# Local Palais-Smale Sequences for the Willmore Functional

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**Abstract :** Using the reformulation in divergence form of the Euler-Lagrange equation for the Willmore functional as it was developed in the second author's paper [Riv2], we study the limit of a local Palais-Smale sequence of weak Willmore immersions with locally square-integrable second fundamental form. We show that the limit immersion is smooth and that it satisfies the *conformal Willmore equation*: it is a critical point of the Willmore functional restricted to infinitesimal conformal variations.

# I Introduction

## I.1 The Willmore Functional and the Willmore Equation

Consider an oriented surface $\Sigma$ without boundary immersed in $\mathbb{R}^m$, for some $m \geq 3$, through the action of a smooth positive immersion $\vec{\Phi}$. We introduce the Gauß map $\vec{n}$, which to every point $x$ in $\Sigma$ assigns the unit $(m-2)$-plane $N_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$ orthogonal to the oriented tangent space $T_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$. This map acts from $\Sigma$ into $Gr_{m-2}(\mathbb{R}^m)$, the Grassmannian of oriented $(m-2)$-planes in $\mathbb{R}^m$. It thus naturally induces a projection map $\pi_{\vec{n}}$ which to every vector $\xi$ in $T_{\vec{\Phi}(x)}\mathbb{R}^m$ associates its orthogonal projection $\pi_{\vec{n}}(\xi)$ onto $N_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$. Let $x$ be a point on $\Sigma$. We denote by $\vec{B}_x$ the second fundamental form of the immersion $\vec{\Phi}$. It is a $N_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$-valued symmetric bilinear form on $T_x\Sigma \times T_x\Sigma$, defined by $\vec{B}_x = \pi_{\vec{n}} \circ d^2\vec{\Phi}$. Having chosen an orthonormal basis $\{e_1, e_2\}$ on $T_x\Sigma$, the mean curvature $\vec{H}(x)$ of the immersion $\vec{\Phi}$ is the vector in $N_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$ given by

$$\vec{H}(x) \;:=\; \frac{1}{2}\,\mathrm{Tr}\,(\vec{B}_x) \;\equiv\; \frac{1}{2}\left(\vec{B}_x(e_1,e_1) \,+\, \vec{B}_x(e_2,e_2)\right).$$

The *Willmore functional* is the Lagrangian

$$W\big(\vec{\Phi}(\Sigma)\big) \;:=\; \int_\Sigma \big|\vec{H}\big|^2\, d\mu_g\,, \tag{I.1}$$

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where $d\mu_g$ is the area form of the metric $g$ induced on $\vec{\Phi}(\Sigma)$ via the canonical metric on $\mathbb{R}^m$.

The critical points of (I.1) for perturbations of the form $\vec{\Phi} + t\,\vec{\xi}$, where $\vec{\xi}$ is an arbitrary compactly supported smooth map on $\Sigma$ into $\mathbb{R}^m$ are known as *Willmore surfaces*. Examples of Willmore surfaces are legions. Any minimal surface, i.e. one for which $\vec{H} \equiv 0$, realizes an absolute minimum of the Willmore functional. Round spheres are also Willmore surfaces, and, more generally, all Willmore surfaces with genus zero were obtained by Robert Bryant [Bry1], [Bry2]. Another important example was devised by Willmore in 1965. It is the torus of revolution obtained through rotating a circle of radius 1 whose center is located at a distance $\sqrt{2}$ from its axis of rotation (equivalently, it is the stereographic projection into $\mathbb{R}^3$ of the Clifford torus). Willmore proved that this torus is indeed a Willmore surface, and he conjectured that it minimizes, up to Möbius transforms, the Willmore energy in the class of smooth and immersed tori. Despite partial answers, this assertion (known as *the Willmore conjecture*) remains unsolved. For more details, the reader is referred to [BK], [LY], [Sim], and [Wil2]. Further examples of Willmore surfaces are profuse in the literature, and we content ourselves with citing [Wil2], [PS], and the references therein.

The Euler-Lagrange equation obtained through varying the Willmore functional as aforementioned was first[1] derived in [Wil1] in the three-dimensional case, and subsequently extended by Weiner [Wei] for general $m \geq 3$. We now recall this equation.
Given any vector $\vec{w}$ in $N_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$, consider the symmetric endomorphism $A_x^{\vec{w}}$ of $T_x\Sigma$ satisfying $g(A_x^{\vec{w}}(\vec{X}),\vec{Y}) = B_x(\vec{X},\vec{Y})\cdot\vec{w}$, where $\cdot$ denotes the standard scalar product in $\mathbb{R}^m$, for every pair of vectors $\vec{X}$ and $\vec{Y}$ in $T_x\Sigma$. The map $A_x : \vec{w} \mapsto A_x^{\vec{w}}$ is a homomorphism from $N_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$ into the linear space of symmetric endomorphisms on $T_x\Sigma$. We next define $\tilde{A}_x = {}^tA_x \circ A_x$, which is an endomorphism of $N_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$. If $\{e_1, e_2\}$ is an orthonormal basis of $T_x\Sigma$, and if $\vec{L}$ is a vector in $N_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$, then it is readily seen that $\tilde{A}(\vec{L}) = \sum_{i,j}\vec{B}_x(e_i,e_j)\vec{B}_x(e_i,e_j)\cdot\vec{L}$. With this notation, as shown in [Wei], $\vec{\Phi}$ is a smooth Willmore immersion if and only if it satisfies Euler-Lagrange equation

$$\Delta_\perp\vec{H} - 2\,|\vec{H}|^2\vec{H} + \tilde{A}(\vec{H}) = 0\,, \tag{I.2}$$

where $\Delta_\perp$ is the negative covariant Laplacian for the connection $D$ in the normal bundle $N\vec{\Phi}(\Sigma)$ derived from the ambient scalar product in $\mathbb{R}^m$. Namely, for every section $\sigma$ of $N\vec{\Phi}(\Sigma)$, one has $D_{\vec{X}}\sigma := \pi_{\vec{n}}(\sigma_*\vec{X})$.

The paternity of the Willmore functional is delicate to establish precisely.

[1] although it seemingly was known to Thomsen and Blaschke decades sooner.

Although it bears the name of T.J. Willmore whom studied it in 1965 [Wil1] thereby initiating its popularization, the Willmore functional had been previously considered in the works of Sophie Germain [Ger], Gerhard Thomsen [Tho], and Wilhelm Blaschke [Bla]. Its wide range of applications includes various areas of science, where it plays an important rôle. Amongst others, the Willmore functional appears in molecular biophysics as the surface energy for lipid bilayers in the Helfrich model [Hef] (cell membranes tend to position themselves so as to minimize the Willmore energy) ; in solid mechanics as the limit-energy for thin plate theory [FJM] ; in general relativity as the main contributing term to the Hawking quasilocal mass (cf. [Haw], [HI]) ; in string theory as an extrinsic string action à la Polyakov [Pol].

The importance of the Willmore functional is largely due to its invariance under conformal transformations of the metric of the ambient space. This remarkable property was first brought into light by White [Whi] in the three-dimensional case, then generalized by B.Y. Chen [Che]. As the second author of the present paper showed in [Riv1], Euler-Lagrange equations arising from a two-dimensional conformally invariant Lagrangian with quadratic growth can be written in divergence form. These "conservations laws" fostered within variational problems involving conformally invariant Lagrangians offer a significant help. In particular, the general ideas introduced in [Riv1] are led to fruition in [Riv2], where conservation laws relative to the Willmore functional are developed and successfully applied to produce a variety of interesting results. Our present work stems from this alternative formulation of the Willmore equation.

## I.2 Weak Willmore Immersions and Conservation Laws

This section is devoted to recalling the formalism introduced in [Riv2] and the results therein established, which compose the foundation of our work.

Owing to the Gauß-Bonnet theorem, we note that the Willmore energy (I.1) may be equivalently expressed as

$$W\big(\vec{\Phi}(\Sigma)\big) \;=\; \int_\Sigma \big|\vec{B}\big|^2\,d\mu_g \;+\; 4\,\pi\,\chi(\Sigma)\,,$$

where $\vec{B}$ is the second fundamental form, and $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$. Since the latter is a topological invariant, from the variational point of view, we infer that Willmore surfaces are the critical points of the energy

$$\int_\Sigma \big|\vec{B}\big|^2\,d\mu_g\,.$$

In studying such surfaces, it thus appears natural to restrict our attention on immersions whose second fundamental forms are locally square-integrable.

More precisely, we work within the framework of *weak immersion with locally $L^2$-bounded second fundamental form*, which we now define.

**Definition I.1** *Let $\vec{\Phi}$ be a $W^{1,2}$-map from a two-dimensional manifold $\Sigma$ into $\mathbb{R}^m$. $\vec{\Phi}$ is called a weak immersion with locally $L^2$-bounded second fundamental form whenever there exist locally about every point an open disk $D$, a positive constant $C$, and a sequence of smooth embeddings $(\vec{\Phi}_k)$ from $D$ into $\mathbb{R}^m$, such that*

*i)* $$\mathcal{H}_2(\vec{\Phi}(D)) \neq 0\,,$$

*ii)* $$\mathcal{H}_2(\vec{\Phi}_k(D)) \leq C < \infty\,,$$

*iii)* $$\int_D |B_k|^2 \; dvol_{g_k} \leq \frac{8\pi}{3}\,,$$

*iv)* $$\vec{\Phi}_k \rightharpoonup \vec{\Phi} \quad \textit{weakly in } W^{1,2}\,,$$

*where $\mathcal{H}_2$ is the two-dimensional Hausdorff measure, $B_k$ is the second fundamental form associated to the embedding $\vec{\Phi}_k$, and $g_k$ denotes the metric on $\vec{\Phi}_k(\Sigma)$ obtained via the pull-back by $\vec{\Phi}_k$ of the induced metric.*

A useful characterization of weak immersions with square-integrable second fundamental form was originally obtained by Tatiana Toro [To1] and [To2], and by Stefan Müller and Vladimír Šveràk in [MS]. In the same spirit, Frédéric Hélein obtained the following statement (whose proof appears as that of Theorem 5.1.1 in [Hel]).

**Theorem I.1** *Let $\vec{\Phi}$ be a weak immersion from a two-dimensional manifold $\Sigma$ into $\mathbb{R}^m$ with locally $L^2$-bounded second fundamental form. Then locally about every point on $\Sigma$, there exist an open disk $D$ and a homeomorphism $\Psi$ of $D$ such that $\vec{\Phi}\circ\Psi$ is a conformal bilipschitz immersion. In this parametrization, the metric $g$ on $D$ induced by the standard metric of $\mathbb{R}^m$ is continuous. Moreover, the Gauß map $\vec{n}$ of this immersion lies in $W^{1,2}(D, Gr_{m-2}(\mathbb{R}^m))$, relative to the induced metric $g$.*

We previously observed that weak immersions with square-integrable second fundamental form are particularly suited for the study of Willmore surfaces. More importantly, in [Riv2], the notion of *weak Willmore immersion* is introduced. It is based on the following theorem, established in the same paper. It provides a reformulation of the Willmore equation in divergence form.

**Theorem I.2** *The Willmore equation (I.2) is equivalent to*

$$d\left(*_g\, d\vec{H} - 3 *_g \pi_{\vec{n}}\big(d\vec{H}\big)\right) - d\star\left(d\vec{n}\wedge\vec{H}\right) = 0\,,$$

*where* $*_g$ *is the Hodge operator on* $\Sigma$ *associated with the induced metric* $g$*, and* $\star$ *is the usual Hodge operator on forms.*

*In particular, a conformal immersion* $\vec{\Phi}$ *from the flat unit-disc* $D^2$ *into* $\mathbb{R}^m$ *is Willmore if and only if*

$$\Delta\vec{H} \, - \, 3 \, div\big(\pi_{\vec{n}}(\nabla\vec{H})\big) \, + \, div \star \left(\nabla^\perp\vec{n}\wedge\vec{H}\right) \, = \, 0 \, , \qquad \text{(I.3)}$$

*where the operators* $\nabla$*,* $\nabla^\perp$*,* $\Delta$*, and* $div$ *are understood with respect to the flat metric on* $D^2$*. Namely,* $\nabla = (\partial_{x_1}, \partial_{x_2})$*,* $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$*,* $\Delta = \nabla\cdot\nabla$*, and* $div = tr\circ\nabla$*.*

We are now ready to define the notion of *weak Willmore immersion with* $L^2$*-bounded second fundamental form.*

**Definition I.2** *A weak immersion* $\vec{\Phi}$ *from a two-dimensional manifold* $\Sigma$ *into* $\mathbb{R}^m$ *with locally* $L^2$*-bounded second fundamental form is Willmore whenever (I.3) holds in the sense of distributions locally about every point in a conformal parametrization from the two-dimensional disk* $D$*, as indicated in Theorem I.1.*

Henceforth, we will work with weak Willmore immersions, which we shall assume to be conformal on the unit-disk.

We introduce the local coordinates $(x_1, x_2)$ for the flat metric on the unit-disk $D^2 = \big\{(x_1, x_2)\in\mathbb{R}^2 \; ; \; x_1^2 + x_2^2 < 1\big\}$. The operators $\nabla$, $\nabla^\perp$, $div$, and $\Delta$ will be understood in these coordinates. Let $\vec{\Phi} : D^2 \to \mathbb{R}^m$ be a conformal immersion. We define the conformal factor $\lambda$ via

$$\partial_{x_1}\vec{\Phi} \, = \, \mathrm{e}^\lambda \, = \, \partial_{x_2}\vec{\Phi} \, .$$

Thanks to the topology of $D^2$, there exists, for almost every $z\in D^2$, a positively oriented orthonormal basis $\{\vec{n}_1, \ldots, \vec{n}_{m-2}\}$ of $N_{\vec{\Phi}(z)}\vec{\Phi}(D^2)$, complement of the tangent plane to $\vec{\Phi}(D^2)$ at $\vec{\Phi}(z)$, such that $\{\vec{e}_1, \vec{e}_2, \vec{n}_1, \ldots, \vec{n}_{m-2}\}$ forms a basis of $T_{\vec{\Phi}(z)}\mathbb{R}^m$. Following the "Coulomb gauge extraction method" exposed in the proof of Lemma 4.1.3 from [Hel], the basis $\{\vec{n}_\alpha\}_{\alpha=1}^{m-2}$ may be chosen to satisfy

$$div\left\langle\nabla\vec{n}_\alpha\, , \vec{n}_\beta\right\rangle \, = \, 0 \, , \qquad \forall \;\; 1\le\alpha\, ,\beta\,\le\, m-2 \, .$$

From the Plücker embedding, which realizes $Gr_{m-2}(\mathbb{R}^m)$ as a submanifold of the projective space of the $(m-2)^{\text{th}}$ exterior power $\mathbb{P}\big(\bigwedge^{m-2}\mathbb{R}^m\big)$, we can represent the Gauß map as the $(m-2)$-vector

$$\vec{n} \, = \, \bigwedge_{\alpha=1}^{m-2}\vec{n}_\alpha \, .$$

Via the Hodge operator $\star$, we may identify vectors and $(m-1)$-vectors in $\mathbb{R}^m$. In particular, there holds

$$\star\,(\vec{n}\wedge\vec{e}_1)\ =\ \vec{e}_2 \qquad \text{and} \qquad \star\,(\vec{n}\wedge\vec{e}_2) = -\,\vec{e}_1\ .$$

With this notation, the second fundamental form $\vec{B}$, which is a symmetric 2-form on $T_{\vec{\Phi}(z)}\vec{\Phi}(D^2)$ into $N_{\vec{\Phi}(z)}\vec{\Phi}(D^2)$, may be expressed as

$$\vec{B}\ =\ \sum_{\alpha,i,j}\ \mathrm{e}^{-2\lambda}\,h^{\alpha}_{ij}\ \vec{n}_\alpha\,dx_i\otimes dx_j\ \equiv\ \sum_{\alpha,i,j}\ h^{\alpha}_{ij}\ \vec{n}_\alpha\,(\vec{e}_i)^*\otimes(\vec{e}_j)^*\ , \tag{I.4}$$

with

$$h^{\alpha}_{ij}\ =\ -\,\mathrm{e}^{-\lambda}\,\vec{e}_i\cdot\partial_{x_j}\vec{n}_\alpha\ .$$

The mean curvature vector $\vec{H}$ is

$$\vec{H}\ =\ \sum_{\alpha=1}^{m-2}H^{\alpha}\,\vec{n}_\alpha\ =\ \frac{1}{2}\sum_{\alpha=1}^{m-2}\big(h^{\alpha}_{11}+h^{\alpha}_{22}\big)\,\vec{n}_\alpha\ .$$

The Weingarten operator $\vec{H}_0$ is

$$\vec{H}_0\ =\ \sum_{\alpha=1}^{m-2}H^{\alpha}_0\,\vec{n}_\alpha\ =\ \frac{1}{2}\sum_{\alpha=1}^{m-2}\big(h^{\alpha}_{11}-h^{\alpha}_{22}\ +\ 2\,i\,h^{\alpha}_{12}\big)\,\vec{n}_\alpha\ .$$

In this framework, the Euler-Lagrange equation (I.2) for the Willmore functional is cast in the form

$$\Delta_\perp\vec{H}\ +\ \sum_{\alpha,\beta,i,j}h^{\alpha}_{ij}\,h^{\beta}_{ij}\,H^{\beta}\,\vec{n}_\alpha\ -\ 2\,\big|\vec{H}\big|^2\vec{H}\ =\ 0\,, \tag{I.5}$$

with

$$\Delta_\perp\vec{H}\ :=\ \mathrm{e}^{-2\lambda}\,\pi_{\vec{n}}\,div\big(\pi_{\vec{n}}(\nabla\vec{H})\big)\ .$$

# II Main Results

## II.1 Local Palais-Smale Sequences

As we stated in the Introduction, T. Willmore conjectured in 1965 that the Willmore torus (defined on page 2) minimizes, up to Möbius transformations, the Willmore energy in the class of smooth immersed tori in $\mathbb{R}^3$. To this day, no satisfactory demonstration of this assertion has been found. Amid the works aimed at solving this problem, one fundamental property of the Willmore functional was brought into light by Leon Simon [Sim]. Namely, for each dimension $m\geq 3$, there exists a compact embedded real analytic torus in $\mathbb{R}^m$ which minimizes the Willmore energy in the class of compact, genus 1, embedded surfaces without boundary[2]. Unfortunately, it remains unknown whether this minimizer is the Willmore torus. Simon's rather sophisticated proof is constructive in nature: an explicit minimizing sequence for the Willmore functional is devised. This is thus one instance in which studying and understanding minimizing sequences of the Willmore functional are essential. Yet more generally, it is interesting to investigate Palais-Smale sequences for the Willmore functional. Such will be our goal in this section.

We open our considerations with an "empirical" observation which will hopefully convince the reader that the results derived in [Riv2] offer a suitable framework to acquire information on Palais-Smale sequences of Willmore surfaces.

Let us consider a conformal weak Willmore immersion $\vec{\Phi}$ from the flat disk $D^2$ into $\mathbb{R}^m$ with bounded square-integrable second fundamental form. Up to an additive constant, it is possible to define a map $\vec{L}$ satisfying[3]

$$\nabla^{\perp}\vec{L} \,:=\, \nabla\vec{H} \,-\, 3\,\pi_{\vec{n}}\big(\nabla\vec{H}\big) \,+\, \star\big(\nabla^{\perp}\vec{n}\wedge\vec{H}\big)\,. \tag{II.1}$$

In [Riv2], it is shown that the following system holds:

$$\left\{\begin{array}{rcl} \nabla\vec{\Phi}\cdot\nabla^{\perp}\vec{L} & = & 0 \\ \nabla\vec{\Phi}\wedge\nabla^{\perp}\vec{L} & = & 2\,(-1)^m\,\nabla\big(\star(\vec{n}\,\llcorner\vec{H})\big)\,\llcorner\nabla^{\perp}\vec{\Phi}\,. \end{array}\right. \tag{II.2}$$

The Hodge operator $\star$ and the contraction operator $\llcorner$ commute with the partial differentiation operators. Accordingly, the terms appearing in the system (II.2) enjoy a peculiar property: more than mere "products of derivatives", they can be factored in divergence form. This structural feature has the analytical advantage of being robust under weak limiting process. Hence it is legitimate to hope that a local Palais-Smale sequence of conformal weak Willmore immersions with, say, uniformly bounded square-integrable second fundamental forms, converges to an element $\vec{\Phi}$ satisfying the system (II.2) for some function $\vec{L}$ (which may or may not be related to $\vec{\Phi}$ via (II.1)).

This is essentially the result that we shall establish. Prior to stating precisely, it is necessary to define the notion of local Palais-Smale sequence for the Willmore functional.

**Definition II.3** *Let $(\vec{\Phi}_k)$ be a sequence of conformal immersions from the unit-disk $D^2$ into $\mathbb{R}^m$ such that*

$$\big\|\vec{\Phi}_k\big\|_{W^{2,2}\,\cap\,W^{1,\infty}} \,\le\, C \tag{II.3}$$

[2] more generally, Simon obtains an analogous statement for each genus $\mathfrak{g}\in\mathbb{N}$.
[3] refer to the Introduction or the Appendix for the notation.

*holds uniformly for some positive constant $C$. Denoting respectively by $\vec{n}_k$ and $\vec{H}_k$ the Gauß map and the mean curvature vector associated with the immersion $\vec{\Phi}_k$, we set*

$$Q_k \;:=\; \nabla\vec{H}_k \,-\, 3\,\pi_{\vec{n}_k}\big(\nabla\vec{H}_k\big) \,+\, \star\big(\nabla^\perp\vec{n}_k\wedge\vec{H}_k\big)\,.$$

*The sequence $(\vec{\Phi}_k)$ is locally Palais-Smale if, in addition to (II.3), it satisfies*

$$div\,Q_k \;\longrightarrow\; 0 \qquad \textit{strongly in} \quad \big(W^{2,2}\cap W^{1,\infty}\big)'(D^2)\,.$$

The following result provides a first description of the limit of a Palais-Smale sequence of the Willmore functional.

**Theorem II.1** *Let $(\vec{\Phi}_k)$ be a local Palais-Smale sequence of conformal immersions from the unit-disk $D^2$ into $\mathbb{R}^m$. There exist a conformal weak immersion $\vec{\Phi}\in(W^{2,2}\cap W^{1,\infty})(D^2)$ and an element $\vec{L}\in L^{2,\infty}(D^2)$ such that, up to extraction of a subsequence,*

$$\vec{\Phi}_k \;\longrightarrow\; \vec{\Phi} \qquad \textit{in}\quad \mathcal{D}'(D^2)$$

*and the system*

$$\left\{\begin{array}{rcl}\nabla\vec{\Phi}\cdot\nabla^\perp\vec{L} &=& 0\\ \nabla\vec{\Phi}\wedge\nabla^\perp\vec{L} &=& -\,2\,\nabla\vec{\Phi}\wedge\nabla\vec{H}\end{array}\right. \tag{II.4}$$

*holds in the sense of distributions, where $\vec{H}$ denotes the mean curvature vector associated with $\vec{\Phi}$.*

The apparent difference between the systems (II.2) and (II.4) is fictitious only. Indeed, one verifies[4] the identity

$$\nabla\vec{\Phi}\wedge\nabla\vec{H} \;=\; (-1)^{m-1}\,\nabla\big(\star(\vec{n}\,\llcorner\vec{H})\big)\llcorner\nabla^\perp\vec{\Phi}\,.$$

## II.2 The Conformal Willmore Equation

The convergence result stated in Theorem II.1 naturally begs the following question: if an immersion $\vec{\Phi}$ satisfies the system (II.4), for some function $\vec{L}$, is it true that $\vec{\Phi}$ is Willmore ? Unfortunately, and perhaps surprisingly, as far as the authors know, the answer is negative. Nevertheless, the following identification can be verified.

**Theorem II.2** *Let $\vec{\Phi}$ be a conformal weak immersion from the unit-disk $D^2$ into $\mathbb{R}^m$ which lies in $W^{2,2}\cap W^{1,\infty}$ and such that*

$$\int_{D^2}\big|\nabla\vec{n}\big|^2 \;\le\; \varepsilon\,,$$

[4] cf. equation (II.49) in the paper [Riv2].

*for some $\varepsilon > 0$ small enough. Then there exists $\vec{L} \in L^{2,\infty}(D^2)$ such that $\vec{\Phi}$ satisfies (II.4) if and only if $\vec{\Phi}$ is smooth and there holds*

$$e^{2\lambda}\left[\Delta_\perp \vec{H} + \sum_{\alpha,\beta,i,j} h^\alpha_{ij}\, h^\beta_{ij}\, H^\beta\, \vec{n}_\alpha - 2\left|\vec{H}\right|^2 \vec{H}\right] = \Im\big(\vec{H}_0\, f\big)\,, \qquad \text{(II.5)}$$

*for some holomorphic function $f$.*

The "if" part of the statement of Theorem II.2 is clear. Indeed, choosing $f \equiv 0$ and $\vec{L}$ as in (II.1), we recover the Willmore equation (I.5). In this case, the validity of the system (II.4) was settled in [Riv2]. We shall thus focus our attention to the "only if" part of the statement of Theorem II.2.

Although the amount of information on the holomorphic function $f$ is rather limited, we note that the way it appears on the right-hand side of (II.5) makes it play the rôle of a Lagrange multiplier in the Willmore equation (I.5). This observation enables one to give a natural geometric interpretation of $f$ in the context of Teichmüller theory (cf. [BPP]).

The Willmore functional being conformally invariant, the left-hand side of (II.5) remains unchanged under a holomorphic change of coordinates. An easy computation reveals that $\vec{H}_0$ is the coefficient[5] of $dz \otimes dz$ in the two-form $2\vec{B}$ given in (I.4). Looking at the right-hand side of (II.5), we deduce that $f$ must be the coordinate of a section $f(z)\,\partial_z \otimes \partial_z$. There is thus a one-to-one correspondence between the vector space to which $f$ belongs and the vector space $H^0(\Sigma)$ of holomorphic quadratic differentials. Let $\mathfrak{g}$ be the genus of the immersed surface $\Sigma$. From the Riemann-Roch theorem (cf. Corollary 5.4.2 in [Jo]), the space $H^0(\Sigma)$ satisfies

$$\dim_{\mathbb{C}} H^0(\Sigma) = \begin{cases} 0 & , \quad \mathfrak{g} = 0 \\ 1 & , \quad \mathfrak{g} = 1 \\ 3(\mathfrak{g}-1) & , \quad \mathfrak{g} \geq 2\,. \end{cases}$$

The equation (II.5) is not a novelty ; it has been studied in various contexts. Immersions which satisfy (II.5) are sometimes known as *constrained Willmore immersions*, such as in [BPP]. There, it is shown that (II.5) is the Euler-Lagrange equation deriving from the Willmore functional (I.1) under smooth compactly supported infinitesimal *conformal* variations. The corresponding critical points are the conformal-constrained Willmore surfaces. This notion clearly generalizes that of a Willmore surface, obtained via all smooth compactly supported infinitesimal variations. Conformal-constrained Willmore surfaces form a Möbius invariant class of surfaces fostering remarkable properties, some of which are studied in [BPP], [BPU],

[5] the variable $z$ is defined as $x_1 + ix_2$, where $(x_1, x_2)$ are the usual coordinates on the unit-disc for the flat metric.

[Bry1], and [Ric]. In the latter, it is in particular established that every constant mean curvature surface in a 3-dimensional space-form is conformal-constrained Willmore. We have chosen to refer to (II.5) as the *conformal Willmore equation*, rather than as the somewhat vaguer term "constrained Willmore equation". There are indeed many ways to constrain the variations of the Willmore functional (e.g. restrictions on the volume and surface area in the Helfrich model). The adjective "conformal" appears more descriptive in our situation.

Global analogues of the local results presented in this paper are currently being investigated by the authors. A study of the global case shall appear in print in the near future.

# III Proofs of the Theorems

## III.1 Proof of Theorem II.1

Let us set

$$Q_k \;:=\; \nabla\vec{H}_k \;-\; 3\,\pi_{\vec{n}}\big(\nabla\vec{H}_k\big) \;+\; \star\big(\nabla^{\perp}\vec{n}_k\wedge\vec{H}_k\big)\,.$$

We suppose that

$$div\,Q_k \;\longrightarrow\; 0 \qquad \text{strongly in} \quad \big(W^{2,2}\cap W^{1,\infty}\big)'(D^2)\,. \tag{III.1}$$

and

$$\|\vec{\Phi}_k\|_{W^{2,2}\cap W^{1,\infty}} \;\le\; C \qquad \text{uniformly, for some constant } C>0\,. \tag{III.2}$$

We begin our study with an elementary result.

**Lemma III.1** *There holds*

$$\vec{\Phi}_k\,div\,Q_k \;\longrightarrow\; 0 \qquad in \qquad \mathcal{D}'(D^2)\,.$$

**Proof.** For notational convenience, we set

$$X \;:=\; W^{2,2}\,\cap\,W^{1,\infty}\,.$$

Let $u$ be an arbitrary element of $\mathcal{D}=C^{\infty}_{c}(D^2)$. The Sobolev embedding theorem guarantees that the elements $\vec{\Phi}_k$ of $X$ are Hölder continuous on $\overline{D^2}$. Since in addition $X$ is an intersection of Sobolev spaces, it is clear that $\vec{\Phi}_k u$ is an element of $X_0$ (i.e. an element of $X$ with null trace on the boundary of the unit disk). More precisely, from (III.2),

$$\big\|\vec{\Phi}_k\,u\big\|_{X_0} \;\lesssim\; \big\|\vec{\Phi}_k\big\|_{X}\,\big\|u\big\|_{\mathcal{D}} \;\le\; C\,\big\|u\big\|_{\mathcal{D}}\,. \tag{III.3}$$

Setting $T_k := div\ Q_k$, we can make sense of $\vec{\Phi}_k\, T_k$ as a distribution via

$$\big\langle \vec{\Phi}_k\, T_k\,,u\big\rangle_{\mathcal{D}',\mathcal{D}} \;:=\; \big\langle T_k\,,\vec{\Phi}_k\, u\big\rangle_{X_0',X_0}\,.$$

It follows immediately from (III.1) and (III.3) that $\vec{\Phi}_k\, T_k$ converges to zero in the sense of distributions.

■

By hypothesis, $T_k := div\, Q_k$ is a bounded linear functional on $X$ (with $X$ as in the proof of the previous lemma). Using (A.2), we see that $T_k$ belongs to $W^{-2,(2,\infty)}$, which is the dual space of $W_0^{2,(2,1)}$. Accordingly, Proposition I.1 grants the existence of an element $P_k$ of $\mathbb{R}^2\otimes W^{-1,(2,\infty)}$ satisfying

$$T_k \;=\; div\, P_k \tag{III.4}$$

and

$$\|P_k\|_{\mathbb{R}^2\otimes W^{-1,(2,\infty)}} \;\le\; \|T_k\|_{W^{-2,(2,\infty)}} \,+\, \delta_k\,, \tag{III.5}$$

for some positive constant $\delta_k$ arbitrarily chosen.

We next establish a useful result.

**Lemma III.2** *There holds*

$$P_k\,\vec{\Phi}_k \;\longrightarrow\; 0 \qquad in \qquad \mathbb{R}^2\otimes\mathcal{D}'(D^2)\,.$$

**Proof.** From (III.5) and the inclusion

$$X \;:=\; W^{2,2}\cap W^{1,\infty} \;\supset\; W_0^{2,(2,1)}\,,$$

there holds

$$\|P_k\|_{\mathbb{R}^2\otimes W^{-1,(2,\infty)}} \;\le\; \|T_k\|_{X'} \,+\, \delta_k\,.$$

Owing to (III.1), the sequence $(P_k)$ converges strongly to zero in $\mathbb{R}^2\otimes W^{-1,(2,\infty)}$. Moreover, from (III.2) and (A.3), it follows that the sequence $(\vec{\Phi}_k)$ is uniformly bounded in $W^{1,(2,1)}$. The desired statement may now be reached by repeating mutatis mutandis the proof of Lemma III.1, and letting $\delta_k$ tend to zero.

■

The identity (III.4) yields

$$div\,\big(Q_k - P_k\big) \;=\; 0\,. \tag{III.6}$$

By definition, the difference $(Q_k - P_k)$ lies in $\mathbb{R}^2\otimes W^{-1,(2,\infty)}$. As proved in Lemma A.1, the equation (III.6) implies the existence of an element $\vec{L}_k$ in $L^{2,\infty}$ satisfying

$$Q_k \,-\, P_k \;=\; \nabla^{\perp}\vec{L}_k\,. \tag{III.7}$$

Since $(\vec{\Phi}_k)$ is uniformly bounded in $W^{2,2}\cap W^{1,\infty}$, it follows that the sequences $(\vec{H}_k)$, $(\vec{n}_k)$, and thus $(Q_k)$ are uniformly bounded, respectively in $L^2$, $W^{1,2}$, and $\mathbb{R}^2\otimes W^{-1,(2,\infty)}$. Likewise, we saw that $(P_k)$ is uniformly bounded $\mathbb{R}^2\otimes W^{-1,(2,\infty)}$. Hence, the sequence $(\nabla^\perp\vec{L}_k)$ is uniformly bounded in $\mathbb{R}^2\otimes W^{-1,(2,\infty)}$.

**Lemma III.3** *There holds*

$$\left\{\begin{array}{rcl}\nabla\vec{\Phi}_k\cdot\nabla^\perp\vec{L}_k & \longrightarrow & 0\\ \nabla\vec{\Phi}_k\wedge\big(\nabla^\perp\vec{L}_k+2\,\nabla\vec{H}_k\big) & \longrightarrow & 0\end{array}\right.\qquad in\qquad \mathcal{D}'(D^2)\,.$$

**Proof.** Observe that

$$P_k\cdot\nabla\vec{\Phi}_k\;=\;div\,\big(P_k\,\vec{\Phi}_k\big)\;-\;\vec{\Phi}_k\,div\,P_k\;=\;div\,\big(P_k\,\vec{\Phi}_k\big)\;-\;\vec{\Phi}_k\,div\,Q_k\,.$$

Whence, the results of Lemma III.1 and Lemma III.2 show that

$$\nabla\vec{\Phi}_k\cdot P_k\longrightarrow 0\qquad\text{in}\qquad\mathcal{D}'(D^2)\,.\tag{III.8}$$

From (A.10) in the Appendix, we know that

$$\nabla\vec{\Phi}_k\cdot Q_k\;=\;0\qquad\text{and}\qquad\nabla\vec{\Phi}_k\wedge Q_k\;=\;-\,2\,\nabla\vec{\Phi}_k\wedge\nabla\vec{H}_k\,.$$

The desired statement now ensues by combining (III.7) and (III.8).

■

From the characterization provided in Proposition I.1, we can always arrange for the $L^{2,\infty}$-norm of $\vec{L}_k$ to be as close as we please to the $W^{-1,(2,\infty)}$-norm of $\nabla^\perp\vec{L}_k$. In particular, the sequence $(\vec{L}_k)$ is uniformly bounded in $L^{2,\infty}$. We may thus extract a weak* convergent subsequence with

$$\left\{\begin{array}{rcllll}\vec{L}_{k'} & \overset{*}{\rightharpoonup} & \vec{L} & \quad\text{in} & L^{2,\infty}\\ \nabla^\perp\vec{L}_{k'} & \overset{*}{\rightharpoonup} & g & \quad\text{in} & \mathbb{R}^2\otimes W^{-1,(2,\infty)}\,,\end{array}\right.\tag{III.9}$$

with $g=\nabla^\perp\vec{L}$ in the sense of distributions.

As the sequence $(\vec{\Phi}_k)$ is uniformly bounded in $W^{2,2}\cap W^{1,\infty}$ by hypothesis, the Banach-Alaoglu theorem provides a subsequence $(\vec{\Phi}_{k'})$ converging weak* in $W^{2,2}\cap W^{1,\infty}$ to some element $\vec{\Phi}$. In turn, the compact embeddings provided by the Rellich-Kondrachov theorem (A.3) enables us to further extract a subsequence, still denoted $(\vec{\Phi}_{k'})$, satisfying the strong convergences

$$\left\{\begin{array}{rcllll}\vec{\Phi}_{k'} & \longrightarrow & \vec{\Phi} & \quad\text{in} & \bigcap_{p<\infty}W^{1,p}\\ \nabla\vec{\Phi}_{k'} & \longrightarrow & \nabla\vec{\Phi} & \quad\text{in} & \mathbb{R}^2\otimes\bigcap_{p<\infty}L^p\,.\end{array}\right.\tag{III.10}$$

The convergences (III.9) and (III.10) yield

**Lemma III.4**

$$\left\{\begin{array}{lll}\nabla\vec{\Phi}_{k'}\cdot\nabla^\perp\vec{L}_{k'} & \longrightarrow & \nabla\vec{\Phi}\,\cdot\nabla^\perp\vec{L}\\[2mm] \nabla\vec{\Phi}_{k'}\wedge\nabla^\perp\vec{L}_{k'} & \longrightarrow & \nabla\vec{\Phi}\,\wedge\nabla^\perp\vec{L}\end{array}\right.\qquad in\qquad \mathcal{D}'(D^2)\,.$$

**Proof.** We shall only establish the first convergence, the second one being obtained *mutatis mutandis*. Owing to the general identity

$$div\big(a\,\nabla^\perp b\big)\;=\;\nabla a\cdot\nabla^\perp b\,,$$

it suffices to show that

$$\vec{\Phi}_{k'}\,\nabla^\perp\vec{L}_{k'}\;\longrightarrow\;\vec{\Phi}\;\nabla^\perp\vec{L}\qquad\text{in}\qquad\mathcal{D}'(D^2)\,.$$

This is what we shall do. For convenience, we set $Y:=\mathbb{R}^2\otimes W^{1,(2,1)}$. Let $g$ be as in (III.9), and $u$ be an arbitrary test-function in $\mathbb{R}^2\otimes\mathcal{D}$. Clearly, $\vec{\Phi}\,u$ and $\vec{\Phi}_{k'}u$ are elements of $Y_0$ (i.e. elements of $Y$ with null trace on the boundary of the unit-disk), with an estimate analogous to (III.3). Note that

$$\begin{aligned}&\Big\langle\vec{\Phi}_{k'}\nabla^\perp\vec{L}_{k'}\;-\;\vec{\Phi}\,g\,,u\Big\rangle_{\mathcal{D}',\mathcal{D}}\\ &\qquad\equiv\;\Big\langle\nabla^\perp\vec{L}_{k'}\,,\,u\,\big(\vec{\Phi}_{k'}-\vec{\Phi}\big)\Big\rangle_{Y_0',Y_0}\;+\;\Big\langle\nabla^\perp\vec{L}_{k'}-g\,,\,u\,\vec{\Phi}\Big\rangle_{Y_0',Y_0}\,.\end{aligned}$$

Whence, from the convergences (III.9) and (III.10), the desired result follows.

■

As explained in Theorem 3.3.8 from [Hel], the fact that[6]

$$\Delta\lambda_k\;=\;-\,\mathrm{e}^{2\lambda_k}K_k$$

is an element of the Hardy space $\mathcal{H}^1$ implies that $(\lambda_k)$ is a sequence of elements in $W^{1,(2,1)}$, uniformly bounded in norm by a constant depending only upon the uniform bound on $\big(\|\vec{\Phi}_k\|_{W^{2,2}\cap W^{1,\infty}}\big)$. Owing to the Rellich-Kondrachov theorem, we may extract a subsequence $(\lambda_{k'})$ satisfying

$$\lambda_{k'}\;\longrightarrow\;\lambda\qquad\text{in}\qquad\bigcap_{p<\infty}L^p\,,\tag{III.11}$$

for some $\lambda$ in the suitable space. Recall that for $j\in\{1,2\}$, there holds

$$\mathrm{e}^{2\lambda_k}\;=\;\big|\partial_{x^j}\vec{\Phi}_k\big|^2\qquad\text{and thus}\qquad\partial_{x^i}\big(\mathrm{e}^{2\lambda_k}\big)\;=\;2\,\partial_{x^j}\vec{\Phi}_k\cdot\partial_{x^jx^i}\vec{\Phi}_k\,.$$

Accordingly, the sequence $(\mathrm{e}^{2\lambda_k})$ is uniformly bounded in $W^{1,2}\cap L^\infty$. From this, and the boundedness of $\lambda_k$, the same is true about the sequences $(\mathrm{e}^{\pm\lambda_k})$.

[6] where $K_k$ denotes the Gaussian curvature associated with the immersion $\vec{\Phi}_k$.

Combining this to the convergences (III.10) and (III.11), we find the following strong convergences

$$\left\{\begin{array}{rcl} \mathrm{e}^{\pm\lambda_{k'}} & \longrightarrow & \mathrm{e}^{\pm\lambda_k} \\ (\vec{e}_j)_{k'} \;:=\; \mathrm{e}^{-\lambda_{k'}}\partial_{x^j}\vec{\Phi}_{k'} & \longrightarrow & \mathrm{e}^{-\lambda}\partial_{x^j}\vec{\Phi} \;=:\; \vec{e}_j \end{array}\right. \qquad \text{in} \quad \bigcap_{p<\infty} L^p\,. \quad \text{(III.12)}$$

By definition, from the uniform boundedness of the sequence $(\vec{\Phi}_k)$ in $W^{2,2}\cap W^{1,\infty}$ and of that of the sequence $(\vec{n}_k)$ in $W^{1,2}$, we see that the sequence $(\vec{H}_k)$ is uniformly bounded in $L^2$. Accordingly, we can extract a weakly convergent subsequence:

$$\vec{H}_{k'} \rightharpoonup \vec{H} \qquad \text{in} \qquad L^2\,. \qquad \text{(III.13)}$$

Altogether, (III.12) and (III.13) show that

$$(\vec{e}_1)_{k'}\wedge(\vec{e}_2)_{k'}\wedge\vec{H}_{k'} \rightharpoonup \vec{e}_1\wedge\vec{e}_2\wedge\vec{H} \qquad \text{in} \qquad \bigcap_{1\le p<2} L^p\,. \qquad \text{(III.14)}$$

Equation (II.42) established in [Riv2] states

$$\star\,(\vec{n}_{k'}\llcorner\vec{H}_{k'}) \;=\; (-1)^{m-1}\,(\vec{e}_1)_{k'}\wedge(\vec{e}_2)_{k'}\wedge\vec{H}_{k'}\,.$$

Consequently, (III.14) is tantamount to

$$\star\,(\vec{n}_{k'}\llcorner\vec{H}_{k'}) \rightharpoonup \star\,(\vec{n}\llcorner\vec{H}) \qquad \text{in} \qquad \bigcap_{1\le p<2} L^p\,.$$

Combining this to (III.10) shows that

$$\star\,(\vec{n}_{k'}\llcorner\vec{H}_{k'})\llcorner\nabla^\perp\vec{\Phi}_{k'} \rightharpoonup \star\,(\vec{n}\llcorner\vec{H})\llcorner\nabla^\perp\vec{\Phi} \qquad \text{in} \qquad \bigcap_{1\le p<2} L^p\,. \quad \text{(III.15)}$$

It takes little effort to verify that

$$div\,(u\llcorner\nabla^\perp v) \;=\; \nabla u\llcorner\nabla^\perp v\,.$$

holds in general for suitable $u$, $v$. In particular, (III.15) yields

$$\nabla\big(\star\,(\vec{n}_{k'}\llcorner\vec{H}_{k'})\big)\llcorner\nabla^\perp\vec{\Phi}_{k'} \;\longrightarrow\; \nabla\big(\star\,(\vec{n}\llcorner\vec{H})\big)\llcorner\nabla^\perp\vec{\Phi} \qquad \text{in} \qquad \mathcal{D}'(D^2)\,.$$

Using identity (II.49) from [Riv2], the latter is equivalent to

$$\nabla\vec{\Phi}_{k'}\wedge\nabla\vec{H}_{k'} \;\longrightarrow\; \nabla\vec{\Phi}\wedge\nabla\vec{H} \qquad \text{in} \qquad \mathcal{D}'(D^2)\,. \qquad \text{(III.16)}$$

Finally, bringing altogether Lemma III.3, Lemma III.4, and (III.16), we conclude as desired that, in the sense of distributions,

$$\left\{\begin{array}{rcl} \nabla\vec{\Phi}\cdot\nabla^\perp\vec{L} & = & 0 \\ \nabla\vec{\Phi}\wedge\nabla^\perp\vec{L} & = & -\,2\,\nabla\vec{\Phi}\wedge\nabla\vec{H}\,. \end{array}\right.$$

## III.2 Proof of Theorem II.2

As done in the work [Riv2], we introduce the functions $S$ and $\vec{R}$ such that

$$\left\{\begin{array}{lcl}\nabla S & := & \nabla\vec{\Phi}\cdot\vec{L}\\ \nabla\vec{R} & := & \nabla\vec{\Phi}\wedge\vec{L}\,+\,2\,\nabla^{\perp}\vec{\Phi}\wedge\vec{H}\,.\end{array}\right. \tag{III.17}$$

Clearly, $S$ and $\vec{R}$ are defined on $D^2$ up to an unimportant additive constant. By hypothesis, $\vec{L}$ is an element of $L^{2,\infty}(D^2)$, while $\vec{\Phi}$ is Lipschitz. Accordingly, $\nabla S$ and $\nabla\vec{R}$ belong to $L^{2,\infty}(D^2)$ and to $\mathbb{R}^2\otimes L^{2,\infty}(D^2)$ respectively. This observation, and the particular structure of the right-hand side of the system (III.17) will enable us to deduce the regularity result we are seeking to obtain.

### III.2.1 Regularity

It is shown in [Riv2] that $S$ and $\vec{R}$ satisfy the system

$$\left\{\begin{array}{lcl}\Delta S & = & \big(\nabla\star\vec{n}\big)\cdot\nabla^{\perp}\vec{R}\\ \Delta\vec{R} & = & (-1)^n\star\big(\nabla\vec{n}\bullet\nabla^{\perp}\vec{R}\big)\,-\,(\nabla\star\vec{n})\,\nabla^{\perp}S\,.\end{array}\right. \tag{III.18}$$

The advantage of these equations lies essentially in their right-hand sides comprising only Jacobians. This peculiar feature will enable us to apply the techniques of integration by compensation, more precisely Wente-type estimates. We recall two results which shall be of use to our proof. They are due in parts to contributions by Wente [Wen], Tartar [Ta2], Coifman *et al.* [CLMS], Bethuel [Be], and Hélein [Hel].

**Lemma III.5** *Let $\Omega$ be an open subset of $\mathbb{R}^2$ with $C^2$-boundary. Suppose that $a$ and $b$ are elements of $W^{1,2}(\Omega)$ and of $W^{1,(2,\infty)}(\Omega)$, respectively. If $u$ satisfies*

$$\left\{\begin{array}{lcll}\Delta u & = & \nabla a\cdot\nabla^{\perp}b & \textit{in }\ \Omega\\ u & = & 0 & \textit{on }\ \partial\Omega\,,\end{array}\right.$$

*then $\nabla u$ belongs to the space $L^2(\Omega,\mathbb{R}^2)$ with the estimate*

$$\big\|\nabla u\big\|_{L^2(\Omega,\mathbb{R}^2)}\;\lesssim\;\big\|\nabla a\big\|_{L^2(\Omega,\mathbb{R}^2)}\,\big\|\nabla b\big\|_{L^{2,\infty}(\Omega,\mathbb{R}^2)}\,,$$

*up to a multiplicative constant depending only on $\Omega$.*

The interested reader will find the proof of this result and further variations on the same theme in [Hel] (Theorem 3.4.5).
Another result (Theorem 3.4.1 in [Hel]) which will be useful to us is the following.

**Lemma III.6** *Let $\Omega$ be an open subset of $\mathbb{R}^2$ with $C^1$-boundary. Suppose that $a$ and $b$ are elements of $W^{1,2}(\Omega)$. If $u$ satisfies*

$$\left\{\begin{array}{rcll}\Delta u & = & \nabla a\cdot\nabla^{\perp}b & \text{in}\ \ \Omega\\ u & = & 0 & \text{on}\ \ \partial\Omega\,,\end{array}\right.$$

*then $u$ belongs to the space $W^{1,(2,1)}(\Omega)\subset C^0(\Omega)$ with the estimate*

$$\big\|\nabla u\big\|_{L^{2,1}(\Omega)}\;\lesssim\;\big\|\nabla a\big\|_{L^2(\Omega,\mathbb{R}^2)}\,\big\|\nabla b\big\|_{L^2(\Omega,\mathbb{R}^2)}\,,$$

*up to a multiplicative constant depending only on $\Omega$.*

Geared with these results, we are prepared to start our proof. Let us define

$$S\;=\;S_0\,+\,S_1\qquad\text{and}\qquad\vec{R}\;=\;\vec{R}_0\,+\,\vec{R}_1\,,\tag{III.19}$$

where the new variables, in accordance with (III.18), satisfy

$$\left\{\begin{array}{rclcrcll}\Delta S_0 & = & 0 & , & \Delta\vec{R}_0 & = & \vec{0} & \text{in}\ \ D^2\\ S_0 & = & S & , & \vec{R}_0 & = & \vec{R} & \text{on}\ \ \partial D^2\,,\end{array}\right.\tag{III.20}$$

and

$$\left\{\begin{array}{rclcrcll}\Delta S_1 & = & \Delta S & , & \Delta\vec{R}_1 & = & \Delta\vec{R} & \text{in}\ \ D^2\\ S_1 & = & 0 & , & \vec{R}_1 & = & 0 & \text{on}\ \ \partial D^2\,.\end{array}\right.\tag{III.21}$$

We saw above that $S$ and $\vec{R}$ are elements of $W^{1,(2,\infty)}(D^2)$, while $\star\vec{n}$ belongs to $W^{1,2}(D^2)$. Since the right-hand sides of (III.18), and thus of (III.21), comprise only Jacobians, Lemma III.5 may be called upon so as to produce the estimates

$$\left\{\begin{array}{rcl}\big\|\nabla S_1\big\|_{L^2(D^2)} & \lesssim & \varepsilon\,\big\|\nabla\vec{R}\big\|_{L^{2,\infty}(D^2)}\\ \big\|\nabla\vec{R}_1\big\|_{L^2(D^2)} & \lesssim & \varepsilon\Big(\big\|\nabla S\big\|_{L^{2,\infty}(D^2)}\,+\,\big\|\nabla\vec{R}\big\|_{L^{2,\infty}(D^2)}\Big)\,,\end{array}\right.\tag{III.22}$$

up to unimportant multiplicative constants depending only on $D^2$. Here $\varepsilon$ denotes the (adjustable) upper bound on the energy:

$$\big\|\nabla\vec{n}\big\|_{L^2(D^2)}\;\leq\;\varepsilon\,.\tag{III.23}$$

Let us fix once and for all some point $p\in D_{1/2}(0)$, some $0<k<1$, and some radius $0<r<1/2$. The flat disk $D_{kr}(p)$ of radius $kr$ and centered on the point $p$ is properly contained in the unit-disk $D^2$. Since $S_0$ and $\vec{R}_0$ are harmonic and satisfy (III.20), it is clear that $\nabla S_0$ and $\nabla\vec{R}_0$ are square-integrable on $D_{kr}(p)$. From (III.22), it follows that $\nabla S$ and $\nabla\vec{R}$

are likewise square-integrable on $D_{kr}(p)$. Because $\star\vec{n}$ is also an element of $W^{1,2}(D_{kr}(p))$, we are now in position of applying Lemma III.6 to the system (III.21), thereby obtaining

$$\left\{\begin{array}{lcl}\|\nabla S_1\|_{L^{2,1}(D_{kr}(p))} & \lesssim & \varepsilon\,\|\nabla\vec{R}\|_{L^2(D_{kr}(p))}\\ \|\nabla\vec{R}_1\|_{L^{2,1}(D_{kr}(p))} & \lesssim & \varepsilon\left(\|\nabla S\|_{L^2(D_{kr}(p))}\,+\,\|\nabla\vec{R}\|_{L^2(D_{kr}(p))}\right),\end{array}\right.$$

up to some unimportant multiplicative constants involving $\varepsilon$ (they are independent of $r$ by homogeneity).
In particular, since $L^{2,1}\subset L^2$ and $k<1$, the latter gives that for some positive constant $C_0$, there holds

$$E_{D_{kr}(p)}(S_1,\vec{R}_1)\;\le\;C_0\,\varepsilon\,E_{D_r(p)}(S,\vec{R})\,, \tag{III.24}$$

where, for notational convenience, we have set

$$E_{D_\rho(p)}(u,\vec{v})\;:=\;\|u\|_{L^2(D_\rho(p))}\,+\,\|\vec{v}\|_{L^2(D_\rho(p))}\,.$$

On the other hand, since $S_0$ and $\vec{R}_0$ are harmonic, a classical "monotonicity" result (cf. Theorem 3.3.12 in [Hel], and [Gia]) applied to (III.20) yields that

$$E_{D_{kr}(p)}(S_0,\vec{R}_0)\;\le\;k\,E_{D_r(p)}(S_0,\vec{R}_0)\;\le\;k\,E_{D_r(p)}(S,\vec{R})\,. \tag{III.25}$$

Via combining altogether (III.24) and (III.25), we obtain

$$E_{D_{kr}(p)}(S,\vec{R})\;\le\;(C_0\,\varepsilon+k)\,E_{D_r(p)}(S,\vec{R})\,. \tag{III.26}$$

We have the freedom to adjust the positive parameter $\varepsilon$ as we please. Because $k\in(0,1)$, we may in particular arrange for the constant $(C_0\,\varepsilon+k)$ to be smaller than 1. Then, iterating (III.26), we infer the existence of some $\gamma\in(0,1)$ such that

$$E_{D_\rho(p)}\;\le\;C\,\rho^\gamma \tag{III.27}$$

holds for all $0<\rho<1/2$, all points $p\in D_{1/2}(0)$, and some constant $C>0$. With the help of the Poincaré inequality, the estimate (III.27) can be used to show that $S$ and $\vec{R}$ are locally Hölder continuous (see [Gia]). We are however interested in another corollary of (III.27). We denote the maximal function by

$$M_{2-\gamma}F(x)\;:=\;\sup_{\rho>0}\,\rho^{-\gamma}\int_{D_\rho(x)}|F(y)|\,dy\,.$$

Going back to (III.18), owing to (III.27), it follows that

$$M_{2-\gamma}\chi_{D_{1/2}(0)}\Delta S(p)\;\le\;\varepsilon\sup_{0<\rho<\frac{1}{2}}\;\rho^{-\gamma}\,E_{D_\rho(p)}\;\le\;C\,\varepsilon\,,\qquad\forall\;p\in D_{1/2}(0)\,.$$

The same estimate clearly holds with $\vec{R}$ in place of $S$. Accordingly,

$$\left\{\begin{array}{lcl} \big\|M_{2-\gamma}\chi_{D_{1/2}(0)}\Delta S\big\|_{L^\infty(D_{1/2}(0))} & < & \infty \\ \big\|M_{2-\gamma}\chi_{D_{1/2}(0)}\Delta \vec{R}\big\|_{L^\infty(D_{1/2}(0))} & < & \infty\,. \end{array}\right.$$

Moreover, it is clear that $\Delta S$ and $\Delta\vec{R}$ belong to $L^1(D_{1/2}(0))$, since $S$, $\vec{R}$, and $\star\vec{n}$ are elements of $W^{1,2}(D_{1/2}(0))$. We may thus call upon Proposition 3.2 in [Ad2] to deduce that

$$\frac{1}{|x|}\ast\chi_{D_{1/2}(0)}\Delta S \qquad \text{and} \qquad \frac{1}{|x|}\ast\chi_{D_{1/2}(0)}\Delta\vec{R}$$

belong to $L^{q,\infty}(D_{1/2}(0))$, with $q=\dfrac{2-\gamma}{1-\gamma}$.

A classical estimate about Riesz kernels states that in general, there holds

$$|\nabla u|(p)\;\le\;C_1\,\frac{1}{|x|}\ast\chi_{D_{1/2}(0)}\Delta u\,+\,C_2\,,\qquad \forall\;p\in D_{1/4}(0)\,,$$

for two constant $C_1$ and $C_2$. Hence, we infer that $\nabla S$ and $\nabla\vec{R}$ are elements of $L^{q,\infty}(D_{1/4}(0))$, with $q$ as above. In particular, because $\gamma\in(0,1)$, it follows that $q>2$, and thus

$$\nabla S\,\in\,L^{2+\delta}(D_{1/4}(0),\mathbb{R}) \qquad\text{and}\qquad \nabla\vec{R}\,\in\,L^{2+\delta}(D_{1/4}(0),\mathbb{R}^2)\,, \tag{III.28}$$

for some $\delta>0$.

Now that we have this result at our disposal, we are ready to implement our regularity proof. It involves a bootstrapping argument, which will be performed on the following identity, established in Lemma A.4:

$$-\,\Delta\vec{\Phi}\;=\;\nabla\vec{R}\bullet\nabla^\perp\vec{\Phi}\,+\,\nabla S\,\nabla^\perp\vec{\Phi}\,. \tag{III.29}$$

By hypothesis, $\vec{\Phi}$ is a Lipschitz function. Combining (III.28) and (III.29) thus shows that

$$\nabla\vec{\Phi}\,\in\,W^{1,2+\delta}(D_{1/4}(0),\mathbb{R}^2)\,. \tag{III.30}$$

Recall next that

$$\star\vec{n}\;=\;\mathrm{e}^{-2\lambda}\,\partial_{x^1}\vec{\Phi}\wedge\partial_{x^2}\vec{\Phi}\qquad\text{with}\qquad 2\,\mathrm{e}^{2\lambda}\;=\;\big|\nabla\vec{\Phi}\big|^2\,. \tag{III.31}$$

It is known that $\lambda$ belongs to $W^{1,(2,1)}(D^2)$. Indeed, in [MS] the authors show that

$$\Delta\lambda\;=\;-\,\mathrm{e}^{2\lambda}\,K\,,\qquad \text{where } K \text{ is the Gaussian curvature}\,,$$

is an element of the Hardy space $\mathcal{H}^1$, thereby implying that $\nabla\lambda$ lies in the Lorentz space $L^{2,1}$ (cf. Theorem 3.3.8 in [Hel]). Accordingly, $\mathrm{e}^{\pm2\lambda}$ are

bounded from above and below.
Suppose now that $\nabla\vec{\Phi}$ belongs to $W^{1,a}$, for some $a>2$. Since by hypothesis $\vec{\Phi}$ is Lipschitz, the second equation in (III.31) implies that $\mathrm{e}^{2\lambda}$ is an element of $W^{1,a}\cap L^\infty$. In turn, because $\mathrm{e}^{\pm\lambda}$ are bounded from above and below, the first equation in (III.31) yields that $\nabla\star\vec{n}$ belongs to $L^a$. In particular, (III.30) shows that

$$\nabla\star\vec{n}\;\in\;L^{2+\delta}(D_{1/4}(0),\mathbb{R}^2)\,. \tag{III.32}$$

For the reader's convenience, we recall

$$\left\{\begin{array}{lcl}\Delta S&=&\big(\nabla\star\vec{n}\big)\cdot\nabla^\perp\vec{R}\\ \Delta\vec{R}&=&(-1)^n\star\big(\nabla\vec{n}\bullet\nabla^\perp\vec{R}\big)\;-\;(\nabla\star\vec{n})\,\nabla^\perp S\,.\end{array}\right.$$

Accounting for (III.28) and (III.32) in this system gives

$$\nabla S\;\in\;W^{1,1+\delta/2}\;\subsetneq\;L^{2+\delta}\,.$$

Naturally, the same statement holds with $\vec{R}$ in place of $S$. Comparing the latter to (III.28) reveals that the regularity has been improved. The process may thus be repeated a finite of number of times, until we reach that $\nabla S$ and $\nabla\vec{R}$ are continuous. The exact same procedure as above will then yield that $\vec{\Phi}$ belongs to $C^1$, and eventually that it is smooth, thereby concluding the proof.

### III.2.2 Conformal Willmore Equation

We open our derivations by introducing some notation. Let $z=x^1+ix^2$ and $\bar{z}$ be its complex conjugate. We set

$$\vec{e}_z\;:=\;\mathrm{e}^{-\lambda}\,\partial_z\vec{\Phi}\;=\;\frac{1}{2}\big(\vec{e}_1-i\,\vec{e}_2\big)\qquad\text{and}\qquad\vec{e}_{\bar{z}}\;:=\;\mathrm{e}^{-\lambda}\,\partial_{\bar{z}}\vec{\Phi}\;=\;\frac{1}{2}\big(\vec{e}_1+i\,\vec{e}_2\big).$$

Since $\vec{\Phi}$ is conformal, there holds

$$\partial_{x_j}\vec{\Phi}\cdot\partial_{x_k}\vec{\Phi}\;=\;\mathrm{e}^{2\lambda}\,\delta_{jk}\qquad\text{for}\qquad(j,k)\;\in\;\{1,2\}\,,$$

and thus

$$\partial_a\vec{\Phi}\cdot\partial_b\vec{\Phi}\;=\;\frac{1}{2}\,\mathrm{e}^{2\lambda}\,\delta_{a\bar{b}}\qquad\text{for}\qquad(a,b)\;\in\;\{z,\bar{z}\}\,.$$

From this, it follows easily that for any triple $(a,b,c)\in\{z,\bar{z}\}$ there holds

$$\begin{aligned}\partial_a\vec{\Phi}\cdot\partial_{bc}\vec{\Phi}\;&\equiv\;\delta_{ca}\big(\partial_a\vec{\Phi}\cdot\partial_{ab}\vec{\Phi}\big)\;+\;\delta_{cb}\big(\partial_a\vec{\Phi}\cdot\partial_{bb}\vec{\Phi}\big)\\ &=\;\frac{1}{2}\big(\delta_{ca}-\delta_{cb}\big)\partial_b\big|\partial_a\vec{\Phi}\big|^2\;+\;\delta_{cb}\,\partial_b\big(\partial_a\vec{\Phi}\cdot\partial_b\vec{\Phi}\big)\;=\;\delta_{cb}\,\delta_{a\bar{b}}\,\mathrm{e}^{2\lambda}\,\partial_b\lambda\,.\end{aligned}$$

Hence,

$$\begin{aligned}\vec{e}_a \cdot \partial_b \vec{e}_c &\equiv \mathrm{e}^{-2\lambda}\left(\partial_a \vec{\Phi}\cdot\partial_{bc}\vec{\Phi} \,-\, (\partial_b\lambda)\,\partial_a\vec{\Phi}\cdot\partial_c\vec{\Phi}\right) \\ &= \frac{1}{2}\left(2\,\delta_{cb}\,\delta_{a\bar{b}} \,-\, \delta_{a\bar{c}}\right)\partial_b\lambda\,. \end{aligned} \tag{III.33}$$

Observe furthermore that

$$\vec{e}_a \cdot \vec{e}_b \;=\; \frac{1}{2}\,\delta_{a\bar{b}}\,. \tag{III.34}$$

Thus, combining (III.33) and (III.34) gives

$$\partial_b \vec{e}_c \;=\; \partial_b\lambda \sum_{a\in\{z,\bar{z}\}}\left(2\delta_{cb}\delta_{ab} - \delta_{ac}\right)\vec{e}_a \;+\; \pi_{\vec{n}}\big(\partial_b\vec{e}_c\big)\,. \tag{III.35}$$

Next, we have

$$\begin{aligned}\vec{n}_\alpha\cdot\partial_{\bar{z}}\vec{e}_z &= -\,\vec{e}_z\cdot\partial_{\bar{z}}\vec{n}_\alpha \;=\; -\,\frac{1}{4}\big(\vec{e}_1 - i\,\vec{e}_2\big)(\partial_{x_1} + i\,\partial_{x_2})\,\vec{n}_\alpha \\ &= \frac{\mathrm{e}^\lambda}{4}\left(h^\alpha_{11} + h^\alpha_{22}\right) \;=\; \frac{\mathrm{e}^\lambda}{2}\,H^\alpha\,,\end{aligned}$$

and similarly

$$\vec{n}_\alpha\cdot\partial_{\bar{z}}\vec{e}_{\bar{z}} \;=\; \frac{\mathrm{e}^\lambda}{2}\,H^\alpha_0\,.$$

Accordingly, we may now deduce from (III.35) that

$$\left\{\begin{aligned}\partial_{\bar{z}}\vec{e}_z &= -\left(\partial_{\bar{z}}\lambda\right)\vec{e}_z \;+\; \frac{\mathrm{e}^\lambda}{2}\,\vec{H} \\ \partial_{\bar{z}}\vec{e}_{\bar{z}} &= \left(\partial_{\bar{z}}\lambda\right)\vec{e}_{\bar{z}} \;+\; \frac{\mathrm{e}^\lambda}{2}\,\vec{H}_0\,.\end{aligned}\right. \tag{III.36}$$

Note that we can also easily obtain from the above computations the identity

$$\partial_{\bar{z}}\vec{n}_\alpha \;=\; -\,\mathrm{e}^\lambda\left(H^\alpha_0\,\vec{e}_z \,+\, H^\alpha\,\vec{e}_{\bar{z}}\right) \,+\, \pi_{\vec{n}}\big(\partial_{\bar{z}}\vec{n}_\alpha\big)\,. \tag{III.37}$$

These expressions shall come helpful in the sequel.

Suppose now that $\vec{\Phi}$ is smooth and satisfies for some $\vec{L}$ the system

$$\left\{\begin{aligned}\nabla\vec{\Phi}\cdot\nabla^\perp\vec{L} &= 0 \\ \nabla\vec{\Phi}\wedge\nabla^\perp\vec{L} &= -\,2\,\nabla\vec{\Phi}\wedge\vec{H}\,.\end{aligned}\right. \tag{III.38}$$

Since $\vec{e}_j = \mathrm{e}^{-\lambda}\,\partial_{x_j}\vec{\Phi}$, the first equation in the system is equivalent to

$$\vec{e}_1\cdot\partial_{x_2}\vec{L} \;=\; \vec{e}_2\cdot\partial_{x_1}\vec{L}\,.$$

Whence, we deduce that $\vec{L}$ satisfies

$$\nabla^\perp \vec{L} \;=\; \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix} + \begin{pmatrix} p_1^\alpha \\ p_2^\alpha \end{pmatrix} \vec{n}_\alpha\,, \tag{III.39}$$

for some suitable coefficients $a$, $b$, $c$, and $p_j^\alpha$.

Substituting this form in the second equation from (III.38) gives

$$\begin{aligned} &\vec{e}_1 \wedge \Big[ 2\,(\vec{e}_1 \cdot \partial_{x_1}\vec{H})\,\vec{e}_1 \,+\, 2\,(\vec{e}_2 \cdot \partial_{x_1}\vec{H})\,\vec{e}_2 \,+\, 2\,(\vec{n}_\alpha \cdot \partial_{x_1}\vec{H})\,\vec{n}_\alpha \\ &\qquad\qquad + a\,\vec{e}_1 \,+\, b\,\vec{e}_2 \,+\, p_1^\alpha\,\vec{n}_\alpha \Big] \\ =\;\; & -\vec{e}_2 \wedge \Big[ 2\,(\vec{e}_1 \cdot \partial_{x_2}\vec{H})\,\vec{e}_1 \,+\, 2\,(\vec{e}_2 \cdot \partial_{x_2}\vec{H})\,\vec{e}_2 \,+\, 2\,(\vec{n}_\alpha \cdot \partial_{x_2}\vec{H})\,\vec{n}_\alpha \\ &\qquad\qquad + c\,\vec{e}_1 \,-\, a\,\vec{e}_2 \,+\, p_2^\alpha\,\vec{n}_\alpha \Big]\,, \end{aligned}$$

thereby yielding

$$c - b \;=\; 2\,(\vec{e}_2 \cdot \partial_{x_1} - \vec{e}_1 \cdot \partial_{x_2})\vec{H} \qquad \text{and} \qquad p_j^\alpha \;=\; -\,2\,\vec{n}_\alpha \cdot \partial_{x_j}\vec{H}\,.$$

However, because $\vec{H} = H^\alpha \vec{n}_\alpha$, and the second fundamental form is symmetric, there holds

$$\begin{aligned} c - b \;&=\; 2\,(\vec{e}_2 \cdot \partial_{x_1} - \vec{e}_1 \cdot \partial_{x_2})\,\vec{H} \;=\; 2\,H^\alpha\,(\vec{e}_1 \cdot \partial_{x_2} - \vec{e}_2 \cdot \partial_{x_1})\,\vec{n}_\alpha \\ &=\; 2\,H^\alpha\,\mathrm{e}^\lambda\,(h_{21}^\alpha - h_{12}^\alpha) \;=\; 0\,. \end{aligned}$$

Whence, (III.39) may be recast in the form

$$\nabla^\perp \vec{L} \;=\; \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix} \,-\, 2\,\pi_{\vec{n}}(\nabla\vec{H})\,.$$

Equivalently, in the $\{\vec{e}_z, \vec{e}_{\bar{z}}\}$-frame, this expression reads

$$\partial_z \vec{L} \;=\; A\,\vec{e}_{\bar{z}} \,-\, 2\,i\,\pi_{\vec{n}}(\partial_z \vec{H})\,, \tag{III.40}$$

where $A := b + ia\,.$

Owing to the identities (III.36), there holds

$$\begin{aligned} &\partial_{\bar{z}}\pi_{\vec{n}}\,\partial_z\vec{H} \,-\, \pi_{\vec{n}}\,\partial_{\bar{z}}\pi_{\vec{n}}\,\partial_z\vec{H} \\ &\qquad\equiv\; 2\,\Big(\vec{e}_z \cdot \partial_{\bar{z}}\pi_{\vec{n}}\,\partial_z\vec{H}\Big)\,\vec{e}_{\bar{z}} \,+\, 2\,\Big(\vec{e}_{\bar{z}} \cdot \partial_{\bar{z}}\pi_{\vec{n}}\,\partial_z\vec{H}\Big)\,\vec{e}_z \\ &\qquad=\; -\,2\,\Big(\partial_{\bar{z}}\vec{e}_z \cdot \pi_{\vec{n}}\,\partial_z\vec{H}\Big)\,\vec{e}_{\bar{z}} \,-\, 2\,\Big(\partial_{\bar{z}}\vec{e}_{\bar{z}} \cdot \pi_{\vec{n}}\,\partial_z\vec{H}\Big)\,\vec{e}_z \\ &\qquad=\; -\,(\mathrm{e}^\lambda\,\vec{H} \cdot \pi_{\vec{n}}\,\partial_z\vec{H})\,\vec{e}_{\bar{z}} \,-\, (\mathrm{e}^\lambda\,\vec{H}_0 \cdot \pi_{\vec{n}}\,\partial_z\vec{H})\,\vec{e}_z \\ &\qquad=\; -\,\mathrm{e}^\lambda\,\Big[(\vec{H} \cdot \partial_z\vec{H})\,\vec{e}_{\bar{z}} \,+\, (\vec{H}_0 \cdot \partial_z\vec{H})\,\vec{e}_z\Big]\,. \end{aligned}$$

From (III.40), the latter, and (III.37), we see that

$$
\begin{aligned}
\partial_{\bar{z}z}\,\vec{L} \;&=\; \partial_{\bar{z}}(A\,\vec{e}_{\bar{z}}) \,-\, 2\,i\,\partial_{\bar{z}}\pi_{\vec{n}}\,\partial_z\vec{H} \qquad\qquad \text{(III.41)}\\
&=\; \partial_{\bar{z}}(A\,\vec{e}_{\bar{z}}) \,+\, 2\,i\,\mathrm{e}^{\lambda}\Big[\big(\vec{H}\cdot\partial_z\vec{H}\big)\,\vec{e}_{\bar{z}} \,+\, \big(\vec{H}_0\cdot\partial_z\vec{H}\big)\,\vec{e}_z\Big]\\
&\qquad\qquad -\,2\,i\,\pi_{\vec{n}}\,\partial_{\bar{z}}\pi_{\vec{n}}\,\partial_z\vec{H}\\
&=\; \mathrm{e}^{-\lambda}\,\partial_{\bar{z}}(\mathrm{e}^{\lambda}A)\,\vec{e}_{\bar{z}} \,+\, 2\,i\,\mathrm{e}^{\lambda}\Big[\big(\vec{H}\cdot\partial_z\vec{H}\big)\,\vec{e}_{\bar{z}} \,+\, \big(\vec{H}_0\cdot\partial_z\vec{H}\big)\,\vec{e}_z\Big]\\
&\qquad\qquad +\,\frac{\mathrm{e}^{\lambda}}{2}\,A\,\vec{H}_0 \,-\, 2\,i\,\pi_{\vec{n}}\,\partial_{\bar{z}}\pi_{\vec{n}}\,\partial_z\vec{H}\,.
\end{aligned}
$$

Because $\Im(\partial_{\bar{z}z}\vec{L}) = 0$, we "project" this last identity to discover

$$
\partial_{\bar{z}}\big(\mathrm{e}^{\lambda}A\big) \;=\; -\,2\,i\,\mathrm{e}^{2\lambda}\,\big(\vec{H}\cdot\partial_z\vec{H} \,+\, \overline{\vec{H}_0}\cdot\partial_{\bar{z}}\vec{H}\big) \qquad \text{(III.42)}
$$

and

$$
4\,\Im\Big[i\,\pi_{\vec{n}}\,\partial_{\bar{z}}\pi_{\vec{n}}\,\partial_z\vec{H}\Big] \;=\; \mathrm{e}^{\lambda}\,\Im\big(A\,\vec{H}_0\big)\,. \qquad \text{(III.43)}
$$

Comparing (III.42) to the Codazzi-Mainardi equation (A.17) shows that

$$
\partial_{\bar{z}}\Big[\mathrm{e}^{\lambda}\big(A \,+\, 2\,i\,\mathrm{e}^{\lambda}\,\overline{\vec{H}_0}\cdot\vec{H}\big)\Big] \;=\; 0\,.
$$

Accordingly, there exists some holomorphic function $f(z)$ such that

$$
A \;=\; -\,\mathrm{e}^{-\lambda}f(z) \,-\, 2\,i\,\mathrm{e}^{\lambda}\,\overline{\vec{H}_0}\cdot\vec{H}\,. \qquad \text{(III.44)}
$$

In conformal coordinates, there holds

$$
\mathrm{e}^{2\lambda}\,\Delta_\perp\vec{H} \;=\; \pi_{\vec{n}}\Big(div\,\pi_{\vec{n}}\big(\nabla\vec{H}\big)\Big) \;\equiv\; 4\,\Im\Big[i\,\pi_{\vec{n}}\,\partial_{\bar{z}}\pi_{\vec{n}}\,\partial_z\vec{H}\Big]\,,
$$

so that (III.43) becomes

$$
\mathrm{e}^{\lambda}\,\Delta_\perp\vec{H} \;=\; \Im\big(A\,\vec{H}_0\big)\,.
$$

Introducing (III.44) in the latter gives thus

$$
\Delta_\perp\vec{H} \,+\, 2\,\Re\big((\overline{\vec{H}_0}\cdot\vec{H})\,\vec{H}_0\big) \;=\; \mathrm{e}^{-2\lambda}\,\Im\big(\vec{H}_0\,f\big)\,. \qquad \text{(III.45)}
$$

To complete our derivation, there remains to observe that[7]

$$
\begin{aligned}
\Re\Big[H_0^{\alpha}\,\overline{H_0^{\beta}}\Big] \;&=\; \frac{1}{4}\Big[\big(h_{11}^{\alpha}-h_{22}^{\alpha}\big)\big(h_{11}^{\beta}-h_{22}^{\beta}\big) \,+\, 4\,h_{12}^{\alpha}\,h_{12}^{\beta}\Big]\\
&=\; \frac{1}{2}\sum_{i,j=1}^{2} h_{ij}^{\alpha}\,h_{ij}^{\beta} \,-\, \frac{1}{4}\,\big(h_{11}^{\alpha}+h_{22}^{\alpha}\big)\big(h_{11}^{\beta}+h_{22}^{\beta}\big)\\
&=\; \frac{1}{2}\sum_{i,j=1}^{2} h_{ij}^{\alpha}\,h_{ij}^{\beta} \,-\, H^{\alpha}H^{\beta}\,.
\end{aligned}
$$

[7]implicit summations over repeated indices are understood wherever appropriate.

Accordingly, (III.45) yields

$$\Delta_\perp \vec{H} \;+\; \sum_{i,j=1}^{2} h^\alpha_{ij}\, h^\beta_{ij}\, H^\beta\, \vec{n}_\alpha \;-\; 2\,\big|\vec{H}\big|^2 \vec{H} \;\;=\;\; \mathrm{e}^{-2\lambda}\, \Im\big(\vec{H}_0\, f\big)\,.$$

This identity is precisely the announced conformal Willmore equation.

# A Appendix

## A.1 Notational Conventions

We append an arrow to all the elements belonging to $\mathbb{R}^m$. To simplify the notation, by $\vec{\Phi} \in X(D^2)$ is meant $\vec{\Phi} \in X(D^2, \mathbb{R}^m)$ whenever $X$ is a functional space. Similarly, we write $\nabla\vec{\Phi} \in \mathbb{R}^2 \otimes X(D^2)$ for $\nabla\vec{\Phi} \in X(D^2, \mathbb{R}^{2m})$ .

Although this custom may seem odd at first, we allow the differential operators classically acting on scalars to act on elements of $\mathbb{R}^m$. Thus, for example, $\nabla\vec{\Phi}$ is the element of $\mathbb{R}^2 \otimes \mathbb{R}^m$ that can be written $(\partial_{x_1}\vec{\Phi}, \partial_{x_2}\vec{\Phi})$. If $S$ is a scalar and $\vec{R}$ an element of $\mathbb{R}^m$, then we let

$$\begin{aligned}
\vec{R}\cdot\nabla\vec{\Phi} \;&:=\; \big(\vec{R}\cdot\partial_{x_1}\vec{\Phi}\,,\, \vec{R}\cdot\partial_{x_2}\vec{\Phi}\big)\,,\\
\nabla^\perp S\,\nabla\vec{\Phi} \;&:=\; \partial_{x_1}S\,\partial_{x_2}\vec{\Phi} \,-\, \partial_{x_2}S\,\partial_{x_1}\vec{\Phi}\,,\\
\nabla^\perp\vec{R}\cdot\nabla\vec{\Phi} \;&:=\; \partial_{x_1}\vec{R}\cdot\partial_{x_2}\vec{\Phi} \,-\, \partial_{x_2}\vec{R}\cdot\partial_{x_1}\vec{\Phi}\,,\\
\nabla^\perp\vec{R}\wedge\nabla\vec{\Phi} \;&:=\; \partial_{x_1}\vec{R}\wedge\partial_{x_2}\vec{\Phi} \,-\, \partial_{x_2}\vec{R}\wedge\partial_{x_1}\vec{\Phi}\,.
\end{aligned}$$

Similar quantities are defined analogously, following the same logic.

Two operations between multivectors are also useful. The interior multiplication $\llcorner$ maps a pair comprising a $q$-vector $\gamma$ and a $p$-vector $\beta$ to a $(q-p)$-vector. It is defined via

$$\langle \gamma \llcorner \beta\,, \alpha\rangle \;=\; \langle \gamma\,, \beta\wedge\alpha\rangle \qquad \text{for each } (q-p)\text{-vector } \alpha.$$

Let $\alpha$ be a $k$-vector. The first-order contraction operation $\bullet$ is defined inductively through

$$\alpha\bullet\beta \;=\; \alpha\llcorner\beta \qquad \text{when } \beta \text{ is a 1-vector}\,,$$

and

$$\alpha\bullet(\beta\wedge\gamma) \;=\; (\alpha\bullet\beta)\wedge\gamma \,+\, (-1)^{pq}\,(\alpha\bullet\gamma)\wedge\beta\,,$$

when $\beta$ and $\gamma$ are respectively a $p$-vector and a $q$-vector.

## A.2 Lorentz and Sobolev-Lorentz Spaces

For the reader's convenience, we recall in this section the fundamentals of Lorentz spaces. Detailed accounts may be found in [BL], [Hun], and [Ta1].

For a real-valued measurable function $f$ on an open subset of $U \subset \mathbb{R}^n$, its belonging to a Lorentz space is determined by a condition involving the non-decreasing rearrangement of $|f|$ on the interval $(0, |U|)$, where $|U|$ denotes the Lebesgue measure of $U$. The non-increasing rearrangement $f^*$ of $|f|$ is the unique function from $(0, |U|)$ into $\mathbb{R}$ which is non-increasing and which satisfies

$$\Big|\big\{x \in U \;\big|\; |f(x)| \geq s\big\}\Big| \;=\; \Big|\big\{t \in (0, |U|) \;\big|\; f^*(t) \geq s\big\}\Big| \,.$$

If $p \in (1, \infty)$ and $q \in [1, \infty]$, the Lorentz space $L^{p,q}(U)$ is the set of measurable functions $f : U \to \mathbb{R}$ for which

$$\int_0^\infty \big(t^{1/p} f^*(t)\big)^q \,\frac{dt}{t} \;<\; \infty \qquad \text{if} \;\; q < \infty\,,$$

or

$$\sup_{t>0} t^{1/p} f^*(t) \;<\; \infty \qquad \text{if} \;\; q = \infty\,.$$

A norm on $L^{p,q}(U)$ is given by

$$\|f\|_{L^{p,q}(U)} = \big\|\, t^{1/p} f^{**}\big\|_{L^q([0,\infty),\, dt/t)} \qquad \text{where} \qquad f^{**}(t) \;:=\; \frac{1}{t}\int_0^t f^*(\tau)\, d\tau\,.$$

One verifies that

$$L^p \;=\; L^{p,p}\,,$$

and that $L^{p,\infty}$ is the weak-$L^p$ Marcinkiewicz space.

Moreover, we have the inclusions

$$L^{p,1} \;\subset\; L^{p,q'} \;\subset\; L^{p,q''} \;\subset\; L^{p,\infty} \qquad \text{for} \qquad 1 < q' < q'' < \infty\,;$$

and if $U$ has finite measure, there holds for all $q$ and $q'$

$$L^{p',q'}(U) \;\subset\; L^{p,q}(U) \qquad \text{whenever} \qquad p < p'\,.$$

Finally, if $q < \infty$, the space $L^{\frac{p}{p-1},\frac{q}{q-1}}$ is the dual of $L^{p,q}$.

Similarly to Lebesgue spaces, Lorentz spaces obey a pointwise multiplication rule and a convolution product rule. More precisely, for $1 < p_1, p_2 < \infty$ and $1 \leq q_1, q_2 \leq \infty$, there holds

$$L^{p_1,q_1} \times L^{p_2,q_2} \;=\; L^{p,q} \qquad \text{with} \qquad \begin{cases} p^{-1} &=& p_1^{-1} + p_2^{-1} \\ q^{-1} &=& q_1^{-1} + q_2^{-1}\,, \end{cases}$$

and

$$L^{p_1,q_1} * L^{p_2,q_2} = L^{p,q} \qquad \text{with} \qquad \begin{cases} p^{-1} &= p_1^{-1} + p_2^{-1} - 1 \\ q^{-1} &= q_1^{-1} + q_2^{-1}\,. \end{cases}$$

An interesting feature of Lorentz spaces is the possibility to generate them via interpolation of Lebesgue spaces. In particular, if $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open subsets, and if $r_0$, $r_1$, $p_0$, $p_1$ are real numbers satisfying

$$1 \le r_0 < r_1 \le \infty \qquad \text{and} \qquad 1 \le p_0 \ne p_1 \le \infty\,,$$

the following interpolation result holds. Let $T$ be a linear operator which[8] maps continuously $L^{r_j}(U)$ into $L^{p_j}(V)$. Then $T$ maps continuously $L^{r,q}(U)$ into $L^{p,q}(V)$ for each $q \in [1,\infty]$ and every pair $(p,r)$ such that

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \text{and} \qquad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}\,,$$

for some $\theta \in (0,1)$.

Lorentz spaces offer the possibility to sharpen the classical Sobolev embedding theorem. More precisely, it can be shown (see [BL] and [Ta3]) that

$$W^{k,q}(\mathbb{R}^m) \subset L^{p,q}(\mathbb{R}^m)$$

is a continuous embedding as long as

$$1 \le q \le p < \infty \qquad \text{and} \qquad \frac{k}{m} = \frac{1}{q} - \frac{1}{p}\,.$$

Our study requires that we introduce Sobolev-Lorentz spaces. These spaces are defined analogously to the "standard" Sobolev spaces, but with the Lebesgue norms replaced by Lorentz norms. In an effort to simplify the presentation, we shall focus only on the two-dimensional unit disk $D^2$. Let $m \in \mathbb{N}$, $p \in (1,\infty)$, and $q \in [1,\infty]$. The (homogeneous) Sobolev-Lorentz space $W^{m,(p,q)}(D^2)$ consists of all locally summable functions $u$ on $D^2$ such that $D^\alpha u$ exists in the weak sense and belongs to the Lorentz space $L^{p,q}(D^2)$ for all multiindex $\alpha$ with $|\alpha| = m$. The norm

$$\|u\|_{W^{m,(p,q)}} := \sum_{|\alpha|=m} \|D^\alpha u\|_{L^{p,q}}$$

clearly makes $W^{m,(p,q)}(D^2)$ into a Banach space. The space $W^{0,(p,q)}$ is understood to be $L^{p,q}$. For notational convenience, we shall from now on omit to precise that we work on the unit disk $D^2$. This shall arise no

[8] for $j \in \{0,1\}$.

confusion. We also focus on a case of particular interest to us, namely $(p,q) = (2,1)$. Because $L^{2,1}$ is a subspace of $L^2$, it follows immediately that for all $m \in \mathbb{N}$ there holds

$$W^{m,(2,1)} \subset W^{m,2}\,, \tag{A.1}$$

where $W^{m,2}$ is the usual (homogeneous) Sobolev space. In [BW], the authors prove[9] that $W^{1,(2,1)}$ is a subspace of $L^\infty \cap C^0$, so that $W^{m,(2,1)}$ is a subspace of $W^{m-1,\infty} \cap C^{m-1}$. Altogether, there holds

$$W^{m,(2,1)} \subset W^{m,2} \cap W^{m-1,\infty} \cap C^{m-1}\,. \tag{A.2}$$

The Rellich-Kondrachov theorem states that $W^{1,2}$ is compact in $L^r$ for all finite $r \geq 1$. By standard interpolation techniques, it ensues that

$$W^{m,2} \subset\subset W^{m-1,(p,q)} \qquad \text{for} \qquad \begin{cases} p = 1 = q \\ 1 < p < \infty\;,\;\; 1 \leq q \leq \infty \end{cases} \tag{A.3}$$

are compact inclusions for every $m \in \mathbb{N}^*$.

The embedding (A.1) shows that each element of $W^{m,(2,1)}$ has a well-defined trace on the boundary of the unit disk, for $m \geq 1$. If that trace is null, the said element belongs to the space $W_0^{m,(2,1)}$. In addition, it is easily seen that $W_0^{m,(2,1)}$ is the closure of $C_c^\infty$ in $W^{m,(2,1)}$.

It is instructive to characterize the dual of the space $W_0^{m,(p,q)}$, which we shall denote $W^{-m,(p',q')}$, where $(p',q')$ is the conjugate pair of $(p,q)$:

$$p' \;=\; \big(1 - p^{-1}\big)^{-1} \qquad \text{and} \qquad q' \;=\; \big(1 - q^{-1}\big)^{-1}. \tag{A.4}$$

**Proposition I.1** *Suppose that $T$ is an element of $W^{-m,(p',q')}(D^2,\mathbb{R})$, so that $T$ is a bounded linear functional on $W_0^{m,(p,q)}$. For every $\delta > 0$, there exists an element $P$ of $W^{1-m,(p',q')}(D^2,\mathbb{R}^2)$ such that*

$$T \;=\; div\; P\,, \tag{A.5}$$

*and*

$$\big\|P\big\|_{W^{1-m,(p',q')}} \;\leq\; \|T\|_{W^{-m,(p',q')}} \;+\; \delta\,. \tag{A.6}$$

**Proof.** This is a mere adaption of the analogous statement for the Lebesgue-Sobolev space $W^{m,p}$, ultimately following from the Hahn-Banach theorem. The reader is referred to Theorem 3.10 in [Ad1] for details.

■

We next bring into light an interesting Hodge decomposition result which follows from standard elliptic theory and the interpolation nature of Lorentz spaces (cf. Proposition 3.3.9 in [Hel]).

[9] see also Theorem 3.3.4 in [Hel].

**Proposition I.2** *To every vector field* $\vec{g} = (g_1, g_2) \in L^1(D^2, \mathbb{R}^2)$ *we associate the functions* $\alpha$*,* $\beta$*, and* $h$ *on* $D^2$ *which are solutions of*

$$\left\{\begin{array}{rcl} \Delta\alpha & = & div\,\vec{g} \\ \alpha & = & 0 \end{array}\right. , \qquad \begin{array}{rcl} \Delta\beta & = & curl\,\vec{g} \\ \beta & = & 0 \end{array} \qquad \begin{array}{l} in\ \ D^2 \\ on\ \ \partial D^2 , \end{array}$$

*and*

$$\left\{\begin{array}{rcl} g_1 & = & \dfrac{\partial\alpha}{\partial x} + \dfrac{\partial\beta}{\partial y} + \dfrac{\partial h}{\partial x} \\[2ex] g_2 & = & \dfrac{\partial\alpha}{\partial y} - \dfrac{\partial\beta}{\partial x} + \dfrac{\partial h}{\partial y} , \end{array}\right.$$

*so that* $h$ *is harmonic on* $D^2$*.*
*Then the operators*

$$\vec{g} \longmapsto \nabla\alpha , \qquad \vec{g} \longmapsto \nabla\beta , \qquad and \qquad \vec{g} \longmapsto \nabla h$$

*map continuously* $L^{p,q}(D^2, \mathbb{R}^2)$ *into itself for every choice of* $p \in (1, \infty)$ *and* $q \in [1, \infty]$*.*

This lemma is the central ingredient of the following result.

**Lemma A.1** *Let* $p \in (1, \infty)$ *and* $q \in (1, \infty]$*. Suppose that* $G$ *is an element of* $W^{-1,(p,q)}(D^2, \mathbb{R}^2)$ *which satisfies (in the functional sense)*

$$div\, G \;=\; 0 \qquad in \quad D^2 .$$

*Then there exists an element* $L$ *in the space* $L^{p,q}(D^2, \mathbb{R})$ *such that*

$$G \;=\; \nabla^{\perp} L . \tag{A.7}$$

**Proof.** Let $p'$ and $q'$ be as in (A.4). We remind the reader that $W^{-1,(p,q)}$ is the dual of $W_0^{1,(p',q')}$.
A classical result of Laurent Schwartz guarantees the existence of $L$ in $\mathcal{D}'$ such that (A.7) holds in the sense of distributions. Owing to the density of $\mathcal{D} = C_c^{\infty}$ in $W_0^{1,(p',q')}$, it thus follows that

$$\int_{D^2} G \cdot F \;=\; -\int_{D^2} L\; div\, F , \qquad \forall\;\, F \,\in\, W_0^{1,(p',q')}(D^2, \mathbb{R}^2) . \tag{A.8}$$

Next, let $f$ be an arbitrary element of $L^{p',q'}(\mathbb{R})$ with null average over the unit-disk. Choosing $\vec{g} = (f, 0)$ in Proposition I.2, we infer the existence of an element $F := (\alpha, \beta)$ in $W_0^{1,(p',q')}(D^2, \mathbb{R}^2)$ satisfying

$$div\, F \;=\; f \qquad \text{and} \qquad \|F\|_{W^{1,(p',q')}} \;\leq\; \|f\|_{L^{p',q'}} . \tag{A.9}$$

Altogether, (A.8) and (A.9) show that $L$ acts linearly on $L^{p',q'}$ with the estimate

$$\left| \int_{D^2} L\, f \right| \;\leq\; \|G\|_{W^{-1,(p,q)}}\, \|f\|_{L^{p',q'}} .$$

Since $L^{p,q}$ is the dual space of $L^{p',q'}$, the desired statement ensues.

■

### A.3 Miscellaneous Identities

**Lemma A.2** *Let $\vec{\Phi}$, $\vec{n}$, and $\vec{H}$ be as in Theorem II.1. We set*

$$Q \;:=\; \nabla\vec{H} \;-\; 3\,\pi_{\vec{n}}\big(\nabla\vec{H}\big) \;+\; \star\big(\nabla^{\perp}\vec{n}\wedge\vec{H}\big)\,.$$

*Then the following identities hold*

$$\nabla\vec{\Phi}\cdot Q \;=\; 0 \qquad \text{and} \qquad \nabla\vec{\Phi}\wedge Q \;=\; -\,2\,\nabla\vec{\Phi}\wedge\nabla\vec{H}\,. \tag{A.10}$$

**Proof.** We first note that[10]

$$\nabla^{\perp}\vec{n}_{\alpha} \;=\; \big\langle\vec{e}_1\,,\nabla^{\perp}\vec{n}_{\alpha}\big\rangle\,\vec{e}_1 \;+\; \big\langle\vec{e}_2\,,\nabla^{\perp}\vec{n}_{\alpha}\big\rangle\,\vec{e}_2 \;+\; \big\langle\vec{n}_{\beta}\,,\nabla^{\perp}\vec{n}_{\alpha}\big\rangle\,\vec{n}_{\beta}\,,$$

so that

$$\star\big(\vec{n}\wedge\nabla^{\perp}\vec{n}_{\alpha}\big) \;=\; \big\langle\vec{e}_1\,,\nabla^{\perp}\vec{n}_{\alpha}\big\rangle\,\vec{e}_2 \;-\; \big\langle\vec{e}_2\,,\nabla^{\perp}\vec{n}_{\alpha}\big\rangle\,\vec{e}_1\,.$$

Whence, since $\vec{n}_{\alpha}\wedge\vec{n}=0$,

$$\star\big(\nabla^{\perp}\vec{n}\wedge\vec{n}_{\alpha}\big) \;=\; \begin{pmatrix} h^{\alpha}_{22} \\ -\,h^{\alpha}_{12}\end{pmatrix}\partial_{x_1}\vec{\Phi} \;+\; \begin{pmatrix} -\,h^{\alpha}_{12} \\ h^{\alpha}_{11}\end{pmatrix}\partial_{x_2}\vec{\Phi}\,. \tag{A.11}$$

Accordingly, we find

$$\nabla\vec{\Phi}\wedge\star\big(\nabla^{\perp}\vec{n}\wedge\vec{n}_{\alpha}\big) \;=\; \big(h^{\alpha}_{12} \;-\; h^{\alpha}_{12}\big)\,\partial_{x_1}\vec{\Phi}\wedge\partial_{x_2}\vec{\Phi} \;=\; 0\,,$$

and

$$\nabla\vec{\Phi}\;\cdot\;\star\big(\nabla^{\perp}\vec{n}\wedge\vec{n}_{\alpha}\big) \;=\; \mathrm{e}^{2\lambda}\,\big(h^{\alpha}_{11} \;+\; h^{\alpha}_{22}\big) \;=\; 2\,\mathrm{e}^{2\lambda}\,H^{\alpha}\,.$$

The last two identities yield

$$\nabla\vec{\Phi}\wedge\star\big(\nabla^{\perp}\vec{n}\wedge\vec{H}\big) \;=\; 0\,, \tag{A.12}$$

and

$$\nabla\vec{\Phi}\;\cdot\;\star\big(\nabla^{\perp}\vec{n}\wedge\vec{H}\big) \;=\; 2\,\mathrm{e}^{2\lambda}\,\big|\vec{H}\big|^{2}\,. \tag{A.13}$$

For three indices $(a,b,c)\in\{x^1,x^2\}$, let

$$F(a,b,c) \;:=\; \partial_a\vec{\Phi}\;\wedge\;\big\langle\vec{e}_b\,,\partial_c\vec{H}\big\rangle\,\vec{e}_b\,.$$

We have

$$\begin{aligned} F(a,b,c) \;&=\; \mathrm{e}^{\lambda}\,\big\langle\vec{e}_b\,,\partial_c\vec{H}\big\rangle\,\vec{e}_a\wedge\vec{e}_b \\ &=\; -\,\mathrm{e}^{\lambda}\,H^{\alpha}\,\big\langle\vec{n}_{\alpha}\,,\partial_c\vec{e}_b\big\rangle\,\vec{e}_a\wedge\vec{e}_b \qquad \text{since } \vec{H}=H^{\alpha}\vec{n}_{\alpha} \\ &=\; -\,\mathrm{e}^{\lambda}\,H^{\alpha}\,h^{\alpha}_{cb}\,\vec{e}_a\wedge\vec{e}_b\,. \end{aligned}$$

[10] implicit summations over repeated indices are understood wherever appropriate.

Using this, we obtain

$$
\begin{aligned}
\nabla\vec{\Phi} \wedge \Big(\nabla\vec{H} - \pi_{\vec{n}}(\nabla\vec{H})\Big) &\equiv \nabla\vec{\Phi} \wedge \Big(\big\langle\vec{e}_1\,,\nabla\vec{H}\big\rangle\,\vec{e}_1 + \big\langle\vec{e}_2\,,\nabla\vec{H}\big\rangle\,\vec{e}_2\Big) \\
&= F(x^1,x^1,x^1) + F(x^1,x^2,x^1) \\
&\qquad + F(x^2,x^1,x^2) + F(x^2,x^2,x^2) \\
&= -\,\mathrm{e}^{\lambda}\,H^\alpha\,h^\alpha_{21}\,\vec{e}_1\wedge\vec{e}_2 - \mathrm{e}^{\lambda}\,H^\alpha\,h^\alpha_{12}\,\vec{e}_2\wedge\vec{e}_1 \\
&= 0\,.
\end{aligned}
$$

Hence

$$
\nabla\vec{\Phi} \wedge \Big(\nabla\vec{H} - 3\,\pi_{\vec{n}}(\nabla\vec{H})\Big) = -\,2\,\nabla\vec{\Phi}\wedge\nabla\vec{H}\,. \tag{A.14}
$$

An evident yet nonetheless useful identity is

$$
\nabla\vec{\Phi}\cdot\pi_{\vec{n}}(\nabla\vec{H}) = 0\,,
$$

thereby giving

$$
\nabla\vec{\Phi} \cdot \Big(\nabla\vec{H} - 3\,\pi_{\vec{n}}(\nabla\vec{H})\Big) = \nabla\vec{\Phi}\cdot\nabla\vec{H}\,. \tag{A.15}
$$

Bringing altogether (A.12) and (A.14) yields the second part of (A.10). Deriving the first part of (A.10) requires a bit more work. Combining (A.13) and (A.15) shows that

$$
\nabla\vec{\Phi}\cdot Q = \nabla\vec{\Phi}\cdot\nabla\vec{H} + 2\,\mathrm{e}^{2\lambda}\,\big|\vec{H}\big|^2\,. \tag{A.16}
$$

For three indices $(a,b,c)\in\{x^1,x^2\}$, let

$$
G(a,b,c) := \partial_a\vec{\Phi}\cdot\big\langle\vec{e}_b\,,\partial_c\vec{H}\big\rangle\,\vec{e}_b\,.
$$

Just as above, we verify easily that

$$
G(a,b,c) = -\,\mathrm{e}^{2\lambda}\,H^\alpha\,h^\alpha_{cb}\,\delta_{ab}\,.
$$

Consequently, there holds

$$
\begin{aligned}
\nabla\vec{\Phi}\cdot\nabla\vec{H} &= G(x^1,x^1,x^1) + G(x^1,x^2,x^1) + G(x^2,x^1,x^2) + G(x^2,x^2,x^2) \\
&= -\,\mathrm{e}^{2\lambda}\,H^\alpha\,h^\alpha_{11} - \mathrm{e}^{2\lambda}\,H^\alpha\,h^\alpha_{22} \\
&= -\,2\,\mathrm{e}^{2\lambda}\,\big|\vec{H}\big|^2\,.
\end{aligned}
$$

Putting this into (A.16) finally gives the first sought after identity in (A.10).

■

**Lemma A.3** *Using the notation introduced in Section III.2.2, the Codazzi-Mainardi identity may be recast in the form*

$$e^{-2\lambda}\,\partial_{\bar z}\big(e^{2\lambda}\,\overline{\vec H_0}\cdot\vec H\big) \;=\; \vec H\cdot\partial_z\vec H \,+\, \overline{\vec H_0}\cdot\partial_{\bar z}\vec H\,. \qquad \text{(A.17)}$$

**Proof.** Consider the 1-form

$$\eta_\alpha \;:=\; \big\langle \vec e_{\bar z}, d\vec n_\alpha\big\rangle\,.$$

On one hand, from (III.37), there holds

$$\eta_\alpha \;=\; \big(\vec e_{\bar z}\cdot\partial_z\vec n_\alpha\big)\,dz \,+\, \big(\vec e_{\bar z}\cdot\partial_{\bar z}\vec n_\alpha\big)\,d\bar z \;=\; -\,\frac{\mathrm{e}^{\lambda}}{2}\,\big(H^\alpha dz + H_0^\alpha d\bar z\big)\,,$$

so that

$$\star\, d\eta_\alpha \;=\; -\,\frac{\mathrm{e}^{\lambda}}{2}\,\big(H_0^\alpha\partial_z\lambda \,-\, H^\alpha\partial_{\bar z}\lambda \,+\, \partial_z H_0^\alpha \,-\, \partial_{\bar z}H^\alpha\big)\,. \qquad \text{(A.18)}$$

On the other hand, with (III.36) and (III.37), we find

$$\begin{aligned}\star\, d\eta_\alpha \;&=\; \star\, d\,\big\langle \vec e_{\bar z}, d\vec n_\alpha\big\rangle \;=\; \partial_z\vec e_{\bar z}\cdot\partial_{\bar z}\vec n_\alpha \,-\, \partial_{\bar z}\vec e_{\bar z}\cdot\partial_z\vec n_\alpha\\ &=\; \frac{\mathrm{e}^{\lambda}}{2}\,\big(H_0^\alpha\partial_z\lambda \,+\, H^\alpha\partial_{\bar z}\lambda \,+\, \vec H\cdot\partial_{\bar z}\vec n_\alpha \,-\, \vec H_0\cdot\partial_z\vec n_\alpha\big)\,.\end{aligned}$$

Comparing the latter to (A.18) yields

$$2\,H_0^\alpha\,\partial_z\lambda \;=\; \partial_{\bar z}H^\alpha \,-\, \partial_z H_0^\alpha \,-\, \vec H\cdot\partial_{\bar z}\vec n_\alpha \,+\, \vec H_0\cdot\partial_z\vec n_\alpha\,. \qquad \text{(A.19)}$$

By antisymmetry, there holds

$$H^\alpha\vec H\cdot\partial_{\bar z}\vec n_\alpha \;=\; H^\alpha H^\beta\,\vec n_\beta\cdot\partial_{\bar z}\vec n_\alpha \;=\; 0\,,$$

and whence

$$H^\alpha\big(\partial_{\bar z}H^\alpha \,-\, \vec H\cdot\partial_{\bar z}\vec n_\alpha\big) \;=\; H^\alpha\big(\partial_{\bar z}H^\alpha \,+\, \vec H\cdot\partial_{\bar z}\vec n_\alpha\big) \;\equiv\; \vec H\cdot\partial_{\bar z}\vec H\,. \qquad \text{(A.20)}$$

Similarly, we find

$$\begin{aligned}H^\alpha\big(\partial_z H_0^\alpha \,-\, \vec H_0\cdot\partial_z\vec n_\alpha\big) \;&=\; \partial_z\big(\vec H_0\cdot\vec H\big) \,-\, H_0^\alpha\partial_z H^\alpha \,-\, H^\alpha\vec H_0\cdot\partial_z\vec n_\alpha\\ &=\; \partial_z\big(\vec H_0\cdot\vec H\big) \,-\, \vec H_0\cdot\partial_z\vec H\,.\end{aligned} \qquad \text{(A.21)}$$

Multiplying (A.19) throughout by $H^\alpha$, summing over $\alpha$, and using (A.20) and (A.21) gives

$$2\,\vec H\cdot\vec H_0\,\partial_z\lambda \;=\; -\,\partial_z\big(\vec H\cdot\vec H_0\big) \,+\, \vec H\cdot\partial_{\bar z}\vec H \,+\, \vec H_0\cdot\partial_z\vec H\,,$$

from which the (complex conjugate of) the desired (A.17) ensues. ■

**Lemma A.4** *Using the notation of Section III.2, there holds*

$$-\Delta\vec{\Phi} \;=\; \nabla\vec{R}\bullet\nabla^{\perp}\vec{\Phi} \;+\; \nabla S\,\nabla^{\perp}\vec{\Phi}\,. \tag{A.22}$$

**Proof.** Note that for any 1-vector $\vec{a}$, we have

$$(\vec{a}\wedge\vec{e}_j)\bullet\vec{e}_i \;=\; (\vec{e}_i\llcorner\vec{a})\wedge\vec{e}_j \;+\; \vec{a}\wedge(\vec{e}_i\llcorner\vec{e}_j) \;=\; (\vec{e}_i\cdot\vec{a})\,\vec{e}_j \;+\; \delta_{ij}\,\vec{a}\,.$$

From this, and $\vec{e}_i := \mathrm{e}^{-\lambda}\partial_{x^i}\vec{\Phi}$, it follows easily that whenever

$$\vec{V} \;:=\; V^i\,\vec{e}_i \;+\; V^{\alpha}\,\vec{n}_{\alpha}\,,$$

then

$$\left\{\begin{array}{rcl} \big(\vec{V}\wedge\nabla^{\perp}\vec{\Phi}\big)\bullet\nabla^{\perp}\vec{\Phi} & = & \mathrm{e}^{2\lambda}\big(3\,V^i\,\vec{e}_i \;+\; 2\,V^{\alpha}\,\vec{n}_{\alpha}\big) \\ \big(\vec{V}\wedge\nabla\vec{\Phi}\big)\bullet\nabla^{\perp}\vec{\Phi} & = & \mathrm{e}^{2\lambda}\big(V^2\,\vec{e}_1 \;-\; V^1\,\vec{e}_2\big) \quad\equiv\quad \big(\vec{V}\cdot\nabla\vec{\Phi}\big)\nabla^{\perp}\vec{\Phi}\,. \end{array}\right.$$

In particular, since $\vec{H} = H^{\alpha}\,\vec{n}_{\alpha}$, we find that

$$\begin{aligned} \nabla\vec{R}\bullet\nabla^{\perp}\vec{\Phi} &\equiv -\big(\vec{L}\wedge\nabla\vec{\Phi} \;+\; 2\,\vec{H}\wedge\nabla^{\perp}\vec{\Phi}\big)\bullet\nabla^{\perp}\vec{\Phi} \\ &= -\big(\vec{L}\cdot\nabla\vec{\Phi}\big)\nabla^{\perp}\vec{\Phi} \;-\; 2\,\mathrm{e}^{2\lambda}\,\vec{H} \\ &= -\nabla S\,\nabla^{\perp}\vec{\Phi} \;-\; 2\,\mathrm{e}^{2\lambda}\,\vec{H}\,. \end{aligned}$$

Hence,

$$\nabla\vec{R}\bullet\nabla^{\perp}\vec{\Phi} \;+\; \nabla S\,\nabla^{\perp}\vec{\Phi} \;=\; -\,2\,\mathrm{e}^{2\lambda}\,\vec{H}\,.$$

Finally, there remains to observe that

$$\Delta\vec{\Phi} \;=\; 2\,\mathrm{e}^{2\lambda}\,\vec{H}$$

to deduce the desired identity (A.22) .

■

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