say that a 2-form $F \in L^2(M, \wedge^2 TM \otimes \mathfrak{g})$ is the curvature of such A if on U_i there exists a measurable gauge change $\gamma_i : U_i \rightarrow G$ such that

$$\gamma_i^{-1} F \gamma_i = dA_i + A_i \wedge A_i.$$

We denote by $\mathcal{A}^{1,2}(M)$ the class of $W^{1,2}$ -connection forms A as above.

Note that the choice of an atlas is immaterial. Moreover note that once we have a definition on Euclidean space, extending it to a manifold does not present big difficulties. For simplicity we reduce to this case for the case of 5-dimensions.

Definition 7.2. We define the space of weak connection classes on singular Sobolev bundles on \mathbb{R}^5 as follows:

$$\mathcal{A}_G(\mathbb{R}^5) := \left\{ \begin{array}{l} A \in L^2(\mathbb{R}^5, \wedge^1 \mathbb{R}^5 \otimes \mathfrak{g}) \text{ such that loc. } dA + A \wedge A \stackrel{\mathcal{D}'}{=} F \\ \text{for some } F \in L^2(\mathbb{R}^5, \wedge^2 \mathbb{R}^5 \otimes \mathfrak{g}) : \\ \text{and } \forall p \in \mathbb{R}^5, \text{ for a.e. } r > 0, A|_{\partial C_r(p)} \in \mathcal{A}^{1,2}(\partial C_r(p)). \end{array} \right\}.$$

Here $C_r(p)$ is the cube of side-length $2r$ and center p with sides parallel to the coordinate axes.

Remark 7.3. In the above definition we use boundaries of cubes, which seems a rather unnatural choice, fixing a dependence on chosen coordinates. A consequence of our density result will be the fact that we can use balls or general boundaries instead of cubes, obtaining the same space in the end. The fact that cubes tile space makes them more suitable for our proofs. See [PR3] for the proof of the density result in the case of balls.

We will use the following class to approximate connections in \mathcal{A}_G :

$$\mathcal{R}_\infty(\mathbb{R}^5) := \mathcal{A}_G(\mathbb{R}^5) \cap \left\{ \begin{array}{l} [A] : \exists x_1, \dots, x_N \text{ such that} \\ A \in [A] \text{ for some } A \in \mathcal{A}^\infty(\mathbb{R}^5 \setminus \{x_1, \dots, x_N\}) \end{array} \right\},$$

where $\mathcal{A}^\infty(X)$ is the class of smooth connection forms of G -bundles over X and F_A is the curvature form of the connection corresponding to A .

Our approximation result is formulated on a bounded domain (the unit ball) for simplicity:

Theorem 7.4. *curvatures of elements of $\mathcal{R}_\infty(\mathbb{B}^5)$ can be approximated by curvatures of elements of $\mathcal{A}_G(\mathbb{B}^5)$ with respect to the following pseudometric:*

$$\text{dist}^2(F, F') := \inf \left\{ \int |g^{-1} F g - F'|^2 : g \in \mathcal{M}(\mathbb{R}^5, G) \right\}, \quad (7.1)$$

where \mathcal{M} indicates the space of measurable functions.

7.2 Choice of grids

Let F be an L^2 weak curvature expressed in a gauge such that A is in L^2 . Fix $\delta > 0$ and $\epsilon > 0$. The first crucial step is a choice of a suitable sequence of grids.

Definition 7.5. Fix a scale $\epsilon > 0$ and a point $a_\epsilon \in [0, \epsilon]^5$. We then define

$$C_{\epsilon, a_\epsilon} := \{C_{\epsilon, a_\epsilon}^{\mathbf{i}} : \mathbf{i} \in \mathbb{Z}^5, C_{\epsilon, a_\epsilon}^{\mathbf{i}} \cap B \neq \emptyset\}, \quad C_{\epsilon, a_\epsilon}^i := a_\epsilon + \epsilon \mathbf{i} + [0, \epsilon]^5$$

and

$$\partial C_{\epsilon, a_\epsilon} := \left(\bigcup_{\mathbf{i} \in \mathbb{Z}^5} \partial C_{\epsilon, a_\epsilon}^{\mathbf{i}} \right) \cap B.$$

The set of indices \mathbf{i} can be re-indexed by a finite set of indices which we will denote by I_{ϵ, a_ϵ} . We often write $C_\epsilon, C_\epsilon^i, i \in I$ omitting the index a_ϵ when this does not generate confusion.

The main result is the possibility of choosing translations a_ϵ as above for which a good control on the boundary of our grids is available. The proof of the next result follows the strategy of [83, 4.2].

Proposition 7.6. Let F be an L^2 weak curvature expressed in a gauge such that A is in L^2 . It is possible to find $a_\epsilon \in [0, \epsilon]^5$ such that the grids $C_\epsilon := C_{\epsilon, a_\epsilon}$ satisfy, for a constant C depending only on the dimension,

$$\epsilon \int_{\partial C_\epsilon} |F|^2 \leq C \int_{\mathbb{B}^5} |F|^2, \quad (F_1)$$

$$\epsilon \int_{\partial C_\epsilon} |A|^2 \leq C \int_{\mathbb{B}^5} |A|^2. \quad (A_1)$$

With the notation $\bar{F} := \sum_{i \in I} \chi_{C_\epsilon^i} \frac{1}{\epsilon^5} \int_{C_\epsilon^i} F$ we can also achieve at the same time

$$\epsilon \int_{\partial C_\epsilon} |F - \bar{F}|^2 = o(\epsilon). \quad (F_2)$$

Proof. We will give a proof for F but we use only the fact that F is an L^2 function and the choice of a_ϵ will result from the use of Chebychev's inequality. Since this will give a quantitative estimate, we can pay the price of doubling the constants in our inequalities in order to obtain them contemporarily for A and F .

Call $\partial C_{\epsilon, 0}^{(k)}$ the union of those sides of the cubes $C_{\epsilon, 0}^i, i \in I_{\epsilon, 0}$ which are parallel to the hyperplane $\{x^k = 0\}$. Let k be the unit vector orthogonal to

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that hyperplane. Then the translates of $\partial C_{\epsilon,0}^{(k)}$ by the displacements vectors belonging the segment $[0, \epsilon_k[$ are covering exactly all the cubes with which we started, and in particular they cover B . We may then use this information to obtain

$$\int_0^\epsilon \int_{\partial C_{\epsilon,0}^{(k)} + t_k} |F|^2 = \int_{\mathbb{B}^5} |F|^2,$$

thus the measure of the “bad” $s \in [0, 1[$ such that the following inequality is false

$$\epsilon \int_{\partial C_{\epsilon,0}^{(k)} + s\epsilon_k} |F|^2 \leq C \int_{\mathbb{B}^5} |F|^2$$

must be smaller than C^{-1} , by Chebychev’s inequality. We then note that if we require the above inequality to be true for all 5 coordinates k , and also with A in the place of F , we obtain an exceptional set 10 times larger, at most. For a_ϵ having all coordinates proportional to (a good) s (F_1) and (A_1) are thus true.

Fix now a smooth approximant G to F as a function in $L^2(B, \Lambda^2 \mathbb{R}^2 \otimes \mathfrak{g})$, and assume that

$$\int_{\mathbb{B}^5} |G - F|^2 \leq \delta$$

for a small constant $\delta > 0$ to be fixed later. We can apply the above argument again to $G - F$ and obtain also

$$\epsilon \int_{\partial C_{\epsilon,0}^{(k)} + s\epsilon_k} |G - F|^2 \leq C \int_{\mathbb{B}^5} |G - F|^2 \leq \delta$$

up to a set of $s \in [0, 1[$ of measure at most C^{-1} . This is true contemporarily for all k up to a set of measure at most $5C^{-1}$. We also note that doing the averages $\overline{G}^\epsilon, \overline{F}^\epsilon$ with respect to the grids translated of a_ϵ (having all coordinates equal to s in the non-exceptional set above) we obtain by Jensen’s inequality

$$\epsilon \int_{\partial C_{\epsilon,a_\epsilon}} |\overline{G}^\epsilon - \overline{F}^\epsilon|^2 = \int_{\cup C_{\epsilon,a_\epsilon}^i} |\overline{G}^\epsilon - \overline{F}^\epsilon|^2 \leq \int_{\cup C_{\epsilon,a_\epsilon}^i} |G - F|^2 \leq \delta.$$

We then estimate

$$\begin{aligned} \epsilon \int_{\partial C_{\epsilon,a_\epsilon}} |F - \overline{F}^\epsilon|^2 &\leq C\epsilon \int_{\partial C_{\epsilon,a_\epsilon}} (|F - G|^2 + |G - \overline{G}^\epsilon|^2 + |\overline{G}^\epsilon - \overline{F}^\epsilon|^2) \\ &\leq 2C\delta + C\epsilon \int_{\partial C_{\epsilon,a_\epsilon}} |G - \overline{G}^\epsilon|^2. \end{aligned}$$

The last term can be estimated in terms of the C^1 -norm of G for example. We see thus that for given δ we may find ϵ, a_ϵ such that $\epsilon \int_{\partial C_{\epsilon,a_\epsilon}} |F - \overline{F}^\epsilon|^2 \leq C\delta$

where C depends just on how many times we used Chebychev's inequality. This reasoning thus allows to satisfy also (F_2) , thereby finishing the proof. \square

From now on we will restrict to grids translated by vectors a_ϵ as in Proposition 7.6 and we will forget the subscript indicating the choice of a_ϵ . We fixed the above properties in a definition:

Definition 7.7. *Consider a ϵ -grid C_ϵ with notations as above. Let*

- A be an L^2 -connection form on ∂C_ϵ such that the distributional curvature $F_A = dA + A \wedge A$ is L^2 on ∂C_ϵ ,
- \overline{F} be an L^2 -form on \mathbb{B}^5 with values in \mathfrak{g} which is constant on each one of the cubes C_ϵ^i .

We call the grid C_ϵ good with respect to A, F_A, \overline{F} if the relation (F_2) holds.

7.3 Bad cubes and good cubes

We now prove that on “good cubes” forming a full measure subset in the limit $\epsilon \rightarrow 0$ we have good estimates which will allow the approximation by true curvatures. The remaining cubes will be called “bad cubes”.

Definition 7.8. *Fix a constant $\delta > 0$ and a size ϵ grid C_ϵ . Let A, F, \overline{F} be as in Definition 7.7. We call a cube C_ϵ^i of the grid δ -good with respect to $A, F, \overline{F}, C_\epsilon$ if the following estimates hold:*

$$\int_{\partial C_\epsilon^i} |F|^2 \leq \delta, \quad (g1)$$

$$\frac{1}{\epsilon^2} \int_{\partial C_\epsilon^i} |A|^2 \leq \delta, \quad (g2)$$

$$\frac{1}{\epsilon^2} \int_{\partial C_\epsilon^i} |F - \overline{F}|^2 \leq o(\epsilon). \quad (g3)$$

If C_ϵ^i is not good we call it δ -bad. We denote \mathcal{G} the set of good cubes and \mathcal{B} the set of bad cubes.

Proposition 7.9. *Fix $F, A, \overline{F}, \delta > 0$ as in Definition 7.8 and a grid C_ϵ which is good with respect to them according to Definition 7.7. Then the number of δ -bad cubes with respect to $A, F, \overline{F}, C_\epsilon$ can then be estimated as follows*

$$\#\mathcal{B} \leq \frac{\|F\|_{L^2}^2}{\delta\epsilon} + \frac{\|A\|_{L^2}^2}{\delta\epsilon^3} + \frac{1}{\epsilon}.$$

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In particular the total volume of the bad cubes tends to zero at least as $o_\epsilon(\epsilon^2)$ as $\epsilon \rightarrow 0$.

Proof. The second statement follows from the first because the volume of each bad cube is ϵ^5 . To prove the estimate on $\#\mathcal{B}$ we separately estimate the sets \mathcal{B}_i of cubes for which (g_i) fails in Definition 7.8.

Using Proposition 7.6 we then obtain

$$\begin{aligned} \delta\#\mathcal{B}_1 &\leq \sum_{C \in \mathcal{B}_1} \int_{\partial C} |F|^2 \leq \frac{1}{\epsilon} \int_{\mathbb{B}^5} |F|^2, \\ \delta\epsilon^2\#\mathcal{B}_2 &\leq \sum_{C \in \mathcal{B}_2} \int_{\partial C} |A|^2 \leq \frac{1}{\epsilon} \int_{\mathbb{B}^5} |A|^2, \\ o_\epsilon(1)\#\mathcal{B}_3 &\leq \sum_{C \in \mathcal{B}_3} \int_{\partial C} |F - \bar{F}|^2 \leq \frac{o_\epsilon(1)}{\epsilon}. \end{aligned}$$

Since $\mathcal{B} = \cup_{i=1}^4 \mathcal{B}_i$ we obtain

$$\#\mathcal{B} \leq \#\mathcal{B}_1 + \#\mathcal{B}_2 + \#\mathcal{B}_3 + \#\mathcal{B}_4 \leq \frac{\|F\|_{L^2}^2}{\delta\epsilon} + \frac{\|A\|_{L^2}^2}{\delta\epsilon^3} + \frac{1}{\epsilon},$$

as desired. \square

7.4 Extension on a good cube

In this section we prove the extension which will help to define our approximating connections on the good cubes. Note that what we will use is just the result of Proposition 7.6 and the properties of good cubes enunciated in Definition 7.8. We will use later the fact that the whole proof does not depend on A, F directly.

Proposition 7.10. *Let $\epsilon > 0$ be fixed and let C_ϵ be a grid of mesh-size ϵ . Assume A, F_A, \bar{F} are as in Definition 7.7 and C_ϵ is good with respect to them.*

There exists a constant $C > 0$ depending only on the dimension such that if $\delta < C$ then it is possible to find a connection form \hat{A} over the union of δ -good cubes $\cup_{i \in \mathcal{G}} C_\epsilon^i$ such that

- $i_{\partial C_\epsilon^i}^* \hat{A} = A|_{\partial C_\epsilon^i}$ for $i \in \mathcal{G}$,

- for $i \in \mathcal{G}$ there exists a measurable $\hat{g} : C_\epsilon^i \rightarrow G$ such that

$$\tilde{A} := \hat{g}^{-1} d\hat{g} + \hat{g}^{-1} \hat{A} \hat{g}$$

is a smooth connection form and

-

$$\int_{\cup_{i \in \mathcal{G}} C_\epsilon^i} |F_{\tilde{A}} - \bar{F}|^2 \leq C(\delta + o_\epsilon(1)).$$

Proof. The scalings in Definition 7.8 are motivated by the fact that they define quantities which are scaling-invariant. By deforming F, A on a *good* cube by a pullback via the affine map $a : C_\epsilon^i \rightarrow [0, 1]^5$ and after a bilipschitz deformation $b : [0, 1]^5 \rightarrow \mathbb{B}^5$, we may work on $\mathbb{B}^5, \mathbb{S}^4$ instead of $C_\epsilon^i, \partial C_\epsilon^i$, and assume that the following estimates hold:

$$\int_{\mathbb{S}^4} |F|^2 < \delta, \quad \int_{\mathbb{S}^4} |A|^2 < \delta \quad .$$

Let g be the change of gauge (given by Uhlenbeck's Theorem 6.4, cfr. [132]) such that

$$\begin{cases} d_{\mathbb{S}^4}^* A_g = d_{\mathbb{S}^4}^* (g^{-1} dg + g^{-1} A g) = 0 \\ \|A_g\|_{W^{1,2}(\mathbb{S}^4)} \leq C \|F\|_{L^2(\mathbb{S}^4)} \end{cases} \quad . \quad (7.2)$$

The Coulomb gauge of Uhlenbeck is given up to the action of a constant element of G . Poincaré inequality gives

$$\|g - \bar{g}\|_{L^2(\mathbb{S}^4)} \leq C \|dg\|_{L^2(\mathbb{S}^4)} \leq C [\|A_g\|_{L^2(\mathbb{S}^4)} + \|A\|_{L^2(\mathbb{S}^4)}] \leq C \delta^{1/2} \quad . \quad (7.3)$$

where $\bar{g} = |\mathbb{S}^4|^{-1} \int_{\mathbb{S}^4} g$. Hence using the mean value formula there exists $x \in \mathbb{S}^4$ such that

$$|g(x) - \bar{g}| \leq C \delta^{1/2}$$

Changing g by $g_0 g$ for a constant rotation g_0 we obtain

$$|\bar{g} - id| \leq C \delta^{1/2} \quad . \quad (7.4)$$

We have using (7.3) and (7.4) and the fact that \bar{F} is constant

$$\int_{\mathbb{S}^4} |g^{-1} i_{\mathbb{S}^4}^* \bar{F} g - i_{\mathbb{S}^4}^* \bar{F}|^2 dvol_{\mathbb{S}^4} \leq 4 |\bar{F}|^2 \int_{\mathbb{S}^4} |g - id|^2 \leq C \delta \|\bar{F}\|_{L^2(\mathbb{B}^5)}^2.$$

Since $F_{A_g} = g^{-1} F g$, using the previous identity we obtain

$$\int_{\mathbb{S}^4} |F_{A_g} - i_{\mathbb{S}^4}^* \bar{F}|^2 dvol_{\mathbb{S}^4} \leq C \delta \|\bar{F}\|_{L^2(\mathbb{B}^5)}^2 + C \int_{\mathbb{S}^4} |F - i_{\mathbb{S}^4}^* \bar{F}|^2 dvol_{\mathbb{S}^4} \quad . \quad (7.5)$$

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Using now the last line of (7.2) we obtain

$$\int_{\mathbb{S}^4} |F_{A_g} - dA_g|^2 dvol_{\mathbb{S}^4} \leq \int_{\mathbb{S}^4} |A_g|^4 dvol_{\mathbb{S}^4} \leq C \|F\|_{L^2(\mathbb{S}^4)}^4 \quad . \quad (7.6)$$

Combining (7.5) and (7.6) we obtain

$$\begin{aligned} \int_{\mathbb{S}^4} |dA_g - i_{\mathbb{S}^4}^* \bar{F}|^2 dvol_{\mathbb{S}^4} &\leq C \delta \|\bar{F}\|_{L^2(\mathbb{B}^5)}^2 + \\ &+ C \int_{\mathbb{S}^4} |F - i_{\mathbb{S}^4}^* \bar{F}|^2 dvol_{\mathbb{S}^4} + C \|F\|_{L^2(\mathbb{S}^4)}^4 \quad . \end{aligned} \quad (7.7)$$

For any 1-form η in $W^{1,2}(\mathbb{S}^4, T^*\mathbb{S}^4 \otimes \mathfrak{g})$ we denote by $\tilde{\eta}$ the unique solution of the following minimization problem

$$\inf \left\{ \int_{\mathbb{B}^5} |dC|^2 + |d^{*\mathbb{R}^5} C|^2 dx^5 \quad C \in W^{1,2}(\mathbb{B}^5, T^*\mathbb{B}^5 \otimes \mathfrak{g}) \quad i_{\mathbb{S}^4}^* C = \eta \right\} \quad . \quad (7.8)$$

By a classical argument, it is uniquely given by

$$\begin{cases} d^{*\mathbb{R}^5} \tilde{\eta} = 0 & \text{in } \mathbb{B}^5 \\ d^{*\mathbb{R}^5} (d\tilde{\eta}) = 0 & \text{in } \mathbb{B}^5 \\ i_{\mathbb{S}^4}^* \tilde{\eta} = \eta & \text{on } \partial\mathbb{B}^5 \end{cases}$$

and one has

$$\|\tilde{\eta}\|_{L^5(\mathbb{B}^5)} \leq C \|\nabla \tilde{\eta}\|_{W^{3/2,2}(\mathbb{B}^5)} \leq C \|\eta\|_{W^{1,2}(\mathbb{S}^4)} \quad . \quad (7.9)$$

Let

$$B := \sum_{i < j} \bar{F}_{ij} \frac{x_i dx_j - x_j dx_i}{2} \quad (7.10)$$

Observe that

$$\begin{cases} d^{*\mathbb{R}^5} B = 0 & \text{in } \mathbb{B}^5 \\ d^{*\mathbb{R}^5} (dB) = 0 & \text{in } \mathbb{B}^5 \end{cases} .$$

Thus B is the solution to (7.8) for its restriction to the boundary : $i_{\mathbb{S}^4}^* B$

$$i_{\mathbb{S}^4}^* \tilde{B} = B \quad .$$

Observe that $\langle B, dr \rangle \equiv 0$ and $d^{*\mathbb{R}^5} B = 0$ therefore

$$d^{*\mathbb{S}^4} (i_{\mathbb{S}^4}^* B) \equiv 0 \quad \text{on } \mathbb{S}^4 \quad .$$

Since $d^{*\mathbb{S}^4}A_g = 0$ as well, we have using (7.7) that

$$\begin{aligned} \|A_g - i_{\mathbb{S}^4}^*B\|_{W^{1,2}(\mathbb{S}^4)}^2 &\leq C \int_{\mathbb{S}^4} |d(A_g - i_{\mathbb{S}^4}^*B)|^2 dvol_{\mathbb{S}^4} \\ &= \int_{\mathbb{S}^4} |dA_g - i_{\mathbb{S}^4}^*\overline{F}|^2 dvol_{\mathbb{S}^4} \leq C \delta \|\overline{F}\|_{L^2(\mathbb{B}^5)}^2 + \\ &\quad + C \int_{\mathbb{S}^4} |F - i_{\mathbb{S}^4}^*\overline{F}|^2 dvol_{\mathbb{S}^4} + C \|F\|_{L^2(\mathbb{S}^4)}^4 . \end{aligned} \quad (7.11)$$

Combining now (7.9) and (7.11) we obtain

$$\begin{aligned} \|d\tilde{A}_g - \overline{F}\|_{L^2(\mathbb{B}^5)}^2 &\leq C \int_{\mathbb{S}^4} |d(A_g - i_{\mathbb{S}^4}^*B)|^2 dvol_{\mathbb{S}^4} \\ &= \int_{\mathbb{S}^4} |dA_g - i_{\mathbb{S}^4}^*\overline{F}|^2 dvol_{\mathbb{S}^4} \leq C \delta \|\overline{F}\|_{L^2(\mathbb{B}^5)}^2 + \\ &\quad + C \int_{\mathbb{S}^4} |F - i_{\mathbb{S}^4}^*\overline{F}|^2 dvol_{\mathbb{S}^4} + C \|F\|_{L^2(\mathbb{S}^4)}^4 . \end{aligned} \quad (7.12)$$

Using (7.9) again, we obtain

$$\|\tilde{A}_g\|_{L^4(\mathbb{B}^5)} \leq \|A_g\|_{W^{1,2}(\mathbb{S}^4)} \leq C \|F\|_{\mathbb{S}^4} \quad (7.13)$$

Combining (7.12) and (7.13) we obtain

$$\begin{aligned} \|d\tilde{A}_g + \tilde{A}_g \wedge \tilde{A}_g - \overline{F}\|_{L^2(\mathbb{B}^5)}^2 &\leq C \delta \|\overline{F}\|_{L^2(\mathbb{B}^5)}^2 + \\ &\quad + C \int_{\mathbb{S}^4} |F - i_{\mathbb{S}^4}^*\overline{F}|^2 dvol_{\mathbb{S}^4} + C \|F\|_{L^2(\mathbb{S}^4)}^4 . \end{aligned} \quad (7.14)$$

Extend now g radially in \mathbb{B}^5 and denote by \hat{g} this extension. We have using (7.3) and (7.4)

$$\begin{aligned} \int_{\mathbb{B}^5} |\hat{g}^{-1}\overline{F}\hat{g} - \overline{F}|^2 &\leq 4 \|\overline{F}\|^2 \int_{\mathbb{B}^5} |\hat{g} - id|^2 dx^5 \\ &\leq C \|\overline{F}\|_{L^2(\mathbb{B}^5)}^2 \int_{\mathbb{S}^4} |g - id|^2 dvol_{\mathbb{S}^4} \leq C \delta \|\overline{F}\|_{L^2(\mathbb{B}^5)}^2 . \end{aligned} \quad (7.15)$$

Combining (7.14) and (7.15) gives

$$\begin{aligned} \|d\tilde{A}_g + \tilde{A}_g \wedge \tilde{A}_g - \hat{g}^{-1}\overline{F}\hat{g}\|_{L^2(\mathbb{B}^5)}^2 &\leq C \delta \|\overline{F}\|_{L^2(\mathbb{B}^5)}^2 + \\ &\quad + C \int_{\mathbb{S}^4} |F - i_{\mathbb{S}^4}^*\overline{F}|^2 dvol_{\mathbb{S}^4} + C \|F\|_{L^2(\mathbb{S}^4)}^4 . \end{aligned}$$

Denote $\hat{A} := \hat{g}\tilde{A}_g\hat{g}^{-1} + \hat{g}d\hat{g}^{-1}$. Observe that with this notation one has

$$F_{\hat{A}} = \hat{g}F_{A_g}\hat{g}^{-1} .$$

This one form \hat{A} extends A in \mathbb{B}^5 , there is a gauge in which it is smooth and we have

$$\begin{aligned} \|d\hat{A} + \hat{A} \wedge \hat{A} - \overline{F}\|_{L^2(\mathbb{B}^5)}^2 &\leq C \delta \|\overline{F}\|_{L^2(\mathbb{B}^5)}^2 + \\ &\quad + C \int_{\mathbb{S}^4} |F - i_{\mathbb{S}^4}^*\overline{F}|^2 dvol_{\mathbb{S}^4} + C \|F\|_{L^2(\mathbb{S}^4)}^4 . \end{aligned}$$

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Going back to the ϵ scale by pull backing all forms to the good cube C_ϵ^i using the dilation map $x \mapsto \epsilon^{-1}x$, denoting $\hat{A}_\epsilon = \epsilon^{-1} \sum_{j=1}^5 \hat{A}_j(\epsilon^{-1}x) dx_j$,

$$\begin{aligned} \int_{C_\epsilon^i} |d\hat{A}_\epsilon + \hat{A}_\epsilon \wedge \hat{A}_\epsilon - \overline{F}|^2 dx^5 &\leq C \delta \int_{C_\epsilon^i} |\overline{F}|^2 dx^5 + \\ &+ C \epsilon \int_{\partial C_\epsilon^i} |F - i_{\partial C_\epsilon^i}^* \overline{F}|^2 dvol_{\partial C_\epsilon^i} + C \epsilon \delta \int_{\partial C_\epsilon^i} |F|^2 dvol_{\partial C_\epsilon^i} \quad . \end{aligned}$$

Summing up over the good cubes - index i - using (F_1) and (F_2) we finally obtain the desired estimate

$$\sum_{i \in \mathcal{G}} \int_{C_\epsilon^i} |d\hat{A}_\epsilon + \hat{A}_\epsilon \wedge \hat{A}_\epsilon - \overline{F}|^2 dx^5 \leq C \delta + o_\epsilon(1).$$

□

Remark 7.11. If $\overline{F} = \sum_i \chi_{C_\epsilon^i} f_{C_\epsilon^i} F$ then we also obtain

$$\|\overline{F} - F\|_{L^2(\mathbb{B}^5)}^2 = o_\epsilon(1),$$

therefore in that case $F_{\hat{A}}$ would be an approximant of F as well.

7.5 Smoothing on the 4-skeleton

We will use the following classical result:

Lemma 7.12. *Let $p \geq n/2$ and let A be a $W^{1,p}$ -connection over an n -dimensional smooth cell complex X . Then there exists a sequence A_η of smooth connections over X such that*

$$\lim_{\eta \rightarrow 0} \|A_\eta - A\|_{W^{1,p}(X)} = 0 \quad \text{and} \quad \lim_{\eta \rightarrow 0} \|F_{A_\eta} - F_A\|_{W^{1,p}(X)} = 0.$$

Proof. If we had just functions $f, f_\eta : X \rightarrow \wedge^1 \mathbb{R}^n \otimes \mathfrak{g}$ in our statement, then the result would be classical (even without the restriction on p) and it would suffice to mollify f in order to obtain approximants $f_\eta = f * \rho_\eta$ where ρ_η is a scale- η smooth mollifier.

The problem which we face is just the fact that A is not globally defined: we have instead local expressions A_i in the chart U_i , and we must mollify A_i to $A_{i,\eta}$ for which $A_{i,\eta} = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1} A_{j,\eta} g_{ij} := g_{ij}(A_{j,\eta})$ are still true. We use a partition of unity $(\theta_i)_i$ adapted to the charts U_i and define

$$(A_\eta)_i = \theta_i A_i * \rho_\eta + \sum_{i' \neq i} \theta_{i'} g_{ii'}(A_{i'} * \rho_\eta).$$

By the cocycle condition $g_{i'j'}g_{i'j} = g_{ij}$ we obtain the desired $(A_\eta)_i = g_{ij}((A_\eta)_j)$. The derivatives of θ_i enter the estimate of $\|A_\eta - A\|_{W^{1,p}(X)}$ introducing a possibly huge L^∞ -factor, however this factor is independent on η . We therefore have $\lim_{\eta \rightarrow 0} \|A_{i,\eta} - A_i\|_{W^{1,p}} = 0$.

The restriction on the exponent p is needed in to prove the convergence of curvatures. This is based on the following inequality:

$$\begin{aligned} \|F_A - F_B\|_{L^p} &\lesssim \|dA - dB\|_{L^p} + \|(A - B) \wedge A\|_{L^p} + \|(A - B) \wedge B\|_{L^p} \\ &\lesssim \|DA - DB\|_{L^p} + \|A - B\|_{L^{2p}}(\|A\|_{L^{2p}} + \|B\|_{L^{2p}}). \end{aligned}$$

We are able to conclude using the $W^{1,p}$ -convergence of the A_η because we have the Sobolev embedding $W^{1,p} \hookrightarrow L^{2p}$ valid precisely when $p \geq n/2$. We leave the details of the proof to the reader. \square

In Definition 7.2 we have assumed that our globally L^2 -integrable connection form A is gauge-equivalent (on almost all boundaries of 5-cubes) to a $W^{1,2}$ -connection as in Definition 7.1. By a small perturbation of our grids we can ensure that each ∂C_ϵ^i has a boundary for which an equivalent $W^{1,2}$ -connection exists. We note that since in all charts the local $W^{1,2}$ -connection forms are equivalent, they give a global connection form B on ∂C_ϵ . We may thus apply Lemma 7.12 on this grid and perturb B to a smooth B_η . If on a chart U_i there holds $B = g_i^{-1}dg_i + g_i^{-1}Ag_i$ then we define A_η by requiring $B_\eta = g_i^{-1}dg_i + g_i^{-1}A_\eta g_i$. Then the following conditions hold for A_η on each cube:

$$\begin{aligned} \text{as } \eta \rightarrow 0 \text{ we have,} & \quad \left\{ \begin{array}{l} \int_{\partial C_\epsilon^i} |F_{A_\eta} - F_A|^2 \leq o_\eta(1), \\ \text{uniformly in } i \\ \text{and at fixed } \epsilon \end{array} \right. & (7.16) \\ & \quad \int_{\partial C_\epsilon} |A_\eta - A|^2 \leq o_\eta(1) \end{aligned}$$

Note that we are still comparing our smoothed curvature to the average of the original one. This is what we need in order to apply Proposition 7.10 and still obtain a good approximant to the original curvature F .

Proof of estimates (7.16). We prove the estimate on A_η by noting that by summing the local estimates in charts on all ∂C_ϵ^i we have $\|B_\eta - B\|_{L^2(\partial C_\epsilon)} \rightarrow 0$. Since differences of connections are gauge invariant we obtain

$$\int_{\partial C_\epsilon^i} |A_\eta - A|^2 = \int_{\partial C_\epsilon^i} |g_i^{-1}(B_\eta - B)g_i|^2 \leq C \int_{\partial C_\epsilon^i} |B_\eta - B|^2,$$

where C depends on the diameter of the group G . The estimate on A_η follows from Lemma 7.12. The estimate on F_η is similar. \square

Using these estimates we see that

Lemma 7.13. *Whenever a grid C_ϵ is good for A, F, \bar{F} then for η small enough C_ϵ is also good for A_η, F_η, \bar{F} and δ -good cubes with respect to A, F, \bar{F} are 2δ -good for A_η, F_η, \bar{F} .*

Proof. Use estimates (7.16) and the triangle inequality. □

Smoothing once more

To motivate the following, note that the proof of Proposition 7.10 extends the candidate connection A_η to a Coulomb gauge $g_{i,\eta}$ on each cube boundary ∂C_ϵ^i . Then the forms $g_{i,\eta}(A_\eta)$ are extended harmonically and \tilde{A}_η^i in the gauges $\hat{g}_{i,\eta}^{-1}$ coincide with A_η . We thus have smoothness inside for the Coulomb gauge of A_η . We also obtain that in the Coulomb gauge A_η is smooth, because we can obtain a $W^{1,2}$ such gauge as in [132], to which we can apply the following result.

Proposition 7.14 ([83] Prop. 3.4). *, Suppose that B is a smooth connection on a 4-dimensional manifold M and that $A_C = g^{-1}dg + g^{-1}Bg$ is a $W^{1,2}$ -Coulomb gauge then also g (and thus B_C) is smooth.*

The proof of the above proposition goes as follows: by Lorentz space theory (see [107]) we obtain that if $A_C, B \in W^{1,2}, d^*A_C = 0$ then $g \in W^{2,2} \cap C^0$ (this is analogue to the 2-dimensional Wentz lemma [137]). This regularity for g allows to apply classical elliptic theory to the elliptic system issued from $d^*(g^{-1}dg) = d^*(g^{-1}A_Cg)$ and to conclude by bootstrap.

Note that A_η is not assured to be smooth yet, we just know that about B_η , while we ignore the smoothness of the gauge used to pass from A_η to B_η . However we can apply Proposition 7.14 to B_η and to the Coulomb gauge of A_η , which are indeed gauge-related forms. We obtain that the Coulomb 1-form $(A_\eta)_g$ obtained during the proof of Proposition 7.10 is smooth.

Thus the extensions on good cubes obtained by applying Prop. 7.10 to A_η, F_η, \bar{F} as in Lemma 7.13 stay smooth up to the boundary.

We still don't control the gauges g_η which pass from B_η back to A_η . It is sufficient to approximate the g_η with smooth gauge changes in $W^{1,2}$. We use the following result:

Proposition 7.15. *Any function $g \in W^{1,2}(X, G)$ from a smooth 4-dimensional cell complex X to a compact Lie group G containing no copies of \mathbb{S}^1 can be approximated strongly in $W^{1,2}$ -norm by smooth functions $g'_\eta \in C^\infty(X, G)$.*

Proof. This follows from the characterization [19, 64, 65] and from the fact that $\pi_1(G) = 0$ because G contains no \mathbb{S}^1 and from the general fact true for compact Lie groups that $\pi_2(G) = 0$. \square

We thus have $g_{\eta, \eta'} \rightarrow g_\eta$ as $\eta' \rightarrow 0$. From

$$A_\eta = g_\eta d(g_\eta^{-1}) + g_\eta B_\eta g_\eta^{-1},$$

since $W^{1,2} \cap L^\infty$ is an algebra (see for instance [79] Sec. 6 for proofs) it follows that (for η' small enough and fixed ϵ, η)

$$A_{\eta, \eta'} := g_{\eta, \eta'} d(g_{\eta, \eta'}^{-1}) + g_{\eta, \eta'} B_\eta g_{\eta, \eta'}^{-1}$$

is close to A_η in L^2 on ∂C_ϵ . This gives the estimate (uniform in i for fixed ϵ, η)

$$\int_{\partial C_\epsilon^i} |A_{\eta, \eta'} - A_\eta|^2 = o'_\eta(1). \quad (7.17)$$

For the estimates on the curvature we note that $g_{\eta, \eta'} \rightarrow g_\eta$ in $W^{1,2}$ and thus also in L^4 , while F_{B_η} is smooth so its L^∞ norm is bounded. On each good cube we estimate as follows:

$$\begin{aligned} \int_{\partial C_\epsilon^i} |F_{A_{\eta, \eta'}} - F_{A_\eta}|^2 &= \int_{\partial C_\epsilon^i} |g_{\eta, \eta'}^{-1} F_{B_\eta} g_{\eta, \eta'} - g_\eta^{-1} F_{B_\eta} g_\eta|^2 \\ &\lesssim \|F_{B_\eta}\|_{L^\infty(\partial C_\epsilon)}^2 \int_{\partial C_\epsilon^i} |g_{\eta, \eta'} - g_\eta|^2, \end{aligned}$$

and the last term converges to zero as $\eta' \rightarrow 0$, uniformly in i at fixed ϵ, η :

$$\int_{\partial C_\epsilon^i} |F_{A_{\eta, \eta'}} - F_{A_\eta}|^2 = o'_\eta(1). \quad (7.18)$$

7.6 Proof of Theorem 7.4

Consider $A, F = F_A$ which are L^2 on the unit ball and correspond to a connection class in \mathcal{A}_G . We desire to approximate this F in L^2 by the curvature form of a smooth connection class which belongs to \mathcal{R}_∞ .

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For each $\epsilon > 0$ we find grids C_ϵ which are good with respect to A, F, \bar{F} , where

$$\bar{F} = \sum_i \chi_{C_\epsilon^i} \sum_{j < k} \left(\frac{1}{|C_\epsilon^i|} \int_{C_\epsilon^i} F_{jk} \right) dx_j \wedge dx_k.$$

The existence of such grids is proved in Proposition 7.6.

We will use δ -good grids for $\delta > 0$ to be fixed later depending on ϵ . From Proposition 7.9 we know that the total volume of bad cubes $\epsilon^5 \#\mathcal{B}_\epsilon$ will go to zero, provided $\epsilon^2 = o(\delta)$.

We will require $\delta < C/4$ for the constant C of Proposition 7.10. Lemma 7.13 provides connections and curvatures A_η, F_η which can substitute A, F in Proposition 7.10. Up to increasing the chosen δ a bit, choosing η small will not change the set of good cubes.

Proceeding as in Section 7.5 we obtain smooth approximations $A_{\eta, \eta'}$ to A_η on ∂C_ϵ such that also their curvatures $F_{\eta, \eta'}$ approximate F_η in L^2 -norm up to an error which is $o_{\eta'}(0)$. We see as in Lemma 7.13 that for η' small enough C_ϵ will still be good with respect to $A_{\eta, \eta'}, F_{\eta, \eta'}, \bar{F}$ and 2δ -good cubes for A_η, F_η, \bar{F} will be 4δ -good for $A_{\eta, \eta'}, F_{\eta, \eta'}, \bar{F}$.

We can then apply Proposition 7.10 to $A_{\eta, \eta'}, F_{\eta, \eta'}, \bar{F}, C_\epsilon$. As described in Section 7.5 we obtain smooth extensions on each δ -good cube. If by abuse of notation we still denote by \hat{A} the connection equal to the \hat{A} obtained this way on each good cube, we have the control

$$\int_{\cup \mathcal{G}} |F_{\hat{A}} - \bar{F}|^2 \lesssim \delta + o_\epsilon(1). \quad (7.19)$$

Let C_ϵ^i be a bad cube. We extend $A_{\eta, \eta'}$ in C_ϵ^i radially: let

$$\pi : C_\epsilon^i \setminus \{\text{center of } C_\epsilon^i\} \rightarrow \partial C_\epsilon^i \quad \text{be the radial projection.}$$

Then define \hat{A} on this C_ϵ^i as

$$\hat{A} = \pi^* A_{\eta, \eta'}.$$

It then follows

$$F_{\hat{A}} = \pi^* F_{\eta, \eta'}$$

and in particular for $C_\epsilon^i \in \mathcal{B}$

$$\int_{C_\epsilon^i} |F_{\hat{A}}|^2 \leq \epsilon \int_{\partial C_\epsilon^i} |F_{\eta, \eta'}|^2 \leq \epsilon \left(\int_{\partial C_\epsilon^i} |\bar{F}|^2 + o_{\eta, \eta'}(1) \right).$$

For η, η' small enough (depending on ϵ) we can obtain that the above $o(1)$ is uniformly small over all bad cubes, and can be absorbed in the first term:

$$\int_{\cup \mathcal{B}} |F_{\hat{A}}|^2 \lesssim \epsilon \int_{\partial \mathcal{B}} |\bar{F}|^2 \lesssim \int_{\cup \mathcal{B}} |\bar{F}|^2 \lesssim \int_{\cup \mathcal{B}} |F|^2, \quad (7.20)$$

where

$$\partial \mathcal{B} := \cup_{i \in \mathcal{B}} \partial C_\epsilon^i.$$

Up to now we have a connection \hat{A} which is smooth outside

$$\partial C_\epsilon \cup \{\text{centers of } C_\epsilon^i, i \in \mathcal{B}\}.$$

The finite set of centers of bad cubes will be the points where our final curvature will be singular. Near ∂C_ϵ instead, we have that \hat{A} is guaranteed to be C^0 so far, while its derivative orthogonal to the boundary could jump. We now mollify \hat{A} near ∂C_ϵ .

Fix $\eta'' > 0$ and consider a smooth mollifier $\rho_{\eta''}$. Note that locally on neighboring cubes C_ϵ^i the gauges in which \hat{A} was proved to be smooth on each of the C_ϵ^i are a continuous extension of the ones in which $A_{\eta, \eta'}$ is smooth. By compactness arguments we can locally (with respect to the charts on ∂C_ϵ) extend these gauges in a 2σ -neighborhood of the whole ∂C_ϵ for some $\sigma > 0$. We then mollify \hat{A} by convolution on scales which decrease away from ∂C_ϵ using partitions of unity on such neighborhood, as the proof of Lemma 7.12. We use the following mollifiers:

$$\rho_{\eta'', x} := \rho_{\eta(x)} \quad \text{for} \quad \eta(x) = \eta''(\sigma - \text{dist}(x, \partial C_\epsilon))^+$$

and we use the convention

$$(f * \rho_0)(z) := f(z).$$

The difference with that lemma is that we don't have $\hat{A} \in W^{1,2}$ but only $\hat{A} \in C^0 \cap L^2$. This allows just to show that the smoothing which locally looks like

$$(\hat{A}_{\eta''})_i = \theta_i \hat{A}_i * \rho_{\eta'', x} + \sum_{i' \neq i} \theta_{i'} g_{ii'} (\hat{A}_{i'} * \rho_{\eta''})$$

is close in C^0 -norm to \hat{A} . However since the curvature $F_{\hat{A}}$ is L^2 it follows that locally (and by compactness reasons also globally) $d\hat{A} \in L^2$, thus the estimate of $d((\hat{A}_{\eta''})_i)$ is still valid, showing together with the C^0 -estimate that locally in a chart U_i there holds $(F_{\hat{A}_{\eta''}})_i \rightarrow (F_{\hat{A}})_i$ in L^2 for $\eta'' \rightarrow 0$. By gauge invariance and using partitions of unity,

$$\|\hat{A}_{\eta''} - \hat{A}\|_{L^2(\mathbb{B}^5)} \rightarrow 0, \quad \|F_{\hat{A}_{\eta''}} - F_{\hat{A}}\|_{L^2(\mathbb{B}^5)} \rightarrow 0 \quad \text{as } \eta'' \rightarrow 0. \quad (7.21)$$

We can now state the estimates which complete our proof of Theorem 7.4:

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Proposition 7.16. *With the above notations, $F_{\eta''} := F_{A_{\eta''}} \in \mathcal{R}_\infty$ approximates F with respect to the distance (7.1) as $\epsilon = \delta \rightarrow 0$ and $\eta, \eta', \eta'' \rightarrow 0$.*

Proof. Because of equation (7.21) it is enough to prove that $\text{dist}(F_{\hat{A}}, F) \rightarrow 0$ as $\epsilon = \delta \rightarrow 0$. We just have to compare $F_{\hat{A}}$ to F on good cubes in the gauge given via Proposition 7.10 and in the original gauge in which A (and $A_{\eta, \eta'}$) is given, on the bad cubes.

We have

$$\begin{aligned} \sum_i \int_{C_\epsilon^i} |F_{\hat{A}} - F|^2 &\lesssim \sum_{i \in \mathcal{G}} \int_{C_\epsilon^i} |F_{\hat{A}} - F|^2 + \sum_{i \in \mathcal{B}} \int_{C_\epsilon^i} |F_{\hat{A}}|^2 + \int_{\cup \mathcal{B}} |F|^2 \\ &\stackrel{(7.20)}{\lesssim} \sum_{i \in \mathcal{G}} \int_{C_\epsilon^i} |F_{\hat{A}} - \bar{F}|^2 + \sum_{i \in \mathcal{G}} \int_{C_\epsilon^i} |F - \bar{F}|^2 + \int_{\cup \mathcal{B}} |F|^2 \\ &\stackrel{(7.19)}{\lesssim} \delta + o_\epsilon(1) + \sum_{i \in \mathcal{G}} \int_{C_\epsilon^i} |F - \bar{F}|^2 + \int_{\cup \mathcal{B}} |F|^2, \end{aligned} \quad (7.22)$$

where the last estimate is true provided we choose η, η' small enough, depending on ϵ .

For an L^2 -function f and for any sequence of ϵ -grids C_ϵ there holds

$$\sum_i \int_{C_\epsilon^i} \left| f - \frac{1}{|C_\epsilon^i|} \int_{C_\epsilon^i} f \right|^2 = o_\epsilon(1)$$

as a consequence of the well-known fact that

$$\int |f(x) - f(x+h)|^2 dx = o(|h|) \quad \text{for } f \in L^2.$$

This suffices to estimate

$$\sum_{i \in \mathcal{G}} \int_{C_\epsilon^i} |F - \bar{F}|^2 \leq \sum_i \int_{C_\epsilon^i} |F - \bar{F}|^2 = o_\epsilon(1). \quad (7.23)$$

For the last term we observe that

$$\chi_{\mathcal{B}} F := \sum_{i \in \mathcal{B}} \chi_{C_\epsilon^i} F \quad \text{satisfies} \quad \begin{cases} \forall \epsilon, |\chi_{\mathcal{B}} F|(x) \leq |F|(x) \text{ a.e. } x, \\ \chi_{\mathcal{B}} F(x) \rightarrow 0 \text{ a.e. } x, \text{ as } \epsilon \rightarrow 0, \end{cases}$$

the last statement following from the fact that the total volume of bad cubes is $O(\epsilon^2/\delta)$ as $\epsilon, \delta \rightarrow 0$ and we took $\delta = \epsilon$. By dominated convergence it follows that

$$\int_{\cup \mathcal{B}} |F|^2 \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \quad (7.24)$$

The estimates (7.22), (7.19) and (7.20) complete the proof. \square

Chapter 8

Weak closure for nonabelian curvatures in 5 dimensions

8.1 The weak closure result

We now prove the following theorem, part of the joint work with my advisor Tristan Rivière [PR3]:

Theorem 8.1 (Weak closure of the class \mathcal{A}_G). *Assume that we have a sequence of L^2 curvature forms F_n corresponding to connection classes*

$$[A_n] \in \mathcal{A}_G(\mathbb{B}^5)$$

such that

$$\sup_n \|F_n\|_{L^2(\mathbb{B}^5)} < \infty$$

and

$$F_n \rightharpoonup F \text{ in } L^2(\mathbb{B}^5, \wedge^2 \mathbb{R}^5 \otimes \mathfrak{g}).$$

Then F also corresponds to some $[A] \in \mathcal{A}_G(\mathbb{B}^5)$ as well.

We can find L^2 -controlled connection forms A_n corresponding to F_n and obtain a weak limit A which will be an L^2 connection form corresponding to F . The main difficulty is to find gauges g in which i^*A belongs to $\mathcal{A}^{1,2}$.

For the proof we use the overall strategy which worked in the abelian case as well and was employed in Chapter 2.

We start by identifying the traces on lower dimensional sets $\partial B_\rho(x_0)$ with elements of a metric space $(\mathcal{Y}, \text{dist})$ such that we have a local control of the

Hölder norm of the slice functions in terms of the L^2 -norms of the F_n . We will use for this the abstract Theorem 8.2.

Mixing a compactness result for slice functions with respect to the distance on \mathcal{Y} with the weak convergence of the A_n we will manage to obtain the convergence of a.e. slice to an element of \mathcal{Y} .

Elements of \mathcal{Y} are by definition also in $\mathcal{A}^{1,2}$, completing the proof.

8.2 The metric space \mathcal{Y}

To prove the weak closure result for \mathcal{A}_G we will use a slicing technique. In the definition of \mathcal{A}_G we required that any weak connection on each slice for which there exist local gauges in which it is represented by a $W^{1,2}$ form. Therefore we consider the following space of possible slices:

$$\mathcal{Y} := \{[A] \in \mathcal{A}^2(\mathbb{S}^4) / \sim : \exists B \in [A] \text{ s.t. } B \in \mathcal{A}^{1,2}(\mathbb{S}^4)\}, \quad (8.1)$$

where the equivalence relation \sim on global L^2 connections is

$$A \sim B \text{ if } \exists g \in W^{1,2}(\mathbb{S}^4, G) \text{ s.t. } g^{-1}dg + g^{-1}Ag = B$$

We define the following gauge-invariant function:

$$\text{“dist”}(A, A') := \left(\inf \left\{ \int_{\mathbb{S}^4} |A - g^{-1}dg - g^{-1}A'g|^2 : g \in W^{1,2}(\mathbb{S}^4, G) \right\} \right)^{\frac{1}{2}}.$$

The gauge invariance implies that “dist” is not a distance on connection forms, but rather restricts to a distance on their gauge equivalence classes. For two connection forms A, A' if $g_A, g_{A'}$ are $W^{1,2}$ -gauges such that

$$B = g_A^{-1}dg_A + g_A^{-1}Ag_A, \quad B' = B = g_{A'}^{-1}dg_{A'} + g_{A'}^{-1}A'g_{A'}$$

then, since $A \mapsto g^{-1}dg + g^{-1}Ag$ is a continuous group action of $\mathcal{G} \cap W^{1,2}$ on $\mathcal{A}^{1,2}$, we have

$$\text{“dist”}(A, A') = \text{“dist”}(B, B').$$

“dist” then descends to a well-defined distance $\text{dist}([A], [A'])$ on equivalence classes of connection forms. Let

$$[A] = \text{image of } A \text{ under the projection } \mathcal{A}^2(\mathbb{S}^4) \rightarrow \mathcal{A}^2(\mathbb{S}^4) / \sim.$$

The natural metric to impose on \mathcal{Y} is the L^2 -distance between (global) gauge orbits (cfr [47]):

$$\text{dist}([A], [B]) = \inf \{ \|A' - B'\|_{L^2(\mathbb{S}^4)} : A' \in [A], B' \in [B] \}. \quad (8.2)$$

On the metric space $(\mathcal{Y}, \text{dist})$ we will study the functional

$$\mathcal{N} : \mathcal{Y} \rightarrow \mathbb{R}^+, \quad \mathcal{N}([A]) = \int_{\mathbb{S}^4} |F_A|^2. \quad (8.3)$$

Note that because the curvature satisfies $F_{g^{-1}dg+g^{-1}Ag} = g^{-1}F_Ag$ and since the norm on 2-forms is G -invariant, we have that $\mathcal{N}([A])$ does not depend on the representative A employed to compute F_A .

8.3 The slice a.e. convergence

We employ the following abstract theorem. See [72] Thm. 9.1 for the original inspiration; for the proof we refer to Theorem 2.13; see Appendix E for another version. We use the notation overlapping with the previous section. The goal will be to justify this overlap in notation subsequently, by proving that the spaces and functions of Section 8.2 satisfy the hypotheses of the theorem.

Theorem 8.2. *Consider a metric space $(\mathcal{Y}, \text{dist})$ on which a function $\mathcal{N} : \mathcal{Y} \rightarrow \mathbb{R}^+$ is defined. Suppose that the following hypothesis is met:*

$$\forall C > 0 \text{ the sublevels } \{\mathcal{N} \leq C\} \text{ are seq. compact.} \quad (H)$$

Suppose $f_n : [0, 1] \rightarrow \mathcal{Y}$ are measurable maps such that

$$\text{dist}(f_n(t), f_n(t')) \leq C|t - t'|^{1/2} \quad (8.4)$$

and that

$$\sup_n \int_0^1 \mathcal{N}(f_n(t)) dt < C.$$

Then f_n have a subsequence which converges pointwise almost everywhere. The limiting function f also satisfies

$$\text{dist}(f(t), f(t')) \leq C|t - t'|^{1/2}, \quad \int_0^1 \mathcal{N}(f(t)) dt < C.$$

8.4 Verifying the hypothesis of Theorem 8.2

we verify that we can apply the abstract theorem 8.2 to our situation, where the goal is to prove weak closure for the class \mathcal{A}_G .

The compactness result (H)

We start by verifying the first statement of the hypothesis (H) for \mathcal{Y}, \mathcal{N} as in Section 8.2:

Proposition 8.3. *Let \mathcal{Y} be the space of slices as in (8.1) and $\mathcal{N} : \mathcal{Y} \rightarrow \mathbb{R}^+$ be the norm of the curvature as in (8.3). Then \mathcal{N} has sublevels which are compact with respect to the distance dist defined in (8.2).*

Proof. We assume that we are given a sequence of curvatures F_n corresponding to connection form classes $[A_n]$, such that

$$\|F_n\|_{L^2(\mathbb{S}^4)} \leq C.$$

The claim of the proposition is that the $[A_n]$ have a convergent subsequence with respect to the distance d .

Up to a global gauge change we may assume that the A_n are controlled globally in L^2 (see Lemma 8.4):

$$\|A_n\|_{L^2(\mathbb{S}^4)} \lesssim \|F_n\|_{L^2(\mathbb{S}^4)}.$$

Up to extracting a subsequence we have that

$$A_n \rightharpoonup A_\infty, F_n \rightharpoonup F_\infty \quad \text{in } L^2(\mathbb{S}^4).$$

Step 1. *Concentration points of the curvature energy and a good atlas.* By usual covering arguments we have that up to extracting a subsequence there exist a finite number of concentration points of the curvature's L^2 -energy a_1, \dots, a_N in \mathbb{S}^4 . In other words there holds

$$\forall \epsilon > 0, \rho_\epsilon := \liminf_{n \rightarrow \infty} \inf \left\{ \rho > 0, x_0 \in \mathbb{S}^4 \setminus \cup B_\epsilon(a_i) \int_{B_\rho^{\mathbb{S}^4}(x_0)} |F_n|^2 \geq \delta \right\} > 0.$$

The number N of such points is $N \leq C/\delta$ where C is the above L^2 -bound on the curvatures.

Up to diminishing ϵ and $\rho := \rho_\epsilon$ we may suppose $\epsilon + \rho_\epsilon < \rho_{\text{inj}}(M)$ and that the balls $B_\epsilon(a_i)$ are disjoint. We can find a cover by the balls $B_\epsilon(a_i)$ and by finitely many balls $B_\rho(x_i)$ such that the maximum number of overlaps of those balls is a universal constant. The $B_\rho(x_i)$'s will be called *good balls* and they will be simply denoted B_i below.

Step 2. *Uhlenbeck Coulomb gauges converge weakly on the good balls.* Using Uhlenbeck's gauge extraction of Theorem 6.4 on each B_i one finds a gauge g_n^i such that $A_n^i := (g_n^i)^{-1}dg_n^i + (g_n^i)^{-1}A_n g_n^i \in W^{1,2}$ and such that

$$d^*A_n^i = 0, \quad \|A_n^i\|_{W^{1,2}} \lesssim \|F_n\|_{L^2} \text{ on } B_i.$$

Therefore up to a diagonal subsequence we also may assume that

$$A_n^i \rightarrow A^i \text{ weakly in } W^{1,2} \text{ and strongly in } L^2. \quad (8.5)$$

By interpolation since the g_n^i are bounded in L^∞ we see that

$$g_n^i \rightarrow g^i \text{ weakly in } W^{1,2} \text{ and strongly in } L^q, \forall q < \infty.$$

This strong convergence in L^q together with the weak convergence of A_n and of the dg_n^i in L^2 implies that

$$A_n = g_n^i d(g_n^i)^{-1} + g_n^i A_n^i (g_n^i)^{-1} \rightarrow g^i d(g^i)^{-1} + g^i A^i (g^i)^{-1} = A \text{ in } \mathcal{D}'$$

and by uniqueness of weak limits the A^i obtained above are the local expressions of the limit A in the limit gauges g^i .

Step 3. *Point removability and strong global gauge convergence on good part.* By Theorem 6.5 the gauge changes $g_n^{ij} := g_n^j (g_n^i)^{-1}$ needed to pass from A_n^i to A_n^j are controlled in $W^{2,2} \cap C_0$. Therefore up to taking a diagonal subsequence we have for all i, j

$$g_n^{ij} \rightarrow g^{ij} \text{ weakly in } W^{2,2}, \text{ strongly in } W^{1,2} \text{ and locally uniformly in } C^0.$$

In particular we can apply the gauge extension procedure of the proof of Theorem 6.2 both to g_n^{ij} and to g^{ij} on balls covering any open contractible subset U^{good} in the complement of the bad balls $B_\epsilon(a_1), \dots, B_\epsilon(a_N)$, obtaining gauge transformations $\overline{g_n^{good}}, g^{good}$. We recall that in this process we multiply gauges by the constants $\overline{g_n^{ij}}$ then truncate the error terms $(\overline{g_n^{ij}})^{-1}g_n^{ij}$ away from $B_i \cap B_j$. We note that up to extracting subsequences we may assume (by compactness of G and finiteness of the balls intersecting U^{good}) that the constants involved also converge:

$$\overline{g_n^{ij}} \rightarrow \overline{g^{ij}}.$$

This implies together with (8.5) that on U^{good}

$$g_n^{good}(A_n) \rightarrow g^{good}(A) \text{ in } L^2(U^{good}).$$

Step 4. *The bad part's contribution.* The last part of the proof consists of noticing that by diminishing ϵ and by letting U^{good} increase to a set of full measure, we may find gauges $g_n^k = (g^{good})^{-1}g_n^{good}$ such that

$$(g_n^k)^{-1}dg_n^k + (g_n^k)^{-1}A_n g_n^k \rightarrow A \text{ in } L^2 \text{ outside a set of measure } \frac{1}{k}.$$

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By extracting a diagonal subsequence we obtain g_n such that

$$g_n^{-1}dg_n + g_n^{-1}A_n g_n \rightarrow A \text{ in } L^2(\mathbb{S}^4).$$

Therefore

$$\text{dist}([A_n], [A]) \rightarrow 0$$

as desired. □

The second hypothesis of Theorem 8.2

We now assume given a sequence of connection forms A_n with $[A_n] \in \mathcal{A}_G$ on \mathbb{B}^5 such that their distributional curvatures F_n are bounded in L^2 and converge weakly in L^2 to a 2-form F . For a fixed center $x_0 \in \mathbb{B}^5$ and for a radii $t \in [r, 2r]$ with $r > 0$, the slices of the connections A_n via spheres $\partial B_t(x_0)$ are defined and taking values in \mathcal{Y} for a.e. t by the assumption that $[A_n] \in \mathcal{A}_G$. We then define (classes of) functions

$$f_n : [r, 2r] \rightarrow \mathcal{Y}, \quad f_n(t) := i_{\partial B_t(x_0)}^* A_n.$$

Notation: We denote $A(s)$ the slice along $\partial B_s(x_0)$ i.e. the pullback of $i_{\partial B_s(x_0)}^* A$ to \mathbb{S}^4 via the homothety $\mathbb{S}^4 \rightarrow \partial B_s(x_0)$ when it exists.

We verify that the f_n satisfy the hypothesis (8.4):

Lemma 8.4. *Assume that $[A] \in \mathcal{F}_{\mathbb{Z}}$ and choose a gauge-representative A which is in L^2 on $B_{2r}(x_0) \setminus B_r(x_0)$. Then there exists a gauge change g such that $A' := g^{-1}dg + g^{-1}Ag$ has no radial component and such that for a.e. $t > t' \in [r, 2r]$*

$$\int_{\mathbb{S}^4} |A'(t) - A'(t')|^2 \lesssim \frac{1}{r^2} |t - t'| \int_{B_t(x_0) \setminus B_{t'}(x_0)} |F|^2$$

for a universal implicit constant.

Proof. We will assume $x_0 = 0$ for simplicity. Note that

$$\int_{t'}^t \|A(t)\|_{L^2(\mathbb{S}^4)}^2 dt = \int_{\mathbb{S}^4} \int_{t'}^t |\rho i_{\partial B_\rho}^* A|^2 \rho^4 d\rho d\omega.$$

Solve the following ODE in polar coordinates:

$$\begin{cases} \partial_\rho g(\omega, \rho) = -A_\rho(\omega, \rho)g(\omega, \rho), & \text{for } \rho \in [t', t], \\ g(\omega, t') = id, & \text{for all } \omega \in \mathbb{S}^4. \end{cases} \quad (8.6)$$

It then follows that for $A' = g^{-1}dg + g^{-1}Ag$ there holds

$$\sum_k \frac{x_k}{\rho} A'_k := A'_\rho = 0$$

therefore at (ω, ρ) we write

$$\sum_k x_k g^{-1} F_{ki} g = \sum_k x_k \partial_k A'_i - \sum_k x_k \partial_i A'_k + \sum_k x_k [A'_k, A'_i] = \partial_\rho(\rho A'_i).$$

In other words

$$\rho \partial_\rho \llcorner (g^{-1} F g) |_{\partial B_s(x_0)} = \partial_\rho(\rho i_{\partial B_\rho}^* A').$$

Integrating in s we have for a.e. $t > t'$ and then in ω we obtain

$$\begin{aligned} \int_{\mathbb{S}^4} |t i_{\partial B_t}^* A' - t' i_{\partial B_{t'}}^* A'|^2 &= \int_{\mathbb{S}^4} \left| \int_{t'}^t \rho \partial_\rho \llcorner (g^{-1} F g) d\rho \right|^2 \\ &\lesssim |t - t'| \int_{\mathbb{S}^4 \times [t', t]} \rho^2 |\partial_\rho \llcorner F|^2. \end{aligned}$$

We used Jensen's inequality and the fact that the norm is G -invariant. Note that for $\omega \in \mathbb{S}^4$ there holds

$$A'(s)(\omega) = s i_{\partial B_s}^* A'(s\omega),$$

therefore from above it follows

$$\int_{\mathbb{S}^4} |A'(t) - A'(t')|^2 \lesssim \frac{|t - t'|}{(t')^2} \int_{B_t \setminus B_{t'}} |F|^2.$$

Since $t' > r$ the thesis follows. \square

In the end the functions $f_n(t)$ which will satisfy (8.4) in our situation will be the slice functions of the connections $A_n(t)$ in the gauges given by Lemma 8.4. Note that as a direct consequence of Lemma 8.4 we have also

$$\text{dist}(A_n(t), A_n(t')) \lesssim \frac{\|F_n\|_{L^2(B_{2r} \setminus B_r)}}{r} |t - t'|^{1/2} \leq \frac{\|F_n\|_{L^2}}{r} |t - t'|^{1/2}. \quad (8.7)$$

8.5 Proof of the Closure Theorem 8.1

We consider a sequence in $\mathcal{A}_G(\mathbb{B}^5)$ as in Theorem 8.1 and we construct representatives of the connection classes A_n such that

$$\int_{\mathbb{B}^5} |A_n|^2 \leq C \int_{\mathbb{B}^5} |F_n|^2$$

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like in Lemma 8.4. We thus have that up to extracting a subsequence there holds

$$A_n \rightharpoonup A \quad \text{in } L^2(\mathbb{B}^5). \quad (8.8)$$

As noted above it suffices that for all centers x_0 and a.e. radius $t > 0$ the homothety pullback to \mathbb{S}^4 of the slice $i_{\partial B_t}^* A$ of the limit connection A is in $\mathcal{A}^{1,2}(\mathbb{S}^4)$ or equivalently to \mathcal{Y} . Fix $x_0 \in \mathbb{B}^5$ and a range of radii $[r, 2r]$. It is sufficient to prove that

$$\text{a.e. } s \in [r, 2r], \quad A(s) \in \mathcal{A}^{1,2}(\mathbb{S}^4). \quad (8.9)$$

We will assume for simplicity that $x_0 = 0$ and we apply Lemma 8.4 obtaining new gauges for the A_n in which (8.7) is valid. From now on we are going to work in these gauges only. For simplicity of notation we still denote the expressions the A_n in these gauges by A_n . Note that we still have the control

$$\|A_n\|_{L^2(B_{2r} \setminus B_r)} \lesssim \|F_n\|_{L^2}$$

if in the proof of Lemma 8.4 for $A = A_n$ we replace the ODE (8.6) by

$$\begin{cases} \partial_\rho g(\omega, \rho) = -(A_n)_\rho(\omega, \rho)g(\omega, \rho), & \text{for } \rho \in [s, t], \\ g(\omega, s) = id, & \text{for all } \omega \in \mathbb{S}^4. \end{cases}$$

for s such that the slice $A_n(s)$ satisfies

$$\|A_n(s)\|_{L^2} \lesssim \frac{1}{r} \|F_n\|_{L^2}.$$

Thus we may still suppose that (8.8) holds on $B_{2r} \setminus B_r$. We next prove that in this case we have a stronger convergence:

Lemma 8.5. *Assume that for a sequence of connection forms $A_n \in L^2(B_{2r} \setminus B_r, \wedge^1 \mathbb{R}^5 \otimes \mathfrak{g})$ there holds*

$$\|A_n(t) - A_n(t')\|_{L^2(\mathbb{S}^4)} \leq C|t - t'|^{1/2}$$

and that

$$A_n \rightharpoonup A \quad \text{weakly in } L^2 \text{ on } B_{2r} \setminus B_r.$$

Then there exists a subsequence n' such that

$$\text{for a.e. } s \in [r, 2r] \text{ there holds } A_{n'}(s) \rightharpoonup A(s) \quad \text{weakly in } L^2(\mathbb{S}^4). \quad (8.10)$$

Proof. The weak convergence hypothesis means that

$$\int A_n \wedge \beta \rightarrow \int A \wedge \beta \quad \text{for all } \beta \in L^2(B_{2r} \setminus B_r, \wedge^3 \mathbb{R}^5 \otimes \mathfrak{g}).$$

Consider an arbitrary 3-forms ω which is L^2 on \mathbb{S}^4 and a test 1-form $\varphi(t)$ on $[r, 2r]$. By taking

$$\beta := h_t^* \omega \wedge \varphi(t) \quad \text{where } h_t : \mathbb{S}^4 \rightarrow \partial B_t \text{ is a homothety}$$

we obtain

$$\int_r^{2r} \int_{\mathbb{S}^4} A_n(t) \wedge \omega \wedge \varphi(t) \rightarrow \int_r^{2r} \int_{\mathbb{S}^4} A(t) \wedge \omega(x) \wedge \varphi(t).$$

If we use the notation

$$f_n^\omega(t) = \int_{\mathbb{S}^4} A_n(t) \wedge \omega$$

then from the first hypothesis it follows that

$$\begin{aligned} |f_n^\omega(t) - f_n^\omega(t')| &\leq \|A_n(t) - A_n(t')\|_{L^2} \|\omega\|_{L^2} \\ &\leq C|t - t'|^{1/2} \|\omega\|_{L^2}. \end{aligned}$$

By Arzelà-Ascoli theorem the f_n^ω have a subsequence which converges uniformly to a 1/2-Hölder function with the same Hölder constant:

$$\sup_{t \in [r, 2r]} |f_n^\omega(t) - f^\omega(t)| \rightarrow 0.$$

By applying this reasoning to a countable L^2 -dense subset D of ω 's in $L^2(\mathbb{S}^4, \wedge^3 T\mathbb{S}^4 \otimes \mathfrak{g})$ and by a diagonal procedure we obtain that

$$\forall \omega \in D, \quad \sup_{t \in [r, 2r]} |f_n^\omega(t) - f^\omega(t)| \rightarrow 0.$$

Since the functionals $\omega \mapsto \int A_n(t) \wedge \omega$ are strongly continuous on L^2 -forms for a.e. t , we obtain that the above convergence holds on all $\omega \in L^2$, completing the proof. \square

We are now ready to conclude the proof of our weak closure result.

End of proof of Theorem 8.1: Consider the global weak limit connection form $A \in L^2(\mathbb{B}^5)$. As said above we prove that a.e. slice of it is in $\mathcal{A}^{1,2}$ by considering separately the groups of slices with center x_0 and radii in $[r, 2r]$. We assumed $x_0 = 0$ for simplicity and we obtained that the A_n have a weakly convergent subsequence on $B_{2r} \setminus B_r$, therefore we may apply Lemma 8.5. We obtain up to extracting a subsequence the slicewise a.e. weak convergence (8.10):

$$\text{for a.e. } s \in [r, 2r] \text{ there holds } A_n(s) \rightharpoonup A(s) \quad \text{weakly in } L^2(\mathbb{S}^4).$$

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Note that in this case the slicewise weak limit $A(s)$ is indeed the slice of the limit connection.

On the other hand we saw in Section 8.4 that the hypotheses of the Theorem 8.2 are verified for our A_n therefore we also have up to another subsequence extraction

$$\text{for a.e. } s \in [r, 2r] \text{ there holds } [A_n](s) \rightharpoonup [A^d](s) \quad \text{in } (\mathcal{Y}, \text{dist}).$$

We have now to compare the slice $A(s)$ of the weak limit with the dist-limit of slices $A^d(s)$. Since

$$\text{dist}([A_n](s), [A^d](s)) = \inf_{g \in W^{1,2}(\mathbb{S}^4, G)} \|g^{-1}dg + g^{-1}A_n(s)g - A^d(s)\|_{L^2}$$

we obtain a sequence $g_n(s) \in W^{1,2}(\mathbb{S}^4, G)$ such that

$$g_n(s)^{-1}dg_n(s) + g_n(s)^{-1}A_n(s)g_n(s) - A^d(s) \rightarrow 0 \quad \text{strongly in } L^2. \quad (8.11)$$

It follows that

$$\|dg_n(s)\|_{L^2} \lesssim \|A^d(s)\|_{L^2} + \|A_n(s)\|_{L^2}.$$

From

$$\|A_n(t) - A_n(t')\|_{L^2} \leq C|t - t'|^{1/2}$$

and from the fact that for all n there exists $s \in [r, 2r]$ such that

$$\|A_n(s)\|_{L^2} \lesssim \|F_n\|_{L^2} \leq C$$

it follows that $A_n(s)$ is bounded in L^2 . Thus $dg_n(s)$ is also bounded in L^2 . Thus up to extracting a subsequence (dependent on t)

$$dg_n(t) \rightharpoonup dg_\infty(t) \quad \text{weakly in } L^2.$$

Since $g_n(s)$ is also bounded in L^∞ we obtain by Rellich's theorem and by interpolation that up to extracting a subsequence $n(t)$

$$g_n(t) \rightarrow g_\infty(t) \quad \text{in } L^q \forall q < \infty.$$

The last two facts together with the convergence $A_n(t) \xrightarrow{L^2} A(t)$ suffice to prove that

$$\begin{aligned} g_n(t)^{-1}A_n(t)g_n(t) &\rightarrow g_\infty(t)^{-1}A(t)g_\infty(t) \text{ in } \mathcal{D}'(\mathbb{S}^4), \\ g_n(t)^{-1}dg_n(t) &\rightarrow g_\infty(t)^{-1}dg_\infty(t) \text{ in } \mathcal{D}'(\mathbb{S}^4). \end{aligned}$$

This is valid for a.e. $t \in [r, 2r]$. Therefore

$$A^d(t) = g_\infty(t)^{-1}dg_\infty(t) + g_\infty(t)^{-1}A(t)g_\infty(t), \quad \text{for a.e. } t \in [r, 2r].$$

Since $A^d(t) \in \mathcal{A}^{1,2}(\mathbb{S}^4)$, this shows that for a.e. t the slice $A(t)$ of the limit connection A belongs to $\mathcal{A}^{1,2}$, as desired. \square

Chapter 9

Global gauges and nonlinear Sobolev spaces

In this chapter we study globally controlled gauges in which a control on connection in the Lorentz space $L^{4,\infty}$ in terms of the Yang-Mills energy is obtained in 4 dimensions, even in the “bubbling” cases where the stronger control provided by Uhlenbeck’s theorem 6.4 fails. We then prove several related controlled extension results for nonlinear Sobolev spaces. This chapter is based on joint work with my advisor Tristan Rivière [PR2].

9.1 Introduction

The use of Hodge decomposition is by now one of the classical tools in the study of elliptic systems and is related to important breakthroughs such as the famous “div-curl”-type theorems [36]. More recently such decomposition has allowed to solve [108] S. Hildebrandt’s conjecture [76], and at the same time establishing an important link to an apparently unrelated fields of geometry, such as the study of conformally invariant geometric problems in 2-dimensions [75] and the study of Yang-Mills bundles and gauge theory [132], with the introduction of controlled Coulomb gauges.

The study of 2-dimensional problems using controlled gauges has already given its fruits, and in connection to the discovery of H. Wente’s inequality (which gave the basis for introducing the Lorentz spaces $L^{(2,\infty)}$ in geometric problems) allowed the successful use of controlled moving frames in the study of harmonic maps and prescribed mean curvature surfaces [75], [94]. We come back to this in Section 9.2.8. Techniques and function spaces related to the

moving frame method also apply to the study of the Willmore functional [109] for immersed surfaces.

The use of controlled gauges especially in relation to Lorentz spaces in dimensions higher than 2 is far less developed. We attempted here a first attack of this completely new area of research, and we obtained some extensions of previous results for the case of Yang-Mills fields on 4 dimensional manifolds.

9.1.1 Yang-Mills theory and controlled gauges

For a $W^{1,2}$ -connection of an L^2 -curvature over a closed 4-manifold it is easy to construct a Coulomb gauge in which we have just an L^2 -control in terms of the curvature. This is done by first obtaining any gauge in which

$$\|A\|_{L^2} \leq C\|F\|_{L^2}$$

and then finding the smallest norm coefficients with respect to that gauge on our manifold M :

$$\min \left\{ \int_M |g^{-1}dg + g^{-1}Ag|^2 dx : g \in W^{1,2}(M, SU(2)) \right\}.$$

A unique minimizer will exist by convexity and it will satisfy the Coulomb equation $d^*A = 0$.

The control of A in the higher norm $W^{1,2}$ is done under a L^2 -smallness hypothesis on F , as we already discussed

Theorem 9.1 (controlled Coulomb gauge under assumption of small energy, [132]). *There exists a constant $\epsilon_0 > 0$ such that if the curvature satisfies $\int_M |F|^2 \leq \epsilon_0$ then there exists a Coulomb gauge $\phi \in W^{2,2}(M, SU(2))$ such that in that gauge the connection satisfies $\|A_\phi\|_{W^{1,2}(M)} \leq C\|F\|_{L^2(M)}$ with $C > 0$ depending only on the dimension.*

The reason why the smallness of the curvature is necessary is that $\|F\|_{L^2(M)}$ being above a certain threshold allows the second Chern number of the bundle to be nontrivial:

$$c_2(E) = \frac{1}{8\pi^2} \int_M \text{tr}(F \wedge F) \neq 0.$$

If for such F the controlled gauge would be *global*, i.e. if we would have a global trivialization in which the connection of the above F is expressed as $d + A$ with

$$\|A\|_{W^{1,2}(M)} \leq C,$$

then by Sobolev and Hölder inequalities we would have enough control on the quantities involved to prove the following formal identity for our A :

$$\text{tr} [(dA + [A, A]) \wedge (dA + [A, A])] = d \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$

Now the right side is an exact form, thus it has integral equal to zero over the boundaryless manifold M , contradicting $c_2(E) \neq 0$.

M. Atiyah-N. Hitchin-I. Singer [10] and C. Taubes [124] constructed instantons with nontrivial Chern numbers as in the above heuristic. To exemplify the phenomena at work consider the simplest instanton, having $c_2(E) = 1$ over $M = \mathbb{S}^4$ (cfr. [58], Ch. 6 for notations and details). Recall that we may use quaternion notation due to the isomorphisms $SU(2) \sim Sp(1)$ and $su(2) \sim Im\mathbb{H}$, under which Pauli matrices correspond to quaternion imaginary units. We then have the following local expression of A over \mathbb{R}^4 (identified by stereographic projection with $\mathbb{S}^4 \setminus \{p\}$) in a trivialization:

$$A = Im \left(\frac{x d\bar{x}}{1 + |x|^2} \right).$$

If Ψ is the inverse stereographic projection then Ψ^*A is smooth away from the pole p , but near p we have $|\Psi^*A|(q) \sim \text{dist}_{\mathbb{S}^4}(p, q)^{-1}$, which is not L^4 in any neighborhood of p .

Such behavior like $\frac{1}{|x|}$ implies that we are in any space L^p for $p < 4$ but not in L^4 . The natural space is the weak- L^4 space, which is strictly contained between all $L^p, p < 4$ and L^4 :

Definition 9.2 (see [63]). *Let X, μ be a measure space. The space $L^{p,\infty}(X, \mu)$ (also called weak- L^p or Marcinkiewicz space) is the space of all measurable functions f such that*

$$\|f\|_{L^{p,\infty}}^p := \sup_{\lambda > 0} \lambda^p \mu\{x : |f(x)| > \lambda\}$$

is finite.

We note immediately that the function $f(x) = \frac{1}{|x|}$ belongs to $L^{4,\infty}$ on \mathbb{R}^4 and the above global gauge gives an $L^{4,\infty}$ 1-form Ψ^*A on \mathbb{S}^4 . Spaces $L^{p,\infty}$ arise naturally in dealing to the critical exponent estimates for elliptic equations. The Green kernel $K_n(x)$ of the Laplacian on \mathbb{R}^n satisfies indeed $\nabla K \in L^{\frac{n}{n-1},\infty}$ but not $\nabla K \in L^{\frac{n}{n-1}}$. Thus $\Delta u = f$ with $f \in L^1$ implies $\nabla u = \nabla K * f \in L^{\frac{n}{n-1},\infty}$ by an extended Young inequality (see [63]), unlike the higher exponent case $f \in L^p, p > 1$, which gives the stronger result $\nabla u \in L^p$.

9.1.2 Controlled global gauges

As shown heuristically by the explicit case of the instanton A above, it is known how to construct $L^{4,\infty}$ global gauges. Our main effort in this work is to obtain a norm-controlled gauges, mirroring Theorem 9.1 by K. Uhlenbeck. The main result is the following:

Theorem 9.3. *Let M^4 be a Riemannian 4-manifold. There exists a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with the following properties.*

Let ∇ be a $W^{1,2}$ connection over an $SU(2)$ -bundle over M . Then there exists a global $W^{1,(4,\infty)}$ section of the bundle (possibly allowing singularities) over the whole M^4 such that in the corresponding trivialization ∇ is given by $d + A$ with the following bound.

$$\|A\|_{L^{4,\infty}} \leq f(\|F\|_{L^2(M)}),$$

where F is the curvature form of ∇ .

This theorem is related to a second main result of this work, namely the introduction of Lorentz-Sobolev extension theorems for nonlinear maps. This result takes most of our efforts and can be stated as follows:

Theorem 9.4. *There exists a function $f_1 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with the following property. Suppose $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$. Then there exists an extension $u \in W^{1,(4,\infty)}(B^4, \mathbb{S}^3)$ of ϕ such that the following estimate holds:*

$$\|\nabla u\|_{L^{4,\infty}(B^4)} \leq f_1(\|\nabla \phi\|_{L^3}).$$

The originality of Theorem 9.4 with respect to the previous results [22] or [93] is that whereas the previous works were concerned with the existence of an extension, in our case a control is provided in term of the boundary value. We will see below that even under the hypothesis $\deg(\phi) = 0$ such that a $W^{1,4}$ -extension surely exists, no energy control will be available.

9.1.3 Strategy of gauge construction

The link between Theorems 9.3 and 9.4 is given by the well-known identification $SU(2) \simeq \mathbb{S}^3$. Therefore Theorem 9.4 can be rephrased as follows:

Theorem 9.5. *Fix a trivial $SU(2)$ -bundle E over the ball B^4 . There exists a function $f_1 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with the following property. If $g \in W^{1,3}(\mathbb{S}^3, SU(2))$ gives a trivialization of the restricted bundle $E|_{\partial B^4}$, then there exists an extension of g to a trivialization $\tilde{g} \in W^{1,(4,\infty)}(B^4, SU(2))$ such that the following estimate holds:*

$$\|\nabla \tilde{g}\|_{L^{4,\infty}(B^4)} \leq f_1(\|\nabla g\|_{L^3(\mathbb{S}^3)}).$$

The proof of the Theorem 9.3 is by a sequence of gauge extensions along the simplices of a suitable triangulation. We use simplices where Uhlenbeck's result 9.1 holds, i.e. F has energy $\lesssim \epsilon_0$. To ensure a lower bound on the size of simplices we cut areas of energy concentration and use induction on the energy, see the summary (9.56).

We discuss the relevance of our theorem, several possible extensions and related phenomena in Section 9.2.

9.1.4 Ingredients used in the construction of $W^{1,(4,\infty)}(B^4, \mathbb{S}^3)$ extensions

The starting new idea was to the use of implicit function theorems and of a limit on the integrability exponent as done in [131] for extension result. The procedure of Appendix F.1 is generalizable to other contexts with no new ingredients, at least as long as a Lie group structure is present.

For the implicit function theorems above we needed here a new product estimate valid in Sobolev spaces, which is presented in Section F.2, partially extending the results of [31], cfr. [111] and [128].

The second idea was to use $L^{(4,\infty)}$ functions such that the L^4 -estimate would fail just near a controlled number of points. Such singular points (where "singular" is meant with respect to the L^4 estimates) are introduced via Lemma 9.30 and Theorem 9.31.

The uniform $L^{(4,\infty)}$ -control is obtainable just in the case where the boundary value has no large energy "hot spots". To deal with the case where energy concentrates we use two tools which are available in the particular case of $\mathbb{S}^3 \simeq SU(2)$: (1) the group operation of $SU(2)$, which gives a continuous product on $W^{1,3}(X, \mathbb{S}^3)$; (2) the Möbius group of \mathbb{S}^3 , coupled with the conformal invariance of the L^3 -norm of the gradient on \mathbb{S}^3 .

Under a balancing condition on the boundary value ϕ we can write $\phi = \phi_1\phi_2$ where the product is taken in $SU(2)$, and the energies of $\phi_i, i = 1, 2$ are strictly less than that of ϕ , allowing an induction on the energy. If the balancing is not valid, we apply a Möbius transformation F_v to \mathbb{S}^3 and either

reduce to a balanced situation for $F_v \circ \phi$ and for some v or provide a substitute $v \in B^4 \mapsto \int_{\mathbb{S}^3} \phi \circ F_v$ to the harmonic extension of ϕ , to which we can now apply the projection trick. The natural parameterization of the Möbius group of \mathbb{S}^3 via vectors in B^4 fits very well in this setting, and we were inspired to use it by the similar use of it in [90].

9.1.5 Other extension results proved in this chapter

We list here, for further reference, the other extension theorems proved in this chapter, and announced in in the introduction of this thesis.

Theorem 9.6 (see Section 9.3). *Suppose $\phi \in W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$ is given. Then there exists $u \in W^{1,(3,\infty)}(B^3, \mathbb{S}^2)$ such that in the sense of traces $u|_{\partial B^3} = \phi$ and such that the following estimate holds, for a constant independent of ϕ .*

$$\|u\|_{W^{1,(3,\infty)}(B^3)} \leq C \|\phi\|_{W^{1,2}(\mathbb{S}^2)} (1 + \|\phi\|_{W^{1,2}(\mathbb{S}^2)}).$$

Theorem 9.7. *Assume $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^2)$. Then there exists a controlled extension $u \in W^{1,(4,\infty)}(B^4, \mathbb{S}^2)$ with the control*

$$\|u\|_{W^{1,(4,\infty)}(B^4, \mathbb{S}^2)} \leq C \|\phi\|_{W^{1,3}(\mathbb{S}^3, \mathbb{S}^2)} (1 + \|\phi\|_{W^{1,3}(\mathbb{S}^3, \mathbb{S}^2)}).$$

If instead we have $\phi \in W^{1,p}(\mathbb{S}^3, \mathbb{S}^2)$ for $9/4 \leq p < 3$ then there exists an extension $u \in W^{1, \frac{4}{3}p}(B^4, \mathbb{S}^2)$ with

$$\|u\|_{W^{1, \frac{4}{3}p}(B^4, \mathbb{S}^2)} \leq C \|\phi\|_{W^{1,p}(\mathbb{S}^3, \mathbb{S}^2)} (1 + \|\phi\|_{W^{1,p}(\mathbb{S}^3, \mathbb{S}^2)}).$$

Proposition 9.8. *Assume $n = 2, m \geq 3$ and $\frac{3m}{m+1} \leq p < \frac{4m}{m+1}$ and consider a $\phi \in W^{1,p}(\mathbb{S}^m, \mathbb{S}^2)$. Then there exists a controlled extension $u \in W^{1, \frac{m+1}{m}p}(B^{m+1}, \mathbb{S}^2)$ with*

$$\|u\|_{W^{1, \frac{m+1}{m}p}(B^{m+1}, \mathbb{S}^2)} \leq C \|\phi\|_{W^{1,p}(\mathbb{S}^m, \mathbb{S}^2)} (1 + \|\phi\|_{W^{1,p}(\mathbb{S}^m, \mathbb{S}^2)}).$$

Appendix F.1 deals with our new “extension” version of Uhlenbeck’s gauge construction and in Section F.2 we prove the needed new product inequality. Appendix F.3 contains computations and notation for the Möbius groups of B^4 and \mathbb{S}^3 .

9.1.6 Plan of this chapter

Section 9.2 contains a list of positive and negative results concerning phenomena parallel to ours, proving that our results are optimal. Section 9.3 contains the proof of Theorem 9.6. In Section 9.4 we prove Theorem 9.4 and in Section 9.5 we prove the Theorem 9.3.

9.2 Controlled and uncontrolled nonlinear Sobolev extensions

Classical Sobolev Space theory features optimal extension theorems in natural trace norms. For example if $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain and $u : \partial\Omega \rightarrow \mathbb{R}$ is a $W^{1,n-1}$ -function then there exists an extension $\bar{u} : \Omega \rightarrow \mathbb{R}$ such that $\bar{u} \in W^{1,n}$ and the estimate

$$\|\bar{u}\|_{W^{1,n}} \leq C\|u\|_{W^{1,n-1}}$$

holds (with C independent of u). This extension theorem is optimal in the sense that for dimensions $n > 2$ the natural trace operator $\bar{u} \in W^{1,n}(\Omega) \mapsto \bar{u}|_{\partial\Omega}$ sends $W^{1,n}$ to the optimal space $W^{1-\frac{1}{n},n}$ (see [123] chapter 40 for the natural appearance of this space), and we have the optimal Sobolev continuous embedding $W^{1-\frac{1}{n},n} \rightarrow W^{1,n-1}$ (see [123]) which brings us back to the original space. A similar result still holds if we replace the codomain \mathbb{R} by \mathbb{R}^m .

However for $n = 2$ the space $W^{1,1}(\mathbb{S}^1, \mathbb{S}^1)$ does not continuously embed in $H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$, making the above reasoning less poignant, see Sec. 9.2.3.

A possible construction of \bar{u} can be done by imitating the following model valid for $\Omega = \mathbb{R}_+^n := \{(x_1, \dots, x_n) | x_n \geq 0\}$:

$$\bar{u}(x_1, \dots, x_{n-1}, \epsilon) := (\rho_\epsilon * u)(x_1, \dots, x_{n-1}),$$

where ρ_ϵ is a usual family of radial smooth compactly supported mollifiers.

An equivalent construction of \bar{u} in terms of function spaces is by harmonic extension. The optimal result is the following

Proposition 9.9 (harmonic extension, cfr. [59] Ch. 10). *Assume $q > 1$ and $u \in W^{1-\frac{1}{q},q}(\partial B^{m+1}, \mathbb{R}^{n+1})$. Then there exists a harmonic extension $\bar{u} \in W^{1,q}(B^{m+1}, \mathbb{R}^{n+1})$ such that*

$$\|\bar{u}\|_{W^{1,q}(B^{m+1}, \mathbb{R}^{n+1})} \leq C_{m,n,q}\|u\|_{W^{1-\frac{1}{q},q}(\partial B^{m+1}, \mathbb{R}^{n+1})}.$$

By Sobolev embedding we have the controlled inclusion $W^{1,p} \hookrightarrow W^{1-\frac{1}{q},q}$ on an m -dimensional bounded open domain (or a compact manifold like ∂B^{m+1}) for $q \leq \frac{m+1}{m}p$, therefore this q is the largest exponent where we can hope to have a control for the extension.

If u is a constrained function with values in a subset of \mathbb{R}^{n+1} (e.g. a curved n -dimensional submanifold like \mathbb{S}^n) then averaging even on a very small scale could push the values of \bar{u} quite far from the constraint obeyed by u . This happens in particular for Sobolev exponents making the dimension “supercritical”, i.e. exponents such that $W^{1,q}(B^{m+1})$ is not constituted of continuous functions. We pass to describe some cases where directly projecting back to \mathbb{S}^n does not destroy the norm control of Prop. 9.9(harmonic extension).

9.2.1 Projection from a well-chosen center

We present in this section a trick which probably appeared for the first time in relation to nonlinear Sobolev extensions in [69] and [70]. For a Lorentz space version cfr. Prop. 9.23(projection trick 2).

Proposition 9.10 (projection trick). *If $f \in W^{1,q}(\Omega, B^{n+1})$ with $q < n + 1$ and Ω is a bounded open simply connected domain of \mathbb{R}^{m+1} then there exists $a \in B_{1/2}^{n+1}$ and a constant C depending only on q, m, n such that if $f_a(x) = \pi_a(f(x))$ where $\pi_a : B^{n+1} \setminus \{a\} \rightarrow \mathbb{S}^n$ is the projection which is constant along the segments $[a, \omega], \omega \in \mathbb{S}^n$, then*

$$\|f_a\|_{W^{1,q}(\Omega, \mathbb{S}^n)} \leq C \|f\|_{W^{1,q}(\Omega, B^{n+1})}.$$

Proof. We have just to estimate the gradient of f_a in terms of that of f since the functions themselves are anyways bounded and Ω is assumed of finite measure. We first note that since $a \in B_{1/2}^{n+1}$ is away from the boundary of B^{n+1} , we have the pointwise estimate

$$|\nabla f_a|(x) \lesssim \frac{|\nabla f|(x)}{|f(x) - a|},$$

where the implicit constant depends only on n . We next consider the following “average” on a :

$$\int_{B_{1/2}^{n+1}} \left(\int_{\Omega} |\nabla f_a|^q(x) dx \right) da \lesssim \int_{\Omega} |\nabla f|^q(x) \left(\int_{B_{1/2}^{n+1}} \frac{da}{|f(x) - a|^q} \right) dx.$$

We note that the inner integral is of the form

$$I(y) := \int_{B_{1/2}^{n+1}} \frac{da}{|y - a|^q},$$

and

$$\max_y I(y) = I(0) = C_n \int_0^{1/2} r^{n+q} dr = C_{n,q} < \infty \text{ since } q < n + 1.$$

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therefore we obtain

$$\int_{B_{1/2}^{n+1}} \|\nabla f_a\|_{L^q}^q da \leq C_{n,q} \|\nabla f\|_{L^q}^q,$$

and the proof is easily concluded. \square

The above proposition together with Prop. 9.9(harmonic extension) and the remark on Sobolev exponents following it gives the following:

Theorem 9.11 (corollary of the projection trick, cfr [70] Thm. 6.2). *Let $m, n \in \mathbb{N}^*$. If $1 \leq p < \frac{n+1}{m+1}m$ then for any $\phi \in W^{1,p}(\partial B^{m+1}, \mathbb{S}^n)$ there exists a nonlinear extension $u \in W^{1, \frac{m+1}{m}p}(B^{m+1}, \mathbb{S}^n)$ satisfying the control*

$$\|u\|_{W^{1, \frac{m+1}{m}p}(B^{m+1}, \mathbb{S}^n)} \leq C_{m,n,p} \|\phi\|_{W^{1,p}(\partial B^{m+1}, \mathbb{S}^n)}.$$

Remark 9.12. *Note that from the same ingredients we obtain also the stronger estimate where for $q := \frac{m+1}{m}p < m$ the weaker space $W^{1-\frac{1}{q}, q}(\partial B^{m+1}, \mathbb{S}^n)$ replaces $W^{1,p}(\partial B^{m+1}, \mathbb{S}^n)$. This was done in [22] and [70]. We stated Theorem 9.11 as above to emphasize the connection with our Theorems 9.4 and 9.6. Indeed taking $m = n$ we see that those Theorems cover the critical exponent $p = n$, for which the projection trick stops working.*

9.2.2 Large integrability exponents

We now consider functions in $W^{1,p}(\mathbb{S}^m, \mathbb{S}^n)$ with $p > m$. The space $C^{0,1-m/p}(\mathbb{S}^m, \mathbb{S}^n)$ continuously embeds in this space. The candidate extension space $W^{1, \frac{m+1}{m}p}(B^{m+1}, \mathbb{S}^n)$ is made of $C^{0,1-m/p}$ -functions as well. Extension problem is guaranteed to have a solution as long as $\pi_m(\mathbb{S}^n) = 0$. This is true for $m < n$ but false for many choices of $m > n$ and for $m = n$.

When an extension exists i.e. for ϕ representing the identity of $\pi_m(\mathbb{S}^n) \neq 0$, a controlled extension can be constructed, based on the fact that a bound on the $C^{0,\alpha}$ -norm for $\alpha > 0$ implies a control on the modulus of continuity.

9.2.3 Extension for maps in $W^{1,1}(\partial \mathbb{S}^1, \mathbb{S}^1)$

For maps with values in \mathbb{S}^3 we are helped by the existence of a well-behaved product structure on \mathbb{S}^3 , i.e. the one which gives the identification $\mathbb{S}^3 \simeq SU(2)$. This is enough to get the analogous result for $n = 1$ as we will see now. It

is however well-known (see [74] 2.3) that this is a very unusual case: a group operation exists on \mathbb{S}^k only for $k = 1, 3$.

We can state a similar extension problem in the 1-dimensional case. This kind of controlled extension result is related to the recent work on Ginzburg-Landau functionals in [119].

Here the main structural ingredients present for \mathbb{S}^3 are again present: namely, we have a group operation on \mathbb{S}^1 (in this case it is even an abelian group) and a Möbius structure on D^2 , restricting to one on \mathbb{S}^1 . We follow the strategy of proof used also for S^3 . The result is:

Theorem 9.13 (1-dimensional version of the extension). *There exists a function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with the following property. If $\phi \in W^{1,1}(\mathbb{S}^1, \mathbb{S}^1)$ then there exists $u \in W^{1,(2,\infty)}(D^2, \mathbb{S}^1)$ with $u|_{\partial D^2} = \phi$ in the sense of traces and we have the norm control*

$$\|u\|_{W^{1,(2,\infty)}(D^2, \mathbb{S}^1)} \leq g(\|\phi\|_{W^{1,1}(\mathbb{S}^1, \mathbb{S}^1)}).$$

We will explain the changes which occur with respect to the proof of Theorem 9.4 (see Sec. 9.4).

Sketch of proof: The procedure is as in Section 9.4 and Appendix F.1, we have just to replace exponents and dimensions 3,4 with 1,2. For the analogue of Proposition 9.34 (balancing \Rightarrow extension) the biharmonic equation (9.45) is replaced by a harmonic equation, while the resulting estimates persist. Perhaps the only main change is Lemma F.5 of Section F.2 changes more drastically. It should be replaced by the following product estimate valid for $f \in W^{1,1}(D^2)$, $g \in L^\infty \cap W^{1,2}(D^2)$:

$$\|fg\|_{W^{1,1}} \leq \|f\|_{W^{1,1}} (\|g\|_{L^\infty} + \|g\|_{W^{1,2}})$$

□

We must however note that the naturality of the space $W^{1,1}(\mathbb{S}^1, \mathbb{S}^1)$ in Theorem 9.13 is less evident, since the trace space $H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$ does not continuously embed in it, unlike what happens in higher dimensions. This is seen by considering

$$u_\epsilon(\theta) = \exp \left(i \min \left\{ 1, \epsilon^{-1} \text{dist}_{\mathbb{S}^1}(\theta, [-\pi/2, \pi/2]) \right\} \right).$$

It is then clear that $\|\nabla u_\epsilon\|_{L^1(\mathbb{S}^1)} = 2$ while we estimate the double integral in θ, θ' giving the $H^{1/2}$ -norm by the contribution of the regions $\theta \in [0, \pi/2]$, $\theta' \in$

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$[\pi/2 + \epsilon, \pi + \epsilon]$. Under these choices $u_\epsilon(\theta) = e^0, u_\epsilon(\theta') = e^i$ and their distance in \mathbb{S}^1 is 1. Thus

$$\begin{aligned} \|u_\epsilon\|_{H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)}^2 &= \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{\text{dist}_{\mathbb{S}^1}(u_\epsilon(\theta), u_\epsilon(\theta'))^2}{\text{dist}_{\mathbb{S}^1}(\theta, \theta')^2} d\theta d\theta' \\ &\leq \int_0^1 \int_0^1 \frac{1}{|x + 2\epsilon/\pi - y|^2} dx dy \\ &\lesssim |\log \epsilon| + 1. \end{aligned}$$

9.2.4 Using controlled liftings to obtain controlled extensions

The control obtained for extensions of maps in $W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ and $W^{1,1}(\mathbb{S}^1, \mathbb{S}^1)$ is exponential in the norms of these maps. In Section 9.3 we describe an approach working for $\phi \in W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$ which is completely different than in dimensions 1, 3 and yields a faster proof and a better control. Such approach was first considered in [72]. This is based on the existence of controlled Hopf lifts. The result is (see Corollary 9.22) that there exists a $L^{2,\infty}$ -controlled lifting $\tilde{\phi} : \mathbb{S}^2 \rightarrow \mathbb{S}^3$ i.e. a function such that $H \circ \tilde{\phi} = \phi$ where $H : \mathbb{S}^3 \rightarrow \mathbb{S}^2$ is the Hopf fibration and we have the control

$$\|\nabla \tilde{\phi}\|_{L^{2,\infty}} \leq C \|\nabla \phi\|_{L^2} (1 + \|\nabla \phi\|_{L^2}).$$

The analogous controlled lift exists also for $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^2)$, whereas for $2 \leq p < 3$ we have a control on the L^p -norm of the lift instead of the $L^{p,\infty}$ one, cfr. Proposition 9.8. This lift allows to prove, along the same lines, Theorem 9.6 and Theorem 9.7.

The gist of the proof is the following. Once we have the controlled lift indeed, the lifted map takes values into a sphere of a higher dimension. This allows a wider range of application to the projection trick of Prop. 9.10 (projection trick) or of its Lorentz space analogue of Prop. 9.23 (projection trick 2).

After having extended the lift, re-projecting the extension to S^2 via the Hopf map maintains the gradient estimates. This is due to the fact that the Hopf fibration is a submersion (cfr. (9.4)) and our lift can be taken such that also the “vertical” component η is controlled.

The existence of nonlinear liftings has been so far very active regarding S^1 -valued maps (see e.g. [28], [23] and the references therein). Looking also

at higher dimensional analogues seems very promising in relation to extension results.

9.2.5 Small energy extension with estimate

As for the case of curvatures over bundles with a compact Lie group, the small energy regime allows a kind of linearization of the problem and gives estimates which are better than what expected in general. We obtain in particular an estimate in $W^{1,4}$ instead of $W^{1,(4,\infty)}$ for the extension, provided that the norm of the boundary trace is small:

Proposition 9.14 (see Thm. 9.29). *There is a constant $\epsilon_0 > 0$ and a finite constant C such that if*

$$\int_{\mathbb{S}^3} |\nabla\phi|^3 \leq \epsilon_0, \phi : \mathbb{S}^3 \rightarrow \mathbb{S}^3,$$

then there exists $u \in W^{1,4}(B^4, \mathbb{S}^3)$ such that

$$u = \phi \text{ on } \partial B^4 \text{ in the sense of traces and } \|\nabla u\|_{L^4(B^4)} \leq C \|\nabla\phi\|_{L^3(\mathbb{S}^3)}.$$

This is part of our proof of Theorem 9.4 and is proved in Section 9.4.2 using a method in the spirit of [132], developed in Appendix F.1.

9.2.6 Existence of $W^{1,4}$ -extension without norm bounds

As for the case of global gauges, we can in general obtain $W^{1,4}(B^4, \mathbb{S}^3)$ -extensions once we give up the requirement to have a norm control of the extension like in Theorem 9.4. This phenomenon represents one example of situations in which function spaces have a behavior which is more complex than what can be detected by only looking at their norms.

Proposition 9.15. *If $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ then its topological degree is well-defined, cfr. [114] and [140]. Suppose then that $\deg\phi = 0$.*

Then there exists $u \in W^{1,4}(B^4, \mathbb{S}^3)$ such that

$$u = \phi \text{ on } \partial B^4 \text{ in the sense of traces.}$$

Proof. We use the extension as in the Section 9.4.1. The construction using Lemma 9.28(Courant-Lebesgue analogue) is done on a series of domains

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$B(x_i, \rho_i) \cap B^4$ where $x_i \in \partial B^4$, $\rho_i \in [\rho_0, 2\rho_0]$ for the choice

$$\rho_0 := \inf \left\{ \rho > 0 \text{ s.t. } \exists x_0 \in \partial B^4, \int_{B(x_0, 2\rho) \cap \partial B^4} |\nabla \phi|^3 \geq \epsilon_0 \right\}.$$

Note that we have no a priori control on how small ρ_0 could get, but it cannot be zero for a fixed ϕ . Then a Lipschitz extension $u : \mathcal{R} \rightarrow \mathbb{S}^3$ to a Lipschitz region \mathcal{R} included between $B^4 \setminus B_{1-2\rho_0}$ and $B^4 \setminus B_{1-\rho_0}$ would exist as in Section 9.4.1 and such u will also be Lipschitz (with constant blowing up at the rate $\sim \rho_0^{-1}$) and would have degree zero (the preservation of degree follows because the extension used in the construction preserves the homotopy type, cfr [140]). In particular we can do a further Lipschitz (thus $W^{1,4}$) extension to the interior of $B^4 \setminus \mathcal{R}$. This provides the desired u . \square

The proof of the above proposition is constructive, and no hint that the construction is optimal is available. In the next section we prove that actually no general bound in $W^{1,4}$ can be achieved, because of the intervention of the topological degree, much as in the case of $SU(2)$ -instantons.

9.2.7 Impossibility of $W^{1,4}$ -bounds for an extension

Proposition 9.16. *There exists no finite function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for each $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ there exists a function $u \in W^{1,4}(B^4, \mathbb{S}^3)$ satisfying*

$$u = \phi \text{ on } \partial B^4 \text{ in the sense of traces and } \|\nabla u\|_{L^4(B^4)} \leq f(\|\nabla \phi\|_{L^3(\mathbb{S}^3)}).$$

Proof. We recall the robustness of degree under strong convergence in $W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ (see [114, 140] and also [32, 33]). Consider $\phi = id_{\mathbb{S}^3}$, which has degree 1. Suppose an extension $u : B^4 \rightarrow \mathbb{S}^3$ to ϕ would exist with $\|u\|_{W^{1,4}} \leq C'$. It will be possible to approximate in $W^{1,4}$ -norm u by functions $u_i \in C^\infty(B^4, \mathbb{S}^3)$, since smooth functions are dense in $W^{1,4}(B^4, \mathbb{S}^3)$. In particular the degrees $\deg(\phi_i)$ of $\phi_i = u_i|_{\partial B^4}$ will have to be zero. Thus it is not possible that $\phi_i \xrightarrow{W^{1,3}} \phi$ because the degree is preserved under strong $W^{1,3}$ -convergence).

This proves the absence of a continuous extension operator. To prove that also boundedness is impossible, we use a slightly different argument.

Consider $\phi_0 \in W^{1,3} \cap C^\infty(\mathbb{S}^3, \mathbb{S}^3)$ which is a perturbation of the identity equal to the south pole S in a neighborhood N_S of S . Then consider a Möbius transformation $F : \mathbb{S}^3 \rightarrow \mathbb{S}^3$ such that $F^{-1}(N_S)$ includes the lower hemisphere, and consider $\phi' = \phi_0 \circ F$, $\phi'' = \phi_0 \circ (-F)$. Then identifying $\mathbb{S}^3 \sim SU(2)$ such

that $S \sim id_{SU(2)}$ use the group operation to define $\phi = \phi' \phi''$. Note that $\|\phi\|_{W^{1,3}} \leq 2\|\phi_0\|_{W^{1,3}}$ since the conformal maps $F, -F$ preserve the energy; moreover ϕ has zero degree.

Let F_n be a family of Möbius transformations symmetric about S and such that they concentrate more and more near S (with the notation of Appendix F.3 we may take $F_n := F_{v_n}$ for $v_n = (1 - 1/n)S$). Define $\phi'_n := \phi' \circ F_n$ and $\phi_n = \phi'_n \phi''$. It is clear by conformal invariance of the $W^{1,3}$ -energy that ϕ_n have constant energy. They converge weakly to ϕ'' and have degree zero.

Call u_n the extension of ϕ_n and suppose that $\|u_n\|_{W^{1,4}} \leq C$ independent of n . We may suppose that $u_n \xrightarrow{W^{1,4}} u_\infty \in W^{1,4}(B^4, \mathbb{S}^3)$ and we obtain $u_\infty|_{\partial B^4} = \phi''$ in the sense of traces. We then apply the result of [140] (see also [114]) which in this case says that the 3-dimensional homotopy class passes to the limit under bounded sequential weak $W^{1,4}(B^4, \mathbb{S}^3)$ -limits. We obtain again a contradiction to boundedness since $\deg(\phi'') = -1$ whereas the same degree is zero for the maps ϕ_n . \square

9.2.8 Moving frames and their gauges

We describe here a lifting problem arising in the theory of moving frames on 2-dimensional surfaces, where the Lorentz spaces appear again in the optimal estimates. The model question is as follows:

Open Problem 13. *Suppose given a map (representing the normal vector of an immersed surface) $\vec{n} \in W^{1,2}(D^2, \mathbb{S}^2)$. Does there exist a $W^{1,2}$ controlled trivialization $\vec{e} = (\vec{e}_1, \vec{e}_2)$ of the pullback bundle $\vec{n}^{-1}T\mathbb{S}^2$? A trivialization is defined by two vector fields $\vec{e}_1, \vec{e}_2 \in W^{1,2}(D^2, \mathbb{S}^2)$ such that the pointwise constraints $|\vec{e}_1| = |\vec{e}_2| = 1, \vec{e}_1 \cdot \vec{e}_2 = 0$ are satisfied almost everywhere and $\vec{n} = \vec{e}_1 \times \vec{e}_2$.*

This problem behaves like the one of global controlled gauges, namely for small energy a lift exists and is controlled, and for large energy lifts can be found but with no general control. Ulenbeck's ϵ -regularity estimate is mirrored in the following Theorem. This result, was proved initially by F. Hélein under the hypothesis $\|\nabla \vec{n}\|_{L^2} \leq C$ and improved by Y. Bernard and T. Rivière who proved that it is enough to assume a smallness condition in weak- L^2 :

Theorem 9.17 ([16] Lemma IV.3, cfr. also [75] Lemma 5.1.4). *There exists ϵ_0 such that if $\|\nabla \vec{n}\|_{L^{2,\infty}} \leq \epsilon_0$ then there exists a trivialization, with the control*

$$\|\nabla \vec{e}_1\|_{L^2} + \|\nabla \vec{e}_2\|_{L^2} \leq C \|\nabla \vec{n}\|_{L^2}.$$

and

$$\|\nabla \vec{e}_1\|_{L^{2,\infty}} + \|\nabla \vec{e}_2\|_{L^{2,\infty}} \leq C \|\nabla \vec{n}\|_{L^{2,\infty}}.$$

Note that for the improvement above, the L^2 -energy might blow up, yet still control the energy of the trivialization, as long as we stay small in Lorentz norm. It would be interesting to explore this kind of phenomenon also for curvatures in higher dimensions like in our setting.

The bad behavior in case of large energy regime starts at the energy level 8π (and this is optimal, see [86]). This number has an evident topological significance, because if \vec{n} is homotopically nontrivial, i.e. parameterizes a non-contractible 2-cell of \mathbb{S}^2 then $4\pi = |\mathbb{S}^2| \leq \int_{D^2} u^*(d\text{Vol}_{\mathbb{S}^2}) \leq \frac{1}{2} \int_{D^2} |\nabla \vec{n}|^2$, so 8π is the smallest energy of a topologically nontrivial \vec{n} . We also have the following lemma, similar to Section 9.2.7:

Lemma 9.18. *For $\int |\nabla \vec{n}|^2 > 8\pi$ there can be no controlled $W^{1,2}$ trivialization \vec{e} .*

Sketch of proof: We choose \vec{n} mapping a neighborhood $D^2 \setminus B_r := N_1$ for small r to the south pole of \mathbb{S}^2 , has degree 1 and equals a conformal map outside a small neighborhood $N_2 \ni N_1$. Such \vec{n} exists with energy as close as desired to 8π , independently of r by conformal invariance of the energy. Supposing a trivialization $\vec{e} = (\vec{e}_1, \vec{e}_2)$ exists, on N_1 it will span the “horizontal” 2-plane of \mathbb{R}^3 which is perpendicular to $S = (0, 0, -1)$. On circles $\partial B_\rho, \rho > r$ by Fubini theorem for almost all ϵ we will have that $\vec{e}_i, i = 1, 2$ will be $W^{1,2}$ thus C^0 and they have values in the equator of \mathbb{S}^2 . By well-posedness of the topological degree and since \vec{n} is nontrivial in homotopy, we obtain that each e_i will make a full turn on each ∂B_r . This gives that $\int_{\partial B_r} |\nabla \vec{e}_i| \geq 1$ on ∂B_r and by Jensen’s inequality we obtain

$$\int_{D^2 \setminus B_r} |\nabla \vec{e}_i|^2 \geq C \int_r^1 \frac{1}{\rho^2} \rho d\rho \geq C \left| \log \frac{1}{r} \right|$$

since there is no positive lower bound of $r > 0$, we see that we cannot have a controlled trivialization. □

There is an analogue also of our $W^{1,(4,\infty)}$ extension result here, and it corresponds to taking the so-called “Coulomb frames”. The result is a general estimate with no restriction on \vec{n} , but with the Lorentz norm $L^{(2,\infty)}$ instead of the L^2 norm (this estimate follows from Wente’s [137] inequality using [1]):

Proposition 9.19 ([109], VII.6.3). *Let $\vec{n} \in W^{1,2}(D^2, \mathbb{S}^2)$. Then there exist a trivialization \vec{e} belonging to $W^{1,(2,\infty)}$ exists, which satisfies the Coulomb condition*

$$\operatorname{div}\langle \vec{e}_1, \nabla \vec{e}_2 \rangle = 0$$

and the control

$$\|\nabla \vec{e}_1\|_{L(2,\infty)} + \|\nabla \vec{e}_2\|_{L(2,\infty)} \lesssim \|\nabla \vec{n}\|_{L^2} + \|\nabla \vec{n}\|_{L^2}^2.$$

9.3 The Hopf lift extension

We prove here the Theorem 9.6. We consider a fixed $\phi \in W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$ and we need to construct an extension $u \in W^{1,(3,\infty)}(B^3, \mathbb{S}^2)$ such that

$$\|u\|_{W^{1,(3,\infty)}(B^3)} \lesssim \|\phi\|_{W^{1,2}(\mathbb{S}^2)}(1 + \|\phi\|_{W^{1,2}(\mathbb{S}^2)}),$$

where the implicit constant is independent of ϕ .

The strategy of proof uses a construction based on the Hopf fibration which has been introduced in [72]. The same strategy has been later on performed in [21] for proving similar lifting results as in [72]. In the smooth case we will first lift $\phi : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ to $\tilde{\phi} : \mathbb{S}^2 \rightarrow \mathbb{S}^3$ such that $H \circ \tilde{\phi} = \phi$ where $H : \mathbb{S}^2 \rightarrow \mathbb{S}^3$ is the Hopf fibration. Then we will extend $\tilde{\phi}$ by using a Lorentz analogue of 9.10 (projection trick), working with similar conditions on dimensions and exponents. projecting back to \mathbb{S}^2 via H will keep the estimates.

Before the proof, we recall some properties of the map H .

9.3.1 Facts about the Hopf fibration

Identifying \mathbb{S}^3 with the unit sphere of \mathbb{C}^2 , with complex coordinates (Z, W) , the Hopf projection is $H(Z, W) = Z/\bar{W}$ and its fibers are maximal circles. This gives a function with values in $\mathbb{C} \cup \{\infty\} \simeq \mathbb{S}^2$. If we look at $\mathbb{S}^3 \subset \mathbb{R}^4$ with the inherited coordinates (x_1, x_2, x_3, x_4) then we can identify

$$H^*\omega_{\mathbb{S}^2} = d\alpha, \quad \text{for } \alpha = \frac{1}{2}(x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3). \quad (9.1)$$

Here $\omega_{\mathbb{S}^2}$ is a constant multiple of the volume form of \mathbb{S}^2 . Since $\mathbb{S}^1 \sim U(1)$ we can regard $\mathbb{S}^3 \xrightarrow{H} \mathbb{S}^2$ as a principal $U(1)$ -bundle $P \rightarrow \mathbb{S}^2$.

Let $\phi : \mathbb{C} \rightarrow \mathbb{S}^2$ be a smooth function. Then $d(\phi^*\omega_{\mathbb{S}^2}) = 0$ because $\Omega^3(\mathbb{R}^2 \simeq \mathbb{C}) = \{0\}$. Since $H_{dR}^2(\mathbb{C}) = 0$ there exists a 1-form η such that

$$d\eta = \phi^*\omega_{\mathbb{S}^2}. \tag{9.2}$$

We also note that for a smooth $\phi : \mathbb{C} \rightarrow \mathbb{S}^2$ the pullback of the $U(1)$ -bundle P is trivial, since \mathbb{R}^2 is contractible. A trivialization of the bundle $\phi^*P \rightarrow \mathbb{C}$ can be identified with a lift $\tilde{\phi}$ of ϕ . From the equation (9.1) we can deduce that $d\eta = \tilde{\phi}^*H^*\omega_{\mathbb{S}^2} = \tilde{\phi}^*d\alpha = d(\tilde{\phi}^*\alpha)$ and again there exists a 1-form $\tilde{\eta}$ as in (9.2), defined by

$$\tilde{\eta} = \tilde{\phi}^*\alpha. \tag{9.3}$$

$\tilde{\eta}$ coincides with η up to adding an exact form $d\theta$: we have $\tilde{\phi}^*\alpha - \eta = d\theta$. If we come back to the bundle point of view then $d\theta$ represents the effect of change of coordinates of the trivialization giving $\tilde{\phi}$, i.e. of a change of gauge. We have then $\eta = \tilde{\phi}^*\alpha - d\theta = (e^{-i\theta}\tilde{\phi})^*\alpha$, where the action of $e^{-i\theta}$ is intended as an $U(1)$ -gauge change and $\theta : \mathbb{C} \rightarrow \mathbb{R}$ is determined up to a constant. Moreover, since DH is an isometry between the orthogonal complement of the tangent space of the fiber $T_pH^{-1}(H(p))$ and $T_p\mathbb{S}^2$, we also obtain the following norm identity:

$$|D\tilde{\phi}|^2 = |\tilde{\eta}|^2 + |D\phi|^2. \tag{9.4}$$

9.3.2 Hopf lift with estimates

We start the proof of Theorem 9.6 with the following first step:

Proposition 9.20. *Suppose $\phi \in W^{1,2}(\mathbb{C}, \mathbb{S}^2)$. Then there exists a lifting $\tilde{\phi} : \mathbb{C} \rightarrow \mathbb{S}^3$ such that $H \circ \tilde{\phi} = \phi$ and there exists a universal constant C such that*

$$\|\nabla\tilde{\phi}\|_{L^{2,\infty}} \leq C\|\nabla\phi\|_{L^2}(1 + \|\nabla\phi\|_{L^2}).$$

Proof of Proposition 9.20: The proof is divided in two steps.

Step 1. *Constructions in the smooth case.* We have seen that, at least in the smooth case, constructing a 1-form η as in (9.2) is equivalent to the construction of a lift $\tilde{\phi} : \mathbb{C} \rightarrow \mathbb{S}^3$. We now observe that such a 1-form can be in turn easily constructed, by inverting the Laplacian on \mathbb{C} , via its Green kernel, which is of the form $K(x) = -\gamma \log|x|$. In particular $K \in W^{1,(2,\infty)}$, which is the reason why this norm appears). First note that $dd^*(K * \beta) = 0$ for a smooth L^1 -integrable 2-form β on \mathbb{C} . We can then use this formula for $\beta = \phi^*\omega_{\mathbb{S}^2}$, and taking into account the fact that ∇K is in $L^{2,\infty}$, by the Lorentz space Young inequality (see [63]) we obtain that the 1-form η defined as

$$\eta := d^*[K * (\phi^*\omega_{\mathbb{S}^2})], \quad \eta \rightarrow 0 \text{ at infinity} \tag{9.5}$$

satisfies (9.2) and the estimates

$$\|\eta\|_{L^{2,\infty}} \lesssim \|\phi^* \omega_{\mathbb{S}^2}\|_{L^1} \lesssim \|D\phi\|_{L^2}^2 \|\phi\|_{L^\infty} \simeq \|D\phi\|_{L^2}^2. \quad (9.6)$$

We have mentioned where to find the proof that η corresponds up to a unitary transformation to a lift $\tilde{\phi}$, and from (9.4) and from (9.6) we also obtain the estimate for $\tilde{\phi}$ which reads as follows:

$$\|D\tilde{\phi}\|_{L^{2,\infty}} \lesssim \|\eta\|_{L^{2,\infty}} + \|D\phi\|_{L^2} \lesssim \|D\phi\|_{L^2}(1 + \|D\phi\|_{L^2}). \quad (9.7)$$

Step 2. *Extending the constructions to $W^{1,2}$.* The results obtained so far apply for $\phi \in C^\infty(\mathbb{C}, \mathbb{S}^2)$. We use the by now well-known fact that while not dense in the strong topology, the functions in $C^\infty(\mathbb{C}, \mathbb{S}^2)$ are instead *dense with respect to the weak sequential convergence* (see [19, 65]). The constraint of u_n having values in \mathbb{S}^2 , as well as the constraint $\tilde{\phi}_n \circ H = \phi_n$ for the $\tilde{\phi}_n$, are pointwise constraints (note indeed that the function H is smooth), so they are preserved under weak convergence $\phi_n \rightharpoonup \phi \in W^{1,2}$. Now we state the only less classical point in the following lemma.

Lemma 9.21. *$L^{2,\infty}$ -estimates are preserved under weak convergence in L^2 . In other words, if $f_n \in L^2$ are weakly convergent to $f \in L^2$ then $\|f\|_{L^{2,\infty}} \leq \liminf_{n \rightarrow \infty} \|f_n\|_{L^{2,\infty}}$.*

Proof of the lemma: We observe that a positive answer to this question cannot directly and trivially be obtained by interpolation, since L^∞ -norm is not lower semicontinuous with respect to weak convergence in L^2 . We thus proceed by duality, namely we note that

$$L^{(2,\infty)} = (L^{(2,1)})' \quad \text{and} \quad L^{(2,1)} \subset L^2.$$

Therefore $\langle f_n, \phi \rangle \rightarrow \langle f, \phi \rangle$ for all $\phi \in L^{(2,1)}$ and by usual Banach space theory we obtain the thesis. \square

Applying the Lemma, we obtain the desired estimate via Bethuel's weak density result. \square

We observe that given a map $\phi \in W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$, we can obtain a map $u : \mathbb{C} \rightarrow \mathbb{S}^2$ having the same norm by composing with the inverse stereographic projection $\Psi^{-1} : \mathbb{C} \rightarrow \mathbb{S}^2$: we use here the facts that the exponent 2 is equal to the dimension, and that Ψ is conformal. In a similar way, having constructed a lift $\tilde{u} : \mathbb{C} \rightarrow \mathbb{S}^3$, we obtain automatically a lift $\tilde{\phi}$ of ϕ by composing back with S . The same reasoning using conformality also implies that the $L^{2,\infty}$ -norm of the gradient of $\tilde{\phi}$ is preserved. This proves the following:

Corollary 9.22. *Suppose $\phi \in W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$. Then there exists a lifting $\tilde{\phi} : \mathbb{S}^2 \rightarrow \mathbb{S}^3$ such that $H \circ \tilde{\phi} = \phi$ and there exists a universal constant C such that*

$$\|\nabla \tilde{\phi}\|_{L^{2,\infty}} \leq C \|\nabla \phi\|_{L^2} (1 + \|\nabla \phi\|_{L^2}).$$

9.3.3 Projection and wise choice of the point

To proceed in our strategy for the proof of Theorem 9.6, we use a version of the projection trick of Section 9.2.1.

Proposition 9.23 (projection trick 2). *Suppose that $\tilde{\phi} \in W^{1,(2,\infty)}(\mathbb{S}^2, \mathbb{S}^3)$. Then there exists a function $\tilde{u} : B^3 \rightarrow \mathbb{S}^3$, such that $\tilde{u}|_{\partial B^3 \setminus \mathbb{S}^2} = \tilde{\phi}$ and satisfying the following bounds for some universal constant C*

$$\|\tilde{u}\|_{W^{1,(3,\infty)}(B^3)} \leq C \|\tilde{\phi}\|_{W^{1,(2,\infty)}(\mathbb{S}^2)}.$$

Proof. We proceed in two steps, of which the first one introduces the $W^{1,(3,\infty)}$ -norm estimate, and the second one ensures that the constraint of having values in \mathbb{S}^3 can be preserved.

Step 1. Harmonic extension. Consider a solution \tilde{u} of the following equation:

$$\begin{cases} \Delta \tilde{u} = 0 \text{ on } B^3, \\ \tilde{u} = \tilde{\phi} \text{ on } \partial B^3. \end{cases} \quad (9.8)$$

By using the Poisson kernel estimates we obtain that $\tilde{u} \in W^{1,(3,\infty)}(B^3, B^4)$ and

$$\|\nabla \tilde{u}\|_{L^{(3,\infty)}} \lesssim \|\nabla \tilde{\phi}\|_{L^{(2,\infty)}}. \quad (9.9)$$

Step 2. Projection in the target. We now correct the fact that \tilde{u} has values not in \mathbb{S}^3 but in its convex hull B^4 . For $a \in B_{1/2}^4$ we note π_a the radial projection $\pi_a : B^4 \rightarrow \mathbb{S}^3$ of center a , i.e.

$$\pi_a(x) := a + t_{a,x}(x - a), \text{ for } t_{a,x} \geq 0 \text{ such that } |\pi_a(x)| = 1.$$

In order to estimate the norm of $u_a := \pi_a \circ \tilde{u}$ we note that

$$|\nabla(\pi_a \circ \tilde{u})|(x) \lesssim \frac{|\nabla \tilde{u}(x)|}{|u(x) - a|},$$

with an implicit constant bounded by 4 as long as $a \in B_{1/2}^4$. We just estimate the L^p -norm of ∇u_a for $p \in [1, 4[$. We note that $\int_{B_{1/2}} |\tilde{u}(x) - a|^{-p} da$ is bounded

for all such p by a number C_p independent of x , therefore by changing the order of integration and applying Fubini, we obtain

$$\int_{B_{1/2}} \int_{B_1} |\nabla u_a(x)|^p dx da \leq C_p \int_{B_1} |\nabla \tilde{u}(x)|^p \int_{B_{1/2}} |\tilde{u}(x) - a|^{-p} da \leq C_p \|\nabla \tilde{u}\|_p^p.$$

In other words, the assignment $a \mapsto u_a$ gives a map whose $L_a^1(B_{1/2}, W_x^{1,p}(B^3, \mathbb{S}^3))$ -norm is bounded by the L^p -norm of $\nabla \tilde{u}$ for $p \in [1, 4[$. First observe that by Lions-Peetre reiteration $L^{(3,\infty)}$ is an interpolation between L^{p_0} and L^{p_1} with $3 \in]p_0, p_1[\subset]1, 4[$. We now use the nonlinear interpolation theorem of Tartar. Call $U(a, x) := \frac{\nabla \tilde{u}(x)}{|\tilde{u}(x) - a|}$. We know that the map $u \mapsto U$ is bounded between W^{1,p_i} and L^{p_i} for $i = 0, 1$. In order to prove that it also satisfies

$$\sup_{\lambda > 0} \lambda^3 \left| \left\{ (x, a) \in B_1 \times B_{1/2} : \frac{|\nabla u(x)|}{|u(x) - a|} > \lambda \right\} \right| = \|U\|_{L^{(3,\infty)}}^3 \lesssim \|\tilde{u}\|_{W^{1,(3,\infty)}}^3 \quad (9.10)$$

we will check the local estimate

$$\left\| \frac{\nabla u(x)}{|u(x) - a|} - \frac{\nabla v(x)}{|v(x) - a|} \right\|_{L^{p_1}} \lesssim \|u - v\|_{L^{p_1}}.$$

This follows since

$$\begin{aligned} & \int_{B_1} \int_{B_{1/2}} \left| \frac{\nabla u(x)}{|u(x) - a|} - \frac{\nabla v(x)}{|v(x) - a|} \right|^{p_1} \\ & \lesssim \int_{B_1} |\nabla u - \nabla v|^{p_1} \int_{B_{1/2}} (|u(x) - a|^{-p_1} + |v(x) - a|^{-p_1}) da dx \end{aligned}$$

and to the second factor the same estimates as before apply, uniformly in x . Thus (9.10) holds. From (9.10) it easily follows that there exists $a \in B_{1/2}$ for which

$$\|\nabla u_a\|_{L^{(3,\infty)}(B_1)} \lesssim \|\tilde{u}\|_{W^{1,(3,\infty)}}. \quad (9.11)$$

Combining (9.9) and (9.11), we obtain the claim of the proposition, for $\hat{u} := u_a$. \square

9.3.4 End of proof

Proof of Theorem 9.6: Apply consecutively Corollary 9.22 and Prop. 9.23 (projection trick 2). For this \hat{u} as in Prop. 9.23 we can then consider $u := H \circ u_a : B^3 \rightarrow \mathbb{S}^2$. Since H is Lipschitz we obtain the pointwise estimate

$$|\nabla u| \lesssim |\nabla u_a|. \quad (9.12)$$

Combining this with the estimates of Corollary 9.22 and Prop. 9.23 (projection trick 2) we obtain the thesis of Theorem 9.6. \square

9.3.5 Modification of proof in the case of $W^{1,p}(\mathbb{S}^m, \mathbb{S}^2)$

In this section we prove Theorem 9.7 and Proposition 9.8.

Proof of Theorem 9.7 and of proposition 9.8. We consider here $n = 2 < m$ and $\frac{3m}{m+1} \leq p < \frac{4m}{m+1}$ as in Proposition 9.8. We will use the fact that such p is always > 2 . The construction of the 1-form η satisfying (9.3) and (9.4) can be done in a completely analogous way if the domain is \mathbb{R}^m , $m \geq 3$. The only difference is that in such case the Laplacian on 2-forms like $\phi^*\omega_{\mathbb{S}^2}$ has the form $\Delta = d^*d + dd^*$ where the first part does not vanish anymore. In this case however we may still solve

$$\begin{cases} d\eta = \phi^*\omega_{\mathbb{S}^2}, \\ d^*\eta = 0, \\ \eta(x) \rightarrow 0, \quad |x| \rightarrow \infty. \end{cases}$$

If $\phi \in W^{1,p}(\mathbb{R}^m, \mathbb{S}^2)$ and since $p > 2$ we then have

$$\|d\eta\|_{L^{p/2}(\mathbb{R}^m)} \leq C\|\phi^*\omega_{\mathbb{S}^2}\|_{L^{p/2}(\mathbb{R}^m)} \leq C\|d\phi\|_{L^p(\mathbb{R}^m)}^2.$$

As before we have (9.4), from which we also obtain $|D\tilde{\phi}|^p \lesssim |\eta|^p + |D\phi|^p$. Passing to \mathbb{S}^m and noting that in dimension $m \geq p$ there holds

$$W^{1,p/2}(\mathbb{S}^m, \mathbb{S}^2) \hookrightarrow L^{\frac{mp}{2m-p}}(\mathbb{S}^m, \mathbb{S}^2) \hookrightarrow L^p(\mathbb{S}^m, \mathbb{S}^2)$$

we obtain

$$\|D\tilde{\phi}\|_{L^p(\mathbb{S}^m, \mathbb{S}^2)} \lesssim \|D\phi\|_{L^p(\mathbb{S}^m, \mathbb{S}^2)}^2 + \|D\phi\|_{L^p(\mathbb{S}^m, \mathbb{S}^2)}.$$

Harmonic extension and Prop. 9.10(projection trick) allow then to obtain an extension $\tilde{u} : B^{m+1} \rightarrow \mathbb{S}^2$ of $\tilde{\phi}$ such that

$$\|\nabla\tilde{u}\|_{L^{\frac{m+1}{m}p}(B^{m+1}, \mathbb{S}^3)} \lesssim \|D\tilde{\phi}\|_{L^p(\mathbb{S}^m, \mathbb{S}^3)},$$

provided $\frac{m+1}{m}p < 4$ (which is the condition appearing in Prop. 9.10(projection trick)). Composing with the Hopf map H at most decreases the norm, thus we obtain that $u := H \circ \tilde{u}$ is the desired controlled extension as in Proposition 9.8 and in Theorem 9.7 (note that for $m = 3$ the condition $\frac{m+1}{m}p < 4$ is equivalent to $p < 3$). \square

9.4 The extension theorem for $W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ maps

This section is devoted to the proof of the following theorem:

Theorem 9.24. *There exists a constant $C > 0$ with the following property. Suppose $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$. then there exists an extension $u \in W^{1,(4,\infty)}(B^4, \mathbb{S}^3)$ of ϕ such that the following estimate holds:*

$$\|\nabla u\|_{L^{4,\infty}(B^4)} \leq C \left(e^{C\|\nabla\phi\|_{L^3}^9} + e^{C\|\nabla\phi\|_{L^3}^6} \|\nabla\phi\|_{L^3} \right). \quad (9.13)$$

9.4.1 Modulus of integrability estimates

In general during our estimates we indicate by C a positive constant, which may change from line to line, and also within the same line. We start by fixing the notation for the main quantity which will be used control the energy concentration of our maps.

Definition 9.25. *If $D \subset \mathbb{R}^4$ and $f : D \rightarrow \mathbb{R}$ is measurable then let $E(f, \rho, D)$ denote the (possibly infinite) modulus of integrability of f , which is defined as*

$$E(f, \rho, D) = \sup_{x \in D} \int_{B_\rho(x) \cap D} |f|.$$

The modulus of integrability fits into a sort of elliptic estimate as follows.

Proposition 9.26 (integrability modulus estimates). *Let $\phi \in W^{1,3}(\partial B^4, \mathbb{S}^3)$ and assume that u is the solution to the following equation:*

$$\begin{cases} \Delta u = 0 & \text{on } B^4, \\ u = \phi & \text{on } \partial B^4. \end{cases}$$

Then there exists a constant C_1 independent of ϕ, ρ such that when $\rho \in]0, 1/4[$ the following inequality holds true:

$$E(|\nabla u|^4, \rho, B^4) \leq C_1 E(|\nabla\phi|^3, 2\rho, \partial B^4)^{1/3} \int_{\partial B^4} |\nabla\phi|^3. \quad (9.14)$$

Proof. We have to prove that for all $x_0 \in B^4$,

$$\int_{B_\rho(x_0) \cap B^4} |\nabla u|^4 \leq C_1 E(|\nabla\phi|^3, 2\rho, \partial B^4) \int_{\partial B^4} |\nabla\phi|^3. \quad (9.15)$$

Step 1. We prove (9.15) for $x_0 \in \partial B^4$.

$$\int_{B_\rho(x_0) \cap B^4} |\nabla u|^4 \leq C_0 E(|\nabla\phi|^3, 2\rho, \partial B^4) \int_{\partial B^4} |\nabla\phi|^3.$$

The function u can be obtained by superposition, using a cutoff function $\eta : \mathbb{S}^3 \rightarrow [0, 1]$ which equals 1 on $B_\rho(x_0) \cap \mathbb{S}^3$ and 0 outside $B_{2\rho}(x_0)$ and satisfies $|\nabla\eta| \lesssim \rho^{-1}$. We will use the functions

$$\begin{cases} \Delta u_1 = 0 & \text{on } B^4, \\ u_1 = \eta\phi := \phi_1 & \text{on } \partial B^4. \end{cases} \quad \begin{cases} \Delta u_2 = 0 & \text{on } B^4, \\ u_2 = (1 - \eta)\phi := \phi_2 & \text{on } \partial B^4. \end{cases}$$

We can estimate these two functions separately because there holds

$$\int_{B_\rho(x_0) \cap B^4} |\nabla u|^4 \lesssim \int_{B_\rho(x_0) \cap B^4} |\nabla u_1|^4 + \int_{B_\rho(x_0) \cap B^4} |\nabla u_2|^4.$$

It is convenient to estimate separately the contributions of u_1 on $S' = B_{2\rho}(x_0) \cap \mathbb{S}^3$ and of u_2 on $S'' = \mathbb{S}^3 \setminus B_\rho(x_0)$; on S'' we use the Poisson formula and on S' we use elliptic estimates.

By elliptic theory and the definition of η ,

$$\int_{B_\rho(x_0) \cap B^4} |\nabla u_1|^4 \lesssim \left(\int_{S'} |\nabla\phi|^3 \right)^{4/3}.$$

Poisson's formula gives

$$u_2(x) = C(1 - |x|^2) \int_{\partial B^4} \frac{\phi_2(y)}{|x - y|^4} dy,$$

therefore (using also the bound on η) we obtain a pointwise bound, in case $x \in B_\rho(x_0) \cap B^4, \rho < 1/4$:

$$|\nabla u_2|(x) \lesssim \rho \int_{S''} \frac{|\nabla\phi|}{|x - y|^4} dy + \int_{S''} \frac{|\phi|}{|x - y|^4} dy \lesssim \rho \int_{S''} \frac{|\nabla\phi|}{|x - y|^4} dy.$$

Patching together the estimates obtained so far, we write

$$\int_{B_\rho(x_0) \cap B^4} |\nabla u|^4 \lesssim \left(\int_{S'} |\nabla\phi|^3 \right)^{4/3} + \rho^8 \left(\int_{S''} \frac{|\nabla\phi|}{|x - y|^4} \right)^4 = I + II, \quad (9.16)$$

where the factor ρ^8 comes from the pointwise estimate for ∇u_2 keeping in mind that $|B_\rho(x_0) \cap B^4| \lesssim \rho^4$.

The first summand is estimated as needed:

$$I \leq \left(\int_{B_{2\rho}(x_0) \cap \partial B^4} |\nabla\phi|^3 \right)^{1/3} \int_{\mathbb{S}^3} |\nabla\phi|^3 \leq E(|\nabla\phi|^3, 2\rho, \partial B^4) \int_{\mathbb{S}^3} |\nabla\phi|^3.$$

To estimate II we consider a cover of S'' by (finitely many) balls $B_\rho^i = B_\rho(x_i)$ such that x_i form a maximal 2ρ -separating net and they are at distance at least ρ from x_0 . We use the estimate

$$\int_{B_{2\rho}^i} |\nabla\phi| \leq |B_{2\rho}^i| \left(\int_{B_{2\rho}^i} |\nabla\phi|^3 \right)^{1/3},$$

and the fact that for $y \in B_{2\rho}^i$ and $x \in B_\rho(x_0) \cap B^4$ there holds $|x - y| \gtrsim \text{dist}(x_i, x_0)$. The second summand of (9.16) can then be estimated as follows:

$$II \lesssim \rho^8 \left(\sum_i \text{dist}^{-4}(x_i, x_0) \rho^3 a_i^{1/3} \right)^4$$

where $a_i = \int_{B_{2\rho}^i} |\nabla\phi|^3$. We can use the expression $1/3 = 1/4 + 1/12$ for the exponent of a_i together with a Hölder inequality to obtain:

$$\begin{aligned} II &\lesssim \rho^8 \sup_i a_i^{1/3} \left(\sum_i \text{dist}^{-4}(x_i, x_0) \rho^3 a_i^{1/4} \right)^4 \\ &\lesssim \rho^{20} \left(\sup_i a_i^{1/3} \right) \left(\sum_i a_i \right) \left(\sum_i \text{dist}^{-\frac{16}{3}}(x_i, x_0) \right)^3. \end{aligned}$$

Now the first parenthesis is estimated by $\rho^{-1} E(|\nabla\phi|^3, 2\rho, \partial B^4)$, the second one by $\rho^{-3} \int_{\mathbb{S}^3} |\nabla\phi|^3$, and for the last factor we have the elementary estimate

$$\sum_i \text{dist}^{-\frac{16}{3}}(x_i, x_0) \lesssim \frac{1}{\rho^3} \int_{\mathbb{S}^3} \frac{dx}{|x - x_0|^{16/3} + \rho^{16/3}} \lesssim \rho^{-\frac{16}{3}}.$$

These new estimates give

$$\begin{aligned} II &\lesssim \rho^{20} \rho^{-1} E(|\nabla\phi|^3, 2\rho, \partial B^4) \rho^{-16} \rho^{-3} \int_{\mathbb{S}^3} |\nabla\phi|^3 \\ &\lesssim E(|\nabla\phi|^3, 2\rho, \partial B^4) \int_{\mathbb{S}^3} |\nabla\phi|^3. \end{aligned}$$

This provides the desired estimate for II , finishing the proof of (9.15) in the case $x_0 \in \partial B^4$. Note that the constants introduced in our inequalities can be chosen independent of ρ and are independent of ϕ . Thus C_0 is also independent of these data.

Step 2. We now observe that we can reduce the case of $|x_0| < 1$ to the treatment of Step 1, up to changing the constant C_0 in our estimate from Step 1.

If $|x_0| < 1 - 2\rho$ then we can directly apply the estimates for the term II of (9.16), since now the denominator $|x - y|$ in the Poisson formula will be at least ρ for all $x \in B_\rho(x_0)$.

The estimate of Step 1 also holds for $\rho > 1/4$ with the same constant. We can cover the case $|x_0| \in]1 - 2\rho, 1[$ with $\rho < 1/4$ by noticing that if $x'_0 = x_0/|x_0|$ then $B_{3\rho}(x'_0) \supset B_\rho(x_0)$ and that the measures $|\nabla\phi|^3 d\sigma, |\nabla u|^4 dx$ are doubling with constants bounded by the packing constants of \mathbb{S}^3 and of B^4 respectively, while the function $E(f, \rho, D)$ is increasing in ρ . Therefore the inequality (9.15) also holds for this last choice of x_0 up to changing C_0 by a factor depending only of the above packing constants. \square

9.4.2 Extension in the case of small energy concentration

The following two lemmas will be used for the harmonic extension of a boundary value $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ under the small concentration hypothesis of Proposition 9.26:

Lemma 9.27. *If $u \in W^{1,4}(B^4, \mathbb{R}^4)$ and $\rho \in]0, 1/2[$, $x_0 \in \partial B^4$ then there exists $\bar{\rho} \in [\rho, 2\rho]$ such that*

$$\bar{\rho} \int_{\text{int}(B^4) \cap \partial B_{\bar{\rho}}(x_0)} |\nabla u|^4 \leq C \int_{B^4 \cap B_\rho(x_0)} |\nabla u|^4.$$

Proof. We just use the mean value theorem together with the following computation:

$$\int_\rho^{2\rho} \int_{\text{int}(B^4) \cap \partial B_{\rho'}(x_0)} |\nabla u|^4 d\rho' = \int_{B_{2\rho} \setminus B_\rho(x_0)} |\nabla u|^4 \leq \int_{B^4 \cap B_\rho(x_0)} |\nabla u|^4.$$

\square

Lemma 9.28 (Courant-Lebesgue analogue). *Fix $\bar{\rho} \in]0, 1[$. There exists a constant $C > 0$ such that if $u \in W^{1,4}(B^4, \mathbb{R}^4)$ is the extension of $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ and if*

$$\bar{\rho} \int_{\text{int}(B^4) \cap \partial B_{\bar{\rho}}(x_0)} |\nabla u|^4 \leq C$$

with $x_0 \in \partial B^4$, then for almost every $x \in \partial(B^4 \cap B_{\bar{\rho}}(x_0))$ there holds

$$\text{dist}(u(x), \mathbb{S}^3) \leq \frac{1}{8}. \tag{9.17}$$

Proof. Note that the hypotheses $x_0 \in \partial B^4, \bar{\rho} < 1$ have the following two geometric consequences: (1) $\partial B^4 \cap \partial B_{\bar{\rho}}(x_0)$ has positive measure; (2) $B^4 \cap B_{\bar{\rho}}(x_0)$ is 2-bilipschitz equivalent to $B_{\bar{\rho}}$. Therefore we may just prove that (9.17) holds true on ∂B_ρ for a function such that

$$\begin{cases} \bar{\rho} \int_{\partial B_{\bar{\rho}}} |\nabla u|^4 < C, \\ |\{x : |u|(x) = 1\}| > 0. \end{cases}$$

To do this note that by definition $u(x) \in \mathbb{S}^3$ for a.e. $x \in \partial B^4$, then use the Sobolev inequality

$$\|u\|_{C^{0,1/4}(\partial B_{\bar{\rho}})}^4 \lesssim \bar{\rho} \int_{\partial B_{\bar{\rho}}} |\nabla u|^4,$$

valid in dimension 3. For C small enough we obtain (9.17). \square

The next theorem is inspired by Uhlenbeck's technique for the removal of singularities of Yang-Mills fields. We postpone its proof to Appendix F.1. See Theorem F.2 (small energy extension) for an equivalent statement.

Theorem 9.29 (Uhlenbeck analogue). *There exist two constants $\delta > 0, C > 0$ with the following property. Suppose $\psi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ such that $\|\nabla \psi\|_{L^3(\mathbb{S}^3)} \leq \delta$. Then there exists an extension $v \in W^{1,4}(B^4, \mathbb{S}^3)$ satisfying the following estimate:*

$$\|v\|_{W^{1,4}(B^4)} \leq C \|\nabla \psi\|_{L^3(\mathbb{S}^3)}.$$

The following lemma will be later applied to the restriction of u to a smaller ball $B_{1-\rho}$, where u , being harmonic, is smooth.

Lemma 9.30 (interior estimate). *Given $u \in W^{1,4} \cap C^1(B^4, B^4)$, there exists a constant C independent of u such that for half of the points $a \in B^4$ there holds*

$$\left\| \frac{1}{|u-a|} \right\|_{L^{4,\infty}(B^4)}^4 \leq C \int_{B^4} |\nabla u|^4.$$

Proof. By the co-area formula we have

$$\begin{aligned} |\{x : |u(x) - a|^{-1} > \Lambda\}| &= |u^{-1}(B_{\Lambda^{-1}}(a))| \\ &= \int_{B_{\Lambda^{-1}}(a)} \text{Card}(u^{-1}(x)) dx \\ &\leq C \int_{B^4} |\nabla u|^4. \end{aligned}$$

We then observe that the measurable positive function $F_u(x) := \text{Card}(u^{-1}(x))$ belongs to $L^1(B^4)$. The maximal function MF_u has $L^{1,\infty}$ -norm bounded by the L^1 -norm of F_u and in particular there exists a constant C independent of u such that for at least half of the points $a \in B^4$ there holds

$$\sup_{\lambda} \frac{1}{\lambda^4} \int_{B_{\lambda}(a)} F_u \leq C \int_{B^4} F_u \leq C \int_{B^4} |\nabla u|^4.$$

For such a we have, after the change of notation $\lambda = \Lambda^{-1}$, the desired estimate

$$|\{x : |u(x) - a|^{-1} > \Lambda\}| \Lambda^4 \leq C \int_{B^4} |\nabla u|^4.$$

□

We now have the right ingredients to prove our first extension result.

Theorem 9.31 (small concentration extension). *There exists a constant $\delta \in]0, 1/4[$ with the following property. For each $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$, such that the following local estimate holds with $\|\nabla\phi\|_{L^3(\mathbb{S}^3)}^3 = E$:*

$$E(|\nabla\phi|^3, 2\rho, \mathbb{S}^3) \leq \frac{\delta}{C_1 E}. \tag{9.18}$$

there exists a function $\tilde{u} \in W^{1,(4,\infty)}(B^4, \mathbb{S}^3)$ which equals ϕ on \mathbb{S}^3 in the sense of traces and satisfies

$$\|\nabla\tilde{u}\|_{L^{4,\infty}} \lesssim \frac{\|\nabla\phi\|_{L^3}^2}{\rho} + \|\nabla\phi\|_{L^3}. \tag{9.19}$$

Proof. Step 1. We first observe that the harmonic extension u of ϕ satisfies

$$|\nabla u|(x) \lesssim \frac{\|\phi\|_{W^{1,3}(\mathbb{S}^3)}}{\rho} \quad \text{for } x \in B_{1-\rho}.$$

A direct way to see this is by estimating via the Poisson formula together with Poincaré's inequality and a good covering by ρ -balls $B_j \subset \mathbb{S}^3$:

$$\begin{aligned} |\nabla u|(x) &\lesssim \rho \left(\int_{\mathbb{S}^3} \frac{|\nabla\phi|}{|x-y|^4} dy + \int_{\mathbb{S}^3} \frac{|\phi|}{|x-y|^4} dy \right) \\ &\lesssim \sum_j \frac{\int_{B_j} |\nabla\phi| + |\phi|}{d_j^4} \rho^4, && \text{where } d_j \sim \text{dist}(B_j, x) \\ &\lesssim \sum_j \left(\frac{\rho}{d_j} \right)^4 \int_{B_j} |\nabla\phi| + 1, && \text{by Poincaré} \\ &\lesssim \left(\sum_j (\rho/d_j)^6 \right)^{2/3} \left(\sum_j \left(\int_{B_j} |\nabla\phi| \right)^3 + 1 \right)^{1/3}, && \text{by Hölder} \\ &\lesssim \frac{\|\phi\|_{W^{1,3}(\mathbb{S}^3)}}{\rho}. \end{aligned}$$

To justify the last passage we observe that $\text{Card}\{j : d_j \sim 2^j \rho\} \sim 2^{4j}$ and thus the first factor in the forelast line is bounded by $\left(\sum_{j \geq 0} 2^{-2j}\right)^{2/3}$, while for the second factor of that line we use Jensen's inequality.

Step 2. We now use Lemma 9.30 and we observe that if $\pi_a : B^4 \setminus \{a\} \rightarrow \mathbb{S}^3$ is the retraction of center a then

$$|\nabla(\pi_a \circ u)| \leq C \frac{|\nabla u|}{|u - a|}.$$

In particular using Step 1 and Lemma 9.30 we obtain

$$\|\nabla(\pi_a \circ u)\|_{L^{4,\infty}} \leq \|\nabla u\|_{L^\infty} \left\| \frac{1}{|u - a|} \right\|_{L^{4,\infty}} \leq C \frac{\|\nabla \phi\|_{L^3}}{\rho} \|\nabla u\|_{L^4}. \quad (9.20)$$

Step 3. Consider a maximal cover $\{B_i\}$ of $\mathbb{S}^3 = \partial B^4$ by 4-dimensional balls of radius ρ and centers on ∂B^4 . It is possible to find a constant C depending only on the dimension such that the collection of balls of doubled radius $\{2B_i\}$ can be written as a union of C families of disjoint balls $\mathcal{F}_1, \dots, \mathcal{F}_C$.

Then apply Lemma 9.27 to each ball $B_i \in \mathcal{F}_1$. This will give a new family of balls $\{B'_i : B_i \in \mathcal{F}_1\}$ with radii between ρ and 2ρ to which it will be possible to apply Lemma 9.28 (Courant-Lebesgue analogue). Thus $\text{dist}(u(x), \partial B^4) < \frac{1}{8}$ on $\partial(B^4 \cap B'_i)$ for all B'_i . Because of the choice of \mathcal{F}_1 it also follows that the balls B'_i are disjoint.

If we choose the projection π_a of Step 2 such that $\text{dist}(a, \partial B^4) > \frac{1}{4}$ then

$$u_1^i := \pi_a \circ (u|_{\partial(B^4 \cap B'_i)}) \text{ satisfies } |\nabla u_1^i| \leq C|\nabla u| \text{ on } \partial B'_i \cap B^4$$

by the estimates of Step 2. Note that a will be fixed during the whole construction.

We extend u_1^i (denoting the extension again by u_1^i) inside $B'_i \cap B^4$ via Theorem 9.29 (Uhlenbeck analogue) obtaining a new function

$$u_1 := \begin{cases} \pi_a \circ u & \text{on } B^4 \setminus \cup B'_i, \\ u_1^i & \text{on } B'_i. \end{cases}$$

Theorem 9.29 implies that u_1 satisfies

$$\|\nabla u_1\|_{L^4(B'_i)} \leq C \left(\int_{\partial B'_i} |\nabla u_1|^3 \right)^{1/3}.$$

We can rewrite this as follows:

$$\begin{aligned} \int_{B_i \cap B^4} |\nabla u_1|^4 &\leq C \left(\int_{B_i \cap \partial B} |\nabla \phi|^3 + \int_{\text{int}(B) \cap \partial B_i} |\nabla u_1^i|^3 \right)^{4/3} \\ &\lesssim \left(\int_{B_i \cap \partial B} |\nabla \phi|^3 \right)^{4/3} + \left(\int_{\text{int}(B) \cap \partial B_i} |\nabla u_1^i|^3 \right)^{4/3}. \end{aligned} \quad (9.21)$$

We note that (using Lemma 9.28)

$$\begin{aligned} \left(\int_{\partial B_i \cap \text{int}(B)} |\nabla u_1^i|^3 \right)^{4/3} &\leq \mathcal{H}^3(\partial B_i)^{1/3} \int_{\partial B_i \cap \text{int}(B)} |\nabla u_1^i|^4 \\ &\lesssim \rho \int_{\partial B_i \cap \text{int}(B)} |\nabla u|^4 \\ &\lesssim \int_{B_i \cap B^4} |\nabla u|^4 \end{aligned} \quad (9.22)$$

therefore u_1 still satisfies (9.14) with a constant C_1 which is now changed by a universal factor.

Step 4. It is possible to repeat the same operation starting from the function u_1 and using the balls of the family \mathcal{F}_2 to obtain a function u_2 , and then do the same iteratively for all the families $\mathcal{F}_2, \dots, \mathcal{F}_C$.

Denote by \mathcal{R} the union of all the perturbed balls B'_i corresponding to the families $\mathcal{F}_1, \dots, \mathcal{F}_C$. Recall that the number of families is equal to the maximal number of overlaps of balls of different families, and depends only on the dimension. Then iterating the estimates (9.21) using (9.22) for all families \mathcal{F}_i we obtain for the last function u_C

$$\begin{aligned} \int_{\mathcal{R}} |\nabla u_C|^4 &\lesssim E(|\nabla \phi|^3, 2\rho, \mathbb{S}^3)^{1/3} \sum_i \int_{B_i \cap \partial B} |\nabla \phi|^3 + \int_{\mathcal{R}} |\nabla u|^4 \\ &\leq \|\nabla \phi\|_{L^3(\mathbb{S}^3)}^3 (E(|\nabla \phi|^3, 2\rho, \mathbb{S}^3)^{1/3} + \|\nabla \phi\|_{L^3(\mathbb{S}^3)}), \end{aligned} \quad (9.23)$$

where for the last inequality we also used the elliptic estimates for u in terms of ϕ .

Step 5. We now collect the estimate (9.20) for the part $B \setminus \mathcal{R} \subset B_{1-\rho}$ and (9.23). Observe that in general $\|f\|_{L^{4,\infty}} \lesssim \|f\|_{L^4}$ and that the $L^{4,\infty}$ -norm satisfies the triangle inequality. We obtain

$$\|\nabla \tilde{u}\|_{L^{4,\infty}} \lesssim \frac{\|\nabla \phi\|_{L^3}^2}{\rho} + \|\nabla \phi\|_{L^3} + \|\nabla \phi\|_{L^3}^{3/4} E(|\nabla \phi|^3, 2\rho, \mathbb{S}^3)^{1/12}. \quad (9.24)$$

Using the trivial estimate $E(|\nabla\phi|^3, 2\rho, \mathbb{S}^3) \leq \int_{\mathbb{S}^3} |\nabla\phi|^3$, the desired estimate follows. \square

9.4.3 The case of large energy concentration

In this section E will denote an *upper bound* for the L^3 -energy of boundary value functions ϕ . Following Theorem 9.31 we are led to divide the set of boundary value functions $W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ into two classes, based on whether or not the energy concentrates. We will do the division based on the following parameters: the energy bound E , a concentration radius ρ_E and an upper bound on the concentration A_E . ρ_E, A_E will be fixed in Section 9.4.4, depending only on E . We introduce the following two classes of “good” and “bad” boundary value functions:

$$\begin{aligned} \mathcal{G}^E &:= \{\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3) : \|\nabla\phi\|_{L^3}^3 \leq E, E_\phi \leq A_E\}, \\ \mathcal{B}^E &:= \{\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3) : \|\nabla\phi\|_{L^3}^3 \leq E, E_\phi > A_E\}. \end{aligned} \quad (9.25)$$

where

$$E_\phi := E(|\nabla\phi|^3, \rho_E, \mathbb{S}^3) \text{ for } \phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3).$$

The precise steps of our extension construction are as follows (see also the scheme (9.26)):

1. Theorem 9.31 gives a good estimate for the boundary values in \mathcal{G}^E .
2. If $\phi \in \mathcal{B}^E$ has average close to zero, i.e.

$$\left| \int_{\mathbb{S}^3} \phi \right| \leq \frac{1}{4},$$

then it is possible to write $\phi = \phi_1\phi_2$ with

$$\int_{\mathbb{S}^3} |\nabla\phi_i|^3 \leq E - A_E/2$$

(the product of \mathbb{S}^3 -valued functions is pointwise the product on $\mathbb{S}^3 \simeq SU(2)$).

3. If we are not in the two cases above, we use the functions

$$F_v(x) := -v + (1 - |v|^2)(x^* - v)^*$$

where $a^* = \frac{a}{|a|^2}$, $v \in B^4$, which form a subset of the Möbius group of B^4 . We have two cases:

(a) $\forall v \in B^4$ there holds $|\int_{\mathbb{S}^3} \phi \circ F_v| > \frac{1}{4}$, in which case

$$\tilde{u}(v) := \pi_{\mathbb{S}^3} \left(\int_{\mathbb{S}^3} \phi \circ F_v \right)$$

gives an extension of ϕ with values in \mathbb{S}^3 and satisfying

$$\|u\|_{W^{1,4}} \lesssim \|\phi\|_{W^{1,3}},$$

(b) $\exists v \in B^4$ such that $|\int_{\mathbb{S}^3} \phi \circ F_v| \leq \frac{1}{4}$, in which case we can apply the reasoning of cases (1), (2) above to $\tilde{\phi} := \phi \circ F_v$. Since F_v is conformal and $|\phi| = |\tilde{\phi}| = 1$ we have

$$\|\nabla \phi\|_{L^3} = \|\nabla \tilde{\phi}\|_{L^3}, \quad \|\phi\|_{W^{1,3}} = \|\tilde{\phi}\|_{W^{1,3}}.$$

Again we reason differently in the two cases $\tilde{\phi} \in \mathcal{G}^E$ and $\tilde{\phi} \in \mathcal{B}^E$.

4. If in case (3b) $\tilde{\phi} \in \mathcal{B}^E$ then we apply case (2) to $\tilde{\phi}$ and we can express

$$\tilde{\phi} = \tilde{\phi}_1 \tilde{\phi}_2$$

and

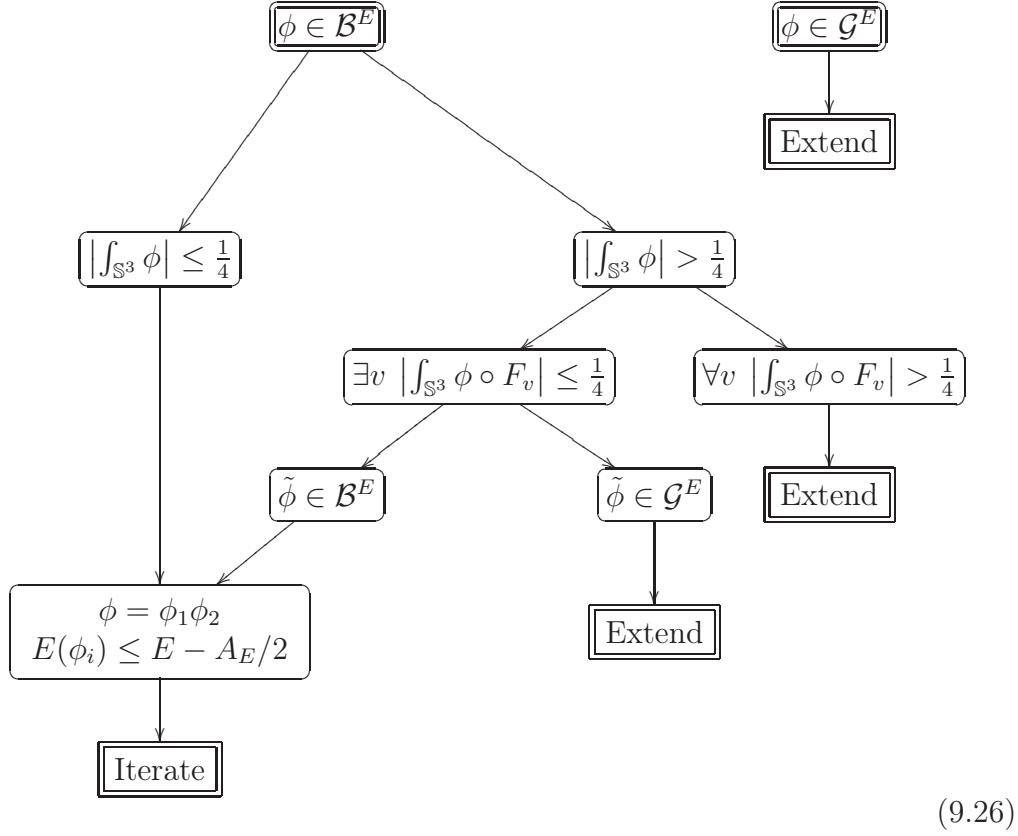
$$\phi = (\tilde{\phi}_1 \circ F_v^{-1})(\tilde{\phi}_2 \circ F_v^{-1}).$$

Then $\phi_i := \tilde{\phi}_i \circ F_v^{-1}$ are as in case (2).

5. If in case (3b) $\tilde{\phi} \in \mathcal{G}^E$ then we apply case (1) to $\tilde{\phi}$. With a careful study of the relation between the position of $v \in B^4$ relative to ∂B^4 and the parameter ρ_E , we construct

$$u \in W^{1,(4,\infty)}(B^4, \mathbb{S}^3) \text{ extending } \phi = \tilde{\phi} \circ F_v^{-1}$$

starting from the extension \tilde{u} of $\tilde{\phi}$ given in case (1).



Proposition 9.32 (balancing \Rightarrow splitting). *There exists a geometric constant C with the following property. Suppose that $\phi \in \mathcal{B}^E$ with the notations of (9.25), and assume $A_E \leq 1/C$ and $\rho_E \leq e^{-C \max\{EA_E, (EA_E)^3\}}$. Further assume that as a function in $W^{1,3}(\mathbb{S}^3, \mathbb{R}^4)$, ϕ satisfies*

$$\left| \int_{\mathbb{S}^3} \phi \right| \leq \frac{1}{4}.$$

Then identifying $\mathbb{S}^3 \sim SU(2)$ there exists a decomposition

$$\phi = \phi_1 \phi_2 \tag{9.27}$$

such that for both $i = 1, 2$ we have that

$$\int_{\mathbb{S}^3} |\nabla \phi_i|^3 < E - A_E/2. \tag{9.28}$$

Proof. We will proceed through several steps.

Step 1. Fix a concentration ball $B = B^{\mathbb{S}^3}(\rho_E, x_0)$ such that

$$\int_B |\nabla \phi|^3 > A_E. \tag{9.29}$$

Step 2. Consider dyadic rings in \mathbb{S}^3 defined as $R_i := 2^{i+1}B \setminus 2^iB$ where we denote $2^iB = B^{\mathbb{S}^3}(2^i\rho_E, x_0)$. We observe that for $N_E < -C \log_2 \rho_E$ the rings with $i \leq N_E$ stay all disjoint (we will fix N_E later). Therefore there holds

$$\sum_{i=1}^{N_E} \int_{R_i} |\nabla\phi|^3 < E.$$

By pigeonhole principle, there exists $i_0 \in \{1, \dots, N_E\}$ such that

$$\int_{R_{i_0}} |\nabla\phi|^3 < \frac{E}{N_E}.$$

Again by pigeonhole principle (using the fact that the cubes are dyadic) there exists then $t \in [2^{i_0+1}\rho_E, 2^{i_0}\rho_E]$ such that

$$t \int_{\partial B^{\mathbb{S}^3}(t, x_0)} |\nabla\phi|^3 < C \frac{E}{N_E}, \tag{9.30}$$

where C is a constant depending only on the geometry of \mathbb{S}^3 .

Step 3. Denote $B_t = B^{\mathbb{S}^3}(t, x_0)$ as in Step 2. We define the function $\tilde{\phi}_1$ via a suitable harmonic extension outside of B_t as follows:

$$\begin{cases} \tilde{\phi}_1 = \phi & \text{on } \partial B_t, \\ \Delta(\tilde{\phi}_1 \circ \Psi) = 0 & \text{on } B_1^{\mathbb{R}^3}, \end{cases}$$

where $\Psi : \mathbb{R}^3 \rightarrow \mathbb{S}^3 \setminus \{x_0\}$ is a stereographic projection composed with a dilation of \mathbb{R}^3 , such that $\Psi(B^{\mathbb{R}^3}(1, 0)) = \mathbb{S}^3 \setminus B_t$. On B_t we define $\tilde{\phi}_1 \equiv \phi$. By Hölder's inequality, using elliptic estimates and the conformality of dilations and inverse stereographic projections, we have

$$\begin{aligned} t \int_{\partial B_t} |\nabla\tilde{\phi}_1|^3 &\geq C \left(\int_{\partial B_t} |\nabla\tilde{\phi}_1|^2 \right)^{3/2} = C \left(\int_{\partial B_1^{\mathbb{R}^3}} |\nabla\tilde{\phi}_1 \circ \Psi|^2 \right)^{3/2} \\ &\geq C \int_{B_1^{\mathbb{R}^3}} |\nabla\tilde{\phi}_1 \circ \Psi|^3 = C \int_{\mathbb{S}^3 \setminus B_t} |\nabla\tilde{\phi}_1|^3. \end{aligned} \tag{9.31}$$

However, note that in general $\tilde{\phi}_1$ will have values in \mathbb{R}^4 but we can insure that they belong to \mathbb{S}^3 only on the ball B_t .

Step 4. We define then

$$\phi_1 = \pi_{\mathbb{S}^3} \circ \tilde{\phi}_1.$$

We claim that if N_E is large enough then ϕ_1 satisfies some estimates like (9.31) where the constants C are worsened just by a factor close to 1. Indeed, (9.30) together with the Sobolev embedding $W^{1,3} \rightarrow C^{0,1/3}$ (valid for 2-dimensional domains like ∂B_t) implies that $\phi|_{\partial B_t}$ stays close to a fixed point of \mathbb{S}^3 as in the proof of lemma 9.28(Courant-Lebesgue analogue). Therefore also $\phi_1 \circ \Psi|_{\partial B_1^{\mathbb{R}^3}}$ does. By mean value theorem, $\phi_1 \circ \Psi|_{B_1^{\mathbb{R}^3}}$ and thus $\tilde{\phi}_1|_{B_t}$ will not have a larger

distance to the same point of \mathbb{S}^3 . Quantitatively, there exists a geometric constant C such that if

$$\frac{E}{N_E} \leq C \quad (9.32)$$

then

$$\text{dist}(\tilde{\phi}_1, \mathbb{S}^3) \leq 1/2.$$

This implies via the pointwise bound

$$|\nabla(\pi_{\mathbb{S}^3} \circ f)| \leq C \frac{|\nabla f|}{|f|}$$

that pointwise a.e. there holds the following estimate

$$|\nabla \phi_1| \leq C |\nabla \tilde{\phi}_1|,$$

which proves our claim. This claim together with the estimates (9.31) and (9.30) implies the following bound, valid under condition (9.32):

$$\int_{\mathbb{S}^3 \setminus B_t} |\nabla \phi_1|^3 \leq C \frac{E}{N_E}. \quad (9.33)$$

Step 5. We now estimate from below the energy of $\phi|_{\mathbb{S}^3 \setminus B_t}$. Denote by $\bar{\phi}_\Omega$ the average of ϕ on a domain $\Omega \subset \mathbb{S}^3$. First we use the Poincaré inequality on $\mathbb{S}^3 \setminus B_t$ and the fact that $|\phi| \equiv 1$ almost everywhere.

$$\begin{aligned} \int_{\mathbb{S}^3 \setminus B_t} |\nabla \phi|^3 &\gtrsim \int_{\mathbb{S}^3 \setminus B_t} |\phi - \bar{\phi}_{\mathbb{S}^3 \setminus B_t}|^3 \gtrsim \left(\int_{\mathbb{S}^3 \setminus B_t} |\phi - \bar{\phi}_{\mathbb{S}^3 \setminus B_t}| \right)^3 \\ &\gtrsim (|\mathbb{S}^3 \setminus B_t| (1 - |\bar{\phi}_{\mathbb{S}^3 \setminus B_t}|))^3. \end{aligned} \quad (9.34)$$

Using the fact that $|\bar{\phi}_{\mathbb{S}^3}| \leq \frac{1}{4}$ and the triangle inequality we have

$$|\mathbb{S}^3 \setminus B_t| |\bar{\phi}_{\mathbb{S}^3 \setminus B_t}| \leq \frac{1}{4} |\mathbb{S}^3| + |B_t| |\bar{\phi}_{B_t}|. \quad (9.35)$$

(9.34) and (9.35) together with the estimate $|\bar{\phi}_{B_t}| \leq 1$ give

$$\int_{\mathbb{S}^3 \setminus B_t} |\nabla \phi|^3 \geq \frac{1}{C} \left(\frac{3}{4} |\mathbb{S}^3| - 2|B_t| \right)^3. \quad (9.36)$$

From this inequality and since we assumed A_E to be small, we obtain

$$\int_{\mathbb{S}^3 \setminus B_t} |\nabla \phi|^3 \geq A_E \quad \text{if } t < C, \quad (9.37)$$

for some geometric constant C .

Step 6. We now define $\phi_2 := \phi_1^{-1}\phi$ where the pointwise product uses the group operation on $\mathbb{S}^3 \sim SU(2)$. Observe that since $|\phi| = |\phi_1| = 1$ a.e.,

$$|\nabla(\phi_1^{-1}\phi)| = |\phi^{-1}\nabla\phi_1\phi_1^{-1}\phi + \phi_1^{-1}\nabla\phi| \leq |\nabla\phi| + |\nabla\phi_1|.$$

We then apply this last inequality together with Hölder's inequality to obtain that if the number of rings N_E in (9.33) is so large that $\|\nabla\phi_1\|_{L^3(\mathbb{S}^3 \setminus B_t)} \leq \|\nabla\phi\|_{L^3(\mathbb{S}^3 \setminus B_t)}$ then

$$\int_{\mathbb{S}^3 \setminus B_t} |\nabla\phi_2|^3 \leq \int_{\mathbb{S}^3 \setminus B_t} |\nabla\phi|^3 + 7 \left(\int_{\mathbb{S}^3 \setminus B_t} |\nabla\phi_1|^3 \right)^{\frac{1}{3}} \left(\int_{\mathbb{S}^3 \setminus B_t} |\nabla\phi|^3 \right)^{\frac{2}{3}}.$$

By using (9.37) and (9.33) we then obtain (under the hypotheses (9.32) and $A_E \leq 1/C$ needed for these inequalities to hold)

$$\int_{\mathbb{S}^3 \setminus B_t} |\nabla\phi_2|^3 \leq \int_{\mathbb{S}^3 \setminus B_t} |\nabla\phi|^3 + C \frac{E}{N_E^{\frac{1}{3}}} \leq E - A_E + C \frac{E}{N_E^{\frac{1}{3}}}. \quad (9.38)$$

Step 7. It is now possible to conclude. The estimate (9.28) for ϕ_2 follows from (9.38) and (9.29), if the last summand in (9.38) is smaller than $A_E/2$. This requirement translates into

$$N_E \geq CE^3 A_E^3. \quad (9.39)$$

The estimate (9.28) for ϕ_1 follows by observing that by construction $\phi_1 \equiv \phi$ on B_t . It follows from (9.37) and (9.33) that

$$\int_{\mathbb{S}^3} |\nabla\phi_1|^3 = \int_{B_t} |\nabla\phi|^3 + \int_{\mathbb{S}^3 \setminus B_t} |\nabla\phi_1|^3 \leq E - A_E + C \frac{E}{N_E}.$$

Therefore the request that the last term is $\leq E - A_E/2$ translates into

$$N_E \geq CE A_E. \quad (9.40)$$

Recall that in Step 2 we connected N_E to ρ_E by the condition $N_E < -C \log_2 \rho_E$, so (9.39), (9.40) translate into the requirement $\rho_E \leq e^{-C \max\{EA_E, (EA_E)^3\}}$ assumed in the thesis. The requirement on A_E was needed for the reasoning of Step 5. \square

Remark 9.33. *The proof of (9.36) in Step 5 gives the following general estimate valid for bounded Sobolev functions on a compact manifold M and for any Poincaré domain $\Omega \subset M$:*

$$\|\nabla\phi\|_{L^p(\Omega)} \geq C_\Omega \left[|M| (\|\phi\|_{L^\infty(M)} - |\bar{\phi}_M|) - 2\|\phi\|_{L^\infty(M)} |M \setminus \Omega| \right], \quad (9.41)$$

where C_Ω is the Poincaré constant of Ω .

Consider now the following conformal transformations of the unit ball B^4 :

$$F_v(x) = -v + (1 - |v|^2)(x^* - v)^*, \text{ where } v \in B^4 \text{ and } a^* = \frac{a}{|a|^2}.$$

We will prove here the following proposition:

Proposition 9.34 (balancing \Rightarrow extension). *Let $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$. Suppose that for all $v \in B^4$ there holds*

$$\left| \int_{\mathbb{S}^3} \phi \circ F_v \right| \geq \frac{1}{4}. \quad (9.42)$$

Then the following function $u : B^4 \rightarrow \mathbb{S}^3$ extends ϕ

$$u(v) := \pi_{\mathbb{S}^3} \left(\int_{\mathbb{S}^3} \phi \circ F_v \right), \text{ where } \pi_{\mathbb{S}^3}(a) = \frac{a}{|a|} \text{ for } a \in \mathbb{R}^4 \setminus \{0\}. \quad (9.43)$$

Moreover, there exists a constant C independent of ϕ such that the following estimate holds:

$$\|\nabla u\|_{L^4(B^4)} \leq C \|\nabla \phi\|_{L^3(\mathbb{S}^3)}. \quad (9.44)$$

Proof. Step 1. We note that after a change of variable there holds

$$\int_{\mathbb{S}^3} \phi \circ F_v(x) dx = \int_{\mathbb{S}^3} \phi(y) |(F_v^{-1})'|^3(y) dy.$$

where $|(F_v^{-1})'|$ is the conformal factor of DF_v^{-1} . We know from Lemma F.6 that

$$|(F_v^{-1})'| (y) = |F'_{-v}|(y) = \frac{1 - |v|^2}{|y + v|^2},$$

therefore

$$\int_{\mathbb{S}^3} \phi \circ F_v = \int_{\mathbb{S}^3} \phi(y) \left(\frac{1 - |v|^2}{|y + v|^2} \right)^3 dy.$$

As follows from [97], in dimension 4 the function

$$K(x, y) = |\mathbb{S}^3|^{-1} \left[\frac{1 - |y|^2}{|x - y|^2} \right]^3$$

is the Poisson kernel for the equation

$$\begin{cases} \Delta^2 u = 0 & \text{on } B^4, \\ \frac{\partial u}{\partial \nu} \Big|_{\partial B^4} = 0, & u|_{\partial B} = \phi. \end{cases} \quad (9.45)$$

Therefore the function

$$\tilde{u}(v) := \int_{\mathbb{S}^3} \phi \circ F_v$$

is equal to the biharmonic extension of ϕ given by equation (9.45).

Step 2. We recall the following classical estimate which holds for equation (9.45):

$$\|\nabla u\|_{L^4(B^4)} \leq C \|\nabla \phi\|_{L^3(B^3)}.$$

For the proof of this estimate see [59], where the stronger and more natural estimate $\|u\|_{W^{1,4}(\Omega)} \leq \|\phi\|_{W^{1-1/4,4}(\partial\Omega)}$ is obtained in Chapter 2.

Step 3. We note that

$$\forall v \in B^4, \quad 1/4 \leq |\tilde{u}(x)| \leq C$$

because of our hypothesis (9.42), $|\phi| \equiv 1$ and by the elementary estimate $\int_{\mathbb{S}^3} \left(\frac{1-|v|^2}{|y+v|^2}\right)^3 dy \leq C$. As in Step 2 of the proof of Theorem 9.31 (in the present case we have $\pi_{\mathbb{S}^3} = \pi_a$ for $a = 0$) we then obtain the pointwise estimate

$$|\nabla(\pi_{\mathbb{S}^3} \circ \tilde{u})| \sim |\nabla \tilde{u}|.$$

From this and Step 2 the estimate (9.44) follows. \square

We next consider the case in which the hypothesis of Proposition 9.34 (balancing \Rightarrow extension) is false, i.e. that

$$\exists v \in B^4 \quad \left| \int_{\mathbb{S}^3} \phi \circ F_v \right| \leq \frac{1}{4}. \quad (9.46)$$

We then denote

$$\tilde{\phi} := \phi \circ F_v \text{ for a fixed } v \text{ satisfying (9.46)}. \quad (9.47)$$

Note that $F_v|_{\mathbb{S}^3}$ is conformal and bijective (see Section F.3) and thus for $A \subset \mathbb{S}^3$

$$\int_A |\nabla \tilde{\phi}|^3 = \int_{F_v^{-1}(A)} |\nabla \phi|^3,$$

in particular $\tilde{\phi}$ has the same energy bound E as ϕ (we use here the notation of (9.25)). We start with an easy result:

Lemma 9.35. *Under the assumption (9.46) and with the notation (9.47), suppose that $\tilde{\phi} \in \mathcal{B}^E$. Then there exist $\phi_1, \phi_2 \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3 \simeq SU(2))$ such that*

$$\phi = \phi_1 \phi_2, \quad \int_{\mathbb{S}^3} |\nabla \phi_i|^3 \leq E - A_E/2 \text{ for } i = 1, 2,$$

with the constant A_E coming from Proposition 9.32 (balancing \Rightarrow splitting).

Proof. We observe that Proposition 9.32 applies to $\tilde{\phi}$ directly, due to our hypotheses. Therefore we can find $\tilde{\phi}_1, \tilde{\phi}_2 \in W^{1,3}(\mathbb{S}^3, SU(2))$ such that

$$\tilde{\phi} = \tilde{\phi}_1 \tilde{\phi}_2, \quad \int_{\mathbb{S}^3} |\nabla \tilde{\phi}_i|^3 \leq E - A_E / \text{ for } i = 1, 2.$$

We then precompose with F_v^{-1} which preserves the pointwise product and the L^3 -energy of the gradients, obtaining the same decomposition for ϕ . \square

The case $\tilde{\phi} \in \mathcal{G}^E$ is a bit more difficult:

Proposition 9.36. *Under the assumption (9.46) and with the notation (9.47), suppose that $\tilde{\phi} \in \mathcal{G}^E$. Then there exists an extension $u \in W^{1,(4,\infty)}(B^4, \mathbb{S}^3)$ of ϕ such that*

$$\|\nabla u\|_{L^{4,\infty}(B^4)} \leq \frac{C}{\rho_E} \|\nabla \phi\|_{L^3(\mathbb{S}^3)}^2 + \|\nabla \phi\|_{L^3(\mathbb{S}^3)}, \quad (9.48)$$

under the assumption that

$$\rho_E \leq \frac{1}{4}. \quad (9.49)$$

Proof. To simplify notations $\rho = \rho_E$ during this proof. We divide the domain B^4 into

$$A := F_v^{-1}(B(0, 1 - \rho)), \quad A' := B^4 \setminus A.$$

Using Lemma F.7 it follows that there exists a geometric constant C and a function $h(v)$ such that for $x \in A$ and under the condition (9.49),

$$\frac{h(v)}{C} \leq |F'_v|(x) \leq Ch(v). \quad (9.50)$$

We can use (9.50) to control the $L^{4,\infty}$ -norm of ∇u restricted to A via the similar norm of $\nabla \tilde{u}$:

$$\begin{aligned} |\{x \in A : |\nabla u|(x) > \Lambda\}| &= |\{x \in A : |\nabla \tilde{u}|(F_v(x)) |F'_v|(x) > \Lambda\}| \\ &\leq |\{x \in A : |\nabla \tilde{u}|(F_v(x)) > \Lambda / (Ch(v))\}| \\ &= \int_{F_v(A) \cap B} |F'_v|^{-4} dy, \quad \text{8 for } B := \{|\nabla \tilde{u}| > \Lambda / (Ch(v))\} \\ &\leq C^4 h^{-4}(v) |\{y \in B_{1-\rho} : |\nabla \tilde{u}| > \Lambda / (Ch(v))\}| \\ &\leq C^8 \Lambda^{-4} \|\nabla \tilde{u}\|_{L^{4,\infty}(B_{1-\rho})}^4. \end{aligned}$$

By bringing Λ to the other side it follows that

$$\Lambda^4 |\{x \in A : |\nabla u|(x) > \Lambda\}| \leq C^8 \|\nabla \tilde{u}\|_{L^{4,\infty}(B(0,1-\rho))}^4. \quad (9.51)$$

On the other hand we can use the conformal invariance, the invertibility of F_v and the usual estimate between $L^{4,\infty}$ and L^4 to complete a first step of the proof:

$$\Lambda^4 |\{x \in A' : |\nabla u|(x) > \Lambda\}| \leq C \|\nabla u\|_{L^4(A')}^4 = C \|\nabla \tilde{u}\|_{L^4(B \setminus B_{1-\rho})}. \quad (9.52)$$

We now sum (9.51) to (9.52) and we take the supremum on $\Lambda > 0$. It follows that up to increasing C ,

$$[\nabla u]_{L^{4,\infty}(B^4)} \leq C(\|\nabla \tilde{u}\|_{L^{4,\infty}(B_{1-\rho})} + \|\nabla \tilde{u}\|_{L^4(B \setminus B_{1-\rho})}). \quad (9.53)$$

The estimate (9.53) together with Theorem 9.31 applied to \tilde{u} gives the desired estimate for the first summand, while for the second summand we proceed as in Step 3 of the proof of Theorem 9.31. We use the small concentration regions B_i for $\tilde{\phi}$, on which we apply the Courant lemma 9.28 which allows to project the values of $u := \tilde{u} \circ F_v^{-1}$ as well on \mathbb{S}^3 , with little change of the gradient of u . We observe that F_v^{-1} is conformal, so the L^3 -energy of \tilde{u} on ∂B_i is the same as the L^3 -energy of u on $\partial F_v^{-1}(B_i)$ and use the Uhlenbeck extension result of Theorem 9.29(Uhlenbeck analogue) for \tilde{u} as in Step 3 of the proof of Theorem 9.31. We obtain:

$$\|\nabla u\|_{L^4(F_v^{-1}(B \setminus B_{1-\rho}))} = \|\nabla \tilde{u}\|_{L^4(B \setminus B_{1-\rho})} \leq C \|\nabla \tilde{\phi}\|_{L^3(\mathbb{S}^3)} = C \|\nabla \phi\|_{L^3(\mathbb{S}^3)}.$$

This and (9.53) conclude the proof. \square

9.4.4 End of the proof of Theorem 9.24

We will refer to the scheme (9.26) for the idea of the proof.

Choice of A_E . In (9.25) take $A_E \leq \frac{\delta}{C_1}$ with the notations of Theorem 9.31 so that it applies to give extensions for the small concentration case (“good” boundary conditions). Here δ is the constant coming from the Uhlenbeck procedure on regions of radius ρ_E near ∂B^4 . If necessary diminish A_E such that the requirement $A_E \leq C^{-1}$ of Proposition 9.32(balancing \Rightarrow splitting) is also satisfied.

Choice of ρ_E . Recall that the constant C appearing there was depending just on the volume of \mathbb{S}^3 . For the radius of concentration ρ_E we need just to impose the bound present in Proposition 9.32, which with the choices of A_E just done becomes $\rho_E \lesssim e^{-C \max(1, E^3)}$.

Estimates for extensions. Consider again the scheme (9.26). Each time we extend some boundary datum ϕ obtained during our constructions via a function $u : B^4 \rightarrow \mathbb{S}^3$, we do so with one of the following estimates:

- In the case of the extensions of Theorem 9.31 or of Proposition 9.36 (which in turn actually depends on Theorem 9.31) we have

$$\|\nabla u\|_{L^{4,\infty}} \lesssim \frac{\|\nabla\phi\|_{L^3}^2}{\rho_E} + \|\nabla\phi\|_{L^3}.$$

- In the case of the biharmonic extension of Proposition 9.34 (balancing \Rightarrow extension) we have the much better

$$\|\nabla u\|_{L^4} \lesssim \|\nabla\phi\|_{L^3}.$$

The number of iterations to be made when we apply the procedure described in scheme (9.26) is bounded by

$$E \Big/ \frac{A_E}{2} \sim E^2.$$

Since each iteration creates two new boundary value functions out of one, in the end we may have a decomposition into no more than

$$e^{CE^2} \text{ boundary value functions.}$$

By the triangle inequality we see that in this case there exists an extension of the initial ϕ satisfying

$$\|\nabla u\|_{L^{4,\infty}} \lesssim e^{C\|\nabla\phi\|_{L^3}^9} \|\nabla\phi\|_{L^3}^2 + e^{C\|\nabla\phi\|_{L^3}^6} \|\nabla\phi\|_{L^3}. \quad (9.54)$$

this gives the estimate (9.13) of Theorem 9.24, finishing the proof. \square

9.5 Controlled global gauges

We now fix a closed Riemannian 4-manifold (M, h) with a connection represented by $A \in W^{1,2}(M, T^*M \otimes su(2))$ whose curvature will be denoted by F . We desire to find a global gauge for A in which $\|A\|_{W^{1,(4,\infty)}} \leq f(E)$ where $E := \int_M |F|^2$.

We will use the following two results. The first one is the restatement of Theorem 9.5 which we repeat for easier reference.

Theorem 9.37. *Fix a trivial $SU(2)$ -bundle E over the ball B^4 . There exists a function $f_1 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with the following property. If $g \in W^{1,3}(\mathbb{S}^3, SU(2))$ gives a trivialization of the restricted bundle $E|_{\partial B^4}$, then there exists an extension of g to a trivialization $\tilde{g} \in W^{1,(4,\infty)}(B^4, SU(2))$ such that the following estimate holds:*

$$\|\nabla \tilde{g}\|_{L^{4,\infty}(B^4)} \leq f_1 \left(\|\nabla g\|_{L^3(\mathbb{S}^3)} \right).$$

The second theorem is the main result of [132].

Theorem 9.38 (Uhlenbeck gauge). *There exists $\epsilon_0 > 0$ such that if the curvature satisfies $\int_{B_1} |F|^2 \leq \epsilon_0$ then there exists a gauge $\phi \in W^{2,2}(B_1, SU(2))$ such that in that gauge the connection satisfies $\|A_\phi\|_{W^{1,2}(B_1)} \leq C\|F\|_{L^2(B_1)}$ with $C > 0$ depending only on the dimension.*

Theorem 9.39. *For each closed boundaryless 4-manifold M^4 there exists a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with the following properties.*

Let ∇ be a $W^{1,2}$ connection for an $SU(2)$ -bundle over M . Then there exists a global $W^{1,(4,\infty)}$ section of the bundle over the whole M^4 such that in the corresponding trivialization ∇ is given by $d + A$ with the following bound.

$$\|A\|_{L^{(4,\infty)}} \leq f \left(\|F\|_{L^2(M)} \right), \tag{9.55}$$

where F is the curvature form of ∇ .

9.5.1 Scheme of the proof

We indicate here the sketch of the proof, before going through the details.

Proof. We will denote the L^2 -norm of F by E . We may assume that a first guess for A (i.e. a fixed trivialization) is already given and belongs to $W^{1,2}$ (if the bound by ϵ_0 on the energy of F is available, we may also assume more, by Uhlenbeck’s result stated above, namely that one controls the $W^{1,2}$ -norm of A by the energy).

It can be seen from the formula of change of gauge that it is equivalent to estimate the gradient of the trivialization g or the gradient of the connection A in that gauge.

We define f by iteration on E . The main steps are as follows (see the scheme (9.56)):

- Uhlenbeck's theorem already gives a gauge, with an L^4 -estimate of the gradient of the trivialization, in case the energy of F is smaller than ϵ_0 . Instead of the desired $L^{4,\infty}$ -estimate, we get the stronger estimate in terms of the L^4 -norm. The difficulty in our proof is to find an estimate without a priori assumptions on the L^2 -smallness of F .
- Let ρ_0 be the largest scale at which no more than $\epsilon_0/2$ of F 's L^2 -norm concentrates.
- In case $\rho_0 \geq \bar{\rho}_0 := C\rho_{inj}(M)2^{-E/\epsilon_1}$ we iteratively extend our gauge on the simplexes of a triangulation where each simplex is well inside a ball of radius $\rho_{inj}(M)$. To do this we iteratively extend with $W^{1,3}$ estimates the change of gauge along the 3-skeleton of the triangulation, then on each simplex we use Theorem 9.24 to extend inside that simplex. See Section 9.5.2. The estimates depend only on M^4 .
- The other alternative is $\rho_0 \leq \bar{\rho}_0$, or more explicitly

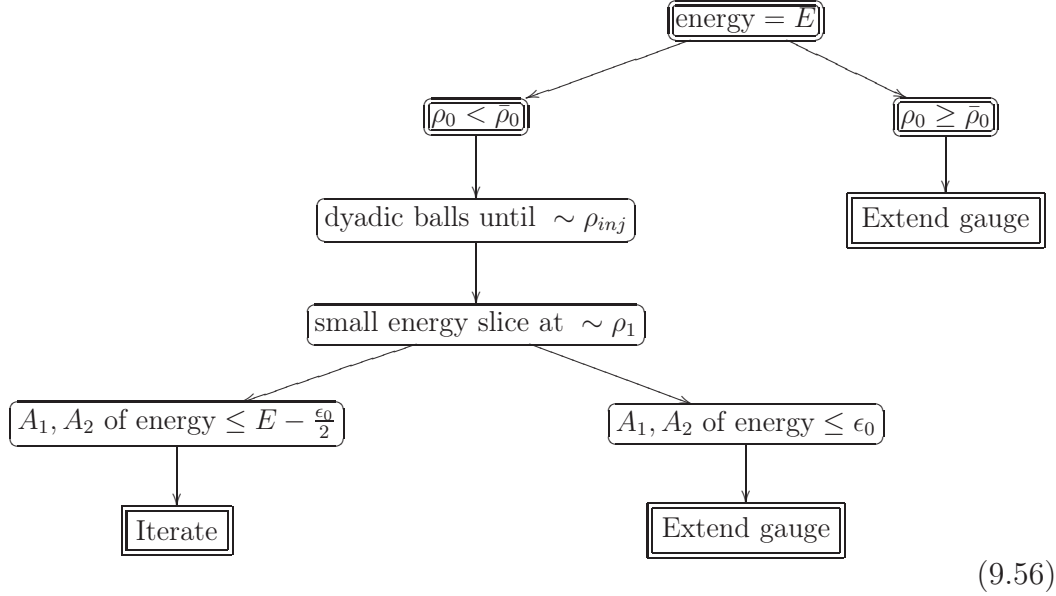
$$\epsilon_1 \log_2 \frac{C\rho_{inj}}{\rho_0} \leq E.$$

Then consider a point x_0 at which $|F|$ concentrates and look at the geodesic dyadic rings

$$R_k := B(x_0, 2^{k+1}\rho_0) \setminus B(x_0, 2^k\rho_0), \quad k \in \{0, \dots, \lfloor \log_2(C\rho_{inj}/\rho_0) \rfloor\}.$$

By pigeonhole principle, in one of these rings D_{k_0} the curvature F has energy less or equal than ϵ_1 . The parameter ϵ_1 can be chosen, depending only on ϵ_0 , in such a way that this estimate of the energy ensures the existence of a small energy slice along a geodesic sphere of radius $t \sim 2^{k_0}\rho_0$. We then have extensions of the connections with curvatures of energy smaller than $E - \frac{\epsilon_0}{2}$. We use Lemma 9.42 (finding good slices). To avoid subtleties about traces we will ensure that these two connections coincide on an open set. The choice of slice is described in Section 9.5.4.

- Then we separately trivialize these two connections using the iterative assumption that the f as described in the claim of our theorem is already defined on $[0, E[$. By iterative assumption we then define $f(E)$ based on $f(E - \epsilon_0/2)$ and on the function f_1 which appears in Theorem 9.4. The detailed bounds are given in Sections 9.5.5 and 9.5.6.



9.5.2 Iterations based on a suitable triangulation

Define, for ϵ_0 as in Theorem 9.38(Uhlenbeck gauge), the following radius:

$$\rho_0 := \inf \left\{ \rho > 0 : \exists x_0 \in M, \int_{B_\rho(x_0)} |F|^2 = \frac{\epsilon_0}{2} \right\}. \quad (9.57)$$

Denote

$$\bar{\rho}_0 := C \rho_{\text{inj}}(M) 2^{-\frac{E}{\epsilon_1}},$$

where $\rho_{\text{inj}}(M)$ is the injectivity radius of M and the constant ϵ_1 will be fixed later and depends only on the geometry of M and on ϵ_0 . Fix then a triangulation on M having in-radius $\gtrsim \bar{\rho}_0$ and size $\lesssim \bar{\rho}_0$, with implicit constants bounded by 4. $C < 1$ in the definition of $\bar{\rho}_0$ can be fixed now, so that each simplex of the triangulation is contained in a ball of radius $\rho_{\text{inj}}(M)/2$. In particular all k -simplexes of the triangulation are bi-Lipschitz equivalent to \mathbb{S}^k with bi-Lipschitz constants which depend just on k .

Theorem 9.38(Uhlenbeck gauge) gives a trivialization ϕ_i associated to each 4-simplex C_i , such that the expression of A in those coordinates

$$A_i = \phi_i^{-1} d\phi_i + \phi_i^{-1} A \phi_i \text{ on } C_i \quad (9.58)$$

satisfies

$$\|A_i\|_{W^{1,2}(C_i)} \leq C \|F\|_{L^2(C_i)}. \quad (9.59)$$

If we call

$$g_{ij} := \phi_j^{-1} \phi_i \quad (9.60)$$

then $g_{ij}g_{jk} = g_{ik}$, in particular we have $g_{ij}^{-1} = g_{ji}$; moreover

$$A_j = g_{ij}dg_{ji} + g_{ij}A_i g_{ji} \text{ on } \partial C_i \cap \partial C_j. \quad (9.61)$$

In particular, it follows from the above expression that $g_{ij} \in W^{1,3}(\partial C_i \cap \partial C_j, SU(2))$. We now state a lemma which will enable us to extend the gauge from one 4-simplex to the next one.

Lemma 9.40 (extension on a sphere). *Let \mathbb{S}_+^3 be the upper hemisphere $\mathbb{S}^3 \cap \{x_3 \geq 0\}$. Then for any $g \in W^{1,3}(\mathbb{S}_+^3, SU(2))$ there exists $\tilde{g} \in W^{1,3}(\mathbb{S}^3, SU(2))$ such that $\tilde{g} = g$ on \mathbb{S}_+^3 and*

$$\|\nabla \tilde{g}\|_{L^3(\mathbb{S}^3)} \leq C \|\nabla g\|_{L^3(\mathbb{S}_+^3)}.$$

Proof. Up to enlarging \mathbb{S}_+^3 to a spherical cap of height $\leq 3/2$, we may assume that for a universal constant $C > 0$

$$\|g|_{\partial \mathbb{S}_+^3}\|_{W^{1,2}(\partial \mathbb{S}_+^3)} \leq C \|g\|_{W^{1,3}(\mathbb{S}_+^3)}. \quad (9.62)$$

We observe that $g|_{\partial \mathbb{S}_+^3 \simeq \mathbb{S}^2} \in W^{1,2}(\mathbb{S}^2, SU(2))$ and we desire to extend this trace inside $B^3 \simeq \mathbb{S}_+^3$ with a good norm estimate. We start with a harmonic extension (identifying $SU(2) \simeq \partial B^4$), namely

$$\begin{cases} \Delta \hat{g} = 0 \text{ on } B^3, \\ \hat{g} = g \text{ on } \partial B^3. \end{cases}$$

Then we have by the usual elliptic estimates

$$\|\hat{g}\|_{W^{1,3}(\mathbb{S}_+^3)} \leq C \|g|_{\partial \mathbb{S}_+^3}\|_{W^{1,2}(\partial \mathbb{S}_+^3)}. \quad (9.63)$$

We then observe that for $a \in B_{1/2}^4$ if g_a is the radial projection of the values of \hat{g} on the boundary with center a , then the following pointwise inequality holds (as in the projection trick of Section 9.2.1)

$$|\nabla g_a| \leq C \frac{|\nabla \hat{g}|}{|\hat{g} - a|}. \quad (9.64)$$

We also have

$$\int_{a \in B_{1/2}^4} \int_{B^3} |\nabla g_a|^3 \leq C \int_{B^3} |\nabla \hat{g}|^3.$$

Therefore there exists $a \in B_{1/2}^4$ such that

$$\|\nabla g_a\|_{L^3(B^3 \simeq \mathbb{S}_+^3)} \leq C \|\nabla \hat{g}\|_{L^3(B^3 \simeq \mathbb{S}_+^3)}. \quad (9.65)$$

Combining the inequalities (9.62), (9.63), (9.64) and (9.65) we obtain the thesis for $\tilde{g} = g_a$ with a as above. \square

Corollary 9.41 (iteration step). *Suppose that on our 4-manifold M a connection A is fixed and an Uhlenbeck gauge ϕ_j is defined on a 4-simplex C_j , i.e. the estimate (9.59) holds with the notation (9.58). Also suppose that a global gauge ϕ_I is defined on a finite union of simplexes $C_I := \cup_{\alpha \in I} C_{i_\alpha}$ and that $\partial C_j \cap C_I^{(3)}$ (where $C_I^{(3)}$ is the simplicial 3-skeleton of C_I) contains some, but not all, 3-faces of C_j . It is then possible to extend the gauge change g_{ij} defined in (9.60) to \tilde{g}_{ij} defined on the whole of ∂C_j with a norm bound*

$$\|\nabla \tilde{g}_{ij}\|_{L^3(\partial C_j)} \leq C \|\nabla g_{ij}\|_{L^3(\partial C_j \cap C_I^{(3)})},$$

where C depends only on M .

Proof. $H := (\partial C_j \setminus C_I^{(3)})_\delta$ is bi-Lipschitz to a ball for δ equal to $2/3$ the smallest in-radius of a face of C_j . Here A_δ is a δ -neighborhood of A inside ∂C_j . Also let $H' := (\partial C_j \setminus C_I^{(3)})_{2\delta}$. Note that the triple $(\partial C_j, H, H')$ is C -bi-Lipschitz equivalent to $(\mathbb{S}^3, \mathbb{S}_-^3, K)$ where K is the spherical cap of height $3/4$ extending \mathbb{S}_-^3 . We may then apply the construction of Lemma 9.40(extension on a sphere) and a bi-Lipschitz deformation, in order to “fill the hole” H extending the gauge g_{ij} with estimates. The bi-Lipschitz constant is bounded by the geometric constraints on our triangulation and is independent of A and of g_{ij} . \square

Given Lemma 9.40(extension on a sphere) and Corollary 9.41(iteration step) we proceed iteratively on the triangulation as follows (the indices labeling the simplexes are re-defined during the whole procedure in a straightforward way):

- Suppose that we already defined the gauge $\tilde{\phi}_{j-1}$ on a set of $j-1$ simplexes C_1, \dots, C_{j-1} , whose union forms a connected set.
- Consider a new simplex C_j extending such connected set. This choice of notation brings us directly under the hypothesis of Corollary 9.41(iteration step) and thus we are able to extend g_{ij} to \tilde{g}_{ij} as in the corollary.
- We next apply Theorem 9.24 and extend \tilde{g}_{ij} to a gauge change h_{ij} defined inside C_j and satisfying

$$\|\nabla h_{ij}\|_{L^{(4,\infty)}(C_j)} \leq f(\|\nabla \tilde{g}_{ij}\|_{L^3(C_j)}) \leq C_0, \tag{9.66}$$

with C_0 depending only on universal constants and on ϵ_0 . The function f is explicitly expressed in the statement of Theorem 9.24.

- On $\cup_{i < j} C_i$ we keep $\tilde{\phi}_j = \tilde{\phi}_{j-1}$, while on C_j we define $\tilde{\phi}_j = \phi_j h_{ij}$.

We see that this construction gives for the local expression \tilde{A}_j corresponding to the gauge $\tilde{\phi}_j$ the bound

$$\|\tilde{A}_j\|_{L^{(4,\infty)}(C_j)} \lesssim \|A_j\|_{L^4(C_j)} + \|\nabla h_{ij}\|_{L^{(4,\infty)}(C_j)} \leq \epsilon_0 + C_0.$$

Iterating this gauge extension strategy for all simplexes of a triangulation we would obtain a global gauge \tilde{A} on the whole of M such that

$$\|\tilde{A}\|_{L^{(4,\infty)}(M)} \leq C(\text{number of simplexes})(C_0 + \epsilon_0) \leq C \frac{\text{Vol}(M)}{\bar{\rho}_0^4}, \quad (9.67)$$

since the volume of each simplex is $\gtrsim \bar{\rho}_0^4$. The above bound depends on the geometry of M and on the energy E of the curvature only. Note that the above reasoning works only as long as $\rho_0 \lesssim \bar{\rho}_0$. As noted before, so far we have little control on ρ_0 , in particular we have no bound from below. For this reason we next consider the case $\rho_0 \geq \bar{\rho}_0$.

9.5.3 Extending the connection with small curvature changes

We now concentrate on proving the following lemma:

Lemma 9.42 (finding good slices). *There exists a constant ϵ_1 with the following properties. If M is a fixed 4-manifold with a $W^{1,2}$ -connection A and if $B_{2t}(x_0) \subset M$ is a geodesic ball with the estimate*

$$t \int_{\partial \tilde{B}_t} |F|^2 \leq \epsilon_1$$

then there exists $\hat{A} \in W^{1,2}(M, T^*M \otimes su(2))$ such that $\hat{A} = A$ on B_t and

$$\int_{M \setminus B_t} |F_{\hat{A}}|^2 \leq C\epsilon_1$$

with a constant C depending only on M . In particular it is possible to ensure $C\epsilon_1 < \frac{\epsilon_0}{4}$, with ϵ_0 as in Theorem 9.38 (Uhlenbeck gauge).

Proof. Up to a change of gauge which does not increase the norm, we may assume the Neumann condition

$$\langle A, \nu \rangle \equiv 0 \text{ on } \partial B_t. \quad (9.68)$$

This is obtained for example by minimizing $\|g^{-1}dg + g^{-1}Ag\|_{L^2(B_t)}$ among gauge functions $g \in W^{2,2}(B_t, SU(2))$.

We next extend A to $B_{2t} \setminus B_t$ by

$$\tilde{A} := \pi^* i_{\partial B_t}^* A, \text{ where } \pi(x) = t \frac{x}{|x|} \text{ and } i_{\partial B_t} \text{ is the inclusion.}$$

Using the hypothesis and the facts that $i_{\partial B_t}^*$ acts on F_A by just forgetting about some of its components and that π is bi-Lipschitz, we obtain

$$\int_{B_{2t} \setminus B_t} \left| d\tilde{A} + \frac{1}{2}[\tilde{A}, \tilde{A}] \right|^2 \leq C\epsilon_1.$$

We can apply a change of gauge $g(\sigma)$ depending only on the angular variable $\sigma \in \partial B^4$ such that

$$d_{\partial B_t}^* A_g|_{\partial B_t} = 0.$$

This preserves the condition (9.68) and also gives the following behavior as $s \rightarrow 0$:

$$C\epsilon_1 \geq \int_{B_s \cap \partial B_t} |dA_g + \frac{1}{2}[A_g, A_g]|^2 \geq \int_{B_s \cap \partial B_t} |dA|^2 - o(s) \int_{B_s \cap \partial B_t} |\nabla A|^2.$$

Therefore $A_g \in W^{1,2}(T^*\partial B_t \otimes su(2))$, $\tilde{A}_g \in W^{1,2}(\wedge^1 B_{2t} \setminus B_t, su(2))$ and both A_g, \tilde{A}_g satisfy (9.68). Therefore \tilde{A}_g extends by A_g in a neighborhood of ∂B_t , giving still a $W^{1,2}$ -gauge. We observe that by Sobolev embedding

$$\int_{\partial B_t} |[A, A]|^2 \lesssim \left(\int_{\partial B_t} |\nabla A|^2 \right)^2,$$

and by Hodge decomposition and using $d_{\partial B_t}^* A = 0$

$$\int_{\partial B_t} |\nabla A|^2 \lesssim \int_{\partial B_t} (|dA|^2 + |d^*A|^2) \lesssim \int_{\partial B_t} |F_A|^2 + \left(\int_{\partial B_t} |\nabla A|^2 \right)^2.$$

The above inequality implies an inequality of the form $X \leq \epsilon_1 + X^2$ by our hypothesis and the gauge invariance of the curvature, with $X = \|\nabla A\|_{L^2(\partial B_t)}^2$.

We may thus assume that

$$t \int_{\partial B_t} |\nabla A|^2 \leq Ct \int_{\partial B_t} |F|^2,$$

which allows us to use a cutoff procedure, defining $\hat{A} := \chi_t A$ for a smooth $[0, 1]$ -valued cutoff function χ_t such that $\chi_t \equiv 1$ on B_t and $\chi_t \equiv 0$ outside B_{2t} . With this choice and the above estimate for ∇A we obtain

$$\int_{B_{2t}} |F_{\hat{A}}|^2 \leq \int_{B_t} |F_A|^2 + C\epsilon_1$$

and we can extend $\hat{A} \equiv 0$ outside B_{2t} obtaining the desired estimate. □

Remark 9.43. *We will use the above lemma only in order to obtain a new connection with a controlled small energy, but the modification from A to \hat{A} will not be used otherwise: we will only be interested to change the gauge on the region where $A = \hat{A}$.*

The above lemma is used to select a radius giving a slice with small energy concentration, and to make an induction on the energy.

9.5.4 Cutting M by a small energy slice

Suppose for this subsection that we are in the case $\rho_0 < \bar{\rho}_0$. We start by defining the following positive number ρ_1 , which uses the same constant C as in the definition of $\bar{\rho}_0$:

$$\rho_1 := \begin{cases} \inf \left\{ \rho \geq \rho_0 : \int_{B_{2\rho} \setminus B_\rho} |F|^2 \leq \frac{\epsilon_1}{4} \right\} & \text{if this is } < C\rho_{\text{inj}}(M), \\ C\rho_{\text{inj}} & \text{else.} \end{cases}$$

Note that because of the hypothesis $\rho_0 < \bar{\rho}_0$ and because of the choice of ϵ_1 , the ρ_1 is rather small, in such a way that $B_{2\rho_1}$ is bi-Lipschitz to B_1 . Thus Lemma 9.42 (finding good slices) applies. More precisely, we will apply the Lemma for two different radii $t_1 \in [\rho_1, 5/4\rho_1]$, $t_2 \in [7/4\rho_1, 2\rho_1]$. Chebychev's theorem implies the existence of $t_i, i = 1, 2$ such that

$$t_i \int_{\partial B_{t_i}} |F|^2 \leq \epsilon_1.$$

We divide the proof into two cases, according to how large $\int_{M \setminus B_{2\rho_1}} |F|^2$ is with respect to ϵ_0 from Theorem 9.38 (Uhlenbeck gauge).

9.5.5 The case $\int_{M \setminus B_{2\rho_1}} |F|^2 \geq \frac{\epsilon_0}{2}$

In this case we split to the regions B_{t_2} and $M \setminus B_{t_1}$ and do induction on the energy in order to find gauges satisfying our estimates on these two overlapping regions.

Lemma 9.42 (finding good slices) gives extensions

$$\begin{cases} \hat{A}_1 \equiv A \text{ on } B_{t_2} \text{ s.t.} & \int_M |F_{\hat{A}_1}|^2 \leq \int_{B_{t_2}} |F_A|^2 + C\epsilon_1, \\ \hat{A}_2 \equiv A \text{ on } M \setminus B_{t_1} \text{ s.t.} & \int_M |F_{\hat{A}_2}|^2 \leq \int_{B_{t_1}} |F_A|^2 + C\epsilon_1. \end{cases} \quad (9.69)$$

In particular \hat{A}_1, \hat{A}_2 are equivalent on $B_{\frac{7}{4}\rho_1} \setminus B_{\frac{5}{4}\rho_1}$ and

$$\int |F_{\hat{A}_i}|^2 \leq \int |F_A|^2 - \frac{\epsilon_0}{4}.$$

If we can find global gauges $g_i^\infty, i = 1, 2$ in which \hat{A}_i have expressions \hat{A}_i^∞ with $L^{(4,\infty)}$ -bounds as in Theorem 9.4, then it is enough to apply

$$g_{12}^\infty := (g_1^\infty)^{-1} g_2^\infty$$

on $R := B_{\frac{7}{4}\rho_1} \setminus B_{\frac{5}{4}\rho_1}$ in order to obtain

$$A_2^\infty = g_{12}^\infty A_1^\infty (g_{12}^\infty)^{-1} + g_{12}^\infty d (g_{12}^\infty)^{-1}.$$

This implies also

$$\|\nabla g_{12}^\infty\|_{L^{(4,\infty)}(R)} \leq f\left(E - \frac{\epsilon_0}{4}\right).$$

Then there exists $t_3 \in [\frac{5}{4}\rho_1, \frac{7}{4}\rho_1]$ such that

$$\int_{\partial B_{t_3}} |\nabla g_{12}^\infty|^3 \leq f\left(E - \frac{\epsilon_0}{4}\right)$$

and thus by Theorem 9.4 we can find a $W^{1,(4,\infty)}$ -extension h_{12}^∞ of g_{12}^∞ to a map from B_{t_3} to $SU(2)$. The estimate for h_{12}^∞ is exactly as in Theorem 9.4. Thus if we call f_1 the function of $\|\nabla\phi\|_{L^3}$ appearing Theorem 9.4 then

$$\|\nabla h_{12}^\infty\|_{L^{(4,\infty)}(B_{t_3})} \leq f_1\left(f\left(E - \frac{\epsilon_0}{4}\right)\right)$$

We then choose the following global gauge:

$$g^\infty := \begin{cases} g_2^\infty & \text{on } M^4 \setminus B_{t_3}, \\ h_{12}^\infty g_1^\infty & \text{on } B_{t_3}. \end{cases} \quad (9.70)$$

∇g^∞ is then estimated by an universal constant times

$$f_1(f(E - \epsilon_0/4)) + f(E - \epsilon_0/4),$$

which allows to define inductively $f(E)$.

9.5.6 The case $\int_{M \setminus B_{2\rho_1}} |F|^2 \leq \frac{\epsilon_0}{2}$

In this case outside B_{ρ_1} we apply directly Uhlenbeck's procedure, i.e. Theorem 9.38(Uhlenbeck gauge), while on $B_{2\rho_1}$ we extend the so-obtained gauge via

Theorem 9.24. If we call A_1, A_2 the so-obtained connections on $B_{2\rho_1}, M \setminus B_{\rho_1}$ respectively, then

$$\exists t \in [\rho_1, 2\rho_1] \text{ s.t. } \int_{\partial B_t} (|A_1|^3 + |A_2|^3) \leq C(f_1(\epsilon_0) + \epsilon_0),$$

thus as above the same bound is true also for the gradient of the change of gauge ∇g_{12} . Then Theorem 9.4 gives the extension h_{12} to a gauge in $W^{1,(4,\infty)}(B_t, SU(2))$. The estimate which we reach is

$$\|\nabla h_{12}\|_{L^{4,\infty}(B_{t_3})} \leq f_1(C(f_1(\epsilon_0) + \epsilon_0)).$$

We then choose

$$g^\infty := \begin{cases} g_2 & \text{on } M^4 \setminus B_{t_3}, \\ h_{12}g_1 & \text{on } B_{t_3}. \end{cases} \quad (9.71)$$

This g^∞ satisfies an estimate independent on E and dependent only on ϵ_0 , again allowing to define $f(E)$ inductively. \square

Appendix A

Smirnov decomposition of currents

A.1 Smirnov's original result

In Chapter 5 we use S. K. Smirnov's decomposition theorem for the proof of the main step of the interior regularity for abelian curvatures on B^3 (see Theorem 5.3). We restate below the result which we used. We will need some definitions before.

Definition A.1. *Let A be a k -current of finite mass. We then define the variation measure of A as follows:*

$$\text{For } X \text{ Borel, } \|A\|(X) = \sup \left\{ \sum_{\sigma} \mathbb{M}(T \llcorner X_{\sigma}) : (X_{\sigma}) \text{ Borel partition of } X \right\}.$$

Alternatively, $\|A\|$ is the infimum of all Borel measures μ controlling A , in the sense that for all smooth k -forms ω

$$\langle A, \omega \rangle \leq \langle \mu, |\omega| \rangle.$$

Definition A.2. *Let A, B, C be k -currents satisfying $A = B + C$. We say that $B + C$ is a decomposition of A if the variation measures satisfy $\|A\| = \|B\| + \|C\|$. We say that $B + C$ is a total decomposition of A if it is a decomposition and also $\partial A = \partial B + \partial C$ is a decomposition.*

Definition A.3. *A current T is a cycle if $\partial T = 0$. If for all total decompositions $T = X + Y$ neither X nor Y is a cycle, then T is called acyclic.*

Definition A.4. *A Lipschitz curve $\gamma : [a, b] \rightarrow \mathbb{R}^n$ is called an arc if it is injective. To an arc γ we may associate the 1-current of integration along γ : For all smooth 1-forms α we define*

$$\langle [\gamma], \alpha \rangle := \int_{\gamma} \alpha.$$

For arcs there holds $\|[\gamma]\| = |\dot{\gamma}|$, \mathcal{L}^1 -a.e. on $[a, b]$ and in particular the mass of $[\gamma]$ is the length of γ . There also holds, with the above notations, $\partial[\gamma] = \delta_{\gamma(b)} - \delta_{\gamma(a)}$ and $\|\partial[\gamma]\| = \delta_{\gamma(b)} + \delta_{\gamma(a)}$.

Definition A.5. *Suppose that $\gamma : \mathbb{R} \rightarrow B^n$ is a 1-Lipschitz curve, assume that the limit*

$$\langle S, \omega \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{\gamma|_{[-T, T]}} \omega$$

exists for all $\omega \in C^\infty(B^n, \wedge^1 \mathbb{R}^n)$ and that the so-defined current S has mass 1 and support included in $\gamma(\mathbb{R})$. Then S is called an elementary solenoid (or a Schwartzman cycle). We denote the space of elementary solenoids by Sol .

We are now ready to state Smirnov's decomposition result:

Theorem A.6 ([120]). *Assume T is a finite mass 1-current on \mathbb{B}^n with finite mass boundary ∂T . Then there exists a total decomposition $T = A + C$ such that $\partial C = 0$ and A is acyclic.*

A can be further decomposed into a superposition of arcs as follows. There exists a finite positive Borel measure μ on the space of arcs such that

$$\langle A, \omega \rangle = \int \langle [\gamma], \omega \rangle d\mu(\gamma), \quad (\text{A.1})$$

$$\langle \|A\|, \phi \rangle = \int \langle \|[\gamma]\|, \phi \rangle d\mu(\gamma), \quad (\text{A.2})$$

$$\langle \partial A, f \rangle = \int \langle \partial[\gamma], f \rangle d\mu(\gamma), \quad (\text{A.3})$$

$$\langle \|\partial A\|, \phi \rangle = \int \langle \|\partial[\gamma]\|, \phi \rangle d\mu(\gamma), \quad (\text{A.4})$$

for all $\omega \in C^\infty(B^n, \wedge^1 \mathbb{R}^n)$, $f \in C^\infty(B^n)$, $\phi \in C^0(B^n)$. C can be decomposed into a superposition of elementary solenoids, i.e. there exists a finite Borel positive measure ν on Sol such that

$$\langle C, \omega \rangle = \int \langle S, \omega \rangle d\nu(S), \quad (\text{A.5})$$

$$\langle \|C\|, \phi \rangle = \int \langle \|S\|, \phi \rangle d\nu(S). \quad (\text{A.6})$$

Ideas of the proof

The proof by S. K. Smirnov is roughly as follows (see [120]). The fact that there exists a maximal cycle, i.e. that we have a total decomposition $T = A + C$, is

consequence of Zorn's lemma.

Smirnov reduces the decomposition problem for A to the one for C as follows: given an acyclic A , consider on the enlarged space $[0, L] \times B^n$ the cycle $C := \delta_1 \times A - \delta_0 \times A + [0, L] \times \partial A$. Next, each elementary solenoid in the decomposition of C can be decomposed into arcs of length L ; such arcs can be decomposed into parts which travel along the segments $[0, L] \times \{p\}$ and parts which live on $\{0, L\} \times B^n$. Restricting the curves with both ends in $]0, 1[\times B^n$ to $\{0\} \times B^n$ gives the desired decomposition of "part of" A i.e. we have a total decomposition $A = A_1 + A'_1$ such that A_1 is decomposed as in the claim of the theorem. Smirnov controls from below the ratio $\mathbb{M}(\partial A_1)/\mathbb{M}(\partial A)$ thus by repeating the procedure on A'_1 and iterating we reach the result.

The case of general C is in turn reduced to the case when $C = C(x)$ is a smooth vector field. In this case the decomposing solenoids will be constructed from flow trajectories of C . In order to better follow the slow trajectories (corresponding to regions where C is small) at the same time as the fast ones, one studies rather the vector field $C'(x, t) = (C(x), 1)$ on $\mathbb{R} \times B^n$. Using the flow of C a shift-invariant measure $\nu = \nu_C$ can be defined on the space \mathcal{S} of possible trajectories of such C' , where we use the shift $\sigma : \gamma(t) \mapsto \gamma(t + 1)$. ν is defined as to satisfy

$$\langle C, \omega \rangle = \int_{\mathcal{S}} \int_{\pi\gamma} \omega d\nu(\gamma).$$

where $\pi : \mathbb{R} \times B^n \rightarrow B^n$ is the projection. From a 1-form $\omega \in C^\infty(B^n, \wedge^1 \mathbb{R}^n)$ we obtain a bounded (hence $L^1(\nu)$) function

$$f_\omega(\gamma) = \int_{\gamma|_{[0,1]}} \pi^* \omega,$$

where $\gamma|_{[0,1]}$ is the restriction of γ to the interval $[0, 1]$ (note that it would be equivalent to restrict the graph of γ to $[0, 1] \times B^n$ since γ has vertical speed 1). By the ergodic theorem the following limit exists ν -a.e. γ :

$$\bar{f}_\omega(\gamma) := \lim_{k \rightarrow \infty} \frac{1}{2^k} \sum_{j=-k}^{k-1} f_\omega(\sigma^j \gamma)$$

and it satisfies

$$\int \bar{f}_\omega d\nu = \int f_\omega d\nu.$$

By repeating this for a dense subset of ω 's it follows that ν -a.e. trajectory γ gives rise to an elementary solenoid corresponding to the curve $\pi\gamma$ and the measure ν decomposes C .

A.2 Decomposition of currents in metric spaces

Recently the above decomposition of 1-currents was generalized to 1-currents on metric spaces, in the sense of Ambrosio and Kirchheim. This result is due to E. Paolini and E. Stepanov. We recall how such currents are defined.

Definition A.7. Let E be a metric space. We denote by $\mathcal{D}^k(E)$ the space of $(k+1)$ -ples (f, π_1, \dots, π_k) of Lipschitz functions from E to \mathbb{R} such that f is also bounded. We also use the notation

$$(f, \pi_1, \dots, \pi_k) = f d\pi.$$

A function $T : \mathcal{D}^k(E) \rightarrow \mathbb{R}$ is called a metric functional if it is subadditive and positively homogeneous with respect to each variable.

Definition A.8. For $\omega = (f, \pi_1, \dots, \pi_k) \in \mathcal{D}^k(E)$ we define the exterior differential by

$$d\omega = (1, f, \pi_1, \dots, \pi_k).$$

We define the boundary ∂T of a metric functional T by $\partial T(\omega) = T(d\omega)$.

Definition A.9. A metric functional T has finite mass if for some finite measure μ on E there holds, for all $\omega = (f, \pi_1, \dots, \pi_k) \in \mathcal{D}^k(E)$,

$$|T(f, \pi_1, \dots, \pi_k)| \leq \prod \text{lip}(\pi_i) \int_E |f| d\mu.$$

We denote by $\|T\|$ the minimal measure satisfying the above inequality.

Definition A.10 (metric currents). A k -dimensional metric functional T of finite mass on E is called a k -dimensional metric current if

- T is multilinear in (f, π_1, \dots, π_k) ,
- $T(f, \pi_1^i, \dots, \pi_k^i) \rightarrow T(f, \pi_1, \dots, \pi_k)$ if the π_h^i converge pointwise and $\text{lip}(\pi_h) \leq C$,
- $T(f, \pi_1, \dots, \pi_k) = 0$ whenever π_j is constant on a neighborhood of $\{f \neq 0\}$.

The class of k -dimensional metric currents is denoted $\mathbb{M}_k(E)$.

One can associate a 1-current to a Lipschitz curve $\gamma : [a, b] \rightarrow E$ by defining

$$[\gamma](f d\pi) = \int_a^b f(\gamma(t)) d\pi(\gamma(t)).$$

We can repeat the same definitions as in the previous section also replacing B^n by a metric space E and for arcs we have the same identifications as in the previous section, once we replace $|\dot{\gamma}|$ by the metric differential of γ . The following decomposition theorem is still true if we use the above definitions:

Theorem A.11 ([99, 100]). *The statements of Theorem A.6 are still valid for 1-currents T on a metric space E , using the above definitions and notations to replace the ones valid for B^n and if instead of testing with $\omega \in C^\infty(B^n, \wedge^1 \mathbb{R}^n)$, $f \in C^\infty(B^n)$, $\phi \in C^0(B^n)$ we now use $\omega = fd\pi \in \mathcal{D}^1(E)$, $f \in Lip(E)$, $\phi \in C^0(E)$.*

Ideas of proof

The strategy in this case is in a sense the opposite to the one of the original proof by S.K. Smirnov summarized above: the decomposition result for cyclic currents C becomes a consequence of the one for acyclic currents A .

The procedure for proving that existence of a decomposition in the acyclic case implies the one in the cyclic case is as follows. One adds one dimension to the space, i.e. one considers on $[0, 1] \times E$ the current $C' = \|[0, 1]\| \times C + [0, 1] \times \|C\|$ (where we identify the interval $[0, 1]$ with the integration current on it). Then C' is acyclic. Using the decomposition for acyclic currents we find a Borel measure $\bar{\mu}$ on the space of Lipschitz arcs on $[0, 1] \times E$ decomposing C' and its boundary. Since $C' \llcorner \{0\} \times E$ and $C' \llcorner \{1\} \times E$ are just a translation of each other, we can use them to paste the arcs together and obtain a shift-invariant measure $\bar{\nu}$ on $Lip(\mathbb{R}, E)$ whose image on $Lip([0, 1], E)$ under restriction gives C as a superposition. It is then possible to apply the ergodic theorem in a way that parallels the one in Smirnov's proof, showing the existence of a limit as in the definition of a solenoid, for each γ in the support of ν by using the functions f_ω as before.

The proof of the decomposition of acyclic 1-currents A for metric spaces follows from the case $E = \mathbb{R}^n$ by approximation. More precisely, any metric space E is isometrically embedded in $\ell^\infty(E)$, which has the *metric approximation property* i.e. the property that that the identity is strongly approximated on compact sets by projections P_n to a sequence of finite-dimensional subspaces. The projected currents $(P_n)_\# T$ converge weakly to T and can be decomposed. The crucial result is then the following one:

Proposition A.12. *Let T_n be a sequence of 1-currents on E such that $T_n \rightharpoonup T$, $\mathbb{M}(T_n) \rightarrow \mathbb{M}(T)$ and T_n are decomposable as in the statement of Theorems*

A.11 and A.6. Then up to subsequence the decomposing measures μ_n given by Theorem A.11 converge weakly to a measure μ which decomposes T in the sense of Theorem A.11.

The above compactness result follows by an exhaustion argument done both on the space E and on the lengths of the decomposing curves. The source of this result is the following quantitative Ascoli-Arzelà result:

Proposition A.13. *Let $\gamma_n : [0, 1] \rightarrow E$ be Lipschitz curves with uniformly bounded Lipschitz constants and such that for all ϵ there exists a compact K such that $\mathcal{L}^1(\gamma_n^{-1}(E \setminus K)) \leq \epsilon$ for all n . Then there exists a subsequence of γ_n uniformly converging to a Lipschitz curve $\gamma : [0, 1] \rightarrow E$.*

A.3 No decomposition without the finite boundary condition

In Section B.22 we prove via an example (Example B.4.1) that there is no direct improvement of the Theorem A.6 in the case where we withdraw the condition of finite boundary mass. We reproduce that example below

A.4 Minimal measures and Schwartzman cycles

There are other related decompositions of weak objects by 1-curves which are related to Smirnov's decomposition. A source of interesting questions is the study of minimization of the norm of cycles in their homology class. One of the most relevant questions related to this topic is the question of *representing homology classes* in $H_1(M, \mathbb{R})$ by weak objects, behaving well enough under weak convergence.

A first approach to this question is **Mather theory** (we follow the treatment of [116] and [13]). The objects considered in this case are probability measures μ on the tangent bundle TM of a closed compact Riemannian manifold (M, h) . We require such μ to be invariant under the geodesic flow and to satisfy the finite moment condition

$$A(\mu) := \frac{1}{2} \int_{TM} |v|^2 d\mu(v) < \infty.$$

If ω is a smooth 1-form on M then the corresponding function $\tilde{\omega} : TM \rightarrow \mathbb{R}$ is automatically μ -integrable. From the fact that μ is invariant it follows that

for $\omega = d\alpha$ there holds $\int_{TM} \tilde{\omega} d\mu = 0$ and therefore μ 's action on 1-forms passes to the quotient $H^1(M, \mathbb{R})$, i.e. μ represents a homology class $[\mu] \in H_1(M, \mathbb{R})$. The problem

$$\min \{A(\mu) : [\mu] = h, \mu \text{ is invariant}\}$$

is well-posed and the minimum is achieved.

A second approach is presented by V. Bangert in [13] and consists in introducing **measures on the set of Lipschitz curves** $\mathcal{L} = \text{Lip}(\mathbb{R}, M)$. On \mathcal{L} we consider the \mathbb{R} -action by time-shift. The probability measures μ on \mathcal{L} which are invariant under this action and satisfy the finiteness condition

$$A(\mu) := \frac{1}{2} \int_{\mathcal{L}} |\dot{\gamma}(0)|^2 d\mu(\gamma) < \infty$$

are denoted by \mathcal{ML} . By shift-invariance the choice of the point where to compute $\dot{\gamma}$ above is immaterial and the action is well-defined. To see that measures in \mathcal{ML} correspond to 1-dimensional homology classes we again exploit the duality with 1-forms ω . This time for a smooth 1-form ω we consider the following action on $\gamma \in \mathcal{L}$:

$$\gamma \mapsto \omega(\dot{\gamma}(0)).$$

This defines a function f_ω which is μ -integrable for all $\mu \in \mathcal{ML}$. For $\omega = d\alpha$ there holds $\int_{\mathcal{L}} f_\omega d\mu = 0$ therefore μ again determines a well-defined homology class on which a minimization can be done.

A third possibility is to minimize mass among finite mass 1-cycles (i.e. a finite mass **1-currents with vanishing boundary**). The connection with the previous setting is as follows. A measure $\mu \in \mathcal{ML}$ gives such a current via the map

$$\nu \mapsto \int_{\mathcal{L}} \delta_{\dot{\gamma}(0)} d\nu(\gamma) = \vec{T}\sigma,$$

where $\sigma = (ev_0)_\# \mu$ is the pushforward measure of μ via the evaluation in zero $\gamma \mapsto \gamma(0)$ and the vector field \vec{T} is determined μ -a.e. by requiring $\vec{T}(\gamma(0)) = \dot{\gamma}(0)$ for μ -a.e. γ .

A.5 Higher dimensional decompositions

The decomposition question for k -currents (or of closely related objects as in the previous section) with $k > 1$ is in general much more difficult than in the

case $k = 1$, with the exception of the case of currents of codimension 1. We describe here some known results and some obstacles and conjectures in these cases.

The first question which we could ask is the following one:

Open Problem 14. *Assume T is a k -current in \mathbb{R}^n such that T and ∂T have finite mass. Can T be represented as a superposition via a finite measure on the space of integral rectifiable k -currents such that the statement of Theorem A.6 holds once we replace 1-forms and functions by k -forms and $(k-1)$ -forms respectively?*

A.5.1 The case of codimension 1

In [71] (see also the explanation in [141]) the following theorem was proven, giving a positive answer to Open Problem 14 in the case $k = n - 1$ if we know that the boundary of the current to be decomposed is rectifiable:

Theorem A.14 (Hardt-Pitts). *Assume that N is a finite mass $(n-1)$ -current in \mathbb{R}^n and ∂N has finite mass and is also integer rectifiable. Then there exists a family of integral $(n-1)$ -currents parameterized by $[0, 1]$ such that the decomposition of Theorem A.6 holds with $\mu = \mathcal{L}^1 \llcorner [0, 1]$.*

The proof of this result uses the fact that there exists a rectifiable $(n-1)$ -current R having the same boundary as N and $N - R$ can be represented as $\partial(\mathbb{E}^n \llcorner f)$ for a real-valued measurable function f . The decomposition result follows by foliating the difference $N - R$ by boundaries of sublevelsets of f . Note that this case is precisely analogous to the 1-dimensional version which was useful for us in Chapter 5: indeed also there we needed to decompose just 1-currents with integral boundary.

A.5.2 The higher codimension case

We need the following definition:

Definition A.15. *A k -vector ξ is called simple if it can be expressed as $\xi = v_1 \wedge \dots \wedge v_k$ for some 1-vectors v_1, \dots, v_k . Similarly a k -vector field is called simple if its values are simple vectors.*

A simple k -vector field is called integrable if the associated k -plane distribution is integrable in the sense of Frobenius' theorem.

We can now formulate a case in which the answer to Open Problem 14 is negative. In [141] the following result was obtained:

Proposition A.16. *Let N be a k -current of finite mass and finite boundary mass represented by a simple vector field $\xi \in L^1(\mathbb{R}^n, \wedge^k \mathbb{R}^n)$. If ξ is not integrable then N cannot be decomposed as in Theorem A.6, even if we withdraw the requirement that the boundary should be also decomposed.*

For example the 2-current of integration along the standard contact distribution in \mathbb{R}^3 is not decomposable.

Remark A.17. *Note that for $N = \xi \mathcal{L}^n$ as above the condition $\partial N = 0$ automatically implies the fact that ξ is integrable, thus the above result does not give information on cycles.*

Still in [141] it was proved that in the case of smooth simple k -vector fields integrability ensures decomposability:

Theorem A.18. *Let N be a k -current of finite mass and finite boundary mass represented by a simple integrable k -vector field $\xi \in C^\infty(\mathbb{R}^n, \wedge^k \mathbb{R}^n)$. Then each point has a neighborhood U such that $N \llcorner U$ is decomposable as in Theorem A.6.*

As (1) many k -vector fields are non-integrable and (2) no general result is available in the non-smooth case (see the two counterexamples in [141], Prop. 2 and 3), an interesting question is the following one:

Open Problem 15. *For $k \in [2, n - 2]$ describe a large enough class of k -currents such that a decomposition result like Theorem A.6 is available and at the same time the class has good weak closure properties.*

The above question is vague and leaves many choices open. We describe here some candidates.

Choice 1. One might argue that while in the 1-dimensional case any current can be decomposed into a superposition of members from two classes, i.e. the one of Schwartzman 1-cycles and the one of Lipschitz arcs, perhaps for currents of higher dimension one should *extend the number of available classes beyond usual integral cycles and integral submanifolds*. Perhaps including some well-behaved classes of non-integrable distributions it could be possible to achieve some elegant decomposition of any k -current.

It is very challenging to find a criterion for classifying such non-integrable distribution classes. In particular it seems that together with such new classes there is a need of introducing more restrictive notions of a “superposition”

suited to non-integrable k -currents. This new notion should be such that a partition refining the one into “cyclic part” and “acyclic part” as in the beginning of Theorem A.6 becomes available in higher dimension.

Choice 2. Since in geometric problems the role of submanifolds is a primary one, one might argue that desiring to decompose general k -currents for $k > 2$ is an overly abstract goal. Therefore it is tempting to restrict to the decomposition of more specialized classes of currents, motivated by particular geometric problems.

In the next section we describe some candidates for the second of the above choices, as available in the literature.

A.5.3 Higher dimensional geometric decompositions

A.5.4 Ruelle-Sullivan currents

The most popular smooth model for a decomposition result which has been used in geometric problems is given by the so-called *generalized Ruelle-Sullivan currents* (see [110]) and the main motivations come from ergodic theory. We start with a definition of a notion of foliated manifolds, the so-called k -solenoids. The reader should be warned that the “solenoids” of Definition A.1 are not directly related to the 1-solenoids of the next definition, but rather indirectly, i.e. they correspond to the 1-dimensional Schwartzman cycles which we treat below.

Definition A.19 ([96]). *A smooth k -solenoid is a compact Hausdorff space endowed with an atlas of flow-boxes (U_i, ϕ_i) such that*

$$\phi_i : U_i \rightarrow D^k \times K(U_i), \quad K(U_i) \subset \mathbb{R}^l,$$

such that the $K(U_i)$ are compact and the smooth changes of charts $\phi_{ij} := \phi_i \circ \phi_j^{-1}$ are of the form

$$\phi_{ij} = (X(x, y), Y(y)).$$

A *transversal* to a k -solenoid S is a set $T \subset S$ which is a finite union of sets $\phi_i^{-1}(K(U_i))$ as above. A *leaf* of a k -solenoid is a connected k -dimensional manifold such whose intersection with all flow-boxes is a finite union of local leaves $\phi_i^{-1}(D^k \times \{p\})$. Given two transversals T_1, T_2 and a path contained in a leaf and connecting a point of T_1 to a point of T_2 one can define a *holonomy*

map $h : T_1 \rightarrow T_2$. Such maps form a pseudo-group. A transversal measure μ to a k -solenoid is the assignment of a local measure μ_T on each transversal T in such a way that for any element of the holonomy pseudogroup $h : T_1 \rightarrow T_2$ there holds $h_{\#}\mu_{T_1} = \mu_{T_2}$.

Definition A.20. Let S, μ be a measured k -solenoid. Let $f : S \rightarrow M$ be an immersion of S , i.e. a smooth map whose differential has maximal rank k on all leaves of S . Consider a partition $S = \cup S_i$ such that each S_i is contained in a flow-box U_i . For ω a smooth k -form on M we may define the following k -current on M :

$$\langle (f, S_{\mu}), \omega \rangle = \sum_i \int_{K(U_i)} \left(\int_{\phi_i^{-1}(D_k \times \{y\}) \cap S_i} f^* \omega \right) d\mu_{K(U_i)}(y).$$

Such current is called a generalized Ruelle-Sullivan k -current.

Note that the above currents are automatically closed, therefore they define homology classes.

A.5.5 Schwartzman cycles

We now come to define the class generalizing the objects of Definition A.1, namely Schwartzman k -cycles:

Definition A.21. Let S be a k -solenoid. We say that S has controlled growth if it has a leaf l and an exhaustion (C_n) of it such that for any flow-box (U, ϕ) in a finite covering of S we have

$$\lim_{n \rightarrow \infty} \frac{\text{Vol}_k(B_n)}{\text{Vol}_k(A_n)} = 0$$

where $C_n \cap U = A_n \cup B_n$ is the partition where A_n is the part made of full disks $\phi^{-1}(D_k \times \{y\})$ and B_k is the part made of incomplete disks.

To a k -solenoid of complete growth as above we can define the following weak limit in duality with the space of k -forms on S

$$[S] = \lim_{n \rightarrow \infty} \frac{[C_n]}{\mathbb{M}([C_n])},$$

where $[C_n]$ is the current of integration along C_n .

In general Schwartzman cycles will not be closed unless we are able to control the boundary growth of the currents $[C_n]$. The way to ensure the good definition of related homology classes is to request the existence of a *sequence of small caps* E_n i.e. of k -dimensional submanifolds such that $[C_n] + [E_n]$ is a cycle and that $\mathbb{M}([E_n]) = o(\mathbb{M}([C_n]))$ so that the choice of the E_n is immaterial for the limit.

Appendix B

An optimal transport problem

B.1 Introduction

In this appendix we explain how Smirnov's decomposition theorem (see Theorem A.6) can be used for studying transport models with congestion effects. We will further describe how the minimization problem for weak $U(1)$ -curvatures can be interpreted as an extended transportation problem. This section is based on work in collaboration with Lorenzo Brasco [BP].

B.1.1 Transport problems with congestion effects and Smirnov decomposition

To start with, we formally define three variational problems which can be settled (for simplicity) on the closure of an open convex subset $\Omega \subset \mathbb{R}^N$ having smooth boundary. For the moment we are a little bit imprecise about the datum f but we will properly settle our hypotheses later. The first problem is the minimization of the total variation of a Radon vector measure under a divergence constraint:

$$\min_V \left\{ \int_{\Omega} d|V| : -\operatorname{div} V = f, V \cdot \nu_{\Omega} = 0 \right\}. \quad (\mathcal{B})$$

The above problem can be connected by duality with the following one, called the Kantorovich problem:

$$\max_{\phi} \{ \langle f, \phi \rangle : \|\nabla \phi\|_{L^{\infty}(\Omega)} \leq 1 \}, \quad (\mathcal{K})$$

where now the variable ϕ is a Lipschitz function and $\langle \cdot, \cdot \rangle$ represents a suitable duality pairing. Finally the third problem is the minimization of the total

length

$$\min_Q \left\{ \int_{\mathcal{P}} \ell(\gamma) dQ(\gamma) : (e_1 - e_0)_\# Q = f \right\}, \quad (\mathcal{M})$$

where \mathcal{P} is the space of Lipschitz continuous paths $\gamma : [0, 1] \rightarrow \Omega$, the *length functional* ℓ is defined by

$$\ell(\gamma) = \int_0^1 |\gamma'(t)| dt,$$

e_0, e_1 are the evaluation functions giving the starting and ending points of a path and the variable Q is a measure concentrated on \mathcal{P} .

The classical setting for the above problems is when f is of the form $f = f^+ - f^-$ where f^+ and f^- are positive measures on Ω having the same mass (for example conventionally one can consider them to be probability measures). We point out that in this case a more familiar formulation of (\mathcal{M}) is the so-called *Monge-Kantorovich problem*

$$\min_\eta \left\{ \int_{\Omega \times \Omega} |x - y| d\eta(x, y) : (\pi_x)_\# \eta = f^+ \quad \text{and} \quad (\pi_y)_\# \eta = f^- \right\}, \quad (\mathcal{M}')$$

where $\pi_x, \pi_y : \Omega \times \Omega \rightarrow \Omega$ stand for the projections on the first and second variable, respectively. It is useful to recall that the link between (\mathcal{M}) and (\mathcal{M}') is given by the fact that if η_0 is optimal for Monge-Kantorovich problem then the measure which concentrates on *transport rays* i.e.

$$Q_0 = \int \delta_{\overline{xy}} d\eta_0(x, y), \quad \text{where } \overline{xy} \text{ stands for the segment connecting } x \text{ and } y,$$

is optimal in (\mathcal{M}) and

$$\int_{\mathcal{P}} \ell(\gamma) dQ_0(\gamma) = \int_{\Omega \times \Omega} |x - y| d\eta_0(x, y).$$

When f has the above mentioned form $f^+ - f^-$ the equivalence of the three problems above is well understood. The equivalence of $(\mathcal{M}') = (\mathcal{M})$ and (\mathcal{K}) is the classical Kantorovich duality (see [82]), while that between (\mathcal{B}) and (\mathcal{K}) seems to have been first identified in [121].

Recently the equivalence of the above three problems has been proved in [26] for f belonging to a wider class, i.e. when f is in the completion of the space of zero-average measures with respect to the norm dual to the C^1 (or flat) norm. This wider space was studied in [68] and characterized recently in [26, 27]. A different point of view is also available in [88], where the space of such f is called $W^{-1,1}$.

Our starting observation is that problem (\mathcal{B}) pertains to a wide class of optimal transport problems introduced by Martin J. Beckmann in [14], which are of the form

$$\min_V \left\{ \int_{\Omega} \mathcal{H}(V) \, dx : -\operatorname{div} V = f, V \cdot \nu_{\Omega} = 0 \right\}, \quad \text{where } c_1 |z|^p \leq \mathcal{H}(z) \leq c_2 |z|^p, \tag{\mathcal{B}_{\mathcal{H}}}$$

for a suitable density-cost convex function $\mathcal{H} : \mathbb{R}^N \rightarrow \mathbb{R}^+$ and $p \geq 1$. For a problem of this type the question of finding equivalent formulations of the form (\mathcal{K}) and (\mathcal{M}) has already been addressed in [29] under some restrictive assumptions on f , like for example

$$f = f^+ - f^- \quad \text{with } f^+, f^- \in L^p(\Omega) \quad \text{and} \quad \int_{\Omega} f^+ = \int_{\Omega} f^- = 0.$$

We will study the problem $(\mathcal{B}_{\mathcal{H}})$ in its natural functional analytic setting, i.e. when f belongs to a dual Sobolev space $W^{-1,p}$ (whose elements are not measures, in general). We will also see that alternative formulations of the type (\mathcal{K}) and (\mathcal{M}) are still possible for $(\mathcal{B}_{\mathcal{H}})$ in this extended setting. These formulations are still well-posed on the dual space $W^{-1,p}$ and equivalence can be proved in this larger space. The problem corresponding to (\mathcal{K}) will now have the form (see Section 3 for more details)

$$\max_{\phi} \left[\langle f, \phi \rangle - \int_{\Omega} \mathcal{H}^*(\nabla \phi) \, dx \right], \tag{\mathcal{K}_{\mathcal{H}}}$$

and the equivalence with $(\mathcal{B}_{\mathcal{H}})$ will just follow by standard convex duality arguments (which are later recalled, for the convenience of the reader). On the contrary, in the proof of the equivalence between $(\mathcal{B}_{\mathcal{H}})$ and its Lagrangian formulation

$$\min_Q \left\{ \int_{\Omega} \mathcal{H}(i_Q) \, dx : (e_1 - e_0)_{\#} Q = f \right\}, \tag{\mathcal{M}_{\mathcal{H}}}$$

some care is needed and we will require to f be a finite measure belonging to $W^{-1,p}$. Here the measure i_Q will be some sort of *transport density*¹ generated by Q , which takes into account the amount of work generated in each region by our distribution of curves Q (see Section B.4 for the precise definition). In particular the proof of this equivalence will point out another not emphasized connection to Geometric Measure Theory.

The main result of this paper can be formulated as follows (see Theorems B.10 and B.19 for more precise statements):

¹When $\mathcal{H}(t) = |t|$ problem $(\mathcal{M}_{\mathcal{H}})$ is again the Monge-Kantorovich one and i_Q for an optimal Q is nothing but the usual concept of transport density, see [25, 40, 55].

Theorem B.1. *Let $1 < p < \infty$. Suppose $\Omega \subset \mathbb{R}^N$ is the closure of a smooth bounded open set, let $f \in W^{-1,p}(\Omega)$ and let \mathcal{H} be a strictly convex function having p -growth. Then the minimum in $(\mathcal{B}_{\mathcal{H}})$ and the maximum in $(\mathcal{K}_{\mathcal{H}})$ are achieved and coincide. Moreover the unique minimizer V of $(\mathcal{B}_{\mathcal{H}})$ and any maximizer v of $(\mathcal{K}_{\mathcal{H}})$ are linked by the relation $\nabla v \in \partial\mathcal{H}(V)$, as specified in Theorem B.10.*

If in addition f is a Radon measure then we have the following relationship among the optimizers of $(\mathcal{B}_{\mathcal{H}})$ and $(\mathcal{M}_{\mathcal{H}})$:

- (i) the unique minimizer of $(\mathcal{B}_{\mathcal{H}})$ corresponds to a minimizer of $(\mathcal{M}_{\mathcal{H}})$ in the sense of Proposition B.17;*
- (ii) each minimizer of $(\mathcal{M}_{\mathcal{H}})$ corresponds to the unique minimizer of $(\mathcal{B}_{\mathcal{H}})$ in the sense of Proposition B.17.*

The connection of the above theorem to Geometric Measure Theory lies in the basic theory of *normal 1-currents*, whose basic steps are recalled in the (long) appendix at the end of the paper. Indeed, in order to obtain equivalence of $(\mathcal{B}_{\mathcal{H}})$ and $(\mathcal{M}_{\mathcal{H}})$ our cornerstone is *Smirnov decomposition theorem* for 1-currents.

For the sake of completeness and in order to neatly motivate the studies performed in this paper it is worth recalling that the proof of this equivalence in [29] was based on the *Dacorogna-Moser construction* to produce transport maps (see [37]), which has revealed to be a powerful tool for optimal transport problems². In a nutshell, this method consists in associating to the “static” vector field V which is optimal for $(\mathcal{B}_{\mathcal{H}})$ the following dynamical system

$$\partial\mu_t + \operatorname{div} \left(\frac{V}{(1-t)f^+ + tf^-} \mu_t \right) = 0, \quad \mu_0 = f^+,$$

i.e. a continuity equation with driving velocity field \tilde{V}_t given by V rescaled by the linear interpolation between f^+ and f^- . Assuming that one can give a sense (either deterministic or probabilistic) to the flow of \tilde{V}_t , the construction of the measure Q_V concentrated on the flow lines of \tilde{V}_t paves the way to the equivalence between the Lagrangian model $(\mathcal{M}_{\mathcal{H}})$ and $(\mathcal{B}_{\mathcal{H}})$.

²It is worth remarking that the first proof of the existence of an optimal transport map for problem (\mathcal{M}') , more than 200 years after Monge stated it, was based on a clever use of this construction (see [50]).

B.2 Well-posedness of Beckmann's problem

Let $\Omega \subset \mathbb{R}^N$ be the closure of an open bounded connected set having smooth boundary. In what follows Ω will always be compact. Given $1 < q < \infty$ we indicate with $W^{1,q}(\Omega)$ the usual Sobolev space of $L^q(\Omega)$ functions whose distributional gradient is in $L^q(\Omega; \mathbb{R}^N)$ as well. We then define the quotient space

$$\dot{W}^{1,q}(\Omega) = \frac{W^{1,q}(\Omega)}{\sim},$$

where \sim is the equivalence relation defined by

$$u \sim v \iff \text{there exists } c \in \mathbb{R} \text{ such that } u(x) - v(x) = c \text{ for a.e. } x \in \Omega.$$

When needed the elements of $\dot{W}^{1,q}(\Omega)$ will be identified with functions in $W^{1,q}(\Omega)$ having zero mean. We endow the space $\dot{W}^{1,q}(\Omega)$ with the norm

$$\|u\|_{\dot{W}^{1,q}(\Omega)} := \left(\int_{\Omega} |\nabla u|^q dx \right)^{\frac{1}{q}}, \quad u \in W^{1,q}(\Omega),$$

then we denote by $\dot{W}^{-1,p}(\Omega)$ its dual space, equipped with the dual norm. The latter is defined as usual by

$$\|T\|_{\dot{W}^{-1,p}(\Omega)} := \sup \left\{ \langle T, \varphi \rangle : \varphi \in \dot{W}^{1,q}(\Omega), \|\varphi\|_{\dot{W}^{1,q}} = 1 \right\},$$

where $p = q/(q - 1)$. We start recalling the following basic fact.

Lemma B.2. *Let $T \in \dot{W}^{-1,p}(\Omega)$. Then*

$$\|T\|_{\dot{W}^{-1,p}(\Omega)} = p^{\frac{1}{p}} \left[\max_{\varphi \in \dot{W}^{1,q}(\Omega)} |\langle T, \varphi \rangle| - \frac{1}{q} \int_{\Omega} |\nabla \varphi|^q dx \right]^{\frac{1}{p}}.$$

Proof. For every $\varphi \in \dot{W}^{1,q}(\Omega)$ we have

$$|\langle T, \varphi \rangle| - \frac{1}{q} \int_{\Omega} |\nabla \varphi|^q dx \leq \sup_{\lambda \geq 0} \left[\lambda |\langle T, \varphi \rangle| - \frac{\lambda^q}{q} \int_{\Omega} |\nabla \varphi|^q dx \right].$$

On the other hand the supremum on the right is readily computed: this corresponds to the choice

$$\lambda = |\langle T, \varphi \rangle|^{\frac{1}{q-1}} \left(\int_{\Omega} |\nabla \varphi|^q dx \right)^{-\frac{1}{q-1}},$$

which gives

$$\sup_{\lambda \geq 0} \left[\lambda |\langle T, \varphi \rangle| - \frac{\lambda^q}{q} \int_{\Omega} |\nabla \varphi|^q dx \right] = \frac{1}{p} \left(\frac{|\langle T, \varphi \rangle|}{\|\varphi\|_{\dot{W}^{1,q}}} \right)^p.$$

Passing to the supremum over $\varphi \in \dot{W}^{1,q}(\Omega)$ and using the definition of the dual norm we get the thesis. \square

We also denote by $\mathcal{E}'_1(\Omega)$ the space of distributions of order 1 with (compact) support in Ω . In what follows we tacitly identify this space with the dual of the space $C^1(\Omega)$, endowed with the norm

$$\|\varphi\|_{C^1(\Omega)} = \sup_{x \in \Omega} |\varphi(x)| + \sup_{x \in \Omega} |\nabla \varphi(x)|.$$

We denote by ν_Ω the outer normal unit vector to $\partial\Omega$. We have the following characterization for the dual space $\dot{W}^{-1,p}(\Omega)$.

Lemma B.3. *Let $p = q/(q-1)$. We say that a vector field $V \in L^p(\Omega; \mathbb{R}^N)$ and $T \in \mathcal{E}'_1(\Omega)$ satisfies*

$$-\operatorname{div} V = T \quad \text{in } \Omega, \quad V \cdot \nu_\Omega = 0 \quad \text{on } \partial\Omega, \quad (\text{B.1})$$

if

$$\int_{\Omega} \nabla \varphi \cdot V \, dx = \langle T, \varphi \rangle, \quad \text{for every } \varphi \in C^1(\Omega).$$

If we set

$$\mathcal{E}'_{1,p}(\Omega) = \{T \in \mathcal{E}'_1(\Omega) : \text{there exists } V \in L^p(\Omega; \mathbb{R}^N) \text{ satisfying (B.1)}\},$$

we then have the identification

$$\dot{W}^{-1,p}(\Omega) = \mathcal{E}'_{1,p}(\Omega).$$

Proof. Let $T \in \dot{W}^{-1,p}(\Omega)$. We observe that then $T \in \mathcal{E}'_1(\Omega)$ as well. Now consider the following maximization problem

$$\sup_{v \in \dot{W}^{1,q}(\Omega)} \langle T, v \rangle - \frac{1}{q} \int_{\Omega} |\nabla v|^q \, dx.$$

By means of the Direct Methods, it is not difficult to see that there exists a (unique) maximizer $u \in \dot{W}^{1,q}(\Omega)$ for this problem. Moreover such a maximizer satisfies the relevant Euler-Lagrange equation given by

$$\int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla \varphi \, dx = \langle T, \varphi \rangle, \quad \text{for every } \varphi \in \dot{W}^{1,q}(\Omega).$$

By taking $V = |\nabla u|^{q-2} \nabla u \in L^p(\Omega; \mathbb{R}^N)$, the previous identity implies $T \in \mathcal{E}'_{1,p}(\Omega)$.

Conversely let us take $T \in \mathcal{E}'_{1,p}(\Omega)$. Then for every $\varphi \in C^1(\Omega)$ equation (B.1) implies

$$|\langle T, \varphi \rangle| = \left| \int_{\Omega} \nabla \varphi \cdot V \, dx \right| \leq \|\varphi\|_{\dot{W}^{1,q}} \|V\|_{L^p(\Omega)}.$$

Using the density of $C^1(\Omega)$ in $\dot{W}^{1,q}(\Omega)$ we obtain that T can be extended in a unique way as an element (that we still denote T for simplicity) of $\dot{W}^{-1,q}(\Omega)$. This extension satisfies

$$\|T\|_{\dot{W}^{-1,p}(\Omega)} \leq \|V\|_{L^p(\Omega)},$$

as can be seen by taking the supremum in the previous inequality. □

Remark B.4. We remark that the elements of $\mathcal{E}'_{1,p}(\Omega)$ have “zero average” i.e.

$$\langle T, 1 \rangle = 0,$$

as follows by testing the weak formulation of (B.1) with $\varphi \equiv 1$. This is coherent with the previous identification $\dot{W}^{-1,p}(\Omega) = \mathcal{E}'_{1,p}(\Omega)$ since by construction the space $\dot{W}^{1,q}(\Omega)$ does not contain any non trivial constant function.

Example B.5. Consider the measure $T = \delta_a - \delta_b$ for two points $a \neq b \in \mathbb{R}^N$. We claim that

$$T = \delta_a - \delta_b \in \dot{W}^{-1,p}(\Omega) \quad \text{if and only if} \quad 1 \leq p < N/(N - 1),$$

where Ω is a sufficiently large ball containing a, b in its interior. We prove this by using the characterization of Lemma B.3.

Suppose indeed that there exists some $V \in L^p(\Omega)$ such that $-\operatorname{div} V = T$. We pick a ball $B_r(a)$ centered at a and having radius r such that $2r < |a - b|$. Then for each $\varepsilon < r$ we consider a $C^1_0(B_r(a))$ function η_ε such that

$$\eta_\varepsilon \equiv 1 \quad \text{in } B_{r-\varepsilon}(a) \quad \text{and} \quad \|\nabla \eta_\varepsilon\|_{L^\infty} \leq C \varepsilon^{-1}.$$

Thanks to our assumption we have

$$1 = \langle T, \eta_\varepsilon \rangle = \int_{B_r(a)} V \cdot \nabla \eta_\varepsilon \, dx,$$

so that

$$\int_{B_r(a) \setminus B_{r-\varepsilon}(a)} |V| \, dx \geq \frac{\varepsilon}{C}.$$

By Hölder inequality this easily implies a lower bound on the L^p norm of V , namely

$$\int_{B_r(a)} |V|^p \, dx \geq \varepsilon^p |B_r(a) \setminus B_{r-\varepsilon}(a)|^{1-p} = C_{N,p} \varepsilon^p r^{N(1-p)} \left[1 - \left(1 - \frac{\varepsilon}{r} \right)^N \right]^{1-p}.$$

We now make the choice $\varepsilon = r/2$, so that from the previous we can infer

$$\int_{B_r(a)} |V|^p \, dx \geq \tilde{C}_{N,p} r^{p+N(1-p)} = \tilde{C}_{N,p} r^{N-p(N-1)}.$$

The previous estimate clearly contradicts the assumption $V \in L^p(\Omega)$ if the exponent $N - p(N - 1)$ is not strictly positive. Therefore we see by Lemma B.3 that $p < N/(N - 1)$ is a necessary condition for $T \in \dot{W}^{-1,p}(\Omega)$.

This condition on p is also sufficient for T to belong to $\dot{W}^{-1,p}(\Omega)$, as we now proceed to prove. Set $2\tau = |a - b|$ and for simplicity assume that $a = (-\tau, 0, \dots, 0)$ and $b = (\tau, 0, \dots, 0)$. We use the notation $x = (x_1, x')$ for a generic point in \mathbb{R}^N , where $x' \in \mathbb{R}^{N-1}$. We define the following vector field

$$V_{a,b}(x) = \begin{cases} \frac{(x_1 + \tau, x')}{(x_1 + \tau)^N}, & \text{if } |x'| \leq \tau \text{ and } |x'| - \tau \leq x_1 \leq 0, \\ \frac{(x_1 - \tau, x')}{(x_1 - \tau)^N}, & \text{if } |x'| \leq \tau \text{ and } \tau - |x'| \geq x_1 \geq 0, \\ (0, \dots, 0), & \text{otherwise.} \end{cases}$$

It is easily seen that $\operatorname{div} V_{a,b} = \delta_a - \delta_b$ and that $V_{a,b}$ is supported on the set

$$D_{a,b} = \left\{ (x_1, x') \in \mathbb{R}^N : \frac{|a - b|}{2} \geq |x'| + |x_1| \right\},$$

which is just the union of two cones centered at a and b having opening 1 and height $\tau = |a - b|/2$. By construction we have

$$\begin{aligned} \int_{Q_{a,b}} |V_{a,b}|^p dx &= 2 \int_{-\tau}^0 \int_{\{|x'| = |x_1 + \tau|\}} \frac{\left(\sqrt{(x_1 + \tau)^2 + |x'|^2}\right)^p}{(x_1 + \tau)^{Np}} dx' dx_1 \\ &= 2^{\frac{p+2}{2}} N \omega_N \int_{-\tau}^0 (x_1 + \tau)^{-Np+p+N-1} dx_1 \end{aligned}$$

so that finally

$$\|V_{a,b}\|_{L^p}^p \leq C_{N,p} |a - b|^{N-p(N-1)},$$

thanks to our assumption $p < N/(N - 1)$.

As a consequence of Lemma B.3 we have the following well-posedness result for Beckmann's problem.

Proposition B.6. *Let $1 < p < \infty$. Let $\mathcal{H} : \Omega \times \mathbb{R}^N$ be a Carathéodory function such that $z \mapsto \mathcal{H}(x, z)$ is convex on \mathbb{R}^N for every $x \in \Omega$. We further suppose that \mathcal{H} satisfies the growth conditions*

$$\lambda(|z|^p - 1) \leq \mathcal{H}(x, z) \leq \frac{1}{\lambda}(|z|^p + 1), \quad (x, z) \in \Omega \times \mathbb{R}^N \quad (\text{B.2})$$

for some $0 < \lambda \leq 1$. Then the following problem

$$\min_{V \in L^p(\Omega; \mathbb{R}^N)} \left\{ \int_{\Omega} \mathcal{H}(x, V) dx : -\operatorname{div} V = T, \quad V \cdot \nu_{\Omega} = 0 \right\} \quad (\text{B.3})$$

admits a minimizer with finite energy if and only if $T \in \dot{W}^{-1,p}(\Omega)$.

Proof. Let $T \in \dot{W}^{-1,p}(\Omega)$. Thanks to Lemma B.3 there exists at least one admissible vector field V_0 with finite energy, thus the infimum (B.3) is finite. If $\{V_n\}_{n \in \mathbb{N}} \subset L^p(\Omega; \mathbb{R}^N)$ is a minimizing sequence then the hypothesis (B.2) on \mathcal{H} guarantees that this sequence is weakly convergent to some $\tilde{V} \in L^p(\Omega; \mathbb{R}^N)$. Thanks to the convexity of \mathcal{H} the functional is weakly lower semicontinuous, i.e.

$$\begin{aligned} \int_{\Omega} \mathcal{H}(x, \tilde{V}) dx &\leq \liminf_{n \rightarrow \infty} \int_{\Omega} \mathcal{H}(x, V_n) dx \\ &= \min_{V \in L^p(\Omega; \mathbb{R}^N)} \left\{ \int_{\Omega} \mathcal{H}(x, V) dx : \begin{array}{l} -\operatorname{div} V = T, \\ V \cdot \nu_{\Omega} = 0 \end{array} \right\}. \end{aligned}$$

Moreover the vector field \tilde{V} is still admissible since

$$\int_{\Omega} \nabla \varphi \cdot \tilde{V} dx = \lim_{n \rightarrow \infty} \int_{\Omega} \nabla \varphi \cdot V_n dx = \langle T, \varphi \rangle, \quad \text{for every } \varphi \in C^1(\Omega),$$

by weak convergence. Therefore \tilde{V} realizes the minimum.

On the other hand suppose that $T \notin \dot{W}^{-1,p}(\Omega)$. Again thanks to Lemma B.3 we have that the set of admissible vector fields is empty so the problem is not well-posed. \square

We need the following definition.

Definition B.7. We say that a vector field $V \in L^1(\Omega; \mathbb{R}^N)$ is acyclic if whenever we can write $V = V_1 + V_2$ with $|V| = |V_1| + |V_2|$ and $\operatorname{div} V_1 = 0$ with homogeneous Neumann boundary condition, namely

$$\int_{\Omega} V_1 \cdot \nabla \varphi dx = 0, \quad \text{for every } \varphi \in C^1(\Omega),$$

there must result $V_1 \equiv 0$.

The following is a mild regularity result for optimizers of (B.3) in the isotropic case i.e. when \mathcal{H} depends on the variable z only through its modulus. This becomes crucial in order to equivalently reformulate (B.3) as a Lagrangian problem, where the transport is described by measures on paths.

Proposition B.8. *Assume that \mathcal{H} satisfies the hypotheses of Proposition B.6. In addition assume that*

$$z \mapsto \mathcal{H}(x, z) \text{ is a strictly convex increasing function of } |z| \text{ for every } x.$$

Then there exists a unique minimizer V for (B.3) and V is acyclic.

Proof. The uniqueness of V follows by strict convexity. We now prove that V is acyclic. Suppose that we can write $V = V_1 + V_2$ for some vector fields $V_1, V_2 \in L^1(\Omega; \mathbb{R}^N)$ such that

$$|V| = |V_1| + |V_2| \quad \text{and} \quad \operatorname{div} V_1 = 0.$$

It follows that $\operatorname{div} V = \operatorname{div} V_2$ and $|V| \geq |V_2|$. Thus V_2 is a competitor for problem (B.3) with energy not larger than that of V thanks to the monotonicity of \mathcal{H} . Since V is the unique minimizer, it must have energy equal to that of V_2 . Thus $|V| = |V_2|$ and $|V_1| = 0$ almost everywhere. This shows that V is acyclic, concluding the proof. \square

B.3 Duality for Beckmann's problem

We need the following general convex duality result (for the proof the reader is referred to [49, Proposition 5, page 89]). The statement has been slightly simplified in order to be directly adapted to our setting.

Theorem B.9. *Let $\mathcal{F} : Y \rightarrow \mathbb{R}$ be a convex lower semicontinuous functional on the reflexive Banach space Y . Let X be another reflexive Banach space and $A : X \rightarrow Y$ a bounded linear operator, with adjoint operator $A^* : Y^* \rightarrow X^*$. Then we have*

$$\sup_{x \in X} \langle x^*, x \rangle - \mathcal{F}(Ax) = \inf_{y^* \in Y^*} \{ \mathcal{F}^*(y^*) : A^*y^* = x^* \}, \quad x^* \in X^*, \quad (\text{B.4})$$

where $\mathcal{F}^* : Y^* \rightarrow \mathbb{R} \cup \{+\infty\}$ denotes the Legendre-Fenchel transform of \mathcal{F} . If the supremum in (B.4) is attained at some $x_0 \in X$ then the infimum in (B.4) is attained as well by a $y_0^* \in Y^*$ such that

$$y_0^* \in \partial \mathcal{F}(Ax_0).$$

Thanks to the above result we obtain that Beckmann's problem admits a dual formulation which is a classical elliptic problem in Calculus of Variations.

Theorem B.10 (Duality). *Let $1 < p < \infty$ and $q = p/(p - 1)$. Let \mathcal{H} be a function satisfying the hypotheses of Proposition B.6 and $T \in \dot{W}^{-1,p}(\Omega)$. Then*

$$\begin{aligned} \min_{V \in L^p(\Omega; \mathbb{R}^N)} \left\{ \int_{\Omega} \mathcal{H}(x, V) dx : \begin{array}{l} -\operatorname{div} V = T, \\ V \cdot \nu_{\Omega} = 0 \end{array} \right\} \\ = \max_{v \in \dot{W}^{1,q}(\Omega)} \left\{ \langle T, v \rangle - \int_{\Omega} \mathcal{H}^*(x, \nabla v) dx \right\}, \end{aligned} \quad (\text{B.5})$$

where \mathcal{H}^* is the partial Legendre-Fenchel transform of \mathcal{H} , i.e.

$$\mathcal{H}^*(x, \xi) = \sup_{z \in \mathbb{R}^N} \xi \cdot z - \mathcal{H}(x, z), \quad x \in \Omega, \xi \in \mathbb{R}^N.$$

Moreover if $V_0 \in L^p(\Omega)$ and $v_0 \in \dot{W}^{1,q}(\Omega)$ are two optimizers for the problems in (B.5) then we have the following primal-dual optimality condition

$$V_0 \in \partial \mathcal{H}^*(x, \nabla v_0) \quad \text{in } \Omega, \quad (\text{B.6})$$

where $\partial \mathcal{H}^*$ denotes the subgradient with respect to the ξ variable, i.e.

$$\partial \mathcal{H}^*(x, \xi) = \{z \in \mathbb{R}^N : \mathcal{H}^*(x, \xi) + \mathcal{H}(x, z) = \xi \cdot z\}, \quad x \in \Omega.$$

Proof. To prove (B.5) it is sufficient to apply the previous result with the choices

$$Y = L^q(\Omega; \mathbb{R}^N), \quad X = \dot{W}^{1,q}(\Omega), \quad \mathcal{F}(\phi) = \int_{\Omega} \mathcal{H}^*(x, \phi(x)) dx \quad \text{and} \quad A(\varphi) = \nabla \varphi.$$

The operator A is bounded since

$$\|A(\varphi)\|_Y = \|\nabla \varphi\|_{L^q(\Omega)} = \|\varphi\|_X, \quad \text{for every } \varphi \in X,$$

and

$$\mathcal{F}^*(\xi) = \int_{\Omega} \mathcal{H}^{**}(x, \xi(x)) dx = \int_{\Omega} \mathcal{H}(x, \xi(x)) dx,$$

since $\xi \mapsto \mathcal{H}(x, \xi)$ is convex and lower semicontinuous, for every $x \in \Omega$. We only need to compute the adjoint operator $A^* : L^p(\Omega; \mathbb{R}^N) \rightarrow \dot{W}^{-1,p}(\Omega)$. Let us define the map $\Psi : L^p(\Omega; \mathbb{R}^N) \rightarrow \mathcal{E}'_{1,p}(\Omega)$ by

$$\Psi(V) \in \mathcal{E}'_1(\Omega) \quad \text{such that} \quad \langle \Psi(V), \varphi \rangle = \int_{\Omega} \nabla \varphi \cdot V dx, \quad \text{for every } \varphi \in C^1(\Omega).$$

Observe that Ψ is a linear operator whose image is contained in $\mathcal{E}'_{1,p}(\Omega) = \dot{W}^{-1,p}(\Omega)$ by construction and by the definition of $\mathcal{E}'_{1,p}(\Omega)$. Moreover for $\varphi \in C^1(\Omega)$ and $V \in L^p(\Omega; \mathbb{R}^N)$ we have

$$\langle A \varphi, V \rangle = \int_{\Omega} \nabla \varphi \cdot V dx = \langle \varphi, \Psi(V) \rangle.$$

By density of $C^1(\Omega)$ in $W^{1,q}(\Omega)$ we obtain that $\Psi = A^*$, thus (B.5) follows from (B.4).

The primal-dual optimality condition (B.6) is a direct consequence of the second part of the convex duality result as well. It is sufficient to observe that the maximum in (B.5) is attained at some $v_0 \in \dot{W}^{1,p}(\Omega)$ by the Direct Methods. Thus, by the above convex duality theorem, a minimizer V_0 of Beckmann's problem has to satisfy

$$V_0 \in \partial\mathcal{F}(\nabla v_0),$$

which implies directly (B.6). \square

A significant instance of the previous result corresponds to $\mathcal{H}(x, z) = |z|^p$. Thanks to Lemma B.2 we have the following result.

Corollary B.11. *For every $T \in \dot{W}^{-1,p}(\Omega)$ we have*

$$\|T\|_{\dot{W}^{-1,p}(\Omega)} = \min_{V \in L^p(\Omega; \mathbb{R}^N)} \left\{ \|V\|_{L^p(\Omega)} : -\operatorname{div} V = T, \quad V \cdot \nu_\Omega = 0 \right\}.$$

Proof. It is sufficient to use (B.5) and Lemma B.2 and to observe that

$$\max_{\varphi \in \dot{W}^{1,q}(\Omega)} |\langle T, \varphi \rangle| - \frac{1}{q} \int_\Omega |\nabla \varphi|^q dx = \max_{\varphi \in \dot{W}^{1,q}(\Omega)} \langle T, \varphi \rangle - \frac{1}{q} \int_\Omega |\nabla \varphi|^q dx.$$

This establishes the thesis. \square

Corollary B.12. *Under the hypotheses of Theorem B.10 we have that the functional*

$$\begin{aligned} \mathfrak{F}_\mathcal{H} : \dot{W}^{-1,p}(\Omega) &\rightarrow \mathbb{R}^+ \\ T &\mapsto \text{minimal value (B.3)} \end{aligned}$$

is convex and weakly lower semicontinuous.

Proof. It is sufficient to observe that thanks to Theorem B.10 the value (B.3) can be written as a supremum of the affine continuous functionals L_φ defined by

$$L_\varphi(T) = \langle T, \varphi \rangle - \int_\Omega \mathcal{H}^*(x, \nabla \varphi) dx, \quad \varphi \in \dot{W}^{1,q}(\Omega).$$

Then the thesis follows. \square

Some comments are in order about the duality result of Theorem B.10.

Remark B.13 (Economic interpretation). *By the so-called Legendre reciprocity formula in Convex Analysis the primal-dual optimality condition (B.6) can be equivalently written as*

$$\nabla v_0 \in \partial \mathcal{H}(x, V_0), \quad \text{in } \Omega, \tag{B.7}$$

so this result is the rigorous justification of the necessary optimality conditions derived in [14, Lemma 2]. Such v_0 is called a Beckmann potential and its economic interpretation is that of an efficiency price, i.e. it represents a system price for moving commodities in the most efficient regime for a transport company. It can be seen as a generalization of a Kantorovich potential to a situation where the cost to move some unit of mass from x to y is not fixed. Indeed it depends on the quantity of traffic generated by the transport V_0 itself. Heuristically observe that in this case the minimal cost is given by the “congested metric”

$$d_{V_0}(x_0, x_1) = \min_{\gamma: \gamma(i)=x_i} \int_0^1 |\nabla \mathcal{H}(\cdot, V_0) \circ \gamma| |\gamma'(t)| dt.$$

In other words each mass particle is charged for the marginal cost it produces, the latter being the derivative of the function \mathcal{H} (we suppose for simplicity that \mathcal{H} possesses a true gradient and not just a subgradient). Then v_0 acts as a Kantorovich potential for the Optimal Transport problem

$$\min \left\{ \int_{\Omega \times \Omega} d_{V_0}(x, y) d\eta(x, y) : (\pi_x)_\# \eta = T_+ \quad \text{and} \quad (\pi_y)_\# \eta = T_- \right\},$$

where we assume for simplicity that $T = T_+ - T_-$, with T_+ and T_- positive measures having equal total masses. It should be remarked that $\nabla \varphi_0$ does not give the direction of optimal transportation in Beckmann’s problem since $\nabla \varphi_0$ and V_0 are only linked through the relation (B.7) and they are not parallel in general. They are guaranteed to be parallel only when the cost function \mathcal{H} is isotropic, i.e. when it just depends on $|V|$ for every admissible vector field V . This is the case studied by Beckmann in his original paper [14].

Remark B.14 (Regularity of optimal vector fields). *We point out that if $z \mapsto \mathcal{H}(x, z)$ is strictly convex then $\xi \mapsto \mathcal{H}^*(x, \xi)$ is C^1 . In this case the optimal V_0 is unique and we have*

$$V_0 = \nabla \mathcal{H}^*(x, \nabla v_0).$$

Then the regularity of the optimal vector field V_0 can be recovered from the regularity of a Beckmann potential, which solves the following elliptic boundary value problem

$$\begin{cases} -\operatorname{div} \nabla \mathcal{H}^*(x, \nabla u) = T, & \text{in } \Omega \\ \nabla \mathcal{H}^*(x, \nabla u) \cdot \nu_\Omega = 0, & \text{on } \partial \Omega. \end{cases} \tag{B.8}$$

For instance if \mathcal{H}^* is uniformly convex “at infinity”, meaning that there exist $C_1, C_2, M > 0$ such that

$$C_1(1 + |z|^2)^{\frac{q-2}{2}} \leq \min_{|\xi|=1} \langle D^2 \mathcal{H}^*(x, z) \xi, \xi \rangle, \quad \text{for every } |z| \geq M, x \in \Omega,$$

and such that

$$|D^2 \mathcal{H}^*(x, z)| \leq C_2(1 + |z|^2)^{\frac{q-2}{2}}, \quad (x, z) \in \Omega \times \mathbb{R}^N,$$

then V is bounded provided that $T \in L^{N+\varepsilon}(\Omega)$, with $\varepsilon > 0$. Indeed, in this case solutions to (B.8) are Lipschitz. These assumptions are verified for example by (see [29])

$$\mathcal{H}^*(z) = \frac{1}{q}(|z| - \delta)_+^q, \quad z \in \mathbb{R}^N,$$

where $(\cdot)_+$ stands for the positive part and where we assume $\delta \geq 0$, but they are violated by anisotropic functions of the type

$$\mathcal{H}^*(z) = \sum_{i=1}^N \frac{1}{q} (|z_i| - \delta_i)_+^q, \quad z = (z_1, \dots, z_N) \in \mathbb{R}^N.$$

B.4 A Lagrangian reformulation

The aim of this section is to introduce a Lagrangian counterpart of Beckmann’s model and to show how the two models turn out to be equivalent. The model we are going to present is a continuous version of a classical discrete model on networks by Wardrop (see [136]). This continuous model has already been addressed in [35] and the equivalence has been discussed in [29]. We prove well-posedness of the Lagrangian problem and equivalence of the models by imposing in addition that the datum T is a finite measure belonging to $\dot{W}^{-1,p}(\Omega)$. The proofs use Smirnov’s decomposition theorem for 1-currents (see Theorem A.6).

Given two Lipschitz curves $\gamma_1, \gamma_2 : [0, 1] \rightarrow \Omega$ we say that they are *equivalent* if there exists a continuous surjective nondecreasing function $\mathbf{t} : [0, 1] \rightarrow [0, 1]$ such that

$$\gamma_2(t) = \gamma_1(\mathbf{t}(t)), \quad \text{for every } t \in [0, 1].$$

We call $\mathcal{L}(\Omega)$ the set of all equivalence classes of Lipschitz paths in Ω . We introduce a topology on this set by defining the following distance

$$d(\gamma_1, \gamma_2) := \max \{ |\hat{\gamma}_1(t) - \hat{\gamma}_2(t)| : t \in [0, 1], \hat{\gamma}_i \text{ equivalent to } \gamma_i \}.$$

Observe that convergence in this metric is nothing but the usual uniform convergence, up to reparameterizations.

We denote the class of finite positive Borel (with respect to the above topology) measures on $\mathcal{L}(\Omega)$ by $\mathcal{M}_+(\mathcal{L}(\Omega))$. For $Q \in \mathcal{M}_+(\mathcal{L}(\Omega))$. We define the corresponding *traffic intensity* by

$$\langle i_Q, \varphi \rangle := \int_{\mathcal{L}(\Omega)} \left(\int_0^1 \varphi(\gamma(t)) |\gamma'(t)| dt \right) dQ(\gamma), \quad \varphi \in C(\Omega),$$

provided that the outer integral converges, in which case we say that “the traffic intensity i_Q exists”. If this is the case then the following integral also converges:

$$\langle \mathbf{i}_Q, \varphi \rangle = \int_{\mathcal{L}(\Omega)} \left(\int_0^1 \varphi(\gamma(t)) \cdot \gamma'(t) dt \right) dQ(\gamma), \quad \varphi \in C(\Omega; \mathbb{R}^N).$$

These definitions do not depend on the particular representative of the equivalence class we chosen, since the integrals in brackets are invariant under time reparameterization.

Remark B.15. *Observe that i_Q counts in a scalar way the traffic generated by Q while \mathbf{i}_Q computes it in a vectorial way. This means that in principle i_Q and $|\mathbf{i}_Q|$ could be very different: in \mathbf{i}_Q two huge amounts of mass going in opposite direction give rise to a lot of cancellations, as the orientation of curves is taken into account. As a simple example suppose to have two distinct points $x_0 \neq x_1$ and consider the measure*

$$Q = \frac{1}{2} \delta_{\gamma_1} + \frac{1}{2} \delta_{\gamma_2},$$

with $\gamma_1(t) = (1 - t)x_0 + tx_1$ and $\gamma_2(t) = (1 - t)x_1 + tx_0$. By computing the traffic intensity we obtain

$$i_Q = \mathcal{H}^1 \llcorner \overline{x_0x_1},$$

which takes into account the intuitive fact that on the segment $\overline{x_0x_1}$ globally there is a non negligible amount of transiting mass. On the other hand it is easily seen that

$$\mathbf{i}_Q \equiv 0.$$

Given a Radon measure T on Ω we define the following space

$$\mathcal{Q}_p(T) := \left\{ Q \in \mathcal{M}_+(\mathcal{L}(\Omega)) : i_Q \in L^p(\Omega) \text{ and } (e_1 - e_0)_\# Q = T \right\},$$

where $e_i : \mathcal{L}(\Omega) \rightarrow \Omega$ is defined by $e_i(\gamma) = \gamma(i)$, for $i = 0, 1$. Now consider a Carathéodory function $\mathcal{H} : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$\lambda (t^p - 1) \leq \mathcal{H}(x, t) \leq \frac{1}{\lambda} (t^p + 1), \quad x \in \Omega, t \in \mathbb{R}^+ \quad (\text{B.9})$$

for some $0 < \lambda \leq 1$ and such that

$$t \mapsto \mathcal{H}(x, t) \quad \text{is convex, for every } x \in \Omega.$$

If $\mathcal{Q}_p(T) \neq \emptyset$ then we define the following minimization problem:

$$\inf_{Q \in \mathcal{Q}_p(T)} \int_{\Omega} \mathcal{H}(x, i_Q(x)) dx. \quad (\text{B.10})$$

Remark B.16. *Similar Lagrangian formulations have been studied in connection with transport problems involving concave costs, e.g. problems where to move a mass m of a length ℓ costs $m^\alpha \ell$ ($0 < \alpha < 1$). For these the reader is referred to the monograph [17], as well as to the papers [101, 134].*

We prove that problem (B.10) is well-posed and equivalent to the one in (B.3). To this aim we use the following result, which is a reformulation of Theorem A.6 for the special case of currents representable by L^p -vector fields.

Proposition B.17. *Let $1 \leq p \leq \infty$. Assume that $V \in L^p(\Omega, \mathbb{R}^N)$ and that it is acyclic. Let $T = -\operatorname{div} V$ be a Radon measure on Ω . It is then possible to find $Q \in \mathcal{M}_+(\mathcal{L}(\Omega))$ such that*

$$(e_0)_\# Q = T_- \quad \text{and} \quad (e_1)_\# Q = T_+.$$

Moreover we have

$$\mathbf{i}_Q = V \quad \text{and} \quad i_Q = |V|.$$

In particular $Q \in \mathcal{Q}_p(T)$.

Thanks to the above result we can prove the following.

Proposition B.18. *Let T be a Radon measure on Ω . The set $\mathcal{Q}_p(T)$ is not empty if and only if $T \in \dot{W}^{-1,p}(\Omega)$.*

Proof. Let us suppose that $T \notin \dot{W}^{-1,p}(\Omega)$ and assume by contradiction that there exists $Q_0 \in \mathcal{Q}_p(T)$. In particular

$$\int_{\Omega} |i_{Q_0}|^p dx < +\infty. \quad (\text{B.11})$$

The vector measure \mathbf{i}_{Q_0} satisfies (B.1), since

$$\begin{aligned} \int_{\Omega} \nabla \varphi \cdot d\mathbf{i}_{Q_0} &= \int_{\mathcal{L}(\Omega)} \left(\int_0^1 \nabla \varphi(\gamma(t)) \cdot \gamma'(t) dt \right) dQ_0(\gamma) \\ &= \int_{\mathcal{L}(\Omega)} [\varphi(\gamma(1)) - \varphi(\gamma(0))] dQ_0(\gamma) = \langle T, \varphi \rangle, \end{aligned}$$

for every $\varphi \in C^1(\Omega)$. Thanks to the fact that $|\mathbf{i}_{Q_0}| \leq i_{Q_0}$ and to (B.11) we have that $\mathbf{i}_{Q_0} \in L^p(\Omega; \mathbb{R}^N)$. This contradicts the fact that $T \notin \dot{W}^{-1,p}(\Omega)$, as desired.

Now take $T \in \dot{W}^{-1,p}(\Omega)$. Then there exists a minimizer V of problem (B.3) with $\mathcal{H}(x, z) = |z|^p$. Thanks to Proposition B.8 we know that V is acyclic. Since T is a Radon measure we can apply Proposition B.17 and we infer the existence of $Q \in \mathcal{Q}_p(T)$. This gives directly the thesis. \square

We now prove our equivalence statement, which is the main result of this section. Observe that we prove at the same time existence of a minimizer for (B.10).

Theorem B.19. *Let $\mathcal{H} : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a Carathéodory function satisfying (B.9) and such that*

$$t \mapsto \mathcal{H}(x, t) \text{ is strictly convex and increasing, } \quad x \in \Omega.$$

If T is a Radon measure belonging to $\dot{W}^{-1,p}(\Omega)$ then we have

$$\inf_{Q \in \mathcal{Q}_p(T)} \int_{\Omega} \mathcal{H}(x, i_Q) dx = \min_{V \in L^p(\Omega; \mathbb{R}^N)} \left\{ \int_{\Omega} \mathcal{H}(x, |V|) dx : \begin{array}{l} -\operatorname{div} V = T, \\ V \cdot \nu_{\Omega} = 0 \end{array} \right\} \quad (\text{B.12})$$

and the infimum on the left-hand side is achieved.

Moreover, if $Q_0 \in \mathcal{Q}_p(T)$ is optimal then $\mathbf{i}_{Q_0} \in L^p(\Omega; \mathbb{R}^N)$ is a minimizer of Beckmann's problem. Conversely, if V_0 is optimal then there exists $Q_{V_0} \in \mathcal{Q}_p(T)$ such that $i_{Q_{V_0}} = |\mathbf{i}_{V_0}|$ minimizes the Lagrangian problem.

Proof. By the previous result the set $\mathcal{Q}_p(T)$ is not empty. For every admissible Q we have $|\mathbf{i}_Q| \leq i_Q$, therefore \mathbf{i}_Q is admissible for Beckmann's problem. Using the monotonicity of $\mathcal{H}(x, \cdot)$ we then obtain

$$\min_{V \in L^p(\Omega; \mathbb{R}^N)} \left\{ \int_{\Omega} \mathcal{H}(x, |V|) dx : \begin{array}{l} -\operatorname{div} V = T, \\ V \cdot \nu_{\Omega} = 0 \end{array} \right\} \leq \inf_{Q \in \mathcal{Q}_p(T)} \int_{\Omega} \mathcal{H}(x, i_Q(x)) dx < +\infty.$$

Now let $V_0 \in L^p(\Omega; \mathbb{R}^N)$ be a minimizer for Beckmann's problem. By Proposition B.8 V_0 is acyclic. Thus by Proposition B.17 there exists $Q_0 \in \mathcal{Q}_p(T)$ such that $|V_0| = i_{Q_0}$, i.e.

$$\min_{V \in L^p(\Omega; \mathbb{R}^N)} \left\{ \int_{\Omega} \mathcal{H}(x, |V|) dx : \begin{array}{l} -\operatorname{div} V = T, \\ V \cdot \nu_{\Omega} = 0 \end{array} \right\} = \int_{\Omega} \mathcal{H}(x, i_{Q_0}(x)) dx.$$

This implies that (B.12) holds true and that the infimum in the left-hand side is indeed a minimum.

The relation between minimizers of the two problems is an easy consequence of the previous constructions. \square

B.4.1 The case of flat currents

In Section B.4 we required $T \in \dot{W}^{-1,p}(\Omega)$ to be a Radon measure. As already mentioned, this further hypothesis permits to identify optimal vector fields for Beckmann's problems with acyclic normal currents. Then well-posedness and equivalence of the problems can be obtained by means of Smirnov's Theorem. However in the setting of Beckmann's problem and of its dual it would be natural to allow T to be a generic element of $\dot{W}^{-1,p}(\Omega)$. If one wishes to extend the analysis of the Lagrangian formulation to this larger space then one is naturally lead to consider a possible extension of Smirnov's result to L^1 vector fields having divergence which is not a Radon measure. Observe that such vector fields correspond to *flat currents*. In this subsection we investigate the possibility to have Smirnov's Theorem for such a class of currents.

We start by observing that the measure Q which decomposes I may not be finite in general.

Example B.20. For $1 \leq p < \frac{N}{N-1}$ we consider an infinite sequence of small dipoles $\{(a_i, b_i)\}_i$ such that

$$\sum_{i=1}^{\infty} |a_i - b_i|^{N-p(N-1)} < +\infty, \quad \text{and } D_{a_i, b_i} \text{ are disjoint,}$$

where the sets D_{a_i, b_i} are defined as in Example B.5. If we consider the vector fields V_{a_i, b_i} as in Example B.5 then the new vector field defined by $V = \sum_{i=1}^{\infty} V_{a_i, b_i}$ verifies

$$\|V\|_{L^p}^p = \left\| \sum_{i=1}^{\infty} V_{a_i, b_i} \right\|_{L^p}^p \leq C_N \sum_{i=1}^{\infty} |a_i - b_i|^{N-p(N-1)} < +\infty,$$

which implies $T := \sum_i (\delta_{a_i} - \delta_{b_i}) \in \dot{W}^{-1,p}(\Omega)$. By observing that $\infty = \mathbb{M}(T) = \int_{\mathcal{L}(\Omega)} dQ$ for any decomposing measure we see that no finite measure Q can be found. On the other hand a σ -finite measure Q can be found, since each V_{a_i, b_i} can be separately decomposed with a measure Q_i of mass 2 and the Q_i have disjoint supports.

Example B.21. We present now another version of Example B.20, which exploits the Sobolev embedding theorem. Let us take again $1 \leq p < N/(N-1)$, that is $q = p/(p-1) > N$. Then $\dot{W}^{1,q}(\Omega)$ can be identified with a space of functions which are Hölder continuous of exponent $\alpha = 1 - N/q$. We consider the following two curves

$$\gamma_1(t) = \frac{1}{t^{2/\alpha}} (\cos t, \sin t) \quad \text{and} \quad \gamma_2(t) = \frac{g(t)}{t^{2/\alpha}} (\cos t, \sin t), \quad t \geq 1,$$

where $g : [1, \infty) \rightarrow \mathbb{R}^+$ is a continuous function such that

$$1 > g(t) > \left(\frac{t}{t + 2\pi} \right)^{2/\alpha} \quad \text{and} \quad t \mapsto \frac{g(t)}{t^{2/\alpha}} \text{ is decreasing.}$$

We define the distribution

$$\langle T, \varphi \rangle = \int_1^\infty [\varphi(\gamma_1(t)) - \varphi(\gamma_2(t))] dt, \quad \varphi \in C^\infty(\Omega).$$

This is an element of $\dot{W}^{-1,p}(\Omega)$ since by Sobolev embedding $[\varphi]_{C^{0,\alpha}} \leq C_\Omega \|\varphi\|_{\dot{W}^{1,q}}$. Thus

$$\begin{aligned} |\langle T, \varphi \rangle| &\leq \int_1^\infty |\varphi(\gamma_1(t)) - \varphi(\gamma_2(t))| dt \leq C \|\varphi\|_{W^{1,q}(\Omega)} \int_1^\infty |\gamma_1(t) - \gamma_2(t)|^\alpha dt \\ &= C \|\varphi\|_{W^{1,q}(\Omega)} \int_1^\infty \frac{|1 - g(t)|^\alpha}{t^2} dt \\ &\leq 2^\alpha C \|\varphi\|_{W^{1,q}(\Omega)}, \quad \varphi \in \dot{W}^{1,q}(\Omega). \end{aligned}$$

We then introduce the measure on paths Q_T defined by

$$Q_T = \int_1^\infty \delta_{\overline{\gamma_1(t)\gamma_2(t)}} dt,$$

where for $t \geq 1$ we indicate by $\overline{\gamma_1(t)\gamma_2(t)}$ the straight segment going from $\gamma_1(t)$ to $\gamma_2(t)$. Observe that for every φ we have

$$\int_{\mathcal{L}(\overline{\Omega})} [\varphi(\gamma(0)) - \varphi(\gamma(1))] dQ_T(\gamma) = \int_1^\infty [\varphi(\gamma_1(t)) - \varphi(\gamma_2(t))] dt = \langle T, \varphi \rangle$$

and

$$\begin{aligned} \int_{\mathcal{L}(\overline{\Omega})} \ell(\gamma) dQ_T(\gamma) &= \int_1^\infty |\gamma_1(t) - \gamma_2(t)| dt = \int_1^\infty \frac{|1 - g(t)|}{t^{2/\alpha}} dt \\ &\leq \frac{2\alpha}{\alpha - 2} t^{1 - \frac{2}{\alpha}} \Big|_1^\infty = \frac{2\alpha}{2 - \alpha} < \infty, \end{aligned}$$

while Q_T is not finite, but just σ -finite.

The previous examples clarify that we cannot hope to give a distributional meaning to the positive and negative parts of the divergence of V . The good definition of $\operatorname{div} V$ as a distribution relies in general on some sort of “almost-cancellation”. Therefore the last no-cancellation requirement $(-\operatorname{div} V)_+ + (-\operatorname{div} V)_- = (e_1 + e_0)_\# Q$ of Smirnov’s Theorem A.6 should be relaxed when we try to extend it to a larger class of V ’s.

Actually we can say more. For general flat currents even the existence of a decomposition Q is not granted, as demonstrated by the following example.

Example B.22. Let $\Omega = [0, 1]^N \subset \mathbb{R}^N$ and let us consider a totally disconnected closed set $E \subset \Omega$ such that $\mathcal{L}^N(E) > 0$. We then pick a vector $\vartheta_0 \in \mathbb{R}^N \setminus \{0\}$ and set

$$V(x) = \vartheta_0 \cdot 1_E(x), \quad x \in \Omega.$$

Of course this is an $L^\infty(\Omega)$ vector field and the associated 1-current I_V is flat. We claim that I_V does not admit a Smirnov decomposition. Indeed, assume by contradiction that a decomposition $Q \in \mathcal{M}_+(\mathcal{L}(\Omega))$ satisfying the following condition exists:

$$\mu_{I_V} = \int_{\mathcal{L}(\Omega)} \mu_{[\gamma]} dQ(\gamma). \quad (\text{B.13})$$

We see that $\text{spt}(\mu_{I_V}) = \text{spt}(I_V)$ and from this we can infer that Q -a.e. curve γ has support included in $\text{spt}(I_V) = E$. Indeed, if the latter were not true then the supports of the two sides of (B.13) would differ, as follows by observing that the left-hand side is a superposition of the positive measures $\mu_{[\gamma]}$.

By knowing now that Q -a.e. curve γ has support in the totally disconnected set E and that the curves γ are connected, we deduce that Q -a.e. curve γ is constant. This implies that

$$\text{for } Q\text{-a.e. curve } \gamma \text{ there holds } [\gamma] = 0.$$

Therefore from (B.13) it follows $\mu_{I_V} = 0$, which contradicts the fact that

$$\mu_{I_V}(\Omega) = \mathbb{M}(I_V) = \int_{\Omega} |V| dx > 0.$$

Since the existence of Q satisfying (B.13) leads to a contradiction, we conclude that no Smirnov decomposition of I_V exists. Note that whether Q is assumed to be finite or only σ -finite is immaterial for this contradiction.

We point out that by defining the distribution T as

$$\langle T, \varphi \rangle = \int_{\Omega} V(x) \cdot \nabla \varphi(x) dx = \int_E \vartheta_0 \cdot \nabla \varphi(x) dx, \quad \text{for every } \varphi \in C^1(\Omega),$$

this can be considered as an element of $\dot{W}^{-1,p}(\Omega)$ for any $1 < p < \infty$, thanks to Lemma B.3.

We also claim that V (and thus I_V) is acyclic. Suppose that we can write $V = V_1 + V_2$ with $|V| = |V_1| + |V_2|$ and $\text{div} V_1 = 0$. This implies that $V_1 = \lambda_1 \vartheta_0 \cdot 1_E$, where $\lambda_1 \in L^1(\Omega)$ and it takes values in $[0, 1]$. In particular as above we have

$$0 = \langle -\text{div} V_1, \varphi \rangle = \int_E \lambda_1(x) \vartheta_0 \cdot \nabla \varphi(x) dx, \quad \text{for every } \varphi \in C^1(\Omega).$$

By taking $\varphi(x) = \vartheta_0 \cdot x$ we observe that the integral is nonzero unless $\lambda_1 \equiv 0$ a.e. on E , in which case $V_1 = 0$. By appealing to Definition B.7 we eventually prove that V is acyclic.

Appendix C

Combinatorial tools for regularity

In this Chapter we focus on the combinatorial part of the proof of the ϵ -regularity result of Chapter 5. The driving idea behind this appendix is that carefully considering combinatorial problems and their solutions one can obtain robust insights into geometric measure theory phenomena. We focus first on the classical maxflow-mincut theorem which we exploited directly (see Section C.1), then we consider its extension to infinite graphs (see Section C.2).

C.1 The classical Maxflow-mincut theorem

C.1.1 From general graphs to weighted graphs

Definition C.1. *We call a collection of elements of a set X a sequence of elements of X , possibly with repetitions, defined up to reordering.*

A general digraph G is given by the data (V, E) where V is a set of vertices and E is a collection of directed edges i.e. elements of $V \times V$. We similarly define general graph.

The underlying graph underlying a general digraph G is the graph $\bar{G} = (V, \bar{E})$ where \bar{E} is the collection of undirected edges corresponding to E , i.e. if $(x_1, x_2), (x_2, y_2), \dots$ is a sequence representing the collection E then the sequence of collections $\{x_1, y_1\}, \{x_2, y_2\} \dots$ is by definition a sequence representing the collection \bar{E} .

A capacity function on a digraph is a positive function $c : E \rightarrow \mathbb{R}^+$, i.e. a

function on the sequences defining the collection E which is invariant under reordering.

The classical definition consists in replacing collections with subsets in the above definition. Finite collections can be identified with *sets with multiplicities in \mathbb{N}* and collections without repetitions can be identified with sets, simplifying the notation. We will mostly consider the finitary setting. Starting from general (di)graphs also makes the introduction of weight functions $c : E \rightarrow \mathbb{R}$ on classical graphs very natural. Finitary general (di)graphs are identified with classical finitary graphs with multiplicities in \mathbb{N} ; as in the passage from the definition of \mathbb{N} to those of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, we can extend this class to the one of weighted graphs.

C.1.2 Flows and cuts

Etymology. The name of a “flow” on a graph models the mental representation where the network represents a network of connected tubes through which water (or another incompressible fluid) is flowing and of which we can measure just in the average flux through each tube. The water could flow through part of the tubes or all of them, between a “source” and a “sink”. To model the constraint on the flow of water given by the section of the tubes one introduces the notion of “capacity”. The word “flow” is also used to denote the total flow of water between the source and the sink. “Cutting” the graph corresponds to obstructing enough tubes that no flow of water is possible anymore.

We now state the mathematical definitions.

Definition C.2. Fix a (classical) connected locally finite digraph $G = (V, E)$ of which two sets $A, B \subset V$, called respectively source and sink sets, are selected and such that a function $c : E \rightarrow [0, \infty]$, called capacity, is given. A flow on G is a function $f : E \rightarrow \mathbb{R}^+$ such that for each vertex $v \in V \setminus (A \cup B)$ the sum

$$\sum_{x:(x,v) \in E} f(x, v) - \sum_{y:(v,y) \in E} f(v, y) = 0$$

and such that

$$\forall e \in E, \quad f(e) \leq c(e).$$

and for $a \in A, b \in B$

$$f(x, a) = 0 \text{ if } (x, a) \in E, \quad f(b, y) = 0 \text{ if } (b, y) \in E.$$

If G is finite then we also have

$$\sum_{x:(x,b) \in E, b \in B} f(x, b) = \sum_{y:(a,y) \in E, a \in A} f(a, y)$$

and this value is called the value of the flow f .

A flow on a (non-directed) locally finite graph G corresponding to data A, B, c as above is given by fixing a direction of the edges of G and a flow f on the so-obtained digraph.

If $G = (V, E)$ is a digraph with two fixed sets $A, B \subset V$ are selected then a cut is a subset $S \subset E$ such that the only flow f corresponding to the capacity c_S which is zero on S and ∞ outside it exists. When G is finitary this is equivalent to saying that all directed paths starting in A and ending in B intersect S .

A cut of a graph $G = (V, E)$ with fixed vertex subsets A, B is a set $S \subset E$ such that for all assignments of directions making G a digraph S is a cut of the resulting digraph.

For a (di)graph G the value of the cut S is given by the sum

$$\sum_{s \in S} c(s).$$

A cut S is saturated by a flow f if $S \subset \{f = c\}$.

C.1.3 The maxflow-mincut theorem

The classical version of the maxflow-mincut theorem is as follows:

Theorem C.3 (Maxflow-Mincut theorem). *Let G be a finite digraph with fixed source and sink sets A, B and everywhere-finite capacity function c . Then the maximum value of a flow between A and B is equal to the minimum value of a cut separating A from B .*

Moreover each maximal flow has a saturating minimal cut and each minimal cut has a saturating maximal flow.

The most direct proof of this result follows from the strong duality theorem of Linear Programming, since the problems of maximizing the value of the flow f and that of minimizing the value of the cut S under the constraint $f \leq c$ are dual to each other. The cut S is saturated by the flow f .

A good grasp of this proof is given by simple geometric considerations. Fix G and denote by $F_{A,B}$ the set of flows with data G, A, B and $c_\infty := \infty$, i.e. forgetting capacity constraint. Note that $F_{A,B}$ is a convex (unbounded) polytope with facets $\{F_{A',B'} : A' \subset A, B' \subset B\}$. Note that for finite c the constraint $f \leq c$ gives a convex bounded polytope P_c . The value of the flow is a linear function, thus it achieves its extrema on a subset of facets of $F_{A,B} \cap P_c$. From this consideration it can be deduced that *the above theorem is equivalent to the spacial case where A, B are singletons.*

The maxflow-mincut problem can also be proved in a somehow more natural combinatorial way, however it gives also algorithms for finding the minimum cut. See the original articles [62] and [98] or the treatment in [129].

The strategy for proving the full statement of Theorem C.3 is close to the ideas of Section C.1.1. We first reduce to the case where A, B are singletons by convexity.

If we prove the theorem for \mathbb{N} -valued capacities and flows then we can obtain general capacities by approximation. The case of \mathbb{N} -valued capacities can be reduced to a proof of a statement for general digraphs:

Theorem C.4 (Menger's theorem, cfr. [129], VI.48). *Let a, b be distinct vertices of a general digraph G as in Definition C.1. Then the maximal cardinality of a collection of paths from a to b with no common edge is equal to the minimal cardinality of a cut.*

The proof of this result contains very similar ideas to the combinatorial part of the ϵ -regularity Theorem 5.3. The fact that the cardinality of path collections substitutes the flow "multiplicity" f bears strong analogies to Smirnov's decomposition theorem A.6.

We include a proof also because our Definition C.1 is unusual and the definition of cut is slightly different than in [129].

Sketch of proof: One first proves that the maximizing path collection X contains no directed cycles. It follows that each cut has cardinality greater or equal to the one of the collection X .

Now assume by contradiction that all cuts had strictly higher cardinality than that of X . Then any cut consisting of one edge for each path of X would still allow a path γ from a to b . If γ intersects a path $\pi \in X$ on one edge e

then we might swap the edge $S \cap \pi$ with e obtaining a new cut of the same cardinality as X which now interrupts both π and γ . If this can be done for all such γ we reach a contradiction to our assumption. If on the contrary there exists a γ not intersecting any $\pi \in X$ then we contradict the maximality of the cardinality of X . \square

From the proof it also follows that path collections are saturated by some cut, and any cut saturates some collection X of paths.

C.2 Generalization to infinite networks

Before applying our combinatorial method to the ϵ -regularity proof in Chapter 5 we have to reduce to the case in which our weak curvature has finitely many charges, i.e. we need to first use the approximation theorem 1.19. One might wonder if this step can be avoided (see Question 3) and a possible approach would be to repeat the combinatorial constructions of Chapter 5 in an infinitary setting. We describe here an analogue of the Maxflow-mincut theorem valid for infinite digraphs, whose possible improvement would provide new tools in such direction. We start with a very general statement generalizing Menger's theorem and closing a conjecture which was open for more than 45 years:

Theorem C.5 ([2]). *Let G be a (possibly infinite) digraph in which we fix two vertex sets A, B . Then there exists a family X of disjoint finite paths from A to B and a cut S consisting of the choice of precisely one edge from each path in X .*

Note that while in the finitary case Menger's theorem implies the Maxflow-Mincut theorem as described above, this is not true for infinite digraphs. However, using techniques similar in spirit to the ones for above theorem the following result can be proved.

Theorem C.6 ([3]). *Let G be a countable digraph with no loops and let a, b be a source and a sink and c be a capacity on G . Then there exists a flow on G, a, b, c and a cut S which saturates f .*

We now describe why this result is not a consequence of Menger's theorem for infinite graphs. In order to connect this result to the infinite Menger's theorem we would need to consider so-called mundane flows, i.e. flows which are superposition of paths. This notion turns out to be incorrect, while a better notion is that of a flow respecting finite cuts

Definition C.7. A flow $f : G \rightarrow \mathbb{R}^+$ on a (possibly infinite) digraph G with source a and sink b is called *mundane* if there exists a family of finite paths $(\pi_i)_{i \in I}$ going from a to b and a family of positive real numbers $(\theta_i)_{i \in I}$ such that for all $x \in E$ there holds $f(x) = \sum_{i \in I} \theta_i \chi_{\pi_i}(x)$

A flow $f : G \rightarrow \mathbb{R}^+$ on a (possibly infinite) digraph G with source a and sink b is said to be *finite-cut respecting* if for each $A \subset V$ such that $a \in A$ and such that $S := \{(a, y) : a \in A, y \notin A\}$ and $S' := \{(x, a) : a \in A, x \notin A\}$ are finite there holds

$$\sum_{s \in S} f(s) = \begin{cases} \sum_{s' \in S'} f(s') & \text{if } a \in A, \\ \sum_{s' \in S'} f(s') + \sum_{x: (a,x) \in E} f(a, x) & \text{if } a \notin A. \end{cases}$$

The second definition has the effect of preventing only infinite paths which “escape to infinity” from contributing to f , whereas in the first one we are excluding categorically all infinite path. The following result taken from [3] implies that the first restriction is too drastic:

Theorem C.8. Let G, a, b, c be as in the previous theorem. Then

- The quantities

$$\inf \left\{ \sum_{s \in S} c(s) : S \text{ is a cut} \right\}$$

and

$$\sup \left\{ \sum_{x: (a,x) \in E} f(a, x) : f \text{ is a mundane flow} \right\}$$

are equal.

- The mincut is achieved.
- There exists a locally finite digraph on which the maxflow is not achieved among mundane flows.
- The maxflow is realized if we replace the class of mundane flows by class of paths respecting finite cuts.

Appendix D

Regularity results for elliptic complexes

In this Appendix we discuss the elliptic regularity theorems needed to prove the local regularity result of Chapter 5 after the proof of ϵ -regularity. Although these results are by now fairly classical, we describe them for the sake of completeness.

The original result is due to K.K. Uhlenbeck [130] who considered the regularity of nonlinear systems of the form

$$\begin{cases} d^*(|\omega|^{p-2}\omega) = 0, \\ d\omega = 0, \end{cases} \quad (\text{D.1})$$

for ω a k -form and an exponent $p \geq 2$. The above system is an element of a wider class of quasilinear elliptic systems for which a unified regularity theory is available. We present this wider setting here and describe the proofs. The regularity for $p < 2$ was proven by P. Tolksdorf [127], who considers systems of the form

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \sim \begin{cases} \operatorname{div}(|X|^{p-2}X) = 0, \\ \operatorname{curl}X = 0, \end{cases} \quad (\text{D.2})$$

Although this system is different from (D.1), the proof of regularity of [127] for it still applies in the wider setting of [130].

D.1 Elliptic complexes

We start with the simplest case of *constant coefficients*.

D.1.1 Constant coefficients

Consider a sequence V_i of finite dimensional vector spaces and linear operators

$$A_{i,k} : V_i \rightarrow V_{i+1}, \quad k = 1, \dots, n.$$

These operators define linear differential operators: if $u : \mathbb{R}^n \rightarrow V_i$ then we can define $A_i u : \mathbb{R}^n \rightarrow V_{i+1}$ as follows:

$$A_i u = \sum_{k=1}^n A_{i,k} \frac{\partial u}{\partial x_k}.$$

For each $\xi \in \mathbb{R}^n$ we then define the symbol $\sigma(A_i, \xi)$ by

$$\sigma(A_i, \xi) = \sum_{k=1}^n \xi_k A_{i,k}.$$

We say that the complex $\{A_i\}$ is elliptic if for all $\xi \neq 0$ the symbol complex

$$\dots \rightarrow V_{i-1} \xrightarrow{\sigma(A_{i-1}, \xi)} V_i \xrightarrow{\sigma(A_i, \xi)} V_{i+1} \xrightarrow{\sigma(A_{i+1}, \xi)} V_{i+2} \rightarrow \dots$$

is exact.

We can also define the dual complex $\{A_i^*\}$ by taking the duals of the operators A_i , defined as

$$A_i^* v = \sum_{k=1}^n A_{i,k}^* \frac{\partial v}{\partial x_k},$$

where $A_{i,k}^* : V_{i+1}^* \rightarrow V_i^*$ is the dual of $A_{i,k}$. In the presence of an inner product on the V_i we can identify $V_i^* = V_i$ and the operators A_i^* are the adjoint operators to the A_i . The adjoint complex

$$\dots \leftarrow V_{i-1}^* \xleftarrow{\sigma(A_{i-1}^*, \xi)} V_i^* \xleftarrow{\sigma(A_i^*, \xi)} V_{i+1}^* \xleftarrow{\sigma(A_{i+1}^*, \xi)} V_{i+2}^* \leftarrow \dots$$

is exact if and only if the original one is.

We can then define the analogues of the Laplacian:

$$\Delta_i := A_i A_i^* + A_{i+1}^* A_{i+1}.$$

The most common examples of elliptic complexes are the following ones, corresponding to the equations given at the beginning of the chapter.

Example D.1. Let $V_i = \wedge^{i-k+1}\mathbb{R}^n$ and $A_i = d$, the usual exterior differentiation. Then $A_i^* = d^*$. The ellipticity follows from the well-known formula $d \circ d = 0$. We obtain $\Delta_1 = dd^* + d^*d$, the usual Hodge Laplacian on k -forms.

Example D.2. We consider now the space V_k of k -forms on \mathbb{R}^n with values in \mathbb{R}^m . In particular we can define $V_{-1} = 0, V_0 = \mathbb{R}^m, V_1 = \mathbb{R}^m \times \mathbb{R}^n, V_2 = \mathbb{R}^m \times \wedge^2\mathbb{R}^n$. We then differentiate componentwise: $A_0u = (du_1, \dots, du_m)$ and for $v : \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}^n, v = (v_1, \dots, v_m)$ we have $A_0^*v = (d^*v_1, \dots, d^*v_m)$. We have also $A_1v = (dv_1, \dots, dv_m)$. If we identify 1-forms with vector fields we can reinterpret $A_0 = \nabla, A_0^* = \text{div}, A_1 = \text{curl}$. Then $\Delta_1 = A_0A_0^* + A_1^*A_1$ is the componentwise Laplacian on functions $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

D.1.2 Variable coefficients

The next case is the one where the linear operators $A_{i,k} : V_i \rightarrow V_{i+1}$ are allowed to vary depending on the point where they are applied, i.e. have the form

$$A_i(x)u(x) = \sum_{k=1}^n A_{i,k}(x) \frac{\partial u}{\partial x_k}(x).$$

We can repeat the above definitions also in this case, and ellipticity will mean that the symbols $\sigma(A_i(x), \xi)$ form an exact complex pointwise in both x, ξ for $\xi \neq 0$. It is however not true that the adjoint A_i^* of A_i is the operator with dual coefficients $\sum [A_{i,k}(x)]^* \partial_{x_k}$. We have indeed the formal computation

$$\begin{aligned} \langle A_i(x)u, v \rangle_{V_{i+1}} &= \sum \langle A_{i,k}(x) \partial_k u, v \rangle_{V_{i+1}} \\ &= \sum \langle \partial_k u, A_{i,k}^*(x)v \rangle_{V_i} \\ &= - \sum \langle u, \partial_k (A_{i,k}^*(x)v) \rangle_{V_i}, \end{aligned}$$

which shows that the difference between the dual and the adjoint is $\sum \partial_k A_{i,k}^*$, which is an operator of order zero, therefore does not appear in the symbol. It is therefore still true that the adjoint complex is elliptic if and only if the original one is. If the $A_{i,k}$ are regular enough then the regularity theory for complexes with variable coefficients can be regarded as a ‘‘perturbation’’ of the one with constant coefficients.

We can similarly extend the above definitions to the case where the functions $u : \mathbb{R}^n \rightarrow V_i$ are replaced by sections of smooth enough vector bundles $E_i \rightarrow M$ such that the fibre of E_i is V_i .

D.2 Uhlenbeck's result

The main theorem of [130] is the following:

Theorem D.3. *Let $\{A_i\}$ be a constant coefficient elliptic system as above, and assume that*

$$A_0 A_0^* + A_1^* A_1 = \Delta,$$

which is the usual Laplacian on functions $u : \mathbb{R}^n \rightarrow V_1$. Assume that $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous differentiable function such that for some constants $K > 0, p \geq 2, \alpha > 0, C \geq 0$ there holds, for $q = \frac{p-2}{2}$,

$$K^{-1}(Q + C)^q \leq \rho(Q) + 2Q\rho'(Q) \leq K(Q + C)^q, \quad (\text{ellipticity})$$

$$|\rho'(Q_1)Q_1 - \rho'(Q_2)Q_2| \leq K(Q_1 + Q_2 + C)^{q-\alpha}(Q_1 - Q_2)^\alpha. \quad (\text{growth})$$

Then any weak solution ω in a domain $D \subset \mathbb{R}^n$ of the equations

$$\begin{cases} A_0^*(\rho(|\omega|^2)\omega) = 0, \\ A_1\omega = 0, \end{cases}$$

which lies in $L^p(D)$ is Hölder-continuous in the interior of D .

Example D.4. *If we apply the theorem to the complex of Example D.1 then we obtain the system (D.1) and if we apply it to the complex of Example D.2 we obtain the system (D.2).*

Sketch of the arguments for Theorem D.3: The main difficulty and difference from weak solutions of the case $p = 2$

$$\begin{cases} A_0^*\omega = 0, \\ A_1\omega = 0, \end{cases} \sim \Delta\omega = 0$$

is the fact that the equation is not anymore uniformly elliptic, i.e. for $p > 2$ the coefficients could become zero if the solution is zero (while for $p < 2$ they could explode). Therefore the case $C = 0$ is significantly more difficult than the case $C > 0$.

Uhlenbeck considers the auxiliary quantity $H = H(|\omega|^2)$ where H is defined via ρ by requiring

$$H'(Q) = \frac{1}{2}\rho(Q) + Q\rho'(Q).$$

This quantity appears when we compute (with the notation $\rho := \rho(|\omega|^2)$):

$$\begin{aligned} \langle \omega, \Delta(\rho\omega) \rangle &= \sum_i (\partial_i \langle \omega, \partial_i(\rho\omega) \rangle - \langle \partial_i \omega, \partial_i(\rho\omega) \rangle) \\ &= \sum_i \partial_i (|\omega|^2 \rho' \partial_i |\omega|^2 + \rho \langle \omega, \partial_i \omega \rangle) \\ &\quad - \sum_i (\rho \langle \partial_i \omega, \partial_i \omega \rangle + \rho' \langle \partial_i \omega, \omega \rangle \partial_i |\omega|^2) \\ &= \Delta H - \sum_i (\rho |\partial_i \omega|^2 + 2\rho' |\langle \partial_i \omega, \omega \rangle|^2). \end{aligned}$$

From the relation between our operators A_i and the Laplacian and the ellipticity hypothesis of the theorem we obtain

$$0 = \langle \omega, \Delta(\rho\omega) \rangle - \langle \omega, A_1^* A_1(\rho\omega) \rangle \geq \Delta H - K^{-1}(|\omega|^2 + C)^q |\nabla \omega|^2 - B_\omega^* B_\omega(\rho)$$

where $B_\omega \phi := \sum_k A_{1,k} \omega \partial_k \phi = A_1(\phi\omega)$ for a scalar function ϕ since by hypothesis $A_1 \omega = 0$. If we write

$$\Delta H - B_\omega^* B_\omega(\rho) = L_\omega H, \quad L_\omega = \sum_{k,j} \partial_k (a_{kj} \partial_j)$$

then we obtain

$$a_{kj} = \delta_{kj} - \frac{\rho'}{H'} \langle A_{1,k} \omega, A_{1,j} \omega \rangle$$

and from the definition of H' and the ellipticity condition on ρ we obtain that L_ω is uniformly elliptic. The above inequality $K^{-1}(|\omega|^2 + C)^q |\nabla \omega|^2 \geq L_\omega H$ implies that H is a L_ω -subsolution and for all $\phi \in C^\infty(D, \mathbb{R}^+)$ there holds

$$\int_D \phi L_\omega H \geq K^{-1} \int \phi (|\omega|^2 + C)^q |\nabla \omega|^2.$$

Uhlenbeck then proves the estimate

$$\int_{B(x,r)} (|\omega|^2 + C)^q |\nabla \omega|^2 \lesssim r^{-2} \int_{B(x,2r)} |\omega|^4 (|\omega|^2 + C)^q$$

by using the equation and a difference quotient method. This allows to use a modification of the Moser iteration technique [92] for subsolutions to obtain the following control on H :

$$\begin{cases} \max_{y \in B(x,r)} H \lesssim r^{-n} \int_{B(x,4r)} (|\omega|^2 + C)^{q+1}, \\ r^{-n} \int_{B(x,3r)} (H(M(4r)) - H) \lesssim H(M(4r)) - H(M(r)), \end{cases}$$

where

$$M(\rho) := \operatorname{ess\,max}_{y \in B(x,\rho)} |\omega|^2(y).$$

This allows to prove the following crucial step of the proof:

Proposition D.5. *There exists $\lambda > 0$ such that the following holds. Let ω be bounded in $B(x, r)$. Then for $\rho \leq r/4$ either*

$$M(\rho) + C \leq (1 - \lambda)(M(4\rho) + C)$$

or there exists a constant ω_0 such that

$$\rho^{-1}(|\omega_0|^2 + C)^q \int_{B(x, \rho)} |\omega - \omega_0|^2 \lesssim \lambda(M(4\rho) + C)^{q+1}$$

and we can choose ω_0 such that $|\omega_0|^2 + C \gtrsim_\lambda M(4\rho) + C$.

In the first case we obtain directly the Hölderianity of ω . In the second case we prove the Hölderianity by “harmonic approximation”. One linearizes the equation $A_0^*(\rho\omega) = 0$ at ω_0 , in which case we obtain the operator

$$\bar{A}_0^* \tilde{\omega} = A_0^* \left(\tilde{\omega} + 2 \frac{\rho'_0}{\rho_0} \langle \tilde{\omega}, \omega_0 \rangle \omega_0 \right)$$

where ρ_0, ρ'_0 are abbreviations of $\rho(|\omega_0|^2), \rho'(|\omega_0|^2)$ respectively. the operator \bar{A}_0^* is the adjoint of A_0 with respect to the inner product

$$\langle \omega_1, \omega_2 \rangle' := \langle \omega_1, \omega_2 \rangle + 2 \frac{\rho'_0}{\rho_0} \langle \omega_1, \omega_0 \rangle \langle \omega_2, \omega_0 \rangle,$$

which is positive definite as long as $\omega_0 \neq 0$. The idea is to compare a solution of \bar{A}_0^* to $\omega - \omega_0$. We use the elliptic complex to simplify the estimates, i.e. we define

$$\begin{cases} A_0 \phi = \omega \\ A_{-1}^* \phi = 0 \end{cases} \quad \begin{cases} A_0 \phi_0 = \omega_0 \\ A_{-1}^* \phi_0 = 0 \end{cases}$$

and we define $\tilde{\omega} := A_0 \tilde{\phi}$ for $\tilde{\phi}$ solving

$$\begin{cases} \bar{A}_0^* A_0 \tilde{\phi} + A_{-1} A_{-1}^* \tilde{\phi} = 0 & \text{on } B(x, 4\rho), \\ \tilde{\phi} = \phi - \phi_0 & \text{on } \partial B(x, 4\rho). \end{cases}$$

Since $\tilde{\phi}$ is also a minimizer of the Dirichlet energy

$$\int_B \langle A_0 \tilde{\phi}, A_0 \tilde{\phi} \rangle' + \langle A_{-1}^* \tilde{\phi}, A_{-1}^* \tilde{\phi} \rangle$$

for fixed boundary value we have the control (using also the equations verified by ϕ, ϕ_0)

$$\begin{aligned} \int_B \langle \tilde{\omega}, \tilde{\omega} \rangle' &\leq \int_B \langle A_0 \tilde{\phi}, A_0 \tilde{\phi} \rangle' + \langle A_{-1}^* \tilde{\phi}, A_{-1}^* \tilde{\phi} \rangle \\ &\leq \int_B \langle A_0 \phi - \phi_0, A_0 \phi - \phi_0 \rangle' + \langle A_{-1}^* \phi - \phi_0, A_{-1}^* \phi - \phi_0 \rangle \\ &= \int_B \langle \omega - \omega_0, \omega - \omega_0 \rangle' \lesssim \int_B |\omega - \omega_0|^2 \end{aligned}$$

and uniform bounds of $\tilde{\omega}, \nabla \tilde{\omega}$ on a smaller ball follow via elliptic estimates.

The growth hypothesis of Theorem D.3 is used to prove a bound using the Taylor remainder

$$\rho\omega - \rho_0\omega_0 - \rho'_0\langle\omega_0, \omega - \omega_0\rangle\omega_0 := \rho_0G(\omega, \omega_0).$$

We obtain that for $u = \tilde{\omega} - (\omega - \omega_0)$ there holds

$$\int_{B(x, 2\rho)} |u|^2 \lesssim \int_{B(x, 2\rho)} |G(\omega, \omega_0)|^2 \lesssim \frac{(M(2\rho) + C)^{2q}}{(|\omega_0|^2 + C)^{2q+\alpha}} \left(\int_{B(x, 4\rho)} |\omega - \omega_0|^2 \right)^{1+\alpha}.$$

From this estimate it follows that on smaller balls we can obtain better and better approximations. The situation up to rescaling such that $B(x, 2\rho)$ becomes $B(1)$ is the following: if

$$\int_{B(1)} |\omega - \omega_0|^2 = \epsilon(M(1) + C), \quad |\omega_0|^2 + C = \eta(M(1) + C)$$

then using the elliptic bounds for $\tilde{\omega}$ and the above estimate for u it is possible to find another constant ω_1 which approximates ω better on the smaller ball $B(r), r < \frac{1}{4}$:

$$\int_{B(r)} |\omega - \omega_1|^2 \lesssim (r^{n+2} + \eta^{-2q}\epsilon^\alpha) \int_{B(2)} |\omega - \omega_0|^2.$$

As expected from the mean value inequality for $\tilde{\omega}$, the correct choice above is ω_1 such that

$$\omega_1 - \omega_0 = \int_{B(1/2)} \tilde{\omega}.$$

We can then iterate the procedure and obtain a Morrey decay, thus Hölderianity. What ensures that the procedure is feasible is the behavior of the constants in Proposition D.5. \square

D.3 Changes for the case $p < 2$

For exponents $p < 2$ the power q appearing in the hypothesis of Theorem D.3 becomes negative. The difference quotient method for obtaining the integrability of $(1 + |\omega|^2)^q |\nabla \omega|^2$ as above can be recovered by a subtler estimate even for $q < 0$ (see [127], Lem. 4.1).

The main obstacle for repeating Uhlenbeck's proof is the fact that the subharmonicity of H does not suffice to directly apply Moser's iteration: we cannot control well the contribution of ω coming from the regions where it is close to zero. The strategy of Tolksdorf [127] is to prove a stronger property of the function $|\omega|^2$, namely its *quasisubharmonicity*. The estimate which can be obtained is the following one (we state a weakened result in order to put the gist of the improvement in evidence):

Proposition D.6. *If ω is as in Theorem D.3 and we further assume that ρ is twice derivable and satisfies $|\rho''(Q)|Q^2 \leq C\rho(Q)$ then $|\omega|^2$ is quasisubharmonic, i.e. for all $\phi \in C_c^\infty(D)$ and for all smooth nondecreasing nonnegative G there holds, abbreviating $\rho := \rho(|\omega|^2)$, $G := G(|\omega|^2)$ as above,*

$$\int_D \rho (|\nabla\omega|^2 G + |\nabla|\omega|^2| G') \phi^2 \leq C \int_D \rho |\nabla|\omega|^2| G |\nabla\phi| |\phi| + l.o.t.$$

The proof of Tolksdorf in the case $\omega = \nabla u$ is via difference quotients, but this strategy is easily repeated in the case of general elliptic systems by determining ϕ which solves the equations $A_0\phi = \omega$, $A_{-1}^*\phi = 0$ and then applying the difference quotient method of Tolksdorf to such ϕ . It is for proving this result that the condition on second derivatives of ρ is needed.

The usefulness of this definition consists in the fact that it gives a way of truncating $|\omega|$ away from zero, by choosing G which is equal to zero in a ball containing the origin. Tolksdorf then uses such G to perform Moser's iteration, therefore this G replaces the H of Uhlenbeck's proof for the case $p < 2$. The rest of the proof can then proceed along the same strategy as Uhlenbeck's proof.

Appendix E

Review of slice distances

In this appendix we recall more in detail the definitions and constructions concerning the control of geometric objects via their slices present in [8] and in [72]. We describe the relation to our results of Chapters 2, 4, 8.

For definitions and notations on currents see [52]; for metric currents see Section A.2 and the references therein.

E.1 Slicing in euclidean space

We start by recalling some results of [52] 4.3:

Proposition E.1. *Suppose that T is a normal k -current in \mathbb{R}^m and that $f : \mathbb{R}^m \rightarrow \mathbb{R}^k$ is Lipschitz. The following statements hold:*

- *For a.e. $y \in \mathbb{R}^k$ there exists a normal 0-current $\langle T, f, y \rangle$ supported on $f^{-1}(y)$ such that for all $\psi \in C_c(\mathbb{R}^k)$ and for all $\omega \in C_c^\infty(\mathbb{R}^m)$*

$$\int_{\mathbb{R}^m} \langle T, f, y \rangle \psi(y) dy = T \llcorner (\psi \circ f) df.$$

- *The total variation map $y \mapsto \|\langle T, f, y \rangle\|(\mathbb{R}^m)$ is integrable and for every $v \in C_c^0(\mathbb{R}^m)$ there holds*

$$\langle \|T \llcorner df\|, v \rangle = \int_{\mathbb{R}^k} \langle \|\langle T, f, y \rangle\|, v \rangle dy.$$

- *If T is integer rectifiable then a.e. $y \in \mathbb{R}^k$ the slice $\langle T, f, y \rangle$ is integer rectifiable.*

- If $\eta : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a bijective Lipschitz map then a.e. $y \in \mathbb{R}^k$ there holds

$$\eta_{\#} \langle T, f, y \rangle = \langle \eta_{\#} T, f \circ \eta^{-1}, y \rangle.$$

- If $\xi : \mathbb{R}^k \rightarrow \mathbb{R}^k$ is a diffeomorphism then a.e. $y \in \mathbb{R}^k$ there holds

$$\langle T, \xi^{-1} \circ f, y \rangle = \langle T, f, \xi(y) \rangle.$$

The fundamental observation present in [81] and [139] and which also helps for more general situation, is the observation that if T is a normal k -current and $f : \mathbb{R}^m \rightarrow \mathbb{R}^k$ is a Lipschitz function then the slice map

$$y \mapsto \langle T, f, y \rangle$$

defined a.e. as above is a (metric) bounded variation function from the space \mathbb{R}^k into the metric space of 0-currents endowed with the flat norm. We consider this situation in the next section.

E.2 Slices for currents in metric spaces

Recall that a subset $A \subset Y$ of a metric space is called *countably \mathcal{H}^k -rectifiable* if it can be covered up to a \mathcal{H}^k -negligible set by a countable union of Lipschitz images of closed subsets of \mathbb{R}^k .

On the space $\text{Lip}_b(E)$ of bounded Lipschitz functions on a metric space E we define the flat norm by

$$\mathbb{F}(\phi) := \sup |\phi| + \text{Lip}(\phi)$$

and on the dual space $\mathbb{M}_0(E)$ we denote by \mathbb{F} the dual norm:

$$\mathbb{F}(T) := \sup\{T(\phi) : \phi \in \text{Lip}_b(E), \mathbb{F}(\phi) \leq 1\}.$$

If E is weakly separable then $\mathbb{M}_0(E)$ embeds isometrically into $\mathbb{M}_0(\ell^\infty)$. This is the dual of the separable space $\text{Lip}_b(\ell^\infty)$ thus is weakly separable.

We now give the definitions of rectifiable and integer rectifiable metric currents:

Definition E.2 (rectifiable metric currents). *Let $k \geq 1$ and $T \in \mathbb{M}_k(E)$. We say that T is rectifiable if $\|T\|$ is concentrated on a countably \mathcal{H}^k -rectifiable set and it vanishes on \mathcal{H}^k -negligible Borel sets.*

We denote by $\mathcal{R}_k(E)$ the class of rectifiable currents.

Definition E.3 (integer rectifiable and integral metric currents). *Let $k \geq 1$. We say that a rectifiable current T is integer rectifiable if for any $\phi \in \text{Lip}(E, \mathbb{R}^k)$ and any open set $A \subset E$ the pushforward current $\phi_{\#}(T \llcorner A)$ is represented by $dy \mathcal{L}^k \llcorner \theta$ for $\theta \in L^1(\mathbb{R}^k, \mathbb{Z})$.*

We denote by $\mathcal{I}_k(E)$ the class of integer rectifiable k -currents.

For $k \geq 1$ the space of integral currents is

$$\mathbb{I}_k(E) := \mathcal{I}_k(E) \cap \mathbb{N}_k(E).$$

The space $\mathbb{I}_0(E)$ is constituted of the currents in $\mathbb{M}_0(E)$ which are finite sums of Dirac masses.

We then have the following:

Proposition E.4. *With the above definitions the analogue of Proposition E.1 is valid also for metric currents.*

E.2.1 Metric bounded variation

In some sense at least in the case of $\Omega \subset \mathbb{R}^m$, BV functions, i.e. functions in $L^1(\Omega)$ such that $\int_{\Omega} |f| + |Df|(\Omega) < \infty$, are the simplest class which has the same functional analytic properties as normal currents. If we associate the current $T : C_0^\infty(\Omega, \wedge^m \mathbb{R}^m) \ni \omega \mapsto \int_{\Omega} f \omega$ to a BV function f we see that the measure $|Df|$ corresponds to the variation measure $\|T\|$. The fact that this correspondence can be extended to the most general situations is what we now pass to address.

We recall here the definitions from [7] and [8].

Definition E.5 (weakly separable metric space). *A metric space (Y, d) is called weakly separable if there exists a countable family $\mathcal{F} \subset \text{Lip}_1 \cap \text{Lip}_b(Y, \mathbb{R})$ such that*

$$\forall x, y \in Y, \quad d(x, y) = \sup_{\phi \in \mathcal{F}} |\phi(x) - \phi(y)|.$$

The space $\ell^\infty(\mathbb{R})$ is weakly separable, a closed subset of a weakly separable space is weakly separable and the map $x \mapsto (\phi(x) - \phi(p))_{\phi \in \mathcal{F}}$ is a distance-preserving embedding of (Y, d) into ℓ^∞ therefore

Proposition E.6. *A metric space (Y, d) is weakly separable if and only if there exists a distance-preserving embedding $Y \mapsto \ell^\infty$.*

Definition E.7 (metric bounded variation). *Let (Y, d) be a weakly separable space and \mathcal{F} be as in the definition of weak separability. We say that $u : \mathbb{R}^k \rightarrow Y$ is a function of metric bounded variation (and we write $u \in MBV(\mathbb{R}^k, Y)$) if for all $\phi \in \mathcal{F}$ we have $\phi \circ u \in BV_{loc}(\mathbb{R}^k)$. We denote by*

$$\|Du\| := \bigvee_{\phi \in \mathcal{F}} |D(\phi \circ u)| < \infty,$$

where for a family of measures μ_i we note

$$\bigvee \mu_i(B) := \sup \left\{ \sum_i \mu_i(B_i) : B_i \text{ are a partition of } B \right\}.$$

We also denote by MDu the maximal function of $\|Du\|$:

$$MDu(x) := \sup_{\rho > 0} \frac{\|Du\|(B_\rho(x))}{|B^{\mathbb{R}^k}(x, \rho)|}$$

We then have

Lemma E.8 ([8], 7.3). *Let Y be as above and $u \in MBV(\mathbb{R}^k, Y)$. Then for a.e. $x, y \in \mathbb{R}^k$ the following inequality holds:*

$$d(u(x), u(y)) \leq C MDu(x)|x - y|.$$

Sketch of proof: Step 1. For usual BV_{loc} functions $w : \mathbb{R}^k \rightarrow \mathbb{R}$ outside the Lebesgue points of w there holds

$$\int_{B(x,r)} \frac{|w(x) - w(y)|}{|x - y|} dy \leq MDw(x)$$

which implies the statement of the theorem for $Y = \mathbb{R}^k$. The above inequality is proved for C^1 -functions via the mean value inequality on segments. It holds for BV by approximation. See [51].

Step 2. We now pass to more general Y . Recall that for a function $f : \mathbb{R} \rightarrow \mathbb{R}$ there holds

$$|Df|(A) = \sup \left\{ \sum_1^n |f(t_i) - f(t_{i-1})| : t_0 < \dots < t_n, t_i \in A \setminus N \right\}$$

for all \mathcal{L}^1 -negligible sets N containing the non-Lebesgue points of f . Consider now a 1-Lipschitz bounded function $\psi : \mathbb{R} \rightarrow Y$ and $u \in MBV(\mathbb{R}, Y)$. For s, t outside a negligible set depending on \mathcal{F} and u we then have

$$|\psi \circ u(s) - \psi \circ u(t)| \leq \sup_{\phi \in \mathcal{F}} |\phi \circ u(s) - \phi \circ u(t)| \leq \|Du\|([t, s]).$$

Therefore we have

$$\|Du\| \geq |D(\psi \circ u)|.$$

In [8] it is proved that the same holds also when we replace the domain \mathbb{R} by \mathbb{R}^k . In particular $MDu \geq MD(\psi \circ u)$ too. Now using Step 1 and taking a supremum we conclude. \square

Remark E.9. *By refining the proof we can also obtain that for all open sets A for a.e. $x, y \in A$ there holds*

$$d(u(x), u(y)) \leq CM^A Du(x)|x - y|,$$

where M^A denotes the relative maximal function defined as follows:

$$M^A \mu(x) := \sup \left\{ \frac{\mu(B')}{|B'|} : B' \text{ is a ball contained in } A \right\}.$$

E.2.2 Rectifiability and closure

We start by proving that slices of normal currents have bounded variation:

Proposition E.10 (The slice function is MBV). *Let $T \in \mathbb{N}_k(E)$ and $f \in Lip_1(E, \mathbb{R}^k)$. Then the map*

$$x \mapsto \langle T, f, x \rangle =: T_x$$

belongs to $MBV(\mathbb{R}^k, Y)$ where Y is $\mathbb{M}_0(E)$ endowed with the flat norm and

$$\|DT_x\| \leq C(f_{\#}\|\partial T\| + f_{\#}\|T\|).$$

Proof. The proof is straightforward. We consider $\phi \in Lip_b(E)$ such that $\mathbb{F}(\phi) \leq 1$ and we prove that for C independent of such ϕ there holds

$$|DT_x(\phi)| \leq C(f_{\#}\|\partial T\| + f_{\#}\|T\|).$$

To do so we use the characterization

$$|DT_x(\phi)| = \sup \left\{ \int_{\mathbb{R}^k} T_x(\phi) \operatorname{div} \psi(x) dx : \psi \in C_c^1(\mathbb{R}^k), \|\psi\|_{L^\infty} \leq 1 \right\}.$$

The desired estimate follows from the following computation, where $df_i := df_1 \wedge \dots \wedge df_{i-1} \wedge df_{i+1} \wedge \dots \wedge df_k$:

$$\begin{aligned} (-1)^{i-1} \int_{\mathbb{R}^k} T_x(\phi) \frac{\partial \psi}{\partial x_i}(x) dx &= (-1)^{i-1} T \llcorner \left(\phi \frac{\partial \psi}{\partial x_i} \circ f \right) df \\ &= T(\phi d(\phi \circ f) \wedge df_i) \\ &= \partial T(\phi(\psi \circ f) df_i) - T(\psi \circ f d\phi \wedge df_i). \end{aligned}$$

\square

We now have to prove that a *MBV*-function f into $\mathbb{M}_0(E)$ has rectifiable graph. Of course we have to assume that the values of f are in \mathcal{R}_0 . We prove the following rectifiability criterion which, we think, is suggestive of the mechanism behind the more general results of [8]:

Proposition E.11. *Let E be a weakly separable metric space and let $Y = \mathbb{M}_0(E)$, endowed with the flat norm. Let $T \in MBV(\mathbb{R}^k, Y)$ and assume that T takes a.e. values in $\mathbb{I}_0(E)$. Then there exists a negligible set $N \subset \mathbb{R}^k$ such that the set*

$$\mathcal{R} := \bigcup_{z \in \mathbb{R}^k \setminus N} \text{spt}(T(z))$$

is countably \mathcal{H}^k -rectifiable.

Proof. By a covering argument we may replace E by $B := B^{\mathbb{F}}(p, 1)$. By Lemma E.8 and by the hypothesis there exists a negligible set $N_0 \subset \mathbb{R}^k$ outside which $T(z) \in \mathbb{I}_0(E)$ and the following formula holds:

$$\mathbb{F}(T(z) - T(z')) \leq C MDT(z)|z - z'|.$$

The set $N_1 := \{MDT = \infty\}$ is also negligible and $\mathbb{R}^k \setminus N_1$ is covered by the sets $\mathcal{R}_n := \{MDT < n/C\}$, on which T is n -Lipschitz.

If $\mathbb{F}(A - A') < 1$ for $A, A' \in \mathbb{I}_0(E)$ then the numbers for Dirac masses representing A and A' are the same (to see this test $A - A'$ on the function $\phi \equiv 1$). The multivalued function corresponding to T is Lipschitz if it is multivalued and T is Lipschitz (this can be seen for example by testing with the calibration of Proposition 2.7 and Remark 2.8). Therefore we can cover each region $(\mathcal{R}_n \setminus N_0) \times B$ by finitely many k -dimensional Lipschitz graphs. \square

A reasoning along the lines of Propositions E.10 and E.11 gives the following result:

Theorem E.12 (Characterization of rectifiability [8]). *Let $T \in \mathbb{N}_k(E)$. Then $T \in \mathcal{R}_k(E)$ if and only if*

$$\text{for any } \pi \in \text{Lip}(E, \mathbb{R}^k), \quad \langle T, \pi, x \rangle \in \mathcal{R}_0(E), \mathcal{L}^k\text{-a.e. } x \in \mathbb{R}^k.$$

Moreover $T \in \mathbb{I}_k(E)$ if and only if the same holds with $\mathbb{I}_0(E)$ in place of $\mathcal{R}_0(E)$.

We present here a possible approach to the closure theorem for rectifiable and integral currents based on Lemma E.8 and on the maximal inequality. Again we present the reasoning in a simpler case, to highlight the main idea. The precise statement is the following.

Theorem E.13. *Let $T_n \in \mathbb{I}_k(E)$ be a sequence of integral currents. If the T_n converge weakly to a current T and there exists a constant C such that $\mathbb{M}(T_n) + \mathbb{M}(\partial T_n) \leq C$ then $T \in \mathbb{I}_k(E)$.*

Sketch of proof: This theorem follows from Propositions E.10 and E.11 and using the characterization of Theorem E.12, once we prove that the slices of the weak limit T by any function $f \in \text{Lip}_1(E, \mathbb{R}^k)$ are $MBV(\mathbb{R}^k, \mathbb{M}_0(E))$ and take a.e. values in $\mathbb{I}_0(E)$.

To obtain this, we use

- the fact that the theorem is true for $k = 0$, implying in particular

$$\{T \in \mathbb{I}_0(E) : \mathbb{M}(T) \leq C\} \text{ is } \mathbb{F}\text{-seq. compact for all } C.$$

- the fact that by Proposition E.10 and the first part of Lemma E.8 with the notation $T_n(y) := \langle T_n, f, y \rangle$ there holds

$$\mathbb{F}(T_n(y) - T_n(y')) \leq C : MDT_n(y)|y - y'|$$

- the fact that by the maximal function inequality and the second part of Lemma E.8

$$\begin{aligned} \sup_{\lambda > 0} \lambda |\{y : MDT_n(y) > \lambda\}| &\leq C \|DT_n\|(\mathbb{R}^k) \\ &\leq \text{Lip}(f)(\mathbb{M}(T_n) + \mathbb{M}(\partial T_n)) \leq C. \end{aligned}$$

The conclusion now follows from the next abstract proposition, modeled on Theorem 2.13. □

Theorem E.14. *Assume that Y is a weakly separable metric space (possibly not complete) and $\mathcal{N} : Y \rightarrow [0, \infty]$ is a function such that $\{y \in Y : \mathcal{N}(y) \leq C\}$ is closed and sequentially compact for all C . If $f_n \in MBV(\mathbb{R}^k, Y)$ are such that*

$$\int_{\mathbb{R}^k} \mathcal{N} \circ f_n + \|Df_n\|(\mathbb{R}^k) \leq C$$

for some C independent of k , then the f_i have a subsequence that converges pointwise almost everywhere. The limit f satisfies the weaker estimate

$$\int_{\mathbb{R}^k} \mathcal{N} \circ f_n + \sup_{\lambda > 0} \lambda |\{M_d f(x) > \lambda\}| \leq C,$$

where

$$M_d f(x) := \text{esssup}_{y \neq x} \frac{d(f(x), f(y))}{|x - y|}.$$

Proof. It is enough to prove that the f_n have a subsequence which is pointwise a.e. *Cauchy convergent*.

Indeed, under this hypothesis for a.e. $y \in \mathbb{R}^k$ there would exist a limit $\hat{f}(y) \in \hat{Y}$, the completion of Y . We then use Fatou and the fact that the \mathcal{N} -energy is bounded to prove that up to subsequence a.e. y

$$\sup_n \mathcal{N}(f_n(y)) \leq \infty.$$

Since sublevels of \mathcal{N} are compact, up to subsequence $f_n(y)$ converges to a point of Y and by uniqueness of the limit this point is $\hat{f}(y)$.

From the estimate of Lemma E.8 and we obtain a.e. x, y

$$d(f_n(x), f_n(y)) \leq C M D f_n(x) |x - y|.$$

This implies that for $B = B(x, r) \subset \mathbb{R}^k$ there holds

$$\begin{aligned} M_d f(x) &:= \operatorname{esssup}_{y \neq x} \frac{d(f(x), f(y))}{|x - y|} \\ &\leq \liminf_n \operatorname{esssup}_{x \neq y \in B} \frac{d(f_n(x), f_n(y))}{|x - y|} \\ &\leq C \liminf_n M D f_n(x). \end{aligned}$$

We note here that from the maximal inequality

$$\sup_{\lambda > 0} \lambda |\{y : M D f_n(y) > \lambda\}| \leq C \|D f_n\|(\mathbb{R}^k) \leq C$$

it follows that for all $\lambda > 0$

$$\lambda |\{x : M_d f(x) > \lambda\}| \leq \liminf_n \lambda |\{M D f_n > \lambda\}| \leq C.$$

Note also that for each cube $Q \subset \mathbb{R}^k$ using the refinement from Remark E.9 we obtain

$$\operatorname{esssup}_{x \neq y, x, y \in Q} \frac{d(f_n(x), f_n(y))}{|x - y|} \leq C M^Q D f_n(x),$$

for which, using the relative version of the maximal inequality, there holds

$$\sup_{\lambda > 0} \lambda |\{y \in Q : M^Q D f_n(y) > \lambda\}| \leq C \|D f_n\|(Q) := \mu_n(Q).$$

The rest of the proof proceeds as the one of Theorem 2.13, with $p = 1$. The difference is that covering by intervals should be replaced by a covering by cubes, but no complications arise. \square

Remark E.15. *A simpler proof of the above theorem, which uses the isometric embedding of weakly separable spaces into ℓ^∞ is present in [42]. We utilize this proof since it demonstrates the importance of the estimate of Lemma E.8 which is easier to verify in applications.*

Remark E.16. *In the above theorem what gives the existence of the pointwise a.e. limit is the control on the “higher order term” $\|Df_n\|$, which is really connected to the behavior of the slices via Proposition E.10.*

The fact that the slice function $y \mapsto \langle T_n, f, y \rangle$ is mass-integrable is used to give the coercivity, and in particular any function having the same sublevels as the mass (e.g. M^α for $\alpha > 0$) will give the same results. This important observation appearing in [72] is at the basis of a generalization of metric currents to the so-called rectifiable scans. The above reasoning shows that similar rectifiability results will be achievable once the correct analogue of mass is chosen.

E.3 Rectifiable scans

In this section we describe the results of R. Hardt and T. Rivière [72] and the related theory which is being developed by R. Hardt and T. De Pauw [42, 43, 44].

Since the review [44] gives a good description of the new phenomena appearing when replacing the mass by its fractional power, we rather focus on describing the natural appearance of scans of graphs of Sobolev functions, [72], which brings better to light the relation to weak curvatures.

E.3.1 Bubbling and minimal connections in supercritical dimensions

We continue here the review of the theory of weak approximations for nonlinear Sobolev spaces from Section 1.6.2. The underlying question which is to be addressed is one of the following kind:

Question 1. *Are functions $C^\infty(X, Y)$ dense in $W^{1,p}(X, Y)$ for the strong topology?*

In general in the critical dimension $p = \dim X$ where locally the strong closure of continuous maps in $W^{1,p}(B^p, Y)$ is still valid in general (see [114,

19, 23]), the so-called *bubbling phenomena* occur, namely a sequence of weakly convergent smooth maps can lose energy in the limit while simplifying its topology. Topological objects (the so-called “bubbles”) can be defined, the topological behavior happens just at an isolated set of points and usually the energy loss is quantized, i.e. while bubbles disappear in the limit, each one of them is responsible of a fixed amount of energy.

In the supercritical case $p < \dim X$ the behavior of the singular set can be determined by looking at an appropriate kind of degree (see e.g. [73]) and the limit map will have a “bubbling set” of dimension $\dim X - [p]$ which will connect the singular set of the limit. The fact that a “connection” appears can be best understood from the point of view of Cartesian currents.

In the simple case of $W^{1,2}(B^3, \mathbb{S}^2)$ the following result is valid.

Theorem E.17 ([61]). *Consider a sequence of maps $u_n \in C^\infty(B^3, \mathbb{S}^2)$ which converges weakly in $W^{1,2}$ to $u \in W^{1,2}(B^3, \mathbb{S}^2)$. Then there exists a finite mass rectifiable 1-current I such that up to extracting a subsequence the 3-dimensional currents of integration on the graphs of u_n, u satisfy*

$$\langle \text{Graph}(u_n), \omega \rangle \rightarrow \langle \text{Graph}(u) - I \times [\mathbb{S}^2], \omega \rangle$$

for all smooth 3-forms on $B^3 \times \mathbb{R}^3$.

Several facts are consequences of this description:

- As a consequence of the fact that “the u_n do not jump” we have

$$\partial \text{Graph}(u_n) \llcorner B^3 \times \mathbb{R}^3 = 0$$

and this is preserved under weak convergence, therefore (since the integration current along \mathbb{S}^2 is closed)

$$\partial \text{Graph}(u) = \partial(I \times [\mathbb{S}^2]) = \partial I \times [\mathbb{S}^2].$$

- Projecting the above identity via $\pi : B^3 \times \mathbb{R}^3 \rightarrow B^3$ we obtain the fact that I connects the topological singularities of u .
- The formula $4\pi \partial I = *d(u^* \omega_{\mathbb{S}^2})$ follows by integrating the above identity against a “vertical” (i.e. constant in the directions parallel to the B^3 -factor) 2-form extending the $\omega_{\mathbb{S}^2}$ to $B^3 \times \mathbb{R}^3$.

E.3.2 Hopf singularities and the appearance of \mathbb{M}^α

Following [72], we now consider the following question which is a special case of Question 1 which appeared in the introduction of the chapter:

Question 2. *Consider a sequence of functions $u_n \in C^\infty(B^4, \mathbb{S}^2)$ which converge weakly in $W^{1,3}$ to a limit $u \in W^{1,3}(B^4, \mathbb{S}^2)$. How can one control the behavior of topological singularities of the limit?*

Recall the case of maps in $W^{1,2}(B^3, \mathbb{S}^2)$: topological singularities modeled on the radial vector field $B^3 \ni x \mapsto \frac{x}{|x|}$ were allowed to appear when the singular set was made of isolated points, i.e. for maps in Betuel's space $\mathcal{R}^{\infty,2}(B^3, \mathbb{S}^2)$ consisting of functions which are smooth outside a finite set. This could be checked by considering the homotopy equivalence classes of maps

$$[B^3 \setminus \{p_1, \dots, p_N\}, \mathbb{S}^2] \sim \left[\bigvee_1^N \mathbb{S}^2, \mathbb{S}^2 \right] \sim \bigoplus_1^N [\mathbb{S}^2, \mathbb{S}^2] \sim \bigoplus_1^N \pi_2(\mathbb{S}^2),$$

because of which topological singularities were classified by $\pi_2(\mathbb{S}^2) \simeq \mathbb{Z}$.

The space $\mathcal{R}^{\infty,3}(B^4, \mathbb{S}^2)$ which is strongly dense in $W^{1,3}(B^4, \mathbb{S}^2)$ is again made of functions which are smooth outside some finite set. We also have that

$$[B^4 \setminus \{p_1, \dots, p_N\}, \mathbb{S}^2] \sim \bigoplus_1^N \pi_3(\mathbb{S}^2),$$

therefore again the problem reduces to the study of local $\pi_3(\mathbb{S}^2)$ -type singularities

Remark E.18. *Note that for a general 4-manifold M the homotopy equivalence classes $[M \setminus \{p_1, \dots, p_N\}, \mathbb{S}^2]$ are not equivalent to $\bigoplus_1^N \pi_3(\mathbb{S}^2)$ in general. This is still true if $H^1(M, \mathbb{Z}) = 0$, otherwise the classification [102] is more complicated. An analogue of the Hopf invariant can however be constructed in the general case. See [9] for the analytical aspects.*

Consider the Hopf fibration H of Section 9.3.1, i.e. identify \mathbb{S}^3 with the unit sphere $\{(z, w) : |z|^2 + |w|^2 = 1\}$ in \mathbb{C}^2 then define the equivalence relation

$$(z, w) \sim (z', w') \text{ if } (z, w) = (\lambda z', \lambda w')$$

for some $\lambda \in \mathbb{C}$ of modulus 1 (an alternative description in quaternion notation is $H(q) = q^{-1}iq$, see [9]). We have a fibration

$$\mathbb{S}^1 \rightarrow \mathbb{S}^3 \xrightarrow{H} \mathbb{C}P^1 \sim \mathbb{S}^2.$$

From the exact homotopy sequence

$$\cdots \rightarrow \pi_k(\mathbb{S}^1) \rightarrow \pi_k(\mathbb{S}^2) \rightarrow \pi_k(\mathbb{S}^2) \rightarrow \pi_{k-1}(\mathbb{S}^1) \rightarrow \cdots$$

since $\pi_k(\mathbb{S}^1) = 0$ for $k > 1$ we obtain $\pi_k(\mathbb{S}^2) = \pi_k(\mathbb{S}^3)$ for $k \geq 3$, in particular $\pi_3(\mathbb{S}^2) = \mathbb{Z}$.

The analogue of the differential invariant $u^*\omega_{\mathbb{S}^2}$ from the case of $W^{1,2}(B^3, \mathbb{S}^2)$ is the Hopf invariant (or Hopf degree) defined for a map $u : B^4 \setminus \{p_1, \dots, p_N\} \rightarrow \mathbb{S}^2$ as

$$\frac{1}{4\pi^2} \int_{\mathbb{S}^3} \eta \wedge u^*\omega_{\mathbb{S}^2}, \quad \text{for } \eta \text{ s.t. } d\eta = u^*\omega_{\mathbb{S}^2}. \quad (\text{E.1})$$

We see that the failure of a map $u \in \mathcal{R}^{\infty,3}(B^4, \mathbb{S}^2)$ to be approximable by smooth maps is detected by

$$\text{Sing}_{\text{top}} u := *d(d^{-1}(u^*\omega_{\mathbb{S}^2}) \wedge u^*\omega_{\mathbb{S}^2}),$$

where $d^{-1}(u^*\omega_{\mathbb{S}^2})$ is a shorthand for any η such that $d\eta = u^*\omega_{\mathbb{S}^2}$ (this notation comes from [73]).

The main difficulty which one faces is that as a consequence of a dipole construction *the formula (E.1) does not extend by strong density to $W^{1,3}(B^4, \mathbb{S}^2)$* , and this is the crucial difference with the “classical” case of $W^{1,2}(B^3, \mathbb{S}^2)$.

We shortly describe what fails. First note that $u^*\omega_{\mathbb{S}^2}$ has the form $udu \wedge du$, therefore is quadratic in ∇u and by Hölder inequality belongs to $L^{3/2}$ in general. By Hodge decomposition the best regularity for η such that $d\eta =$ is obtained by imposing the Coulomb condition $d^*\eta = 0$ and satisfies $\|d\eta\|_{L^{3/2}} \lesssim \|u^*\omega_{\mathbb{S}^2}\|_{L^{3/2}}$. By Sobolev embedding in 4 dimensions we have $\eta \in L^p$ for $p \leq \frac{4 \cdot \frac{3}{2}}{4 - \frac{3}{2}} = \frac{12}{5}$. In general thus we have $\eta \wedge u^*\omega_{\mathbb{S}^2} \in L^{\frac{3}{2}} \cdot L^{\frac{12}{5}}$ but $\frac{2}{3} + \frac{5}{12} > 1$ thus in general this product will not belong to L^1_{loc} . From [106] it follows that indeed for $d \rightarrow \infty$

$$\inf \left\{ \int_{\mathbb{S}^3} |\nabla \phi|^3 : \phi : \mathbb{S}^3 \rightarrow \mathbb{S}^2, \text{ of Hopf degree } j \right\} \sim j^{\frac{3}{4}}.$$

In particular *no minimal connection of finite mass can exist for general $u \in W^{1,3}(B^4, \mathbb{S}^2)$* :

Example E.19. *We can construct $u \in W^{1,3}(B^4, \mathbb{S}^2)$ such that no current of finite mass can connect the singularities of u .*

u will be constructed by adding small dipoles concentrated near segments $I_j \subset \mathbb{R}^4$ of lengths l_j . These small dipoles are constructed as follows (this is a construction similar in spirit to [6] and uses the conformal invariance of the energy in 3-dimensions).

Foliate a very small neighborhood of I_j by 3-disks $D_j^t, t \in I_j$ contained in the orthogonal hyperplanes to I_j , with centers on I_j . Identify each half-radius disk $\frac{1}{2}D_j^t$ to $\mathbb{S}^3 \setminus B_\epsilon^{\mathbb{S}^3}$ via a conformal diffeomorphism and extend this map to $\sigma : D_j^t \rightarrow \mathbb{S}^3$ so that it equals the center of $B_\epsilon^{\mathbb{S}^3}$ (which we may in turn assume to be the south pole S of \mathbb{S}^3) near ∂D_j^t .

Then define $u|_{D_j^t} = \phi_j \circ \sigma$ with ϕ_j a map of almost minimal 3-energy among maps $\phi : \mathbb{S}^3 \rightarrow \mathbb{S}^2$ of Hopf degree j . Then (by conformal invariance of the 3-energy in 3-dimensions) up to an arbitrary small error we will have $\int_{D_j^t} |\nabla u|^3 \sim j^{\frac{3}{4}}$. This is independent on how small the radii of D_j^t are. We may now require the following constraints to hold:

- $\sum_j j l_j = \infty, \sum_j j^{3/4} l_j < \infty$ (in particular $\sum_j l_j < \infty$),
- $u \equiv S$ outside $\cup_j U_j$ with U_j a small neighborhood of I_j ,
- the U_j are disjoint, in particular

$$\begin{aligned} \int_{B^4} |\nabla u|^3 &= \sum_j \int_{U_j} |\nabla u|^3 = \sum_j \int_{I_j} \left(\int_{D_j^t} |\nabla u|^3 \right) dt \\ &\lesssim \sum_j l_j j^{\frac{3}{4}} < \infty. \end{aligned}$$

- the I_j are so well separated that

$$\text{dist}(\cup_{k \neq j} I_k, I_j) > \sum_j l_j.$$

Because of this construction, if a minimal connection of the singularities of u existed, it would have to be $\sum_j j \partial[I_j]$, but this connection has infinite mass.

Due to the above behavior it seems natural to introduce the fractional mass $\mathbb{M}^{\frac{3}{4}}$ in order to “measure” the length of minimal connections.

E.3.3 Definition of scans

Similarly to our approach for the controlled extension of maps belonging to $W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$ the authors of [72] then proceed to consider the controlled Hopf lift of $W^{1,3}$ -maps $u : B^4 \rightarrow \mathbb{S}^2$, obtaining $\tilde{u} : B^4 \rightarrow \mathbb{S}^3$ such that $H \circ \tilde{u} = u$. More importantly, the topological degree of \tilde{u} equals the Hopf degree of u and \tilde{u} can be found such that its 3-energy is equivalent to that of u .

In order to define the analogue of the Cartesian currents

$$\text{Graph}(u), u \in W^{1,2}(B^3, \mathbb{S}^2)$$

for the functions \tilde{u} , in view of the representation of currents as *MBV*-functions, we have to introduce a family of slicing sets of dimension k and a class of functionals

$$S : \{\text{allowed slicing sets}\} \rightarrow \{0\text{-currents on } B^4 \times \mathbb{R}^3\}.$$

Such functionals have to meet the following requirements:

- There are *enough slicing sets* to allow the scan corresponding to $\text{Graph}(\tilde{u})$ to uniquely determine \tilde{u} . This is equivalent to a locality condition, or to the fact that in a natural sense the scans commute with restriction operations in the sense of currents.
- A *notion of boundary* is defined, in such a way that for smooth \tilde{u} the boundary of the scan of its graph is supported on $\partial B^4 \times \mathbb{S}^3$.
- A *condition which ensures closure of rectifiable scans* on S is imposed, i.e. such that a compactness theorem like Theorem E.14 is valid, for a functional $\mathcal{N} : \{0\text{-currents on } B^4 \times \mathbb{R}^3\} \rightarrow [0, \infty]$ such that the sum of dipoles of Example E.19 always give a scan with

$$\int_{\{\text{allowed slicing sets}\}} \mathcal{N}(S(x)) dx < \infty.$$

This last condition is the one implying that we cannot require that F be *MBV*, we must relax this requirement. The constraint that a compactness theorem like Theorem E.14 be valid is instead telling us that we cannot relax it too much.

The choices which have been done in [72] and [42, 43] are as follows:

- In the case of [72] we have a perfect candidate for \mathcal{N} , namely $\mathbb{M}^{\frac{3}{4}}$. The sublevels of this functional are the same as those of the mass,

therefore Theorem E.14 holds without problems, and we can define the “rectifiability-respecting” condition for scans which replaces MBV to be

$$\|DF\| \leq CM^{\frac{3}{4}}. \quad (\text{respectively } \alpha \in]0, 1] \text{ in general})$$

The corresponding topology on k -currents is the one given by the α -flat distance

$$\mathcal{F}_\alpha(T, T') = \inf\{\mathbb{M}_\alpha(R) + \mathbb{M}_\alpha(S) : T - T' = R + \partial S\},$$

where \mathbb{M}_α on k -currents is defined by integrating \mathbb{M}^α over the slices.

- A natural way for meeting the first requirement is to take inspiration from integral geometry (see e.g. [54] and the related case [34] for the case of curved slices analogous to our slices for the classes \mathcal{A}_G). The simplest and most useful choice is to take as allowed slicing sets all k -planes, endowed with the normalized isometry-invariant measure. Note however that this is slightly redundant, and it would be enough to consider just a finite number of families of hyperplanes such that at each point they span the Grassmannian of k -planes (this observation is made in [72]).
- The definition of boundary is the most tricky for general choices of allowed slicing sets (see [34]). In the case where we allow exactly all k -planes Π the vanishing of the boundary of usual currents is expressible in terms of the sliced currents T_Π :

$$\partial T = 0 \Leftrightarrow \text{a.e. } \Pi, \quad T_\Pi(1) = 0,$$

thus we may take this definition as a vanishing-boundary definition. This gives a good definition in the relevant case where the boundary of our scan is more regular, i.e. it equals that of a current.

Appendix F

The Uhlenbeck method for nonlinear extensions

F.1 Uhlenbeck small energy extension

We now use the strategy which Uhlenbeck [132] employed for the proof of controlled coulomb gauges under a small curvature requirement to prove Theorem 9.29(Uhlenbeck analogue). We note that the analogy is in the method of proof more than in the result.

First observe that the following infimum is attained, as soon as the class on which we minimize is not empty (recall that $W^{1,2}(X, \mathbb{S}^3) = W^{1,2}(X, \mathbb{R}^4) \cap \{u : u(x) \in \mathbb{S}^3 \text{ a.e.}\}$):

$$\inf \left\{ \int_{B^4} |\nabla P|^2 : P \in W^{1,2}(B^4, \mathbb{S}^3), P = P_0 \text{ on } \partial B^4 \right\}. \quad (\text{F.1})$$

Indeed a minimizing sequence will have a $W^{1,2}$ -weakly convergent subsequence, which will automatically also converge pointwise everywhere. In particular the constraint $u(x) \in \mathbb{S}^3$ a.e. is preserved. By weak lowersemicontinuity a minimizer exists, and by convexity it is unique. The minimizer P verifies the following equation in the sense of distributions:

$$\operatorname{div}(P^{-1}\nabla P) = 0. \quad (\text{F.2})$$

In the language of differential forms we can rewrite

$$d^*(P^{-1}dP) = 0. \quad (\text{F.3})$$

This P will be our extension inside the domain, and we will now prove some estimates which prove useful later.

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Lemma F.1 (a priori estimates). *There exists $\epsilon > 0$ with the following property. Let P with $\|P - I\|_{W^{1,4}(B^4)} \leq \epsilon$ be an extension of $P_0 \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ which satisfies also (F.2). We identify \mathbb{S}^3 with the Lie group $SU(2)$. Then there exists a constant C_ϵ such that*

$$\|P - I\|_{W^{4/3,3}(B^4)} \leq C_\epsilon \|\nabla P_0\|_{L^3(\mathbb{S}^3, \mathbb{S}^3)}. \quad (\text{F.4})$$

Proof. We will start by a L^2 -Hodge decomposition of $P^{-1}dP$: this 1-form can be written in the form

$$P^{-1}dP = dU + d^*V, \quad (\text{F.5})$$

where a description of V is as the unique minimizer of

$$\min \left\{ \int_{B^4} |d^*V - P^{-1}dP|^2, *V|_{\partial B^4} = 0, dV = 0 \right\}.$$

The existence of a minimizer follows easily by convexity as for (F.1). The Euler-Lagrange equation is

$$\begin{cases} \Delta V = dd^*V = dP^{-1} \wedge dP, \\ dV = 0, \\ *V = 0. \end{cases}$$

The fact that $\Delta V = (d^*d + dd^*)V$ coincides with dd^*V is a consequence of the constraint $dV = 0$. We claim that the following estimate holds:

$$\|\nabla V\|_{L^3(\partial B^4)} \lesssim \epsilon \|P - I\|_{W^{1,4}(B^4)}. \quad (\text{F.6})$$

To see this, observe that by elliptic, Hölder and Poincaré estimates (observe that $d(P^{-1}) = P^{-1}dP P^{-1}$ and $P, P^{-1} \in L^\infty$ with norm equal to 1):

$$\begin{aligned} \|\nabla V\|_{W^{1,2}(B^4)} &\lesssim \|dP^{-1} \wedge dP\|_{L^2(B^4)} \lesssim \|d(P^{-1})\|_{L^4(B^4)} \|dP\|_{L^4(B^4)} \\ &\lesssim \|dP\|_{L^4(B^4)} \|P^{-1}\|_{L^\infty}^8 \|\nabla P\|_{L^4(B^4)} \\ &\lesssim \epsilon \|P - I\|_{W^{1,4}(B^4)}. \end{aligned} \quad (\text{F.7})$$

Then we use the trace and Sobolev embedding inequalities:

$$\|V\|_{L^p(\partial B^4)} \lesssim \|V\|_{W^{1-\frac{1}{q},q}(\partial B^4)} \lesssim \|V\|_{W^{1,q}(B^4)},$$

where in general, in dimension n large enough,

$$p = \frac{qn}{n - \left(1 - \frac{1}{q}\right)q}$$

so that for $n = 4, q = 2$ we obtain $p = 3$. Therefore we can concatenate the two last chains of inequalities and we obtain (F.6).

Using the trace of the Hodge decomposition formula (F.5) on the boundary, we obtain from (F.6) that

$$\|dU - P_0^{-1}dP_0\|_{L^3(\partial B^4)} \lesssim \epsilon \|P - I\|_{W^{1,4}(B^4)}. \quad (\text{F.8})$$

As for V , for U we have the following equation:

$$\Delta U = d^*dU = d^*(P^{-1}dP) = 0.$$

To justify the last passage recall (F.3).

We apply the elliptic estimates for U to obtain:

$$\|dU\|_{W^{1/3,3}(B^4)} \lesssim \|\nabla U\|_{L^3(\partial B^4)}, \quad (\text{F.9})$$

while the triangle inequality and the fact that $\|P_0\|_{L^\infty} = 1$ give together with (F.8):

$$\begin{aligned} \|U\|_{L^3(\partial B^4)} &\lesssim \|dU - P_0^{-1}dP_0\|_{L^3(\partial B^4)} + \|P_0^{-1}dP_0\|_{L^3(\partial B^4)} \\ &\lesssim \epsilon \|P - I\|_{W^{1,4}(B^4)} + \|dP_0\|_{L^3(\partial B^4)}. \end{aligned} \quad (\text{F.10})$$

We now use again (F.5), the triangle inequality and the estimates (F.7), (F.9),(F.10):

$$\begin{aligned} \|P^{-1}dP\|_{W^{1/3,3}(B^4)} &\lesssim \|d^*V\|_{W^{1/3,3}(B^4)} + \|dU\|_{W^{1/3,3}(B^4)} \\ &\lesssim \epsilon \|P - I\|_{W^{1,4}(B^4)} + \|dP_0\|_{L^3(\partial B^4)}. \end{aligned} \quad (\text{F.11})$$

We write $dP = P P^{-1}dP$ and observe that $P \in L^\infty \cap W^{1,4}$ since \mathbb{S}^3 is bounded, while $P^{-1}dP \in W^{1/3,3}$ from (F.11). We now use Lemma F.5 for the product fg with $f = P, g = P^{-1}dP$ and we obtain

$$\|dP\|_{W^{1/3,3}(B^4)} \lesssim \|P^{-1}dP\|_{W^{1/3,3}} (\|P\|_{L^\infty} + \|P - I\|_{W^{1,4}(B^4)}). \quad (\text{F.12})$$

Note again that $\|P\|_{L^\infty} = 1$ and deduce then from (F.11), (F.5) and Poincaré inequality that

$$\|P - I\|_{W^{4/3,3}(B^4)} \leq C \|dP_0\|_{L^3(\mathbb{S}^3)} + C\epsilon \|P - I\|_{W^{1,4}(B^4)}. \quad (\text{F.13})$$

Using the Sobolev inequality related to the continuous embedding $W^{4/3,3}(B^4) \rightarrow W^{1,4}(B^4)$ we can absorb the $\|P - I\|$ -term to the left and we obtain the thesis. \square

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We are now ready for the proof of the small energy extension result of Theorem 9.29. We restate the same result with a slight change of notation and more details.

Theorem F.2 (small energy extension). *There exist two constants $\delta > 0, C > 0$ with the following property. Suppose $Q \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ such that $\|dQ\|_{L^3(\mathbb{S}^3)} \leq \delta$. Then there exists an extension $P \in W^{1,4}(B^4, \mathbb{S}^3)$ satisfying the following estimate:*

$$\|P - I\|_{W^{1,4}(B^4)} \leq C\|dQ\|_{L^3(\mathbb{S}^3)}.$$

Proof. Define the following two sets:

$$\mathcal{G}_\epsilon^\alpha = \{Q \in W^{1,3+\alpha}(\mathbb{S}^3, SU(2)) : \|\nabla Q\|_{L^3} \leq \epsilon\} \tag{F.14}$$

$$\mathcal{F}_{\epsilon,C}^\alpha = \left\{ \begin{array}{l} Q \in \mathcal{G}_\epsilon^\alpha : \exists P \in W^{1,4+\alpha}(B^4, SU(2)), \\ \left\{ \begin{array}{ll} \operatorname{div}(P^{-1}\nabla P) = 0 & \text{on } B^4 \\ P = Q & \text{on } \partial B^4, \end{array} \right. \\ \|P - I\|_{W^{1,4}(B^4)} \leq K\|\nabla Q\|_{L^3(\partial B^4)} \\ \|P - I\|_{W^{1,4+\alpha}(B^4)} \leq C\|\nabla Q\|_{L^{3+\alpha}(\partial B^4)} \end{array} \right\}. \tag{F.15}$$

The constant $K > 0$ will be fixed later. In this language, the theorem states that a P with estimates similar to the definition of $\mathcal{F}_{\epsilon,C}^\alpha$ can be constructed to extend any $Q \in \mathcal{G}_\delta^0$ when δ is small enough. The strategy of the proof is to use the supercritical spaces $\mathcal{G}_\epsilon^\alpha, \alpha > 0$ to approximate \mathcal{G}_ϵ^0 . We divide the proof in five steps, paralleling Uhlenbeck’s paper [132].

- **Claim 1:** $\mathcal{G}_\epsilon^\alpha$ is connected for all $\epsilon, \alpha \geq 0$.
- **Claim 2:** $\mathcal{F}_{\epsilon,C}^\alpha$ is closed (in $\mathcal{G}_\epsilon^\alpha$) with respect to the $W^{1,3+\alpha}$ -norm for $\alpha \geq 0$ and for any $C > 0$.
- **Claim 3:** For $\epsilon > 0$ small enough and $\alpha > 0$, there exists $C = C_\alpha$ such that the set $\mathcal{F}_{\epsilon,C}^\alpha$ is open in $\mathcal{G}_\epsilon^\alpha$ with respect to the $W^{1,3+\alpha}$ -topology.
- **Claim 4:** \mathcal{G}_ϵ^0 is contained in the $W^{1,3}$ -closure of $\cup_{\alpha>0} \mathcal{G}_{2\epsilon}^\alpha$.

Proof of Claim 1. This is straightforward since $\mathcal{G}_\epsilon^\alpha$ is actually convex.

Proof of Claim 2. Consider a family $Q_j \in \mathcal{F}_{\epsilon,C}^\alpha$ with associated P_j as in (F.15) which converge to Q in $W^{1,3+\alpha}$. We can extract a weakly convergent subsequence of the P_j and the estimate passes to the limit by weak lower-semicontinuity (and by convergence of the Q_j). Similarly, the equations pass to

weak limits, since they are intended in the weak sense.

Ideas for Claim 3. For the proof we need to study the behavior of solutions to the equation $\operatorname{div}(P^{-1}\nabla P) = 0$, which is regarded here as an equation $\mathcal{N}_\alpha(P) = 0$, with P close to the constant I which is a zero of \mathcal{N}_α . The equation considered is elliptic. The proof of the claim is thus done by linearization of \mathcal{N} near I and by implicit function theorem. Ellipticity of the equation translates into invertibility of this linearized operator. The estimate of the $W^{1,4}$ -norm will follow from the a priori estimate of Lemma F.1 once we choose for example $K \leq C_\epsilon/2$. See Lemma F.4 for the complete proof.

Proof of Claim 4. Consider $Q \in G_\epsilon^0$. By density arguments we find a sequence $Q_i \in C^\infty(\mathbb{S}^3, SU(2))$ such that $Q_i \rightarrow Q$ in $W^{1,3}(\mathbb{S}^3, SU(2))$. The density of smooth functions in the Sobolev space $W^{1,p}(X, Y)$ where X, Y are smooth compact manifolds was studied in [19], [64, 65, 66, 67], and this density is always true for $p \geq \dim(X)$; see the cited papers and the references therein for more general results. As in the cited proofs of the density, the case $p = \dim(X)$ is obtain by a limiting procedure on $p \rightarrow (\dim(X))^+$, which for us means that we may assume as well $Q_i \in \mathcal{G}_{\epsilon_i}^{\alpha_i}$, for some sequence $\alpha_i \rightarrow 0^+$. We note that the L^3 -norm of a function f can be obtained as

$$\lim_{q \rightarrow 3^+} \|f\|_{L^q}$$

so in particular we may assume up to extracting a subsequence that $\epsilon_i \leq 2\epsilon$.

End of proof. Consider Q as in the statement of the theorem. In other words, $Q \in \mathcal{G}_\delta^0$. We use Claim 4 to approximate Q in $W^{1,3}$ -norm by $Q_i \in \mathcal{G}_{2\delta}^{\alpha_i}$ with $\alpha_i > 0$. From the first three claims above it follows that there exist functions $P_i \in W^{1,4+\alpha_i}(B^4, SU(2))$ such that

$$\|P_i - I\|_{W^{1,4}(B^4)} \leq K \|dQ_i\|_{L^3(\mathbb{S}^3)} \leq 2K\delta.$$

The P_i have a weakly convergent subsequence whose limit P satisfies

$$\begin{cases} \operatorname{div}(P^{-1}\nabla P) = 0 & \text{on } B^4 \\ P = Q & \text{on } \mathbb{S}^3 \end{cases} \quad \text{and } \|P - I\|_{W^{1,4}(B^4)} \leq 2K\delta.$$

We now use the a priori estimates, Lemma F.1. For this, we will choose $\delta > 0$ such that $2K\delta \leq \epsilon$ for ϵ as in Lemma F.1. We can then apply that lemma and obtain that

$$\|P - I\|_{W^{1,4}(B^4)} \leq c \|P - I\|_{W^{4/3,3}(B^4)} \leq cC_\epsilon \|Q\|_{L^3(\mathbb{S}^3)}.$$

This concludes the proof. □

Remark F.3 (Need for a priori estimates). *In the proof of Claim 3 of the above proof we use the fact that for $\alpha > 0$ we have the Sobolev inequality (valid on compact 3-dimensional manifolds) $\|Q\|_{C^0} \leq c_\alpha \|Q\|_{W^{1,3+\alpha}}$. The dependence of the resulting constant C_α on α comes from this inequality, in particular $C_\alpha \rightarrow \infty$ for $\alpha \rightarrow 0^+$. The a priori estimate of Lemma F.1 used in the last step of the proof is crucial precisely for this reason.*

We now use the inverse function theorem for the operator $P \mapsto \operatorname{div}(P^{-1}\nabla P)$.

Lemma F.4. *There exist $\epsilon > 0, K > 0$ such that for all $\alpha > 0$ there exists $C_\alpha > 0$ with the following properties.*

Let $Q_0 \in W^{1,3+\alpha}(\mathbb{S}^3, SU(2))$ and let $P_0 \in W^{1,4+\alpha}(B^4, SU(2))$ be an extension of Q_0 which satisfies $\operatorname{div}(P_0^{-1}\nabla P_0) = 0$. If the following estimates hold:

$$\|dQ_0\|_{W^{1,3}(\mathbb{S}^3)} < \epsilon, \quad (\text{F.16})$$

$$\|P_0 - I\|_{W^{1,4}(B^4)} \leq K \|dQ_0\|_{W^{1,3}(\mathbb{S}^3)}, \quad (\text{F.17})$$

$$\|P_0 - I\|_{W^{1,4+\alpha}(B^4)} \leq C_\alpha \|dQ_0\|_{W^{1,3+\alpha}(\mathbb{S}^3)}, \quad (\text{F.18})$$

then for some $\delta > 0$ depending on Q_0 , for all Q satisfying

$$\|Q - Q_0\|_{W^{1,3+\alpha}(\mathbb{S}^3, SU(2))} < \delta, \quad (\text{F.19})$$

there exists an extension P of Q satisfying the same equation $\operatorname{div}(P^{-1}\nabla P) = 0$ and such that (F.16), (F.17), (F.18) hold with P, Q in place of P_0, Q_0 .

Proof. We fix Q satisfying (F.19) and (F.16). The proof is divided in two parts:

- **Claim 1:** *For $\delta n > 0$ small enough and for Q satisfying (F.19) there exists an extension P of Q solving $\operatorname{div}(P^{-1}\nabla P) = 0$ and such that (F.18) holds.*
- **Claim 2:** *The function P of Claim 1 satisfies (F.17).*

Proof of Claim 1. First note that $V = \exp^{-1}(Q_0^{-1}Q)$ is well defined for $\alpha > 0$ because in that case we have an estimate of the form

$$\|Q - Q_0\|_{W^{1,3+\alpha}} \geq c_\alpha \|Q - Q_0\|_{L^\infty} \Leftrightarrow \|Q_0^{-1}Q - I\|_{L^\infty} \leq \epsilon/c_\alpha$$

and \exp^{-1} is well-defined in a neighborhood of the identity.

We consider the problem of extending $Q_0 \exp(V)$ inside B^4 to a function $P = P_0 \exp(U)$ satisfying (F.20). Instead of considering the extension as a perturbation of P_0 only, we first extend V to \tilde{V} such that $\Delta \tilde{V} = 0$ inside B^4 .

We look for a P of the form $P_0 \exp(\tilde{V}) \exp(U)$. We thus consider the equation

$$\mathcal{N}(U, V) := d^* \left(\exp(-U) \exp(-\tilde{V}) P_0^{-1} d(P_0 \exp(\tilde{V}) \exp(U)) \right) = 0. \quad (\text{F.20})$$

In order to solve (F.20) it is interesting to look at the operator

$$\mathcal{N}(V, U) : W_0^{1,4+\alpha}(B^4, su(2)) \rightarrow W^{-1,4+\alpha}(B^4, su(2)). \quad (\text{F.21})$$

We have to prove that for $\delta > 0$ small enough for each Q satisfying $d^*(P^{-1}dP) = 0$ (i.e. for each small enough V), there exists a unique U such that $\mathcal{N}(V, U) = 0$. Therefore it will be enough to prove that $\partial \mathcal{N} / \partial U$ is an isomorphism between the two spaces above. It will be enough to restrict to the case where V, U have norms $\leq C\delta$. Our estimates will prove that $\mathcal{N}(U, V)$ is C^1 near the couple $(0, 0)$ and that $\partial \mathcal{N} / \partial U(0, 0)$ is an isomorphism, given the existence of $\delta > 0$ as desired.

A simple calculation gives:

$$\begin{aligned} \frac{\partial \mathcal{N}}{\partial U} \cdot \eta &= \left. \frac{\partial}{\partial t} \right|_{t=0} \mathcal{N}(U + t\eta, V) \\ &= d^* d\eta - d^* \left[\eta, \exp(-U) \exp(-\tilde{V}) P_0^{-1} d(P_0 \exp(\tilde{V})) \exp(U) \right] \\ &:= \Delta \eta - L\eta. \end{aligned}$$

We observe that $d^*d = \Delta$ is an isomorphism between the spaces above, so it will be enough to prove that for U, \tilde{V} small enough in the $W^{1,4+\alpha}$ -norm the commutator term $L\eta$ is just a small perturbation of Δ (with respect to the norms present in (F.21)). First note that we can write

$$\begin{aligned} L\eta &= [\nabla \eta, X] + [\eta, \operatorname{div} X], \\ X &:= \exp(-U) \exp(-\tilde{V}) P_0^{-1} d(P_0 \exp(\tilde{V})) \exp(U) \end{aligned}$$

Estimate for $[\nabla \eta, X]$. First note that by the Sobolev, Hölder and triangle inequalities

$$\|[\nabla \eta, X]\| W^{-1,4+\alpha} \lesssim \|[\nabla \eta, X]\|_{L^{p_\alpha}} \lesssim \|\nabla \eta\|_{L^{4+\alpha}} \|X\|_{L^4}$$

where

$$\frac{1}{p_\alpha} = \frac{1}{4+\alpha} + \frac{1}{4}.$$

We then observe

$$X = \exp(-U)\exp(-\tilde{V})P_0^{-1}d(P_0\tilde{V})\exp(\tilde{V})\exp(U)$$

and note $|\exp A| = 1$ therefore

$$\|X\|_{L^4} = \|d(P_0\tilde{V})\|_{L^4} \lesssim \|dP_0\|_{L^4} + \|d\tilde{V}\|_{L^4} \lesssim \epsilon + \delta.$$

We thus have the first desired estimate

$$\|[\nabla\eta, X]\|_{W^{-1,4+\alpha}} \lesssim (\epsilon + \delta)\|\eta\|_{W^{1,4+\alpha}}.$$

Estimate for $[\eta, \operatorname{div}X]$. Here we start with

$$\|[\eta, \operatorname{div}X]\|_{W^{-1,4+\alpha}} \lesssim \|\eta\|_{L^\infty}\|\operatorname{div}X\|_{L^{p_\alpha}}.$$

Note that $\|\eta\|_{L^\infty} \lesssim \|\eta\|_{W^{1,4+\alpha}}$ by the Sobolev embedding. We start the computations for the second fact or above. Note

$$\nabla(P_0\exp\tilde{V}) = (\nabla P_0)\exp\tilde{V} + P_0\nabla(\exp\tilde{V})$$

and then expand:

$$\begin{aligned} \operatorname{div}X &= \operatorname{div}\left[\exp(-U)\exp(-\tilde{V})P_0^{-1}\nabla(P_0\exp(\tilde{V}))\exp(U)\right] \\ &= \nabla\left(\exp(-U)\right)\exp(-\tilde{V})P_0^{-1}\nabla(P_0\exp(\tilde{V}))\exp(U) \\ &\quad + \exp(-U)\nabla\left(\exp(-\tilde{V})\right)P_0^{-1}\nabla(P_0\exp(\tilde{V}))\exp(U) \\ &\quad + \exp(-U)\exp(-\tilde{V})\operatorname{div}\left(P_0^{-1}\nabla P_0\right)\exp(\tilde{V})\exp(U) \\ &\quad + \exp(-U)\exp(-\tilde{V})P_0^{-1}P_0\operatorname{div}\nabla\left(\exp(\tilde{V})\right)\exp(U) \\ &\quad + \exp(-U)\exp(-\tilde{V})P_0^{-1}\nabla P_0\nabla\left(\exp(\tilde{V})\right)\exp(U) \\ &\quad + \exp(-U)\exp(-\tilde{V})P_0^{-1}\nabla(P_0\exp(\tilde{V}))\nabla\left(\exp(U)\right) \end{aligned}$$

We have $\operatorname{div}(P_0^{-1}\nabla P_0) = 0$ and $\operatorname{div}\nabla(\exp(\tilde{V})) = 0$ so two terms cancel. Note also the fact that $\|P_0^{-1}\nabla P_0\|_{L^4} \leq \|\nabla P_0\|_{L^4} \leq \epsilon$. Recall again that $|\exp A| = 1$ for all $A \in su(2)$. For estimating $\nabla(\exp(\pm\tilde{V}))$ observe that \tilde{V} satisfies a Dirichlet boundary value problem therefore we assumed the estimate $\|\tilde{V}\|_{W^{1,4+\alpha}} \lesssim \delta$, and $\|U\|_{W^{1,4+\alpha}} \lesssim \delta$ which by the smoothness of \exp

imply $\|\nabla(\exp(\pm\tilde{V}))\|_{L^{4+\alpha}} \lesssim \delta$ and $\|\nabla(\exp(\pm U))\|_{L^{4+\alpha}} \lesssim \delta$. From all this it follows that we can estimate

$$\begin{aligned} \|\operatorname{div} X\|_{L^{p\alpha}} &\lesssim \|\nabla(\exp(-U))\|_{L^{4+\alpha}} \|\nabla(P_0 \exp \tilde{V})\|_{L^4} \\ &\quad + \|\nabla(\exp(-\tilde{V}))\|_{L^{4+\alpha}} \|\nabla(P_0 \exp \tilde{V})\|_{L^4} \\ &\quad + \|\nabla P_0\|_{L^4} \|\nabla(\exp(\tilde{V}))\|_{L^{4+\alpha}} \\ &\quad + \|\nabla(\exp(U))\|_{L^{4+\alpha}} \|\nabla(P_0 \exp \tilde{V})\|_{L^4} \\ &\lesssim \delta \|\nabla(P_0 \exp \tilde{V})\|_{L^4} + \epsilon \delta \\ &\lesssim \delta(\epsilon + \delta). \end{aligned}$$

We thus again combine all the estimates and obtain the desired smallness result

$$\|[\eta, \operatorname{div} X]\|_{W^{-1,4+\alpha}} \lesssim \delta(\epsilon + \delta) \|\eta\|_{W^{1,4+\alpha}}.$$

Step 3. We now have that

$$\|L\eta\|_{W^{-1,4+\alpha}} \lesssim (\delta + 1)(\epsilon + \delta) \|\eta\|_{W^{1,4+\alpha}}$$

while

$$\|\Delta\eta\|_{W^{-1,4+\alpha}} \gtrsim \|\eta\|_{W^{1,4+\alpha}}.$$

Therefore for small enough ϵ, δ we have also

$$\|(\Delta - L)\eta\|_{W^{-1,4+\alpha}} \gtrsim \|\eta\|_{W^{1,4+\alpha}}.$$

This concludes the proof. □

F.2 A product estimate with only one bounded factor

Lemma F.5 (cf. [31]). *Let Ω be a smooth compact 4-manifold. If $f \in W^{1/3,3}(\Omega)$ and $g \in W^{1,4} \cap L^\infty(\Omega)$ then we have the following estimate, with the implicit constant depending only on Ω :*

$$\|fg\|_{W^{1/3,3}(\Omega)} \lesssim \|f\|_{W^{1/3,3}(\Omega)} (\|g\|_{L^\infty(\Omega)} + \|g\|_{W^{1,4}(\Omega)})$$

Proof. The estimates for the non-homogeneous part of the norms are trivial, so we concentrate on the homogeneous part.

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We use the Littlewood-Paley decompositions $f = \sum_{j=0}^{\infty} f_j, g = \sum_{k=0}^{\infty} g_k$, and we recall that the $W^{s,p}$ -norm is equivalent to the Triebel-Lizorkin $\dot{F}_{4,2}^1$ -norm and the $W^{\theta,4}$ -norm is equivalent to the $F_{p,2}^s$ -norm, where in general the following definition holds

$$\|f\|_{\dot{F}_{p,q}^s} = \left\| \left| 2^{ks} f_k(x) \right|_{\ell^q} \right\|_{L^p}.$$

We use different notations $\|\cdot\|, |\cdot|$ for the different norms just to facilitate the reading of formulas. As is usual in the theory of paraproducts, we estimate separately the following three contributions (where $g^k := \sum_{i=0}^k g_i$ and similarly for f^k)

$$fg = \sum_i f_i g^{i-4} + \sum_{|k-l|<4} f_k g_l + \sum_i f^{i-4} g_i := I + II + III.$$

The support of $\widehat{(f_i g^{i-4})}$ is included in $B_{2^{i+2}} \setminus B_{2^{i-2}}$ thus there holds

$$\|I\|_{W^{\frac{1}{3},3}} = \left\| \sum_i f_i g^{i-4} \right\|_{W^{\frac{1}{3},3}} \sim \left[\int_{\Omega} \left(\sum_i 2^{\frac{2i}{3}} |f_i g^{i-4}|^2 \right)^{\frac{3}{2}} \right]^{\frac{1}{3}}. \tag{F.22}$$

and analogously for $III = \sum_i f^{i-4} g_i$. Regarding the term II we will estimate only $II' := \sum_i f_i g_i$ because the same estimate will apply also to the finitely many contributions of the form $\sum_i f_i g_{i+l}$ with $0 < |l| < 4$.

We start with the most difficult term III . From above we have

$$\begin{aligned} \|III\|_{W^{\frac{1}{3},3}} &\sim \left[\int \left(\sum_i 2^{\frac{2i}{3}} |f^{i-4} g_i|^2 \right)^{\frac{3}{2}} \right]^{\frac{1}{3}} \\ &\leq \left[\int \left(\sum_i 2^{-\frac{4i}{3}} |f^{i-4}|^2 \right)^{\frac{3}{2}} \left(\sum_i 2^{2i} |g_i|^2 \right)^{\frac{3}{2}} \right]^{\frac{1}{3}} \\ &\leq \left[\int \left(\sum_i 2^{-\frac{4i}{3}} |f^{i-4}|^2 \right)^6 \right]^{\frac{1}{12}} \left[\int \left(\sum_i 2^{2i} |g_i|^2 \right)^2 \right]^{\frac{1}{4}} \\ &\leq \|f\|_{W^{-\frac{2}{3},12}} \|g\|_{W^{1,4}} \\ &\leq \|f\|_{W^{\frac{1}{3},3}} \|g\|_{W^{1,4}}. \end{aligned}$$

For the term I we have

$$\begin{aligned} \|I\|_{W^{\frac{1}{3},3}} &\sim \left[\int \left(\sum_i 2^{\frac{2i}{3}} |f_i g^{i-4}|^2 \right)^{\frac{3}{2}} \right]^{\frac{1}{3}} \\ &\lesssim \|g\|_{L^\infty} \|f\|_{W^{\frac{1}{3},3}} \end{aligned}$$

because of the estimate $\|g^{i-4}\|_{L^\infty} \lesssim \|g\|_{L^\infty}$. Finally we estimate II' as promised. We prove it by duality, namely we prove that II' is bounded as a linear functional on the unit ball of the dual $W^{-\frac{1}{3},\frac{3}{2}}$. Consider therefore h in this ball. We note that the support of $(f_i g_i)$ is included in $B_{2^{i+2}}$ therefore some terms cancel

$$\begin{aligned} \int h \cdot II' &\sim \sum_{k,i} \int h_k f_i g_i = \sum_{k \leq i+4} \int h_k f_i f_j = \sum_i \int h^{i+4} f_i g_i \\ &\leq \left| \sum_i \int 2^{-\frac{i}{3}} h^{i+4} 2^{\frac{i}{3}} f_i g_i \right| \\ &\leq \|g\|_{B_{\infty,\infty}^0} \int \left(\sum_i 2^{-\frac{2i}{3}} |h^{i+4}|^2 \right)^{\frac{1}{2}} \left(\sum_i 2^{\frac{2i}{3}} |f_i|^2 \right)^{\frac{1}{2}} \\ &\leq \|g\|_{W^{1,4}} \|h\|_{W^{-\frac{1}{3},\frac{3}{2}}} \|f\|_{W^{\frac{1}{3},3}} \end{aligned}$$

The last estimate follows recalling that

$$\|g\|_{B_{\infty,\infty}^0} := \sup_i \|g_i\|_{L^\infty}$$

and that in dimension 4 we have continuous embeddings

$$W^{1,4} \hookrightarrow \text{BMO} \hookrightarrow B_{\infty,\infty}^0.$$

Summing up the different terms we conclude. □

F.3 Computations for the Möbius group

We call the Möbius group of \mathbb{R}^n the group $M(\mathbb{R}^n)$ generated by all similarities and the inversion with respect to the unit sphere. Recall that a similarity is an affine map of the form

$$x \mapsto \lambda Kx + b \text{ with } \lambda > 0, K \in O(n), b \in \mathbb{R}^n,$$

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and the inversion $i_{c,r}$ with respect to the sphere $\partial B(c,r)$ is the map

$$x \mapsto c + r^2 \frac{x - c}{|x - c|^2}.$$

The formula $i_{c,r} = (r^2 Id + c) \circ i_{0,1} \circ (Id - c)$ implies that all inversions belong to $M(\mathbb{R}^n)$. We use the following abridged notation:

$$x^* := i_{1,0}(x) = x/|x|^2.$$

The Möbius group of B^{n+1} is the subgroup $M(B^{n+1})$ of all transformations belonging to $M(\mathbb{R}^n)$ and which preserve B^{n+1} . Similarly we define the Möbius group $M(\mathbb{S}^n)$ of the unit sphere $\mathbb{S}^n \subset \mathbb{R}^n$. The general form of an element $\gamma \in M(B^{n+1})$ is

$$\gamma = K \circ F_v, \text{ with } K \in O(n), v \in B^4, F_v := -v + (1 - |v|^2)(x^* - v)^*.$$

We use the following basic properties of the functions F_v which can be found in [4], Chap. 2:

Lemma F.6. • *There holds*

$$|F'_v|(x) = \frac{1 - |v|^2}{[x, v]}$$

$$\text{where } [x, y] = |x||x^* - y| = |y||y^* - x|.$$

- F_v is conformal. We have $F_v^{-1} = F_{-v}$, $F_v(0) = -v$ and $F_v(v) = 0$.
- The conformal factor $|F'_v|(x)$ is explicitly computed as

$$|F'_v|(x) = \frac{1 - |v|^2}{1 + |x|^2|v|^2 - 2x \cdot v} = \frac{|v^*|^2 - 1}{|x - v^*|^2}.$$

- The restriction $F_v|_{\mathbb{S}^3}$ belongs to $M(\mathbb{S}^3)$, in particular $F_v|_{\mathbb{S}^3}$ is a conformal involution and

$$|(F_v|_{\mathbb{S}^3})'(x) = \frac{1 - |v|^2}{|x - v|^2}.$$

The next lemma gives the estimate need in Lemma 9.35 for the case when v is close to ∂B^4 :

Lemma F.7. *Suppose that*

$$\rho \leq \frac{1}{4}.$$

Then on $F_v^{-1}(B_{1-\rho})$ the following estimate holds with a geometric constant C :

$$\frac{h(v)}{C} \leq |F'_v|(x) \leq Ch(v).$$

Proof. We will calculate

$$\frac{\max\{|F'_v|(y) : y \in F_v^{-1}(B_{1-\rho})\}}{\min\{|F'_v|(y') : y' \in F_v^{-1}(B_{1-\rho})\}} = \max\left\{\frac{|F'_v|(y)}{|F'_v|(y')} : y, y' \in F_v^{-1}(B_{1-\rho})\right\}$$

and we prove that this quantity is bounded. The following equalities hold:

$$\begin{aligned} \max\left\{\frac{|F'_v|(x)}{|F'_v|(x')} : x, x' \in B_{1-\rho}\right\} &= \max\left\{\frac{|F'_{-v}|(x)}{|F'_{-v}|(x')} : x, x' \in B_{1-\rho}\right\} \\ &= \max\left\{\frac{|(F_v^{-1})'(x)|}{|(F_v^{-1})'(x')|} : x, x' \in B_{1-\rho}\right\} \\ &= \min\left\{\frac{|F'_v|(F_v^{-1}(x'))}{|F'_v|(F_v^{-1}(x))} : x, x' \in B_{1-\rho}\right\} \\ &= \min\left\{\frac{|F'_v|(y')}{|F'_v|(y)} : y, y' \in F_v^{-1}(B_{1-\rho})\right\}. \end{aligned}$$

From the formula of the previous lemma it follows that

$$\nabla_x |F'_v|(x) = 2 \frac{|v^*|^2 - 1}{|v^* - x|^4} (v^* - x),$$

therefore $|F'_v|$ achieves its extrema on $B_{1-\rho}$ at $\pm(1-\rho)\frac{v}{|v|}$. The maximum M and the minimum m of $|F'_v|$ satisfy

$$\begin{aligned} M &= \frac{1 - |v|^2}{1 + |v|^2(1-\rho)^2 - 2(1-\rho)|v|} = \frac{1 - |v|^2}{(1 - (1-\rho)|v|)^2}, \\ m &= \frac{1 - |v|^2}{1 + |v|^2(1-\rho)^2 + 2(1-\rho)|v|} = \frac{1 - |v|^2}{(1 + (1-\rho)|v|)^2}, \\ \frac{M}{m} &= \left(\frac{1 + (1-\rho)|v|}{1 - (1-\rho)|v|}\right)^2 \sim (1 - (1-\rho)|v|)^{-2} \sim 1, \end{aligned}$$

which finishes the proof. □

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General Informations

Born on 13/02/1985, Italian citizenship

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Current employment

Teaching assistant, Mathematics Department, ETH Zürich.

Education

2008–present **Ph.D. student in Mathematics**, ETH Zürich, Ph.D research topic: L^p -curvatures on weak bundles, Advisor: Prof. Tristan Rivière.

2008 **Master degree in Mathematics**, Università di Pisa and Scuola Normale Superiore di Pisa, Thesis Title: *Differential Inclusions and the Euler Equation*, Advisor: Prof. Luigi Ambrosio.

Grade: 110/110 cum laude

2006 **Bachelor degree in Mathematics**, Università di Pisa and Scuola Normale Superiore di Pisa, Thesis Title: *Gromov's Non-Squeezing Theorem*, Advisor: Prof. Alberto Abbondandolo.

Grade: 110/110 cum laude

Teaching experience at ETH Zürich

Fall 2012 **Calculus I**, (1st year course for Electrical Engineers), coordination of exercises.

Spring 2012 **Introduction to Elliptic PDEs**, (master course for Mathematicians), main lecturer.

Fall 2011 **Critical and Optimal Surfaces**, (master course for Mathematicians), teaching assistant.

Fall 2011 **Calculus III**, (2nd year course for Electrical Engineers), coordination of the exercises.

Spring 2011 **Functional Analysis II**, (3rd year course for Mathematicians), coordination of the exercises.

Fall 2010 **Partial Differential Equations**, (2nd year course for Electrical Engineers), coordination of the exercises.

Spring 2010 **Calculus II**, (1st year course for Electrical Engineers), coordination of the exercises.

Spring 2010 **Calculus II**, (1st year course for Electrical Engineers), exercise classes.

Fall 2009 **Hidden Convexity in some Nonlinear PDE's from Geometry and Physics**, (*graduate lectures by Prof. Yann Brenier, Institute of Mathematical Research*), preparation of the lecture notes of the course.

Fall 2009 **Calculus I**, (*1st year course for Informaticians*), exercise classes.

Grants and Awards

2010 **Swiss National Science Foundation Graduate Research Fellowship**, 200021-126489, for the project *The analysis of singular or blow up sets for solutions to PDE's arising in physics and geometry*.

2006 **Grant from Scuola Normale Superiore di Pisa**, for a 3 month stay at ENS Paris.

2003-2008 **Scholarship of Scuola Normale Superiore di Pisa**.

Invited talks

31/01/2013 "*The Yang-Mills Lagrangian in supercritical dimensions*", Geometric Analysis and PDE seminar, Cambridge University, Cambridge

31/01/2013 "*Singular bundles and the Yang-Mills lagrangian in supercritical dimensions*", Interview talk, Imperial College, London

01/10/2012 "*Combinatorial regularity for supercritical curvatures*", Arbeitsgemeinschaft Analysis, University of Zürich

05/09/2012 " *L^p vector fields with topological defects: geometric, variational and combinatorial aspects*", Meeting on applied Mathematics and the Calculus of Variations, Rome

16/06/2012 "*Constructing nontrivial $U(1)$ -bundles by variational methods*", Riemannian Topology Meeting, University of Fribourg

06/06/2012 "*A new glimpse of optimal transport from the geometric analysis viewpoint: regularity theory for minimizing abelian curvatures*", Conference for the centenary of L. Kantorovich, EIMI St. Petersburg

15/05/2012 "*A Plateau problem for $U(1)$ -bundles in 3 dimensions*", Analysis seminar, ETH, Zürich

11/05/2011 "*A variational problem with unboundedly many charges in 3 dimensions*", Centre de Mathématiques appliquées, École Polytechnique, Paris

03/03/2009 "*What is a Micro-structure?*", Zürich Graduate Colloquium, ETH, Zürich

Referee work

Annali della Scuola Normale Superiore

Journal of Functional Analysis

Discrete and Continuous Dynamical Systems

Languages

English: fluent in writing and speaking

French: fluent in writing and speaking

German: good in writing and speaking

Italian: mother tongue

Romanian: mother tongue