RECTIFIABILITY OF ENTROPY MEASURES IN ONE SPACE DIMENSION

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Abstract. We show that entropy solutions to 1 dimensional scalar conservation laws for totally nonlinear fluxes and for arbitrary measurable bounded data have a structure similar to the one of BV maps without being always BV. The singular set -shock waves- of such solutions is contained in a countable union of C^1 curves and $\mathcal{H}^1$ almost everywhere along these curves the solution has left and right approximate limits. The entropy production is concentrated on the shock waves and can be explicitly computed in terms of the approximate limits. The solution is approximately continuous $\mathcal{H}^1$ almost everywhere outside this union of curves.

1. Introduction

Let $u : \mathbf{R}^+_t \times \mathbf{R}_x$ be a bounded entropy solution of $\partial_t u + \partial_x[f(u)] = 0$ and assume f is strictly convex. Since the classical results of Lax and Oleinik, it is known that u is locally a BV function, even when the initial data $u(0,\cdot)$ are very irregular. We recall that a bounded distributional solution of $\partial_t u + \partial_x[f(u)] = 0$ is an entropy solution if and only if

- $\partial_t[q(u)] + \partial_x[\eta(u)]$ is a nonpositive measure for every convex entropy–entropy flux pair (q,η), i.e. for every (q,η) such that q is convex and $q'(t) = \eta'(t)f'(t)$ $\mathscr{L}^1$–a.e..

When f is not convex the solution of the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x[f(u)] \;=\; 0 \\ \\ u(0,\cdot) = u_0(\cdot) \end{cases} \tag{1}$$

is BV_{loc} if $u_0 \in BV_{loc}(\mathbf{R})$. But for less regular u_0, u is not, in general, a BV function. In [LPT] the authors have introduced a kinetic formulation for (1) and, using velocity averaging lemmas, they have proved that u belongs always to some fractional Sobolev space $W^{\alpha,q}$, even if u_0 is not better than L^∞ (the exponents α and q depending on the nonlinearity of the flux f). We refer to the book [P] for an account of the rich literature on kinetic formulations for conservation laws. See also [DW] for some examples concerning the optimal regularity of u.

However, the fractional Sobolev spaces $W^{\alpha,q}$ with $\alpha < 1$ do not provide good information on the "structure" of the singularities of the function u. The meaning of structure is explained by the following examples. First, let u be a C^1 solution of (1) and (η, q) a C^1 entropy–entropy flux pair. Then

$$T \;:=\; \partial_t[\eta(u)] + \partial_x[q(u)] \;=\; \eta'(u)u_t + (q'(u))u_x \;=\; q'(u)[u_t + f'(u)u_x] \;=\; 0\,. \tag{2}$$

Instead, let u be discontinuous but *piecewise* C^1. In particular assume the existence of a smooth 1–dimensional set J_u such that u is C^1 on $\mathbf{R}^2 \setminus J_u$ and has left and right traces (denoted by $u^\pm$) on J_u. Then the distribution T does not vanish any more, but it is a measure concentrated on J_u. Indeed, if $(1,s)/\sqrt{1+s^2}$ denotes the tangent to J_u and $\mathcal{H}^1$ denotes the 1–d Hausdorff measure, then

$$\langle T, \varphi \rangle \;=\; \int_{J_u} \left[\frac{s\big[\eta(u^+) - \eta(u^-)\big] - \big[q(u^+) - q(u^-)\big]}{\sqrt{1+s^2}} \right] \varphi \, d\mathcal{H}^1 \,. \tag{3}$$

For BV solutions, the BV structure theorem and Vol'pert chain rule (see [AFP]) give a fairly good understanding of what happens. Indeed they imply the existence of a *rectifiable* set J_u such that

(i) u is approximately continuous outside J_u and has left and right traces on J_u;
(ii) for every entropy–entropy flux pair (η, q) the distribution T is still given by (3).

In this note we prove that, under some regularity assumptions on the flux f, the same structure holds for *every* entropy solution u

Theorem 1.1. *Let $f \in C^2(\mathbf{R}, \mathbf{R})$ and $\{x | f''(x) = 0\}$ be locally finite. If u is an entropy solution of (1), then there is a rectifiable 1–d. set $J \subset \mathbf{R}^2$ s.t.*

(a) *Every $y \notin J$ is a Lebesgue point for u.*
(b) *u has right and left traces $\mathcal{H}^1$–a.e. on J.*
(c) *For any smooth entropy–entropy flux pair (η, q), the entropy production is concentrated on J and can be computed "classically" as*

$$\partial_t[\eta(u(t,x))] + \partial_x[q(u(t,x))] \;=\; \frac{s\big[\eta(u^+) - \eta(u^-)\big] - \big[q(u^+) - q(u^-)\big]}{\sqrt{1+s^2}} \, \mathcal{H}^1 \llcorner J \,. \tag{4}$$

Remark 1.2. We stress on the fact that such solutions u are not, in general, in BV. Indeed, let $f(v) = |v|^p$, with $p > 2$. Clearly, f satisfies all the assumptions above. Then, there are *entropy solutions* to $\partial_t u + \partial_x |u|^p = 0$ such that $u \notin W^{\alpha,q}_{loc}$ for any $\alpha > \frac{1}{p-1}$ (and any q); cp. Proposition 3.4. of [DW].

Remark 1.3. In view of the fact that u is an *entropy* solution we actually expect that u is *continuous* outside J_u. Indeed this is known to be true for strictly convex fluxes (see Chapter XI of [Da]).

Much is known about the regularity of solutions to scalar conservation laws in one–dimension and, after all, if the initial data are BV, the solution is BV. Indeed our interest comes from a more general question in measure theory, which arises naturally in different areas of PDE.

1.1. **The general measure–theoretic question.**

Problem 1.4. *Let $\mathcal{E} \subset C^1(\mathbf{R}^k, \mathbf{R}^n)$ and $u \in L^\infty(\mathbf{R}^n, \mathbf{R}^k)$. Assume that $\mu_\Phi := \mathrm{div}[\Phi(u)]$ is a Radon measure for every $\Phi \in \mathcal{E}$.*

(i)' *Does there exist a codimension 1 rectifiable set J_u such that u is approximately continuous outside J_u and has left and right traces on J_u?*
(ii)' *If the answer to (i) is yes and $\Phi, \Psi \in \mathcal{E}$, can we relate the measures μ_Ψ, μ_Φ, and the pointwise information on u by "chain–rule" formulas?*

We can give more specific versions of this quite general problem by simply assuming more information on the μ_Φ's (i.e. that some are nonnegative measures, or that some vanish): Indeed in many concrete examples we know more about μ_Φ.

Note that the classical structure theorem of BV functions is a positive answer to (i)' when $\mathcal{E}$ is the class of linear mappings $L : \mathbf{R}^k \to \mathbf{R}^n$. In this case the information of (i)' are summarized in the so called *precise representative of* u. Vol'pert chain–rule is a positive answer to (ii)' when $\mathcal{E}$ *contains* the linear maps L. In this case, for any $\Phi \in C^1$, Vol'pert chain–rule provides an explicit formula for $\mathrm{div}[\Phi(u)]$ in terms of the measures $\partial_j u^i$ and of the precise representative of u.

Thus, Problem 1.4 can be considered as a non–linear version of the theory of fine properties of BV functions. Recently, some papers (see [AKLR], [DO], [DOW], [LR]) have given a positive answer to (i)' for many examples of classes $\mathcal{E}$ related to PDE problems. To our knowledge this note provides the first positive answer to (ii)' in a case where there is no BV regularity. Moreover the answer to (i)' given in the papers cited above is not complete: their results do not prove that outside J_u the function is approximate continuous, but they yield a milder property (cp. (a') in subsection 2.1 and (a) in Theorem 1.1). In the particular case considered here we are also able to fill this gap.

1.2. **Applications to PDEs.** The link with the theory of scalar conservation laws is transparent. In this case u is an L^∞ entropy solution of (1) and $\mathcal{E}$ is the set of convex entropy–entropy flux pairs (η, q). This framework is available also for multi–d scalar equations, where Kruzkov's theory provides existence and uniqueness of entropy solutions to the Cauchy problem. Even for 2×2 systems in 1 spatial dimension one can show, via compensated compactness, the existence of global L^∞ entropy solutions for any bounded initial data (this approach was pioneered in [DP] in the system of isentropic gas dynamics; we refer to [Se] for the general treatment of 2×2 systems). However, except for some isolated examples, nothing is known about the regularity and the structure of these solutions. In this case an answer to Problem 1.4 would be much more relevant, since even when the initial data are BV, there are no global–in–time BV estimates when starting from large data. For small data, the recent remarkable work [BB] give BV estimates when the entropy solution achieved by compensated compactness is generated by the vanishing viscosity limit.

Besides the area of conservation laws, there is another active field in which Problem 1.4 has interesting applications. In recent years, models arising from different areas of physics (such as micromagnetism, liquid crystals, film–blistering) have raised the issues of understanding the asymptotic behavior of certain second–order functionals of Ginzburg–Landau type (see for example [AG], [DKMO1] and [GO]). It turns out that the Γ–limit of these functionals (i.e. the appropriate limiting variational problem) can be properly understood in classes of functions which satisfy certain PDE's and for which the divergence of certain nonlinear quantities are Radon measures (see [ADM], [DKMO2], [JK] and [RS1]). Indeed the total variation of these Radon measures is controlled by the limit functional. It turns out, however, that this control does not give BV bounds and these classes of functions are strictly larger than BV (see [ADM] and [DW]).

In these variational problems the papers [DO] and [AKLR] provide, by giving a partial answer to (i)', a regularity theory for the functions in the domain of the conjectured Γ–limits. A positive answer to (ii)', which is still lacking, would give nice formulas for the conjectured

Γ–limits and, potentially, could lead to complete proofs of the Γ–convergence results (see [ADM] and [RS1]).

1.3. **Links to kinetic theory.** Most of the PDE problems mentioned above enjoy a kinetic formulation (for the variational cases this formulation was introduced by [JP] and [RS2]). We give the kinetic formulation for entropy solutions of (1) and we refer to the book [P] for an account of the various kinetic formulations of the problems mentioned above. Let u be an entropy solution of (1) and assume (for simplicity) that u is nonnegative. Define the maxwellian $\chi : \mathbf{R}_v \times \mathbf{R}_t \times \mathbf{R}_x$ as

$$\chi(v,t,x) \;=\; \begin{cases} +1 & \text{if } 0 < v \leq u(t,x) \\ 0 & \text{otherwise} \end{cases}$$

Then χ satisfies, in the sense of distributions, the *kinetic* equation

$$\partial_t \chi + f'(v)\partial_x \chi \;=\; \partial_v \mu \tag{5}$$

where μ is a Radon measure on $\mathbf{R}_v \times \mathbf{R}^2_{t,x}$. Moreover, if we set

$$K(v,u) \;=\; \begin{cases} v & \text{if } u \geq v \\ u & \text{otherwise} \end{cases} \tag{6}$$

we then have $\mu(v,t,x) = \partial_t\big[K(v,u(t,x))\big] + \partial_x\big[f(K(v,u(t,x))\big]$. Thus, a characterization of the measures

$$\mu_v \;:=\; \partial_t\big[K(v,u(t,x))\big] + \partial_x\big[f(K(v,u(t,x))\big] \tag{7}$$

is equivalent to characterize the r.h.s. of (5). Indeed, in all the cases where a kinetic formulation is available, point (ii)' of Problem 1.4 reduces essentially to prove that the r.h.s. of the corresponding kinetic equation is concentrated on the set J_u. We point out that the problem of proving concentration estimates for the entropy measure μ was first mentioned in [LPT] (cf. the first open question listed in section 1.13 of [P]).

Finally we remark that some technical lemmas proved in this note yield new results even in the kinetic theory. Indeed:

(1) Thanks to a regularity result of [Ch] we prove that for μ in (5), $\partial^2_v \mu$ is a measure (see Proposition 4.1). This information can be combined with suitable modifications of the velocity averaging lemmas in [JP2] to improve the Sobolev regularity of u known up to now. However we do not pursue this issue.
(2) In Section 6 we derive a new averaging lemma for solutions of the transport equation (43). To our knowledge this is the first example of an averaging lemma where no L^p bounds in the transported values are assumed.

2. Outline of the proof

2.1. **Previous results.** From theorem 2.4 and remark 2.5 of [DOW] we know the existence of a rectifiable set J such that (b) above holds and:

(a') In every $y \notin J$ the mean oscillation of u vanishes.
(c') For any smooth entropy–entropy flux pair (η, q), the entropy production is given by $\zeta + \alpha$, where ζ is the right hand–side of (4) and α satisfies the following condition:

$$\alpha(K) \;=\; 0 \qquad \text{for every Borel set } K \text{ with } \mathcal{H}^1(K) < \infty\,. \tag{8}$$

Hence, our tasks are to improve (a') and (c') to (a) and (c). A crucial role will be played by the following theorem of [Ch]:

Theorem 2.1. *There is a constant C (depending on $\|u\|_\infty$ and f) such that*

$$\left\|\partial_x\big[f'(u(T,\cdot))\big]\right\|\big([a,b]\big) \;\leq\; C\left(1+\frac{b-a}{T}\right). \tag{9}$$

Actually, the author in [Ch] gives an explicit proof of Theorem 2.1 when $|\{f''=0\}|\leq 2$ and at the end of the paper remarks that this proof can be generalized to the case when the set $\{f''=0\}$ is locally finite (cp. section 6 of [Ch]).

2.2. **Strategy of the proof.** We first establish some notation which will be used throughout the paper. If ν is a Radon measure on Ω, then ν^+ and ν^- denote its positive and negative part ($\nu=\nu^+-\nu^-$). $\|\nu\|$ denotes the measure $\nu^++\nu^-$ and $\|\nu\|_{\mathcal{M}(\Omega)}$ denotes the total variation of ν on Ω (that is, $\|\nu\|(\Omega)$). $B_r(y)$ denotes the ball of radius r centered at y.

Proof of (a) This is based on the following remark. Assume that at point (t_0,x_0) the mean oscillation of u vanishes, but u is not approximate continuous. This implies that the averages of u on the balls of radius r oscillates between two values $a<b$ as $r\downarrow 0$. By a Fubini–Tonelli argument, this oscillation will take place in most of the lines passing through (t_0,x_0). A linear change of variables and Theorem 2.1 give that this oscillation cannot take place if the lines are space–like. The detailed proof is given in Section 3.

Proof of (c) Everything boils down to show that the measure μ on the r.h.s of (5) is concentrated on J.

Using Theorem 2.1, in Section 4 we prove that $\partial_v^2\mu$ is a measure. Denote by ν the non-negative measure on $\mathbf{R}^2$ which is the (x,t)–marginal of the total variation of $\partial_v^2\mu$. Then the estimate on $\partial_v^2\mu$ allows to write μ as $g(v,t,x)\nu$, where $\partial_v^2 g(\cdot,t,x)$ is a measure in v for ν–a.e. (t,x) (see Lemma 5.1). Thus our claim is equivalent to show that ν is concentrated on J. We argue by contradiction and assume that $\nu(\mathbf{R}^2\setminus J)>0$. Take a "typical" point which lies outside J but which "sees" the measure ν (for the precise meaning compare with the set A defined in Proposition 5.3). In what follows, this point will be called *base point* and for simplicity we assume that it is the origin.

We look at the rescaled kinetic equations satisfied by the rescaled functions $\chi_r(v,t,x):=\chi(v,rt,rx)$, that is

$$\partial_t\chi_r+f'(v)\partial_x\chi_r \;=\; \partial_v\frac{\tilde{\mu}^r}{r}\,. \tag{10}$$

Here the $\tilde{\mu}^r$ are the appropriate rescalings of the measure μ. We divide (10) by the quantity $\alpha_r=\nu(B_r)/r$, thus getting

$$\partial_t\frac{\chi_r}{\alpha_r}+f'(v)\partial_x\frac{\chi_r}{\alpha_r} \;=\; \partial_v\frac{\tilde{\mu}^r}{\nu(B_r)} \;=:\; \partial_v\mu^r\,. \tag{11}$$

By (c') it follows that J coincides (up to ν–negligible sets) with the set of points y where

$$\limsup_{r\downarrow 0}\frac{\nu(B_r(y))}{r} \;>\; 0\,.$$

Thus, "typically" $\alpha_r\downarrow 0$, since our base point is out of J.

By an infinite version of Radon–Nykodim Theorem (see Proposition 5.3), the convergent subsequences μ^{r_n} are converging to a measure of the form $g(v)\mathscr{L}^1 \times \nu_\infty$, where ν^∞ is a nonnegative measure on $\mathbf{R}^2$. This product structure is a consequence of a very general fact and similar remarks have already been used in [DO], [DOW] and [AKLR]. Note that, since the base point is "typical" and sees the measure ν, we have that $g(v)\mathscr{L}^1 \times \nu_\infty$ is not the trivial measure.

Take an arbitrary $T \in [-1, 1]$ and consider the solution χ^f_n of the free transport equation

$$\begin{cases} \partial_t \chi^f_n + f'(v)\partial_x \chi^f_n \;=\; 0 \\ \\ \chi^f_n(v, T, x) \;=\; \chi_{r_n}(v, T, x)\,. \end{cases} \tag{12}$$

Define

$$F_n(v, t, x) \;:=\; \frac{\chi_{r_n}(v, t, x) - \chi^f_n(v, t, x)}{\alpha_{r_n}}$$

and note that they solve the transport equation

$$\begin{cases} \partial_t F_n + f'(v)\partial_x F_n \;=\; \partial_v \mu^{r_n} \\ \\ F_n(v, T, x) \;=\; 0\,. \end{cases} \tag{13}$$

Formally, in the limit we get a distribution L which solves

$$\begin{cases} \partial_t L + f'(v)\partial_x L \;=\; g'(v)\nu_\infty \\ \\ L(v, T, x) \;=\; 0\,. \end{cases} \tag{14}$$

The $\chi_{r_n}(v, \cdot)$ are rescalings of the $\chi(v, \cdot)$, which are the characteristic functions of the v–sublevel sets of u. Since our base point does not belong to J, (a) applies and hence the rescalings of u around the base point are converging to a constant (recall that the base point is the origin and thus this constant is $u(0)$). Thus $\chi_{r_n}(v, \cdot)$ is converging to the constant 1 if $0 < v < u(0)$ and to the constant 0 otherwise. The χ^f_n, being solutions of a free transport equation, take value in $\{0, 1\}$. Thus one could hope that the distribution L satisfies the sign condition:

$$L \leq 0 \text{ on }]u(0), +\infty[\, \times \mathbf{R}^2 \text{ and } L \geq 0 \text{ on }]0, u(0)[\, \times \mathbf{R}^2. \tag{15}$$

This may not be the case, since L is the limit $F_n = \frac{\chi_{r_n} - \chi^f_n}{\alpha_{r_n}}$ and $\alpha_{r_n} \downarrow 0$. However, recall the estimate on $\partial^2_v \mu$. In a "typical point" this estimate translates into a uniform estimate for the measures $\partial^2_v \mu^{r_n}$. This is used in section 6 to prove an averaging lemma (see Lemma 6.1) for the functions F_n. This lemma is, to our knowledge, new and provides sufficiently strong information in order to derive (15). Then, playing with the arbitrariness of T in (14), with (15) and with the condition $\nu_\infty \geq 0$, we can prove that L and ν_∞ must vanish identically. This gives a contradiction since we have fixed a typical point which "sees" the measure ν (that is, ν_∞ cannot vanish identically).

3. From VMO to Lebesgue points

In this section we use Theorem 2.1 to show (a). Let us fix $y \notin J$ and assume $y \in \{t > 0\}$. For simplicity, assume that $y = (T, 0)$ and recall that u is an entropy solution in $\{t > 0\}$.

Set

$$\overline{u}^r \;=\; \frac{1}{\pi r^2}\int_{B_r(y)} u(t,x)\,dtdx\,.$$

From (a') we get that

$$\lim_{r\downarrow 0}\frac{1}{\pi r^2}\int_{B_r(y)}\left|u(t,x)-\overline{u}^r\right|dtdx \;=\; 0\,. \tag{16}$$

Thus we have to prove that $a := \liminf_{r\downarrow 0}\overline{u}_r = \limsup_{r\downarrow 0}\overline{u}_r =: b$.

Step 1 Assume, by contradiction, that $a < b$ and fix the following convention:

- If ℓ is a half–line starting at y and $y_1 \neq y_2 \in \ell$, then we say that $y_1 > y_2$ if $|y_1| > |y_2|$.
- We parameterize the family of all half–lines ℓ's using vectors of $\mathbf{S}^1$ in the usual way.

Applying Fubini–Tonelli Theorem in polar coordinates, we get the following:

(Co) Let $\delta > 0$ be given, $N > 0$ be any given natural number and $I_1, I_2 \subset]a,b[$ two given intervals. Then for $\mathcal{H}^1$–a.e. ℓ, there exist $2N$ points $y_1,\ldots,y_{2N} \in \ell\cap B_\delta(y)$ with
  (i) $y_1 > y_2 > \ldots > y_{2N}$;
  (ii) All y_i's are Lebesgue points for u and $u(y_{2i}) \in I_2$, $u(y_{2i+1}) \in I_1$.

Fix now two intervals $I_1, I_2 \subset [a,b]$ such that $f'(I_1) \leq c < d \leq f'(I_2)$ (this is certainly possible since f'' vanishes only in finitely many points). Note that, if for N large enough one of the ℓ's above were the x axis, we would have a contradiction. Indeed we would have $TV(f'(u(T,\cdot))) \geq N(d-c)$ and for large N's this would contradict (9). In the next step we will modify this idea using half–lines ℓ which are close to the horizontal one.

Step 2 Let us make a linear change of coordinates by putting $\xi = x - \varepsilon t$. In these new coordinates the conservation law becomes

$$\partial_t[u+\varepsilon f(u)] + \partial_\xi[f(u)] \;=\; 0\,.$$

Note that for ε sufficiently small the function $g_\varepsilon(v) = v + \varepsilon f(v)$ is invertible in the range of u (the range of u is bounded). We define

$$f_\varepsilon :]-C_1, C_1[\,\to \mathbf{R} \qquad \text{as } f_\varepsilon(v) = f(g_\varepsilon^{-1}(v))\,,$$

and $w_\varepsilon \;=\; u+\varepsilon f(u)$, where C_1 is a suitable constant. Note that w_ε is a distributional solution of $\partial_t w_\varepsilon + \partial_\xi[f_\varepsilon(w_\varepsilon)] = 0$. Actually it is not difficult to see that w_ε is an *entropy* solution. Moreover the following straightforward computations show that the numbers of zeros of f_ε and f are the same (cf. (17) below). From $f_\varepsilon(g_\varepsilon(v)) = f(v)$ and $g'_\varepsilon(v) \;=\; 1+\varepsilon f'(v)$, we get

$$\begin{aligned} f'_\varepsilon(g_\varepsilon(v)) \;&=\; \frac{f'(v)}{1+\varepsilon f'(v)} \;=\; \frac{1}{\varepsilon}\left[1-\frac{1}{1+\varepsilon f'(v)}\right] \\ f''_\varepsilon(g_\varepsilon(v)) \;&=\; \frac{1}{g'_\varepsilon(v)}\left[\frac{f''(v)}{(1+\varepsilon f'(v))^2}\right] \;=\; \frac{f''(v)}{(1+\varepsilon f'(v))^3} \end{aligned} \tag{17}$$

We are in the conditions of applying Theorem 2.1 with w_ε in place of w and f_ε in place of f. In order to simplify the notation, we will use the following convention: If $S \subset \mathbf{R}^2$ is any segment and $g : \mathbf{R}^2 \to \mathbf{R}$, then $TV(g,S)$ is the total variation of the restriction of g to S.

Define S^ε_δ as the segment joining $y = (T,0)$ and the point $y + \delta(1,\varepsilon)/\sqrt{1+\varepsilon^2}$. Denote by h_ε the function $h_\varepsilon(v) \;=\; f'_\varepsilon(g_\varepsilon(v))$. Apply Theorem 2.1 to w_ε and f_ε in place of w and f and translate the BV estimate in the old coordinates (t, x). It is immediate to check that we get the following: There exists a constant K such that, if δ and ε are small enough, then

$$TV\big(h_\varepsilon(u), S^\varepsilon_\delta\big) \;\leq\; K\,. \tag{18}$$

Recall c, d, I_1 and I_2 defined in Step 1. Clearly, for ε sufficiently small we have

$$h_\varepsilon(I_1) \leq \kappa_1 < \kappa_2 \leq h_\varepsilon(I_2)\,. \tag{19}$$

Now choose N large enough so that $2N(\kappa_2 - \kappa_1) > K$ and select ε so that S^δ_ε contains $2N$ points $y_1, \ldots, y_{2N}$ satisfying (i) and (ii) of (Co) in the previous step. Then we would have $TV(h_\varepsilon(u(t,x)), S^\delta_\varepsilon) \geq 2N(\kappa_1 - \kappa_2) > K$, which contradicts (18).

4. Estimate for $\partial^2_v \mu$

Proposition 4.1. *Let u and f be as in Theorem 1.1 and let $y = (T, z) \in \mathbf{R}^+ \times \mathbf{R}$. There is a constant C_1 (depending on $\|u\|_\infty$, f and T) s.t.*

$$\|\partial^2_v \mu\|\big(\mathbf{R}_v \times B_{T/2}(y)\big) \;\leq\; C_1\,. \tag{20}$$

Proof. It is sufficient to prove (20) when $u \in BV_{loc}$. Indeed, assume that (20) holds for BV solutions and fix an entropy solution u. Choose a sequence $\{v_n\} \subset BV_{loc}(\mathbf{R})$ s.t.

$$v_n(\cdot) \to u(0,\cdot) \text{ in } L^1_{loc} \qquad \text{and} \qquad \|v_n\|_\infty \leq \|u\|_\infty\,.$$

Let u_n be the entropy solution of

$$\begin{cases} \partial_t u_n + \partial_x[f(u_n)] \;=\; 0 \\ \\ u_n(0,\cdot) = v_n(\cdot) \end{cases}$$

By the maximum principle, $\|u_n\|_\infty \leq \|v_n\|_\infty \leq \|u\|_\infty$. By the L^1 contraction principle (see Theorems 6.2.2 and 6.2.3 of [Da]), $u_n \in BV_{loc}(\mathbf{R}^2)$ and $u_n \to u$ in $L^1_{loc}(\mathbf{R}^+ \times \mathbf{R})$. Thus

$$\partial^2_v \mu_n \;=\; \partial_v(\partial_t \chi_n + f'(u)\partial_x \chi_n) \;\to\; \partial^2_v \mu \qquad \text{in the sense of distributions}\;.$$

Since $\|\partial^2_v \mu_n\|\big(\mathbf{R}_v \times B_{T/2}(y)\big) \leq C_1$, by semicontinuity of the total variation we get (20).

The case $u \in BV_{loc}$

For $u \in BV_{loc}$, we prove (20) using Vol'pert chain rule. Denote by J the jump set of u and by $\xi = (1, s)/\sqrt{1+s^2}$ the tangent to J. Then Vol'pert chain rule implies

$$\|\partial_x f'(u)\| \;\geq\; \frac{|f'(u^+) - f'(u^-)|}{\sqrt{1+s^2}} \mathcal{H}^1 \llcorner J\,. \tag{21}$$

We calculate μ using (7). Vol'pert chain rule gives $\mu = g(v, u^+, u^-, s)\mathcal{H}^1 \llcorner J$, with

$$g(v, u^+, u^-, s) \;=\; \frac{1}{\sqrt{1+s^2}} \Big\{\big[f(K(u^+, v)) - f(K(u^-, v))\big] - s\big[K(u^+, v) - K(u^-, v)\big]\Big\}\,.$$

Assume, for the sake of simplicity, that $u^+ > u^-$. Then

$$\begin{aligned} h(s, u^+, u^-, v) \;&:=\; [f(K(u^+, v)) - f(K(u^-, v))] - s[K(u^+, v) - K(u^-, v)] \\ &=\; [f(u^+) - f(v) - s(u^+ - v)]\mathbf{1}_{[u^+, u^-]}(v)\,. \end{aligned}$$

For each t, x, consider the function

$$h_{t,x}(v) \;=\; h(s(t,x), u^+(t,x), u^-(t,x), v)\,.$$

Clearly $h_{t,x} \in C^2([u^-,u^+])$. The Rankine–Hugoniot condition gives

$$s \;=\; \frac{f(u^+)-f(u^-)}{u^+-u^-}\,.$$

Hence $h_{t,x}(u^+) = h_{t,x}(u^-) = 0$. Note that

$$\partial^2_v \mu \;=\; \Big[h''_{t,x}(v) + h'_{t,x}(u^+)\delta_{u^+}(v) - h'_{t,x}(u^-)\delta_{u^-}(v)\Big]\mathcal{H}^1 \llcorner J\,.$$

Thus, set $B = B_{T/2}(y)$ and compute

$$\|\partial^2_v\mu\|(\mathbf{R}_v \times B) \;=\; \int_{J\cap B_{T/2}(y)} \left[\int_{u^-}^{u^+} |h''_{t,x}(v)|\,dv + |h'_{t,x}(u^+)| + |h'_{t,x}(u^-)|\right] d\mathcal{H}^1(t,x)\,. \tag{22}$$

To estimate (22), we split $J\cap B$ into two parts. Fix ε so small that

$$\Big\{\; v_1 \neq v_2,\; f''(v_1) = f''(v_2) = 0 \text{ and } |v_i| \leq \|u\|_\infty \;\Big\} \quad\Longrightarrow\quad |v_1 - v_2| > \varepsilon \tag{23}$$

and define the sets

$$\begin{array}{lcl} J^l & := & \Big\{(t,x) \in J\cap B:\; |u^+(t,x) - u^-(t,x)| > \varepsilon\Big\} \\ J^s & := & \Big\{(t,x) \in J\cap B:\; |u^+(t,x) - u^-(t,x)| \leq \varepsilon\Big\}\,. \end{array}$$

Clearly there is a $C(\varepsilon)$ such that, if $|u^+ - u^-| > \varepsilon$, then

$$\int_{u^-}^{u^+} |h''_{t,x}(v)|\,dv + |h'_{t,x}(u^+)| + |h'_{t,x}(u^-)| \;\leq\; C(\varepsilon)\int_{u^-}^{u^+} |h_{t,x}|(v)\,dv\,. \tag{24}$$

Thus

$$\|\partial^2_v\mu\|\,(\mathbf{R}_v \times J^l) \;\leq\; C(\varepsilon)\,\|\mu\|\,(\mathbf{R}_v\times J^l) \;\leq\; C(\varepsilon)\,\|\mu\|\,(\mathbf{R}_v \times B)\,. \tag{25}$$

Fix $(t,x) \in J^s$. Since $h''_{t,x}(v) = -f''(v)/\sqrt{1+s^2}$, (23) implies that $h''_{t,x}$ changes sign at most once in $[u^-,u^+]$. Recall that

$$h_{t,x}(u^-) \;=\; h_{t,x}(u^+) \;=\; 0$$

and that, since $\mu \geq 0$, we have $h_{t,x} \geq 0$ on $[u^-,u^+]$. All these conditions imply that $h''_{t,x} \leq 0$ on $[u^-,u^+]$ (which in turn implies $f'' \geq 0$). Moreover there exists a $v \in [u^-,u^+]$ such that $h'_{t,x}(v) = 0$. Thus

$$\begin{aligned} \int_{u^-}^{u^+} |h''_{t,x}(v)|\,dv \;+\; |h'_{t,x}(u^+)| \;+\; |h'_{t,x}(u^-)| \;&\leq\; 3\int_{u^-}^{u^+} |h''_{t,x}(v)|\,dv \\ &=\; \frac{3}{\sqrt{1+s^2}}\int_{u^-}^{u^+} f''(v)\,dv \;=\; \frac{3(f'(u^+) - f'(u^-))}{\sqrt{1+s^2}}\,. \end{aligned} \tag{26}$$

(21) implies that

$$\big\|\partial^2_v\mu\big\|\;(\mathbf{R}_v \times J^s) \;\leq\; 3\,\big\|\partial_x[f'(u)]\big\|\,(J^s) \;\leq\; 3\,\big\|\partial_x[f'(u)]\big\|\,(B)\,. \tag{27}$$

Adding (25) and (27) we get

$$\|\partial^2_v\mu\|\,(\mathbf{R}_v \times B) \;\leq\; C(\varepsilon)\,\big\|\mu\big\|\,(\mathbf{R}_v\times B) \;+\; 3\,\|\partial_x[f'(u)]\|\,(B)\;.$$

The first part of the right hand side is bounded by a constant depending only on $\|u\|_\infty$ and f. The second part can be bounded using Theorem 2.1. This concludes the proof. $\square$

5. Blow–up of measures

Let μ, u and χ be as in Subsection (1.3). We denote by ν the x,t–marginal of $\|\partial_v^2\mu\|$, i.e. the measure of $\mathcal{M}(\mathbf{R}^2)$ defined as

$$\nu(A) \;:=\; \|\partial_v^2\mu\|(\mathbf{R}_v\times A) \qquad \text{for all Borel sets } A\subset\mathbf{R}^2\,. \tag{28}$$

Note that we can give a "pointwise"–in v meaning to the measure μ. More precisely, thanks to equation (7), the distribution

$$\mu_v \;:=\; \partial_t K(v,u(t,x)) + \partial_x\Big[f\big(K(v,u(t,x))\big)\Big] \tag{29}$$

is a measure for each v and

$$\int \varphi(v,t,x)\,d\mu(v,t,x) \;=\; \int_{\mathbf{R}}\left[\int_{\mathbf{R}^2}\varphi(v,t,x)\,d\mu_v(t,x)\right]dv \qquad \text{for all } \varphi\in C_c(\mathbf{R}^3)\,. \tag{30}$$

Lemma 5.1. *There exists a bounded Borel function g s.t. $\mu(v,t,x)=g(v,t,x)\nu$ and*

$$\textit{for } \nu\textit{–a.e. } (t,x),\; \partial_v^2 g(\cdot,t,x) \textit{ is a measure on } \mathbf{R}_v \textit{ with } \|\partial_v^2 g(\cdot,t,x)\|_{\mathcal{M}(\mathbf{R}_v)}=1\,. \tag{31}$$

Proof. Fix $v>0$ and a ball $B\subset\mathbf{R}^2$. Take a sequence of functions $\{\varphi_n\}\subset C^\infty_c(\mathbf{R})$, with

$$\|\varphi_n\|_{L^1} \;=\; 1 \qquad\text{and}\qquad \varphi_n\to\delta_v \text{ in the sense of distributions}\,.$$

Choose a sequence

$$\{\Phi_m\}\subset C^\infty_c(B) \quad \text{with } \Phi_m\uparrow \mathbf{1}_B \text{ pointwise everywhere.}$$

Using (30), (29) and the nonnegativity of μ_v, we easily get

$$\mu_v(B) \;=\; \lim_{m\uparrow\infty}\lim_{n\uparrow\infty}\int \varphi_n(v)\Phi_m(t,x)\,d\mu(v,t,x)\,. \tag{32}$$

Recall that $\|u\|_\infty<\infty$ by assumption and that $\mu_v\equiv 0$ for $v\notin[-\|u\|_\infty,\|u\|_\infty]$. Choose ψ_n such that $\psi_n''=\varphi_n$ and $\|\psi_n\|_{L^\infty(I)}\leq 2\|u\|_\infty$. Then we have

$$\left|\int \varphi_n(v)\Phi_m(t,x)\,d\mu(v,t,x)\right| \;=\; \left|\int \psi_n(v)\Phi_m(t,x)\,d[\partial_v^2\mu](v,t,x)\right| \;\leq\; 2\|u\|_\infty\nu(B)\,. \tag{33}$$

Combining (32) and (33) we conclude

$$\mu_v(B) \;\leq\; 2\|u\|_\infty\nu(B)\,.$$

By the arbitrariness of B and by Radon–Nykodim Theorem, $\mu_v=g_v(t,x)\nu$ for some $g_v\in L^1(\mathbf{R}^2,\nu)$. We set $g(v,t,x)=g_v(t,x)$, getting $\mu=g(v,t,x)\nu$. Clearly,

$$\int_U\int_{\mathbf{R}}|g(v,t,x)|\,dv\,d\nu(t,x) \;=\; \|\mu\|(\mathbf{R}\times U) \;<\; \infty \qquad \text{for every bounded set } U.$$

Thus the function $g_{t,x}(v):=g(v,t,x)$ is in $L^1(\mathbf{R},\mathscr{L}^1)$ for ν–a.e. (t,x). Hence the distribution $g''_{t,x}\in\mathcal{D}'(\mathbf{R}_v)$ is well defined (for ν–almost every (t,x)) and

$$\int \Psi\,d[\partial_v^2\mu] \;=\; \int_{\mathbf{R}^2}\big\langle \Psi(\cdot,t,x), g''_{t,x}(\cdot)\big\rangle\,d\nu(t,x) \tag{34}$$

for every $\Psi \in C_c^\infty(\mathbf{R}^3)$. Since ν is the t,x marginal of $\|\partial_v^2\mu\|$, standard theorems in the disintegration of measures (see for instance [AFP]) imply the existence of a map $\xi : \mathbf{R}^2 \ni (t,x) \to \xi_{t,x} \in \mathcal{M}(\mathbf{R})$ such that:

$$\|\xi_{t,x}\|_{\mathcal{M}(\mathbf{R})} \;=\; 1 \text{ for } \nu\text{–a.e. } (t,x); \tag{35}$$

$$\int \Psi\, d[\partial_v^2\mu] \;=\; \int_{\mathbf{R}^2}\int_{\mathbf{R}} \Psi(v,t,x)\, d\xi_{t,x}(v)\, d\nu(t,x) \qquad \text{for every } \Psi \in C_c^\infty(\mathbf{R}^3). \tag{36}$$

Comparing (34) and (36) we get easily that $g''_{t,x} = \xi_{t,x}$ for ν–a.e. t,x. □

We now want to study a particular class of rescalings of the measure μ. We first set a bit of notation on tangent measures

Definition 5.2. *Let $\nu \in \mathcal{M}(\mathbf{R}^2)$, $\mu \in \mathcal{M}(\mathbf{R}_v \times \mathbf{R}^2)$ and $y \in \mathbf{R}^2$. We define the measures $\nu^{y,r}$, $\mu^{y,r}$ as*

$$\nu^{y,r}(A) \;=\; \frac{\nu(y+rA)}{\nu(B_r(y))} \qquad \textit{for all bounded Borel sets } A \subset \mathbf{R}^2$$

$$\mu^{y,r}(C\times A) \;=\; \frac{\mu(C\times(y+rA))}{\mu(\mathbf{R}\times B_r(y))} \qquad \textit{for all bounded Borel sets } A \subset \mathbf{R}^2, C \subset \mathbf{R}_v\,.$$

The sets of tangent measures $T(y,\nu)$ (resp. $T(y,\mu)$) are defined as the limits of all sequences $\{\nu^{y,r_n}\}_{r_n\downarrow 0}$ (resp. $\{\mu^{y,r_n}\}_{r_n\downarrow 0}$) which are convergent in the sense of measures.

We come to the main goal of this section.

Proposition 5.3. *Let ν, μ and g be as in Lemma 5.1. For every $y=(t,x)$ denote by ξ_y the measure $g_y(v)\mathscr{L}^1$ of $\mathcal{M}(\mathbf{R})$. Then there is a Borel set A with $\nu(\mathbf{R}^2\setminus G)=0$ such that for every $y \in A$ the following holds:*

$$\textit{If } \nu^\infty \in T(y,\nu) \textit{ then the } \text{product measure } \xi_y\times\nu^\infty \textit{ is in } T(y,\mu); \tag{37}$$

$$\textit{If } \mu^\infty \in T(y,\mu)\textit{, then there is } \nu^\infty \in T(y,\nu) \textit{ such that } \mu^\infty = \xi_y\times\nu^\infty\,. \tag{38}$$

Warning 5.4. *We stress on the fact that $\xi_y\times\nu^\infty$ is a product, that is*

$$\int \varphi(v)\psi(t,x)\, d[\xi_y\times\nu^\infty](v,t,x) \;=\; \int \varphi(v)\, d\xi_y(v) \int \psi(t,x)\, d\nu^\infty(t,x)\,.$$

Proof. First of all select a countable set $\{\varphi_n\} \subset C_c(\mathbf{R})$ which is dense in the uniform topology on compact subsets. We define the functions

$$\omega(t,x) \;:=\; \|g_{t,x}\|_{L^1} \qquad \omega_k(t,x) \;:=\; \int \varphi_k(v) g_{t,x}(v)\, dv\,.$$

We define the set

$$G = \{y|\, y \text{ is a } \nu\text{–Lebesgue point for } \omega \text{ and } \omega_k, \text{ and } \omega(y)\neq 0\}\,.$$

Thanks to Lemma 5.1 we have $\nu(\mathbf{R}^2\setminus G)=0$. We prove only (37), the proof of (38) being analogous. Fix $y \in G$ and $\nu^\infty \in T(y,\nu)$. Thus there exists a sequence ν^{y,r_n} of rescaled measures converging to ν^∞. Let $\Phi \in C_c(\mathbf{R}^2)$. Note that

$$\int \varphi_k(v)\Phi(t,x)\, d\mu^{y,r_n}(v,t,x) \;=\; \int \omega_k(t,x)\Phi(t,x)\, d\nu^{y,r_n}(t,x) \tag{39}$$

and that, since y is ν–Lebesgue point for ω_k,

$$\lim_{r_n\downarrow 0}\int \big|(\omega_k(t,x)-\omega_k(y))\Phi(t,x)\big|\,d\nu^{y,r_n}(t,x) \;=\; 0\,,$$

$$\begin{aligned}\lim_{r_n\downarrow 0}\int \omega_k(t,x)\Phi(t,x)\,d\nu^{y,r_n}(t,x)(t,x) \;&=\; \lim_{r_n\downarrow 0}\int \omega_k(y)\Phi(t,x)\,d\nu^{y,r_n}(t,x)\\ &=\; \int \omega_k(y)\Phi(t,x)\,d\nu^{\infty}(t,x)\,. \end{aligned}\tag{40}$$

Choose a subsequence of $\{r_n\}$ such that μ^{y,r_n} has a limit μ^∞. Being y a ν–Lebesgue point for ω, we have that that

$$\lim_{r_n\downarrow 0}\frac{\mu(\mathbf{R}\times B_{r_n}(y))}{\nu^\infty(B_{r_n}(y))} \;=\; \omega(y)\;\neq\; 0\,. \tag{41}$$

Then (40) and (41) imply

$$\int \varphi_k(v)\Phi(t,x)\,d\mu^\infty(v,t,x) \;=\; \int \varphi_k(v)d\xi_y(v)\int \Phi(t,x)\,d\nu^\infty(t,x)\,. \tag{42}$$

Recall that $\{\varphi_k\}$ is dense in $C_c(\mathbf{R})$. Hence, (42) holds for every $\varphi\in C_c(\mathbf{R})$ in place of φ_k. The arbitrariness of φ and Φ gives (37). $\square$

6. An averaging lemma

In this section we prove an averaging lemma which will be used in the proof of point (c) of Theorem 1.1.

Lemma 6.1. *Let $F_n:\mathbf{R}_v\times\mathbf{R}_t\times\mathbf{R}_x\to\mathbf{R}$ be L^1 solutions of the transport equations*

$$\left\{\begin{array}{l} \partial_t F_n + f'(v)\partial_x F_n \;=\; \partial_v\mu^n\\[2ex] F(v,0,x)\;=\;0\,. \end{array}\right. \tag{43}$$

Assume that

- *$F_n,\mu^n\equiv 0$ on $(\mathbf{R}\setminus L)\times\mathbf{R}^2_{t,x}$ for some bounded interval L;*
- *$\partial^2_v\mu^n$ are all Radon measures;*
- *$\|\partial^2_v\mu^n\|(\mathbf{R}\times U)$ is a bounded sequence for every bounded open set $U\subset\subset\mathbf{R}^2$.*

Let I be an interval such that $\inf_I|f''|>0$ and let $\psi\in C^\infty_c(I)$. Then:

$$\|F_n\|_{L^1_{loc}(\mathbf{R}\times U)}\quad \textit{is a bounded sequence for every } U\subset\subset\mathbf{R}^2. \tag{44}$$

$$\textit{The functions }\;\Xi_n(t,x)\;:=\;\int\psi(v)F_n(v,t,x)\,dv\;\textit{ are weakly precompact in } L^1_{loc} \tag{45}$$

Proof

6.1. L^1 **bound.** In this subsection we prove (44). Choose balls $B \subset B' \subset \mathbf{R}^2$. Since $\partial_v^2 \mu^n$ is a measure and $\mu^n \equiv 0$ on $(\mathbf{R} \setminus L) \times \mathbf{R}^2_{t,x}$, it is immediate to check that $\|\partial_v \mu^n\|(\mathbf{R} \times B')$ is bounded. The ball B' will be chosen later.

By standard arguments (e.g. using convolution kernels in t, x) for every n we can find L^1 functions G_n and g_n satisfying the following conditions:

- For $\mathscr{L}^1$–a.e. v, the functions $G_n(v,\cdot), g_n(v,\cdot) \in C^\infty(\mathbf{R}^2)$ and satisfy the transport equation
$$\begin{cases} \partial_t G_n + f'(v)\partial_x G_n \;=\; g_n \\ G_n(v,0,x) = 0 \end{cases} \tag{46}$$
- $\|F_n - G_n\|_{L^1(\mathbf{R}\times B')} \leq \frac{1}{n}$ and $\|g_n\|_{L^1\mathbf{R}\times B')} \leq \|\partial_v \mu^n\|(\mathbf{R} \times B') + \frac{1}{n}$.

Since $g_n(v,\cdot)$ is smooth we can explicitly compute
$$G_n(v,t,x) \;=\; \int_0^t g_n\big(x + (\tau - t)f'(v), \tau, v\big)\, d\tau\,. \tag{47}$$
Take the absolute value and integrate in t and x. Recall that $f' \in C^1(L)$ and thus is bounded on L. Then there exists a constant C such that, if the ball B' is large enough, then
$$\int_B |G_n(v,t,x)|\, dt\, dx \;\leq\; C\|\partial_v \mu^n(v,\cdot)\|_{L^1(B')} \qquad \text{for } v \in L. \tag{48}$$
Note also that the size of B' depends only on the size of B and on $\sup_L |f'|$. Integrating (48) in v and recalling that $G_n \equiv 0$ on $\mathbf{R} \setminus L$, we get $\|G_n\|_{L^1(\mathbf{R}\times B)} \leq C\|\partial_v \mu_n\|_{L^1(\mathbf{R}\times B')}$.

6.2. **Weak** L^1**–precompactness.** It remains to show that $\{\Xi_n\}$ is weakly precompact. Define g_n and G_n as in subsection 6.1. Our claim reduces to the local weak L^1 precompactness of the functions
$$\Omega_n(t,x) \;:=\; \int_I \psi(v) G_n(v,t,x)\, dv\,,$$
We restrict to a compact set of $\mathbf{R}^2$, say a ball B. To show the weak L^1–precompactness of Ω_n in B, it is sufficient to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that
$$\text{If } E \subset B \text{ satisfies } \mathscr{L}^2(E) < \delta, \text{ then we have } \lim_{n\uparrow\infty} \left| \int_E \Omega_n \right| \;\leq\; \varepsilon\,. \tag{49}$$
Recall that the g_n's are supported in $L' \times \mathbf{R}^2$ for some bounded L'. Thus, since the velocity of propagation of the transport equation is bounded, we can truncate g_n smoothly to 0 outside a compact set of $\mathbf{R}^2$, without affecting the value of Ω_n in the ball B. Hence, we assume that the g_n are supported in $I \times [-C, C]^2$, for some constant C.

We split E into that $E^+ = E \cap \{t \geq 0\}$ and $E^- = E \cap \{t \leq 0\}$. Since the estimate is the same, we only show the one for E^+ and for simplicity we drop the plus. Using (47) we compute
$$\int_A \Omega_n \;=\; \iint_{\mathbf{R}^2} \mathbf{1}_E(t,x) \int_{\mathbf{R}} \psi(v) \int_0^t g_n\big(v, \tau, x + (\tau - t)f'(v)\big)\, d\tau\, dv\, dx\, dt\,.$$
We rewrite the integral as
$$\iiiint \mathbf{1}_E(t,x)\, \mathbf{1}_{[0,t]}(\tau)\, \psi(v)\, g_n\big(v, \tau, x + (\tau - t)f'(v)\big)\, d\tau\, dv\, dx\, dt\,.$$

We change variable by putting $y = x + (\tau - t) f'(v)$ and we get

$$\iiiint \mathbf{1}_E\big(t, y + (\tau - t) f'(v)\big)\, \mathbf{1}_{[0,t]}(\tau)\, \psi(v)\, g_n(v,\tau,y)\, d\tau\, dv\, dy\, dt$$

We now integrate by parts in v and get

$$-\iiiint \left[\int_\eta^v \mathbf{1}_E\big(t, y + (t-\tau) w\big) dw\right] \mathbf{1}_{]0,t]}(\tau)\, \big[\psi'(v) g_n(v,\tau,y) + \psi(v) \partial_v g_n(v,\tau,y)\big]\, d\tau\, dv\, dy\, dt\,,$$

where η is the left endpoint of the interval I. The functions

$$\Theta_n(v,\tau,y) \;:=\; \big[\psi'(v) g_n(v,\tau,y) + \psi(v) \partial_v g_n(v,\tau,y)\big]\,.$$

are supported in a compact set $I \times [-C, C]^2$. Thus we rewrite the integral as

$$\iiint \left[\int \left(\int_\eta^v \mathbf{1}_E\big(t, y + (t-\tau) w\big) dw\right) \mathbf{1}_{]\tau,C]}(t)\, dt\right] \Theta_n(v,\tau,y)\, dy\, d\tau\, dv\,.$$

Recall that the L^1 norm of the Θ_n's is bounded. Thus, if we define the functions

$$\Psi(v,\tau,y) \;:=\; \int_\tau^C \int_\eta^v \mathbf{1}_E\big(t, y + (t-\tau) f'(w)\big)\, dw\, dt\,. \tag{50}$$

we just need to prove that for any $\varepsilon > 0$, there exists δ s.t.:

$$\mathscr{L}^2(E) \;\le\; \delta \quad\Longrightarrow\quad \sup_{(v,\tau,y)\in I\times[-C,C]^2} |\Psi(v,\tau,y)| \;\le\; \varepsilon\,. \tag{51}$$

Since the sets E and $E + (0, y)$ have the same area, it suffices to show (51) when $y = 0$. By changing coordinates with $\sigma = t - \tau$, this reduces to estimating

$$\sup_{v\in I, \tau\in[-C,C]} \int_0^{C-\tau} \int_\eta^v \mathbf{1}_E(\sigma + \tau, \sigma f'(w))\, dw\, d\sigma\,. \tag{52}$$

Hence, it is sufficient to bound

$$\sup_{v\in I} \int_0^{2C} \int_\eta^v \mathbf{1}_E(\sigma + \tau, \sigma f'(w))\, dw\, d\sigma\,.$$

Since E and $E + (\tau, 0)$ have the same area, it suffices to bound

$$\sup_{v\in I} \int_0^{2C} \int_\eta^v \mathbf{1}_E(\sigma, \sigma f'(w))\, dw\, d\sigma\,. \tag{53}$$

Recall that $\inf_I |f''| \ge \kappa > 0$. Thus we can change variable by putting $z = \sigma f'(w)$, getting

$$\sup_{wvinI} \kappa^{-1} \int_0^{2C} \left| \int_{\sigma f'(\eta)}^{\sigma f'(v)} \mathbf{1}_E(\sigma, z)\, \frac{dz}{\sigma} \right| d\sigma\,.$$

We split E into two parts: $E_\lambda := E \cap \{\sigma < \lambda\}$ and $E \setminus E_\lambda$. Then

$$\begin{aligned}
\kappa^{-1} \int_0^{2C} \left| \int_{\sigma f'(\eta)}^{\sigma f'(v)} \mathbf{1}_{E_\lambda}(\sigma, z)\, \frac{dz}{\sigma} \right| d\sigma \;&\le\; \kappa^{-1} \int_0^{2C} \left| \int_{\sigma f'(\eta)}^{\sigma f'(v)} \mathbf{1}_{\{\sigma<\lambda\}}(\sigma, z)\, \frac{dz}{\sigma} \right| d\sigma \\
&\le\; \frac{\lambda}{\kappa} \sup_{v\in I} |f'(v) - f'(\eta)| \;=\; C_1 \lambda\,.
\end{aligned}$$

Whereas,

$$\kappa^{-1}\int_0^{2C}\left|\int_{\sigma f'(\eta)}^{\sigma f'(v)}\mathbf{1}_{E\setminus E_\lambda}(\sigma,z)\,\frac{dz}{\sigma}\right|d\sigma\;\leq\;\frac{\mathscr{L}^2(E)}{\kappa\lambda}$$

Thus, for every $\varepsilon>0$, we first choose λ so that $C_1\lambda\leq\varepsilon/2$ and then we choose δ such that $\delta/(\kappa\lambda)\leq\varepsilon/2$. Clearly, $\mathscr{L}^2(E)<\delta$ implies

$$\sup_{v\in I}\kappa^{-1}\int_0^{2C}\int_{\sigma f'(\eta)}^{\sigma f'(v)}\mathbf{1}_E(\sigma,z)\,\frac{dz}{\sigma}d\sigma\;\leq\;\varepsilon\,,$$

which gives (51). This completes the proof. □

7. Concentration–rectifiability

We now come to the proof of (c) of Theorem 1.1. Recall the definition of the (convex) functions $K(v,\cdot):\mathbf{R}\to\mathbf{R}^+$ given by (6). Define the set Kr as the pairs (η,q) such that there exist real numbers $v_1,\ldots,v_n$, $\alpha_1,\ldots,\alpha_n$ such that

$$\eta(\cdot)\;:=\;\sum_{i=1}^n\alpha_iK(v_i,\cdot)\qquad q(\cdot)\;=\;\sum_{i=1}^n\alpha_if\big(K(v_i,\cdot)\big)\,.$$

It is not difficult to see that for any convex entropy–entropy flux pair (η,q) there is a sequence $\{(\eta_i,q_i)\}\subset\{\mathrm{Kr}\}$ such that $\eta_i\to\eta$ and $q_i\to q$ uniformly on compact sets. Thus it is enough to prove that (c) holds for the entropies of $\{\mathrm{Kr}\}$. By linearity, it is sufficient to prove (c) for $(K(v,\cdot),f(K(v,\cdot)))$ for each v. Thanks to (c'), it is sufficient to show that each μ_v of (29) is concentrated on J. Recall that

$$\partial_t\chi+f'(v)\partial_x\chi\;=\;\partial_v\mu\,.$$

Thanks to Lemma 5.1 (and to the continuity in v of $K(v,\cdot)$) we only need to show that ν is concentrated in J, where ν is the x,t–marginal of μ (see Section 5).

7.1. Setting and blow–up.

We argue by contradiction using a blow–up argument. Let A be the set of Proposition 5.3. If ν is not concentrated on J, then there exists $y\in A\setminus J$ such that $T(y,\mu)\neq\{0\}$. From (a) in Theorem 1.1, we know that

$$y\text{ is a Lebesgue point for }u.\tag{54}$$

Without loss of generality, assume that

$$y\;=\;0\qquad\text{and}\qquad u(0)\;=\;1\,.\tag{55}$$

So fix a $\nu^\infty\in T(0,\nu)$ which is non–trivial and a sequence $r_n\downarrow 0$ such that $\nu^{0,r_n}\to\nu^\infty$, in the sense of measures. Thanks to Proposition 5.3,

$$\mu^{0,r_n}\text{ converge to }g_0(v)\mathscr{L}^1\times\nu^\infty(t,x).\tag{56}$$

Moreover, since by Lemma 5.1 $g''_{t,x}$ is a measure for ν–a.e. A, without loosing our generality we can assume that g''_0 is a measure. Let us go back to the kinetic equation $\partial_t\chi+f'(v)\partial_x\chi=\partial_v\mu$. We make a radial change of coordinates $(t,x)\to(r_nt,r_nx)$. We denote by χ_n the function χ in the rescaled coordinates, that is, $\chi_n(v,t,x):=\chi(v,r_nt,r_nx)$ and for simplicity we put $\mu^n=\mu^{0,r_n}$. Then, we can rewrite the kinetic equation as

$$\partial_t\frac{\chi_n}{\alpha_n}+f'(v)\partial_x\frac{\chi_n}{\alpha_n}\;=\;\partial_v\mu^n\,,\tag{57}$$

where α_n are suitable constants.

7.2. **Comparison with the free transport.** Since g_0'' is a measure (and is supported on a compact set), g_0' is BV. Hence, g_0' is continuous except for an (at most) countable set. Moreover $g_0 \neq 0$, otherwise $T(0,\mu)$ would be the trivial set $\{0\}$. Thus we can fix an interval I such that

$$g_0' \neq 0 \text{ on } I. \tag{58}$$

For the sake of simplicity, assume

$$g_0' < 0 \text{ on } I = [\eta, \xi] \text{ and } 0 < \eta < \xi < 1 = u(0) \tag{59}$$

(it is easy to see that in the other cases we can argue similarly). Since f'' vanishes finitely many times, we can assume

$$\inf_I |f''| \ > \ 0\,. \tag{60}$$

Finally, without loosing our generality, we can impose that

$$\nu^\infty \text{ is non trivial in the ball } B_1(0), \text{ that is } \nu^\infty(B_1(0)) \ > \ 0. \tag{61}$$

Recall that the χ_n are the characteristic functions of sublevel sets of rescalings of the initial function u. Thus, using Fubini–Tonelli Theorem and the monotonicity in v of χ_n we have:

$$\text{For almost every } T < -1,\ \chi_n(v,\cdot) \text{ has a trace on the line } \{t = T\} \text{ for each } v. \tag{62}$$

T will be chosen later so to fulfill appropriate requirements (see subsection 7.3). We denote by χ_n^f the solution of the free transport equation

$$\left\{\begin{array}{rcl} \partial_t \chi_n^f + f'(v)\partial_x \chi_n^f &=& 0 \\ \\ \chi_n^f(v,T,x) &=& \chi_n(v,T,x)\,. \end{array}\right. \tag{63}$$

We define the functions

$$F_n(v,t,x) \ := \ \frac{\chi_n(v,t,x) - \chi_n^f(v,t,x)}{\alpha_n}$$

and note that they solve the transport equation

$$\left\{\begin{array}{rcl} \partial_t F_n + f'(v)\partial_x F_n &=& \partial_v \mu^n \\ \\ F(v,T,x) &=& 0\,. \end{array}\right. \tag{64}$$

7.3. **Contradiction.** In the next subsection we will prove that there is a subsequence $n(k)$ such that

$$\text{On } I \times \mathbf{R}^2, \text{ the } F_{n(k)}\text{'s converge, in the sense of measures, to a nonnegative } \omega. \tag{65}$$

Here we show how (65) yields a contradiction. Fix a segment a on $\{t = T\}$ and a line $\ell = \{t = T'\}$. Both T, T' and a will be chosen later. For each w, consider the two adjacent segments (say b_w and d_w) parallel to the vector $(1, f'(w))$, starting at the endpoints of a and ending when they meet ℓ. Finally we denote by c_w the segment of ℓ which, together with a_w, b_w and c_w, forms a parallelogram P_w (see Fig. 1).

Denote by $\eta < \xi$ the two endpoints of I and consider the three–dimensional $S := \bigcup_{w\in]\eta,\xi[} P_w$. The set S is bounded by the four planes $\{t = T\}$, $\mathbf{R}\times\ell$, $\{v = \eta\}$ and $\{v = \xi\}$

and by two ruled surfaces Γ_1 and Γ_2. We first choose a nonnegative function $\varphi \in C^1(\overline{S})$ with the following properties:

$$\varphi = 0 \text{ in a neighborhood of } \mathbf{R} \times \ell \text{ and is constant in a neigh. of } \{t = T\}; \tag{66}$$
$$(\partial_t + f'(w)\partial_x)\varphi \leq 0 \text{ everywhere on } S; \tag{67}$$
$$\varphi \geq 1 \text{ on } I \times B_1(0). \tag{68}$$

It is easy to construct φ "slice–by–slice", i.e. constructing each $\varphi(v, \cdot) \in C^1(P_v)$, provided that: $\{t = T\}$ and $\ell = \{t = T'\}$ are sufficiently far from $B_1(0)$ and a is sufficiently large; see Fig. 1. This choice can be clearly made (recall that a.e. $T < -1$ satisfies the trace–condition (62)).

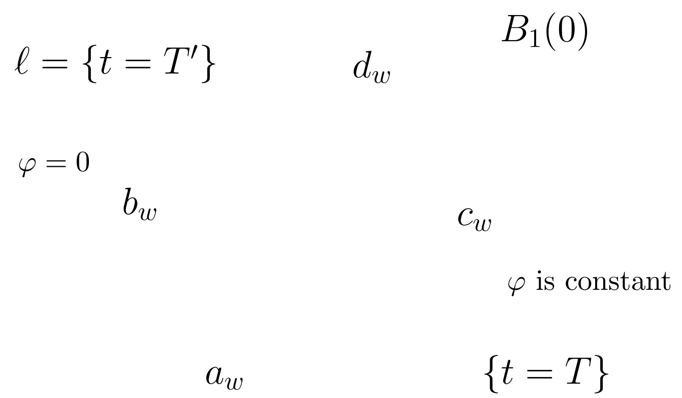

FIGURE 1. The parallelogram P_w and the shape of (a typical) φ on P_w. In the rectangular region φ grows from 0 to a constant and depends only on $t + f'(w)x$

Next, we choose a nonnegative function $\psi \in C^1(\overline{S})$ such that

$$\psi = 0 \text{ on } \Gamma_1 \text{ and } \Gamma_2 \text{ and } \psi = 1 \text{ on } I \times B_1(0)$$
$$(\partial_t + f'(w)\partial_x)\psi = 0 \text{ everywhere on } S.$$

Moreover, we fix a smooth nonnegative bump function ζ supported on I and equal to 1 on some interval L. Thus, the nonnegative function $\Phi = \zeta\psi\varphi \in C^1(\overline{S})$ satisfies the following conditions:

$$(\partial_t + f'(w)\partial_x)\Phi \leq 0 \text{ on } S \text{ and } (\partial_t + f'(w)\partial_x)\Phi = 0 \text{ on } \partial S \tag{69}$$
$$\Phi = 0 \text{ in a neighborhood of } \partial S \setminus \{t = T\} \text{ and } \Phi = 1 \text{ on } L \times B_1(0) \tag{70}$$

Finally, we claim that T can be chosen so that

$$\nu^\infty(\{t = T\}) \;=\; 0\,. \tag{71}$$

Since the T's for which $\nu^\infty(\{t = T\})$ are countably many, this is certainly possible.

Test (29) with the function Φ. Since Φ vanishes on a neighborhood of $\partial S \setminus \{t = T\}$ and $F_n = 0$ on $\{t = T\}$, we can integrate by parts and get

$$-\int_S \Big[\big(\partial_t + f'(v)\partial_x\big)\Phi(v,t,x)\Big]\, F_n(v,t,x)\, dw\, dt\, dx \;=\; \int_S \Phi(v,t,x)\, d(\partial_v \mu^n)(v,t,x)\,. \tag{72}$$

Since $(\partial_t + f(v)\partial_x)\Phi$ vanishes in a neighborhood of ∂S, thanks to (65), we can pass to the limit in the left hand side and we conclude that this limit is

$$\int_S \Big[- \big(\partial_t + f'(v)\partial_x\big)\Phi\Big]\, d\omega(v,t,x)\,. \tag{73}$$

Since ω is a nonnegative measure and the integrand in (73) is nonnegative, the number (73) is nonnegative.

Note that $\partial_v \mu^n$ converges, in the sense of measure, to $\partial_v\mu^\infty = g_0' \mathscr{L}^1 \times \nu^\infty$. Moreover, by (71), $\nu^\infty(\{t=T\}) = 0$, whereas Φ vanishes in a neighborhood of $\partial S \setminus \{t=T\}$. By classical theorems on the weak convergence of measures, these conditions imply that the right hand side of (72) converges to

$$\int_S \Phi\, d[\partial_v \mu^\infty]\,. \tag{74}$$

Recall that, because of (59), $\partial_v\mu^\infty$ is a nonpositive measure on S and that, by (61), we have $\partial_v\mu^\infty(L\times B_1(0)) < 0$ for every interval $L\subset I$. For one such interval, we have $\Phi = 1$ on $L\times B_1(0)$. Since $\Phi\geq 0$, this implies that (74) is a negative number. By (72), (73) should be equal to (74), which is a contradiction.

7.4. F_N **converge to a nonnegative measure on** $I\times\mathbf{R}^2$**.** It remains to show (65). Since (60) holds, we can apply Lemma 6.1 to get

$$\|F_n\|_{L^1_{loc}(\mathbf{R}\times U)} \quad \text{is a bounded sequence for every } U\subset\subset \mathbf{R}^2. \tag{75}$$

Thanks to (75) we can extract a subsequence which is converging in the sense of measures to a measure ω. Fix a nonnegative $\psi\in C^\infty_c(I)$. Again thanks to Lemma 6.1 we have:

$$\Xi_n(t,x) \;:=\; \int \psi(v) F_n(v,t,x)\, dv \quad \text{are weakly precompact in } L^1_{loc}. \tag{76}$$

We will show below that this implies:

$$\text{If } \Xi_\infty \text{ is limit of a subsequence of } \Xi_n, \text{ then } \Xi_\infty \geq 0\,. \tag{77}$$

Note that (77) gives $\int \psi(v)\varphi(t,x)\, d\omega(v,t,x)\geq 0$ for all nonnegative functions $\psi\in C^\infty_c(I)$, $\varphi\in C^\infty_c(\mathbf{R}^2)$. By a standard density argument, we get $\int \Phi d\omega \geq 0$ for every $\Phi\in C_c(I\times\mathbf{R}^2)$. This gives (65). We now come to the proof of (77). Recall the following facts

$$F_n \;=\; (\chi_n - \chi^f_n)/\alpha_n \tag{78}$$

$\chi_n(v,\cdot)$ is the characteristic of the v–sublevel of a suitable rescaling of u; (79)

χ^f_n is defined via (63); thus its range is contained in $\{0,1\}$; (80)

0 is a Lebesgue point for u, $I=[\eta,\xi]$ and $0<\eta<\xi<1=u(0)$. (81)

Define the set

$$A_n \;:=\; \{x\in\mathbf{R}^2 | \chi_n(\eta,t,x)\geq 1\}$$

and fix any compact set $K\subset\mathbf{R}^2$. (79) and (81) imply that

$$\mathscr{L}^2(K\setminus A_n)\;\downarrow\; 0 \quad \text{for } n\uparrow\infty.$$

Moreover (79) implies $\chi_n(v,\cdot)\leq\chi_n(w,\cdot)$ for every $0<v\leq w$. Hence $\chi_n(v,\cdot)=1$ on A_n for *every* $v\in I$. This, together with (78) and (80), implies

$$F_n(v,t,x)\;\geq\; 0 \qquad \text{for every } v\in I \text{ and every } (t,x)\in A_n.$$

Hence $\Xi_n \geq 0$ on A_n. Thanks to the weak L^1–precompactness of $\{\Xi_n\}$ we have

$$\lim_n \int_{A_n} \left|\Xi_n(t,x)\right| dtdx \;=\; 0.$$

This implies $\Xi_\infty \geq 0$ for any Ξ_∞ which is limit of a subsequence of $\{\Xi_n\}$.

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