Néel and Cross-Tie Wall Energies for Planar Micromagnetic Configurations

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Abstract

We study a two-dimensional model for micromagnetics, which consists in an energy functional over S^2-valued vector fields.

Bounded-energy configurations tend to be planar, except in small regions which can be described as vortices (Bloch lines in physics). As the characteristic "exchange-length" tends to 0, they converge to planar divergence-free unit norm vector fields which jump along line singularities. We derive lower bounds for the energy, which are explicit functions of the jumps of the limit. These lower bounds are proved to be optimal and are achieved by one-dimensional profiles called Néel walls if the jump is small enough (less than $\pi/2$ in angle), and by two-dimensional profiles called cross-tie walls if the jump is bigger.

We also establish other lower bounds and compactness properties on different quantities which provide a good notion of convergence and cost of vortices.

key-words : micromagnetics, thin films, cross-tie walls, Gamma-convergence.

I Introduction

I.1 General presentation

This paper deals with the analysis of ferromagnetic materials. The general model for such materials is given by Brown [Br] where it is considered that the sample occupies a region $\Omega \subset \mathbb{R}^3$ and there exists a distribution of magnetization $m(x) \in \mathbb{R}^3$ in Ω which is constrained to be of constant magnitude M_S, and to which is associated an energy of the form

$$E(m) = A\int_\Omega |\nabla m|^2 + K\int_\Omega \varphi(m) + \int_{\mathbb{R}^3} |H|^2. \tag{I.1}$$

The first term is usually called the exchange energy and penalizes strong local variations of m.

The second term is called the anisotropy energy and accounts for prefered directions for m, which are namely the wells of the smooth function $\varphi : S^2 \to \mathbb{R}^+$. In real materials (see [HS]), several anisotropies do exist, such as uniaxial (one prefered direction), cubic (3 mutually orthogonal prefered directions), or planar (see [HS] p. 406 and 115).

Eventually, the last term is called the demagnetizing energy. The vector field H is the stray field induced by the distribution in the whole space, given by (up to a physical factor)

$$H = -\nabla \Delta^{-1} \mathrm{div}\ (m\mathbf{1}_\Omega), \tag{I.2}$$

where $\mathbf{1}_\Omega$ denotes the characteristic function of Ω. It turns out that this term vanishes whenever m is divergence-free and tangent to the boundary of Ω.

One can remark that A must be proportional to the square of a length. This is the exchange length, typical lengthscale of order one variations of m.

There is a very large variety of configurations observed by physicists, which is partly explained by the competition between all these terms, the shape and size of the domain and various material constants and anisotropies. For instance, several possible limits are presented in detail in [DKMO2] with a particular focus on thin films.

There is also a fairly large mathematics literature on different variants of the energy (some include an applied magnetic field). First, Visintin [Vi] proved the existence of a solution of the system. The question of the smoothness of critical points was addressed in [Ca], [HK]. Many studies have focused on the "no-exchange" case where the exchange energy term is dropped (A set to zero): James and Kinderlehrer [JaKi], De Simone [DeS] (who justified the limit $A \to 0$ in the case of large bodies), De Corogna-Fonseca [DF]...
Very recently, in [DKMO3], DeSimone, Kohn, Müller and Otto have derived rigorously a 2D model as a thin-film limit of the full 3D model (the film thickness being much smaller than the exchange length $\sqrt{A}$).

I.2 The model

The original model is 3D but we reduce to a 2D model by focusing on the case of an infinite cylindrical vertical body of section Ω and considering translation-invariant configurations. We also assume there is a forcing term which amounts to planar anisotropy, and that the lengthscale $\sqrt{A}$ is small compared to the size of the body. Mathematically, we are thus led to consider the functional

$$E_\varepsilon(u) = \frac{1}{2}\left(\varepsilon \int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_{\mathbb{R}^2} |H|^2 + \frac{1}{c_\varepsilon}\int_\Omega |u_3|^2\right), \tag{I.3}$$

over Ω a smooth simply connected region of $\mathbb{R}^2$, where $u = (u_1, u_2, u_3) \in S^2$ is a unit vector field ($u = \frac{m}{M_S}$), and ε is a small lengthscale. The demagnetizing field $H : \Omega \to \mathbb{R}^3$ is solution to

$$\begin{cases} \mathrm{curl}\ H = 0 & \text{in } \mathbb{R}^2 \\ \mathrm{div}\ (H + u\mathbf{1}_\Omega) = 0 & \text{in } \mathbb{R}^2, \end{cases} \tag{I.4}$$

and using the vertical translation invariance, one can easily observe that (I.4) implies that H_3 is uniform in space. Since H belongs to $L^2(\mathbb{R}^2)$, we get $H_3 = 0$ and (I.4) rewrites into the two-dimensional model $H = (H_1, H_2)$,

$$\left\{ \begin{array}{ll} \partial_1 H_2 - \partial_2 H_1 = 0 & \text{in } \mathbb{R}^2 \\ \partial_1 (H_1 + u_1 \mathbf{1}_\Omega) + \partial_2 (H_2 + u_2 \mathbf{1}_\Omega) = 0 & \text{in } \mathbb{R}^2. \end{array} \right. \tag{I.5}$$

It seems that such a model can also be obtained on a formal level, following the type of argument of [DKMO2], as a limiting thick-film regime for thin films (in the sense that the thickness is small but much larger than the exchange length $\sqrt{A}$).

We want to study the asymptotics $\varepsilon \to 0$ (corresponding to a small lengthscale of variations of u), in the regime $c_\varepsilon \ll \varepsilon$, i.e. the planar anisotropy is stronger than the demagnetizing energy. For technical convenience, we will in fact assume

$$c_\varepsilon \leq \varepsilon^{1+\delta} \qquad \text{for some } \ \delta > 0. \tag{I.6}$$

In the limit $\varepsilon \to 0$, sequences of configurations of bounded energy $E_\varepsilon(u^\varepsilon) \leq C$ (such sequences do actually exist) converge to a limiting field u that satisfies

$$\left\{ \begin{array}{l} u \in S^1 \text{ i.e. } u = (u_1, u_2), \ |u| = 1 \\ \text{div } u = 0 \text{ in } \Omega \\ u \cdot n = 0 \ \text{ on } \partial\Omega. \end{array} \right. \tag{I.7}$$

Indeed, observe that $u_3 \to 0$, $H \to 0$ and then plug it into (I.5). Writing $u = -\nabla^\perp g$ leads to the eikonal equation

$$\left\{ \begin{array}{ll} |\nabla g| = 1 & \text{in } \Omega \\ g = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{I.8}$$

Such fields u typically have line singularities on which the energy concentrates. This kind of phenomenon has been studied for related energies, see [AG1, AG2, ADM, DKMO1, JP] for the gradient theory of phase transitions or "Aviles-Giga energy", and [RS1, RS2] for a 2D micromagnetic toy model. Beside the compactness proofs (i.e. $u^\varepsilon \to u$ in $\cap L^p$) the essential problem is always to understand the structure and costs of jumps across the singular lines at the limit $\varepsilon \to 0$, a question which was more particularly addressed in [ADM, JK, RS1, RS2].

I.3 The results of [RS1, RS2]

Let us explain in further detail the mathematical model studied by Rivière and Serfaty in [RS1, RS2]. It consists in studying the same energy under the additional constraint that u belongs to S^1 (i.e. $u_3 \equiv 0$ and the anisotropy term drops out). In that case, any $u^\varepsilon \in H^1(\Omega, S^1)$ possesses a lifting $\varphi_\varepsilon \in H^1(\Omega, \mathbb{R})$ such that

$$u^\varepsilon = e^{i\varphi_\varepsilon}.$$

The sequence (φ_ε) is shown to be also compact and the authors exhibit a limiting energy which is written in terms of φ, the limit of φ_ε:

$$E_0(u, \varphi) = \int_\Omega |\text{div } (\varphi u + u^\perp)|. \tag{I.9}$$

Here, $u^\perp = (-u_2, u_1)$, and div $(\varphi u + u^\perp)$ is a bounded Radon measure supported on the singular set of φ ($\int_\Omega |\text{div } (\varphi u + u^\perp)|$ is its total mass). In the case $\varphi \in BV(\Omega, \mathbb{R})$, E_0 can be written in a more explicit way

$$E_0(u, \varphi) = \int_{\Sigma_\varphi} 2|\sin X - X \cos X|, \tag{I.10}$$

wher Σ_φ is the one-dimensional jump set of φ and $X = \frac{[\varphi]_{\Sigma_\varphi}}{2}$ is half the value of the jump of φ across Σ_φ (this formula is valid for $|X| \leq \pi$ and can be extended to all X, see [RS2]). This lower bound was proved to be optimal (at least for $|X| \leq \pi$) and achieved by *one-dimensional profiles* at the scale ε called Néel walls in the physics literature.

Moreover, it is also proved in [RS1] that

$$\min E_0(u, \varphi) = |\partial\Omega|, \qquad \min_{u_\varepsilon \in H^1(\Omega, S^1)} E_\varepsilon(u^\varepsilon) \to |\partial\Omega|, \tag{I.11}$$

and a minimizer of E_0 is given by

$$u_* = \pm\nabla^\perp \text{dist}(x, \partial\Omega), \tag{I.12}$$

where $\nabla^\perp = (-\partial_2, \partial_1)$. Taking a minimizing sequence of the energy of [RS1] as a test-function in E_ε, we get as a first upper bound that $\min E_\varepsilon \leq |\partial\Omega| + o(1)$.

I.4 Vortices

Let us point out that the energy computed in [RS1] includes a cost (see (I.10)) for jumps of 2π in φ (while u does not jump), and such jumps have to occur because of the topological constraint $u^\varepsilon(x) \in S^1$ and $u \cdot n = 0$ on $\partial\Omega$. The model that we study here allows us to release this artificial constraint $u^\varepsilon \in S^1$, and thus lower energies can be achieved. We only have $u \in S^1$ at the limit, and this makes the situation quite different. Indeed, one can consider the planar projection of u^ε

$$u^\varepsilon_{||} = (u^\varepsilon_1, u^\varepsilon_2).$$

It does not satisfy $u^\varepsilon_{||} \in S^1$, and admits a local lifting φ_ε such that

$$u^\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon},$$

which cannot be defined globally since $\rho_\varepsilon = |u^\varepsilon_{||}|$ can vanish at a point x_0 while φ_ε can have a nonzero degree around x_0. This is precisely what is called a vortex, and such vortices are frequently seen in physical situations and called "Bloch lines". Mathematically, they can be compared to the vortices arising in the Ginzburg-Landau type models (see the book of Bethuel, Brezis and Hélein [BBH]). Here, they carry only a vanishing fraction of the energy, and they allow to get rid of jumps of 2π of the phase. In fact, the functionals

$$\int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{c_\varepsilon}|u_3|^2$$

with $u \in H^1(\Omega, S^2)$, corresponding to dropping $\int_{\mathbb{R}^2} |H|^2$ in E_ε, have been studied by Sandier [Sa], André and Shafrir [AS], and more recently by Hang [Ha]. It leads to an analysis and to results that are very similar to [BBH]. Here, in contrast, because of the presence of the demagnetizing energy, which forces configurations to be almost divergence-free, u not only has vortex (point) singularities, but also line singularities. Also we will see that the notion of vortex is weaker here, and, because u has line singularities, the vortices cannot be defined from the knowledge of u only, but rather as limits of those of u^ε, and will thus depend on the sequence $u^\varepsilon \to u$.

Let us now focus on the crucial example of $\Omega = B(0, R)$, the limiting divergence-free configuration $u_* = \nabla^\perp \mathrm{dist}(x, \partial\Omega) = e^{i\theta}$ in polar coordinates, can be approximated by a smooth $u^\varepsilon_{||} = \rho_\varepsilon e^{i\theta}$ with $\rho_\varepsilon(0) = 0$, and such that the energy tends to 0 as $\varepsilon \to 0$. (In fact, the result of [JOP] is that the limiting energy can only be zero in the case of Ω being a disc and $u = \pm u_*$.) On the other hand, trying to approximate u_* by smooth S^1-valued u^ε gives rise to a concentration of the energy along a radius of $B(0, R)$, along which u^ε has to wind once around S^1. In that case (which is the case of [RS1]), the energy is bounded below by $2\pi R$ (perimeter of the ball). This illustrates the main difference between both models and it turns out to be a major feature in the richness of limiting configurations, as well as new technical difficulties. For instance, the upper bound on the energy $E_\varepsilon(u^\varepsilon) \leq C$ is not sufficient to control the number of vortices at the limit, contrarily to cases such as [BBH]. However, the mathematical quantities and methods introduced in [RS1] turn out to be relevant for analyzing this new problem (the jump singularities and their cost).

I.5 Main result

The spirit is, exactly as in [RS1, RS2], that of Gamma-convergence : we look for relevant lower bounds for the energy and for corresponding upper bounds, thus deriving a limiting energy in the asymptotics $\varepsilon \to 0$.

Let us start from a family $u^\varepsilon \in H^1(\Omega, S^2)$ such that $E_\varepsilon(u^\varepsilon) \leq C$. The L^p compactness of u^ε follows easily from previous studies on similar problems (for example the proof of [JP], see Lemma II.1), and u^ε converges strongly to a limiting S^1-valued u which satisfies (I.7), and which generically has line-singularities. If u is assumed to be in $BV(\Omega)$, then we can get a lower bound for the energy in terms of the jumps of u, which is of the type of (I.10), but a different formula. The structure theorem of BV allows to define a rectifiable one-dimensional jump set of u, Σ_u, and traces u_+, u_- on both sides of Σ_u. Then we define

$$X = \frac{1}{2} \min \left| \widehat{(u_\pm; u_\mp)} \right| \in [0, \frac{\pi}{2}]$$

i.e. the *geometric* half-angle between u_+ and u_-.

Theorem 1 *Let $\varepsilon_n \to 0$ and u^{ε_n} be sequences of maps from Ω into S^2 satisfying $E^{\varepsilon_n}(u^{\varepsilon_n}) \leq C$. Then, up to extraction, u^{ε_n} converges in $\cap_{p<\infty} L^p(\Omega)$ to some limit u which satisfies (I.7). If, in addition, $u \in BV(\Omega, S^1)$, then*

$$\liminf_{n\to\infty} E_{\varepsilon_n}(u^{\varepsilon_n}) \geq \int_{\Sigma_u} A(X) d\mathcal{H}^1 \qquad \text{(I.13)}$$

where

$$A(X) = 2|\sin X - X\cos X| \qquad \textit{for } X \in [0, \frac{\pi}{4}] \tag{I.14}$$

$$A(X) = 2\left|\left(X - \tfrac{\pi}{2}\right)\cos X - \sin X + \sqrt{2}\right| \quad \textit{for } X \in [\frac{\pi}{4}, \frac{\pi}{2}]. \tag{I.15}$$

Here, $\mathcal{H}^1$ denoted the one-dimensional Hausdorff measure.

One notices that for jumps less than $\frac{\pi}{4}$ (I.14), one recovers (I.10) and thus the (purely planar) Néel walls found in [RS1] are still optimal. On the contrary, for jumps bigger than $\frac{\pi}{2}$, they are no longer optimal, the lower bound (I.15) is smaller than that of (I.10), due to the extra degree of freedom $u^\varepsilon \in S^2$ and the possibility of vortices. (I.15) is obtained by analytical means but turns out to be also optimal, which is one of our main results. Thus $\int_{\Sigma_u} A(X)$ can be seen as the limiting energy for E_ε.

Theorem 2 *For all $X \in [\frac{\pi}{4}, \frac{\pi}{2}]$, there exists a cross-tie wall configuration u over $\mathbb{R}^2 = \{(x,y)\}$ such that*

$$\begin{aligned} \lim\nolimits_{y\to+\infty} u(x,y) = u_+ = e^{i(\frac{\pi}{2}-X)} \\ \lim\nolimits_{y\to-\infty} u(x,y) = u_- = e^{i(\frac{\pi}{2}+X)}, \end{aligned}$$

and that its energy per unit-length of x is (after rescaling at any scale going to 0)

$$A(X) + o(1) = 2\left|\left(X - \frac{\pi}{2}\right)\cos X - \sin X + \sqrt{2}\right| + o(1).$$

The optimal configurations are this time two-dimensional configurations (i.e. u depends on both x and y) and have an *unbounded* number of vortices. They are called in physics "cross-tie walls" and are observed in thin film experiments (see [NUH, HS, RH]). In the particular case $X = \frac{\pi}{2}$ the pattern is that of Figure 1.

The configuration is periodic and each cell contains one degree +1 and one degree -1 vortex. This particular configuration is strikingly similar to the numerical simulations and experimental results presented in [NUH]. This picture can be generalized to all jumps $X \in [\frac{\pi}{4}, \frac{\pi}{2}]$, see Section IV.2.

One may then use these 2D cross-tie profiles to construct test-configurations which achieve equality in (I.13). Jump-lines of the cross-tie walls have to be regularized as 1D Néel walls (the jumps are $\le \frac{\pi}{2}$). Since the energy per unit length of the cross-tie pattern is *scale-invariant*, one can blow them down and, given a u with given jump-lines and jumps, paste them along the jump-lines at any scale which tends to 0 as $\varepsilon \to 0$ and which is $\gg \varepsilon$. This will regularize u into a family $u^\varepsilon \to u$ with equality in (I.13).

The method to obtain the lower bound is inspired by the following "trick" of [RS1]

$$E_\varepsilon(u^\varepsilon) \ge \frac{1}{2}\int_\Omega \varepsilon|\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon}|H_\varepsilon|^2 \ge \int_\Omega |\nabla \varphi_\varepsilon \cdot H_\varepsilon|. \tag{I.16}$$

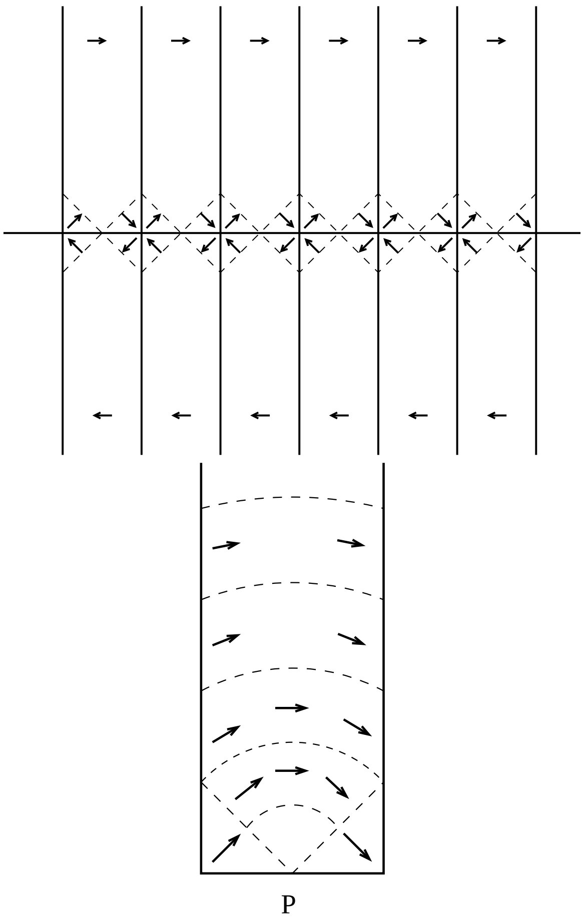

Figure 1: The optimal cross-tie wall.

The problem here is that the phase φ_ε of u^ε is not univalued, so we use the fact that

$$E_\varepsilon(u^\varepsilon) \geq \int_\Omega |f(\varphi_\varepsilon)\nabla\varphi_\varepsilon \cdot H_\varepsilon| = \int_\Omega |\nabla F(\varphi_\varepsilon) \cdot H_\varepsilon|$$

for all $|f| \leq 1$ and $F' = f$. We then find such functions F such that $F(\varphi_\varepsilon)$ depends only on φ_ε modulo 2π, hence only on u^ε. As in [RS1], one can write $F(u_\varepsilon) \cdot H_\varepsilon$ as a divergence and pass to the limit (see Section II).

I.6 Lower bounds involving the vortices

This second approach gives a result which is not always optimal but does not require any BV assumption on u and allows to take into account the vortices. The lower bound of Theorem 1 depended only on the limiting u. However, the result we are going to state shows that two sequences u^ε converging to the same u do not necessarily have the same limiting energy. Their vortex-structure, which is not given by the knowledge of u (because u has line singularities) has to be taken into account. One must thus add the knowledge of the vortices of the sequence $u^\varepsilon \to u$ and find a suitable notion of convergence of those vortices.

Let us start from a family $u^\varepsilon \in H^1(\Omega, S^2)$ such that $E_\varepsilon(u^\varepsilon) \le C$, converging to u in L^p. By a certain transformation (see Lemma II.2), one may reduce to configurations $u^\varepsilon \in H^1(\Omega\backslash\{a_1^\varepsilon, \cdots, a_{n(\varepsilon)}^\varepsilon\}, S^1)$, S^1-valued except at a finite number of singular points. First, from the equation (I.5) defining H, there exists g_ε such that

$$u^\varepsilon \mathbf{1}_\Omega + H_\varepsilon = -\nabla^\perp g_\varepsilon. \tag{I.17}$$

Notice that $g_\varepsilon \to g$ defined in (I.8). Away from its singular points, u^ε admits locally an H^1-lifting $\varphi_\varepsilon \in \mathbb{R}$ i.e. $u^\varepsilon = e^{i\varphi_\varepsilon}$, by the result of [BZ]. The vector field $V_\varepsilon = \nabla\varphi_\varepsilon$ is curl-free in $\Omega\backslash \cup_i \{a_i\}$. Let also d_i^ε be the degree associated to a_i^ε, i.e. the winding number of φ_ε around a_i.

The approach to the lower bound is again to start with (I.16). The idea of [RS1] was to write

$$\nabla\varphi_\varepsilon \cdot H_\varepsilon = \nabla\varphi_\varepsilon \cdot (H_\varepsilon + u^\varepsilon) - \nabla\varphi_\varepsilon \cdot u^\varepsilon = -\nabla\varphi_\varepsilon \cdot \nabla^\perp g_\varepsilon - \text{curl } u^\varepsilon. \tag{I.18}$$

Now in the presence of vortices, we have

$$-\nabla\varphi_\varepsilon \cdot \nabla^\perp g_\varepsilon - \text{curl } u^\varepsilon = \text{curl } (g_\varepsilon V_\varepsilon - u^\varepsilon) - 2\pi \sum_i d_i^\varepsilon g_\varepsilon(a_i^\varepsilon)\delta_{a_i^\varepsilon} \tag{I.19}$$

where δ_{a_i} is the Dirac mass at the point a_i. If the vortices (a_i, d_i) are fixed for example, one may write

$$\begin{aligned} \liminf_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon) \ge \liminf_{\varepsilon\to 0} \int_\Omega |\nabla\varphi_\varepsilon \cdot H_\varepsilon| \;&\ge\; \int_\Omega |\text{curl } (g_\varepsilon V_\varepsilon - u^\varepsilon) - 2\pi \sum_i d_i g_\varepsilon(a_i)\delta_{a_i}| \\ &= \int_{\Omega\backslash\cup_i\{a_i\}} |\text{curl } (g_\varepsilon V_\varepsilon - u^\varepsilon)|. \end{aligned} \tag{I.20}$$

In the case where g has a constant sign in Ω, then $u\cdot\tau$ does not change sign on $\partial\Omega$ and formally passing to the limit $\varepsilon \to 0$,

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon) \ge \liminf_{\varepsilon\to 0} \left| \int_\Omega \text{curl } (g_\varepsilon V_\varepsilon - u^\varepsilon) - 2\pi \sum_i d_i g_\varepsilon(a_i)\delta_{a_i} \right| = \left| |\partial\Omega| - 2\pi \sum_i d_i g(a_i) \right|. \tag{I.21}$$

Thus the measure curl $(g\nabla\varphi - u)\lfloor_{\Omega\backslash\{a_i\}}$ plays the same role as the measure div $(\varphi u + u^\perp)$ in [RS1]. It carries a cost of the jumps of u and φ along the singular lines of u. But contrarily to the case of the S^1 constraint, some jumps of 2π of φ have now been removed by the presence of vortices. This can be made rigorous in the case of prescribed vortices for example. Nevertheless, we are still able to get a rigorous result without any assumption on the vortices.

Let us introduce a few more notations. $D_\varepsilon(t)$ will be the total degree of u^ε on the level-set $\{g_\varepsilon = t\}$, i.e.

$$D_\varepsilon(t) = \sum_{i/g_\varepsilon(a_i^\varepsilon)\geq t} d_i^\varepsilon \qquad \in \mathbb{Z}, \tag{I.22}$$

and

$$M_\varepsilon(t) = 2\pi \int_0^t D_\varepsilon(s)\, ds. \tag{I.23}$$

We also define

$$p_\varepsilon(t) = \int_{\{g_\varepsilon=t\}\cap\Omega} |\nabla g_\varepsilon| \tag{I.24}$$

(extended by 0 for $t \geq \max\, g_\varepsilon$ or $t \leq \min\, g_\varepsilon$). We denote

$$p(t) = \mathcal{H}^1(\{g = t\}) \tag{I.25}$$

extended by the value 0 for $t > \max\, g$, and $|\partial\Omega| = \mathcal{H}^1(\partial\Omega)$ for $t < 0$. In fact, p is in L^1 and possibly has jumps. We will show $p_\varepsilon \rightharpoonup p$ in $L^1(\mathbb{R}^+)$. With an abuse of notation, we define the BV norm of a function of $\mathbb{R}$ by

$$\|f\|_{BV} = \int_{\mathbb{R}} |f'| := \sup_{\substack{\zeta\in C_0^\infty(\mathbb{R})\\ |\zeta|\leq 1}} \int_{\mathbb{R}} \zeta' f.$$

Theorem 3 *Assume $E_\varepsilon(u^\varepsilon) \leq C$, $u^\varepsilon \to u$ in $\cap L^p(\Omega)$ and $g \geq 0$ in Ω. Then, as $\varepsilon \to 0$,*

$$p_\varepsilon + M_\varepsilon \text{ is bounded in } BV_{loc}(\mathbb{R}^+). \tag{I.26}$$

Extracting subsequences if necessary, there exist $M \in L^1(\mathbb{R}^+)$ and $D = \dfrac{M'}{2\pi}$ in $W^{-1,1}(\mathbb{R}^+)$ such that

$$p_\varepsilon + M_\varepsilon \rightharpoonup p + M \quad \text{in } BV_{loc}(\mathbb{R}^+) \tag{I.27}$$

i.e., denoting by $\mathcal{M}(\mathbb{R}^+)$ the dual of $C_0^0(\mathbb{R}^+)$,

$$p'_\varepsilon + 2\pi D_\varepsilon \rightharpoonup p' + 2\pi D \quad \text{in } \mathcal{M}(\mathbb{R}^+). \tag{I.28}$$

Moreover,

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon) \geq E_0(u, D) = \|p + M\|_{BV(\mathbb{R}^+)} = \int_0^{+\infty} |p' + 2\pi D|. \tag{I.29}$$

This lower bound is obtained by integrating the quantity $\nabla^\perp g_\varepsilon \cdot \nabla\varphi_\varepsilon + \operatorname{curl} u^\varepsilon$ over regions $\{t \le g_\varepsilon \le t+dt\}$. Thus we reduce to an integral over $\mathbb{R}$, the parameter t being the value of g, is a relevant parameter, even though it only gives information transversally to the level-sets of g.

The limiting energy E_0 does not depend only on u, but on the couple (u, D). It appears that it is the quantity $D(t)$, rather than the vortices themselves, which is relevant. In addition, let us point out that D is not necessarily integer-valued, because vortices of opposite signs can accumulate (counter-examples can be easily constructed); D is only a bounded measure. If however limiting vortices (a_i, d_i) can be defined, e.g. if they are fixed a priori, then $D \in \mathbb{Z}$ and $D(t) = \sum_{i/g(a_i)\ge t} d_i$ and, going back to the lower bound (I.29), one may write

$$
\begin{aligned}
\int_0^{+\infty} |p' + 2\pi D| &\ge \left|\int_0^{+\infty} p' + 2\pi D\right| = \left|p(\infty) - p(0) + 2\pi \sum_i d_i g(a_i)\right| \\
&\ge \left||\partial\Omega| - 2\pi \sum_i d_i g(a_i)\right|, \qquad \text{(I.30)}
\end{aligned}
$$

using the fact that $g = 0$ on $\partial\Omega$ hence $p(0) \ge |\partial\Omega|$; hence one recovers the formula (I.21).

Conversely, taking $u = u_* = -\nabla^\perp g_0$ where $g_0 = \operatorname{dist}(., \partial\Omega)$, and putting one vortex a of degree 1 at a point of maximum of g_0, one would get equality in formula (I.30) and an energy $|\partial\Omega| - 2\pi g_0(a) = |\partial\Omega| - 2\pi \max_\Omega g$. It follows that

$$
\limsup_{\varepsilon\to 0} \min_{H^1(\Omega,S^2)} E_\varepsilon \le |\partial\Omega| - |B|, \qquad \text{(I.31)}
$$

where $|B|$ is the perimeter of a maximal disc B inscribed in Ω. This provides an upper bound which is already smaller than $|\partial\Omega|$, that of [RS1] and of pure S^1 configurations. If the domain is not convex, it is more interesting to put vortices of degree +1 at local maxima of g_0 and vortices of degree -1 at saddle points of g_0 in such a way that the total degree on each connected component of a level-set of g is 1.

The question of finding and describing an optimal configuration (which is probably not unique), i.e. one which minimizes the limiting energy $\int_{\Sigma_u} A(X)$, is open and seems difficult in general. This is another main difference with the purely S^1-case of [RS1], where u_* was a minimizer. We refer to Section IV.1 for more examples and counter-examples.

The paper is organized as follows : Section II contains the proof of Theorem 1, Section III the proof of Theorem 3. Both these sections are more analytical and may be skipped in a first reading. Section IV.1 contains examples and Section IV.2 the construction of optimal cross-tie wall profiles.

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II The limiting jump energy

We consider any family $u^\varepsilon \in H^1(\Omega, S^2)$ such that $E_\varepsilon(u^\varepsilon) \le C$ and wish to prove convergence and lower bounds. Here, ε will stand for any sequence ε_n going to 0.

Lemma II.1 *Let u^ε be in $H^1(\Omega, S^2)$ such that $E_\varepsilon(u^\varepsilon) \le C$. There exists $u \in \cap_{p<\infty} L^p(\Omega, S^1)$ and $g \in \cap_{p<\infty} W_0^{1,p}(\Omega)$ such that (up to extraction)*

$$\text{(II.1)} \qquad \begin{gathered} u^\varepsilon \to u \quad \textit{in } \cap_{p<\infty} L^p(\Omega) \\ \begin{cases} u = -\nabla^\perp g & \textit{in } \Omega \quad |\nabla g| = 1 \\ g = 0 & \textit{on } \partial\Omega \end{cases} \end{gathered}$$

(i.e. the limiting u is a divergence-free, S^1-valued vector field tangent to $\partial\Omega$.)

Proof: As written in the introduction (equation (I.5)), H is defined by

$$\text{(II.2)} \qquad \begin{cases} \partial_1(u_1\mathbf{1}_\Omega + H_1) + \partial_2(u_2\mathbf{1}_\Omega + H_2) = 0 \\ \partial_1 H_2 - \partial_2 H_1 = 0, \end{cases}$$

and we have the standard estimate

$$\text{(II.3)} \qquad \forall 1 < p < \infty,\ \exists C_p \quad \|H_\varepsilon\|_{L^p(\mathbb{R}^2)} \le C\|u^\varepsilon\|_{L^p(\Omega)} \le C_p.$$

Also $H_\varepsilon \to 0$ in $L^2(\mathbb{R}^2)$ hence $H_\varepsilon \to 0$ in $\cap_p L^p(\mathbb{R}^2)$.

We can write a two-dimensional Hodge decomposition of $u_{||}^\varepsilon \mathbf{1}_\Omega$ (where $u_{||} = (u_1, u_2)$):

$$\text{(II.4)} \qquad u_{||}^\varepsilon \mathbf{1}_\Omega = -H_\varepsilon - K_\varepsilon$$

with $K_\varepsilon = \nabla^\perp g_\varepsilon$ a divergence-free vector-field in $\cap_{p<\infty} L^p(\mathbb{R}^2)$ and $\|K_\varepsilon\|_{L^2(\mathbb{R}^2)} \le C$, $\|g_\varepsilon\|_{H^1(\mathbb{R}^2)} \le C$. Then,

$$\|\Delta g_\varepsilon\|_{L^2(\Omega)} = \|\mathrm{curl}\ u_{||}^\varepsilon\|_{L^2(\Omega)} \le C\|\nabla u^\varepsilon\|_{L^2(\Omega)}.$$

Let now Ω' be any open subset whose closure is in Ω. By elliptic regularity,

$$\|g_\varepsilon\|_{H^2(\Omega')} \le C\|\nabla u^\varepsilon\|_{L^2(\Omega)} \le \frac{C}{\sqrt{\varepsilon}}$$

in view of the energy upper bound $E_\varepsilon(u^\varepsilon) \le C$. Thus,

$$\text{(II.5)} \qquad \|\nabla K_\varepsilon\|_{L^2(\Omega')} \le \frac{C}{\sqrt{\varepsilon}}.$$

On the other hand,

$$||K_\varepsilon| - 1| = \left| |H_\varepsilon + u_{||}^\varepsilon| - \frac{|u_{||}^\varepsilon|}{|u_{||}^\varepsilon|} \right| \le |H_\varepsilon| + \left| u_{||}^\varepsilon - \frac{u_{||}^\varepsilon}{|u_{||}^\varepsilon|} \right| = |H_\varepsilon| + ||u_{||}^\varepsilon| - 1|,$$

hence

$$\|1-|K_\varepsilon|\|_{L^2(\Omega)} \le \|H_\varepsilon\|_{L^2(\Omega)} + \||u_{||}^\varepsilon|-1\|_{L^2(\Omega)}.$$

But we also have

$$\int_\Omega ||u_{||}^\varepsilon|-1|^2 \le C\int_\Omega |u_3^\varepsilon|^2 \le Cc_\varepsilon \le C\varepsilon, \tag{II.6}$$

in view of the energy bound, so

$$\|1-|K_\varepsilon|\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}. \tag{II.7}$$

Combining this with (II.5), we obtain

$$\|\nabla K_\varepsilon\|_{L^2(\Omega')}\|1-|K_\varepsilon|\|_{L^2(\Omega')} \le C,$$

while div $K_\varepsilon = 0$. The result of [JP] allows to conclude that K_ε is compact in $\cap_{p<\infty}L^p(\Omega')$. In the meantime, $H_\varepsilon \to 0$ in $\cap_p L^p(\mathbb{R}^2)$ hence from (II.4), $u_{||}^\varepsilon$ is also compact in $\cap_p L^p(\Omega')$. From the energy bound $\int_\Omega |u_3^\varepsilon|^2 \to 0$ while $|u_3^\varepsilon| \le 1$, thus we can say that $\|u_3^\varepsilon\|_{L^p(\Omega)} \to 0$ for all $p<\infty$; consequently u^ε is also compact in $\cap_{p<s\infty}L^p(\Omega')$ and converges to some u. Then, up to extraction, $u^\varepsilon \to u$ a.e. in Ω and $|u^\varepsilon|$ is bounded by 1, hence by Lebesgue's theorem $u^\varepsilon \to u$ in $L^p(\Omega)$. u is S^1-valued because $u_3^\varepsilon \to 0$ in $\cap L^p$ and thus a.e.
Passing to the limit in (II.2), we get that

$$\text{div }(u\mathbf{1}_\Omega) = 0 \quad \text{in } W^{-1,p}(\mathbb{R}^2),$$

and thus we can find $g \in W^{1,\infty}(\mathbb{R}^2)$ such that $u\mathbf{1}_\Omega = \nabla^\perp g$. g will be constant in $\mathbb{R}^2\backslash\Omega$ hence can be chosen to be zero in $\mathbb{R}^2\backslash\Omega$. This proves the lemma. □

We then have the following lemma, whose proof is postponed until the end of the section.

Lemma II.2 *Let u^ε be such that $E_\varepsilon(u^\varepsilon) \le C$. We can find $\tilde{u^\varepsilon} \in \cap_{1\le p<2}W^{1,p}(\Omega, S^1) \cap H^1(\Omega\backslash \cup_{i=1}^{n(\varepsilon)}\{a_i^\varepsilon\}, S^1)$ where for each ε, $\{a_1^\varepsilon, \cdots, a_{n(\varepsilon)}^\varepsilon\}$ is a finite set of points of Ω; and satisfying the following:*

$$\|\tilde{u^\varepsilon} - u^\varepsilon\|_{L^p(\Omega)} \to 0 \quad \forall p<\infty \tag{II.8}$$

$$\|\tilde{H}_\varepsilon - H_\varepsilon\|_{L^p(\Omega)} \to 0 \quad \forall p<\infty, \tag{II.9}$$

where $\tilde{H}_\varepsilon$ is the magnetic field generated by $\tilde{u^\varepsilon}$ according to (II.2), and

$$E_\varepsilon(u^\varepsilon) \ge \int_\Omega |H_\varepsilon||\nabla u_{||}^\varepsilon| = \int_\Omega |\tilde{H}_\varepsilon||\nabla \tilde{u^\varepsilon}| + o(1). \tag{II.10}$$

We then introduce notations for maps in $BV(\Omega, S^1)$. We recall the definition :

$$BV(\Omega, S^1) := \left\{u \in BV(\Omega, \mathbb{R}^2), \quad u(x) \in S^1 \text{ a.e. } x \in \Omega\right\}.$$

Recall that a map $u \in BV(\Omega, \mathbb{R}^2)$ is approximately continuous in the whole of Ω except on a 1-dimensional rectifiable subset of Ω called the *jump set* of u that we will denote here by Σ_u (see [EG] section 5.9).

For $\mathcal{H}^1-$a.e. $x \in \Sigma_u$ we denote by $\nu_u(x)$ the approximate unit normal for a given regular choice of orientation of the at most countable union of C^1 curves containing $\mathcal{H}^1$-almost the whole of Σ_u. Denote

$$H^{\pm}_{\nu_u} := \left\{y \in \mathbb{R}^2, \quad (y - x) \cdot \nu_u(x) \geq 0 \text{ (resp. } \leq 0)\right\}$$

Finally, recall that $\mathcal{H}^1-$a.e. point in Σ_u admits a left and right approximate limit along Σ_u : there exists two measurable functions u_+ and u_- on Σ_u such that for $\mathcal{H}^1-$a.e. point x in Σ_u we have

$$\lim_{r \to 0} \frac{1}{r^2} \int_{B_r(x) \cap H^{\pm}_{\nu_u}} |u(y) - u_{\pm}(x)| \, dy = 0$$

That the following bound holds

$$\int_{\Sigma_u} |u_+ - u_-| \, d\mathcal{H}^1 \leq \sum_{i=1}^{2} \int_{\Sigma_u} |u^i_+ - u^i_-| \, d\mathcal{H}^1 \leq \sum_{i=1}^{2} \|u_i\|_{BV} \leq \sqrt{2}\|u\|_{BV} \tag{II.11}$$

We recall that X is the measurable function from Σ_u into $[0, \pi/2]$ given by

$$X = \frac{1}{2} \min \left|\widehat{(u_{\pm}, u_{\mp})}\right|.$$

We shall now establish the following result.

Theorem 1 *Let $\varepsilon_n \to 0$ and u^{ε_n} be sequences of maps from Ω into S^2 satisfying $E^{\varepsilon_n}(u^{\varepsilon_n}) \leq C$. Then, up to extraction, u^{ε_n} converges in $\cap_{p<\infty} L^p(\Omega)$ to some limit u which satisfies (I.7). If in addition $u \in BV(\Omega, S^1)$, then*

$$\liminf_{n \to \infty} E_{\varepsilon_n}(u^{\varepsilon_n}) \geq \int_{\Sigma_u} A(X) d\mathcal{H}^1 \tag{II.12}$$

where

$$\begin{aligned} A(X) &= 2|\sin X - X \cos X| && \textit{for } X \in [0, \frac{\pi}{4}] \\ A(X) &= 2\left|\left(X - \tfrac{\pi}{2}\right) \cos X - \sin X + \sqrt{2}\right| && \textit{for } X \in [\frac{\pi}{4}, \frac{\pi}{2}]. \end{aligned}$$

Proof : ε_n will be simply denoted ε.
-*Step 1:* Using the result of Lemma II.2, we have

$$E_\varepsilon(u^\varepsilon) \geq \int |\tilde{H}_\varepsilon||\nabla \tilde{u^\varepsilon}| - o(1).$$

We introduce

$$\begin{cases} g(s) = s & \text{for } s \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \\ g(s) = \frac{\pi}{2} - s & \text{for } s \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right], \end{cases}$$

and complete it to $\mathbb{R}^2$ into a continuous π-periodic function. Observe that $|g'| \leq 1$. We then define for $\sigma = e^{is_0} \in S^1$,

$$f_\sigma(e^{is}) = g(s - s_0),$$

(again $|f'| \leq 1$), and the vector-field

$$\text{(II.13)} \qquad V_\sigma(t) = \begin{pmatrix} -\displaystyle\int_{s_0}^{t} g'(\tau - s_0)\cos\tau\, d\tau = -\int_0^{t-s_0} g'(\tau)\cos(\tau + s_0)\, d\tau \\ -\displaystyle\int_{s_0}^{t} g'(\tau - s_0)\sin\tau\, d\tau = -\int_0^{t-s_0} g'(\tau)\sin(\tau + s_0)\, d\tau \end{pmatrix}.$$

Observe that $\int g'\cos(\tau + s_0)$ and $\int g'\sin(\tau + s_0)$ are 2π-periodic, therefore V_σ is a 2π-periodic vector-field, hence can be considered as a map from S^1 to $\mathbb{R}^2$. f_σ and V_σ are constructed so that for any *smooth* map v into S^1 we have

$$\text{(II.14)} \qquad \text{div } (V_\sigma(v)) + \nabla(f_\sigma(v)) \cdot v = 0$$

We claim that for any $\sigma = e^{is_0} \in S^1$, we have

$$\text{(II.15)} \qquad \text{div } \Big(f_\sigma(\tilde{u}^\varepsilon)\ (\tilde{u}^\varepsilon + \tilde{H}_\varepsilon) + V_\sigma(\tilde{u}^\varepsilon)\Big) = \frac{f'_\sigma(\tilde{u}^\varepsilon)}{i\tilde{u}^\varepsilon}\nabla\tilde{u}^\varepsilon \cdot \tilde{H}_\varepsilon \quad \text{in } \mathcal{D}'(\Omega).$$

Then, assuming (II.15), since $|\frac{f'_\sigma(\tilde{u}^\varepsilon)}{i\tilde{u}^\varepsilon}| \leq 1$, we deduce that
(II.16)

$$\forall \sigma \in S^1, \quad E_\varepsilon(u^\varepsilon) \geq \int_\Omega |\nabla\tilde{u^\varepsilon}||\tilde{H}_\varepsilon| - o(1) \geq \int_\Omega \left|\text{div } \Big(f_\sigma(\tilde{u}^\varepsilon)\, (\tilde{u}^\varepsilon + \tilde{H}_\varepsilon) + V_\sigma(\tilde{u}^\varepsilon)\Big)\right| - o(1),$$

which replaces the lower bound $\int_\Omega |\text{div } (\varphi u + u^\perp)|$ in the S^1-valued case. The result will follow by passing to the limit $\varepsilon \to 0$.

- *Step 2 :* Proof of (II.15). Observe first that both sides of the equality have a meaning since, by construction (Lemma II.1), $\tilde{H}_\varepsilon \in \cap_{p<+\infty} L^p$ and $\nabla\tilde{u}^\varepsilon \in L^q$ for any $q < 2$. Because of (II.14), since smooth maps are strongly dense in $W^{1,2}(\Omega', S^1)$, where Ω' is an open simply connected subset of Ω containing no vortices of $\tilde{u}^\varepsilon$, we deduce that (II.15) holds once we replace Ω by such an Ω'. Since $\tilde{u}^\varepsilon$ has only finitely many vortices $a_1^\varepsilon \cdots a_{n(\varepsilon)}^\varepsilon$, the distribution

$$\text{(II.17)} \qquad \text{div } \Big(f_\sigma(\tilde{u}^\varepsilon)\ (\tilde{u}^\varepsilon + \tilde{H}_\varepsilon) + V_\sigma(\tilde{u}^\varepsilon)\Big) - f'_\sigma(\tilde{u}^\varepsilon)\ (i\tilde{u}^\varepsilon)^{-1}\nabla\tilde{u}^\varepsilon \cdot \tilde{H}_\varepsilon$$

supported at the isolated points a_j^ε is a combination of $\partial^k\delta_{a_j}$ and since

$$\text{div } \Big(f_\sigma(\tilde{u}^\varepsilon)\ (\tilde{u}^\varepsilon + \tilde{H}_\varepsilon) + V_\sigma(\tilde{u}^\varepsilon)\Big) \in W^{-1,p}$$

for any $1 < p < +\infty$, it has to vanish and (II.15) is proved.

- *Step 3:* Clean-up of Σ_u. Let $\delta > 0$. The rectifiability property of Σ_u means that $\Sigma_u \subset \cup_{k=1}^{+\infty} \Gamma^k \cup \Gamma^0$ where Γ^k for $k \geq 1$ are C^1 embedded curves and $\mathcal{H}^1(\Gamma^0) = 0$. Choose an integer N such that

$$\text{(II.18)} \qquad \int_{\Sigma_u \cap \cup_{k=N+1}^{+\infty} \Gamma^k} |u_+ - u_-| \, d\mathcal{H}^1 \leq \delta$$

We may always assume that $(\overline{\Gamma}^k)_{k=1\cdots N}$ is made of disjoint embedded C^1 curves. Since u is left and right approximately continuous $\mathcal{H}^1$ a.e. on Σ_u, and since u is also approximately continuous $\mathcal{H}^1$ a.e. on $\Omega \setminus \Sigma_u$ we extend the $\mathcal{H}^1$ measurable functions u_+ and u_- on $\Sigma_u \cap \cup_{k=1\cdots N} \overline{\Gamma}^k$ by the approximate limit of u on the whole of $\cup_{k=1\cdots N} \overline{\Gamma}^k$.

On $\cup_{k=1\cdots N} \overline{\Gamma}^k$ we consider the subset L^N of Lebesgue points for u_+ and u_- ($\mathcal{H}^1(\cup_{k=1\cdots N} \overline{\Gamma}^k \setminus L^N) = 0$). Let E^N be a compact subset of L^N such that

$$\text{(II.19)} \qquad \int_{E^N} |u_+ - u_-| \, d\mathcal{H}^1 \geq \int_{\cup_{k=1\cdots N} \overline{\Gamma}^k} |u_+ - u_-| \, d\mathcal{H}^1 - \delta$$

The existence of such a compact set for the given Radon measure $|u_+ - u_-| \, d\mathcal{H}^1 \lfloor L^N$ is given by the standard approximation of measurable sets by compact sets (see for instance [EG] section 1.1). To any $x \in E^N$ we assign a positive radius ρ_x such that

$$\text{(II.20)} \qquad \frac{1}{|\cup_{k=1\cdots N} \overline{\Gamma}^k \cap B_{\rho_x}(x)|} \int_{\cup_{k=1\cdots N} \overline{\Gamma}^k \cap B_{\rho_x}(x)} |u_\pm(y) - u_\pm(x)| \, d\mathcal{H}^1 \leq \frac{\delta}{\mathcal{H}^1(\cup_{k=1}^N \overline{\Gamma}^k)}$$

(by definition of a Lebesgue point). From $(B_{\rho_x}(x))_{x \in E^N}$ we may extract a Besicovitch subcovering of E^N $(B_{\rho_j}(x_j))$ made of at most countably many balls, each of them intersecting at most C_0 other balls, where C_0 is a universal number (see [EG] section 1.5.2). Observe that here this covering has to be finite since E^N is compact. Consider a partition of unity $(\phi_j)_j$ modelled on this subcovering so that $\phi_j \in C_0^\infty(B_{\rho_j}(x_j))$, $\phi_j \geq 0$ and $\sum_j \phi_j \equiv 1$ on E^N, and π the projection from a tubular neighborhood of $\cup_k \overline{\Gamma}^k$ to $\cup_k \overline{\Gamma}^k$. We thus have reduced to the case of a smooth curve.

- *Step 4:* We choose σ in $supp(\phi_i)$ to be the unit normal $\nu(x_j)$ at x_j to the oriented set of curves $\cup_k \overline{\Gamma}^k$ so that $\nu(x_j)$ belongs to the smallest arc joining $u_+(x_j)$ and $u_-(x_j)$. Then, from (II.16),

$$\text{(II.21)} \qquad E_\varepsilon(u_\varepsilon) \geq \sum_j \int_\Omega \phi_j(\pi(x)) \left| \text{div} \, \left(f_{\nu(x_j)}(\tilde{u}^\varepsilon) \; (\tilde{u}^\varepsilon + \tilde{H}_\varepsilon) + V_{\nu(x_j)}(\tilde{u}^\varepsilon) \right) \right| - o(1)$$

Since $\tilde{u^\varepsilon} \to u$ in $\cap_p L^p(\Omega)$,

$$f_{\nu(x_j)}(\tilde{u}^\varepsilon) \; (\tilde{u}^\varepsilon + \tilde{H}_\varepsilon) + V_{\nu(x_j)}(\tilde{u}^\varepsilon) \to f_{\nu(x_j)}(u)u + V_{\nu(x_j)}(u) \quad \text{in } \cap_p L^p.$$

Thus,

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon) \geq \sum_j \int_\Omega \phi_j(\pi(x)) \left|\text{div } \left(f_{\nu(x_j)}(u)u + V_{\nu(x_j)}(u)\right)\right|. \tag{II.22}$$

Since u is assumed to be BV, and div $u = 0$, by the BV chain rule, we find that div $\left(f_{\nu(x_j)}(u)u + V_{\nu(x_j)}(u)\right)$ is supported only in Σ_u (see (II.14)) and

$$\begin{aligned} &\text{(II.23)}\quad \text{div } \left(f_{\nu(x_j)}(u)u + V_{\nu(x_j)}(u)\right) \\ &= \left(f_{\nu(x_j)}(u^+(x))u^+(x) + V_{\nu(x_j)}(u^+(x)) - f_{\nu(x_j)}(u^-(x))u^-(x) - V_{\nu(x_j)}(u^-(x))u^-(x)\right)\cdot\nu(x)\mathcal{H}^1\lfloor_{\Sigma_u}. \end{aligned}$$

But in view of (II.20),

$$\begin{aligned} \sum_j \int_{\Sigma_u} \phi_j(\pi(x))| &\left(f_{\nu(x_j)}(u^\pm(x))u^\pm(x) + V_{\nu(x_j)}(u^\pm(x))\right)\cdot\nu(x) \\ &- \left(f_{\nu(x_j)}(u^\pm(x))u^\pm(x) + V_{\nu(x_j)}(u^\pm(x))u^\pm(x)\right)\cdot\nu(x_j)| \leq C\delta. \end{aligned} \tag{II.24}$$

On the other hand, a simple computation yields the crucial relation

$$\begin{aligned} &| \left(f_{\nu(x_j)}(u^+)(x_j)\ u^+(x_j) + V_{\nu(x_j)}(u^+)(x_j)\right)\cdot\nu(x_j) \\ &- \left(f_{\nu(x_j)}(u^-)(x_j)\ u^-(x_j) + V_{\nu(x_j)}(u^-)(x_j)\right)\cdot\nu(x_j)| = A(X(x_j)). \end{aligned} \tag{II.25}$$

Indeed, in the orthonormal frame to Σ_u at x_j, the situation is that of a jump from $-X(x_j)$ to $X(x_j)$ with $\nu(x_j) = e^{i0}$. If $X \leq \frac{\pi}{4}$, $f(u_\pm) = \pm X$ and $V(u_\pm) = u_\pm^\perp$ with $u_\pm = (\cos\pm X, \sin X)$, hence the jump is

$$X\cos X - \sin X + X\cos X - \sin X = 2(X\cos X - \sin X).$$

If $X \in [\frac{\pi}{4}, \frac{\pi}{2}]$, then $f(u_+) = \frac{\pi}{2} - X$, $f(u_-) = -\frac{\pi}{2} + X$, $V(u_+) = -u_+^\perp + (-\sqrt{2}, \sqrt{2})$, $V(u_-) = -u_-^\perp + (\sqrt{2}, -\sqrt{2})$ and the cost is

$$2\left(\frac{\pi}{2} - X\right)\cos X + 2(\sin X - \sqrt{2}).$$

Combining (II.22)—(II.25), we obtain that

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \geq \int_{E^N} A(X) - C\delta \tag{II.26}$$

and using (II.19) and (II.18) we obtain the result. $\square$

We finish with the
Proof of Lemma II.2 :

We recall that we assume $c_\varepsilon \leq \varepsilon^{1+\delta}$ for some $\delta > 0$ arbitrarily small. For any $\alpha \in S^2$ such that $\alpha_3 > \frac{3}{4}$ we define a map π_a from S^2 to S^1 as follows:

(II.27)

if $\lvert x_3\rvert \leq \frac{1}{4}$	$\pi_\alpha(x) = \left(\dfrac{x_1}{(x_1^2+x_2^2)^{\frac{1}{2}}}, \dfrac{x_2}{(x_1^2+x_2^2)^{\frac{1}{2}}}\right)$
if $x_3 \geq \frac{1}{2}, x \neq \alpha$	π_α is the intersection with the equator of the unique geodesic joining α to the equator and containing x
if $x_3 \leq \frac{1}{2}, x \neq -\alpha$	π_α is the intersection with the equator of the unique geodesic joining $-\alpha$ to the equator and containing x

For $\frac{1}{4} \leq |x_3| \leq \frac{1}{2}$, π_α is extended in such a way that the following holds :

$$\text{for } x \neq \alpha, -\alpha, \quad |\nabla \pi_\alpha(x)| \leq \frac{C}{\inf(|x-\alpha|, |x+\alpha|)} \tag{II.28}$$

$$|x - \pi_\alpha(x)| \leq C x_3. \tag{II.29}$$

Let $p < 2$. For almost every α such that $\alpha_3 > \frac{3}{4}$, α is a regular value of u^ε i.e. $(u^\varepsilon)^{-1}(\alpha)$ is a finite set of points where the jacobian determinant of u^ε does not vanish. Using (II.28), and ω being a subset of Ω,

$$\begin{aligned}\int_{\alpha_3 > \frac{3}{4}} \int_\omega |\nabla(\pi_\alpha(u^\varepsilon(x)))|^p \, dx \, d\alpha &\leq C \int_{\alpha_3 > \frac{3}{4}} \int_\omega |D\pi_\alpha(u^\varepsilon)|^p |\nabla u^\varepsilon|^p(x) \, dx \, d\alpha \\ &\leq C \int_{\alpha_3 > \frac{3}{4}} \int_\omega \frac{|\nabla u^\varepsilon|^p}{|u^\varepsilon - \alpha|^p}.\end{aligned}$$

Using Fubini's theorem, this implies that

$$\begin{aligned}\int_{\alpha_3 > \frac{3}{4}} \int_\omega |\nabla(\pi_\alpha(u^\varepsilon(x)))|^p \, dx \, d\alpha &\leq C \int_\omega |\nabla u^\varepsilon(x)|^p \left(\int_{\alpha \in S^2} \frac{d\alpha}{|u^\varepsilon(x) - \alpha|^p}\right) dx \\ &\leq C \int_\omega |\nabla u^\varepsilon|^p,\end{aligned}$$

(Indeed, since $p < 2$, $\int_{\alpha \in S^2} \frac{d\alpha}{|y-\alpha|^p}$ converges for every $y \in S^2$.) Using the mean-value theorem, it follows that there exists $\alpha^\varepsilon \in S^2$ such that $\alpha_3^\varepsilon > \frac{3}{4}$ and

$$\int_\omega |\nabla(\pi_\alpha(u^\varepsilon))|^p \leq C \int_\omega |\nabla u^\varepsilon|^p \tag{II.30}$$

and such that $(u^\varepsilon)^{-1}(\alpha^\varepsilon)$ is a finite set. We then set

$$\tilde{u^\varepsilon}(x) = \pi_\alpha(u^\varepsilon(x)). \tag{II.31}$$

$\tilde{u^\varepsilon}$ is S^1-valued and has the same regularity as u^ε i.e. H^1 except for $x \in (u^\varepsilon)^{-1}(\alpha^\varepsilon)$. We denote by $a_1, \cdots, a^\varepsilon_{n(\varepsilon)}$ the set of singular points of $\tilde{u^\varepsilon}$ i.e. $(u^\varepsilon)^{-1}(\alpha^\varepsilon)$. Since $\tilde{u^\varepsilon}$ is H^1 away from these points, d_i^ε, the degree of $\tilde{u^\varepsilon}$ at a_i^ε is well-defined. Moreover, as soon as $|u^\varepsilon(x)| \le \frac{1}{4}$, $\tilde{u^\varepsilon} = \frac{u_{\|}^\varepsilon(x)}{|u_{\|}^\varepsilon(x)|}$ ($u_{\|}$ denotes (u_1,u_2)) hence $\tilde{u^\varepsilon}$ has the same degree as $\tilde{u^\varepsilon} = \frac{u_{\|}^\varepsilon(x)}{|u_{\|}^\varepsilon(x)|}$ outside of the set $\{|u^\varepsilon(x)| \le \frac{1}{4}\}$ which is a small set: in view of the energy bound $\frac{1}{c_\varepsilon}\int_\Omega |u_3^\varepsilon|^2 \le C$, we have

$$\text{vol}\left(\left\{|u^\varepsilon(x)| \le \frac{1}{4}\right\}\right) \le O(c_\varepsilon). \tag{II.32}$$

Let us prove that for some $p>1$, we have

$$\int_\Omega |\nabla(\tilde{u^\varepsilon} - u_{\|}^\varepsilon)|^p \to 0. \tag{II.33}$$

Indeed, in view of (II.30),

$$\int_{|u_3^\varepsilon|\ge\frac{1}{4}} |\nabla(\tilde{u^\varepsilon} - u_{\|}^\varepsilon)|^p \le C\int_{|u_3^\varepsilon|\ge\frac{1}{4}} |\nabla\tilde{u^\varepsilon}|^p + |\nabla u^\varepsilon|^p \le C\int_{|u_3^\varepsilon|\ge\frac{1}{4}} |\nabla u^\varepsilon|^p.$$

Using Hölder's inequality,

$$\int_{|u_3^\varepsilon|\ge\frac{1}{4}} |\nabla u^\varepsilon|^p \le \text{vol}\left(\{|u^\varepsilon(x)| \le \frac{1}{4}\}\right)^{1-\frac{p}{2}} \|\nabla u^\varepsilon\|_{L^2(\Omega)}^p.$$

But in view of the energy bound, we have $\|\nabla u^\varepsilon\|_{L^2(\Omega)} \le \frac{C}{\varepsilon^{\frac{1}{2}}}$ and, using (II.32), we are led to

$$\int_{|u_3^\varepsilon|\ge\frac{1}{4}} |\nabla u^\varepsilon|^p \le O\left(\frac{c_\varepsilon^{1-\frac{p}{2}}}{\varepsilon^{\frac{p}{2}}}\right).$$

Since $c_\varepsilon \le \varepsilon^{1+\delta}$, this implies that we can choose $p-1>0$ small enough compared to δ so that

$$\int_{|u_3^\varepsilon|\ge\frac{1}{4}} |\nabla(\tilde{u^\varepsilon} - u_{\|}^\varepsilon)|^p \le o(1). \tag{II.34}$$

Next, we bound $\int_{|u_3^\varepsilon|\le\frac{1}{4}} |\nabla(\tilde{u^\varepsilon} - u_{\|}^\varepsilon)|^p$. Recalling that $\tilde{u^\varepsilon} = \frac{u_{\|}^\varepsilon(x)}{|u_{\|}^\varepsilon(x)|}$ in the set $\{|u_3^\varepsilon| \le \frac{1}{4}\}$, we have

$$|\nabla(\tilde{u_1^\varepsilon} - u_1^\varepsilon)| = \left|\nabla\left(\frac{u_1^\varepsilon}{\sqrt{(u_1^\varepsilon)^2 + (u_2^\varepsilon)^2}} - u_1^\varepsilon\right)\right| = \left|\nabla\left(u_1^\varepsilon\left(\frac{1}{\sqrt{1-(u_3^\varepsilon)^2}} - 1\right)\right)\right|.$$

We observe that in $\{|u_3^\varepsilon| \le \frac{1}{4}\}$,

$$\left|\frac{1}{\sqrt{1-(u_3^\varepsilon)^2}} - 1\right| \le C|u_3^\varepsilon|^2 \quad\text{and}\quad \left|\nabla\frac{1}{\sqrt{1-(u_3^\varepsilon)^2}}\right| \le C|u_3^\varepsilon||\nabla u_3^\varepsilon|.$$

Thus, in $\{|u_3^\varepsilon| \le \frac{1}{4}\}$,

$$|\nabla(\tilde{u_1^\varepsilon} - u_1^\varepsilon)| \le C(|u_3^\varepsilon|^2 |\nabla u_1^\varepsilon| + |u_3^\varepsilon||\nabla u_3^\varepsilon|) \le C|u_3^\varepsilon||\nabla u^\varepsilon|.$$

Similarly, $|\nabla(u_2^\varepsilon - \tilde{u_2^\varepsilon})| \le C|u_3^\varepsilon||\nabla u^\varepsilon|$. Hence,

$$\begin{aligned}
\int_{|u_3^\varepsilon|\le\frac{1}{4}} |\nabla(\tilde{u^\varepsilon} - u_{||}^\varepsilon)|^p &\le C\int_\Omega |u_3^\varepsilon|^p |\nabla u^\varepsilon|^p \\
&\le C\left(\int_\Omega |u_3^\varepsilon|^2\right)^{1-\frac{p}{2}} \left(\int_\Omega |\nabla u^\varepsilon|^2\right)^{\frac{p}{2}} \\
&\le C\frac{c_\varepsilon^{1-\frac{p}{2}}}{\varepsilon^{\frac{p}{2}}} \qquad \text{(II.35)}
\end{aligned}$$

where we have used again Hölder's inequality and the energy upper bound. Choosing again $p-1>0$ small enough, we can ensure (combining (II.34) and (II.35)) that (II.33) holds.

Then, from (II.29), we have for $2 \le q < \infty$,

$$\int_\Omega |\tilde{u^\varepsilon} - u^\varepsilon|^q \le C\int_\Omega |u_3^\varepsilon|^q \le C\int_\Omega |u_3^\varepsilon|^2 \le O(c_\varepsilon).$$

Hence, for all $2 \le q < \infty$, $\|\tilde{u^\varepsilon} - u^\varepsilon\|_{L^q(\Omega)} \le O(c_\varepsilon^{q})$ which proves (II.8). In addition, we deduce the estimate

$$\|H_\varepsilon - \tilde{H}_\varepsilon\|_{L^q(\mathbb{R}^2)} \le C\|\tilde{u^\varepsilon} - u^\varepsilon\|_{L^q(\Omega)} \le O(c_\varepsilon^{q}) \qquad \text{(II.36)}$$

where $\tilde{H}_\varepsilon$ is the demagnetizing field induced by $\tilde{u^\varepsilon}$ (see (II.2)). Thus, in view of (II.33),

$$\int_\Omega |\nabla(\tilde{u^\varepsilon} - u_{||}^\varepsilon)||\tilde{H}_\varepsilon| \le \|\nabla(\tilde{u^\varepsilon} - u_{||}^\varepsilon)\|_{L^p(\Omega)}\|\tilde{H}_\varepsilon\|_{L^q(\mathbb{R}^2)} \le C\|\nabla(\tilde{u^\varepsilon} - u_{||}^\varepsilon)\|_{L^p(\Omega)} = o(1)$$

while, using (II.36) and the energy bound

$$\int_\Omega |\nabla u_{||}^\varepsilon||H_\varepsilon - \tilde{H}_\varepsilon| \le \|\nabla u^\varepsilon\|_{L^2(\Omega)}\|H_\varepsilon - \tilde{H}_\varepsilon\|_{L^2} \le C\left(\frac{c_\varepsilon}{\varepsilon}\right)^{\frac{1}{2}} \to 0.$$

We conclude that

$$\int_\Omega |\nabla u_{||}^\varepsilon||H_\varepsilon| = \int_\Omega |\nabla\tilde{u^\varepsilon}||\tilde{H}_\varepsilon| + o(1).$$

□

Remark II.1: It is not possible to define vortices of u^ε directly because, for example, u_3^ε can be equal to 1 on a subset of Ω of nonempty interior. The construction of $\tilde{u^\varepsilon}$ allows to isolate points next to which u^ε passes through the North or South pole, the vortices which are relevant being those with nonzero degree, i.e. those which are topological. This definition of vortices (and the choice of $\tilde{u^\varepsilon}$) may not seem intrinsic since they depend on a choice of the point α^ε. However, the vortex-locations can only vary very little in space (on an $o(1)$ scale) and their degree is intrinsic.

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III Lower bounds involving the vortices

Let $u^\varepsilon \in H^1(\Omega, S^2)$ be such that $E_\varepsilon(u^\varepsilon) \leq C$ and $u^\varepsilon \to u$ in $\cap_p L^p$. With Lemma II.2, we may use (II.10) to start bounding the energy from below and hence, replacing u^ε by $\tilde{u^\varepsilon}$, work on $\int_\Omega |\tilde{H}_\varepsilon||\nabla \tilde{u^\varepsilon}|$. Then, we can define $g_\varepsilon \in \cap_{p<\infty} W^{1,p}(\mathbb{R}^2)$ such that

$$\tilde{u^\varepsilon}\mathbf{1}_\Omega + \tilde{H}_\varepsilon = -\nabla^\perp g_\varepsilon.$$

Since $u^\varepsilon \to u$ strongly in $\cap_{p<\infty} L^p(\Omega)$ and $H_\varepsilon \to 0$ too, in view of (II.8) and (II.9) we have $\tilde{u^\varepsilon} \to u$ strongly in $\cap_{p<\infty} L^p(\Omega)$ and $\tilde{H}_\varepsilon \to 0$ in $\cap_{p<\infty} L^p(\mathbb{R}^2)$. Therefore, $g_\varepsilon \to g$ (defined in (II.1)) strongly in $\cap_{p<\infty} W^{1,p}(\mathbb{R}^2)$ (hence uniformly). From now on, we will assume that u^ε has been replaced by $\tilde{u^\varepsilon} \in H^1(\Omega \backslash \cup_{i=1}^{n(\varepsilon)} \{a_i^\varepsilon\}, S^1)$ and write u^ε instead of $\tilde{u^\varepsilon}$, H_ε instead of $\tilde{H}_\varepsilon$. We thus have a family of S^1-valued vector fields with a finite number of singular points or "vortices" (for fixed ε). Since u^ε is in H^1 away from these points, we can define its degree around a_i^ε, and write

$$d_i^\varepsilon = \deg(u^\varepsilon, a_i^\varepsilon). \tag{III.1}$$

Locally in $\Omega\backslash\{a_1^\varepsilon, \cdots, a_{n(\varepsilon)}^\varepsilon\}$, from the result of [BZ], u^ε admits an H^1-lifting φ_ε and we may define the vector-field

$$V_\varepsilon = < \nabla u^\varepsilon, iu^\varepsilon > = \nabla \varphi_\varepsilon, \tag{III.2}$$

which is well-defined a.e. in Ω, curl-free in $\Omega\backslash\{a_1^\varepsilon, \cdots, a_{n(\varepsilon)}^\varepsilon\}$. As in [RS1], the lower bound of the energy will follow from bounding $\int_\Omega |H_\varepsilon \cdot V_\varepsilon|$.

In a first step, we shall assume that g defined in (II.1) has a distinguished sign in Ω, i.e., without loss of generality, that

$$\begin{cases} g \geq 0 & \text{in } \Omega \\ g = 0 & \text{on } \partial\Omega. \end{cases}$$

We define

$$D_\varepsilon(t) = \frac{1}{2\pi} \int_{\{g_\varepsilon = t\}\cap\Omega} V_\varepsilon \cdot \tau = \sum_{i/g_\varepsilon(a_i^\varepsilon)\geq t} d_i^\varepsilon \in \mathbb{Z} \tag{III.3}$$

where the normal-vector n to $\{g_\varepsilon = t\}$ is oriented towards the increasing g_ε. D_ε is an integer-valued function on $\mathbb{R}^+$, equal to the total degree on the level-set $\{g_\varepsilon = t\}$. It is defined almost everywhere (for t such that $\{g_\varepsilon = t\}$ is regular and does not contain any vortex-point). It can be defined for all t through the formula $\sum_{i/g_\varepsilon(a_i^\varepsilon)\geq t} d_i^\varepsilon$ and extended by 0 for $t < 0$. Then, we recall that

$$M_\varepsilon(t) = 2\pi \int_0^t D_\varepsilon(s)\, ds, \tag{III.4}$$

and p_ε and p are defined in (I.24) and (I.25). We have the following result

Lemma III.1

$$p_\varepsilon \rightharpoonup p \quad in\ L^1(\mathbb{R}^+).$$

Before proving this lemma, let us state the main result.

Theorem 3 *Assume $E_\varepsilon(u^\varepsilon) \leq C$, $u^\varepsilon \to u$ in $\cap L^p(\Omega)$ and $g \geq 0$ in Ω. Then, as $\varepsilon \to 0$,*

$$p_\varepsilon + M_\varepsilon \text{ is bounded in } BV_{loc}(\mathbb{R}^+). \tag{III.5}$$

Extracting subsequences if necessary, there exist $M \in L^1(\mathbb{R}^+)$ and $D = \frac{M'}{2\pi}$ in $W^{-1,1}(\mathbb{R}^+)$ such that

$$p_\varepsilon + M_\varepsilon \rightharpoonup p + M \quad in\ BV_{loc}(\mathbb{R}^+) \tag{III.6}$$

i.e., denoting by $\mathcal{M}(\mathbb{R}^+)$ the dual of $C_0^0(\mathbb{R}^+)$,

$$p'_\varepsilon + 2\pi D_\varepsilon \rightharpoonup p' + 2\pi D \quad in\ \mathcal{M}(\mathbb{R}^+). \tag{III.7}$$

Moreover,

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon) \geq E_0(u, D) = \|p + M\|_{BV(\mathbb{R}^+)} = \int_0^{+\infty} |p' + 2\pi D|. \tag{III.8}$$

If in addition we assume that u^ε does not have vortices in a set of the form $\{g \leq \delta\}$ for some $\delta > 0$ and has a lifting φ_ε in that set uniformly bounded in L^∞, then the lower bound can be improved as follows:

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon) \geq E_0(u, D) = \|p + M\|_{BV(\mathbb{R})} = \int_{\mathbb{R}} |p' + 2\pi D| \tag{III.9}$$

(D being extended by 0 on $\mathbb{R}^-$ and p by $|\partial\Omega|$).

The result (III.9) is a stronger lower bound than (III.8). However, it assumes that vortices do not accumulate near the boundary, which is not necessarily true.

Corollary 1 *When the assumptions of [RS2] are satisfied, i.e. $u^\varepsilon \in S^1$ ($u_3^\varepsilon \equiv 0$) with a lifting φ_ε, $\|\varphi_\varepsilon\|_{L^\infty} \leq N$, then $D_\varepsilon \equiv 0$, and we get*

$$p_\varepsilon \text{ bounded in } BV_{loc}(\mathbb{R}^+) \qquad p_\varepsilon \rightharpoonup p \text{ in } BV_{loc}(\mathbb{R}^+)$$

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon) \geq \|p\|_{BV(\mathbb{R}^+)} \geq \left| \int_0^{+\infty} p' \right| = |p(0)| \geq |\partial\Omega|. \tag{III.10}$$

Remark III.1: This is consistent with the lower bound obtained in [RS1] which was

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon) \geq \int_\Omega |\text{div } (\varphi u + u^\perp)|.$$

Indeed, this can be proved rather simply. For any $f \in C^\infty(\mathbb{R})$ such that $|f| \leq 1$ and $f(0) = 0$,

$$\int_\Omega f(g)\text{div } (\varphi u + u^\perp) = -\int_\Omega f'(g)$$

(after integrating by parts and using $u = -\nabla^\perp g$, $|\nabla g| = 1$), Using the coarea formula of Federer,

$$-\int_\Omega f'(g) = -\int_{\mathbb{R}} f'(t)p(t)\, dt.$$

Taking the supremum over all such f, we obtain that

$$\int_\Omega |\text{div } (\varphi u + u^\perp)| \geq \|p\|_{BV(\mathbb{R}^+)}.$$

In addition, there is equality in (III.10) if and only if $p' \leq 0$ on $\mathbb{R}^+$ (for $g \geq 0$) or $p' \leq 0$ on $\mathbb{R}^+$ (for $g \leq 0$), which is to be compared to the sign condition obtained in [RS1, RS2].

Remark III.2: If vortices a_i with degrees d_i are fixed a priori, then the lower bound (III.8) of Theorem 3 implies the "nice" formula

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon) \geq \left|\int_{\mathbb{R}} p' + 2\pi D\right| = \left| |\partial\Omega| - 2\pi \sum_i d_i g(a_i)\right|. \tag{III.11}$$

As a corollary of Theorem 3, we can derive a lower bound in the general case when g changes sign.

We define Ω_+ to be the interior of the set $\{g \geq 0\}$ and Ω_- the interior of $\{g \leq 0\}$,

$$p_+(t) = \mathcal{H}^1(\{g = t\}) \quad \text{for } t > 0$$

extended by the values 0 for $t > \max g$, $|\partial\Omega_+|$ for $t < 0$;

$$p_-(t) = \mathcal{H}^1(\{g = t\}) \quad \text{for } t < 0$$

extended by 0 for $t < \min g$, $|\partial\Omega_-|$ for $t > 0$. For $t \geq 0$,

$$D_\varepsilon^+(t) = \frac{1}{2\pi}\int_{\{g_\varepsilon = t\}\cap\Omega_+} V_\varepsilon \cdot \tau = \sum_{i/g_\varepsilon(a_i^\varepsilon)\geq t} d_i^\varepsilon \ \in \mathbb{Z},$$

and for $t \leq 0$,

$$D_\varepsilon^-(t) = \frac{1}{2\pi}\int_{\{g_\varepsilon = t\}\cap\Omega_-} V_\varepsilon \cdot \tau = \sum_{i/g_\varepsilon(a_i^\varepsilon)\leq t} d_i^\varepsilon \ \in \mathbb{Z},$$

and we set

$$M_\varepsilon^\pm = 2\pi \int_0^t D_\varepsilon^\pm(s)\, ds.$$

Obviously, the contributions of energy on Ω_+ and Ω_- can be added up and we deduce from Theorem 3, applied successively on Ω_+ and Ω_-.

Theorem 3' *Assume $E_\varepsilon(u^\varepsilon) \leq C$, then as $\varepsilon \to 0$,*

$$\begin{array}{c} p_\varepsilon \rightharpoonup p_+ \quad \text{in } L^1(\mathbb{R}^+) \\ p_\varepsilon \rightharpoonup p_- \quad \text{in } L^1(\mathbb{R}^-) \\ p_\varepsilon + M_\varepsilon^+ \text{ is bounded in } BV_{loc}(\mathbb{R}^+) \\ p_\varepsilon + M_\varepsilon^- \text{ is bounded in } BV_{loc}(\mathbb{R}^-), \end{array}$$

there exist $M^+, M^- \in L^1$, $D^+ = \frac{(M^+)'}{2\pi}$, $D_- = \frac{(M^-)'}{2\pi}$, such that

$$\begin{array}{c} p_\varepsilon + M_\varepsilon^+ \rightharpoonup p_+ + M_+ \quad \text{in } BV_{loc}(\mathbb{R}^+) \\ p_\varepsilon + M_\varepsilon^- \rightharpoonup p_- + M_- \quad \text{in } BV_{loc}(\mathbb{R}^-) \\ p'_\varepsilon + 2\pi D_\varepsilon^+ \rightharpoonup p'_+ + 2\pi D \text{ in } \mathcal{M}(\mathbb{R}^+) \text{ (respectively } -), \end{array}$$

and we have

$$\begin{aligned} \liminf_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon) &\geq \|p^+ + M^+\|_{BV(\mathbb{R}^+)} + \|p^- + M^-\|_{BV(\mathbb{R}^-)} \\ &= \int_0^{+\infty} |(p^+)' + 2\pi D^+| + \int_{-\infty}^0 |(p^-)' + 2\pi D^-|. \end{aligned} \tag{III.12}$$

If we have the extra assumption on a set $\{-\delta \leq g \leq \delta\}$, then

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon) \geq \|p^+ + M^+\|_{BV(\mathbb{R})} + \|p^- + M^-\|_{BV(\mathbb{R})}.$$

We start with the

Proof of Lemma III.1:

First of all, let us show that the weak limit of p_ε is p. Let $f \in C^0(\mathbb{R}^+)$ be a test-function. Using the coarea formula, we have

$$\begin{aligned} \int_{\mathbb{R}^+} f(t) p_\varepsilon(t)\, dt &= \int_{\mathbb{R}^+} f(t) \left(\int_{\{g_\varepsilon = t\}\cap\Omega} |\nabla g_\varepsilon| \right) dt \\ &= \int_{\{g_\varepsilon \geq 0\}\cap\Omega} f(g_\varepsilon) |\nabla g_\varepsilon|^2. \end{aligned}$$

But we know that $|\nabla g_\varepsilon| \to 1$ in $\cap_{p<\infty} L^p$, thus

$$\int_{\mathbb{R}^+} f(t) p_\varepsilon(t)\, dt = \int_{\{g_\varepsilon \geq 0\}\cap\Omega} f(g_\varepsilon) + o(1),$$

while $g_\varepsilon \to g$ uniformly, hence $f(g_\varepsilon)\mathbf{1}_{g_\varepsilon \geq 0} \to f(g)\mathbf{1}_\Omega$ a.e. It follows by dominated convergence that

$$\int_{\mathbb{R}^+} f(t)p_\varepsilon(t)\,dt \to \int_\Omega f(g).$$

On the other hand, using once again the coarea formula,

$$\int_\Omega f(g) = \int_\Omega f(g)|\nabla g| = \int_{\mathbb{R}^+} f(t)p(t)\,dt. \tag{III.13}$$

Hence,

$$\int_{\mathbb{R}^+} f(t)p_\varepsilon(t)\,dt \to \int_{\mathbb{R}^+} f(t)p(t)\,dt. \tag{III.14}$$

Therefore, the weak limit of p_ε is p. In order to prove weak L^1 convergence, we just prove that p_ε is equiintegrable in $\mathbb{R}^+$.

First of all, $p \in L^1(\mathbb{R}^+)$ because $\int_{\mathbb{R}^+} p(t)\,dt = \int_\Omega 1 = |\Omega|$ as in (III.14). Hence, for $\eta > 0$ given, there exists δ such that

$$|b-a| < \delta \Longrightarrow \int_a^b p(t)\,dt < \eta. \tag{III.15}$$

Secondly,

$$\begin{aligned} \int_a^b p_\varepsilon(t)\,dt = \int_{a \leq g_\varepsilon \leq b} |\nabla g_\varepsilon|^2 &\leq \|\nabla g_\varepsilon\|_{L^4}^2 \mathrm{vol}(a \leq g_\varepsilon \leq b)^{\frac{1}{2}} \\ &\leq C\mathrm{vol}(a \leq g_\varepsilon \leq b)^{\frac{1}{2}}. \end{aligned}$$

By strong convergence of g_ε to g, for ε small enough, and for all $a, b \in \mathbb{R}^+$, we have (with the same coarea formula again)

$$\begin{aligned} \mathrm{vol}(a \leq g_\varepsilon \leq b) &\leq \mathrm{vol}\left(a - \frac{\delta}{4} \leq g \leq b + \frac{\delta}{4}\right) \\ &= \int_{a-\frac{\delta}{4} \leq g \leq b+\frac{\delta}{4}} |\nabla g| = \int_{a-\frac{\delta}{4}}^{b+\frac{\delta}{4}} p(t)\,dt. \end{aligned}$$

Thus, in view of (III.15), if $|b-a| < \frac{\delta}{2}$, we have

$$\int_a^b p_\varepsilon(t)\,dt \leq C\eta^{\frac{1}{2}}.$$

This characterizes the equiintegrability of p_ε, which implies that $p_\varepsilon \rightharpoonup p$ in $L^1(\mathbb{R}^+)$. $\square$

Proof of Theorem 3:

As already mentioned, we are going to use the fact that $E_\varepsilon(u^\varepsilon) \geq \int_\Omega |H_\varepsilon \cdot V_\varepsilon|$ and bound from below $\int_\Omega |H_\varepsilon \cdot V_\varepsilon|$. We first can write $H_\varepsilon \cdot V_\varepsilon$ as follows (using $u^\varepsilon \mathbf{1}_\Omega + H_\varepsilon = -\nabla^\perp g_\varepsilon$):

$$H_\varepsilon \cdot V_\varepsilon = -(u^\varepsilon + \nabla^\perp g_\varepsilon) \cdot V_\varepsilon \quad \text{in } \Omega. \tag{III.16}$$

One can also check that, since u^ε is S^1-valued and locally equal to $e^{i\varphi_\varepsilon}$, we have

$$\text{curl } u^\varepsilon = u^\varepsilon \cdot V_\varepsilon \quad \text{in } \Omega, \tag{III.17}$$

and thus, using curl $H_\varepsilon = 0$,

$$H_\varepsilon \cdot V_\varepsilon = -\nabla^\perp g_\varepsilon \cdot V_\varepsilon - \text{curl } u^\varepsilon = -\nabla^\perp g_\varepsilon \cdot V_\varepsilon + \text{curl } \nabla^\perp g_\varepsilon \quad \text{in } \Omega. \tag{III.18}$$

This decomposition is the same trick as the one that we used in [RS1] to obtain the lower bound. Let us now take a test-function $f \in C^1(\mathbb{R}^+)$ and evaluate

$$\int_\Omega f(g_\varepsilon)(-V_\varepsilon \cdot \nabla^\perp g_\varepsilon + \text{curl } \nabla^\perp g_\varepsilon).$$

Then, we will use

$$\int_\Omega |H_\varepsilon \cdot V_\varepsilon| \geq \sup_{|f|\leq 1} \int_\Omega f(g_\varepsilon)(-V_\varepsilon \cdot \nabla^\perp g_\varepsilon + \text{curl } \nabla^\perp g_\varepsilon)$$

to obtain the lower bound.
Let $\delta > 0$ be fixed. We recall that we are in the case $g \geq 0$ and $g = 0$ on $\partial\Omega$; therefore, by uniform convergence of g_ε to g, for ε small enough, $\{g_\varepsilon \geq \delta\} \subset \Omega$. Using F, some primitive of f,

$$\int_{g_\varepsilon \geq \delta} f(g_\varepsilon)(V_\varepsilon \cdot \nabla^\perp g_\varepsilon - \text{curl } \nabla^\perp g_\varepsilon) = \int_{g_\varepsilon \geq \delta} V_\varepsilon \cdot \nabla^\perp F(g_\varepsilon) - f(g_\varepsilon)\text{curl } \nabla^\perp g_\varepsilon. \tag{III.19}$$

Now we can observe that

$$\begin{aligned} V_\varepsilon \cdot \nabla^\perp F(g_\varepsilon) &= \text{curl } (F(g_\varepsilon)V_\varepsilon) - F(g_\varepsilon)\text{curl } V_\varepsilon \\ &= \text{curl } (F(g_\varepsilon)V_\varepsilon) - 2\pi \sum_{i=1}^{n(\varepsilon)} d_i^\varepsilon F(g_\varepsilon(a_i^\varepsilon))\delta_{a_i^\varepsilon} \end{aligned} \tag{III.20}$$

in the sense of distributions, where the $(a_i^\varepsilon, d_i^\varepsilon)$ are the vortices of u^ε (cf. (III.1)). Hence,

$$\int_{g_\varepsilon \geq \delta} V_\varepsilon \cdot \nabla^\perp F(g_\varepsilon) = \int_{g_\varepsilon = \delta} F(g_\varepsilon) V_\varepsilon \cdot \tau - 2\pi \sum_{i/g_\varepsilon(a_i^\varepsilon)\geq\delta} d_i^\varepsilon F(g_\varepsilon(a_i^\varepsilon)). \tag{III.21}$$

But in view of definition (III.3), this is

$$2\pi F(\delta) D_\varepsilon(\delta) - 2\pi \sum_{i/g_\varepsilon(a_i^\varepsilon)\geq\delta} d_i^\varepsilon F(g_\varepsilon(a_i^\varepsilon)). \tag{III.22}$$

(Without loss of generality, we may assume that $\{g_\varepsilon = \delta\}$ does not contain any vortex.) Meanwhile, we examine the other terms. Integrating by parts, we have

$$\int_{g_\varepsilon \geq \delta} f(g_\varepsilon)\mathrm{curl}\ \nabla^\perp g_\varepsilon = \int_{g_\varepsilon=\delta} f(g_\varepsilon)\frac{\partial g_\varepsilon}{\partial n_{ext}} - \int_{g_\varepsilon\geq\delta} f'(g_\varepsilon)|\nabla g_\varepsilon|^2.$$

On the level-set $\{g_\varepsilon = \delta\}$, we have $\frac{\partial g_\varepsilon}{\partial n_{ext}} = -|\nabla g_\varepsilon|$, hence

$$\text{(III.23)} \qquad \int_{g_\varepsilon \geq \delta} f(g_\varepsilon)\mathrm{curl}\ \nabla^\perp g_\varepsilon = -f(\delta)p_\varepsilon(\delta) - \int_{g_\varepsilon\geq\delta} f'(g_\varepsilon)|\nabla g_\varepsilon|^2.$$

Using the coarea formula, the second term is equal to $-\int_\delta^\infty f'(t)p_\varepsilon(t)\,dt$. Inserting (III.22) and (III.23) into (III.19), we obtain

$$\text{(III.24)} \quad \int_{g_\varepsilon\geq\delta} f(g_\varepsilon)(V_\varepsilon\cdot\nabla^\perp g_\varepsilon - \mathrm{curl}\ \nabla^\perp g_\varepsilon) = 2\pi F(\delta)D_\varepsilon(\delta) - 2\pi\sum_{i/g_\varepsilon(a_i^\varepsilon)\geq\delta} d_i^\varepsilon F(g_\varepsilon(a_i^\varepsilon)) + f(\delta)p_\varepsilon(\delta) + \int_\delta^\infty f'(t)p_\varepsilon(t)\,dt.$$

Taking $f\equiv 1$ and letting $\delta\to 0$, one formally obtains the result mentioned in Remark III.2 from this equation.
We can transform the second term in the sum as follows: noticing that $D_\varepsilon(t) = \sum_{g_\varepsilon(a_i^\varepsilon)\geq t} d_i^\varepsilon$, let us denote by $t_k (k = 1,\cdots,K)$ the increasing sequence of $t\geq\delta$ such that $\{g_\varepsilon = t\}$ contains a vortex. Then,

$$\begin{aligned} 2\pi\sum_{i/g_\varepsilon(a_i^\varepsilon)\geq\delta} d_i^\varepsilon F(g_\varepsilon(a_i^\varepsilon)) &= 2\pi\sum_{k=1}^K\left(\sum_{i/g_\varepsilon(a_i^\varepsilon)=t} d_i^\varepsilon\right)F(t_k) \\ \text{(III.25)} \qquad &= 2\pi\sum_{k=1}^K\left(D_\varepsilon(t_k) - D_\varepsilon(t_{k+1})\right)F(t_k). \end{aligned}$$

After a standard re-summation process, (III.25) becomes

$$2\pi\sum_{i/g_\varepsilon(a_i^\varepsilon)\geq\delta} d_i^\varepsilon F(g_\varepsilon(a_i^\varepsilon)) = 2\pi\sum_{k=2}^K D_\varepsilon(t_k)f(s)\,ds + 2\pi D_\varepsilon(t_1)F(t_1).$$

We observe that $D_\varepsilon(\delta) = D_\varepsilon(t_1)$ and D_ε is equal to $D_\varepsilon(t_k)$ on $(t_{k-1}, t_k]$; therefore,

$$\begin{aligned} 2\pi F(\delta)D_\varepsilon(\delta) - 2\pi\sum_{i/g_\varepsilon(a_i^\varepsilon)\geq\delta} d_i^\varepsilon F(g_\varepsilon(a_i^\varepsilon)) &= -2\pi\int_{t_1}^{t_K} D_\varepsilon(s)f(s)\,ds - 2\pi D_\varepsilon(t_1)\left(F(t_1)-F(\delta)\right) \\ \text{(III.26)} \qquad &= -2\pi\int_\delta^\infty D_\varepsilon(t)f(t)\,dt. \end{aligned}$$

(with the convention $p_\varepsilon = 0$ for $t > \max g_\varepsilon$). Plugging this into (III.24), we conclude that

$$\begin{aligned}\int_{g_\varepsilon \geq \delta} f(g_\varepsilon)(V_\varepsilon \cdot \nabla^\perp g_\varepsilon - \text{curl } \nabla^\perp g_\varepsilon) &= f(\delta)p_\varepsilon(\delta) + \int_\delta^\infty f'(t)p_\varepsilon(t)\,dt - 2\pi \int_\delta^\infty D_\varepsilon(t) f(t)\,dt \\ &= -\int_\delta^\infty f(t)\left(p'_\varepsilon(t) + 2\pi D_\varepsilon(t)\right)\,dt. \end{aligned} \tag{III.27}$$

Going back to (III.18), taking the supremum over all f's such that $|f| \leq 1$, we deduce that $p'_\varepsilon + 2\pi D_\varepsilon$ is bounded in the sense of measures, or that $p_\varepsilon + M_\varepsilon$ is bounded in $BV([\delta, +\infty))$. Since this is true for all $\delta > 0$, we can say it is bounded in $BV_{loc}(\mathbb{R}^+)$. Moreover, we have

$$\int_{g_\varepsilon \geq \delta} |H_\varepsilon \cdot V_\varepsilon| \geq \int_\delta^\infty |p'_\varepsilon + 2\pi D_\varepsilon|,$$

and using the lower semi-continuity of the mass,

$$\liminf_{\varepsilon \to 0} \int_{g_\varepsilon \geq \delta} |H_\varepsilon \cdot V_\varepsilon| \geq \int_\delta^\infty |p' + 2\pi D|. \tag{III.28}$$

Since this is true for all $\delta > 0$, we conclude that

$$\liminf_{\varepsilon \to 0} E_\varepsilon(u^\varepsilon) \geq \liminf_{\varepsilon \to 0} \int_{g_\varepsilon \geq \delta} |H_\varepsilon \cdot V_\varepsilon| \geq \int_0^\infty |p' + 2\pi D|,$$

which provides the desired result.

Next, we consider the particular case in which u^ε does not have vortices in a set of the form $\{g \leq \delta\}$ for some small δ, and thus has a lifting φ_ε which is assumed to be uniformly bounded in L^∞ in this neighborhood. Since $g_\varepsilon \to g$ uniformly, we have $\cup_{\varepsilon_k} \cap_{\varepsilon < \varepsilon_k} \{g_\varepsilon < \delta\} \cap \Omega = \{g < \delta\}$, and we may assume that u^ε has no vortex and has the lifting φ_ε in the set $\{g_\varepsilon < \delta\}$ for ε small enough. Then, we are in the framework of [RS1] and

$$\begin{aligned} H_\varepsilon \cdot V_\varepsilon &= H_\varepsilon \cdot \varphi_\varepsilon = -u^\varepsilon \cdot \nabla \varphi_\varepsilon - \nabla^\perp g_\varepsilon \cdot \nabla \varphi_\varepsilon \\ &= -\text{div } (\varphi_\varepsilon \nabla^\perp g_\varepsilon) - \text{curl } u^\varepsilon \\ &= \text{div } (-\varphi_\varepsilon \nabla^\perp g_\varepsilon + u^{\varepsilon\perp}). \end{aligned} \tag{III.29}$$

Then,

$$\begin{aligned} C \geq \int_{\{g_\varepsilon < \delta\} \cap \Omega} |H_\varepsilon \cdot V_\varepsilon| &= \int_{\{g_\varepsilon < \delta\} \cap \Omega} |\text{div } (-\varphi_\varepsilon \nabla^\perp g_\varepsilon + u^{\varepsilon\perp})| \\ &\geq \int_{\cup_{\varepsilon_k} \cap_{\varepsilon < \varepsilon_k} \{g_\varepsilon < \delta\} \cap \Omega = \{g < \delta\}} |\text{div } (-\varphi_\varepsilon \nabla^\perp g_\varepsilon + u^{\varepsilon\perp})|. \end{aligned} \tag{III.30}$$

φ_ε being bounded in L^∞, it converges weakly in $\cap_p L^p$ to some φ; on the other hand $-\nabla^\perp g_\varepsilon \to -\nabla^\perp g = u$ strongly in $\cap_{p<\infty} L^p$ (see Lemma II.1), thus div $(-\varphi_\varepsilon \nabla^\perp g_\varepsilon + u^{\varepsilon\perp}) \lfloor \{g < \delta\}$ which remains bounded in L^1 converges weakly to div $(\varphi u + u^\perp)$, and

$$\liminf_{\varepsilon \to 0} \int_{g_\varepsilon < \delta} |H_\varepsilon \cdot V_\varepsilon| \geq \int_{g < \delta} |\text{div } (\varphi u + u^\perp)|. \tag{III.31}$$

Following exactly the proof of [RS1] Lemma 5.2, we obtain that

$$\int_{g<\delta} |\text{div } (\varphi u + u^{\perp})| \geq |p(\delta) - |\partial\Omega||, \tag{III.32}$$

(which may not go to 0 as $\delta \to 0$). Then,

$$E_\varepsilon(u^\varepsilon) \geq \int_{g_\varepsilon<\delta} |H_\varepsilon \cdot V_\varepsilon| + \int_{g_\varepsilon\geq\delta} |H_\varepsilon \cdot V_\varepsilon|$$

and using (III.28) together with (III.31) and (III.32), we are led to

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon) \geq \int_\delta^{+\infty} |p' + 2\pi D| + |p(\delta) - |\partial\Omega||.$$

This is true for all $\delta > 0$, hence it implies that

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon) \geq \int_{-\infty}^{+\infty} |p' + 2\pi D|,$$

(in the sense of the BV norm of $p + M$), where p has been extended by $|\partial\Omega|$ for $t < 0$ and D by 0 for $t < 0$. This concludes the proof of Theorem 3, from which Theorem 3' also follows. □

Remark III.3: The lower bounds of Theorem 3 and 3' are not always optimal. To improve them, one should first sum up the contributions of $p'_j + 2\pi D_j$ over all the connected components $\mathcal{C}_j$ of g_ε. We also believe that an optimal bound should involve $\int_{\mathbb{R}^2} |p' + 2\pi\hat{D}| + 2\pi|D - \hat{D}|$ where $\hat{D}$ is the truncation of D : $\hat{D} = D$ if $|D| \leq 1$, $\hat{D} = sign(D)$ otherwise. Thus a total degree one on the boundary of each connected component of g_ε should be favored.

IV Examples and optimal wall jumps

In this section, we would like to explain on several examples the different formulas obtained before. We begin with very simple situations in which the theory is easily explained, and afterwards, we try to consider more and more complex situations in order to get an intuition on the characteristics of the minimizers. Each time, we try to compare the different formulas and sometimes give the main differences between this situation and the one studied in [RS1, RS2]. We also show how the optimal energy can be reached using the so-called "cross-tie" walls or Néel walls depending on the magnitude of the jump across the wall.

We start with the lower bound of Section II (for $u \in BV(\Omega)$)

$$E_0(u) = \int_{\Sigma_u} A(X)\, d\mathcal{H}^1, \tag{IV.1}$$

where A is given by

$$A(X) = 2|\sin X - X\cos X| \qquad \text{for } X \in [0, \frac{\pi}{4}] \tag{IV.2}$$

$$A(X) = 2\left|\left(X - \tfrac{\pi}{2}\right)\cos X - \sin X + \sqrt{2}\right| \quad \text{for } X \in [\frac{\pi}{4}, \frac{\pi}{2}]. \tag{IV.3}$$

We will show that this formula is optimal, and that such bounds are achieved for Néel walls in the first case and “cross-tie” walls in the second case.

We remark that this formula doesn't say anything about the possible convergence of the vortices and optimal places where they should be put. To see that, we can use the following two formulas (only valid if the vortices are of finite number and converge to the points a_i with degree d_i).

$$E_0(u, \{a_i\}, \{d_i\}) = |\partial\Omega| - 2\pi \sum_i d_i g(a_i), \tag{IV.4}$$

which may be useful if we know the function g (e.g. it is the distance to the boundary), or equivalently the formula using the perimeter of the level-sets of g

$$E_0(u, D) = \int |p'(t) + 2\pi D(t)|\, dt. \tag{IV.5}$$

In particular, if we know that the configuration we are looking at is approximated by a sequence (u^ε) for which the number of vortices is bounded, then u^ε can be considered as purely planar away from these points and as in [RS1], the formula (I.10) applies. In other words, if we want to reach the optimal wall energy for big jumps given by the second formula (IV.3) we need to consider sequences with more and more vortices.

IV.1 Limiting energy on simple configurations

IV.1.1 The unit disc

Let us begin with the simplest one vortex situation. Namely, we consider the case where Ω is the unit disc and the limiting (divergence free) vectorfield u_0 is given by

$$u_0 = -\nabla^\perp g,$$

where $g(x) = \mathrm{dist}(x, \partial\Omega)$.

As already explained in the introduction, it is possible to find a sequence (u^ε) such that

$$\begin{aligned}
&u^\varepsilon \in H^1(\Omega, S^2),\\
&u^\varepsilon \longrightarrow u_0 \text{ in } L^2 \text{ strong as } \varepsilon \longrightarrow 0,\\
&E_\varepsilon(u^\varepsilon) \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0.
\end{aligned}$$

This case is clearly already very different from [RS1] where the energy is always bounded from below by $|\partial\Omega|$.

However, since u_0 possesses no line singularity, the formula (IV.1) gives the correct lower bound for the energy since the singular set has no $\mathcal{H}^1$ measure and thus the integration reduces to zero. In [JOP], it is proved that this is the only case where a zero limiting energy can be achieved. Moreover, it is now possible to get different lower bounds according to the way the vortices converge. Indeed, imagine that u^ε possesses a degree one vortex that converges to some point P inside the disc. Then, applying the formula (IV.4), we get the lower bound

$$\begin{aligned}\liminf_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon) &\geq |\partial\Omega| - 2\pi g(P),\\ &= 2\pi \mathrm{dist}(O,P),\end{aligned}$$

where O stands for the center of the disc.

This amount of remaining energy comes from the fact that in order for (u^ε) to converge to u_0, a concentration of energy must occur on a line joining P to O. Along this line, the vectorfield possesses a phase jump of 2π which costs at least the energy given above.

An explicit computation of the lower bound is also possible using the formula (IV.5). Indeed, for a value $t \in (0,1]$, the level set $g^{-1}(t)$ is precisely the circle of center O and radius t whose perimeter is

$$p(t) = 2\pi(1-t).$$

Now, we must compute the degree of these level sets. It is clear that if P is inside the level set $g^{-1}(t)$ then the degree $D(t)$ will be equal to one. Otherwise the degree will be equal to zero.

Finally, P is inside $g^{-1}(t)$ for all $t \in [0, g(P))$ and outside for all $t \in (g(P), 1]$. Putting all together in the formula, we get the lower bound

$$\begin{aligned}\liminf_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon) &\geq \int_0^1 |p'(t) + 2\pi D(t)|\,dt,\\ &= \int_0^{g(P)} 0\,dt + \int_{g(P)}^1 2\pi\,dt,\\ &= 2\pi(1-g(P)),\end{aligned}$$

which is the correct answer.

IV.1.2 The unit square

Consider now the case where Ω is the unit square of $\mathbb{R}^2$, and u_0 is still given by

$$u_0 = -\nabla^\perp g,$$

where $g(x) = \mathrm{dist}(x, \partial\Omega)$. Figure 2 shows the vectorfield u_0.

Since u_0 possesses line singularities, we expect the lower bound of the energy to be strictly nonnegative.

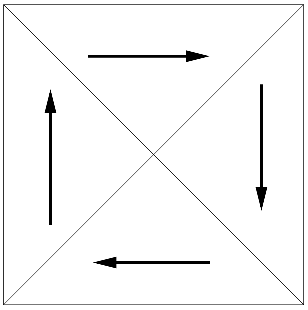

Figure 2: The unit square viscosity solution

Let us use the formula (IV.1) to compute the value of the smallest lower bound. On each diagonal segment, u_0 has a jump of $\frac{\pi}{2}$ in the phase, so that the energy is computable by

$$\begin{aligned}\liminf_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon) &\geq \frac{8\sqrt{2}}{2}|\sin(\frac{\pi}{4}) - \frac{\pi}{4}\cos(\frac{\pi}{4})|,\\ &= 4-\pi\\ &> 0.\end{aligned}$$

Using the formula (IV.4) and assuming that the sequence (u^ε) possesses a degree one vortex that converges to some point P inside Ω, we get

$$\begin{aligned}\liminf_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon) &\geq \left||\partial\Omega| - 2\pi g(P)\right|,\\ &= (4 - 2\pi \mathrm{dist}(P,\partial\Omega)).\end{aligned}$$

In the case where P is equal to the center of the square, we of course recover the preceding result ($\mathrm{dist}(P,\partial\Omega) = \frac{1}{2}$).

Using the formula (IV.5) yields the same answer, and we leave the reader doing the computation.

IV.1.3 The 2×1 rectangle

This geometry is quite famous in the micromagnetic community since it has been used as a benchmark for computing rectangular thin films. A recent paper by W. Rave and A. Hubert [RH] is for instance devoted to the systematic comparison of different configurations in terms of energy, depending on the size and the thickness of the film.

Here, we would like to compare from the point of view of our theory the two usually called Landau-Lifshitz and diamond configurations (see Fig. 3).

For the diamond configuration, there is essentially no problem since all the jumps involved are of magnitude less than $\pi/2$. It is thus easily seen that in that case, we must use the first formula (IV.2). We can also remark that the diamond configuration is only two

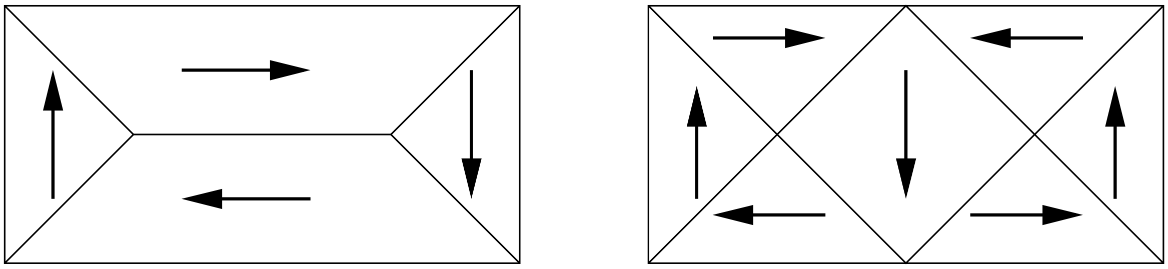

Figure 3: Landau Lifshitz and diamond configurations on a rectangle.

squares glued together (see Figure 4) from which we deduce that the optimal achievable energy is twice that of one square

$$E_{\text{diamond}} = 2(4 - \pi) = 8 - 2\pi.$$

For the Landau-Lifshitz configuration, the situation is slightly more complicated. In order to compute its energy, we must first look at the way the sequence (u^ε) converges to this state. Indeed, in the case the number of vortices is bounded, or there are no vortices at all, we cannot use the second formula (IV.3) and we are led with (IV.2) to

$$E_{LL} = 4 - \pi + 2$$

(the first term is the energy concentrated along the four diagonal and the last one is the energy along the horizontal line).

A too quick conclusion could lead to the misleading inequality

$$E_{LL} > E_{\text{diamond}}.$$

However, allowing the approximate configurations (u^ε) to have an unbounded number of vortices gives with (IV.3) the better lower bound

$$E'_{LL} = 4 - \pi + 2(\sqrt{2} - 1).$$

A numerical evaluation leads to

$$E'_{LL} < E_{\text{diamond}} < E_{LL}.$$

We will see in the next section that E'_{LL} corresponds to putting a cross-tie wall in the horizontal line, whereas the first computation of the energy (E_{LL}) corresponds to putting a Néel wall in this line. This is exactly the kind of behavior that was found in [RH] (Figure 5) where the cross-tie configuration possesses an energy very close to but smaller than the one of the diamond configuration and the Landau-Lifshitz configuration without a cross-tie wall has a much bigger energy.

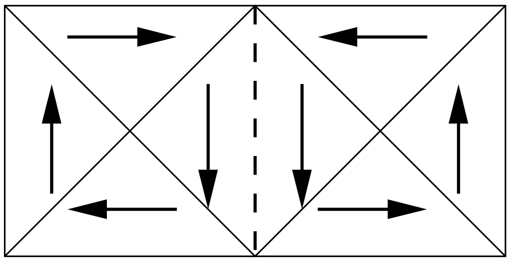

Figure 4: Rectangle as two glued squares.

IV.2 Optimal walls

To conclude this section, we show that the lower bounds given in the previous sections are indeed optimal. The nice phenomenon that occurs is that the energy density given in (IV.2) is achieved by a Néel wall (this was already known from the previous work [RS1]), which is a one-dimensional structure in the wall, but only for jumps in the phase less than $\pi/2$. For bigger jumps, the Néel wall is no longer optimal and the energy given by (IV.3) is achieved with a two-dimensional structure, the "cross-tie walls". As already mentioned, they are currently seen in thin films experiments [HS, NUH]. They are also under investigation by mathematicians [DKMO4]. Here, we would like to give a cross-tie wall configuration which is optimal for our model. We begin with the up-down jump where the situation is somewhat simpler, and generalize afterwards to the case of any jump.

IV.2.1 The up-down situation

In the up-down jump, we have to describe a microscopic structure inside a wall like the one drawn in Figure 5.

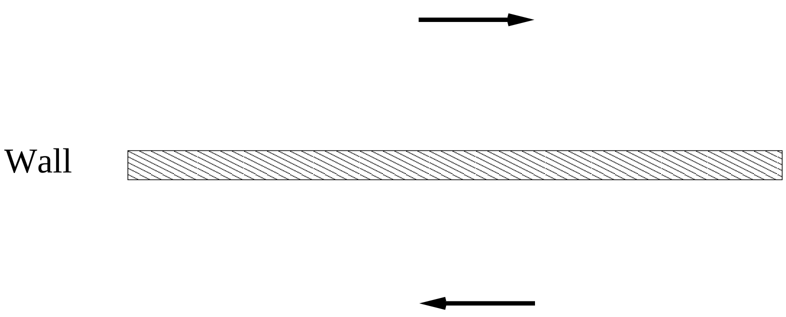

Figure 5: The up-down wall.

We propose the following construction drawn in Figure 6. We decompose the domain into several strips perpendicular to the wall. In each strip, the configuration will be the same, so that the whole configuration is periodic. In the upper half of each strip, we split the domain into three parts (see Figure 7): the lower left triangle Ω_L , the lower right triangle Ω_R and the rest of the domain Ω (the angles being $\frac{\pi}{4}$). We also call P the point on the wall in the middle of the strip. Then, the configuration is given by (see Figure 7)

$$u(x) = \begin{cases} e^{i\frac{\pi}{4}} \text{ on } \Omega_L, \\ e^{-i\frac{\pi}{4}} \text{ on } \Omega_R, \\ -\nabla^{\perp}\text{dist}(x,P) \text{ on } \Omega. \end{cases} \tag{IV.6}$$

On the lower part of the strip, the configuration is simply extended by symmetry around P.

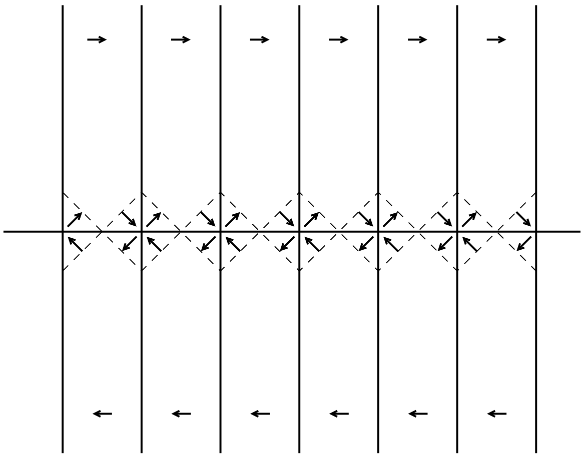

Figure 6: The optimal cross-tie wall.

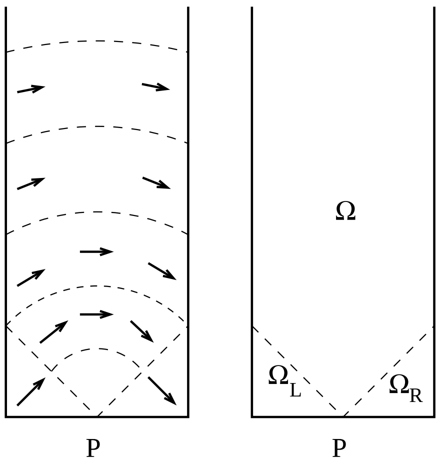

Figure 7: Zoom of the upper half strip.

Now, we compute the energy of this configuration in order to verify that it is equal to $2(\sqrt{2}-1)$ per unit length of the wall (this is what formula (IV.3) gives for an angle of jump equal to π). Since the configuration is periodic, we only need to compute the energy concentrated on the vertical and horizontal lines in one strip assuming the width of the strip is equal to one. As all the jumps are of angle less than $\pi/2$, and may be achieved by the 1D planar profiles of [RS1]. We thus compute the energy using the formula of [RS1] for the energy density

$$A(X) = 2|\sin X - X\cos X|,$$

where X is half the jump in the phase accross the wall.

To do so, we parametrize the vertical discontinuity by s. Since the distance between two neighbouring vortices is assumed to be one, we easily get the half-angle jump X accross the vertical line:

$$X(s) = \begin{cases} \frac{\pi}{4} \text{ if } s \in [-\frac{1}{2}, \frac{1}{2}], \\ \left|\arctan\left(\frac{1}{2s}\right)\right| \text{ otherwise.} \end{cases}$$

On the other hand, the half-angle jump on the horizontal line jump is constant and equal to $\pi/4$.

Putting everything together gives an energy equal to

$$\begin{aligned} E &= 4\left(\frac{\sqrt{2}}{2} - \frac{\pi}{4}\frac{\sqrt{2}}{2}\right) + 2\int_{\frac{1}{2}}^{+\infty} 2\left(\sin(X(s)) - X(s)\cos(X(s))\right) ds \\ &= \frac{\sqrt{2}}{2}(4-\pi) + 4\int_{\frac{1}{2}}^{+\infty} \left(\frac{1}{\sqrt{1+4s^2}} - \arctan\left(\frac{1}{2s}\right)\frac{2s}{\sqrt{1+4s^2}}\right) ds \\ &= \frac{\sqrt{2}}{2}(4-\pi) + 2\left[-\arctan\left(\frac{1}{2s}\right)\sqrt{1+4s^2}\right]_{\frac{1}{2}}^{+\infty} \\ &= \frac{\sqrt{2}}{2}(4-\pi) + 2(\arctan(1)\sqrt{2} - 1) \\ &= 2(\sqrt{2}-1). \end{aligned}$$

We also remark that it is possible to do the computation in a more geometrical way. Indeed, we first truncate the domain by two discs centered in P_1 and P_2 and of radius R as in Figure 8 and then observe that extending in these two regions the configuration by

$$u(x) = \begin{cases} -\nabla^{\perp}\mathrm{dist}(x, P_1) \text{ in the left-hand disc,} \\ -\nabla^{\perp}\mathrm{dist}(x, P_2) \text{ in the right-hand disc,} \end{cases}$$

it does not change the total energy since no discontinuity line has been added.

In order to further simplify the computation, we try to recognize u as the orthogonal gradient of the distance function to the boundary of a domain on each side of the vertical discontinuity line. This is not very difficult and a few attempts lead to the egg-like domain drawn in Fig. 9. The very last remark before the computation is that if one exchanges the left-hand part of the egg-like domain with its right-hand side. More precisely, we have

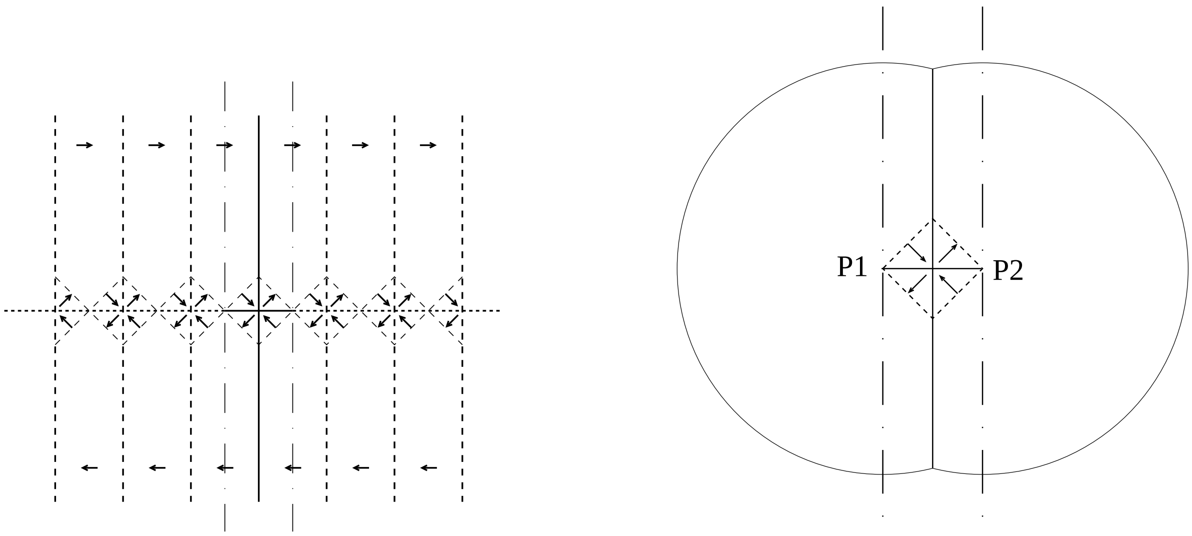

Figure 8: .

to compute the energy of a configuration obtained by gluing two left-hand parts of the egg-like domain. If we decide to glue two left-hand parts (and obtain the diamond-like domain presented in figure (IV.2.1)), the configuration obtained by taking the orthogonal gradient of the distance function to the boundary of this new domain possesses exactly the same energy as the original one since the jumps are exactly of the same amount and precisely occur in the same place.

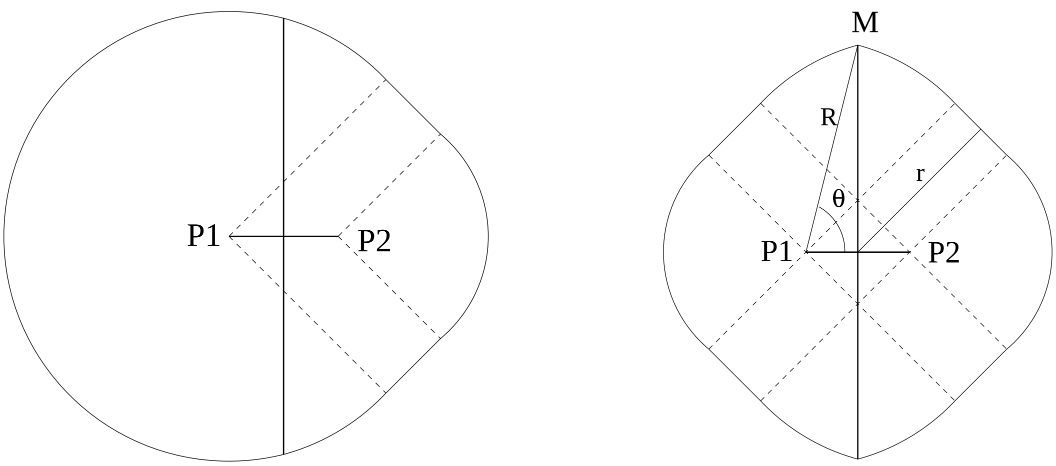

Figure 9: Egg-like domain and diamond-like domain.

Eventually we observe that now, the domain is convex and the potential is simply the distance function to the boundary. Therefore, we are allowed to use formula (IV.4) to compute its energy or at least an upper bound. This leads to the value

$$E = |\partial\Omega| - 2\pi r, \tag{IV.7}$$

where r is the radius of the biggest disc inscribed in the domain.

Recording that $\mathrm{dist}(P_1, P_2) = 1$, calling θ the angle $\widehat{P_2P_1M}$ (see Figure 9) and $R = \dfrac{1}{2\cos(\theta)}$, we get for the perimeter

$$|\partial\Omega| = 4\left(R\left(\theta - \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}} + \frac{\pi}{4}\left(R - \frac{1}{\sqrt{2}}\right)\right).$$

On the other hand, the biggest inscribed disc has a radius (see Figure 9)

$$r = R - \frac{1}{2\sqrt{2}}.$$

Therefore, the energy of the configuration is given by

$$\begin{aligned} E &= |\partial\Omega| - 2\pi r, \\ &= 4\left(R\theta + \frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}\right) - 2\pi R + \frac{\pi}{\sqrt{2}}, \\ &= 2\sqrt{2} + \frac{2\theta - \pi}{2\cos(\theta)}. \end{aligned}$$

Now letting θ tend to $\dfrac{\pi}{2}$ gives

$$\lim_{\theta \to \frac{\pi}{2}} E(\theta) = 2(\sqrt{2} - 1),$$

which is the desired result.

The computation of this energy is also possible via the level-set theory described in the previous section in a slightly more formal way. Indeed, if we compute the level-sets of the potential g associated to u, we find for their degrees

$$D(t) = \begin{cases} -1 \text{ if } t \in \left[0, \frac{\sqrt{2}}{4}\right), \\ 0 \text{ if } t \in \left(\frac{\sqrt{2}}{4}, +\infty\right), \end{cases}$$

and for their lengths

$$p(t) = \begin{cases} (4+\pi)t \text{ if } t \in \left[0, \frac{\sqrt{2}}{4}\right), \\ \text{a non increasing function of } t \text{ if } t \in \left(\frac{\sqrt{2}}{4}, +\infty\right). \end{cases}$$

The lower bound is now easily computable by

$$
\begin{aligned}
E &= \int_0^{+\infty} |p'(t) + 2\pi D(t)|\, dt, \\
&= \int_0^{\frac{\sqrt{2}}{4}} |4 + \pi - 2\pi|\, dt + p\left(\frac{\sqrt{2}}{4}\right) - p(+\infty), \\
&= 2\sqrt{2} - 2.
\end{aligned}
$$

This computation is quite formal since everything has been restricted to only one strip but it is easy to see that doing the computation on the extended configuration drawn in Figure 8 leads exactly to the same result.

Eventually, we realize that making a change of scale of this cross tie configuration doesn't change its energy since the energy behaves like a length. It is therefore possible to draw a microstructure cross-tie wall which possesses the same energy per unit length.

IV.2.2 The situation with a jump in the phase less than π

The optimal cross-tie configuration is much more delicate to understand in this situation. Indeed, we must find a configuration where all the angle jumps are less than $\pi/2$, otherwise these jumps could be replaced by other cross-tie patterns in another smaller scale. Keeping that in mind, we propose the following picture. Around the saddle points we take the same configuration as before (any change here would lead to a bad angle according to the preceding remark)

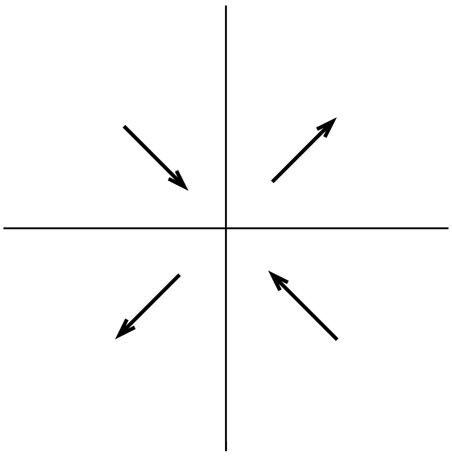

Figure 10: Around the saddle point the configuration consists in 4 domains with constant magnetization.

The only remaining freedom is the relative size of these constantly magnetized subdomains. It is possible to consider triangular domains as in Figure IV.2.2 and try to extend the configuration outside.

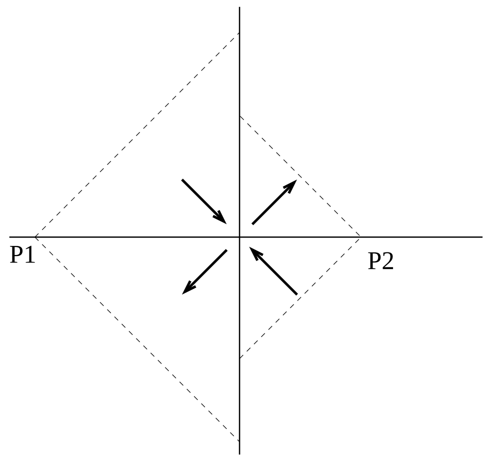

Figure 11: triangles with different sizes.

On the right hand side of the domain, the solution is taken to be

$$u(x) = -\nabla^{\perp}\mathrm{dist}(x, P_2),$$

and we conversely take on the left hand side

$$u(x) = -\nabla^{\perp}\mathrm{dist}(x, P_1).$$

Now the problem becomes to find the two curves separating these two solutions in a way which is consistent with the fact that u is divergence-free. An analytical computation of these curves is possible in the same spirit than when one searches the shock curve of a weak solution of a hyperbolic scalar partial differential equation from Rankine-Hugoniot relation. However, a very simple argument shows that, in our case, this discontinuity line consists of a piece of parabola and a piece of hyperbola as in Figure 12. Indeed, the first part of this curve separates two regions where the characteristic lines are straight lines of direction $\pi/4$ on the left hand side and straight lines emerging from P_2 on the right hand side. Noticing that the tangent to the discontinuity line bissects the angle made by the characteristic on both sides, we get that this discontinuity curve exactly transforms rays emerging from P_2 into rays of direction $\pi/4$ by Fermat's reflection principle. It is therefore a parabola of focal point P_2 and direction $\pi/4$. Farther, the curve transforms light rays emerging from P_2 into light rays emerging from P_1 and is therefore a hyperbola of focal points P_1 and P_2.

In order to compute the energy of this configuration, we proceed as in the preceding case, using the geometric point of view (the direct computation of the energy density along the discontinuity lines is really awkward). Therefore, we will find a convex domain in which the orthogonal gradient of the distance to the boundary function possesses the same jumps as u, and we will use the formula

$$E(u) = |\partial\Omega| - 2\pi r, \tag{IV.8}$$

where r is the radius of the biggest disc inscribed in Ω to obtain an upper bound (see Section III) of the energy. As before, the upper bound obtained will be equal to the lower bound obtained by the formula (IV.4) proving the optimality.

We begin by cutting one strip of the pattern and close it by extending the solution to two discs. One centered in P_1 and of radius $R+C$ and the other centered in P_2 and of radius R. The constant C is tuned in order that both circles intersect at the hyperbola. In other words, the equation of the hyperbola is

$$\mathrm{dist}(x, P_1) = \mathrm{dist}(x, P_2) + C.$$

As before, interchanging the right hand parts of the egg-like domains with the left-hand parts leads to a domain drawn in Figure 13, which is still convex and on which we have to compute the energy of the orthogonal gradient of the distance to the boundary function.

We now use the formula (IV.8). In view of the shape of the domain (see Figure 13 for notations), we get for the left hand part of the boundary

$$|(\partial\Omega)_L| = 2\left(\left(\frac{3\pi}{4} - \theta_2\right) R + \frac{1}{\sqrt{2}} + \frac{\pi}{4}\left(R - \frac{1}{\sqrt{2}}\right)\right), \tag{IV.9}$$

and for the right hand part

$$|(\partial\Omega)_R| = 2\left(\left(\theta_1 - \frac{\pi}{4}\right)(R+C) + \frac{1}{\sqrt{2}} + \frac{\pi}{4}\left(R + C - \frac{1}{\sqrt{2}}\right)\right). \tag{IV.10}$$

Moreover, a quick computation shows that the radius r of the biggest disc inscribed in Ω is given by

$$r = R - \frac{1 - C\sqrt{2}}{2\sqrt{2}}. \tag{IV.11}$$

Putting (IV.9), (IV.10) and (IV.11) together yields

$$\begin{aligned} E(u) &= |(\partial\Omega)_L| + |(\partial\Omega)_R| - 2\pi r, \\ &= 2\left(-\theta_2 R + \sqrt{2} + (R+C)\theta_1 - \frac{\pi C}{2}\right), \\ &= 2\left(\sqrt{2} + R(\theta_1 - \theta_2) + C\left(\theta_1 - \frac{\pi}{2}\right)\right). \end{aligned}$$

Now, letting R tends to $+\infty$, we get the expression of the parameters in terms of the half-angle of the jump

$$C = \cos(X),$$

$$\lim_{R\to+\infty} R(\theta_1 - \theta_2) = -\sin(X),$$

and

$$\lim_{R\to+\infty} \theta_1 = X,$$

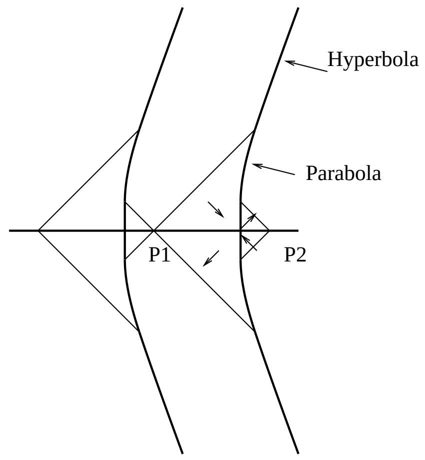

Figure 12: Oblique cross-tie wall (zoom).

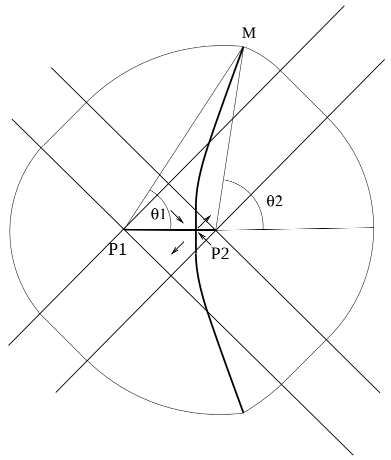

Figure 13: egg-like domain in the oblique case

from which we get the expression for the energy of one cross-tie strip

$$E(u) = 2\left(\sqrt{2} - \sin(X) - \left(\frac{\pi}{2} - X\right)\cos(X)\right),$$

which is the claim.

We have drawn in Figure 14 the pattern obtained by gluing together oblique strips constructed before and in Fig. 15 and 16 reproduced several pictures of cross-tie wall (both straight and oblique) obtained on thin permalloy films and extracted from [HS]. They are strikingly similar.

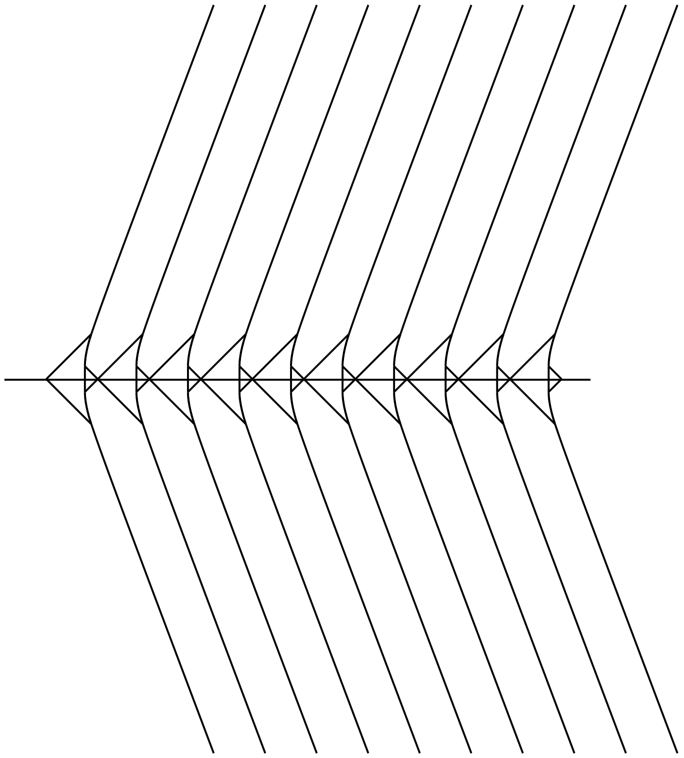

Figure 14: Oblique cross-tie wall.

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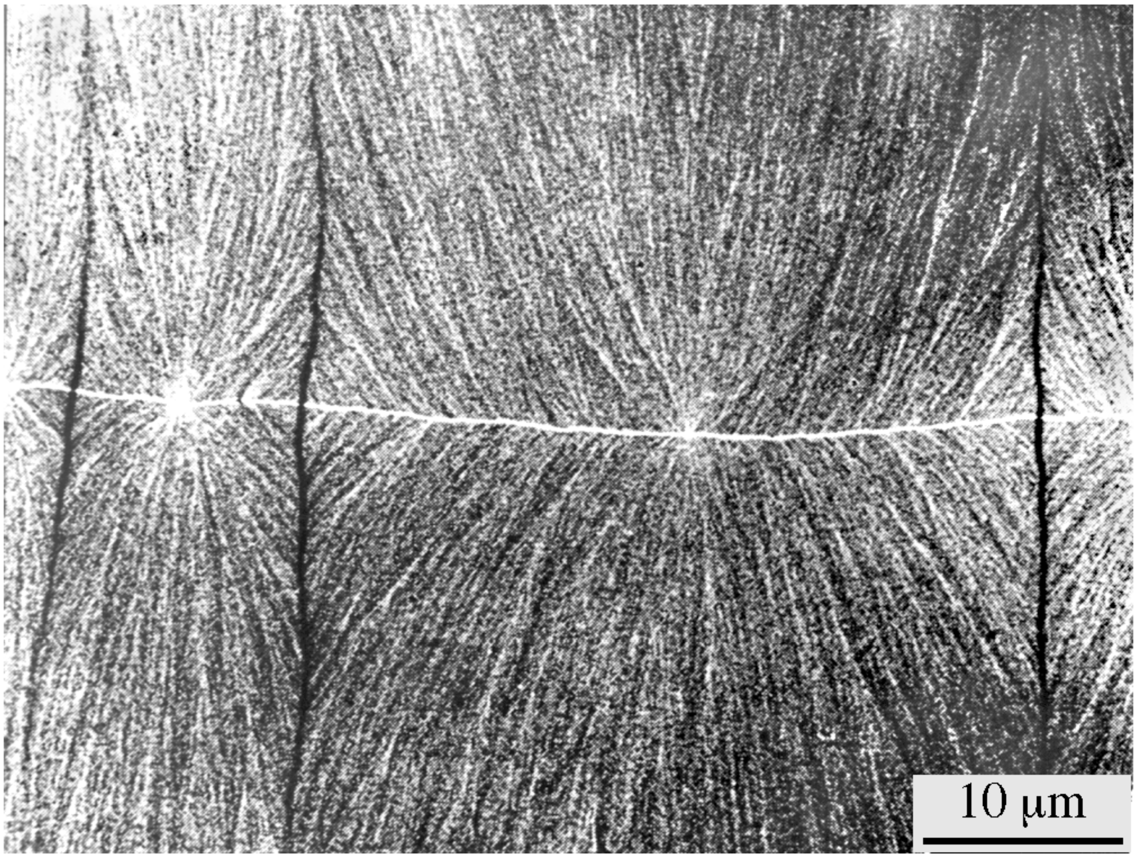

Figure 15: A cross-tie wall in a Permalloy sample (Courtesy R. Schäfer)

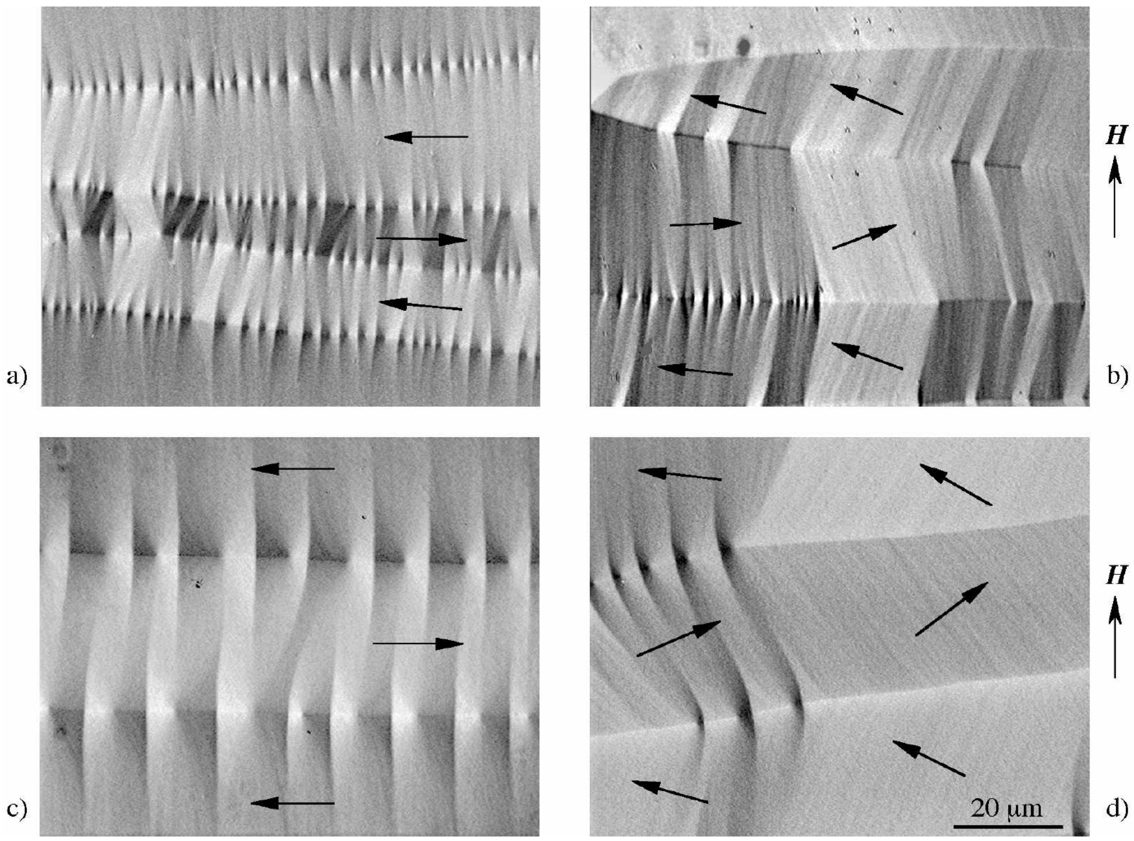

Figure 16: Oblique cross-tie walls in Permalloy (Courtesy R. Schäfer)

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