Regularity for micromagnetic configurations having zero jump energy

Myriam Lecumberry *and Tristan Rivière †

1 Introduction

In this paper, we consider the following family of energy-functionals, related to micromagnetics (see [10] for a more detailed presentation of the physical relevance of the model):

$$E_\epsilon(u) = \int_\Omega \frac{\epsilon}{2}\,|\nabla u|^2 + \frac{1}{2\epsilon}\int_{\mathbb{R}^2} H_u^2$$

where Ω is a bounded simply connected open domain of $\mathbb{R}^2$, $u \in H^1(\Omega, S^1)$ is the vector field corresponding to magnetization and H_u the demagnetizing vector field, defined by

$$\begin{cases} \operatorname{div}(\bar{u} + H_u) = 0 & \text{in } \mathbb{R}^2 \\ \operatorname{curl}(H_u) = 0 & \text{in } \mathbb{R}^2 \end{cases}$$

where $\bar{u}$ is the extension of u by 0 out of Ω.

One of the main issues is to understand the asymptotic behaviour of configurations having uniformly bounded energy E_ϵ while $\epsilon \to 0$. Let us recall the compactness result, proved in [11]:

Let $\epsilon_n \to 0$ and $u_n \in H^1(\Omega, S^1)$ such that u_n has a lifting $\phi_n \in H^1(\Omega, \mathbb{R})$ (ie $u_n = e^{i\phi_n}$ a.e.). Assume that $E_{\epsilon_n}(u_n) \leq C$ and $\|\phi_n\|_{L^\infty} \leq N$. Then there exist u and ϕ in L^p, $\forall\, p < \infty$ such that, up to an extraction

*Laboratoire de Mathématiques, UMR 6629, Université de Nantes, 2 rue de la Houssinière 44322 Nantes Cedex 03, France. E-mail: Myriam.Lecumberry@math.univ-nantes.fr

†* Centre de Mathématiques, CNRS UMR 7640, Ecole Polytechnique, 91128 Palaiseau Cedex, France. E-mail: Tristan.Riviere@math.polytechnique.fr

$\phi_n \to \phi$ and $u_n \to u$ strongly in L^p, $\forall\, p < \infty$.

Moreover, u and ϕ verify:

i) $\operatorname{div} \bar{u} = 0$ in $\mathcal{D}'(\mathbb{R}^2)$

ii) $u = e^{i\phi}$ a.e. in Ω

iii) $\operatorname{div}(\phi u + u^{\perp})$ is a bounded Radon measure on Ω

We denote by $u^{\perp}$ the rotation by $\frac{\pi}{2}$ of $u = (u_1, u_2)$, i.e $u^{\perp} = (-u_2, u_1)$.

$\mathcal{C}$ will denote the class of couples (u, ϕ), $u : \Omega \to S^1$ and $\phi \in L^1(\Omega, \mathbb{R})$, satisfying i), ii) and iii).

The measure $\operatorname{div}(\phi u + u^{\perp})$, that vanishes as long as u is a smooth divergence-free unit vector field, was introduced in [10] as being the most probable candidate for the Γ.convergence of E_ϵ. In particular, it was proved in [10]:

$$\liminf_{\epsilon \to 0} E_\epsilon(u) = \inf_{u \in \mathcal{C}} \int_{\Omega} |\operatorname{div}(\phi u + u^{\perp})|$$

Nevertheless, it is not impossible that $\mathcal{C}$ may be larger than the set of configurations which can be reached by the above convergence process. Precisely, it could happen that $\mathcal{C}_L \subsetneq \mathcal{C}$, where $\mathcal{C}_L$ denotes the set of (u, ϕ) such that u and ϕ are limits in L^1 of sequences (u_n) and (ϕ_n) in $H^1(\Omega)$ which satisfy $u_n = e^{i\phi_n}$, a.e. in Ω, $E_{\epsilon_n}(u_n) \leq C$, and $\|\phi_n\|_{L^\infty} \leq N$, for some C,$N > 0$.

The measure $\operatorname{div}(\phi u + u^{\perp})$, for any $(u, \phi) \in \mathcal{C}_L$, admits the following interpretation in terms of truncations:
The truncated vector field $T^a u$ is defined by

$$\begin{cases} T^a \phi = \inf(\phi, a) \\ T^a u = e^{iT^a \phi} \end{cases}$$

It was proved in [11] that (u, ϕ) in $\mathcal{C}_L$ satisfies the following equality,

$$\operatorname{div}(\phi u + u^{\perp}) = -\int_{\mathbb{R}} \operatorname{div} T^a u \, da \tag{1}$$

It is not clear that (1) holds for any (u, ϕ) in $\mathcal{C}$. It would be perhaps more relevant to replace $\mathcal{C}$ by the class $\mathcal{C}^*$ of (u, ϕ) satisfying i), ii) and

iv) $\operatorname{div} T^a u$ is a bounded Radon measure on $\Omega \times \mathbb{R}$

Let us recall the following inequality, proved in [11]:

Proposition 1.1 *If $(u, \phi) \in \mathcal{C}_L$, then*

$$\int |div(\phi u + u^{\perp})| \leq \iint |div T^a u|\, da \leq \liminf_{n \to \infty} \int |\nabla \phi_n . H_n|$$

for all sequences $u_n = e^{i\phi_n}$, $\phi_n \in H^1(\Omega)$, such that $E_{\epsilon_n}(u_n)$ and $\|\phi_n\|_{L^\infty}$ are uniformly bounded, and $\phi_n \to \phi$ in L^1 (we denote H_{u_n} by H_n).

Therefore, we have

$$\mathcal{C}_L \subset \mathcal{C}^* \subset \mathcal{C} \tag{2}$$

But, we don't know whether these inclusions are strict or not.

At this stage of the development of the understanding of our problem, it seems more reasonable to expect $\iint |\operatorname{div} T^a u| da$ to be the Γ.limit of E_ϵ in $\mathcal{C}^*$ instead of $\int |\operatorname{div}(\phi u + u^{\perp})|$. In this paper, we explore how far $\iint |\operatorname{div} T^a u| da$ characterizes the lack of smoothness of configurations in $\mathcal{C}_L$. Precisely, our main result is the following:

Theorem 1.1 *If $(u, \phi) \in \mathcal{C}_L$, then there is an equivalence between:*

1. *$div T^a u = 0$, in $\mathcal{D}'(\Omega)$, a.e. $a \in \mathbb{R}$.*
2. *$\phi \in H^{1/2}(\Omega)$.*

Moreover, if 1. (and 2.) holds, then for any $\omega \subset\subset \Omega$, ϕ is Lipschitz in ω and level sets of ϕ in ω are straight segments that cross each other nowhere in ω.

The proof of the equivalence between 1. and 2. in Theorem 1.1 is based on a kinetic interpretation of the problem, given in [11]. Let us define χ on $\Omega \times \mathbb{R}$ by $\chi(x, a) = 1$ if $\phi(x) \leq a$ and $\chi(x, a) = 0$ else. Then, the following kinetic equation holds for every (u, ϕ) in $\mathcal{C}_L$:

$$\nabla_x \chi(x, a).(e^{ia})^{\perp} = -\partial_a(\operatorname{div} T^a u), \text{ in } \mathcal{D}'(\Omega \times \mathbb{R}) \tag{3}$$

This was inspired by a kinetic interpretation, made in [8], of the "Aviles-Giga problem". This last problem, which was also studied in [1], [4], and [7], has many similarities with the one we consider here.

The local Lipschitz regularity of ϕ is obtained by rewriting equation for ϕ in suitable coordinates so that it becomes, under assumptions 1. (or 2.) of Theorem 1.1, an entropic solution to Burger equation.

2 Proof of theorem 1.1

2.1 Sobolev regularity

Let us prove the equivalence in theorem 1.1.

Step 1:

Let $(u, \phi) \in \mathcal{C}_L$ and assume that $\phi \in H^{1/2}(\Omega)$.
Let us prove that for any $h \in C_c^\infty(\mathbb{R})$ and $\xi \in C_c^\infty(\Omega)$,

$$\int_{\mathbb{R}} \int_{\Omega} \operatorname{div} T^a u \, h(a) da \, \xi(x) dx = 0 \tag{4}$$

We proceed to a standard regularization. Let (ϕ_n) be a sequence of smooth functions which tend to ϕ in $H^{1/2}(\Omega)$. Then, $u_n \to u$ in $H^{1/2}(\Omega)$, and thus in L^1. Since T^a is continuous from L^1 to L^1 uniformly in $a \in \mathbb{R}$, then

$$\int_{\mathbb{R}} \int_{\Omega} \operatorname{div} T^a u_n \, \xi(x) dx \, h(a) da \to \int_{\mathbb{R}} \int_{\Omega} \operatorname{div} T^a u \, \xi(x) dx \, h(a) da$$

Let H be a primitive of h,

$$\begin{aligned} \int_{\mathbb{R}} \int_{\Omega} \operatorname{div} T^a u_n \, \xi(x) dx \, h(a) da &= \int_{\Omega} \int_{-\infty}^{\phi_n(x)} \operatorname{div} u_n \, h(a) da \, \xi(x) dx \\ &= \int_{\Omega} H(\phi_n(x)) \operatorname{div} u_n \, \xi(x) dx \end{aligned}$$

But, since $u_n \to u$ in $H^{1/2}(\Omega)$ and u is divergence-free, $\operatorname{div} u_n \to 0$ in $H^{-1/2}(\Omega)$. Moreover, $H(\phi_n) \to H(\phi)$ in $H^{1/2}(\Omega)$, so $\int_\Omega H(\phi_n)(\operatorname{div} u_n)\xi \to 0$. Therefore, (4) holds.
Since h and ξ are arbitrary, we have $\operatorname{div} T^a u = 0$ in $\mathcal{D}'(\Omega)$, a.e. $a \in \mathbb{R}$.

Step 2:

Now, let $(u, \phi) \in \mathcal{C}_L$ and assume that $\operatorname{div} T^a u = 0$, a.e. in $a \in \mathbb{R}$. Since $a \mapsto \operatorname{div} T^a u$ is continuous from $\mathbb{R}$ into $\mathcal{D}'(\Omega)$ (see [11]), we have $\operatorname{div} T^a u = 0$ in $\mathcal{D}'(\Omega \times \mathbb{R})$. Then the kinetic equation (3) becomes

$$(e^{ia})^\perp . \nabla_x \chi(x, a) = 0 \tag{5}$$

We prove now that (5) implies that $\phi \in H^{1/2}$. Precisely, we have,

Lemma 2.1 *(Kinetic averaging lemma)*
Let $f, g \in L^2(\mathbb{R}^2 \times \mathbb{R})$ satisfying $(e^{ia})^{\perp}.\nabla_x f(x,a) = g(x,a)$, let $h \in C_c^\infty(\mathbb{R})$, and $\bar{f}$ is defined by $\bar{f}(x) = \int_{\mathbb{R}} f(x,a)h(a)da$, $\forall x \in \mathbb{R}^2$, then $\bar{f} \in H^{1/2}(\mathbb{R}^2)$.

Let $x_0 \in \Omega$ and $r > 0$ such that $B_{2r}(x_0) \subset \Omega$, let us define χ_T by $\chi_T = \chi\theta$ where $\theta \in C^\infty(\mathbb{R}^2)$, $\theta = 1$ on $B_r(x_0)$, $\mathrm{supp}(\theta) \subset B_{2r}(x_0)$. Then, χ_T satisfies $(e^{ia})^{\perp}.\nabla_x \chi_T = g$ where $g = (e^{ia})^{\perp}.(\nabla_x\theta)\chi \in L^2(\mathbb{R}^2 \times \mathbb{R})$. From the lemma we have,

$$\forall\, h \in C_c^\infty(\mathbb{R}),\ \bar{\chi_T}(x) = \int_{\mathbb{R}} \chi_T(x,a)h(a)da \in H^{1/2}(\mathbb{R}^2).$$

Let us choose h such that $h(a) = 1$, $\forall\, a \in [-N-1, N+1]$ with $N = \|\phi\|_\infty$ and $\mathrm{supp}(h) \subset [-N-2, N+2]$.
$\forall x \in B_r(x_0)$, $\bar{\chi_T}(x) = \int_{\mathbb{R}} \chi(x,a)h(a)da = \int_{a<\phi(x)} h(a)da = H(\phi(x))$ where H is a primitive of h. As $H(a) = a$, $\forall\, a \in [-N-1, N+1]$,
$\bar{\chi_T}(x) = \phi(x)$, $\forall\, x \in B_r(x_0)$. Hence, $\phi \in H^{1/2}(B_r(x_0))$, $\forall x_0 \in \Omega$.
The equivalence in Theorem 1.1 is proved.

■

Proof of the lemma 2.1:
Kinetic averaging lemmas were introduced by F. Golse, P.L. Lions, B. Perthame and R. Sentis in [6]. We give here a simpler proof of this regularity result, adapted to our situation, which was communicated to us by B. Perthame.
$\hat{f}$ is the partial Fourier transform in x of f: $\hat{f}(\xi,a) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ix.\xi} f(x,a)dx$
Taking the partial Fourier transform of the equation (5), we obtain:

$$i((e^{ia})^{\perp}.\xi)\hat{f}(\xi,a) = \hat{g}(\xi,a) \qquad \text{a.e. in } (\xi,a) \in \mathbb{R}^2 \times \mathbb{R}$$

and,

$$\hat{f}(\xi,a) = \frac{\hat{g}(\xi,a) + \hat{f}(\xi,a)}{1 + i((e^{ia})^{\perp}.\xi)}$$

Since $\hat{\bar{f}}(\xi) = \int_{\mathbb{R}} \hat{f}(\xi,a)h(a)da$, we have

$$|\hat{\bar{f}}(\xi)|^2 \leq \int_{\mathbb{R}} (|\hat{f}(\xi,a)|^2 + |\hat{g}(\xi,a)|^2)da \int_{\mathbb{R}} \frac{h^2(a)}{|1 + i((e^{ia})^{\perp}.\xi)|^2}da$$

But,

$$\begin{aligned}\int_{\mathbb{R}} \frac{h^2(a)}{|1 + i((e^{ia})^{\perp}.\xi)|^2}da = \int_{\mathbb{R}} \frac{h^2(a)}{1 + ((e^{ia})^{\perp}.\xi)^2}da &\leq C(h)\int_{-\pi}^{\pi} \frac{d\theta}{1 + (|\xi|cos\theta)^2} \\ &\leq \frac{\tilde{C}(h)}{1+|\xi|}\end{aligned}$$

where $\tilde{C}(h)$ depends only on h.
Since $\|\bar{f}\|^2_{H^{1/2}(\mathbb{R}^2)} = \int_{\mathbb{R}^2}(1+|\xi|)|\hat{\bar{f}}(\xi)|^2 d\xi$, we have
$\|\bar{f}\|^2_{H^{1/2}(\mathbb{R}^2)} \leq \tilde{C}(h)(\|\hat{f}\|^2_{L^2} + \|\hat{g}\|^2_{L^2}) = \tilde{C}(h)(\|f\|^2_{L^2} + \|g\|^2_{L^2})$.
Then, $\bar{f} \in H^{1/2}(\mathbb{R}^2)$. ■

Actually, from (5) we can bootstrap to improve the regularity of ϕ obtained above. Precisely the following proposition holds.

Proposition 2.1 *Let $(u,\phi) \in \mathcal{C}_L$, if $div\, T^a u = 0$, a.e. in $a \in \mathbb{R}$, then $\phi \in H^s(\Omega)$, $\forall s < 1$.*

Proof of Proposition 2.1:
The regularity of ϕ and χ are linked as follows:

$$\phi \in W^{\sigma,1}(\Omega) \iff \chi \in L^2(\mathbb{R}, H^{\frac{\sigma}{2}}(\Omega)) \tag{6}$$

Indeed, we have

$$\begin{aligned} \int_a \|\chi(.,a)\|^2_{H^{\frac{\sigma}{2}}(\Omega)} da &= \int_a \int_\Omega \int_\Omega \frac{|\chi(x,a) - \chi(y,a)|^2}{|x-y|^{2+\sigma}} dx dy da \\ &= \int_\Omega \int_\Omega \frac{|\phi(x) - \phi(y)|}{|x-y|^{2+\sigma}} dx dy \end{aligned}$$

The last term in the above identity is finite if and only if $\phi \in W^{\sigma,1}(\Omega)$, which implies the equivalence (6).

Since $H^{1/2}(\Omega) \hookrightarrow W^{\frac{1}{2},1}(\Omega)$, if $\phi \in H^{1/2}(\Omega)$, then $\chi \in L^2(\mathbb{R}, H^{\frac{1}{4}}(\Omega))$.
Let $x_0 \in \Omega$, $r > 0$ such that $B_{2r}(x_0) \subset \Omega$ and χ_T defined above.
Set $\chi_1 = \mathcal{F}_x^{-1}((1+|\xi|^2)^{1/8}\hat{\chi_T}(\xi,a))$ where $\mathcal{F}_x^{-1}$ is the inverse Fourier transform in x.
Then, we easily check that $\chi_1 \in L^2(\mathbb{R}^2 \times \mathbb{R})$ and χ_1 satisfies a kinetic equation $(e^{ia})^\perp.\nabla_x \chi_1 \in L^2(\mathbb{R}^2 \times \mathbb{R})$. From the lemma 2.1, $\bar{\chi_1} \in H^{1/2}(\mathbb{R}^2)$.
But,

$$\hat{\bar{\chi_1}}(\xi) = \int (1+|\xi|^2)^{1/8} \hat{\chi}(\xi,a) h(a) da = (1+|\xi|^2)^{1/8} \hat{\bar{\chi}}(\xi)$$

Thus, $\bar{\chi_1} \in H^{1/2}(\mathbb{R}^2) \Leftrightarrow \bar{\chi} \in H^{\frac{1}{2}+\frac{1}{4}}(\mathbb{R}^2)$.
Therefore, $\phi \in H^{\frac{1}{2}+\frac{1}{4}}(\Omega)$.
Iterating the argument above, we have $\phi \in H^s(\Omega)$, $\forall\, s < 1$.
■

2.2 BV regularity of ϕ

Here, we assume that 1. (and 2.) in theorem 1.1 hold.
The limit field u satisfies:

$$\begin{cases} \operatorname{div} u = 0 & \text{in } \Omega \\ u.\nu = 0 & \text{on } \partial\Omega, \text{ where } \nu \text{ is the outer unit-normal of } \partial\Omega \\ |u| = 1 & \text{in } \Omega \end{cases}$$

Moreover we assume that $\operatorname{div} T^a u = 0$, a.e. in $a \in \mathbb{R}$, which implies that $\operatorname{div}(\phi u + u^\perp) = 0$ by (1). Then, there exist two Lipschitz functions g and h satisfying,

$$\begin{cases} u = \nabla^\perp g & \text{in } \Omega \\ \phi u + u^\perp = \nabla^\perp h & \text{in } \Omega \end{cases}$$

where $\nabla^\perp = (-\frac{\partial}{\partial y}, \frac{\partial}{\partial x})$.

$$\text{Let } \Psi : \begin{array}{ccc} \Omega & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & (g,h) \end{array}$$

Since u and ϕ are in L^∞, Ψ is Lipschitz and at every point where $D\Psi$ exists, $\operatorname{Jac}(\Psi) = \nabla g.\nabla^\perp h = u^\perp.(\phi u + u^\perp) = 1$, because $|u| = 1$.
Let ρ_n be a regularizing kernel in $\mathbb{R}^2$, and set $\Psi_n = \Psi * \rho_n$. Then,

$$\Psi_n \to \Psi \text{ uniformly on every compact subset of } \Omega. \tag{7}$$

Since $\nabla\Psi \in L^p(\Omega)$, $\forall\, p \leq \infty$, Jac $\Psi_n \to$ Jac Ψ in $L^1_{loc}(\Omega)$, and we have

$$\text{Jac } \Psi_n(x,y) \to \text{Jac } \Psi(x,y), \quad \text{a.e. } (x,y) \in \Omega \tag{8}$$

Let $(x_0, y_0) \in \Omega$ satisfying (8). There exists $N \in \mathbb{N}$ such that

$$\text{Jac } \Psi_n(x_0,y_0) \geq \frac{1}{2}, \quad \forall\, n \geq N$$

By the inverse function theorem, for every $n \geq N$, there exist V_n an open neighborhood of (x_0, y_0) and W_n an open neighborhood of $\Psi_n(x_0, y_0)$, such that $\Psi_n : V_n \to W_n$ is inversible. The convergences (7) and (8) allow us to choose V_n and W_n such that $\bigcap_{n \geq N} W_n$ contains an open neighborhood W of $(g_0, h_0) = \Psi(x_0, y_0)$.
By a diagonal argument, since Ψ_n is smooth, we have, up to an extraction, Jac $\Psi_n(x,y) \geq \frac{1}{2}$, $\forall\, n \in \mathbb{N}$, $\forall\,(x,y) \in \Omega$. Then, $\|\text{Jac } \Psi_n^{-1}\|_{L^\infty(W)} \leq 2$ and (Ψ_n^{-1}) is bounded in $W^{1,\infty}(W)$. Since the injection $W^{1,\infty}(W) \subset C(\bar{W})$ is compact, there exists $\Psi^* \in C(W)$ such that

$$\Psi_n^{-1} \to \Psi^* \text{ uniformly on every compact subset of } W. \tag{9}$$

Moreover, since $\|\text{Jac } \Psi_n^{-1}\|_{L^\infty(W)}$ is uniformly bounded, then Ψ^* is Lipschitz. Finally, (7) and (9) imply that $\Psi_n \circ \Psi_n^{-1} \to \Psi \circ \Psi^*$ a.e. in W and then, $\Psi \circ \Psi^* = \text{Id}$ on W.
Similarly, we have $\Psi^* \circ \Psi = \text{Id}$ on $\Psi^{-1}(W)$.
If we set $V := \Psi^{-1}(W)$, V is an open neighborhood of (x_0, y_0), W is an open neighborhood of $(g_0, h_0) = \Psi(x_0, y_0)$ and $\Psi : V \to W$ is inversible. Moreover, $\Psi^{-1} = \Psi^*$ is Lipschitz.

For any function f defined on V, we will denote by $\tilde{f}$ the function $f \circ \Psi^{-1}$ defined on W. Since Ψ is Lipschitz, the BV regularity of $\tilde{\phi}$ will imply the BV regularity of ϕ.

Proposition 2.2 *$\tilde{\phi}$ is a weak solution to Burger equation ie*

$$\forall\, \tilde{v} \in C_c^\infty(W), \quad \int_W \tilde{\phi}\frac{\partial \tilde{v}}{\partial g} + \frac{\tilde{\phi}^2}{2}\frac{\partial \tilde{v}}{\partial h} = 0 \tag{10}$$

Moreover, for every $S \in C^2(\mathbb{R}, \mathbb{R})$,

$$\forall\, \tilde{v} \in C_c^\infty(W), \quad \int_W S(\tilde{\phi})\frac{\partial \tilde{v}}{\partial g} + F(\tilde{\phi})\frac{\partial \tilde{v}}{\partial h} = 0, \tag{11}$$

where F is defined by $F'(t) = tS'(t)$, $\forall\, t \in \mathbb{R}$.

Proof of Proposition 2.2:

Step 1:

Let us show that $\tilde{\phi}$ is a weak solution of Burger equation.
Let $\rho \in C_c^\infty(\mathbb{R}^2)$ such that $supp(\rho) \subset B(0,1)$, the unit ball of $\mathbb{R}^2$, $\rho \geq 0$ and $\int \rho = 1$ and let $\rho_n(x) = n^2\rho(nx)$, then $\phi_n = \phi * \rho_n \to \phi$ in $H^{1/2}(V)$.
Since Ψ^{-1} is Lipschitz, using the change of variables formula of [5], chapter 3, we also have $\tilde{\phi_n} \to \tilde{\phi}$ in $H^{1/2}(W)$.

Let $\tilde{v} \in C_c^\infty(W)$ and $v = \tilde{v} \circ \Psi$. Let $u_n = e^{i\phi_n}$. Since v is Lipschitz, then $v \in H^{1/2}$, and since $\operatorname{div} u = 0$ in $H^{-1/2}$ we have,

$$\int (\operatorname{div} u_n) v \longrightarrow \int (\operatorname{div} u) v = 0$$

$$\text{But, } \int \frac{\partial}{\partial x}(\cos\phi_n)v = \int \left[\left(\frac{\partial}{\partial g}(\cos\tilde{\phi_n}) \circ \Psi \right) \frac{\partial g}{\partial x} + \left(\frac{\partial}{\partial h}(\cos\tilde{\phi_n}) \circ \Psi \right) \frac{\partial h}{\partial x} \right] v$$
$$= \int \left[\left(\frac{\partial}{\partial g}(\cos\tilde{\phi_n}) \circ \Psi \right) \sin\phi + \left(\frac{\partial}{\partial h}(\cos\tilde{\phi_n}) \circ \Psi \right) (\phi\sin\phi + \cos\phi) \right] v$$

Making the change of variables $(x,y) \mapsto \Psi(x,y) = (g,h)$, we get

$$\begin{aligned} \int_V \frac{\partial}{\partial x}(\cos\phi_n)v &= \int_W \left[\frac{\partial}{\partial g}(\cos\tilde{\phi_n})\sin\tilde{\phi} + \frac{\partial}{\partial h}(\cos\tilde{\phi_n})(\tilde{\phi}\sin\tilde{\phi} + \cos\tilde{\phi}) \right] \tilde{v} \\ &= \int_W \Bigg[-\sin\tilde{\phi_n}\sin\tilde{\phi}\frac{\partial\tilde{\phi_n}}{\partial g} \\ &\qquad -\sin\tilde{\phi_n}(\tilde{\phi}\sin\tilde{\phi} + cos\tilde{\phi})\frac{\partial\tilde{\phi_n}}{\partial h} \Bigg] \tilde{v}. \end{aligned} \tag{12}$$

In one hand, $\frac{\partial\tilde{\phi_n}}{\partial g} \rightharpoonup \frac{\partial\tilde{\phi}}{\partial g}$ in $H^{-1/2}$. In the other hand, for every function $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$, $f(\tilde{\phi_n}, \tilde{\phi}) \to f(\tilde{\phi}, \tilde{\phi})$ in $H^{1/2}(W)$, so we have

$$\int -\sin\tilde{\phi_n}\sin\tilde{\phi}\frac{\partial\tilde{\phi_n}}{\partial g}\tilde{v} \longrightarrow \int -(\sin\tilde{\phi})^2\frac{\partial\tilde{\phi}}{\partial g}\tilde{v}$$

The same argument applied to the second term in (12) gives

$$\int_V \frac{\partial}{\partial x}(\cos\phi_n)v \longrightarrow \int_W \left[-(\sin\tilde{\phi})^2\frac{\partial\tilde{\phi}}{\partial g} - \sin\tilde{\phi}(\tilde{\phi}\sin\tilde{\phi} + \cos\tilde{\phi})\frac{\partial\tilde{\phi}}{\partial h} \right] \tilde{v} \tag{13}$$

Similarly, one can show that

$$\int_V \frac{\partial}{\partial y}(\sin\phi_n)v \longrightarrow \int_W \left[-(\cos\tilde{\phi})^2\frac{\partial\tilde{\phi}}{\partial g} - \cos\tilde{\phi}(\tilde{\phi}\cos\tilde{\phi} - \sin\tilde{\phi})\frac{\partial\tilde{\phi}}{\partial h} \right] \tilde{v} \tag{14}$$

Then, from (13) and (14), we have,

$$\int_W \left(-\frac{\partial\tilde{\phi}}{\partial g} - \tilde{\phi}\frac{\partial\tilde{\phi}}{\partial h} \right) \tilde{v} = 0 \tag{15}$$

Since $\tilde{\phi} \in H^{1/2}$, we have

$$\int \tilde{\phi}\frac{\partial\tilde{\phi}}{\partial h}\tilde{v} = \int -\frac{\tilde{\phi}^2}{2}\frac{\partial\tilde{v}}{\partial h}$$

Indeed, $\int \tilde{\phi_n}\frac{\partial\tilde{\phi_n}}{\partial h}\tilde{v} = \int -\frac{\tilde{\phi_n}^2}{2}\frac{\partial\tilde{v}}{\partial h}$. Since $\tilde{\phi_n} \to \tilde{\phi}$ in L^2, $\frac{\tilde{\phi_n}^2}{2}\frac{\partial\tilde{v}}{\partial h} \to \frac{\tilde{\phi}^2}{2}\frac{\partial\tilde{v}}{\partial h}$.
And since $\tilde{\phi_n} \to \tilde{\phi}$ in $H^{1/2}$, $\int \tilde{\phi_n}\frac{\partial\tilde{\phi_n}}{\partial h}\tilde{v} \to \int \tilde{\phi}\frac{\partial\tilde{\phi}}{\partial h}\tilde{v}$.
Finally,

$$\int_W \left(\tilde{\phi}\frac{\partial\tilde{v}}{\partial g} + \frac{\tilde{\phi}^2}{2}\frac{\partial\tilde{v}}{\partial h}\right) = 0 \quad \forall\, \tilde{v} \in C_c^\infty(W).$$

Step 2:

The second part of proposition 2.2 is a simple consequence of the fact that $\tilde{\phi}$ is a weak solution to Burger equation on W and $\tilde{\phi} \in H^{1/2}(W)$.
Let $S, F \in C^2(\mathbb{R}, \mathbb{R})$ be such that

$$F'(t) = tS'(t), \quad \forall\, t \in \mathbb{R}. \tag{16}$$

$\forall\, \tilde{v} \in C_c^\infty(W)$,

$$\int_W S(\tilde{\phi})\frac{\partial\tilde{v}}{\partial g} + F(\tilde{\phi})\frac{\partial\tilde{v}}{\partial h} = \int_W -S'(\tilde{\phi})\frac{\partial\tilde{\phi}}{\partial g}\tilde{v} - F'(\tilde{\phi})\frac{\partial\tilde{\phi}}{\partial h}\tilde{v} \tag{17}$$

Indeed, let $\tilde{\phi}_n$ be a sequence of functions in $C^\infty(W)$ which tends to $\tilde{\phi}$ in $H^{1/2}(W)$. (17) holds for $\tilde{\phi}_n$, $\forall\, n \in \mathbb{N}$. Since $f(\tilde{\phi}_n) \to f(\tilde{\phi})$ in $H^{1/2}(W)$, $\forall\, f \in C^1$, and since $D\tilde{\phi}_n \to D\tilde{\phi}$ in $H^{-1/2}(W)$, where D denotes any first-order derivative, then we can pass to the limit in the right-hand side of the equality to obtain (17).
By (16), we have,

$$\int_W S(\tilde{\phi})\frac{\partial\tilde{v}}{\partial g} + F(\tilde{\phi})\frac{\partial\tilde{v}}{\partial h} = \int_W -S'(\tilde{\phi})\left(\frac{\partial\tilde{\phi}}{\partial g} + \tilde{\phi}\frac{\partial\tilde{\phi}}{\partial h}\right)\tilde{v} \tag{18}$$

Applying (15) to the test function $S'(\tilde{\phi}_n)\tilde{v}$, we have

$$\int_W \left(\frac{\partial\tilde{\phi}}{\partial g} + \tilde{\phi}\frac{\partial\tilde{\phi}}{\partial h}\right) S'(\tilde{\phi}_n)\tilde{v} = 0, \ \forall\, n \in \mathbb{N}$$

Passing to the limit, the above equality holds for $S'(\tilde{\phi})$, then the right-hand side in (18) vanishes, and (18) yields to (11). This proves Proposition 2.2.

$\blacksquare$

Lemma 2.2 *There exists $T \subset W$, a neighborhood of (g_0, h_0), such that $\tilde{\phi} \in BV(T)$*

Proof of Lemma 2.2:

Assume $W =]g_0 - \alpha, g_0 + \alpha[\times]h_0 - \beta, h_0 + \beta[, \ \ \alpha, \beta > 0$.
For any $u \in L^1(W)$, by Fubini and Lebesgue theorems, the following limit

$$\int_{h_0-\beta}^{h_0+\beta} u(g,h)\,dh = \lim_{\epsilon\to 0}\int_0^\epsilon \int_{h_0-\beta}^{h_0+\beta} u(g+\sigma,h)\,dh\,d\sigma$$

holds $\forall\, g \in J_u$, where

$$J_u = \{g \text{ Lebesgue point of } g \mapsto u(g,h), \text{ for a.e. } h \in]h_0-\beta, h_0+\beta[\}$$

Note that $\mathcal{L}^1(]g_0 - \alpha, g_0 + \alpha[\setminus J_u) = 0$.

Let $N := \|\phi\|_\infty$ and let us define the trapezoid

$$T := \{(g,h) \in \mathbb{R}^2 \,|\, -r - N(g_0 + r - g) < h - h_0 < r + N(g_0 + r - g) \text{ and } g \geq g_1\}$$

where $r > 0$ and g_1 is any value picked in $J_{\tilde{\phi}}$ such that $T \subset W$.
From now on, g_1 will denote the origin of the g-axis, we will write $g_1 = 0$. We set $W^+ := W \cap \{g \geq 0\}$.

Since $\frac{\partial \tilde{\phi}}{\partial g} + \frac{\partial}{\partial h}\frac{\tilde{\phi}^2}{2} = 0$ in $\mathcal{D}'(W)$ and since $0 \in J_{\tilde{\phi}}$, we have

$$\forall\, \tilde{v} \in C_c^\infty(W^+), \ \int_{W^+}\left(\tilde{\phi}\frac{\partial \tilde{v}}{\partial g} + \frac{\tilde{\phi}^2}{2}\frac{\partial \tilde{v}}{\partial h}\right) + \int_{h_0-\beta}^{h_0+\beta} \tilde{\phi}(0,h)\tilde{v}(0,h)dh = 0 \quad (19)$$

Indeed, let $\chi_\epsilon \in C(\mathbb{R})$, $\chi_\epsilon(g) = 1$ for $g > \epsilon$, $\chi_\epsilon(g) = 0$ for $g < 0$ and $\chi_\epsilon(g) = \frac{g}{\epsilon}$ for $g \in [0, \epsilon]$. Then, $\forall\, \tilde{v} \in C_c^\infty(W^+)$,

$$0 = \int_W \left(\frac{\partial \tilde{\phi}}{\partial g} + \frac{\partial}{\partial h}\frac{\tilde{\phi}^2}{2}\right)\chi_\epsilon(g)\tilde{v}(g,h)dgdh = \int_W -\left(\tilde{\phi}\frac{\partial \tilde{v}}{\partial g} + \frac{\tilde{\phi}^2}{2}\frac{\partial \tilde{v}}{\partial h}\right)\chi_\epsilon - \tilde{\phi}\tilde{v}\chi'_\epsilon \quad (20)$$

Since $0 \in J_{\tilde{\phi}}$,

$$\int_W \tilde{\phi}\,\tilde{v}\,\chi'_\epsilon = \frac{1}{\epsilon}\int_0^\epsilon \int_{h_0-\beta}^{h_0+\beta} \tilde{\phi}(g,h)\tilde{v}(g,h)\,dh\,dg \underset{\epsilon\to 0}{\longrightarrow} \int_{h_0-\beta}^{h_0+\beta} \tilde{\phi}(0,h)\tilde{v}(0,h)\,dh$$

Hence, letting $\epsilon \to 0$ in (20) yields to (19).

Similarly, since $J_{\tilde{\phi}} = J_{S(\tilde{\phi})}$ for S Lipschitz, the second result of Proposition 2.2 implies that $\forall S, F \in C^2(\mathbb{R},\mathbb{R})$ satisfying (16),

$$\int_{W^+} \left(S(\tilde{\phi})\frac{\partial \tilde{v}}{\partial g} + F(\tilde{\phi})\frac{\partial \tilde{v}}{\partial h} \right) + \int_{h_0-\beta}^{h_0+\beta} S(\tilde{\phi})(0,h)\tilde{v}(0,h)\,dh = 0. \tag{21}$$

Let us recall a classical uniqueness result in the theory of conservation laws for one-dimensional space. The following theorem is contained in theorem 6.2.2 of [3] :

Theorem 2.1 *Let $u \in L^1(W^+)$ satisfying*

1. $0 \in J_u$
2. $\forall\, S, F \in C^2(\mathbb{R},\mathbb{R})$, *$S$ convex, F satisfying (16),* $\forall v \in C_c^\infty(W^+)$, $v \geq 0$,

$$\int_{W^+} \left(S(u)\frac{\partial v}{\partial g} + F(u)\frac{\partial v}{\partial h} \right) + \int_{h_0-\beta}^{h_0+\beta} S(u)(0,h)v(0,h)\,dh \geq 0 \tag{22}$$

Assume that $\bar{u} \in L^1(W^+)$ also satisfies 1. and 2.
Then, $\forall\, g \in J_u \cap J_{\bar{u}}$,

$$\int_{T_g} |u(g,h) - \bar{u}(g,h)|\,dh \leq \int_{T_0} |u(0,h) - \bar{u}(0,h)|\,dh$$

where $T_g :=]-M(g_0+r-g)+h_0-r, M(g_0+r-g)+h_0+r[$ is the section of T at the abscisse g.
Therefore, if the right-hand side of the inequality is equal to zero, u and $\bar{u}$ coincide a.e. in T.

Observe that $\tilde{\phi}$ satisfies 1. and 2.

Now, let us consider $\psi \in L^1 \cap L^\infty(\mathbb{R}^+,\mathbb{R})$ such that $\forall v \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$, $v \geq 0$, $\forall\, S, F \in C^2(\mathbb{R},\mathbb{R})$, S convex, F satisfying (16),

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(S(\psi)\frac{\partial v}{\partial g} + F(\psi)\frac{\partial v}{\partial h} \right) + \int_{\mathbb{R}} S(\psi_0)(h)v(0,h)\,dh \geq 0 \tag{23}$$

with $\psi_0(h) = \tilde{\phi}(0,h)$ if $h \in [h_0-\beta, h_0+\beta]$, $\psi_0(h) = 0$ else.
Such a ψ exists and is unique (see [3], chapter VI). Moreover, $\psi \in C^0([0,+\infty), L^1_{loc}(\mathbb{R}))$. Then, ψ satisfies 1. and 2. Hence,

$$\psi = \tilde{\phi} \quad \text{a.e. in } T, \tag{24}$$

and $\tilde{\phi}$ inherits the local properties of ψ. In particular, $\psi \in BV_{loc}(\mathbb{R}_+^* \times \mathbb{R})$ and $g \mapsto \psi(g,.)$ is Lipschitz from $]0,\sigma[$ to $L^1_{loc}(\mathbb{R})$, $\forall\, \sigma < +\infty$ (see [12], [13]).
Hence $\tilde{\phi} \in BV(T)$, which proves the lemma. Since $\tilde{\phi}$ is in BV_{loc}, then ϕ is in $BV_{loc}(\Omega)$.
Moreover, $g \mapsto \|\tilde{\phi}(g,.)\|_{L^1(T_g)}$ is Lipschitz in $]0, g_0 + \eta[$ which will be useful in the following section.

2.3 Lipschitz regularity of ϕ

The regularity of $\tilde{\phi}$ may be improved as follows:

Lemma 2.3 *$\tilde{\phi}$ is Lipschitz in T where T is the neighborhood of (g_0, h_0) given by Lemma 2.2.*

Proof of Lemma 2.3:

The weak entropy solution to Burger equation on $\mathbb{R}^+ \times \mathbb{R}$, ψ, has the following property (see [13]) : $\exists\, C > 0$ s.t.

$$\forall\, g > 0, \forall\, \sigma > 0, \text{ a.e. } h \in \mathbb{R}, \quad \frac{\psi(g, h+\sigma) - \psi(g,h)}{\sigma} < \frac{C}{g}$$

Thus, the application $h \mapsto \psi(g,h) - \frac{C}{g}h$ is nonincreasing and $h \mapsto \psi(g,h)$, as a sum of monotone functions, is $BV_{loc}(\mathbb{R})$, $\forall\, g > 0$. Then, the right limit $\psi(g,h_+)$ and the left limit $\psi(g,h_-)$ exist $\forall\, h \in \mathbb{R}$, $\forall\, g > 0$. Moreover we have $\psi(g,h_-) \geq \psi(g,h_+)$. Therefore, one can apply to ψ the method of generalized characterisitics, developped in [2], which gives the following results:

A generalized characteristic for ψ is a Lipschitz curve $g \in [a,b] \mapsto \xi(g)$ such that for a.e. $g \in [a,b]$, $\dot{\xi}(g) \in [\psi(g,\xi(g)_+), \psi(g,\xi(g)_-]$.
$\forall\, (\bar{g},\bar{h}) \in \mathbb{R}_+^* \times \mathbb{R}$, there exist at least one backward characteristic and a unique forward characteristic.

A characteristic η defined on a maximal interval $[a,b)$ satisfying $\forall\, g \in [a,b)$, $\psi(g,\eta(g)_-) > \psi(g,\eta(g)_+)$, is called a shock.
If $\psi(\bar{g},\bar{h}_-) > \psi(\bar{g},\bar{h}_+)$, then the unique forward characteristic through $(\bar{g},\bar{h})$ is a shock.

The set of shocks is at most countable and ψ has limits on each side of any shock. On the complement of the set of shocks, ψ is continuous. Then,

the Rankine-Hugoniot condition holds on any shock.
But, ψ satisfies entropy equalities in T because of (21) and (24), so there can't be any shock in T. Indeed, suppose η is a shock included in T, then $\forall\,(S,F)\in C^2(\mathbb{R},\mathbb{R})$, satisfying (16), entropy equality (21) yields to the following Rankine-Hugoniot condition

$$F(g,\eta(g)_-)-F(g,\eta(g)_+)=\dot{\eta}(g)\,[S(g,\eta(g)_-)-S(g,\eta(g)_+)]$$

This equality holds for all $(S,F)\in C^2(\mathbb{R},\mathbb{R})$ satisfying (16) only if $(g,\eta(g)_-)=(g,\eta(g)_+)$, which contradicts with the definition of a shock.
Therefore, $\forall\,(\bar{g},\bar{h})\in T$, $\psi(\bar{g},\bar{h}_-)=\psi(\bar{g},\bar{h}_+)$. In this case, there is a unique backward characteristic through $(\bar{g},\bar{h})$. It is defined on $[0,\bar{g}]$ by

$$\left\{\begin{array}{rcl}\dot{\xi}(g) & = & \psi(\bar{g},\bar{h})\\ \xi(\bar{g}) & = & \bar{h}\end{array}\right.$$

Moreover, $\psi(g,\xi(g))=\psi(\bar{g},\bar{h})$, $\forall\,g\in[0,\bar{g}]$, and distinct characteristics can't intersect each other in T (see [2], Theorem 3.3 and Corollary 3.2).
Thus, the characteristics of $\tilde{\phi}$ are defined in T, they are straight lines which do not intersect each other in T.

Let $g\in[\frac{d}{2},g_0+\frac{d}{2}]$ and $h_1\neq h_2$ in T_g. Let ξ_1, resp. ξ_2, the characteristic of ψ passing through (g,h_1), resp. (g,h_2). Since ξ_1 and ξ_2 do not intersect each other in T, then $|h_1-h_2|>\frac{d}{2}|p_1-p_2|$ where $p_i=\tilde{\phi}(g,h_i)$ is the slope of ξ_i, $i=1,2$.
Hence, $\forall\,g\in[\frac{d}{2},g_0+\frac{d}{2}]$,

$$\forall\,h_1,h_2\in T_g,\quad |\tilde{\phi}(g,h_1)-\tilde{\phi}(g,h_2)|<\frac{2}{d}|h_1-h_2|$$

Recalling that $g\mapsto\|\tilde{\phi}(g,.)\|_{L^1(T_g)}$ is Lipschitz in $]0,g_0+\eta[$, we deduce finally that $\tilde{\phi}$ is Lipschitz in T.

$\blacksquare$

Since Ψ^{-1} is Lipschitz on W, $\phi=\tilde{\phi}\circ\Psi$ is Lipschitz in $\Psi^{-1}(T)$. Then, for almost every $(x_0,y_0)\in\Omega$, there exists a neighborhood of (x_0,y_0) in which ϕ is Lipschitz, hence ϕ is locally Lipschitz in Ω. Let $\omega\subset\subset\Omega$, $\nabla\phi$ is defined a.e. in ω. Thus, the equality div $u=\nabla\phi.(e^{i\phi})^\perp=0$ in $\mathcal{D}'(\omega)$ is actually satisfied a.e. in ω.
Let $(x_0,y_0)\in\omega$, if $\nabla\phi(x_0,y_0)$ exists, then it is orthogonal to $(e^{i\phi})^\perp$. The slope at (x_0,y_0) of the level set $\{\phi=\phi(x_0,y_0)=a\}$ is $(e^{i\phi})^\perp=(e^{ia})^\perp$ and then is constant.

Therefore, level sets of ϕ are straight lines which do not intersect each other since ϕ is Lipschitz. The proof of theorem 1.1 is complete.

$\blacksquare$

2.4 A global result

Let us recall that every (u, ϕ) in $\mathcal{C}_L$ satisfies the limit condition

$$u.\nu = 0 \text{ on } \partial\Omega \text{ where } \nu \text{ denotes the outer unit normal of } \partial\Omega$$

Taking into account this condition, Theorem 1.1 has the following corollary:

Corollary 2.1 *Let $\bar{\Omega}$ be any bounded, regular (C^2), simply connected, closed domain of $\mathbb{R}^2$. Then, there doesn't exist any couple (u, ϕ) in $\mathcal{C}_L$ satisfying div $T^a u = 0$ in $\mathcal{D}'(\Omega)$, a.e. in $a \in \mathbb{R}$.*

Proof of Corollary 2.1:
Let (u, ϕ) in $\mathcal{C}_L$ satisfying div $T^a u = 0$ in $\mathcal{D}'(\Omega)$, a.e. in $a \in \mathbb{R}$. By Theorem 1.1, level sets of ϕ are straight segments which don't cross each other in the interior of Ω.
Γ will denote the set of points $x_0 \in \partial\Omega$ such that there is a unique level set $\{\phi = a\}$ crossing $\partial\Omega$ at x_0.
If $x_0 \in \Gamma$, the limit $\phi(x_0) = \lim\limits_{r\to 0}\frac{1}{r^2}\int_{B_r(x_0)} \phi(x)dx$ exists and is equal to a. Since $e^{i\phi(x_0)}.\nu(x_0) = 0$, where $\nu(x_0)$ is the outer unit-normal of $\partial\Omega$ at x_0. Then, the level set of ϕ which crosses $\partial\Omega$ at x_0 is included in its normal at x_0.
One can easily see that for any $x_0 \in \partial\Omega$, there is at least one level set of ϕ which crosses $\partial\Omega$ at x_0.
Let $x_0 \in \partial\Omega\backslash\Gamma$, there is at least two level sets $\{\phi = a\}$ and $\{\phi = b\}$ with $a \neq b$, which cross $\partial\Omega$ at x_0. Then, for all $c \in [a, b]$, the level set $\{\phi = c\}$ crosses $\partial\Omega$ at x_0, since it can't cross any other level set in Ω. One can see that the set $\partial\Omega\backslash\Gamma$ is $\mathcal{L}^1$.negligible.
Thus, for every $x_0 \in \Gamma$, the segment $S = \{x = x_0 - s\nu(x_0), 0 \leq s \leq t\}$, where t is such that $S \cap \partial\Omega = \{x_0, x_0'\}$, is included in the level set $\{\phi = \phi(x_0)\}$. Suppose that for every $s \in]\frac{t}{2}, t[$, $x = x_0 - s\nu(x_0)$, there exists $x' \neq x_0' \in \partial\Omega$, such that $\text{dist}(x, \partial\Omega) = |x - x'|$. Since $\mathcal{L}^1(\partial\Omega\backslash\Gamma) = 0$, x can be chosen such that there is a point x' in Γ which realizes $\text{dist}(x, \partial\Omega)$. Therefore, x is on the normal of $\partial\Omega$ at $x' \in \Gamma$ which is included in a level set of ϕ. Then, two level sets of ϕ cross each other at $x \in \Omega$, which is impossible.
Now, suppose that for every $x_0 \in \Gamma$, there exists $s \in]\frac{t}{2}, t[$ such that $\text{dist}(x, \partial\Omega) = |x - x_0'|$ where $x = x_0 - s\nu(x_0)$. Then, either $x_0' \in \partial\Omega\backslash\Gamma$ or S is

normal to $\partial\Omega$ at x_0 and x_0'. Therefore, $\partial\Omega$ must be a circle and the normals of $\partial\Omega$ (and then the level sets of ϕ) cross each other at the center of the circle, which is impossible. The corollary is proved.

Once this work was completed, we learnt that similar results for Aviles-Giga problem were independently established by P.E. Jabin, F. Otto and B. Perthame in [9], using different methods.

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