Domain Wall Energy in a Problem Related to Micromagnetism

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Abstract

We study the asymptotic limit of a family of functionals related to the theory of micromagnetism in two dimensions. We prove a compactness result for families of uniformly bounded energy. After studying the corresponding one-dimensional profiles, we exhibit the Γ-limit ("wall energy"), which is a variational problem on the folding of solutions of the eikonal equation $|\nabla g| = 1$. We prove that the minimal wall energy is twice the perimeter. © 2001 John Wiley & Sons, Inc.

1 Introduction

Let Ω be a magnetic body in $\mathbb{R}^3$ and $u : \Omega \to \mathbb{S}^2$ its magnetization per unit volume. Following [11, p. 148], the energy associated to such a magnetization is given as

$$F(u) = w^2 \int_\Omega |\nabla u|^2 + \int_{\mathbb{R}^3} |H_u|^2 + Q \int_\Omega \Phi(u) - 2 \int_\Omega h_e \cdot u \tag{1.1}$$

where H_u is the demagnetizing field induced by the magnetization u and given by the following Maxwell equation:

$$\begin{cases} \operatorname{div}(\tilde{u} + H_u) = 0 & \text{in } \mathbb{R}^2 \\ \operatorname{curl} H_u = 0 & \text{in } \mathbb{R}^2 , \end{cases} \tag{1.2}$$

and $\tilde{u}$ is the extension of u by 0 out of the domain Ω. (In fact, H_u is defined as $-\nabla\Delta^{-1}(\operatorname{div}\tilde{u})$). h_e is some applied external field that will be zero in the present paper. Finally, w^2 and Q are, respectively, the exchange constant and the quality parameter that depend only on the material. $F(u)$ can be decomposed as follows: $w^2 \int_\Omega |\nabla u|^2$ is the exchange energy, $\int_{\mathbb{R}^3} |H_u|^2 = -\int_\Omega u \cdot H_u$ the demagnetizing energy, $Q \int_\Omega \Phi(u)$ the anisotropy energy, and lastly $-2\int_\Omega h_e \cdot u$ the interaction energy. The reader is invited to consult the paper by A. DeSimone, R. Kohn,

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S. Müller, and F. Otto for a general presentation of the model together with related open mathematical questions.

In this paper we will restrict ourselves to the simplified model in which the body is an infinite cylinder and the magnetization is invariant under translations all along the axis of the cylinder so that we can reduce the problem to the study of the cross section still denoted $\Omega \subset \mathbb{R}^2$. Then we assume that there is a very strong planar anisotropy so that the u is forced to take values into $\mathbb{S}^1$. Although such materials with strong planar anisotropy do exist (see [11]), the restriction on u to be strictly in $\mathbb{S}^1$, which is mathematically convenient, induces new topology that will generate perhaps nonphysically relevant parts in the limiting energy. Nevertheless, we believe that the local part of the limiting energy that we will exhibit in Theorem 1.8 should arise in the original problem with strong planar anisotropy but with u free to evolve in all of $\mathbb{S}^2$, and thus it has some physical relevance. So we are led to study the following family of energy functionals:

Insertion of "the problem" OK? Original was unclear.

$$E_\varepsilon(u) = \int_\Omega \varepsilon|\nabla u|^2 + \int_{\mathbb{R}^2} \frac{1}{\varepsilon}|H_u|^2 \,, \tag{1.3}$$

depending on the positive parameter ε. We are interested in the special regime $\varepsilon \to 0$ where the magnetization can vary very rapidly. This corresponds to the limit $w = 0$ $\Omega \subset \mathbb{R}^2$ is a smooth, bounded, simply connected domain on which we shall make additional assumptions and E_ε is defined over $H^1(\Omega, \mathbb{S}^1)$, where $\mathbb{S}^1$ is the unit circle in $\mathbb{C} \simeq \mathbb{R}^2$. Any $u : \Omega \to \mathbb{S}^1$ is extended to

Seems like there is text missing between $w = 0$ and $\Omega \subset \mathbb{R}^2$. Please correct. Or is there simply a missing period?

$$\tilde{u} = \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^2 \setminus \Omega \,, \end{cases}$$

and H_u is a vector field that is deduced from u by the equations

$$\begin{cases} \operatorname{div}(\tilde{u} + H_u) = 0 & \text{in } \mathbb{R}^2 \\ \operatorname{curl} H_u = 0 & \text{in } \mathbb{R}^2 \,. \end{cases} \tag{1.4}$$

There has been some work on this functional for a fixed ε, notably by G. Carbou [6], and then by R. Hardt and D. Kinderlehrer [10], who studied the regularity of minimizers for $\Omega = B_1(0)$. Critical points of E_ε satisfy the following Euler equation:

$$-\Delta u - u|\nabla u|^2 = \frac{1}{\varepsilon^2}(H_u \cdot u^\perp)u^\perp \,, \tag{1.5}$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^2$ and $u^\perp = iu$. Carbou proved that a critical u_ε is continuous in Ω.

Now, let us explain briefly the expected behavior of minimizers. More generally, we can consider families u_ε such that $E_\varepsilon(u_\varepsilon) \le C$. The term $\varepsilon \to 0$ forces the term $\int_{\mathbb{R}^2} |H_u|^2$ to tend to 0. In fact, we will prove more here, since we will show that $\min E_\varepsilon$ remains bounded above as $\varepsilon \to 0$. On the other hand, since $|u_\varepsilon| = 1$, we can extract a subsequence converging weakly in L^∞ to some u_0. One of the first questions that arises, and that we will answer positively, is to know whether the equality

$$|u_0| = 1 \tag{1.6}$$

holds. Then, since $H_u \to 0$ in $L^2(\mathbb{R}^2)$, in view of equation (1.2), u_0 must satisfy, in the sense of distributions,

$$\operatorname{div} \tilde{u}_0 = 0 \quad \text{in } \mathbb{R}^2\,. \tag{1.7}$$

This is equivalent to

$$\begin{cases} \operatorname{div} u_0 = 0 & \text{in } \Omega \\ u_0 \cdot n = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.8}$$

where n is the exterior unit normal to $\partial\Omega$. Therefore, the possible limiting fields lie among all divergence-free (in a weak sense) unit vector fields tangent to $\partial\Omega$, which is a very large class of fields. Since (1.8) holds, there exists a $C_0^{0,1}(\Omega)$ real-valued function g such that

$$u_0 = \nabla^\perp g$$

where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. Thus, in other words, the limit $\varepsilon \to 0$ leads to solutions of the eikonal equation

$$\begin{cases} |\nabla g| = 1 \text{ a.e.} & \text{in } \Omega \\ g = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.9}$$

The difficulty is to understand which of these solutions can be reached through this limiting process. Then we can see the similarity between this problem and another problem, first raised by Aviles and Giga [2], which has also been studied independently by L. Ambrosio, C. De Lellis, and C. Mantegazza in [1], by A. DeSimone, R. Kohn, S. Müller, and F. Otto in [9], and more recently by W. Jin and R. Kohn in [13], consisting in minimizing the energy functionals

$$F_\varepsilon(\psi) = \int_\Omega \varepsilon|\nabla\nabla\psi|^2 + \frac{1}{\varepsilon}(1 - |\nabla\psi|^2)^2\,, \tag{1.10}$$

or more generally, in studying sequences such that $F_\varepsilon(\psi_\varepsilon) \le C$. In this problem (1.10), the fields studied $\nabla\psi$ are constrained to be curl-free, which is analogous, up to a rotation of $\pi/2$, to being divergence-free. In the present problem, they are only divergence-free in the limit. On the other hand, while our fields are constrained to be of unit norm, the fields $\nabla\psi$ in (1.10) only tend to be of norm 1 in the limit $\varepsilon \to 0$. Hence, in some sense, the limiting processes are reversed. But as we shall see, there are other important differences between both problems. First, F_ε is a local functional, whereas E_ε contains a *nonlocal term* $\int_{\mathbb{R}^2} |H_u|^2$, and the same is then true for the Euler equations. Second, the expected Γ-limits for these functionals are quite different.

It is clear by a degree argument that there are no regular divergence-free unit vector fields that are tangent to $\partial\Omega$ for a general simply connected domain. All admissible vector fields have line singularities (except in the case of the ball, where the field $\frac{1}{2\pi}\frac{\partial}{\partial\theta}$ has only a point singularity at the origin). In the case of problem (1.10), the limiting fields also have line singularities. In fact, the conjecture made by Aviles and Giga (see also [3, 13]) is that the energy functionals F_ε Γ-converge to

$$\frac{1}{3}\int\limits_{J_{\nabla\psi}} \left|\nabla^+\psi - \nabla^-\psi\right|^3 d\mathcal{H}^1\,, \tag{1.11}$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure, $J_{\nabla\psi}$ is the jump set of $\nabla\psi$, which is expected to be countably rectifiable, and $\nabla^+\psi$ and $\nabla^-\psi$ are the traces on both sides of the jump set. Thus, the energy E_ε is also expected to concentrate on lines at a scale ε around the lines (because of the scaling in the functional), and the limit $\varepsilon \to 0$ corresponds to allowing sharper and sharper jumps. These lines would correspond in three dimensions to jumps across surfaces, called "domain walls" in the theory of micromagnetism.

Let us state our main results. For reasons that will become clearer later, we need to work on smaller functional spaces involving liftings of u. These definitions can be skipped in a first reading.

DEFINITION 1.1 We call a collection $\mathcal{U}$ of $((U_i)_{i\in I}, (k_{ij})_{i,j\in I})$ an *admissible covering* of Ω where

(i) $\operatorname{Card} I < \infty$;
(ii) U_i is open, $\Omega \subset \bigcup_{i\in I}\overline{U_i}$, $\forall i, j \in I$, and $U_i \cap U_j$ is diffeomorphic to $\mathbb{R}^2$; and
(iii) $\forall i, j \in I$, $k_{ij} \in \mathbb{Z}$, and they satisfy the cocycle relations $k_{ij} = -k_{ji}$, $k_{ij} + k_{jl} + k_{li} = 0$.

For any admissible $\mathcal{U}$, we define $\Lambda_{\mathcal{U}}$ to be the set of $u \in H^1(\Omega, \mathbb{S}^1)$ such that $\forall i \in I$, there exists $l_i \in [0, 2\pi]$ and $\psi_i \in H^1(U_i, [l_i, l_i + 2\pi])$ with

(i) $u = e^{i\psi_i}$ on U_i, $\forall i \in I$, and
(ii) $\forall i, j \in I$, $\psi_i - \psi_j = 2\pi k_{ij}$ in $U_i \cap U_j$.

We will say that the $u \in \Lambda_{\mathcal{U}}$ satisfy the "locally bounded phase condition," or LBP condition.

LEMMA 1.2 *If $u \in \Lambda_{\mathcal{U}}$, there exists $\varphi \in H^1(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$ such that*

$$u = e^{i\varphi} \quad \textit{and} \quad \|\varphi\|_{L^\infty(\Omega)} \le C_{\mathcal{U}}\,,$$

where $C_{\mathcal{U}}$ is a constant that depends only on $\mathcal{U}$.

This comes from the assumption that Ω is simply connected and that $\mathcal{U}$ is an admissible covering. Thus our locally constant presheave is globally constant; see [5, p. 141].

These definitions mean that we restrict ourselves to the set of u that have a $H^1(\Omega, \mathbb{R})$-lifting, locally taking its values in an interval of length $\le 2\pi$ of $\mathbb{R}$. This assumption, which is mainly required in the proof of Theorem 1.5, is not only there to make our life easier; it also seems to have a fundamental mathematical relevance in our problem: Indeed, we will prove later that any solution of the limiting profile equation (1.19) solves the LBP condition: The smooth lifting has to be locally included in an interval of size 2π. The variational problem is well-posed in these spaces; i.e., for any admissible $\mathcal{U}$, E_ε admits a minimum in $\Lambda_{\mathcal{U}}$ that we denote $I_\varepsilon^{\mathcal{U}}$.

THEOREM 1.3 *If $\partial\Omega$ is a finite union of analytic curves, then*

$$\limsup_{\varepsilon\to 0} \min_{H^1(\Omega,S^1)} E_\varepsilon \le 2|\partial\Omega|\,,$$

where $|\partial\Omega|$ denotes the perimeter of Ω. In addition, there exists an admissible covering $\mathcal{U}$ such that

$$\limsup_{\varepsilon\to 0} I_\varepsilon^{\mathcal{U}} \le 2|\partial\Omega|\,.$$

(*We call such a $\mathcal{U}$ a* "*suitable covering of* Ω.")

The second result is a compactness result, in the spirit of that of [1], together with a type of Γ-convergence result. We start by defining the "micromagnetism domain wall configuration space" $\mathcal{C}$, which is the suitable limiting configuration set.

DEFINITION 1.4 $\mathcal{C}$ is the class of couples (u,φ) such that

(i) $u:\Omega\to\mathbb{S}^1$,
(ii) $\operatorname{div}\tilde{u}=0$ in $\mathcal{D}'(\mathbb{R}^2)$,
(iii) $\varphi\in L^1(\Omega,\mathbb{R})$ and $u=e^{i\varphi}$ a.e. in Ω, and
(iv) $\mu_{u,\varphi}:=\operatorname{div}(\varphi u+u^\perp)$ is a bounded Radon measure on Ω.

We denote by $\|\mu_{u,\varphi}\|$ the total mass of $\mu_{u,\varphi}$, $\int_\Omega|\mu_{u,\varphi}|$.

Note that $\mu_{u,\varphi}$ does not change if a constant is added to φ.

We are going to show that the limiting "functional" associated to E_ε is the following "domain wall energy":

$$\|\mu_{u,\varphi}\| = \int_\Omega |\operatorname{div}(\varphi u + u^\perp)|$$

defined over $\mathcal{C}$. This energy is a good candidate for making a selection among the solutions of the eikonal equation (1.9), and this selection should be compared to conjecture 6.2 of [8]. We expect solutions of the eikonal equation belonging to $\mathcal{C}$ to satisfy nice properties (see the comments at the end of the introduction). Our second main result is the following compactness property:

THEOREM 1.5 (without assumptions on $\partial\Omega$) *Let $\mathcal{U}$ be an admissible covering of Ω, $\varepsilon_n\to 0$, and $u_n\in\Lambda_{\mathcal{U}}$ such that $E_{\varepsilon_n}(u_n)\le C$ for some constant C independent of n, and let φ_n be a lifting of u_n given by Lemma* 1.2. *Then, after extraction of a subsequence, there exists u and φ in $\bigcap_{1\le q<\infty} L^q$ such that*

$$u_n\to u \quad \textit{and} \quad \varphi_n\to\varphi \quad \textit{in } L^q(\Omega) \quad \forall 1\le q<\infty\,,$$

In addition, $(u, \varphi) \in \mathcal{C}$ *and*

$$\liminf_{n\to\infty} E_{\varepsilon_n}(u_n) \geq 2\|\mu_{u,\varphi}\| .$$

Thus, a sequence whose energy E_{ε_n} is uniformly bounded (which, from Theorem 1.3, is the case for a sequence of minimizers) is compact in $\bigcap L^q$, and its limits (after extraction) are necessarily of unit norm. $\mathcal{C}$ appears to be the right limiting configuration space. The LBP assumption is crucial in our proof of compactness.

The proof of the connection between E_ε and $\|\mu_{u,\varphi}\|$ at the limit can be sketched in the following way: Consider $\varepsilon_n \to 0$ and u_n such that $E_{\varepsilon_n}(u_n) \leq C$ and u_n has a $H^1(\Omega, \mathbb{R}^2)$-lifting φ_n in Ω ($u_n = e^{i\varphi_n}$). Denoting by H_n the demagnetizing fields associated to u_n by (1.2), one has

$$\begin{aligned} C \geq E_{\varepsilon_n}(u_n) &= \int_\Omega \varepsilon_n |\nabla u_n|^2 + \int_{\mathbb{R}^2} \frac{1}{\varepsilon_n}|H_n|^2 \\ &\geq \int_\Omega \varepsilon_n |\nabla u_n|^2 + \frac{1}{\varepsilon_n}|H_n|^2 \geq 2\int_\Omega |\nabla u_n||H_n| . \end{aligned}$$

But $|\nabla u_n| = |\nabla \varphi_n|$ in Ω; hence we can write

$$C \geq E_{\varepsilon_n}(u_n) \geq 2\int_\Omega |\nabla \varphi_n \cdot H_n| . \tag{1.12}$$

We thus find a quantity $\mu_n = \nabla\varphi_n \cdot H_n$, which remains bounded in the sense of measures. This measure has the particular property that it is a divergence; indeed,

$$\mu_n = \nabla\varphi_n \cdot H_n = -\nabla\varphi_n \cdot u_n + \nabla\varphi_n \cdot (u_n + H_n) . \tag{1.13}$$

But if u_n is regular enough, it is easy to check that

$$\operatorname{curl} u_n = \partial_{x_1}(\sin\varphi_n) - \partial_{x_2}(\cos\varphi_n) = \nabla\varphi_n \cdot u_n . \tag{1.14}$$

On the other hand, from (1.2), $\operatorname{div}(u_n + H_n) = 0$ in Ω; hence

$$\nabla\varphi_n \cdot (u_n + H_n) = \operatorname{div}\big(\varphi_n(u_n + H_n)\big) . \tag{1.15}$$

Combining (1.13), (1.14), and (1.15), we have

$$\mu_n = \operatorname{div}\big(\varphi_n u_n + u_n^\perp + \varphi_n H_n\big) .$$

Since in addition $H_n \to 0$, $\varphi_n u_n + u_n^\perp + \varphi_n H_n$ is the sum of a local term $\varphi_n u_n + u_n^\perp$ and a term that vanishes and can be neglected. We have thus reduced a *nonlocal* problem to a *local* one, and in first approximation, we can write in view of (1.12):

$$C \geq E_{\varepsilon_n}(u_n) \geq 2\int_\Omega |\mu_n| . \tag{1.16}$$

Since μ_n is bounded in measures, it has a weak limit that is a measure μ, and we can expect that

$$\liminf_{n\to\infty} E_{\varepsilon_n}(u_n) \geq 2 \liminf_{n\to\infty} \int_\Omega |\mu_n| \geq 2 \int_\Omega |\mu|$$

with $\mu = \operatorname{div}(\varphi u + u^\perp)$ associated to the limit $u = e^{i\varphi}$ of u_n.

Next, we state a result on minimizers of the limiting problem.

DEFINITION 1.6 $u_\star$ is the unit vector field $\nabla^\perp(\operatorname{dist}(\cdot, \partial\Omega))$.

We will prove in Section 3 that there exists $\varphi_\star$ such that $(u_\star, \varphi_\star)$ belongs to $\mathcal{C}$.

THEOREM 1.7 *If $\partial\Omega$ is analytic by parts,*

$$\min_{(u,\varphi)\in\mathcal{C}} \|\mu_{u,\varphi}\| = |\partial\Omega| \,.$$

The minimum is achieved by $(u_\star, \varphi_\star)$.

We conjecture that the only minimizers are $u_\star$ and $-u_\star$. We give a heuristic justification of this fact in the last part of the introduction. These considerations have to be compared with the recent nonuniqueness result of [13] for F_ε. As a consequence of Theorems 1.3, 1.5, and 1.7, we get the global understanding of minimizers of E_ε under the LBP condition.

THEOREM 1.8 *Assume $\partial\Omega$ is analytic by parts. Let $\mathcal{U}$ be a suitable covering of Ω (i.e., $\limsup_{\varepsilon\to 0} I_\varepsilon^{\mathcal{U}} \leq 2|\partial\Omega|$); then*

$$\lim_{\varepsilon\to 0} I_\varepsilon^{\mathcal{U}} = 2 \min_{(u,\varphi)\in\mathcal{C}} \|\mu_{u,\varphi}\| = 2|\partial\Omega| \,,$$

and a sequence of minimizers $u_\varepsilon = e^{i\varphi_\varepsilon}$ of E_ε over $\Lambda_{\mathcal{U}}$ converges strongly in $\bigcap L^q$ (both u_ε and φ_ε converge) to a minimizer (u, φ) of $\|\mu_{u,\varphi}\|$ over $\mathcal{C}$.

To give an idea of the lower bound $\|\mu_{u,\varphi}\| \geq |\partial\Omega|$ for $u \in \mathcal{C}$, recall that $u \cdot n = 0$ on $\partial\Omega$, and assume, for instance, that $u \cdot \tau = -1$ on $\partial\Omega$, where τ is the unit tangent to $\partial\Omega$ (with positive orientation). Then, integrating by parts, we formally have

$$\int_\Omega |\operatorname{div}(\varphi u + u^\perp)| \geq \int_\Omega \operatorname{div}(\varphi u + u^\perp) = \int_{\partial\Omega} \varphi u \cdot n - u \cdot \tau = |\partial\Omega| \,.$$

The proof needs to be refined when $u \cdot \tau$ changes sign on $\partial\Omega$, but the result is still true.

If $(u, \varphi) \in \mathcal{C}$, as already mentioned, u has singularities, and they are expected to be line singularities. Since, u is divergence-free, its normal component across such singularity lines is preserved. Wherever u and φ are smooth, one has

$$\operatorname{div}(\varphi u) = \nabla\varphi \cdot u = \operatorname{curl} u = -\operatorname{div} u^\perp \,.$$

Hence $\mu_{u,\varphi} = 0$, and $\mu_{u,\varphi}$ is thus supported on the singularities of u. Unfortunately, the condition that $\operatorname{div}(\varphi u + u^\perp)$ be a measure does not guarantee that φ or u is in

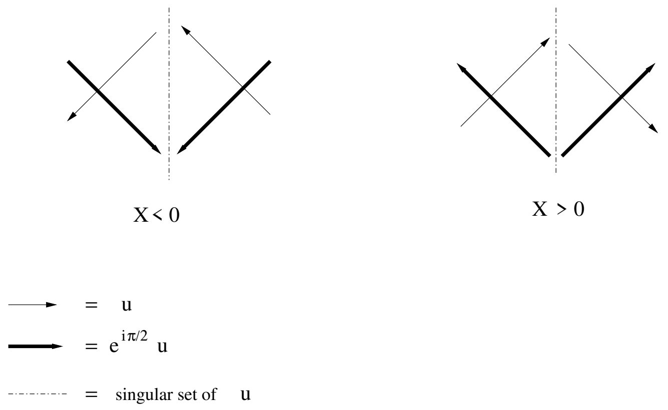

FIGURE 1.1. X is positive when iu is pointing *inwards*; the singular set of u and X is negative when iu is pointing *outwards* from the singular set of u.

$\mathrm{BV}(\Omega)$ (one can see this from the ideas of the counterexamples of [1]): the class of possible limits $\mathcal{C}$ is quite large. Nevertheless, we may expect the measure $\mu_{u,\varphi}$ to be carried on a countably rectifiable set.

We have a more precise result in the case where $\varphi \in \mathrm{BV}(\Omega)$ (where φ is a lifting of u given by the definition of $\mathcal{C}$). Then by definition, $D\varphi$ is a Radon measure, and it is standard that $D\varphi$ can be split into three mutually singular parts

$$D\varphi = \nabla\varphi\mathcal{L}^2 + (\varphi^+ - \varphi^-) \otimes n\mathcal{H}^1 \lfloor_S + D_c\varphi \tag{1.17}$$

where $\mathcal{L}^2$ is the Lebesgue measure, $\mathcal{H}^1$ is the one-dimensional Hausdorff measure, $D_c\varphi$ is the Cantor part of $D\varphi$, S is the jump set of φ, n is the normal to $\mathbb{S}$ pointing from $\mathbb{S}$ into the $+$ half-space, and φ^+ and φ^- are the approximate limits of φ on both "sides", $+$ and $-$, of $\mathbb{S}$; see Section 5.2 for more details.

THEOREM 1.9 (i) *Let $(u, \varphi) \in \mathcal{C}$ be such that $\varphi \in \mathrm{BV}(\Omega)$. Then,*

$$\mu_{u,\varphi} = \operatorname{div}(\varphi u + u^\perp) = \big((\varphi^+ - \varphi^-)(u \cdot n) - (u^+ - u^-) \cdot n^\perp\big)\mathcal{H}^1 \lfloor_S .$$

Thus

$$\|\mu_{u,\varphi}\| = \int_S \big|(\varphi^+ - \varphi^-)(u \cdot n) - (u^+ - u^-) \cdot n^\perp\big| d\mathcal{H}^1 .$$

If $|\varphi^+ - \varphi^-| < 2\pi$ in $\Omega' \subset \Omega$, we define the angle $X \in (-\pi, \pi)$ in the following way (*see Figure* 1.1):

$$\begin{cases} |X| = \frac{|\varphi^+ - \varphi^-|}{2} \\ \operatorname{sgn}(X) = -\operatorname{sgn}(u^+ \cdot n^\perp) . \end{cases}$$

Then we have

$$\mu_{u,\varphi} \lfloor_{\Omega'} = 2(\sin X - X \cos X)\mathcal{H}^1 \lfloor_{S \cap \Omega'} .$$

The density of $\mu_{u,\varphi}$ *has the sign of* X.

(ii) *If* (u,φ) *is a bounded family in* BV *and* $u_\varepsilon = e^{i\varphi_\varepsilon}$ *is such that* $E_\varepsilon(u_\varepsilon) \le C$, *then, up to extraction, there exists* $(u,\varphi) \in \mathcal{C}$ *such that* u *and* φ *are in* BV, *for which* $\varphi_\varepsilon \rightharpoonup \varphi$ *in* BV, $u_\varepsilon \rightharpoonup u$ *in* BV, *and*

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \ge 2\|\mu_{u,\varphi}\| \ge 2|\partial\Omega|\,.$$

Remark 1.10. Observe that the sign of X and then the sign of the measure $\operatorname{div}(\varphi u + u^\perp)$ is independent of the liftings φ^+ and φ^- chosen but only depends on u^+ and u^- as long as we ensure $|\varphi^+ - \varphi^-| \le 2\pi$.

Thus we have a more quantitative description of $\mu_{u,\varphi}$ in terms of the jumps of the phase of u. Observe that when X varies in $[0,\pi]$, $\sin X - X\cos X$ increases from 0 to π, and observe also that for $X \in [-\pi,\pi]$, $\sin X - X\cos X$ has the sign of X. In addition, when X is small

$$\sin X - X\cos X \sim \frac{X^3}{6}\,.$$

This is reminiscent of the case of (1.10) in [1] where the energy cost of the jumps in $\nabla\psi$ is given by (1.11). The two problems should thus have the same qualitative properties for small jumps; see also the discussion about lower semicontinuity in [1]. On the other hand, there is a major difference since we exhibit a cost of the jumps of φ and not only of u. If φ jumps by 2π, for example, u remains continuous, but there is a cost in $\|\mu_{u,\varphi}\|$. This can be very well illustrated in the case of $\Omega = B_1(0)$. The limiting field $u_\star = \frac{1}{2\pi}\frac{\partial}{\partial\theta}$ has a singularity only at 0, but its phase must have a line singularity joining 0 to ∂B_1, and this costs $2\pi = |\partial B_1|$ in $\|\mu_{u_\star,\varphi_\star}\|$, whereas in that case, for F_ε of [1] or [9], the limiting energy is 0.

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For general domains Ω, we prove that $\int_{\mathbb{S}^2}(\sin X - X\cos X) = |\partial\Omega|$ for the field $u_\star = \nabla^\perp \operatorname{dist}(\cdot,\partial\Omega)$. Thus, from the micromagnetism energy, we are led to a limiting functional that contains some of the geometric features of the domain, and we are led to unexpected results on the function $\operatorname{dist}(\cdot,\partial\Omega)$; see Section 5 and Remark 5.3.

The sign of X should play a role in the uniqueness result we expect for the minimizer for $\|\mu_{u,\varphi}\|$ in $\mathcal{C}$ (up to a reverse of sign). Formally following the kind of arguments of the proof of Lemma 5.1, we can see that, for a minimizer, $\mu_{u,\varphi}$ has to be either positive or negative. Then X is either positive everywhere or negative everywhere. Changing u to $-u$ if necessary, we can assume X to be positive, and u to be $\nabla^\perp g$ for some $g \in C_0^{0,1}(\Omega)$. The uniqueness of the minimizer can be sketched the following way: Starting from any point in the domain away from the singular set, we may follow the ray given by $u^\perp$. Because of the positivity of X, we see that it is not possible to cross the singular set; see Figure 1.1. Then the ray reaches the boundary without crossing the singular set. Thus, $g = \operatorname{dist}(x,\partial\Omega)$ and $u = u_\star$.

It is tempting to imagine that there should exist a statement similar to that of Theorem 1.9 for general $(u,\varphi) \in \mathcal{C}$ (or at least the ones that are limits of bounded

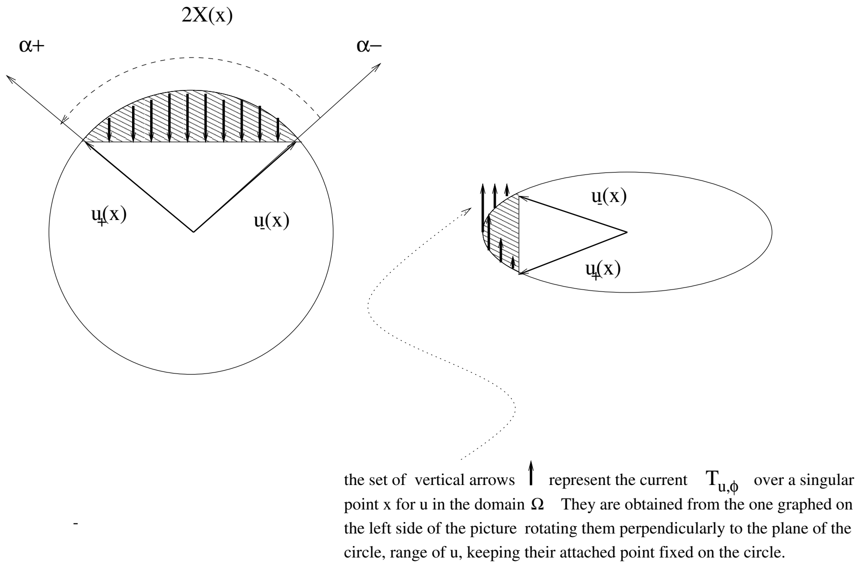

FIGURE 1.2. The current $T_{u,\varphi}$: a geometric interpretation of $\operatorname{div}(\varphi u + u^{\perp})$.

sequences $(u_\varepsilon, \varphi_\varepsilon)$ for E_ε) without the assumption that $\varphi \in \mathrm{BV}(\Omega)$. The main difficulty is to prove that the measure $\operatorname{div}(\varphi u + u^{\perp})$ is supported on some set that has a nice structure, such as countable rectifiability (countable union of smooth curves) in order to be able to define the notions of $+$ and $-$ side, unit normal, etc. The example of [1] seems to indicate that it is too optimistic to expect the jump set to be rectifiable (finite mass). This comes from the fact that we control only X^3 (see discussion above) and not X for X small.

An adapted object to work with that also gives another geometric interpretation of the measure $\operatorname{div}(\varphi u + u^{\perp})$ for solutions of the eikonal equation is the following current: Assume $(u, \varphi) \in \mathcal{C}$ is the L^q limit of a bounded sequence $(u_\varepsilon, \varphi_\varepsilon)$ for E_ε. We introduce the current $T_{u_\varepsilon,\varphi_\varepsilon}$ in $\Omega \times \mathbb{R}^3$ equal to

$$T_{u_\varepsilon,\varphi_\varepsilon} = \Big\{(x, u_\varepsilon(x), t H_{u_\varepsilon} \cdot e) \in \Omega \times \mathbb{R}^3 : (x, t) \in \Omega \times [0, 1]\ , \\ H_{u_\varepsilon} \text{ given by (1.2)},\ e \text{ the unit oriented by } \nabla u_\varepsilon\Big\} .$$

This current has a uniformly bounded mass as $\varepsilon \to 0$. In the case where φ_ε is uniformly bounded in BV, it converges to the following rectifiable current:

$$T_{u,\varphi} = \Big\{(x, e^{i\theta}, t \operatorname{sgn}(u^+ \cdot n^+) h(\theta, x)) \in \Omega \times \mathbb{R}^3 : (x, t) \in \mathbb{S} \times [0, 1], \\ \varphi_- \le \theta \le \varphi_+,\ h(\theta, x) \text{ the distance between } e^{i\theta} \text{ and the line } u^+, u^-\Big\}$$

(see Figure 1.2). It is not difficult to see that

$$M(T_{u,\varphi}) = \int_\Omega \left|\operatorname{div}(\varphi u + u^{\perp})\right| = \|\mu_{u,\varphi}\| . \tag{1.18}$$

In the case where φ is BV, we clearly have $M(\partial T_{u,\varphi}) < +\infty$. Again because of the example in [1], we do not expect this control of the mass of the boundary of the limit of $T_{u_\varepsilon,\varphi_\varepsilon}$ to hold in general; nevertheless, we still expect the limit of $T_{u_\varepsilon,\varphi_\varepsilon}$, which has a finite mass, to be rectifiable, which should imply the countable rectifiability of the jump set of u on Ω.

We now sketch the outline of the paper. Section 2 contains some preliminary results and the study of the one-dimensional problem: We expect variations of u_ε, minimizing E_ε, to be concentrated near S (jump set of $u_\star$) at a scale ε in the direction perpendicular to S. Therefore, we are led, after blowup at the scale ε, to the one-dimensional ODE corresponding to solutions of (1.3) that depend only on one direction, which we prove to be solutions of

$$\begin{cases} \varphi' = \sin\varphi - \sin\alpha_+ \\ \varphi(+\infty) = \alpha_+ \\ \varphi(-\infty) = \alpha_- \\ \sin\alpha_+ = \sin\alpha_-,\ 0 \le \alpha_+ - \alpha_- \le 2\pi\ . \end{cases} \tag{1.19}$$

This equation is the *profile* equation for the model. It is interesting to notice that (1.19) has no solution if the jump $\alpha_+ - \alpha_- > 2\pi$. This is probably related to our assumption $u \in \Lambda_{\mathcal{U}}$ for compactness.

In Section 3 we prove the upper bound for $\min E_\varepsilon$ by constructing a test configuration: It is the function $u_\star$ wherever it is smooth, and we paste the *profiles* (1.19) at a scale ε along its jump set. Then we have to bound $\int_\Omega |H|^2/\varepsilon$ from above. This is done through a "projection lemma" (Lemma 2.3), which permits us to bound the *nonlocal* term $\int_\Omega |H|^2$ by *local* integrals. The assumption that $\partial\Omega$ is analytic by parts is needed to use a result of Choi, Choi, and Moon [7], which asserts that in this case, the singular set of $u_\star$, called the "medial axis" of Ω, is also a finite union of analytic curves, while if $\partial\Omega$ is assumed to be only C^∞, the medial axis can be very pathological; see [7]. Yet we believe that all the quantitative results ($\min E_\varepsilon \sim 2|\partial\Omega|$) would still be true without this assumption on $\partial\Omega$.

In Section 4 we prove the compactness result. Its proof is inspired by that of [1]. We use a convexity-type relation, the div-curl lemma, and Young measures to prove that the limiting u must satisfy $|u| = 1$ a.e., and deduce strong convergence in $\bigcap_{q<\infty} L^p$. Section 5 is devoted to Theorems 1.7 and 1.8 using the method that was already sketched and to Theorem 1.9.

Notation. C is always a positive constant, independent of ε. $\mathcal{M}$ is the space of Radon measures on Ω. $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$, $u^\perp = iu = (-u_2, u_1)$. $B_r(x)$ denotes the ball of radius r centered at x.

2 A Few Preliminary Results

2.1 Preliminary Results on Demagnetizing Fields

In this section we consider a general vector field V on $\mathbb{R}^2$, bounded with compact support. To V is associated the induced demagnetizing field H_V defined by

$$H_V = -\nabla\Delta^{-1}\operatorname{div} V, \tag{2.1}$$

where the operator Δ^{-1} is defined as the convolution with the kernel $2\pi\log|x-y|$. Clearly, H_V satisfies

$$\begin{cases} \operatorname{div}(H_V + V) = 0 & \text{in } \mathbb{R}^2 \\ \operatorname{curl} H_V = 0 & \text{in } \mathbb{R}^2\,. \end{cases} \tag{2.2}$$

LEMMA 2.1 (i) *$\forall p < \infty$, there exists C_p depending only on the support of V such that $\|H_V\|_{L^p(\mathbb{R}^2)} \le C_p\|V\|_{L^\infty}$.*

(ii) *For any $x \in \mathbb{R}^2$ such that* $\operatorname{dist}(x, \operatorname{supp} V) \ge 1$, $|H_V(x)| \le \frac{C}{|x|^2}\|V\|_{L^\infty}$, *and* $|\Delta^{-1}\operatorname{div} V(x)| \le (C/|x|)\|V\|_{L^\infty}$, *where the constants C depend only on the support of V.*

(iii)

$$\int_{\mathbb{R}^2} |H_V|^2 = \int_{\mathbb{R}^2} (\operatorname{div} V)(\Delta^{-1}\operatorname{div} V) = -\int_{\mathbb{R}^2} V\cdot H_V\,. \tag{2.3}$$

PROOF OF LEMMA 2.1: $\forall p < \infty$, $\|V\|_{L^p}$ is bounded by a constant that depends only on the support of V. Thus, standard results on singular integrals give that $\operatorname{div} V = -\operatorname{div} H_V$ is bounded in $W^{-1,p}$ and H_V in L^p.

For any x such that $\operatorname{dist}(x, \operatorname{supp} V) \ge 1$, we see by using integration by parts that

$$\begin{aligned} \left|\Delta^{-1}\operatorname{div} V(x)\right| &= \left|\int_{\mathbb{R}^2} 2\pi\log|x-y|\operatorname{div} V(y)dy\right| \\ &= \left|-2\pi\int_{\mathbb{R}^2} V(y)\cdot\nabla_y\log|x-y|dy\right| \\ &\le 2\pi\int_{\mathbb{R}^2}\frac{|V(y)|}{|x-y|}dy \\ &\le \frac{C}{|x|}\|V\|_{L^\infty}|\operatorname{supp} V|\,, \end{aligned} \tag{2.4}$$

where $|\operatorname{supp} V|$ is the volume of $\operatorname{supp} V$. Similarly,

$$\nabla\Delta^{-1}\operatorname{div} V(x) = \int_{\mathbb{R}^2} L(x,y)\operatorname{div} V(y)dy\,, \tag{2.5}$$

where

$$L(x, y) = 2\pi\left(\frac{x_1 - y_1}{|x - y|^2}, \frac{x_2 - y_2}{|x - y|^2}\right)$$

and

$$\begin{aligned}\left|\nabla\Delta^{-1}\operatorname{div} V(x)\right| &= \left|\int_{\mathbb{R}^2} L(x, y)\operatorname{div} V(y)dy\right| = \left|-\int_{\mathbb{R}^2} V(y)\cdot\nabla_y L(x, y)dy\right| \\ &\le \frac{C}{|x|^2}\|V\|_{L^\infty}|\operatorname{supp} V|\,. \end{aligned} \tag{2.6}$$

As a consequence, H_V is in $L^2(\mathbb{R}^2)$ and assertions (i) and (ii) are true.

Then, using the definition of H_V and several integrations by parts,

$$\begin{aligned}\int_{\mathbb{R}^2} |H_V|^2 = \int_{\mathbb{R}^2} H_V\cdot\nabla\Delta^{-1}\operatorname{div} V &= \int_{\mathbb{R}^2} -(\operatorname{div} H_V)\Delta^{-1}\operatorname{div} V \\ &= \int_{\mathbb{R}^2} (\operatorname{div} V)\Delta^{-1}\operatorname{div} V \\ &= -\int_{\mathbb{R}^2} V\cdot\nabla\Delta^{-1}\operatorname{div} V = -\int_{\mathbb{R}^2} V\cdot H_V\,.\end{aligned}$$

Hence, assertion (iii) is proved. □

Remark 2.2. Applying this lemma, we deduce that for any sequence of unit vector fields u_ε, $\|H_{u_\varepsilon}\|_{L^p}$ is uniformly bounded, $\forall p < \infty$.

Next we state what we call the "projection lemma." This lemma allows us to estimate integrals of demagnetizing fields of the type $\int_{\mathbb{R}^2}|H_V|^2$ knowing V without having to compute H_V and comparing it only with *local* integrals.

LEMMA 2.3 (Projection Lemma) *For any V bounded with compact support, we have*

$$\int_{\mathbb{R}^2} |H_V|^2 = \min_{g\in H^1(\mathbb{R}^2,\mathbb{R})}\int_{\mathbb{R}^2} |V - \nabla^\perp g|^2\,. \tag{2.7}$$

This lemma is in fact the Hodge projection theorem in L^2. In other words, the result amounts to the fact that H_V is the L^2-orthogonal projection of V on the subspace of curl-free vector fields.

Of course, these lemmas apply to the test functions for the problem of minimizing E_ε. From them, we can deduce a restatement of the original problem. Defining over $H^1(\mathbb{R}^2, \mathbb{R})$ the functional

$$G_\varepsilon(g) = \min_{v\in H^1(\Omega,S^1)}\int_\Omega \varepsilon|\nabla v|^2 + \int_{\mathbb{R}^2}\frac{1}{\varepsilon}|\tilde{v} - \nabla^\perp g|^2\,, \tag{2.8}$$

using comparison arguments, one can check that

$$\min_{H^1(\Omega,S^1)} E_\varepsilon = \min_{H^1(\mathbb{R}^2,\mathbb{R})} G_\varepsilon\,.$$

2.2 Study of the One-Dimensional Problem

Heuristic Derivation of the Problem

Consider some family u_ε of uniformly bounded energy E_ε. Its limiting field is a unitary divergence-free vector field u, and it should have line singularities. Across such a line, since u is divergence-free, $u \cdot n$ remains continuous. Therefore, the argument of u in the orthonormal frame relative to the line singularity can only jump from some α_+ to some α_-, with $\sin\alpha_+ = \sin\alpha_-$. Returning to the original field u_ε, we rescale it around x_0, a point of singularity of u, and set $\hat{u}_\varepsilon(x) = u_\varepsilon(x_0 + \varepsilon x)$. We expect $\hat{u}_\varepsilon$ to behave like a function u that depends only on one variable x_2, which minimizes

$$\int_{\mathbb{R}} |\nabla u|^2 + |H|^2 \tag{2.9}$$

and satisfies

$$\lim_{x_2\to+\infty} \operatorname{Arg} u = \alpha_+\,, \qquad \lim_{x_2\to-\infty} \operatorname{Arg} u = \alpha_-\,.$$

The rescaled demagnetizing field H at the limit also depends only on one variable x_2 and is the solution of

$$\begin{cases} \operatorname{div}(H+u) = 0 \\ \operatorname{curl} H = 0\,. \end{cases} \tag{2.10}$$

Writing $u = e^{i\varphi} = (\cos\varphi, \sin\varphi)$, since u depends only on x_2, $\operatorname{div} u = (\cos\varphi)\frac{d\varphi}{dx_2}$; hence u must solve

$$\begin{cases} \frac{dH_2}{dx_2} = -(\cos\varphi)\frac{d\varphi}{dx_2} \\ \frac{dH_1}{dx_2} = 0\,. \end{cases} \tag{2.11}$$

Thus, we see that

$$\begin{cases} H_1 = 0 \\ H_2 = \sin\alpha_+ - \sin\varphi \end{cases} \tag{2.12}$$

is the only solution if we require that H tend to 0 as $|x_2| \to \infty$.

We are thus led to the following one-dimensional problem: Minimize

$$\int_{\mathbb{R}} |\varphi'|^2 + |\sin\varphi - \sin\alpha_+|^2\,. \tag{2.13}$$

We will now consider problem (2.13) independently.

Study of the Ordinary Differential Equation

In the rest of this section, we consider angles α_+ and α_- such that

$$\begin{cases} \alpha_+ = -\pi - \alpha_- \\ \alpha_- \in \left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \quad \alpha_+ \in \left(-\frac{\pi}{2}, -\frac{\pi}{2}\right]. \end{cases} \tag{2.14}$$

We shall write $\sin\alpha_- = \sin\alpha_+ = \sin\alpha$ for simplicity. We consider the energy functional

$$F(\varphi) = \int_{\mathbb{R}} |\varphi'|^2 + |\sin\varphi - \sin\alpha|^2 . \tag{2.15}$$

To this functional is naturally associated the ordinary differential equation:

$$\varphi'' = (\sin\varphi - \sin\alpha)\cos\varphi , \tag{2.16}$$

and if we add to it the limit conditions we are interested in, we get the problem

$$\begin{cases} \varphi'' = (\sin\varphi - \sin\alpha)\cos\varphi \\ \varphi(-\infty) = \alpha_- , \quad \varphi(+\infty) = \alpha_+ . \end{cases} \tag{2.17}$$

We prove the following:

PROPOSITION 2.4 *Let α_+ and α_- satisfy $\alpha_- < \alpha_+$ and $\sin\alpha_+ = \sin\alpha_-$.*

(i) *If $\alpha_+ - \alpha_- > 2\pi$, then* (2.17) *has no solution of finite energy F.*
(ii) *If α_+ and α_- satisfy* (2.14), *then* (2.17) *admits one solution φ_α such that $F(\varphi_\alpha) < \infty$ and $\varphi_\alpha(0) = -\frac{\pi}{2}$. φ_α has the following properties*:

$$\varphi'_\alpha = \sin\alpha - \sin\varphi_\alpha . \tag{2.18}$$

There exists a constant C independent of α_+ and α_- such that

$$\begin{cases} |\varphi_\alpha - \alpha_+| \le \frac{C}{|x|} & \text{for } x \ge 0 \\ |\varphi_\alpha - \alpha_-| \le \frac{C}{|x|} & \text{for } x \le 0 . \end{cases} \tag{2.19}$$

If $\alpha_+ - \alpha_- < 2\pi$, then there exists a constant C_α depending on α_+ and α_- such that

$$\begin{cases} |\varphi_\alpha - \alpha_+| \le C_\alpha e^{-x} & \text{for } x \ge 0 \\ |\varphi_\alpha - \alpha_-| \le C_\alpha e^{x} & \text{for } x \le 0 . \end{cases} \tag{2.20}$$

Denoting by $X = \frac{\alpha_+ - \alpha_-}{2} = -\frac{\pi}{2} - \alpha_-$, we have $0 < X \le \pi$ and

$$F(\varphi_\alpha) = \int_{\mathbb{R}} (\varphi'_\alpha)^2 + (\sin\varphi_\alpha - \sin\alpha)^2 = 4(\sin X - X\cos X) . \tag{2.21}$$

PROOF OF PROPOSITION 2.4: The proof has six steps.

Step 1. Equation (2.16) has trivial constant solutions $\varphi \equiv \alpha_+ + 2k\pi$, $\varphi \equiv \alpha_- + 2k\pi$, $k \in \mathbb{Z}$. Consider any nonconstant finite-energy solution φ of (2.16). Necessarily,

$$|\varphi''| = |\sin\varphi - \sin\alpha||\cos\varphi| \le |\sin\varphi - \sin\alpha|\,;$$

hence $\varphi'' \in L^2(\mathbb{R})$. This implies that $\varphi' \in C^{0,1/2}(\mathbb{R})$ and φ' is uniformly continuous on $\mathbb{R}$. It also implies that $\varphi'\varphi'' \in L^1(\mathbb{R})$ and thus $(\varphi')^2(x) = \int^x 2\varphi'\varphi''$ remains bounded. We can thus assert that $(\varphi')^2$ is uniformly continuous on $\mathbb{R}$ because φ' is and it is bounded. Consequently, since $\int_{\mathbb{R}} |\varphi'|^2 < \infty$, we have $\varphi' \to 0$ at ∞.

Similarly, $(\sin\varphi - \sin\alpha)^2$ is uniformly continuous and integrable, and hence must tend to 0 at ∞.

Multiplying (2.16) on both sides by φ' and integrating between x and ∞, we are led to

$$(\varphi')^2 = (\sin\varphi - \sin\alpha)^2\,. \tag{2.22}$$

Step 2. We claim that φ' does not vanish. If there existed an x_0 such that $\varphi'(x_0) = 0$, then from (2.22), we would have $\sin\varphi(x_0) = \sin\alpha$, and thus $\varphi''(x_0) = 0$ from (2.16). Then, $\psi \equiv \varphi(x_0)$ would be a constant solution of (2.16), which would satisfy

$$\begin{cases} \psi'' = (\sin\psi - \sin\alpha)\cos\psi \\ \psi(x_0) = \varphi(x_0) \\ \psi'(x_0) = 0\,. \end{cases} \tag{2.23}$$

But φ is also a solution of (2.23), and using the Cauchy-Lipschitz theorem on ordinary differential equations, problem (2.23) has a unique solution. Thus φ would be the constant solution, and we are led to a contradiction, since it was assumed to be nonconstant. The claim is proved.

Step 3. We deduce assertion (i) of the proposition. If $\alpha_+ - \alpha_- > 2\pi$ and φ solves (2.17), then $\varphi(+\infty) - \varphi(-\infty) > 2\pi$; therefore, by continuity of φ, there exists a $x_0 \in \mathbb{R}$ such that $\sin\varphi(x_0) = \sin\alpha$. But φ satisfies (2.22); hence $\varphi'(x_0) = 0$, which is impossible from step 2.

Step 4. We prove the existence result. First, we recall that α_+ and α_- are assumed to satisfy (2.14). We set

$$\Phi(t) = \int_{-\pi/2}^{t} \frac{du}{\sin\alpha - \sin u}\,. \tag{2.24}$$

Φ is defined on (α_-, α_+). When $u \to \alpha_+$,

$$\sin u - \sin\alpha_+ \sim (u - \alpha_+)\cos\alpha_+ - \tfrac{1}{2}(u - \alpha_+)^2 \sin\alpha_+\,; \tag{2.25}$$

hence $\Phi \to +\infty$ as $t \to \alpha_+$. Similarly, $\Phi \to -\infty$ as $t \to \alpha_-$. Furthermore,

$$\Phi'(t) = \frac{1}{\sin\alpha - \sin t} > 0 \quad \text{on } (\alpha_-, \alpha_+)\,;$$

thus Φ is increasing and defines a diffeomorphism between (α_-, α_+) and $\mathbb{R}$. We can thus consider its inverse function $\varphi(x) = \Phi^{-1}(x)$. φ is increasing from $\mathbb{R}$ to (α_-, α_+) and satisfies

$$\lim_{x\to+\infty} \varphi = \alpha_+ , \qquad \lim_{x\to-\infty} \varphi = \alpha_- .$$

In addition,

$$\varphi'(x) = \frac{1}{\Phi'(\Phi^{-1}(x))} = \sin\alpha - \sin\varphi(x) . \tag{2.26}$$

Consequently, φ solves (2.17). In addition, from (2.14), φ takes the value $-\frac{\pi}{2}$. But, for any $c \in \mathbb{R}$, $\varphi(\cdot + c)$ is also a solution of (2.17); therefore we can find a solution φ_α of (2.17) such that $\varphi_\alpha(0) = -\frac{\pi}{2}$.

Rewording to "a solution" OK? Original was a bit vague.

Step 5. We prove the stated decay of φ_α. If $\cos\alpha_+ \neq 0$, then from (2.25), $\sin u - \sin\alpha_+ \sim_{\alpha_+} (u - \alpha_+)\cos\alpha_+$. Hence

$$\Phi \sim_{\alpha_+} \frac{-1}{\cos\alpha_+} \log(\alpha_+ - x) ,$$

and therefore $\varphi_\alpha - \alpha_+$ decays exponentially as $x \to +\infty$. The argument is similar near $-\infty$. If $\cos\alpha_+ = 0$, then from (2.25), $\sin u - \sin\alpha_+ \sim_{\alpha_+} -\frac{1}{2}(u - \alpha_+)^2 \sin\alpha_+$ with $\sin\alpha_+ = 1$. Hence

$$\Phi \sim_{\alpha_+} \frac{2}{\alpha_+ - x}$$

and $\varphi_\alpha - \alpha_+$ decays like $1/x$ as $x \to +\infty$, and similarly as $x \to -\infty$. It thus has a decay rate in $1/|x|$ independently from α_+ and α_-. We deduce that $(\varphi_\alpha')^2$ and $(\sin\varphi_\alpha - \sin\alpha)^2$ are in $L^1(\mathbb{R})$ and $F(\varphi_\alpha) < \infty$.

Step 6. There remains to calculate $F(\varphi_\alpha)$. From (2.26),

$$F(\varphi_\alpha) = \int_{\mathbb{R}} (\varphi_\alpha')^2 + (\sin\varphi_\alpha - \sin\alpha)^2 = 2\int_{\mathbb{R}} (\sin\alpha - \sin\varphi_\alpha)\varphi_\alpha' .$$

Performing the change of variables $\varphi_\alpha(x) = t$, we get

$$\begin{aligned} F(\varphi_\alpha) &= 2\int_{\alpha_-}^{\alpha_+} (\sin\alpha - \sin t)dt \\ &= 2\big[\cos\alpha_+ - \cos\alpha_- + (\alpha_+ - \alpha_-)\sin\alpha\big] = 4\cos\alpha_+ + 2(\alpha_+ - \alpha_-)\sin\alpha . \end{aligned}$$

Using the notation $X = (\alpha_+ - \alpha_-)/2 = \alpha_+ + \pi/2$, we have $0 < X \leq \pi$, $\cos\alpha_+ = \sin X$, $\sin\alpha = -\cos X$, and the previous equality becomes

$$F(\varphi_\alpha) = 4(\sin X - X\cos X) .$$

$\square$

3 Upper Bound for the Micromagnetism Energy

Here we prove Theorem 1.3 by constructing a suitable test configuration. This test function is obtained by deformation of the field $u_\star = \nabla^\perp \operatorname{dist}(\cdot, \partial\Omega)$, using the one-dimensional profiles described in Section 2.2.

3.1 Studying $u_\star$

We now mention some important results on $u_\star$ relying on [7] by H. I. Choi, S. W. Choi, and H. P. Moon. $u_\star$ is obviously divergence-free, tangent to $\partial\Omega$ (hence $\operatorname{div} \tilde{u_\star} = 0$), and of unit norm wherever $\operatorname{dist}(\cdot, \partial\Omega)$ is differentiable. If p is a point in Ω that admits a unique nearest-point projection on $\partial\Omega$, then $\operatorname{dist}(\cdot, \partial\Omega)$ is differentiable near p and $u_\star$ is regular near p. Thus, $u_\star$ is singular only at points that do not admit a unique nearest-point projection on $\partial\Omega$. In [7] they study this set, called the medial axis of Ω. They define it as

$$
\begin{aligned}
\Sigma = \{p \in \Omega : \exists r \geq 0, \overline{B_r(p)} \subset \overline{\Omega} \text{ and if } \overline{B_s(q)} \subset \overline{\Omega} \\
\text{and } \overline{B_r(p)} \subset \overline{B_s(q)}, \text{ then } \overline{B_r(p)} = \overline{B_s(q)}\}\,.
\end{aligned}
\tag{3.1}
$$

One can check that Σ coincides with the set of points that do not admit a nearest-point projection on $\partial\Omega$ and is also the set of the centers of maximal disks inscribed in Ω.

If Ω is a ball, then Σ is reduced to the center. If not, and if $\partial\Omega$ is a finite union of analytic curves (which we shall sum up by "analytic by parts"), then they prove that Σ is path-connected and is a finite union of real analytic curves of finite length, plus a finite number of endpoints or branch points called vertices. We will call $\mathcal{V}$ the set of vertices.

LEMMA 3.1 *If $\partial\Omega$ is analytic by parts, there exists $\varphi_\star \in \mathrm{BV}(\Omega)$ smooth except on a finite union of analytic curves S such that the jump of $\varphi_\star$ along S is in $[0, 2\pi]$ and $u_\star = e^{i\varphi_\star}$.*

PROOF OF LEMMA 3.1: If $\partial\Omega$ has " corners," then Σ is adherent to $\partial\Omega$. If $\partial\Omega$ is analytic, consider a point p on $\partial\Omega$ such that the geodesic curvature of $\partial\Omega$ at p is positive (such a point always exists) and the normal to $\partial\Omega$ at p. Following this normal direction inside Ω, one crosses Σ. Let us denote by T this segment joining p to Σ. $S = T \cup \Sigma$ is still a finite union of analytic curves and a finite number of vertices and is path-connected. Since the topological degree of $u_\star$ from $\partial\Omega$ to $\mathbb{S}^1$ is 1, it is possible to define a real-valued $\varphi_\star$ on $\partial\Omega \setminus \{p\}$ such that $u_\star = e^{i\varphi_\star}$ on $\partial\Omega$ and that $\varphi_\star$ is continuous except for a jump of 2π at p. It is then possible to extend $\varphi_\star$ to Ω into a BV function that is regular on $\Omega \setminus (T \cup \Sigma)$. We do it the following way: If $x \in \Omega \setminus (T \cup \Sigma)$, then x has a nearest-point projection y on $\partial\Omega$, and we set $\varphi_\star(x) = \varphi_\star(y)$. Then we have

$$
e^{i\varphi_\star} = u_\star = \nabla^\perp\big(\operatorname{dist}(\cdot, \partial\Omega)\big) \quad \text{in } \Omega \setminus S\,,
$$

and $\varphi_\star$ is C^∞ in $\Omega \setminus S$.

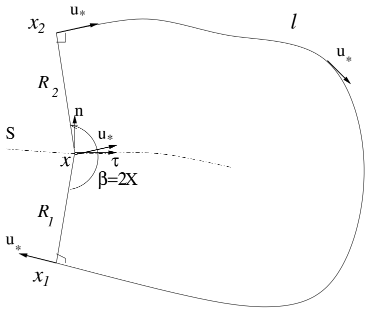

FIGURE 3.1

For example, in the case of the ball, just consider $\varphi_\star(r,\theta) = \theta + \frac{\pi}{2}$ in polar coordinates, where the argument θ lies in $[0, 2\pi)$. Then, $\varphi_\star$ is regular everywhere except on the segment $\theta = 0$ and for $r = 0$. We easily check that $u_\star \in \mathcal{C}$.

In the general case, the jump of $\varphi_\star$ on T is exactly 2π by construction. We then consider $x \in \Sigma$ and wish to evaluate the discontinuity of $\varphi_\star$ at x across Σ. By definition of Σ, there exist at least two rays coming from $\partial\Omega$ crossing at x. By "ray," we mean a segment with one endpoint on $\partial\Omega$ and the other on Σ, normal to $\partial\Omega$ and included in Ω. By construction, $\varphi_\star$ is constant on all rays. Let then R_1 and R_2 be two such rays crossing at x, and let x_1 and x_2 be their endpoints on $\partial\Omega$. We write $\beta = (\overrightarrow{xx_1}, \overrightarrow{xx_2}) \in (0, 2\pi)$. Then let l be the connected component of $\partial\Omega \setminus \{x_1, x_2\}$ that does not contain p. We thus have a closed curve, union of l, $[x, x_1]$ and $[x, x_2]$. We assume, for example, that going from x_1 to x_2 on l and then from x_2 to x_1 via the segments, we follow the trigonometric orientation; see Figure 3.1. We can write that the integral of the geodesic curvature K_g along this curve is equal to 2π:

$$\int_l K_g + \frac{\pi}{2} + (\pi - \beta) + \frac{\pi}{2} = 2\pi\ .$$

Here, we have used the fact that $[x, x_1]$ and $[x, x_2]$ are orthogonal to $\partial\Omega$. Then,

$$\int_l K_g = \pi - (\pi - \beta) = \beta \in (0, 2\pi)\,.$$

But this integral is equal to $-\int_l \frac{d\tau}{ds} \cdot n$ where τ is the unit tangent on l, n is the outer unit normal to l, and s is the parametrization with respect to the arc length on $\partial\Omega$. But $u_\star = -\tau$ on $\partial\Omega$; hence the previous integral is

$$\int_l \frac{\partial u_\star}{\partial s} \cdot n = \int_l \frac{\partial u_\star}{\partial s} \cdot u_\star^\perp = \int_l \frac{\partial \varphi_\star}{\partial s} = \varphi_\star(x_2) - \varphi_\star(x_1)\,.$$

We thus have obtained that

$$0 < \varphi_\star(x_2) - \varphi_\star(x_1) < 2\pi\ . \tag{3.2}$$

In addition, the jump of $\varphi_\star$ at x is exactly equal to $|\varphi_\star(x_2)-\varphi_\star(x_1)|$ by construction; hence we have the desired result. $\square$

Definitions of $X, \alpha_+,$ *and* α_- *for* $u_\star$. If $x \in \Sigma\backslash\mathcal{V}$, then x has exactly two nearest-point projections on $\partial\Omega$, x_1 and x_2 (see [7]) and the direction tangent to Σ separates x_1 and x_2. We can thus choose n to be the unit normal vector to Σ at x pointing towards x_2, i.e., $n\cdot\overrightarrow{xx_2}\geq 0$. If τ is the unit tangent to Σ such that (τ, n) is a direct orthonormal frame, then this means that

$$u_\star\cdot\tau\geq 0 \quad \text{on } (x, x_2]\,. \tag{3.3}$$

If $x\in T$ we orient $-\tau$ pointing towards $\partial\Omega$. By construction of T, $u_\star$ is normal to T and $u_\star = n = \tau^\perp$.

Then, at each point $x_0\in S\setminus\mathcal{V}$, we have defined a unit normal vector n, and we denote by $\varphi_\star^+$ and $\varphi_\star^-$ the values of $\varphi_\star$ for $x\to x_0(\overrightarrow{x_0x}\cdot n>0)$ and $x\to x_0(\overrightarrow{x_0x}\cdot n<0)$, respectively. In view of the previous proof, on Σ we have $\varphi_\star^+=\varphi_\star(x_2)$, $\varphi_\star^-=\varphi_\star(x_1)$. Hence,

$$0<\varphi_\star^+-\varphi_\star^-\leq 2\pi. \tag{3.4}$$

On T, $\varphi_\star^+-\varphi_\star^-=2\pi$; hence (3.4) is verified $\mathcal{H}^1$-a.e. on S. If s still denotes the parameter with respect to the arc length on S, we can then choose $\theta(s)$ such that $e^{-i\theta}\tau=(1,0)$ and

$$\theta-\tfrac{1}{2}\big(\varphi_\star^+ + \varphi_\star^- -\pi\big)\in[-\pi,\pi)\,. \tag{3.5}$$

We distinguish again two cases. If $x\in T$, then there exists $k\in\mathbb{Z}$ such that $\varphi_\star^+-\theta=\frac{\pi}{2}+2k\pi$ and $\varphi_\star^-=\varphi_\star^+-2\pi=\theta+\frac{\pi}{2}+2(k-1)\pi$. In view of (3.5), the only possibility is $k=0$. We set

$$\alpha_+(s)=\varphi_\star^+-\theta=\frac{\pi}{2}\,,\quad \alpha_-(s)=\varphi_\star^--\theta=-\frac{3\pi}{2}\,.$$

If $x\in\Sigma\setminus\mathcal{V}$, then, since $u_\star$ is divergence-free, its normal component is preserved across Σ; hence

$$\sin(\varphi_\star^+-\theta)=\sin(\varphi_\star^--\theta)\,,$$

therefore

$$\exists k\in\mathbb{Z}\quad \varphi_\star^+-\theta=\pi-(\varphi_\star^--\theta)+2k\pi\ .$$

But from (3.3) we also have

$$\cos(\varphi_\star^+-\theta)\geq 0\,,$$

and combining this with (3.4) and (3.5), the only possibility is $k=-1$ and

$$\varphi_\star^+-\theta=-\pi-(\varphi_\star^--\theta)\,,$$

$$\varphi_\star^+-\theta\in\big(-\tfrac{\pi}{2},\tfrac{\pi}{2}\big]\,,\quad \varphi_\star^--\theta\in\big(\tfrac{3\pi}{2},-\tfrac{\pi}{2}\big]\,.$$

We set in all cases $\alpha_+(s)=\varphi_\star^+-\theta$, $\alpha_-(s)=\varphi_\star^--\theta$. They satisfy (2.14). In addition, $\alpha_-(s)$, $\alpha_+(s)$, and $\theta(s)$ are C^1 on S except on a finite number of points. If some of these exceptional points are not in $\mathcal{V}$, we add them to $\mathcal{V}$. Denoting

$X = (\varphi_\star^+ - \varphi_\star^-)/2$, we have $X \in (0, \pi]$ and $X = (\alpha_+ - \alpha_-)/2$, $\cos X = -\sin\alpha$, $\sin X = -\cos\alpha_-$.

Remark 3.2. We recall that if $\partial\Omega$ is not assumed to be a finite union of analytic curves, but only C^∞, for example, then it seems more difficult to construct such a lifting of $u_\star$ because there exist such Ω's for which Σ either is not of finite length or has an infinite number of branch points; see [7] for counterexamples.

Remark 3.3. If Ω is convex, then $\varphi_\star$ takes its values in $[l, l+2\pi]$ for some $l \in [0, 2\pi]$.

3.2 Constructing the Test Functions

We use the notation of the previous subsection, and the orthonormal frame (τ, n) along the line of singularities S of $u_\star$ wherever possible. We choose a γ such that $\frac{1}{2} < \gamma < 1$. We recall that S is a C^1 curve except on a finite set of points $\mathcal{V}$ and denote by S_ε the subset of S composed of all the points that are at a distance $\geq \varepsilon^\gamma$ from $\mathcal{V}$. Thus S is locally C^1 near any point of S_ε. We then define the set B_ε to be

$$B_\varepsilon = \{x = x_0 + \sigma n,\ |\sigma| \leq \varepsilon^\gamma,\ x_0 \in S_\varepsilon,\ n \text{ the normal to } S_\varepsilon \text{ at } x_0\}\,.$$

For ε small enough, B_ε is thus an ε^γ-tubular neighborhood of S_ε, and each point of B_ε can be determined uniquely by coordinates (s, σ) where s corresponds to the parameter with respect to the arc length on S and σ to the distance to S_ε. We also denote $\Omega_\varepsilon = \Omega \setminus \{x : \text{dist}(x, S) \leq \varepsilon^\gamma\}$ and $V_\varepsilon = \Omega \setminus (\Omega_\varepsilon \cup B_\varepsilon)$.

We recall that the argument of $\varphi_\star$ in the local frame is $\alpha_+(s)$ as $\sigma \to 0^+$ and $\alpha_-(s)$ as $\sigma \to 0^-$, with α_+ and α_- satisfying (2.14). Let us also denote by $\beta_+(s)$ the angle of $\varphi_\star$ in the orthonormal frame (τ, n) at the point (s, ε^γ), and $\beta_-(s)$ the one at $(s, -\varepsilon^\gamma)$, $\beta_+(s) := \varphi_\star(s, \varepsilon^\gamma) - \theta(s)$, $\beta_-(s) := \varphi_\star(s, -\varepsilon^\gamma) - \theta(s)$. Of course, these β, unlike the α's, depend on ε. Along Σ, since $\alpha_+ - \alpha_- > 0$, for ε small enough we have $\beta_+ > -\frac{\pi}{2}$ and $\beta_- < -\frac{\pi}{2}$. Since T is a ray starting at a point of $\partial\Omega$ where the curvature is positive, it is not difficult to see that for ε small enough, we also have $\beta_+ < \frac{\pi}{2}$ and $\beta_- > -\frac{3\pi}{2}$, although $\alpha_+ = \frac{\pi}{2}$ and $\alpha_- = -\frac{3\pi}{2}$. Then, from the study of equation (2.16), we know the existence of $\tilde\alpha^+(s) \in [-\frac{\pi}{2}, \frac{\pi}{2})$ and $\tilde\alpha^-(s) \in [-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\varphi_{\tilde\alpha^+(s)}(\varepsilon^\gamma/\varepsilon) = \beta_+(s)$ and $\varphi_{\tilde\alpha^-(s)}(\varepsilon^\gamma/\varepsilon) = \beta_-(s)$ (a priori there is no reason why $\tilde\alpha^+(s)$ and $\tilde\alpha^-(s)$ should coincide). By using the C^1 behavior of $u_\star$ in $\Omega \setminus S$, it is clear that the following bound holds:

$$|\alpha_\pm - \beta_\pm| \leq C\varepsilon^\gamma\,, \tag{3.6}$$

and using (2.19),

$$|\tilde\alpha^\pm(s) - \alpha_+| \leq C\varepsilon^{1-\gamma}\,. \tag{3.7}$$

We then define the test function u_ε in the local coordinates (s, σ) as

$$
\begin{cases}
u_\varepsilon(\sigma,\sigma) = \big(\cos\big(\varphi_{\tilde{\alpha}^\pm(s)}\big(\tfrac{\sigma}{\varepsilon}\big)\big)\big)\tau + \big(\sin(\varphi_{\tilde{\alpha}^\pm(s)}\big(\tfrac{\sigma}{\varepsilon}\big)\big)\big)n \\
\qquad\qquad = e^{i(\psi+\theta)} \text{ for } 0 \le \pm\sigma \le \varepsilon^\gamma \text{ in } B_\varepsilon \\
u_\varepsilon = e^{i\varphi_\star} \quad \text{in } \Omega_\varepsilon\,.
\end{cases} \tag{3.8}
$$

In V_ε, we extend the phase of u_ε in such a way that it remains continuous and that $|\nabla u_\varepsilon| \le C/\varepsilon$. We also define in B_ε

$$
\begin{cases}
K_\varepsilon(s,\sigma) = \big(\cos\big(\varphi_{\tilde{\alpha}^\pm(s)}\big(\tfrac{\sigma}{\varepsilon}\big)\big)\big)\tau + \big(\sin\big(\beta_\pm\big(\tfrac{\sigma}{\varepsilon^\gamma}\big) + \alpha_\pm\big(1-\tfrac{\sigma}{\varepsilon^\gamma}\big)\big)\big)n \\
\qquad\qquad \text{for } 0 \le \pm\sigma \le \varepsilon^\gamma \quad \text{in } B_\varepsilon \\
K_\varepsilon = u_\star \quad \text{in } \Omega_\varepsilon\,.
\end{cases} \tag{3.9}
$$

Notice that K_ε is not unit-valued. We extend K_ε on V_ε in the same way as u_ε with $|K_\varepsilon| \le 1$. We thus end up with two vector fields u_ε and K_ε that are continuous on Ω and have the following properties:

LEMMA 3.4

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \le \frac{C}{\varepsilon}\,,$$

$$|\operatorname{div} K_\varepsilon| \le O(1) \text{ *in* } \Omega\setminus V_\varepsilon \le \frac{1}{\sup(\varepsilon, \operatorname{dist}(x,S))} \quad \textit{for } \operatorname{dist}(x,S) \le \varepsilon^\gamma\,.$$

PROOF OF LEMMA 3.4: The upper bound for ∇u is straightforward from the construction. In B_ε, we have, for $0 \le \pm\sigma \le \varepsilon^\gamma$,

$$
\begin{aligned}
\operatorname{div} K_\varepsilon = {} & \nabla\left(\cos\varphi_{\tilde{\alpha}^\pm(s)}\left(\frac{\sigma}{\varepsilon}\right)\right)\cdot\tau + \left(\cos\varphi_{\tilde{\alpha}^\pm(s)}\left(\frac{\sigma}{\varepsilon}\right)\right)\operatorname{div}\tau \\
& + \nabla\left(\sin\left(\beta_\pm\frac{\sigma}{\varepsilon^\gamma} + \alpha_\pm\left(1-\frac{\sigma}{\varepsilon^\gamma}\right)\right)\right)\cdot n \\
& + \sin\left(\beta_\pm\frac{\sigma}{\varepsilon^\gamma} + \alpha_\pm\left(1-\frac{\sigma}{\varepsilon^\gamma}\right)\right)\operatorname{div} n\,,
\end{aligned}
$$

and from this expression, it is clear using (3.6) that, as $\varepsilon \to 0$, $|\operatorname{div} K_\varepsilon| = O(1)$ in $\Omega \setminus V_\varepsilon$. $\square$

3.3 Evaluation of Their Energy: Proof of Theorem 1.3

The proof of Theorem 1.3 is divided into several lemmas.

LEMMA 3.5

$$\frac{1}{\varepsilon}\int_\Omega |u_\varepsilon - K_\varepsilon|^2 + \varepsilon\int_{B_\varepsilon} |\nabla u_\varepsilon|^2 \le \int_S 4\big(\sin X(s) - X(s)\cos X(s)\big)ds + o(1)\,.$$

PROOF OF LEMMA 3.5: First, observe that in Ω_ε, $u_\varepsilon = K_\varepsilon = u_\star$; hence

$$\frac{1}{\varepsilon}\int_\Omega |u_\varepsilon - K_\varepsilon|^2 = \frac{1}{\varepsilon}\int_{\Omega\setminus\Omega_\varepsilon} |u_\varepsilon - K_\varepsilon|^2\,,$$

and, since $|u_\varepsilon - K_\varepsilon| \le 2$ on V_ε and $\gamma > \frac{1}{2}$,

$$\frac{1}{\varepsilon}\int_\Omega |u_\varepsilon - K_\varepsilon|^2 \le \frac{1}{\varepsilon}\int_{B_\varepsilon} |u_\varepsilon - K_\varepsilon|^2 + C\frac{\varepsilon^{2\gamma}}{\varepsilon} \le \frac{1}{\varepsilon}\int_{B_\varepsilon} |u_\varepsilon - K_\varepsilon|^2 + o(1)\,. \tag{3.10}$$

From definitions (3.8) and (3.9) of u_ε and K_ε,

$$\begin{aligned}\frac{1}{\varepsilon}\int_{B_\varepsilon} |u_\varepsilon - K_\varepsilon|^2 \le \sum_\pm \frac{1}{\varepsilon}\int_{0\le\pm\sigma\le\varepsilon^\gamma} &\left|\sin\varphi_{\tilde\alpha^\pm(s)}\left(\frac{\sigma}{\varepsilon}\right) - \sin\tilde\alpha^\pm(s)\right|^2\\ + \sum_\pm \frac{1}{\varepsilon}\int_{0\le\pm\sigma\le\varepsilon^\gamma} &\left|\sin\tilde\alpha^\pm(s) - \sin\left(\beta_\pm\frac{\sigma}{\varepsilon^\gamma} + \alpha_\pm\left(1-\frac{\sigma}{\varepsilon^\gamma}\right)\right)\right|^2.\end{aligned} \tag{3.11}$$

Using (3.7) we have

$$\frac{1}{\varepsilon}\int_{0\le\pm\sigma\le\varepsilon^\gamma} \left|\sin\tilde\alpha^\pm(s) - \sin\left(\beta_\pm\frac{\sigma}{\varepsilon^\gamma} + \alpha_\pm\left(1-\frac{\sigma}{\varepsilon^\gamma}\right)\right)\right|^2 \le \frac{C}{\varepsilon}\,|B_\varepsilon|\,\left|\varepsilon^{1-\gamma}\right|^2 = O(\varepsilon^{1-\gamma})$$

and (3.11) becomes

$$\frac{1}{\varepsilon}\int_{B_\varepsilon} |u_\varepsilon - K_\varepsilon|^2 \le \sum_\pm \frac{1}{\varepsilon}\int_{0\le\pm\sigma\le\varepsilon^\gamma} \left|\sin\varphi_{\tilde\alpha^\pm(s)}\left(\frac{\sigma}{\varepsilon}\right) - \sin\tilde\alpha^\pm(s)\right|^2 + o(1)\,. \tag{3.12}$$

On the other hand, we have

$$\begin{aligned}&\int_{0\le\pm\sigma\le\varepsilon^\gamma} \varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\left|\sin\varphi_{\tilde\alpha^\pm(s)}\left(\frac{\sigma}{\varepsilon}\right) - \sin\tilde\alpha^\pm(s)\right|^2\\ &= \int_{0\le\pm\sigma\le\varepsilon^\gamma} \frac{1}{\varepsilon}\left|\nabla\varphi_{\tilde\alpha^\pm(s)}\left(\frac{\sigma}{\varepsilon}\right)\right|^2 + \frac{1}{\varepsilon}\left|\sin\varphi_{\tilde\alpha^\pm(s)}\left(\frac{\sigma}{\varepsilon}\right) - \sin\tilde\alpha^\pm(s)\right|^2 ds\,d\sigma\\ &+ \int_{0\le\pm\sigma\le\varepsilon^\gamma} \varepsilon\left|\frac{\partial}{\partial s}\left(\varphi_{\tilde\alpha^\pm(s)}\left(\frac{\sigma}{\varepsilon}\right)\right)\right|^2 ds\,d\sigma\,.\end{aligned} \tag{3.13}$$

Now,

$$\frac{\partial}{\partial s}\left(\varphi_{\tilde\alpha^\pm(s)}\left(\frac{\sigma}{\varepsilon}\right)\right) = \frac{d\tilde\alpha^\pm(s)}{ds}\frac{\partial\varphi_{\alpha(s)}}{\partial\alpha}\left(\frac{\sigma}{\varepsilon}\right)$$

remains bounded, since α is C^1 on $S\setminus\mathcal{V}$. Furthermore, by the definition of φ_α,

$$\nabla\varphi_{\alpha(s)}(x) = \sin\alpha(s) - \sin\varphi_{\alpha(s)}(x)\,. \tag{3.14}$$

Inserting this into (3.13), we obtain

$$\int_{0\le\pm\sigma\le\varepsilon^\gamma} \varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\left|\sin\varphi_{\tilde\alpha^\pm(s)}\left(\frac{\sigma}{\varepsilon}\right) - \sin\tilde\alpha^\pm(s)\right|^2 =$$
$$\frac{2}{\varepsilon}\int_{0\le\pm\sigma\le\varepsilon^\gamma}\left(\sin\tilde\alpha^\pm(s) - \sin\varphi_{\tilde\alpha^\pm(s)}\left(\frac{\sigma}{\varepsilon}\right)\right)\nabla\varphi_{\tilde\alpha^\pm(s)}\left(\frac{\sigma}{\varepsilon}\right) ds\, d\sigma + o(1)\,,$$

where we have used a co-area formula relative to π_S, the orthogonal projection from B_ε into S, and the fact that $||\pi_S^* ds| - 1| \le C_S\varepsilon^\gamma$ and that $\gamma > \frac{1}{2}$ (where C_S involves information on the curvature of S). Performing the change of variable $y = \sigma/\varepsilon$,

(3.15)
$$\begin{aligned}
&\int_{|\sigma|\le\varepsilon^\gamma} \varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\left|\sin\varphi_{\tilde\alpha^{\mathrm{sgn}(\sigma)}(s)}\left(\frac{\sigma}{\varepsilon}\right) - \sin\tilde\alpha^{\mathrm{sgn}(\sigma)}(s)\right|^2 \\
&= 2\int_s\int_{|y|\le\varepsilon^{\gamma-1}} \left(\sin\tilde\alpha^{\mathrm{sgn}(y)}(s) - \sin\varphi_{\tilde\alpha^{\mathrm{sgn}(y)}(s)}(y)\right)\nabla\varphi_{\tilde\alpha^{\mathrm{sgn}(\sigma)}(y)}(y)\, dy\, ds + o(1) \\
&\le 2\int\left(\int_{-\infty}^{+\infty}\left(\sin\tilde\alpha^{\mathrm{sgn}(y)}(s) - \sin\varphi_{\tilde\alpha^{\mathrm{sgn}(y)}(s)}(y)\right)\nabla\varphi_{\tilde\alpha^{\mathrm{sgn}(y)}(s)}(y)\, dy\right) ds + o(1) \\
&\le \int_s^s 4\left(\sin X(s) - X(s)\cos X(s)\right) ds + o(1)\,,
\end{aligned}$$

where we have used (2.21) and the fact that $\tilde\alpha^\pm + \frac{\pi}{2}$ tend to $X = \alpha_+ + \frac{\pi}{2}$ as ε tends to 0.

Combining (3.10), (3.12), and (3.15), we get the result. □

LEMMA 3.6 *Denoting by H_{u_ε} the magnetic field induced by $\tilde{u_\varepsilon}$, we have*

$$\frac{1}{\varepsilon}\int_{\mathbb{R}^2}|H_{u_\varepsilon}|^2 \le \frac{1}{\varepsilon}\int_\Omega |u_\varepsilon - K_\varepsilon|^2 + o(1)\,.$$

We postpone the proof of this lemma until later in order to state the last lemma.

LEMMA 3.7

$$\varepsilon\int_{\Omega\setminus B_\varepsilon}|\nabla u_\varepsilon|^2 = o(1) \quad \textit{as } \varepsilon \to 0\,.$$

PROOF OF LEMMA 3.7: First, by definition, $u_\varepsilon = u_\star$ in Ω_ε; hence

$$\varepsilon\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^2 \to 0\,.$$

Second, the contribution coming from V_ε is negligible because there $|\nabla u_\varepsilon| \le C\varepsilon^{-1}$, and hence

$$\varepsilon \int\limits_{V_\varepsilon} |\nabla u_\varepsilon|^2 \le C\varepsilon \frac{\varepsilon^{2\gamma}}{\varepsilon^2} = \varepsilon^{2\gamma-1} = o(1)\,.$$

This completes the proof of the lemma. □

If we assume the result of Lemma 3.6 to be true, then in view of Lemmas 3.5 and 3.7, we have constructed some $u_\varepsilon \in H^1(\Omega, S^1)$ for which

$$\varepsilon \int\limits_{\Omega} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\int\limits_{\mathbb{R}^2} |H_{u_\varepsilon}|^2 \le \int\limits_{S} 4\big(\sin X(s) - X(s)\cos X(s)\big)ds + o(1) \quad \text{as } \varepsilon \to 0\,. \tag{3.16}$$

If we assume that $\int_S 4(\sin X - X\cos X) = 2|\partial\Omega|$, which shall be proved in Lemma 5.4, then the first part of Theorem 1.3 is established.

PROOF OF LEMMA 3.6: The idea relies on the projection lemma, Lemma 2.3. For simplicity, we drop the subscripts ε. We denote by H_K the magnetic field induced by the configuration $\tilde{K}_\varepsilon$ (which is K_ε extended by 0 outside Ω). Let g achieve

$$\min_{g\in H^1(\mathbb{R}^2,\mathbb{R})} \int\limits_{\mathbb{R}^2} \big|\tilde{K} - \nabla^\perp g\big|^2\,;$$

then, as seen in Section 2, $H_K = \tilde{K} - \nabla^\perp g$. Furthermore, the projection lemma ensures that

$$\begin{aligned}
\Big(\int\limits_{\mathbb{R}^2} |H_u|^2\Big)^{1/2} &\le \Big(\int\limits_{\mathbb{R}^2} |\tilde{u} - \nabla^\perp g|^2\Big)^{1/2} \\
&\le \Big(\int\limits_{\mathbb{R}^2} |\tilde{u} - \tilde{K}|^2\Big)^{1/2} + \Big(\int\limits_{\mathbb{R}^2} |\tilde{K} - \nabla^\perp g|^2\Big)^{1/2} \\
&\le \Big(\int\limits_{\Omega} |u - K|^2\Big)^{1/2} + \Big(\int\limits_{\mathbb{R}^2} |H_K|^2\Big)^{1/2}\,. \qquad (3.17)
\end{aligned}$$

Consequently, the lemma shall be proved if we show that

$$\frac{1}{\varepsilon}\int\limits_{\mathbb{R}^2} |H_K|^2 \to 0\,. \tag{3.18}$$

We now prove (3.18). From Lemma 3.4, $\|\operatorname{div} K\|_{L^\infty} \leq C\varepsilon^{-1}$; on the other hand, $\operatorname{div} K = 0$ in Ω_ε; hence K satisfies the hypotheses of Lemma 2.1. We decompose $\Omega \setminus \Omega_\varepsilon$ as $B_\varepsilon \cup V_\varepsilon$. As seen in Lemma 2.1,

$$\begin{aligned}
&\left|(\Delta^{-1} \operatorname{div} K)(x)\right| \\
&\quad = \left| \int_{\Omega\setminus\Omega_\varepsilon} 2\pi \log|x-y| \operatorname{div} K(y) dy \right| \\
&\quad \leq \int_{B_\varepsilon} \left|2\pi \log|x-y| \operatorname{div} K(y) dy\right| + \int_{V_\varepsilon} \left|2\pi \log|x-y| \operatorname{div} K(y) dy\right| .
\end{aligned}$$

If we assume for simplicity that B_ε is a rectangle in Cartesian coordinates (x_1, x_2) of the form $\{(x_1, x_2) : |x_1| \leq L,\ |x_2| \leq \varepsilon^\gamma\}$; then we deduce, using the L^∞ bounds on $\operatorname{div} K$ of Lemma 3.4,

$$\begin{aligned}
\int_{B_\varepsilon} \left|2\pi \log|x-y| \operatorname{div} K(y) dy\right| &\leq C \left| \int_{y_2=-L}^{L} \int_{y_1=0}^{\varepsilon^\gamma} \log|x-y| dy \right| \\
&\leq C\varepsilon^\gamma |\log\varepsilon| . \qquad (3.19)
\end{aligned}$$

The contribution of V_ε can be bounded as follows:

$$\begin{aligned}
\int_{V_\varepsilon} \left|2\pi \log|x-y| \operatorname{div} K(y) dy\right| &\leq C \int_{y_2=-\varepsilon^\gamma}^{\varepsilon^\gamma} \int_{y_1=0}^{\varepsilon^\gamma} \frac{|\log|x-y||}{\sup(\varepsilon, y_1)} \, dy_1\, dy_2 \\
&\leq \varepsilon^\gamma |\log\varepsilon|^2 . \qquad (3.20)
\end{aligned}$$

Combining (3.19) through (3.20), we get the following L^∞ bound:

$$|(\Delta^{-1} \operatorname{div} K)(x)| \leq C\varepsilon^\gamma |\log\varepsilon|^2 \quad \text{in } \Omega \setminus \Omega_\varepsilon ; \qquad (3.21)$$

the general case follows easily from a change of coordinates. On the other hand, as seen in Lemma 2.1,

$$\int_{\mathbb{R}^2} |H_K|^2 = \int_{\mathbb{R}^2} (\operatorname{div} K)(\Delta^{-1} \operatorname{div} K) = \int_{\Omega\setminus\Omega_\varepsilon} (\operatorname{div} K)(\Delta^{-1} \operatorname{div} K) ,$$

a result of the fact that $\operatorname{div} K \equiv 0$ in Ω_ε. But, thanks to (3.21),

$$\begin{aligned}
\int_{V_\varepsilon} (\operatorname{div} K)(\Delta^{-1} \operatorname{div} K) &\leq C\varepsilon^{2\gamma} \|\operatorname{div} K\|_{L^\infty} \|\Delta^{-1} \operatorname{div} K\|_{L^\infty(\Omega\setminus\Omega_\varepsilon)} \\
&\leq C\varepsilon^{2\gamma}\varepsilon^{-1}\varepsilon^\gamma |\log\varepsilon|^2 \leq C\varepsilon^{3\gamma-1}|\log\varepsilon|^2 = o(\varepsilon) ,
\end{aligned}$$

since $\gamma > \frac{1}{2}$. Similarly, from (3.21) again,

$$\int_{B_\varepsilon} (\operatorname{div} K)(\Delta^{-1} \operatorname{div} K) \leq C\varepsilon^\gamma \varepsilon^\gamma |\log\varepsilon|^2 .$$

Since $\gamma > \frac{1}{2}$, we conclude that

$$\int\limits_{\Omega\setminus\Omega_\varepsilon} (\operatorname{div} K)(\Delta^{-1}\operatorname{div} K) = o(\varepsilon)\,;$$

hence

$$\frac{1}{\varepsilon}\int\limits_{\mathbb{R}^2} |H_K|^2 = o(1)\,.$$

□

In order to complete the proof of Theorem 1.3, there remains to prove the following:

LEMMA 3.8 *There exists an admissible covering $\mathcal{U}$ such that the u_ε constructed in* (3.8) *belong to $\Lambda_{\mathcal{U}}$ for ε small enough.*

PROOF OF LEMMA 3.8: Let us return to the $\varphi_\star$ defined in Lemma 3.1 so that $u_\star = e^{i\varphi_\star}$. We recall that, on Σ, $\varphi_\star^+ - \varphi_\star^- \in (0, 2\pi)$; hence if $x_0 \in \Sigma$, it has an open neighborhood U_{x_0} such that

$$\sup_{U_{x_0}} \varphi_\star - \inf_{U_{x_0}} \varphi_\star \le 2\pi\,.$$

We also choose T to be such that the geodesic curvature of $\partial\Omega$ at $T\cap\partial\Omega$ is positive. Thus, we can make sure that there exists an open neighborhood U_T of T in which $\sup_{U_T}\varphi_\star - \inf_{U_T}\varphi_\star \le 2\pi$. It is also obvious that, since $\varphi_\star$ is continuous in $\Omega\setminus S$, $\Omega\setminus S$ can be covered by open sets U on which $\sup_U \varphi_\star - \inf_U \varphi_\star \le 2\pi$. We can then extract a finite covering $\bigcup_{i\in I} U_i$ of Ω such that $\forall i \in I$, $\sup_{U_i}\varphi_\star - \inf_{U_i}\varphi_\star \le 2\pi$, and such that $\forall i, j \in I$, $U_i \cap U_j$ is diffeomorphic to $\mathbb{R}^2$. This condition guarantees that $u_\star$ has local liftings $\psi_i \in L^\infty(U_i)$ satisfying

$$u_\star = e^{i\psi_i} \text{ in } U_i\,, \quad \psi_i \in [l_i, l_i + 2\pi] \text{ with } l_i \in [0, 2\pi]\,.$$

Of course, $\psi_i - \psi_j$ is constant on $U_i \cap U_j$ connected and equal to $2\pi k_{ij}$ for some $k_{ij} \in \mathbb{Z}$. The k_{ij} satisfy the cocycle relations of Definition 1.1.

On the other hand, u_ε is defined by (3.8); hence it has a lifting

$$\begin{cases} \varphi_\varepsilon = \psi + \theta & \text{in } B_\varepsilon \\ \varphi_\varepsilon = \varphi_\star & \text{in } \Omega_\varepsilon\,. \end{cases} \tag{3.22}$$

By monotonicity of φ_α (see (2.21)), we have $\beta_- \le \varphi_{\tilde{\alpha}^{\operatorname{sgn}(\sigma)}}(\sigma/\varepsilon) \le \beta_+$. By construction of ψ (see (3.8)), we thus have

$$\sup_{B_\varepsilon} \psi - \inf_{B_\varepsilon} \psi \le \sup_s \big(\beta_+(s) - \beta_-(s)\big)\,.$$

Therefore, on each U_i such that $U_i \cap S \ne \varnothing$, for ε small enough,

$$\sup_{U_i} \varphi_\varepsilon - \inf_{U_i} \varphi_\varepsilon \le \sup_{U_i} \varphi_\star - \inf_{U_i} \varphi_\star \le 2\pi\,. \tag{3.23}$$

(The phase of u_ε can be extended to V_ε in such a way that this property also holds.) If U_i does not intersect S, then $u_\varepsilon = u_\star$ on U_i (for ε small enough); hence (3.23) also holds. This property suffices to ensure that $u_\varepsilon \in \Lambda_{\mathcal{U}}$ for ε small enough. $\square$

Remark 3.9. If Ω is convex, then the test function u_ε also satisfies the same property as $\varphi_\star$, $\varphi_\star \in [l, l+2\pi]$, and Ω is a "suitable covering." In the general case, the suitable covering depends only on $u_\star$, and hence on Ω.

4 The Compactness Result: Proof of Theorem 1.5

The proof of compactness relies on a similar method to that developed in [1]. We identify a Jacobian structure in order to use the compensated-compactness lemma and combine it with the use of Young measures. Let us consider a sequence of functions $u_n \in \Lambda_{\mathcal{U}}$ for some $\mathcal{U}$ such that $E_{\varepsilon_n}(u_n)$ remains bounded where ε_n is a sequence converging to 0. Let φ_n be the $H^1(\Omega) \cap L^\infty$ lifting of u_n given by Lemma 1.2. From (1.12) and (1.16) we have

$$\|H_n\|_{L^2(\mathbb{R}^2)} \to 0\,, \tag{4.1}$$

$$C \geq E_{\varepsilon_n}(u_n) \geq 2\int_\Omega |\nabla\varphi_n \cdot H_n| \geq 2\int_\Omega \left|\operatorname{div}\left(\varphi_n u_n + u_n^\perp + \varphi_n H_n\right)\right|\,. \tag{4.2}$$

The assumption of boundedness on $E_{\varepsilon_n}(u_n)$ can be replaced as well by the two conditions $H_n \to 0$ in some L^p and $\int_\Omega |\nabla\varphi_n \cdot H_n|$ bounded without any change in the proof and with the same conclusions.

The quantity to study is

$$v_n = (H_n + u_n)\varphi_n + u_n^\perp\,, \tag{4.3}$$

which is equal to the sum of a local term $u_n\varphi_n + u_n^\perp$ and a nonlocal term $H_n\varphi_n$, which tends to 0 in L^q, $q < \infty$. The hypothesis on v_n is

$$\int_\Omega |\operatorname{div} v_n| \leq C\,. \tag{4.4}$$

Since $|u_n| = 1$, u_n is uniformly bounded in $L^\infty(\Omega)$; hence, extracting a subsequence if necessary, we can assume that it converges weakly in $L^\infty(\Omega)$ to some u. We are going to prove that $|u| = 1$ a.e. in Ω. This constitutes the following compactness result:

Claim 4.1. If $|u| = 1$ a.e. in Ω, then $u_n \to u$ strongly in $L^q(\Omega)$ ($\forall 1 \leq q < \infty$).

PROOF OF CLAIM 4.1: If $|u| = 1$ a.e., then

$$\int_\Omega |u_n| = |\Omega| = \int_\Omega |u|\,.$$

Thus $\|u_n\|_{L^2(\Omega)} \to \|u\|_{L^2(\Omega)}$ while $u_n \rightharpoonup u$ in $L^2(\Omega)$. It is standard that this implies strong convergence of u_n to u in L^2 and in L^q ($\forall q < \infty$) from the bound $|u_n| \leq 1$. $\square$

In view of this claim, it suffices to prove $|u| = 1$ a.e. in Ω.

LEMMA 4.2 *For almost every $x_0 \in \Omega$, there exist sequences u'_k, φ'_k, and H'_k in $B_1(0) = B_1$ such that*

$$|u'_k| = 1 \text{ a.e. in } B_1\,, \tag{4.5}$$

$$u'_k \rightharpoonup u(x_0) \text{ in } L^\infty(B_1)\,, \tag{4.6}$$

$$u'_k = e^{i\varphi'_k} \text{ with } \varphi'_k \in [l_k, l_k + 2\pi] \quad \text{and} \quad \lim_{k\to\infty} l_k = l\,, \tag{4.7}$$

$$\operatorname{div}(H'_k + u'_k) = 0 \quad \text{and} \quad \operatorname{curl} H'_k = 0 \text{ in } B_1\,, \tag{4.8}$$

$$\|H'_k\|_{L^p(B_1)} \to 0 \quad \forall 1 \le p < \infty\,, \tag{4.9}$$

$$\int_{B_1} \left|\operatorname{div}\left(u'_k\varphi'_k + (u'_k)^\perp + H'_k\varphi'_k\right)\right| \to 0\,. \tag{4.10}$$

PROOF OF LEMMA 4.2: This lemma relies on a blowup argument. Let v_n be as in (4.3). From Lemma 2.1, (H_n) is uniformly bounded in $\bigcap_p L^p$. Since $u_n \in \Lambda_{\mathcal{U}}$, φ_n is uniformly bounded in $L^\infty(\Omega)$, and since x_0 is in some U_j, $j \in I$, we can assume that φ_n takes its values in $[l_n, l_n + 2\pi]$ in a neighborhood of x_0. Therefore, v_n is bounded in L^2, for example, and we can assume that $v_n \rightharpoonup v$ in $L^2(\Omega)$. From (4.4), $\operatorname{div} v_n$ is bounded in $\mathcal{M}$ (the space of Radon measures); hence, modulo a subsequence, it converges weakly to some measure in $\mathcal{M}$, and this measure is necessarily $\operatorname{div} v$ by uniqueness of the distribution limit. Thus $\operatorname{div} v \in \mathcal{M}$ and

$$\int_\Omega |\operatorname{div} v| \le \liminf_{n\to\infty} \int_\Omega |\operatorname{div} v_n| \le C\,. \tag{4.11}$$

For almost every $x_0 \in \Omega$,

$$\limsup_{r\to 0} \frac{1}{r^2} \int_{B_r(x_0)} |\operatorname{div} v| < \infty\,. \tag{4.12}$$

If this was false, we could contradict (4.11) by a covering argument. From now on, we consider that x_0 satisfies this condition and also x_0 is chosen to be a Lebesgue point for the limit u.

We define the dilated functions, for any $r > 0$,

$$\begin{cases} u_n^r(x) = u_n(x_0 + rx) \\ v_n^r(x) = v_n(x_0 + rx) \\ \varphi_n^r(x) = \varphi_n(x_0 + rx) \\ H_n^r(x) = H_n(x_0 + rx)\,. \end{cases} \tag{4.13}$$

Observe that

$$\begin{cases} \operatorname{div} v_n^r = r \operatorname{div} v_n(x_0 + rx) \\ \operatorname{div} u_n^r = r \operatorname{div} u_n(x_0 + rx) = -r \operatorname{div} H_n(x_0 + rx) = -\operatorname{div} H_n^r\,, \end{cases} \tag{4.14}$$

and

$$\int_{B_1} |\text{div } v_n^r| = \frac{1}{r} \int_{B_r(x_0)} |\text{div } v_n| \,. \tag{4.15}$$

Since $\text{div } v_n \to \text{div } v$ in $\mathcal{M}$, we have for a fixed r

$$\lim_{n\to\infty} \int_{B_1} |\text{div } v_n^r| = \frac{1}{r} \lim_{n\to\infty} \int_{B_r(x_0)} |\text{div } v_n| \leq \frac{1}{r} \int_{B_{2r}(x_0)} |\text{div } v| \,. \tag{4.16}$$

Hence, in view of (4.12), there exists $C > 0$ such that

$$\forall r > 0\,, \quad \lim_{n\to\infty} \int_{B_1} |\text{div } v_n^r| \leq Cr \,. \tag{4.17}$$

By assumption, $H_n \to 0$ in $L^2(\Omega)$, and it is bounded in all L^p and hence converges to 0 in $\bigcap_{p<\infty} L^p$.

Let $p > 2$; for fixed $r < r_0$ and for r_0 small enough

$$\int_{B_1} |H_n(x_0 + rx)|^p = \frac{1}{r^2} \int_{B_r(x_0)} |H_n|^p \leq \frac{1}{r^2} \int_{B_{r_0}(x_0)} |H_n|^p \to 0 \quad \text{as } n \to \infty \,. \tag{4.18}$$

Let us now take any sequence $r_k \to 0$. In view of (4.17) and (4.18), we may choose a sequence $n_k \to \infty$ such that

$$\lim_{k\to\infty} \int_{B_1} |\text{div } v_{n_k}^{r_k}| = 0 \tag{4.19}$$

and

$$\lim_{k\to\infty} \int_{B_1} |H_{n_k}^{r_k}|^p = \lim_{k\to\infty} \int_{B_1} |H_{n_k}(x_0 + r_k x)|^p = 0 \,. \tag{4.20}$$

We could also have chosen n_k such that $u_{n_k}^{r_k} \rightharpoonup u(x_0)$. Indeed, let e_j be a Hilbert basis for the L^2 scalar product on B_1 such that $e_j \in L^\infty$. We have

(4.21)

$$\begin{aligned}
&\int_{B_1} e_j(x)\big[u_n^{r_k}(x) - u(x_0)\big] \\
&\quad = \frac{1}{{r_k}^2} \int_{B_{r_k}(x_0)} e_j\left(\frac{x}{r_k}\right)\big[u_n(x) - u(x_0)\big] \\
&\quad = \frac{1}{r_k^2} \int_{B_{r_k}(x_0)} e_j\left(\frac{x}{r_k}\right)\big[u_n(x) - u(x)\big] + \frac{1}{r_k^2} \int_{B_{r_k}(x_0)} e_j\left(\frac{x}{r_k}\right)\big[u(x) - u(x_0)\big] \,.
\end{aligned}$$

Since x_0 is a Lebesgue point of u,

$$\frac{1}{r_k^2}\int\limits_{B_{r_k}(x_0)} e_j\left(\frac{x}{r_k}\right)[u(x)-u(x_0)] \to 0 \quad \text{as } k\to\infty\,.$$

Then, using the weak convergence of u_n to u, we deduce that for a fixed $r>0$,

$$\lim_{n\to+\infty}\int\limits_{B_1} e_j(x)\big[u_n^r(x)-u(x_0)\big] = \frac{1}{r^2}\int\limits_{B_r(x_0)} e_j\left(\frac{x}{r}\right)\big[u(x)-u(x_0)\big]$$

so that we may find a subsequence n_k^1 such that

$$\int\limits_{B_1} e_1(x)\big[u_{n_k^1}^{r_k}(x)-u(x_0)\big]\to 0\,.$$

From n_k^1, we extract a subsequence n_k^2 such that the previous affirmation holds for e_2 and e_1, and so on. Then, following the standard diagonal argument, for $n_k=n_k^k$, we have

$$\forall j<+\infty \qquad \int\limits_{B_1} e_j(x)\big[u_{n_k}^{r_k}(x)-u(x_0)\big]\longrightarrow 0\,.$$

Hence, $u_{n_k}^{r_k}$ converges weakly in $L^2(B_1)$ to $u(x_0)$ as $k\to\infty$.

Then we define

$$\begin{cases} u_k' = u_{n_k}^{r_k}\\ \varphi_k' = \varphi_{n_k}^{r_k}\\ H_k' = H_{n_k}^{r_k}\,, \end{cases} \tag{4.22}$$

and we have $u_k' = e^{i\varphi_k'}$. In addition, $\varphi_k'(x)\in[l_{n_k}, l_{n_k}+2\pi]$ for k large and for $x\in B_1$. Extracting again if necessary, we can assume that l_{n_k} converges to some $l\in\mathbb{R}$ as $k\to\infty$. Assertions (4.5), (4.6), (4.7), (4.8), and (4.9) of the lemma are verified. Assertion (4.10) follows from the definition of H_k' and from (4.19). Indeed,

$$\begin{aligned}(\operatorname{div} v_{n_k}^{r_k})(x) = r_k(\operatorname{div} v_{n_k})(x_0+r_kx) &= r_k \operatorname{div}\big(\varphi_{n_k}u_{n_k}+u_{n_k}^\perp+\varphi_{n_k}H_{n_k}\big)(x_0+r_kx)\\ &= \operatorname{div}\big(\varphi_k'u_k'+(u_k')^\perp+\varphi_k'H_k'\big)(x)\,,\end{aligned}$$

and in view of (4.19),

$$\int\limits_{B_1}\big|\operatorname{div}(\varphi_k'u_k'+(u_k')^\perp+\varphi_k'H_k')\big|\to 0\,,$$

which is (4.10). $\square$

Under the same hypotheses, we may now find l_0 such that

$$l_0\in[l,l+2\pi] \tag{4.23}$$

where l satisfies (4.7) and

$$e^{-il_0}u(x_0) = (\alpha, 0)\,, \quad \alpha = |u(x_0)|\,. \tag{4.24}$$

LEMMA 4.3 *Let (u'_k, φ'_k, H'_k) be given by Lemma 4.2, and define*

$$\begin{cases} A_k = e^{-il_0}(u'_k)^{\perp} \\ B_k = e^{-il_0}\varphi'_k u'_k + e^{-il_0}(u'_k)^{\perp} \\ C_k = e^{-il_0}\varphi'_k u'_k\,; \end{cases} \tag{4.25}$$

then, modulo a subsequence, we have

$$\begin{cases} A_k \rightharpoonup (0, \alpha) = A \\ C_k \rightharpoonup (\beta, \gamma) \\ B_k \rightharpoonup (\beta, \gamma + \alpha) = B \end{cases} \qquad \textit{weakly in } L^\infty(B_1)\,. \tag{4.26}$$

Moreover,

$$1 = A_k \cdot B_k \rightharpoonup A \cdot B = \alpha(\gamma + \alpha) \quad \textit{in } \mathcal{D}'(B_1)\,. \tag{4.27}$$

PROOF OF LEMMA 4.3: From (4.6) and (4.24), $A_k \rightharpoonup (0, \alpha) = A$ weakly in $L^\infty(B_1)$. $\varphi_n u_n$ is bounded in $L^\infty(\Omega)$ and so converges weakly to some function f up to extraction. Since $C_k = e^{-il_0}\varphi'_k u'_k$, if x_0 is chosen to be a Lebesgue point of f, we can assume exactly as for u that C_k converges weakly in $L^\infty(B_1)$ to $e^{-il_0}f(x_0) = (\beta, \gamma)$. We immediately deduce that $B_k \rightharpoonup (\beta, \gamma + \alpha)$ weakly in $L^\infty(B_1)$.

We prove (4.27). It is a direct consequence of the compensated-compactness lemma of Murat and Tartar (see [14, 16]), which says that if $A_k \rightharpoonup A$ in L^2, $B_k \rightharpoonup B$ in L^2, and curl A_k and div B_k are compact in H^{-1}, then $A_k \cdot B_k \rightharpoonup A \cdot B$. We just need to check that these hypotheses hold for our A_k and B_k. Their weak convergence in $L^2(B_1)$ has already been established. Furthermore,

$$\operatorname{curl} A_k = -e^{-il_0} \operatorname{div} u'_k = e^{-il_0} \operatorname{div} H'_k \to 0 \quad \text{in } H^{-1}(B_1)$$

from the fact that $\|H'_k\|_{L^2(B_1)} \to 0$.

Please integrate this displayed equation into the rest of the text. Right now it is a complete sentence.

$$\begin{aligned} \operatorname{div} B_k &= e^{-il_0} \operatorname{div}\left(\varphi'_k u'_k + (u'_k)^{\perp}\right) \\ &= e^{-il_0} \operatorname{div}\left(\varphi'_k u'_k + (u'_k)^{\perp} + \varphi'_k H'_k\right) - e^{-il_0} \operatorname{div}(\varphi'_k H'_k)\,. \end{aligned}$$

In view of (4.10), $\operatorname{div}(\varphi'_k u'_k + (u'_k)^{\perp} + \varphi'_k H'_k) \to 0$ in $L^1(B_1)$. On the other hand, this quantity is bounded in $W^{-1,p}$ for all $p < \infty$; hence by a result of Murat (see [15]), it is compact in H^{-1}. Meanwhile, $\operatorname{div}(\varphi'_k H'_k) \to 0$ in $H^{-1}(B_1)$ as previously. Hence, div $B_k \to 0$ in $H^{-1}(B_1)$, and we conclude that A_k and B_k satisfy the desired properties. We have thus established that $A_k \cdot B_k \rightharpoonup A \cdot B$ in $\mathcal{D}'(B_1)$. $\square$

LEMMA 4.4 *$\alpha = 1$ and the Young measure generated by φ'_k at 0 is supported in $\{l_0, l_0 + 2\pi\}$ or $\{l_0 - 2\pi, l_0\}$.*

PROOF OF LEMMA 4.4: We are now in a position to prove that $\gamma\alpha + \alpha^2 = 1$ implies that $\alpha = 1$. Following the approach of [1], since φ_k' is uniformly bounded in $L^\infty(B_1)$, we can introduce ν, the Young measure it generates at 0. Recall that from (4.7), φ_k' takes values in $[l_k, l_k + 2\pi]$ with $l_k \to l$. Therefore, ν is supported in $[l, l+2\pi]$. Thus, in view of (4.23), ν is supported in $[l_0 - 2\pi, l_0 + 2\pi]$. By the definition of Young measures, we have the following relations (in view of (4.25)–(4.26)):

$$\int_{l_0-2\pi}^{l_0+2\pi} d\nu = 1\,, \tag{4.28}$$

$$\alpha = \int_{l_0-2\pi}^{l_0+2\pi} \cos(y - l_0)d\nu(y) = \int_{-2\pi}^{2\pi} (\cos t)d\nu(t + l_0)\,, \tag{4.29}$$

$$0 = \int_{l_0-2\pi}^{l_0+2\pi} \sin(y - l_0)d\nu(y) = \int_{-2\pi}^{2\pi} (\sin t)d\nu(t + l_0)\,, \tag{4.30}$$

$$\gamma = \int_{l_0-2\pi}^{l_0+2\pi} y\sin(y - l_0)d\nu(y) = \int_{-2\pi}^{2\pi} (t\sin t)d\nu(t + l_0)\,. \tag{4.31}$$

We claim that

$$\forall t \in [-2\pi, 2\pi] \quad t\sin t + 2\cos t \le 2\,. \tag{4.32}$$

This can be checked by looking at the variations of $t \mapsto t\sin t + 2\cos t$ on $[-2\pi, 2\pi]$. Inserting (4.32) into (4.31), we are led to

$$\begin{aligned}\gamma = \int_{-2\pi}^{2\pi} (t\sin t)d\nu(t + l_0) &\le 2\int_{l_0-2\pi}^{l_0+2\pi} d\nu - 2\int_{-2\pi}^{2\pi} (\cos t)d\nu(t + l_0)\\ &\le 2 - 2\alpha\,, \end{aligned}\tag{4.33}$$

where we have used the fact that ν is a probability measure and hence positive. It then suffices to insert this relation into $\gamma\alpha + \alpha^2 = 1$ to get $1 \le 2\alpha - 2\alpha^2 + \alpha^2$ equivalent to $(1 - \alpha)^2 \le 0$. Therefore, $\alpha = |u(x_0)| = 1$. In view of (4.29), ν is necessarily supported in $\{l_0, l_0 - 2\pi, l_0 + 2\pi\}$. Since it is also supported in $[l, l + 2\pi]$, its support is either $\{l_0, l_0 + 2\pi\}$ or $\{l_0, l_0 - 2\pi\}$. From Lemma 4.1, this is true for almost every $x_0 \in \Omega$; thus $|u| = 1$ a.e. and $u_n \to u$ strongly in $\bigcap_{1\le q<\infty} L^q$. □

Let us consider a general $u \in C^1(B_1, \mathbb{S}^1)$ and $\varphi \in C^1(B_1, \mathbb{R})$ such that $u = e^{i\varphi}$. Then we define for any $l \in \mathbb{R}$ and $\eta > 0$ the truncated function

$$\begin{cases} T_\eta\varphi = \varphi & \text{if } \varphi \in [l + \eta, l + 2\pi - \eta]\\ T_\eta\varphi = l + \eta & \text{if } \varphi \le l + \eta\\ T_\eta\varphi = l + 2\pi - \eta & \text{if } \varphi \ge l + 2\pi - \eta\,. \end{cases}\tag{4.34}$$

We will write

$$T_\eta u = \exp(iT_\eta\varphi)\,. \tag{4.35}$$

LEMMA 4.5 *Let* $u_k \in H^1(B_1, \mathbb{S}^1)$ *and* $\varphi_k \in H^1(B_1, \mathbb{R})$ *be such that* $u_k = e^{i\varphi_k}$ *and that* (4.8) *through* (4.10) *hold. Then, for any choice of* $l \in \mathbb{R}$ *and for any* $\eta_0 > 0$*, there exists a sequence* $\eta_k \in [\frac{1}{2}\eta_0, \eta_0]$ *such that (for a subsequence)* $T_{\eta_k}\varphi_k$ *converges strongly in* $\bigcap_{1\le q<\infty} L^q$ *and*

$$T_{\eta_k} u_k \to v \ \ in \bigcap_{1\le q<\infty} L^q(B_1) \quad with \ \operatorname{div} v = 0\,,\ |v| = 1\,.$$

PROOF OF LEMMA 4.5: Let us denote $U_k^\eta = \{x \in B_1 : l + \eta \le \varphi_k \le l + 2\pi - \eta\}$. We claim we can choose $\eta_k \in [\frac{1}{2}\eta_0, \eta_0]$ such that

$$\int_{\partial U_k^{\eta_k}} |H_k \cdot n| \to 0 \quad \text{as } k \to \infty\,. \tag{4.36}$$

Indeed, using the co-area formula, we have

$$\int_{\varphi_k \in [l+\frac{1}{2}\eta_0, l+\eta_0] \cup [2\pi + l + \frac{1}{2}\eta_0, 2\pi + l + \eta_0]} |\nabla\varphi_k \cdot H_k| = \int_{\frac{1}{2}\eta_0}^{\eta_0} d\eta \int_{\partial U_k^\eta} |H_k \cdot n|\,.$$

But the left-hand side tends to zero by (4.10). Indeed, $\nabla\varphi \cdot H = \operatorname{div}(u\varphi + u^\perp + H\varphi)$ for H^1 functions, because C^∞ functions are dense in $H^1(\Omega, \mathbb{S}^1)$. Hence, using the mean-value theorem on the right-hand side, we get (4.36).

We now prove that $\|\operatorname{div} T_{\eta_k} u_k\|_{W^{-1,q}(B_1)} \to 0$. Indeed, let $\xi \in C_0^\infty(B_1)$. Since $T_{\eta_k}\varphi_k$ is constant in $B_1 \setminus U_k^{\eta_k}$,

$$\int_{B_1} \operatorname{div}(T_{\eta_k} u_k)\xi = \int_{U_k^{\eta_k}} \operatorname{div}(u_k)\xi\,.$$

From (4.8),

$$\int_{B_1} \operatorname{div}(T_{\eta_k} u_k)\xi = -\int_{U_k^{\eta_k}} \operatorname{div}(H_k)\xi = \int_{U_k^{\eta_k}} H_k \cdot \nabla\xi - \int_{\partial U_k^{\eta_k}} \xi(H_k \cdot n)\,. \tag{4.37}$$

Let μ_k denote the measure defined by $\mu_k(\xi) = \int_{\partial U_k^{\eta_k}} \xi(H_k \cdot n)$. By (4.36), $\mu_k \to 0$ in the sense of measures; hence it converges to 0 in $W^{-1,q}(B_1)$ for any $q < 2$. Also, from condition (4.9),

$$\left| \int_{U_k^{\eta_k}} H_k \cdot \nabla\xi \right| \le o(1)\|\nabla\xi\|_{L^q(B_1)} \quad \forall q < \infty\,.$$

Hence, we deduce from (4.37) that $\operatorname{div}(T_{\eta_k} u_k)$ converges to 0 in $W^{-1,q}(B_1)$ for any $q < 2$. Since $|T_{\eta_k} u_k| = 1$, it is also bounded in $W^{-1,q}(B_1)$ for any $q < \infty$; hence it converges to 0 in $W^{-1,q}(B_1)$ for any $q < \infty$.

Let χ_k be the characteristic function of U_k and

$$D_k = T_{\eta_k}\varphi_k(T_{\eta_k}u_k + H_k) + (T_{\eta_k}u_k)^{\perp} - T_{\eta_k}\phi_k\ H_k\ (1-\chi_k)\,.$$

We prove that div $D_k \to 0$ in the sense of measures.

Let $\xi \in C_0^\infty(B_1)$. We have to consider

$$\int_{B_1} \xi \operatorname{div} D_k = \int_{U_k} \xi(\nabla\varphi_k \cdot u_k + \operatorname{div}(u_k)^{\perp}) + \int_{B_1} \xi T_{\eta_k}\varphi_k \operatorname{div}(T_{\eta_k}u_k + H_k)$$

$$+ \int_{B_1} \xi\varphi_k \cdot H_k - \int_{B_1\setminus U_k} \nabla\xi \cdot H_k\ T_{\eta_k}\phi_k\,. \tag{4.38}$$

The first integral of the three identically vanishes since

$$T_{\eta_k}u_k = \exp(iT_{\eta_k}\varphi_k)$$

and all the functions involved are regular enough. On the other hand, $\operatorname{div}(T_{\eta_k}u_k) = -\operatorname{div} H_k$ in $U_k^{\eta_k}$ and 0 in $B_1 \setminus U_k^{\eta_k}$. Hence, we can rewrite (4.38) as

$$\int_{B_1} \xi \operatorname{div} D_k = -\int_{B_1\setminus U_k^{\eta_k}} \xi T_{\eta_k}\varphi_k \operatorname{div} H_k + \int_{B_1} \xi\nabla T_{\eta_k}\varphi_k \cdot H_k\,,$$

$$-\int_{B_1\setminus U_k^{\eta_k}} T_{\eta_k}\varphi_k\nabla\phi_k \cdot H_k = -\int_{\partial U_k^{\eta_k}} \xi(T_{\eta_k}\varphi_k)H_k \cdot n + \int_{U_k^{\eta_k}} \xi\nabla T_{\eta_k}\varphi_k \cdot H_k\,, \tag{4.39}$$

where we have used both the fact that $T_{\eta_k}\varphi_k$ is constant in each connected component of $B_1 \setminus U_k^{\eta_k}$ and integration by parts. We observe that first

$$\left| \int_{B_1\setminus U_k^{\eta_k}} T_{\eta_k}\varphi_k\nabla\phi \cdot H \right| \le C\|\nabla\xi\|_\infty \int_{B_1} |H| \longrightarrow 0\,;$$

moreover,

$$\int_{\partial U_k^{\eta_k}} |(T_{\eta_k}\varphi_k)H_k \cdot n| = C \int_{\partial U_k^{\eta_k}} |H_k \cdot n| \to 0$$

by (4.36). On the other hand,

$$\int_{U_k^{\eta_k}} |\nabla T_{\eta_k}\varphi_k \cdot H_k| \le \int_{B_1} |\nabla\varphi_k \cdot H_k| \to 0$$

by (4.10). Hence, we deduce from (4.39) that div D_k tends to 0 in the sense of measures.

Let $E_k = T_{\eta_k}\phi_k H_k(1-\chi_k)$. We have div $E_k \to 0$ in $H^{-1}(B_1)$. Indeed,

$$\left|\int_{B_1} \xi \ \operatorname{div}(E_k)\right| = \left|\int_{B_1\setminus U_k^{\eta_k}} T_{\eta_k}\varphi_k \nabla\phi_k \cdot H_k\right| \le C\|\xi\|_{H^1}\ \|H_k\|_{L^2} \to 0\,.$$

Since D_k in bounded in all L^q, div D_k is also bounded in $W^{-1,q}(B_1)$ for any $q<\infty$, and applying again the result of [15], we deduce with the previous step that it converges to 0 strongly in $H^{-1}(B_1)$. Thus div $D_k + E_k$ converges to 0 strongly in $H^{-1}(B_1)$. Denoting $G_k = (T_{\eta_k}u_k)^{\perp}$, curl G_k converges to 0 strongly in $H^{-1}(B_1)$ from step 2. Arguing as in step 2 of the proof of Lemma 4.2 and using the div-curl lemma, we obtain that $G_k \cdot D_k \rightharpoonup G\cdot D + E$, where G, D, and E are the weak limits of G_k, D_k, and E_k. Since $G_k\cdot D_k + E_k \to 1$, we deduce $G\cdot D + E = 1$.

By construction, $T_{\eta_k}\varphi_k$ takes its values in $[l+\frac{1}{2}\eta_0, l+2\pi-\frac{1}{2}\eta_0]$. Following the arguments used above, we can deduce that the Young measure it generates at 0 is a Dirac measure; hence $T_{\eta_k}\varphi_k$ converges strongly in $L^q(\Omega)$ for all $q<\infty$. We deduce that $T_{\eta_k}u_k$ also converges strongly to some v in $\bigcap_{1\le q<\infty} L^q(B_1)$. Necessarily $|v| = 1$ a.e., and div $v = 0$ from step 1. □

We now complete the proof of Theorem 1.5. Applying Lemma 4.5 to the sequences u_k', φ_k', and H_k' of Lemma 4.2 and to the l given in (4.7), we obtain strong convergence of $T_{\eta_k}\varphi_k'$ and $T_{\eta_k}u_k'$ for any choice of $\eta_0 > 0$. We may consider that η_k converges to some η. Denoting again by ν the Young measure φ_k' generates at 0, the Young measure generated by $T_{\eta_k}(\varphi_k')$ at 0 is $T_\eta^*\nu$. On the other hand, from Lemma 4.5, ν is supported, say, in $\{l_0, l_0+2\pi\}$; hence it is $t\delta_{l_0} + (1-t)\delta_{l_0+2\pi}$ for some $t\in[0,1]$. But by strong convergence of $T_{\eta_k}\varphi_k'$, $T_\eta^*\nu$ is a Dirac mass; hence $T_\eta^*(t\delta_{l_0} + (1-t)\delta_{l_0+2\pi})$ is a Dirac mass, and we conclude that t must be 0 or 1, and ν is a Dirac mass too. We deduce that φ_k' converges a.e. to its weak limit φ; hence by Lebesgue's theorem, it converges in L^q for all $q<\infty$. Assuming Proposition 5.2, we deduce Theorem 1.5.

5 Study of the Γ-Limit

5.1 The Limiting Problem and the Energy Lower Bound

We recall that $\mathcal{C}$ was defined in Definition 1.4; it is the natural set of admissible limiting configurations as explained in the introduction. In particular, it contains $u_\star$. Let $u : \Omega \mapsto \mathbb{S}^1$ satisfy div $\tilde{u} = 0$. Then there exists a Lipschitz-continuous function $g \in C^{0,1}(\Omega,\mathbb{R})$ such that

$$\begin{cases} u = \nabla^{\perp} g & \text{in } \Omega \\ g = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{5.1}$$

Indeed, div $\tilde{u} = 0$ means that div $u = 0$ in Ω and $u\cdot n = 0$ on $\partial\Omega$, where n is the outer unit normal to $\partial\Omega$. The existence of g satisfying $u = \nabla^{\perp} g$ in Ω is quite standard. g is then constant on $\partial\Omega$ and can be chosen to be 0 on $\partial\Omega$.

LEMMA 5.1 *Assume* $\varphi \in L^1(\Omega, \mathbb{R})$ *and* $u : \Omega \to \mathbb{S}^1$ *are such that* $\operatorname{div} \tilde{u} = 0$ *and* $\operatorname{div}(\varphi u + u^\perp) \in \mathcal{M}$*; then*

$$\|\operatorname{div}(\varphi u + u^\perp)\| \geq |\partial\Omega| .$$

Notice that for this lemma, u and φ are not necessarily related.

PROOF OF LEMMA 5.1: Let us consider

$$\begin{cases} f_\varepsilon(s) = -1 & \text{if } s \leq -\varepsilon \\ f_\varepsilon(s) = 1 & \text{if } s \geq \varepsilon \\ f_\varepsilon(s) = \frac{s}{\varepsilon} & \text{if } -\varepsilon \leq s \leq \varepsilon . \end{cases} \tag{5.2}$$

Since $\operatorname{div}(\varphi u + u^\perp) \in \mathcal{M}$ and $f_\varepsilon \in L^\infty$, we can consider

$$\left| \int_\Omega f_\varepsilon \operatorname{div}(\varphi u + u^\perp) \right| \leq \left\| \operatorname{div}(\varphi u + u^\perp) \right\| . \tag{5.3}$$

Since $g = 0$ on $\partial\Omega$ and $f_\varepsilon(0) = 0$, we may write, after an integration by parts,

$$\int_\Omega f_\varepsilon(g) \operatorname{div}(\varphi u + u^\perp) = -\int_\Omega (f_\varepsilon)'(g) \nabla g \cdot (\varphi u + u^\perp) . \tag{5.4}$$

The right-hand side is well-defined; indeed, since $\varphi \in L^1$, $\varphi u + u^\perp \in L^1$, while $f_\varepsilon'(g)\nabla g \in L^\infty$. Now, $f_\varepsilon'(s) = \frac{1}{\varepsilon}\mathbf{1}_{[-\varepsilon,\varepsilon]}$ and $u = \nabla^\perp g$; hence (5.4) becomes

$$\int_\Omega f_\varepsilon(g) \operatorname{div}(\varphi u + u^\perp) = \frac{1}{\varepsilon} \int_{-\varepsilon \leq g \leq \varepsilon} |\nabla g|^2 . \tag{5.5}$$

We may use the co-area formula, and since $|\nabla g| = 1$, this can be rewritten as

$$\int_\Omega f_\varepsilon(g) \operatorname{div}(\varphi u + u^\perp) = \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \mathcal{H}^1(\{g = s\} \cap \Omega) ds . \tag{5.6}$$

Then we claim that

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon \mathcal{H}^1\big(\{|g| = s\} \cap \Omega\big) ds \geq |\partial\Omega| . \tag{5.7}$$

Indeed, let V_ε be $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leq \varepsilon\}$. Since $|\nabla g| = 1$ a.e. by the mean value theorem, $|g| \leq \varepsilon$ in V_ε. Hence

$$\int_{V_\varepsilon} |\nabla g| \leq \int_{|g| \leq \varepsilon} |\nabla g| ,$$

and applying the co-area formula again,

$$\frac{1}{\varepsilon}|V_\varepsilon| \leq \frac{1}{\varepsilon} \int_{|g| \leq \varepsilon} |\nabla g| = \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \mathcal{H}^1(\{g = s\} \cap \Omega)\, ds .$$

Therefore

$$\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\mathcal{H}^1(\{g=s\}\cap\Omega)\geq \liminf_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}|V_\varepsilon|\right)\geq|\partial\Omega|\,;$$

thus we have (5.7). On the other hand,

$$\forall\varepsilon>0\quad\left|\int_\Omega f_\varepsilon(g)\operatorname{div}(\varphi u+u^\perp)\right|\leq\|\operatorname{div}(\varphi u+u^\perp)\|\,. \tag{5.8}$$

Thus, combining (5.3), (5.6), and (5.7), we have

$$\|\operatorname{div}(\varphi u+u^\perp)\|\geq|\partial\Omega|\,. \tag{5.9}$$

□

In particular, $\inf_{(u,\varphi)\in\mathcal{C}}\|\mu_{u,\varphi}\|\geq|\partial\Omega|$. The fact that $u_\star$ and $-u_\star$ achieve this minimum shall be proved in the next subsection, Lemma 5.4.

We have the following lower semicontinuity result:

PROPOSITION 5.2 *Let $u_\varepsilon\in\Lambda_{\mathcal{U}}$ be such that $E_\varepsilon(u_\varepsilon)\leq C$, $u_\varepsilon\to u$ in L^q $(\forall q<\infty)$, and $\varphi_\varepsilon\to\varphi$ in L^q $(\forall q<\infty)$ where φ_ε is a lifting of u_ε given by Lemma* 1.2*. Then $(u,\varphi)\in\mathcal{C}$ and*

$$\liminf_{\varepsilon\to 0}E_\varepsilon(u_\varepsilon)\geq 2\|\mu_{u,\varphi}\|\geq 2|\partial\Omega|\,.$$

Again, the hypotheses here could be replaced by

$$\|H_\varepsilon\|_{L^2}\to 0\quad\text{and}\quad\int_\Omega|\nabla u_\varepsilon||H_\varepsilon|\leq C\,.$$

PROOF OF PROPOSITION 5.2: Let φ_ε be the $H^1(\Omega,\mathbb{R})$-lifting of $u_\varepsilon\in\Lambda_{\mathcal{U}}$. The hypotheses of the proposition imply that $H_\varepsilon\to 0$ in $L^2(\Omega)$. Clearly this implies that $\operatorname{div}(\tilde{u_\varepsilon})=-\operatorname{div}H_\varepsilon\to 0$ in H^{-1}; hence $\operatorname{div}\tilde u=0$. Since $u_\varepsilon\in\Lambda_{\mathcal{U}}$, φ_ε is uniformly bounded in L^∞; hence, up to extraction, it converges weakly to some φ. Again as in (1.11), the boundedness of $E_\varepsilon(u_\varepsilon)$ implies $\int_\Omega|\mu_\varepsilon|\leq C$, where

$$\mu_\varepsilon=\operatorname{div}(\varphi_\varepsilon u_\varepsilon+u_\varepsilon^\perp+\varphi_\varepsilon H_\varepsilon)\,.$$

Since u_ε and φ_ε converge strongly in L^q, and H_ε strongly in L^2, we have

$$\varphi_\varepsilon u_\varepsilon+u_\varepsilon^\perp+\varphi_\varepsilon H_\varepsilon\rightharpoonup\varphi u+u^\perp\quad\text{in }\mathcal{D}'(\Omega)\,.$$

Hence $\mu_\varepsilon\rightharpoonup\operatorname{div}(\varphi u+u^\perp)$ in $\mathcal{D}'(\Omega)$. On the other hand, μ_ε is bounded in $\mathcal{M}$ by assumption; hence it converges weakly in $\mathcal{M}$ to $\operatorname{div}(\varphi u+u^\perp)$. Thus $\operatorname{div}(\varphi u+u^\perp)\in\mathcal{M}$ and $(u,\varphi)\in\mathcal{C}$. With Lemma 5.1,

$$|\partial\Omega|\leq\int_\Omega|\operatorname{div}(\varphi u+u^\perp)|\leq\liminf_{\varepsilon\to 0}\int_\Omega|\mu_\varepsilon|\leq\frac{1}{2}\liminf_{\varepsilon\to 0}E_\varepsilon(u_\varepsilon)\,.$$

□

We deduce Theorem 1.8 from Proposition 5.2 and Theorem 1.5.

Remark 5.3. Using integration by parts similar to that used in (5.3) with the measure $\mu_{u_\star,\varphi_\star}$, we obtain the following isoperimetric-type inequality:

$$|\Omega| = \int_\Omega \operatorname{dist}(x, \partial\Omega) \operatorname{div}(\varphi_\star u_\star + u_\star^\perp) \le |\partial\Omega| \max_\Omega \operatorname{dist}(\cdot, \partial\Omega) .$$

5.2 The Limiting Problem Revisited: BV Functions

In the case of BV functions, the limiting functional $u \mapsto \|\mu_{u,\varphi}\|$ can be expressed much more simply. Suppose $\varphi \in \mathrm{BV}$. We use the notation of [1, section 2].

The approximate limit of φ at x, denoted by $\tilde{\varphi}(x)$, is, when it exists, the unique $z \in \mathbb{R}$ satisfying

$$\lim_{r\to 0} \frac{1}{r^2} \int_{B_r(x)} |\varphi(y) - z| dy = 0 .$$

We denote by S' the set of approximate discontinuity points, i.e., the set of points where the approximate limit does not exist. The one-sided approximate limits $\varphi^+(x)$ and $\varphi^-(x)$ are numbers a and b in $\mathbb{R}$ such that

$$\lim_{r\to 0} \int_{B_r^+(x)} |\varphi(y) - a| dy = 0 \quad \text{and} \quad \lim_{r\to 0} \int_{B_r^-(x)} |\varphi(y) - b| dy = 0 ,$$

where $B_r^\pm = \{y \in B_r(x) : \pm\langle y - x, n\rangle \ge 0\}$ are the two half-balls corresponding to some unit vector n. We denote by $S \subset S'$ the set of approximate jump points, i.e., points of S' for which $\varphi^\pm$ exist for some unit vector $n(x)$. The Radon measure $D\varphi$ can be split into three mutually singular parts:

$$D\varphi = \nabla\varphi \mathcal{L}^2 + D_c\varphi + (\varphi^+ - \varphi^-) \otimes n\mathcal{H}^1 \lfloor_S , \tag{5.10}$$

where $\mathcal{L}^2$ is the Lebesgue measure and $D_c\varphi$ is the Cantor part of $D\varphi$. Also $\mathcal{H}^1(S' \setminus S) = 0$. If $D_c\varphi = 0$, then φ belongs to the subspace $\mathrm{SBV}(\Omega)$ of $\mathrm{BV}(\Omega)$.

Proof of Theorem 1.9: If $\varphi \in \mathrm{BV}$, according to Volpert's chain rule (cf. [1, proposition 3.6], for instance), then $\cos\varphi$ and $\sin\varphi$ are in BV; hence $u \in \mathrm{BV}$. It is then easy to check that if, in addition, $\operatorname{div} u = 0$ in Ω, $\operatorname{div}(\varphi u + u^\perp)$ is automatically a Radon measure with $\operatorname{div}(\varphi u) = D\varphi \cdot u$. Indeed, consider $\xi \in C_0^\infty(\Omega)$,

$$\int_\Omega \xi \operatorname{div}(\varphi u) = -\int_\Omega \varphi u \cdot \nabla\xi = -\int_\Omega \varphi \operatorname{div}(\xi u) . \tag{5.11}$$

(Since $\operatorname{div} u = 0$, one can check that $\forall\xi \in C_0^\infty(\Omega)$, $u \cdot \nabla\xi = \operatorname{div}(\xi u)$ in $\mathcal{D}'(\Omega)$.) Since $\xi u \in L^\infty$, by definition of BV,

$$\left| \int_\Omega \varphi \operatorname{div}(\xi u) \right| \le \|\xi u\|_{L^\infty} \int_\Omega |D\varphi| \le \|\xi\|_{L^\infty} \int_\Omega |D\varphi| \tag{5.12}$$

and

$$-\int_\Omega \varphi\, \mathrm{div}(\xi u) = \int_\Omega \xi (D\varphi \cdot u)\,. \tag{5.13}$$

From (5.11) and (5.12), we deduce that $\mathrm{div}(\varphi u)$ is a Radon measure on Ω and from (5.13), that, in the frame of BV functions,

$$\mathrm{div}(\varphi u) = D\varphi \cdot u\,. \tag{5.14}$$

Since $u \in \mathrm{BV}$, $\mathrm{div}(u^\perp)$ is automatically a Radon measure with $\mathrm{div}(u^\perp) = D_2 u_1 - D_1 u_2$. From Volpert's chain rule, we have

$$\begin{cases} D(\cos\varphi) = -\sin\varphi \nabla\varphi - \sin\tilde{\varphi} D_c\varphi + (\cos\varphi^+ - \cos\varphi^-) \otimes n\mathcal{H}^1\lfloor_S \\ D(\sin\varphi) = \cos\varphi\nabla\varphi + \cos\tilde{\varphi} D_c\varphi + (\sin\varphi^+ - \sin\varphi^-) \otimes n\mathcal{H}^1\lfloor_S\,. \end{cases} \tag{5.15}$$

The condition $\mathrm{div}\, u = 0$ means that $D_1(\cos\varphi) + D_2(\sin\varphi) = 0$. Since $D(\cos\varphi)$ and $D(\sin\varphi)$ are split into mutually singular measures, we can consider their jump part (which is supported on S) and deduce that

$$\big((\cos\varphi^+ - \cos\varphi^-)n_1 + (\sin\varphi^+ - \sin\varphi^-)n_2\big)\mathcal{H}^1\lfloor_S = 0$$

where n_1 and n_2 are the Cartesian coordinates of n. In other words, $((u^+ - u^-) \cdot n)\mathcal{H}^1\lfloor_S = 0$, or

$$u^+ \cdot n = u^- \cdot n \quad \mathcal{H}^1\text{- a.e. on } S\,. \tag{5.16}$$

Hence we can write $u \cdot n = u^+ \cdot n = u^- \cdot n$ on S without ambiguity. This means simply that for a divergence-free vector field, the normal component is preserved across a jump line.

Returning to (5.14), we have

$$\mathrm{div}(\varphi u) = D\varphi \cdot u = \nabla\varphi \cdot u + D_c\varphi \cdot \tilde{u} + (\varphi^+ - \varphi^-)(n \cdot u)\mathcal{H}^1\lfloor_S\,.$$

On the other hand, in view of (5.15),

$$\begin{aligned} \mathrm{div}\, u^\perp = D_2 u_1 - D_1 u_2 &= D_2 \cos\varphi - D_1 \sin\varphi \\ &= -u \cdot \nabla\varphi - u \cdot D_c\varphi \\ &\quad + \big((\cos\varphi^+ - \cos\varphi^-)n_2 - (\sin\varphi^+ - \sin\varphi^-)n_1\big)\mathcal{H}^1\lfloor_S\,. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{div}(\varphi u + u^\perp) &= \big((\varphi^+ - \varphi^-)(u \cdot n) \\ &\quad + (\cos\varphi^+ - \cos\varphi^-)n_2 - (\sin\varphi^+ - \sin\varphi^-)n_1\big)\mathcal{H}^1\lfloor_S \\ &= \big((\varphi^+ - \varphi^-)(u \cdot n) + (u^+ - u^-) \cdot n^\perp\big)\mathcal{H}^1\lfloor_S\,. \end{aligned}$$

Hence $\mu_{u,\varphi}$ is only supported on the jump set S of φ.

If $|\varphi^+ - \varphi^-| < 2\pi$, then necessarily $u^+ \neq u^-$ pointwise (otherwise $\varphi^+ = \varphi^-$ and there would be no jump), and $u^+ \cdot n^\perp = -u^- \cdot n^\perp$. It is left to the reader that in this case $u \cdot n = \cos X$, and $(u^+ - u^-) \cdot n^\perp = 2u^+ \cdot n^\perp = 2\sin X$, where $X = -\frac{1}{2}\,\mathrm{sgn}(u^+ \cdot n^\perp)|\varphi^+ - \varphi^-|$.

If φ_ε is bounded in BV and $E_\varepsilon(u_\varepsilon) \le C$ where $u_\varepsilon = e^{i\varphi_\varepsilon}$, u_ε is also bounded in BV. Since BV embeds compactly into L^p for all $p < 2$, we can assume, up to extraction, that $u_\varepsilon \to u$ strongly in L^p $(p < 2)$. Since $|u_\varepsilon| = 1$ a.e., we deduce that $u_\varepsilon \to u$ in $\bigcap_{1\le q<\infty} L^q$. Similarly, we can assume $\varphi_\varepsilon \rightharpoonup \varphi$ weakly in BV and $\varphi_\varepsilon \to \varphi$ strongly in L^p $(p < 2)$. Up to extraction, $\varphi_\varepsilon \to \varphi$ almost everywhere; hence $e^{i\varphi_\varepsilon} \to e^{i\varphi}$ a.e. By Lebesgue's theorem $u_\varepsilon \to e^{i\varphi}$ in L^1; hence $u = e^{i\varphi}$ a.e. As in Proposition 5.2, we are led to

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \ge \liminf_{\varepsilon\to 0} \int_\Omega |\mu_\varepsilon| \ge \|\operatorname{div}(\varphi u + u^\perp)\| = \|\mu_{u,\varphi}\| \,.$$

□

We can then prove the following:

LEMMA 5.4 *For* $u_\star = e^{i\varphi_\star}$ *constructed in Lemma* 3.1*,* $\varphi_\star \in \mathrm{BV}$ *and*

$$\begin{aligned} \mu_{u_\star,\varphi_\star} &= 2(\sin X - X\cos X)\mathcal{H}^1\lfloor_S \quad \textit{where } X = \tfrac12(\varphi_\star^+ - \varphi_\star^-) \ge 0\,,\\ \mu_{u_\star,\varphi_\star} &\ge 0\,,\\ \|\mu_{u_\star,\varphi_\star}\| &= |\partial\Omega|\,. \end{aligned}$$

PROOF OF LEMMA 5.4: We return to the notation of Lemma 3.1. The singular set of $\varphi_\star$ is made of two parts: the medial axis Σ and the ray T. If $x \in \Sigma$ is not a vertex, then we recall that we chose n to be the normal to Σ pointing towards x_2; i.e.,

$$n \cdot \frac{\overrightarrow{xx_2}}{|xx_2|} \ge 0\,,$$

and thus $\varphi_\star^+ > \varphi_\star^-$ in the frame of (5.10) with $X = \frac12(\varphi_\star^+ - \varphi_\star^-) \in\,]0,\pi[$. On the other hand, from (3.3), $u_\star^+ \cdot n^\perp \le 0$ and $u_\star^- \cdot n^\perp = -u_\star^+ \cdot n^\perp \ge 0$,

$$(u_\star^+ - u_\star^-)\cdot n^\perp = 2u_\star^+ \cdot n^\perp = -2\cos(\varphi_\star^+ - \theta) = -2\cos\alpha_+ = -2X\sin X\,,$$

while

$$(\varphi_\star^+ - \varphi_\star^-)(u_\star\cdot n) = (\alpha_+ - \alpha_-)\sin\alpha = 2X\cos X\,.$$

Thus, $x \mapsto \sin x - x\cos x$, which is nonnegative on $[0,\pi]$,

$$\big((\varphi_\star^+ - \varphi_\star^-)(u_\star\cdot n) - (u_\star^+ - u_\star^-)\cdot n^\perp\big)\mathcal{H}^1\lfloor_\Sigma = 2(\sin X - X\cos X)\mathcal{H}^1\lfloor_\Sigma \ge 0\,.$$

It remains to show that $\mu_{u_\star,\varphi_\star} \ge 0$ on T. On T, we chose $n = u_\star$, $\varphi_\star^+ - \varphi_\star^- = 2\pi$, and

$$\mu_{u_\star,\varphi_\star}\lfloor_T = (\varphi_\star^+ - \varphi_\star^-)(u_\star\cdot n)\mathcal{H}^1\lfloor_T = 2\pi\,\mathcal{H}^1\lfloor_T = 2(\sin X - X\cos X)\mathcal{H}^1\lfloor_T \ge 0$$

with $X = \frac12(\varphi_\star^+ - \varphi_\star^-) = \pi$. We conclude that $\mu_{u_\star,\varphi_\star} \ge 0$ since $\mu_{u_\star,\varphi_\star}$ is supported on $S = T \cup \Sigma$. Consequently, arguing as previously,

$$\int_\Omega |\mu_{u_\star,\varphi_\star}| = \int_\Omega \mu_{u_\star,\varphi_\star} = \int_\Omega \operatorname{div}(\varphi_\star u_\star + u_\star^\perp)\,.$$

Since $\varphi_\star \in \mathrm{BV}$, this can be integrated as

$$\int\limits_{\partial\Omega} (\varphi_\star u_\star + u_\star^\perp) \cdot n = \int\limits_{\partial\Omega} -u_\star \cdot \tau = |\partial\Omega|$$

in view of the definition of $u_\star$ ($u_\star \cdot \tau = -1$ on $\partial\Omega$). We conclude that

$$\|\mu_{u_\star,\varphi_\star}\| = 2 \int\limits_S |\sin X - X\cos X| d\mathcal{H}^1 = |\partial\Omega| \, .$$

□

Theorem 1.7 is a consequence of Lemmas 5.1 and 5.4.

Remark 5.5. In Lemma 3.1 there were many possibilities for choosing $\varphi_\star$ since T could be any ray joining Σ to $\partial\Omega$. The results of Lemma 5.1 are independent of this choice, which is interesting in itself: The result $\mu_{u_\star,\varphi_\star} \geq 0$, $\mu_{u_\star,\varphi_\star} = \int_S 2\sin X - X\cos X = |\partial\Omega|$, is true independently of the choice of T. This is made possible by the fact that, as we change T, the contribution of T to the limiting energy (which is 2π times its length) can vary, but the contribution along Σ varies also to compensate this change. Indeed, although u^+ and u^- are independent of T along Σ, as we move T, X changes between the two contact points of the initial and final T. Observe that the sign of X remains unchanged.

Rewording to "many possibilities" OK? Original was awkward.

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