The rectifiability of shock waves for the solutions of genuinely non-linear scalar conservation laws in dimension 1+1

Myriam Lecumberry and Tristan Rivière

1 Introduction

In the late fifties, P.D. Lax and O. Oleinick observed a spectacular regularisation phenomena for the entropic solutions of genuinely non-linear scalar conservation laws in one-space dimension. They proved that for any measurable, bounded initial data, an entropic solution becomes immediately BV for all positive time, and therefore, the shock waves, where the entropic jump measure is concentrated, are rectifiable.
Let $F \in C^2(\mathbf{R})$ be strictly convex. Let us introduce the following notation, for any $a, b \in \mathbf{R}$:

$$\Delta(a,b) := \frac{(a-b)^2\left[\frac{F(a)+F(b)}{2}\right] - (a-b)\int_b^a F(t)\,dt}{[(a-b)^2 + (F(a)-F(b))^2]^{\frac{1}{2}}}$$

The result of Lax-Oleinick reads as follows:

Theorem 1.1 *[La][Ol] Let $F \in C^2(\mathbf{R})$ be strictly convex. Let $\phi_0(x) \in L^\infty \cap L^1(\mathbf{R})$ and $\phi(x,t)$ be a solution of*

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \dfrac{\partial}{\partial x}[F(\phi)] = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}_+ \times \mathbf{R}) \\ \phi(0,x) = \phi_0(x) \end{cases}$$

such that $\forall a \in \mathbf{R}$ the following entropic condition is satisfied

$$m(x,t,a) = \frac{\partial}{\partial t}(\phi \wedge a) + \frac{\partial}{\partial x}[F(\phi \wedge a)] \geq 0 \quad , \qquad (1)$$

then ϕ is in $BV_{loc}(\mathbf{R}_+^* \times \mathbf{R})$ and there exists a countably $\mathcal{H}^1$ rectifiable subset J_ϕ of $\mathbf{R}_+ \times \mathbf{R}$ such that

$$\mu(x,t) = \int_{a\in\mathbf{R}} |m(x,t,a)| \, da = \Delta(\phi_+, \phi_-) \ \mathcal{H}^1 \llcorner J_\phi$$

In this paper, we state a rectifiability result in the more general situation where the jump distributions (1) are only signed measures. Our result is the following:

Theorem 1.2 *Let $F \in C^2(\mathbf{R})$ be strictly convex. Let $\phi_0(x) \in L^\infty \cap L^1(\mathbf{R})$ and $\phi(x,t)$ be a solution of*

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \dfrac{\partial}{\partial x}[F(\phi)] = 0 & \text{in } \mathcal{D}'(\mathbf{R}_+ \times \mathbf{R}) \\ \phi(0,x) = \phi_0(x) \end{cases} \tag{2}$$

such that

$$m(x,t,a) = \frac{\partial}{\partial t}(\phi \wedge a) + \frac{\partial}{\partial x}[F(\phi \wedge a)] \in \mathcal{M}_{loc}(\mathbf{R} \times \mathbf{R}_+^* \times \mathbf{R}), \tag{3}$$

where $\mathcal{M}$ denotes the space of Radon measures. Then, there exists a countably $\mathcal{H}^1$ rectifiable subset J_ϕ of $\mathbf{R}_+ \times \mathbf{R}$ such that

$$\mu_\phi(x,t) = \int_{a\in\mathbf{R}} |m(x,t,a)| \, da = \Delta(\phi_+, \phi_-) \ \mathcal{H}^1 \llcorner J_\phi + \delta_\phi \tag{4}$$

where δ_ϕ is the Cantor part of μ_ϕ satisfying

$$\forall B \ \ \text{Borel} \qquad \mathcal{H}^1(B) < +\infty \Longrightarrow \delta_\phi(B) = 0 \tag{5}$$

The hypothesis of Theorem 1.2 ($m \in \mathcal{M}_{loc}$ instead of the usual entropic assumption $m \geq 0$) was motivated by Γ-limits of variational problems for models of micromagnetism (see [Ri] for a survey of these various variational problems). The question to know whether any ϕ satisfying the assumption of Theorem 1.2 is in BV or not remained open until recently. In [ADM], under corresponding hypothesis in a variational micromagnetic problem, L.Ambrosio, C. De Lellis and C. Mantegazza gave an example of a solution which is not in BV. The present situation shares a common property with the one studied in [ADM]: the expected jump density $\Delta(\phi^+, \phi^-)$ behaves like $(\phi^+ - \phi^-)^3$ in a neighborhood of 0. Indeed, this quantity $\Delta(\phi^+, \phi^-)$ gives, up to a bounded coefficient, the difference between the area between the trapeze $(\phi^-, 0)$, $(\phi^+, 0)$, $(\phi^+, F(\phi^+))$ and $(\phi^-, F(\phi^-))$ and the area under the graph of F between ϕ^- and ϕ^+. This property plays a crucial role in [ADM] to produce solutions not in BV and it is natural to think that their construction should give a solution of (2)-(3) which is also not in BV. Therefore, the proof of Theorem 1.2 is somehow independant of BV theory.

Let us introduce some notations. Let us set $X(\phi) := (\phi, F(\phi))$, then (2) implies that

$$\text{div } X(\phi) = 0, \quad \text{ in } \mathcal{D}'(\mathbf{R}_+ \times \mathbf{R})$$

and (3) is equivalent to

$$\forall a \in \mathbf{R}, \quad \text{div } X(\phi \wedge a) \in \mathcal{M}_{\text{loc}}(\mathbf{R}_+^* \times \mathbf{R}).$$

In [AKLR], in collaboration with Luigi Ambrosio and Bernd Kirchheim, the authors studied the class of functions $\phi \in L^\infty(\Omega, \mathbf{R})$, where Ω is a bounded subset of $\mathbf{R}^2$, such that div $e^{i\phi} = 0$ and div $e^{i(\phi \wedge a)}$ is a Radon measure in $\Omega \times \mathbf{R}$. By analogy with [AKLR], for any open domain $\Omega \subset \mathbf{R}^2$, we define

$$\mathcal{M}_{\text{div}}(\Omega) := \left\{ \begin{array}{c} \phi \in L^\infty(\Omega) \text{ s.t. div } X(\phi) = 0 \text{ and div } X(\phi \wedge a) \\ \text{is a finite Radon measure in } \Omega \times \mathbf{R} \end{array} \right\}$$

Let $\phi \in \mathcal{M}_{\text{div}}(\Omega)$, the total jump distribution μ_ϕ of ϕ is defined as follows:

$$\forall\, B \text{ Borel subset of } \Omega, \quad \mu_\phi(B) := \int_{\mathbf{R}} |\text{div } X(\phi \wedge a)|(B) da. \tag{6}$$

We proved in [AKLR] the corresponding statement to Theorem 1.2, that is the 1-dimensional part of μ_ϕ is rectifiable. Our proof of Theorem 1.2 follows the main steps of the proof in [AKLR]. One of the main ingredient is that a blow-up solution to (2)-(3) (i.e. a limit ϕ_∞ of a sequence of dilated functions $\phi_{r_i}(x) = \phi(x_0 + r_i x)$) solves the following "blow-up equation":

$$\begin{cases} \dfrac{\partial \phi_\infty}{\partial t} + \dfrac{\partial}{\partial x}[F(\phi_\infty)] = 0 & \text{in } \mathcal{D}'(\mathbf{R} \times \mathbf{R}) \\ \dfrac{\partial}{\partial t}(\phi_\infty \wedge a) + \dfrac{\partial}{\partial x}[F(\phi_\infty \wedge a)] = h(a)\nu(x) & \end{cases} \tag{7}$$

where h is a Lipschitz function and ν a positive measure. It is then proved below that such solutions, where the jump distributions are decomposed as $h(a)\nu(x)$, are very particular and satisfy

$$\nu(x) = c\mathcal{H}^1 \llcorner K, \tag{8}$$

where K is a line or a half-line in $\mathbf{R}^2$. That will give, for h non identically equal to 0, the direction of the tangent plane to the jump set J_ϕ of ϕ.

In [DO], C. De Lellis and F. Otto proved independently a similar statement to Theorem 1.2 for a Γ-limit of another micromagnetic model. The approach they adopted is also based on a similar blow-up argument which yields to a decomposition of jump distributions as (7). Our proof of (8) relied strongly on uniqueness and BV properties of entropic solutions to (2). In [DO], such a BV property does not hold, but they nevertheless succeeded in providing a proof of (8) (where K can also be a segment) without passing through BV property of solution to (7). For that reason, it is natural to expect our present result to be extended to higher dimensional problems and more general non linearities. Consider $u : \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ solving

$$\begin{cases} \dfrac{\partial u}{\partial t} + \displaystyle\sum_{i=1}^{N} \dfrac{\partial}{\partial x_i} [F_i(u)] = 0 & \text{in } \mathcal{D}'(\mathbf{R}_+^* \times \mathbf{R}^N) \\ \dfrac{\partial}{\partial t}(u \wedge a) + \displaystyle\sum_{i=1}^{N} \dfrac{\partial}{\partial x_i} [F_i(u \wedge a)] & \text{in } \mathcal{M}_{loc}(\mathbf{R}_+^* \times \mathbf{R}^N \times \mathbf{R}) \end{cases} \tag{9}$$

In the case when compactness holds (see the conditions on (F_i) given by P-L. Lions, B. Perthame and E. Tadmor in [LPT]), it is natural to expect the conclusion corresponding to (4).

Using kinetic formulations and kinetic averaging methods, P-E. Jabin and B. Perthame succeeded to prove, in [JP], that solutions to (2)-(3) are in $W^{\sigma,p}$ with $\sigma < \frac{1}{3}$ and $p < \frac{2}{3}$. This regularity result, although being a nice achievement of linear functional analysis, is nevertheless far from giving corresponding regularity result as the one we obtained above for the solution ϕ. Actually, we present here an alternative approach to the use of linear functional analysis, consisting in studying directly the singular set in order to get a particular geometric structure of this set and then to extend this regularity result on the singular set to the solution itself. This general philosophy happens to be more relevant to the analysis of certain non-linear elliptic equations (see the work of Leon Simon and Fanghua Lin on harmonic applications between manifolds and minimal surfaces).
This is perhaps in the works of R. DiPerna and C.M. Dafermos (see for instance [Di] and [Da]) where this idea of adapting tools from the geometric measure theory to hyperbolic non-linear conservation laws appeared first. But, nevertheless only BV solutions are considered in these works. The goal of the paper is somehow to provide a structure theorem, independent of BV structure theorem, for the more general solutions to hyperbolic laws whose jump distribution is a Radon measure.

2 Some results on scalar conservation laws theory in dimension 1+1

2.1 Definitions and notations

- Let $\phi \in L^\infty(\Omega; \mathbf{R})$, where Ω is an open subset of $\mathbf{R}^2$, let $x \in \Omega$, we say that ϕ has an approximate limit at x if there exists $a \in \mathbf{R}$ such that

$$\lim_{r \to 0^+} \frac{1}{\pi r^2} \int_{B_r(x)} |\phi(y) - a| dy = 0$$

  a is unique and is called the approximate limit of ϕ at x. We denote by S_ϕ the set of points where ϕ has no approximate limit.

- We define the set of jump points of ϕ, J_ϕ, by $x \in J_\phi$ if and only if there exists $a^+ \neq a^- \in \mathbf{R}$ and $\nu_x \in S^1$ such that

$$\lim_{r \to 0^+} \frac{2}{\pi r^2} \int_{B_r^\pm(x)} |\phi(y) - a^\pm| dy = 0$$

  where $B_r^\pm(x) := \{y \in B_r(x) | \pm (y - x) \cdot \nu_x > 0\}$. $a^\pm$ are uniquely determined up to change ν_x into $-\nu_x$ and permute a^+ and a^-. ν_x is a unit normal vector of J_ϕ at x and $a^\pm$ are the approximate limits of ϕ on each side of J_ϕ, denoted by $\phi^\pm(x)$.

- $\mathcal{M}_{\rm loc}(\Omega)$ denotes the space of real Radon measure in Ω, and $\mathcal{M}^+_{\rm loc}(\Omega)$ denotes the space of positive Radon measure in Ω.

- For any $\mu \in \mathcal{M}^+_{\rm loc}(\Omega)$, for any $x \in \Omega$, there exist a sequence $r_i \to 0^+$ and a positive measure σ such that

$$\mu_{x,r_i} \to \sigma \quad \text{in} \quad \mathcal{M}_{\rm loc}(\mathbf{R}^2), \tag{10}$$

$$\text{i.e.} \quad \int_{\mathbf{R}^2} f \, d\mu_{x,r_i} \to \int_{\mathbf{R}^2} f \, d\sigma, \quad \forall f \in C^0_c(\mathbf{R}^2),$$

  where μ_{x,r_i} is the dilated measure of μ at x defined by $\forall\, B$ Borel subset of $\mathbf{R}^2$ such that $x + r_i B \subset \Omega$,

$$\mu_{x,r_i}(B) = \frac{1}{r_i} \mu(x + r_i B).$$

  The set of all σ satisfying (10) for any sequence $r_i \to 0^+$ is denoted by $\mathrm{Tan}(\mu, x)$. Let us notice that $\mathrm{Tan}(\mu, x)$ is not empty as soon as

$\limsup_{r\to 0}\frac{\mu(B_r(x))}{r} < +\infty$.
Let $\nu \in \mathcal{M}_{\rm loc}(\Omega)$ absolutely continuous with respect to μ, then by the Radon-Nikodym Theorem, there exists $f \in L^1(\Omega,\mu)$ such that $\nu = f\mu$. Moreover, if x is a Lebesgue point of f with respect to μ, then

$$\nu_{x,r_i} \to f(x)\sigma \quad \text{in } \mathcal{M}_{\rm loc}(\mathbf{R}^2).$$

2.2 Entropic solution of a scalar conservation law

Let ϕ be an entropic solution of (2) i.e. ϕ satisfies (2)-(3).
A well-known result is that for a given initial data $\phi_0 \in L^\infty(\mathbf{R})$, there is a unique entropic solution ϕ of (2) satisfying $\phi(0,.) = \phi_0$ (see for example [Da]). Moreover, $t \mapsto \phi(t,.) \in C^0([0,+\infty); L^1_{loc}(\mathbf{R}))$. This result is true even if we don't assume that F is convex and if ϕ takes values in $\mathbf{R}^m$ with $m \geq 1$. But, in our situation, where F is stricly convex and ϕ takes values in $\mathbf{R}$, we have a stronger regularity: for any initial data in L^∞, the entropic solution is in $BV_{\rm loc}(\mathbf{R}_+^* \times \mathbf{R})$. Therefore, we can apply the method of characteristic for the case of one space-dimension. Indeed, this method requires at least a BV regularity and it yields to the following results about the structure of the entropic solution.

- The set of jump points of ϕ (which is called the set of shocks in the theory of conservation laws) is a countable union of Lipschitz curves.

- Each Lipschitz curve (called a shock) can be parametrized by t and is defined on an unbounded interval. Therefore, we have

$$J_\phi := \bigcup_{p\in\mathbf{N}} \{(t,\eta_p(t)); t \in [t_{0,p},+\infty)\}$$

- Moreover, $\forall p \in \mathbf{N}$, η_p is differentiable from the right everywhere in $[t_{0,p},+\infty)$ and the right derivative is given by

$$D^+\eta_p(t) = \frac{F(\phi^+(t,\eta_p(t))) - F(\phi^-(t,\eta_p(t)))}{\phi^+(t,\eta_p(t)) - \phi^-(t,\eta_p(t))}, \quad \forall t \in [t_{0,p},+\infty) \quad (11)$$

- The entropic conditions (3) implies that if $x \in J_\phi$ and if we choose for the unit normal of J_ϕ at $x = (t,\eta_p(t))$ the vector

$$\nu_\phi = \frac{1}{\|(-D^+\eta_p(t),1)\|}(-D^+\eta_p(t),1),$$

  then $\phi^-(x) > \phi^+(x)$.

- Finally, out of the set of shocks, ϕ has an approximate limit.

2.3 The associated Hamilton-Jacobi equation

Let $\phi \in L^\infty(\mathbf{R}^2)$ satisfy

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}\big(F(\phi)\big) = 0 \quad \text{in } \mathcal{D}'(\mathbf{R}^2) \tag{12}$$

and the entropy condition

$$\forall a \in \mathbf{R}, \quad \frac{\partial}{\partial t}(\phi \wedge a) + \frac{\partial}{\partial x}\big(F(\phi \wedge a)\big) \quad \in \quad \mathcal{M}^+_{\text{loc}}(\mathbf{R}^2). \tag{13}$$

By (12), there exists a function $g \in W^{1,\infty}(\mathbf{R}^2)$ such that $\nabla g = (-F(\phi), \phi)$. g satisfies the Hamilton-Jacobi equation:

$$\frac{\partial g}{\partial t} + F\Big(\frac{\partial g}{\partial x}\Big) = 0 \tag{14}$$

We assume that F is strictly convex. Since ϕ is an entropic solution of (12), then g is a viscosity solution of (14) (see [E]). Moreover, there is an explicit formula for g, the Hopf-Lax formula: for any $t_0 \in \mathbf{R}$,

$$g(t,x) = \inf_{y \in \mathbf{R}} \Big\{ g(t_0, y) + (t - t_0) F^*\big(\frac{x-y}{t-t_0}\big) \Big\}, \quad \forall t > t_0, \, \forall x \in \mathbf{R} \tag{15}$$

where F^* is the dual convex function of F:

$$F^*(z) := \sup_{y \in \mathbf{R}} \{zy - F(y)\}, \quad \forall z \in \mathbf{R}.$$

For any $t_0 \in \mathbf{R}$, the function g defined by (15) is semiconcave in $[t_0+\eta, +\infty) \times \mathbf{R}$, $\forall \eta > 0$, and the semiconcavity constant is proportional to $1/\eta$ (see [E]). Since g is a viscosity solution of (14) on the whole $\mathbf{R}^2$, then g is concave. Indeed, $\forall t_1 \in \mathbf{R}$, g is semiconcave in $[t_1, +\infty) \times \mathbf{R}$ with a semiconcavity constant less than $1/\eta$, $\forall \eta > 0$ (choosing $t_0 = t_1 - \eta$). Letting $\eta \to +\infty$, we have that g is concave in $[t_1, +\infty) \times \mathbf{R}$, $\forall t_1 \in \mathbf{R}$. Hence, g is concave in $\mathbf{R}^2$. Let us remark that we recover that $\phi \in BV_{\text{loc}}(\mathbf{R}^2)$. Indeed, g concave in $\mathbf{R}^2$ implies that $\nabla g \in BV_{\text{loc}}(\mathbf{R}^2)$ and then its second coordinate $\phi \in BV_{\text{loc}}(\mathbf{R}^2)$.

3 Proof of Theorem 1.2

Since the vector field $X(\phi)$ depends continuously on ϕ, one can adapt most of the results of [AKLR] obtained for the vector field $e^{i\phi}$. For instance, as in [AKLR], Theorem 3.4, we have the absolute continuity of μ_ϕ with respect to $\mathcal{H}^1$. Indeed, the proof is exactly the same.

Proposition 3.1 *Let $\phi \in \mathcal{M}_{\rm div}(\mathbf{R}_+^* \times \mathbf{R})$, then the measure μ_ϕ defined by (6) is absolutely continuous with respect to $\mathcal{H}^1$.*

What we need to apply the blow-up technics of [AKLR] is a compactness result similar to Theorem 3.6 in [AKLR]. It is given by L.Tartar (see [Ta] Theorem 26).

Proposition 3.2 *For any constant $M \geq 0$, for any bounded set Ω, the set*

$$\left\{\phi \in L^\infty(\Omega); \quad \|\phi\|_\infty + |\mu_\phi|(\Omega) \leq M\right\}$$

is compact in $L^1(\Omega)$ with respect to the strong topology.

Since μ_ϕ is absolutely continuous with respect to $\mathcal{H}^1$, then for μ_ϕ-almost every $x_0 \in \mathbf{R}_+ \times \mathbf{R}$, there exists $C > 0$ such that

$$\limsup_{r \to 0} \frac{\mu_\phi(B_r(x_0))}{r} \leq C. \tag{16}$$

Let $(\phi_n)_{n\in\mathbf{N}}$ be a sequence of dilated functions of ϕ at x_0, i.e. given $(r_n)_{n\in\mathbf{N}}$ a sequence of positive numbers tending to 0, ϕ_n is defined by $\phi_n(x) := \phi(x_0 + r_n x)$, $\forall x \in \mathbf{R}^2$ such that $x_0 + r_n x \in \mathbf{R}_+ \times \mathbf{R}$, $\forall n \in \mathbf{N}$. By (16), $(\mu_{\phi_n}(B_R))$ is uniformly bounded $\forall R > 0$. Since $\|\phi_n\|_\infty \leq \|\phi\|_\infty$ $\forall n \in \mathbf{N}$, by Proposition 3.2, and by a diagonal argument, the sequence $(\phi_n)_{n\in\mathbf{N}}$ is compact in $L^1_{loc}(\mathbf{R}^2)$ for the strong topology.
Now, the following proposition can be proved in the same way as Theorem 6.1 in [AKLR] and we omit the proof here.

Proposition 3.3 *Let $\phi \in \mathcal{M}_{\rm div}(\mathbf{R}_+^* \times \mathbf{R})$. For μ_ϕ-almost every $x_0 \in \mathbf{R}_+^* \times \mathbf{R}$, from any sequence $r_n \to 0^+$ one can extract a subsequence r_i such that the dilated functions at x_0 ϕ_{r_i} converge to ϕ_∞ in $L^1_{\rm loc}(\mathbf{R}^2)$. Moreover,*

i/ There exists a non-negative Radon measure ν on $\mathbf{R}^2$ and a Lipschitz map $h : \mathbf{R} \to \mathbf{R}$ such that

$$div\ X(\phi_\infty \wedge a) = h(a)\nu.$$

ii/ There exists a finite or countable family of open segments $I_l = (b_l, c_l)$ such that:

(a) $\mathbf{R} \setminus \cup_{l\in\mathbf{N}} I_l$ has an empty interior,

(b) $\forall l \in \mathbf{N}$, $div\ X((\phi_\infty \wedge c_l) \vee b_l) = 0$,

(c) $\forall l \in \mathbf{N}$, either $div\ X((\phi_\infty \wedge a) \vee b_l) \geq 0$ $\forall a \in \mathbf{R}$, or $div\ X((\phi_\infty \wedge a) \vee b_l) \leq 0$ $\forall a \in \mathbf{R}$.

Let us set $\phi_l := (\phi \wedge c_l) \vee b_l$. By (b) and (c), ϕ_l satisfies div $X(\phi_l) = 0$ and the measures div $X(\phi_l \wedge a)$ are either nonnegative $\forall a \in \mathbf{R}$ or nonpositive $\forall a \in \mathbf{R}$. In terms of conservation laws, this means that either ϕ_l or $-\phi_l$ is an entropic solution of (12). Let us remark that if ϕ_l satisfies div $X(\phi_l \wedge a) = 0$, $\forall a \in \mathbf{R}$, then the set of shocks is empty. Let us note $L := \{l \in \mathbf{N} \mid \text{div } X(\phi_l \wedge a) \neq 0 \text{ for at least one } a \in \mathbf{R}\}$. Let us note J_l the set of jump points of ϕ_l. If $l \in \mathbf{N} \setminus L$, then $J_l = \emptyset$.

<u>Claim</u>: $\forall\, l, m \in L$, $\mathcal{H}^1(J_l \Delta J_m) = 0$ and $\forall\, l \in L$, J_l is contained in one straight line.

Since ϕ_l or $-\phi_l$ is an entropic solution of (12), by Section 2.3 we have that $\phi_l \in BV_{\text{loc}}(\mathbf{R}^2)$. Therefore, using the Vol'Pert Chain Rule, we find an explicit formula for div $X(\phi_l \wedge a)$:

$$\text{div } X(\phi_l \wedge a) = \chi_{\{\phi_l^- < a < \phi_l\}}(X(a) - X(\phi_l^-)) \cdot \omega_l \mathcal{H}^1 \llcorner J_l$$

where ω_l denotes the unit normal vector of J_l and is chosen such that $\phi_l^- < \phi_l^+$.
Moreover, by Proposition 3.3, we have

$$\text{div } X(\phi_l \wedge a) = \begin{cases} h(a)\nu & \text{if } a \in I_l \\ 0 & \text{else} \end{cases}$$

These two formulas allows us to show (see [AKLR]) that ϕ_l^+ is constant $\mathcal{H}^1$-a.e. on J_l equal to c_l and ϕ_l^- is constant $\mathcal{H}^1$-a.e. on J_l equal to b_l. Moreover, one can show that $\forall l, m \in L$, $\mathcal{H}^1(J_l \setminus J_m) = 0$ (the proof is exactly the same as in [AKLR]).
Now, let us show that $\forall\, l \in L$, J_l is contained in one straight line. Using the properties of ϕ_l as an entropic solution of (12), the proof is actually simpler than in [AKLR].
Let $l \in L$. Since ϕ_l^- and ϕ_l^+ are constant $\mathcal{H}^1$-a.e. on J_l, then for every shock $\{(t, \eta(t)); t \in [t_0, +\infty)\}$, we have that $D^+\eta(t) = \frac{F(\phi_l^+) - F(\phi_l^-)}{\phi_l^+ - \phi_l^-}$ is constant for almost every $t \in [t_0, +\infty)$ and doesn't depend on η. Hence, every shock is a half-line or a line and the shocks are parallel to each other.
Let us show that there can be only one shock. Let us assume that there exist $x_1, x_2 \in J_l$ such that $(x_2 - x_1) \cdot \omega_l \neq 0$ where ω_l is chosen such that $\phi_l^- < \phi_l^+$. Let us assume that div $X(\phi_l \wedge a) \geq 0$, $\forall a \in \mathbf{R}$, so that ϕ_l is an entropic solution of (12). By Section 2.2, ω_l is given by

$$\omega_l = \frac{v_l}{|v_l|} \quad \text{where} \quad v_l = \begin{pmatrix} \frac{F(\phi_l^+) - F(\phi_l^-)}{\phi_l^+ - \phi_l^-} \\ -1 \end{pmatrix}$$

Let us note $\omega := \frac{x_2-x_1}{|x_2-x_1|}$ and suppose that $\omega \cdot \omega_l > 0$.
Let $g_l \in W^{1,\infty}(\mathbf{R}^2)$ such that $\nabla g_l = X(\phi_l)^\perp$. g_l is a viscosity solution of the Hamilton-Jacobi equation (14) on $\mathbf{R}^2$. Then, by Section 2.3, g_l is concave on $\mathbf{R}^2$. We denote the left (resp. right) directional derivative along ω of g_l by $\nabla^-_\omega g_l$ (resp. $\nabla^+_\omega g_l$). The concavity of g_l implies that

$$\nabla^+_\omega g_l(x_1) \geq \nabla^-_\omega g_l(x_2).$$

But, we have $\nabla^\pm_\omega g_l(x) = \omega \cdot \nabla^\pm g_l(x)$, $\forall x \in J_l$ (see Proposition 4.1 in [AKLR]). Then

$$\omega \cdot X(\phi_l^+(x_1))^\perp \geq \omega \cdot X(\phi_l^-(x_2))^\perp.$$

Since $\phi_l^\pm$ are constant on J_l, then

$$\omega \cdot X(\phi_l^+)^\perp \geq \omega \cdot X(\phi_l^-)^\perp. \tag{17}$$

But,

$$\begin{aligned}
\omega \cdot (X(\phi_l^+)^\perp - X(\phi_l^-)^\perp) &= -\omega^\perp (X(\phi_l^+) - X(\phi_l^-)) \\
&= -\omega^\perp \cdot \big((X(\phi_l^+) - X(\phi_l^-)) \cdot \omega_l \, \omega_l \\
&\quad + (X(\phi_l^+) - X(\phi_l^-)) \cdot \omega_l^\perp \, \omega_l^\perp \big) \\
&= -(\omega^\perp \cdot \omega_l^\perp) \big(X(\phi_l^+) - X(\phi_l^-) \big) \cdot \omega_l^\perp
\end{aligned}$$

Indeed, since div $X(\phi_l) = 0$, then $X(\phi_l^+) \cdot \omega_l = X(\phi_l^-) \cdot \omega_l$ $\mathcal{H}^1$-a.e. on J_l. By assumption, $\omega^\perp \cdot \omega_l^\perp > 0$, and $\big(X(\phi_l^+ - X(\phi_l^-)\big) \cdot \omega_l^\perp$ has the sign of

$$\big(X(\phi_l^+ - X(\phi_l^-)\big) \cdot v_l^\perp = (\phi_l^+ - \phi_l^-) \Big[1 + \Big(\frac{F(\phi_l^+) - F(\phi_l^-)}{\phi_l^+ - \phi_l^-} \Big)^2 \Big] > 0$$

Then, $\omega \cdot \big(X(\phi_l^+)^\perp - X(\phi_l^-)^\perp \big) < 0$ which contradicts (17). Thus, J_l must be contained in one straight line which does not depend on $l \in L$. Actually, J_l is a shock and must be a half-line or a line, by Section 2.2.
We will note K this line or half-line. If $\forall l \in \mathbf{N}$, $\forall a \in \mathbf{R}$, div $X(\phi_l \wedge a) = 0$, we will set $K = \emptyset$.
Using Lemma 2.1 in [AKLR] and the last result of section 2.2, we have that ϕ_∞ has an approximate limit $\mathcal{H}^1$-a.e. in $\mathbf{R}^2 \setminus K$ and $\mathcal{H}^1$-a.e. point of K is a jump point of ϕ_∞. Moreover, since $\forall l \in L$, $\phi_l^\pm$ are constant on K, then $\phi_\infty^\pm$ are constant on J_{ϕ_∞}.

Let us define the set

$$\Sigma := \Big\{ x \in \mathbf{R}_+^* \times \mathbf{R} \mid \limsup_{r \to 0} \frac{\mu_\phi(B_r(x))}{r} > 0 \Big\}.$$

Let us now prove the rectifiability of the set Σ. As in [AKLR] we will proceed in two steps:

1. First, we prove that the subset of Σ,

$$\Sigma^- := \Big\{ x \in \mathbf{R}_+^* \times \mathbf{R} \mid \liminf_{r\to 0} \frac{\mu_\phi(B_r(x))}{r} > 0 \Big\}$$

is countably $\mathcal{H}^1$ rectifiable. By Theorem 5.3 in [AKLR], it suffices to show that for μ_ϕ a.e. $x \in \Sigma^-$, there exists a unit vector $\xi(x)$ such that $\forall \sigma \in \mathrm{Tan}(\mu_\phi, x)$, σ is concentrated on a line parallel to ξ.

2. Secondly, we prove that $\mu_\phi(\Sigma\setminus\Sigma^-) = 0$, using Theorem 5.2 in [AKLR]. Indeed, if $x \in \Sigma$, then $x \in \Sigma^-$ as soon as x satisfies the following properties:

   i/ $f(r) = \frac{\mu_\phi(B_r(x))}{r}$ is continuous in $(0,\delta)$, where $\delta > 0$;

   ii/ $\limsup_{r\to 0} \frac{\mu_\phi(B_r(x))}{r} < +\infty$;

   iii/ $\exists\, c_x > 0$ s.t. $\forall \sigma \in \mathrm{Tan}(\mu_\phi, x)\setminus\{0\}$, $\exists\, c \geq c_x$ and there exists either a line or a half-line noted by L such that $\sigma = c\mathcal{H}^1 \llcorner L$.

   Properties i/ and ii/ are satisfied by μ_ϕ a.e. $x \in \mathbf{R}_+^* \times \mathbf{R}$ (the argument is given in [AKLR] in the proof of Theorem 6.3). Then, we only have to prove that iii/ is also satisfied by μ_ϕ a.e. $x \in \mathbf{R}_+^* \times \mathbf{R}$.

Let $x_0 \in \mathbf{R}_+^* \times \mathbf{R}$ satisfying i/ and ii/. Let $\sigma \in \mathrm{Tan}(\mu_\phi, x_0)\setminus\{0\}$. Then, there exists a sequence $r_n \to 0$ such that $(\mu_\phi)_{x_0,r_n} \to \sigma$ in the sense of measure. By Proposition 3.3, there exist a subsequence (r_i) of (r_n) and $\phi_\infty \in L^1_{loc}(\mathbf{R}^2)$ such that $\phi_{r_i} \to \phi_\infty$ in L^1_{loc}, where $\phi_{r_i}(x) = \phi(r_i x + x_0)$. Moreover, there exists K the empty set, a line or a half-line such that $\mathcal{H}^1(K \Delta J_{\phi_\infty}) = 0$. Let us note ω_K the unit normal vector of K. We use Lemma 3.5 in [AKLR] since div $X(\phi_\infty \wedge a)$ is a Radon measure for all a and div $X(\phi_\infty) = 0$, since $\mathcal{H}^1(K \cap S_{\phi_\infty}) = 0$ where K is countably $\mathcal{H}^1$ rectifiable. In one hand,

$$\begin{aligned} \text{div } X(\phi_\infty)\llcorner K &= \big(X(\phi_\infty)^+ - X(\phi_\infty^-)\big)\cdot \omega_K \mathcal{H}^1 \llcorner K \\ &= \big(X(\phi_\infty^+) - X(\phi_\infty^-)\big)\cdot \omega_K \mathcal{H}^1 \llcorner K = 0 \end{aligned}$$

Hence, $\big(X(\phi_\infty^+) - X(\phi_\infty^-)\big)\cdot \omega_K = 0$, $\mathcal{H}^1$-a.e. on K, and

$$\omega_K = \frac{|\phi_\infty^+ - \phi_\infty^-|}{\|X(\phi_\infty^+) - X(\phi_\infty^-)\|} \begin{pmatrix} \frac{F(\phi_\infty^+) - F(\phi_\infty^-)}{\phi_\infty^+ - \phi_\infty^-} \\ -1 \end{pmatrix}$$

On the other hand,

$$\text{div } X(\phi_\infty \wedge a)\llcorner K = \big(X(\phi_\infty^+ \wedge a) - X(\phi_\infty^- \wedge a)\big)\cdot \omega_K \mathcal{H}^1 \llcorner K.$$

But, $X(\phi_\infty \wedge a)$ is divergence free on K, $\forall a \in \mathbf{R}$, as explained in [AKLR]. Therefore,

$$\text{div } X(\phi_\infty \wedge a) = \big(X(\phi_\infty^+ \wedge a) - X(\phi_\infty^- \wedge a)\big) \cdot \omega_K \mathcal{H}^1 \llcorner K. \tag{18}$$

For any $h \in C_c(\mathbf{R})$, the measure $m_h := \int_{\mathbf{R}} h(a) \text{ div } X(\phi \wedge a)\, da$ is absolutely continuous with respect to μ_ϕ. Therefore, there exists
$b_h \in L^1(\mathbf{R}_+^* \times \mathbf{R}; \mu_\phi)$ such that $m_h = b_h \mu_\phi$. If $x \in \mathbf{R}_+^* \times \mathbf{R}$ is a Lebesgue point of b_h (with respect to the measure μ_ϕ), then

$$(\mu_\phi)_{x,r_i} \to b_h(x)\sigma \quad \text{ in } \mathcal{M}_{loc}(\mathbf{R}_+^* \times \mathbf{R}).$$

Let us notice that for any $h \in C_c(\mathbf{R})$, μ_ϕ a.e. $x \in \mathbf{R}_+^* \times \mathbf{R}$ is a Lebesgue point of b_h. Suppose that for a fixed $h \in C_c(\mathbf{R})$, x_0 is a Lebesgue point of b_h. The convergence of the dilated functions (ϕ_{r_i}) to ϕ_∞ in L^1_{loc} implies that

$$(m_h)_{x_0,r_i} = \int_{\mathbf{R}} h(a) \text{ div } X(\phi_{r_i} \wedge a) da \to \int_{\mathbf{R}} h(a) \text{ div } X(\phi_\infty \wedge a) da \quad \text{ in } \mathcal{D}'$$

By uniqueness of the limit, we have

$$\int_{\mathbf{R}} h(a) \text{ div } X(\phi_\infty \wedge a) da = b_h(x_0)\sigma \tag{19}$$

But, using (18), one can compute:

$$\int_{\mathbf{R}} h(a) \text{ div } X(\phi_\infty \wedge a) da = \lambda_\infty \int_{\phi_\infty^-}^{\phi_\infty^+} \Big[F'(a) - \frac{F(\phi_\infty^+) - F(\phi_\infty^-)}{\phi_\infty^+ - \phi_\infty^-}\Big] H(a) da$$

where H is a primitive of h and

$$\lambda_\infty := \frac{|\phi_\infty^+ - \phi_\infty^-|}{\|X(\phi_\infty^+) - X(\phi_\infty^-)\|}.$$

Remark that $\lambda_\infty > 0$ since $\phi_\infty^+ \neq \phi_\infty^-$.
Let us consider the functions $h_1 = 1$ and $h_2 = F'$, and let us denote their associated functions $b_{h_1} = b_1$ and $b_{h_2} = b_2$. Let us define the following countable family of functions

$$H_{k,l}(a) := 2^l \Psi(2^l(a - q_k)) \quad \forall a \in \mathbf{R}, \forall k, l \in \mathbf{N},$$

where $(q_k)_{k \in \mathbf{N}} = \mathbf{Q}$ and $\Psi \in C_c^\infty(\mathbf{R})$ is nonnegative with compact support in $[-1, 1]$ and such that $\int_{\mathbf{R}} \Psi = 1$.
Let us denote $b_{k,l}$ the function of $L^1(\mathbf{R}_+^* \times \mathbf{R}; \mu_\phi)$ associated to $H'_{k,l}$.

We now take x_0 a Lebesgue point of b_1, b_2 and $b_{k,l}$ $\forall k, l \in \mathbf{N}$. An easy computation yields to

$$\int_{\mathbf{R}} \text{div } X(\phi_\infty \wedge a) da = \lambda_\infty \int_{\phi_\infty^-}^{\phi_\infty^+} \Big[\frac{F(\phi_\infty^+) + F(\phi_\infty^-)}{2} - F(a)\Big] da \mathcal{H}^1 \llcorner K$$

Let us notice that $\int_{\phi_\infty^-}^{\phi_\infty^+} \big[\frac{F(\phi_\infty^+)+F(\phi_\infty^-)}{2} - F(a)\big] da > 0$ since it measures the area between the graph of F and the segment joining the points $(\phi_\infty^+, F(\phi_\infty^+))$ and $(\phi_\infty^-, F(\phi_\infty^-))$. Since F is strictly convex, it must be positive. Moreover, we have

$$\lambda_\infty \int_{\phi_\infty^-}^{\phi_\infty^+} \Big[\frac{F(\phi_\infty^+) + F(\phi_\infty^-)}{2} - F(a)\Big] da \mathcal{H}^1 \llcorner K = b_1(x_0)\sigma. \tag{20}$$

Therefore, $b_1(x_0) > 0$ and since $\phi_\infty^\pm$ are constant on K, then there exists c such that

$$\sigma = c\mathcal{H}^1 \llcorner K. \tag{21}$$

One also can get the equality

$$\int_{\mathbf{R}} F'(a) \text{ div } X(\phi_\infty \wedge a) da = \frac{F(\phi_\infty^+) - F(\phi_\infty^-)}{\phi_\infty^+ - \phi_\infty^-} \int_{\mathbf{R}} \text{div } X(\phi_\infty \wedge a) da.$$

On one hand, by (19) and (21), we have

$$b_2(x_0) = \frac{F(\phi_\infty^+) - F(\phi_\infty^-)}{\phi_\infty^+ - \phi_\infty^-} b_1(x_0).$$

Thus, $\frac{F(\phi_\infty^+) - F(\phi_\infty^-)}{\phi_\infty^+ - \phi_\infty^-}$ only depends on x_0.

On the other hand, the vector-valued measure

$$\int_{\mathbf{R}} \begin{pmatrix} 1 \\ F'(a) \end{pmatrix} \text{ div } X(\phi_\infty \wedge a) da = \begin{pmatrix} H_1(x_0) \\ H_2(x_0) \end{pmatrix} \sigma$$

is oriented by $\omega_K^\perp$: indeed,

$$\Big[\int_{\phi_\infty^-}^{\phi_\infty^+} \Big(\frac{F(\phi_\infty^+) + F(\phi_\infty^-)}{2} - F(a)\Big) da\Big] \omega_K^\perp \mathcal{H}^1 \llcorner K = \begin{pmatrix} H_1(x_0) \\ H_2(x_0) \end{pmatrix} \sigma.$$

Hence, $\omega_K^\perp$ only depends on x_0, and the first step is achieved.

Since $\frac{F(\phi_\infty^+) - F(\phi_\infty^-)}{\phi_\infty^+ - \phi_\infty^-}$ only depends on x_0, then there is a unique real t_0 (depending only on x_0) such that $F'(t_0) = \frac{F(\phi_\infty^+) - F(\phi_\infty^-)}{\phi_\infty^+ - \phi_\infty^-}$ and either $\phi_\infty^- < t_0 < \phi_\infty^+$ or

$\phi^+_\infty < t_0 < \phi^-_\infty$. Moreover, ϕ^-_∞ is uniquely determined by the data of ϕ^+_∞ and $\frac{F(\phi^+_\infty)-F(\phi^-_\infty)}{\phi^+_\infty-\phi^-_\infty}$. Let us recall that $\forall k,l \in \mathbf{N}$ we have

$$\lambda_\infty \int_{\phi^-_\infty}^{\phi^+_\infty} \Big[F'(a) - \frac{F(\phi^+_\infty)-F(\phi^-_\infty)}{\phi^+_\infty-\phi^-_\infty}\Big] H_{k,l}(a)da = b_{k,l}(x_0)c.$$

Then, we have $\forall k,l \in \mathbf{N}$,

$$\frac{\int_{\phi^-_\infty}^{\phi^+_\infty} \big[F'(a) - \frac{F(\phi^+_\infty)-F(\phi^-_\infty)}{\phi^+_\infty-\phi^-_\infty}\big] H_{k,l}(a)da}{\int_{\phi^-_\infty}^{\phi^+_\infty} \big[\frac{F(\phi^+_\infty)+F(\phi^-_\infty)}{2} - F(a)\big]da} = \frac{b_{k,l}(x_0)}{b_1(x_0)}.$$

Let us assume that there exist two subsequences (r_i) and (r_j) of (r_n) such that $\phi_{r_i} \to \phi_{\infty,1}$ and $\phi_{r_j} \to \phi_{\infty,2}$ in L^1_{loc} where $\{\phi^+_{\infty,1}, \phi^-_{\infty,1}\} \neq \{\phi^+_{\infty,2}, \phi^-_{\infty,2}\}$. Suppose for instance that $\phi^-_{\infty,2} < \phi^-_{\infty,1} < t_0 < \phi^+_{\infty,1} < \phi^+_{\infty,2}$. For any $k,l \in \mathbf{N}$ such that supp $H_{k,l} \subset (\phi^+_{\infty,1}, \phi^+_{\infty,2})$, we have

$$\int_{\phi^-_{\infty,1}}^{\phi^+_{\infty,1}} \Big[F'(a) - \frac{F(\phi^+_\infty)-F(\phi^-_\infty)}{\phi^+_\infty-\phi^-_\infty}\Big] H_{k,l}(a)da = 0$$

Then, $b_{k,l} = 0$ and

$$\begin{aligned} 0 &= \int_{\phi^-_{\infty,2}}^{\phi^+_{\infty,2}} \Big[F'(a) - \frac{F(\phi^+_\infty)-F(\phi^-_\infty)}{\phi^+_\infty-\phi^-_\infty}\Big] H_{k,l}(a)da \\ &= \int_{\phi^+_{\infty,1}}^{\phi^+_{\infty,2}} \Big[F'(a) - \frac{F(\phi^+_\infty)-F(\phi^-_\infty)}{\phi^+_\infty-\phi^-_\infty}\Big] H_{k,l}(a)da. \end{aligned}$$

Therefore, for a.e. $a \in (\phi^+_{\infty,1}, \phi^+_{\infty,2})$, $F'(a) = \frac{F(\phi^+_\infty)-F(\phi^-_\infty)}{\phi^+_\infty-\phi^-_\infty}$. This contradicts the strict convexity of F. Thus, ϕ^+_∞ and ϕ^-_∞ only depend on x_0.
Then, by the equality (20), $\sigma = c\mathcal{H}^1 \llcorner K$ where c only depends on x_0, and the second step is achieved.

Let us conclude the proof of Theorem 1.2. Since Σ is countably $\mathcal{H}^1$ rectifiable, by the Radon-Nikodym theorem, we have $\mu_\phi = f\mathcal{H}^1 \llcorner \Sigma$ for some measurable function f. Thus, for $\mathcal{H}^1$ a.e. $x_0 \in \Sigma$, any blow-up limit ϕ_∞ has its jump values $\phi^\pm_\infty$ uniquely determined by x_0. then, $\mathcal{H}^1$ a.e. $x_0 \in \Sigma$ is a jump point of ϕ. It is not hard to see that $J_\phi \subset \Sigma$, then J_ϕ is countably $\mathcal{H}^1$ rectifiable. One can now use the lemma 3.5 of [AKLR] to get

$$\mu_\phi \llcorner J_\phi = \int_{\mathbf{R}} |\text{div } X(\phi \wedge a)| \, da \llcorner J_\phi = \Delta(\phi^+, \phi^-)\mathcal{H}^1 \llcorner J_\phi.$$

Let us define $\delta_\phi := \mu_\phi \llcorner (\mathbf{R}_+^* \times \mathbf{R} \setminus J_\phi) = \mu_\phi \llcorner (\mathbf{R}_+^* \times \mathbf{R} \setminus \Sigma)$. By a standard argument of geometric measure theory, this measure satisfies (5).
Theorem 1.2 is proved.
■

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