A Lower-epiperimetric Inequality for Area Minimizing Surfaces .

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Abstract

The Epiperimetric inequality introduced by E.R.Reifenberg in [Re] gives a rate of decay, at a point, for the decreasing $k-$density of area of an area minimizing integral $k-$cycle. While dilating the cycle at that point this rate of decay holds once the configuration is close to a tangent cone configuration and *above* the limiting density corresponding to that configuration. This is why we propose to call the Reifenberg Epiperimetric inequality an *"Upper-epiperimetric"* inequality. A direct consequence of this Upper-epiperimetric inequality is the statement that any point posses a unique tangent cone. The Upper-epiperimetric inequality was proved by B.White in [Wh] for area minimizing $2-$cycles in $\mathbb{R}^n$. In the present paper we introduce the notion of *"Lower-epiperimetric"* inequality. This inequality gives this time a rate of decay for the decreasing $k-$density of area of an area minimizing integral $k-$cycle, while dilating the cycle at a point, once the configuration is close to a tangent cone configuration and *below* the limiting density corresponding to that configuration. Our main result in the present paper is to prove the Lower-epiperimetric inequality for area minimizing $2-$cycles in $\mathbb{R}^n$. As a consequence of this inequality we prove the "splitting before tilting" phenomenon for calibrated 2-rectifiable cycles which plays a crucial role in the proof of the regularity of 1-1 integral currents in [RT].

1 Introduction

We adopt the standard notations of Geometric Measure Theory (see [Fe]) for instance $M(B)$ for the mass of the current B, $\mathcal{F}(A)$ for the flat norm of a current A (recall $\mathcal{F}(A) = \inf\{M(S) + M(R) \text{ s.t. } A = \partial S + R\}$)...etc. let μ_r be the dilation in $\mathbb{R}^n$ with ratio $1/r$ we shall denote Σ_r the push-forward of a current Σ by the map μ_r (i.e. the dilation of Σ with coefficient $1/r$). The following result is the **upper-epiperimetric inequality** for minimizing 2-rectifiable current proved by B. White

Theorem 1.1 *[Wh] Given an integer Q, there exists a positive number $\varepsilon_Q > 0$, such that, for any Σ area minimizing 2-rectifiable integral current in $B_2^n(0)$, satisfying $\partial\Sigma \llcorner B_2^n = 0$, assuming there exists Q flat oriented 2-disks $D_1 \dots D_Q$ passing through zero such that*

$$\mathcal{F}\left(\Sigma_2 \llcorner B_1^n - \sum_{i=1}^{Q}[D_i]\right) \le \varepsilon_Q \quad , \tag{1.1}$$

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(where $[D_i]$ is the integral on the oriented 2-disks D_i) then

$$M(\Sigma\llcorner B_1^n)-\pi Q\le(1-\varepsilon_Q)\ \left(\frac{1}{2}M\left(\partial(\Sigma\llcorner B_1^n)\right)-\pi Q\right)\quad . \tag{1.2}$$

As explained in [Wh], this epiperimetric inequality, first introduced by E.Reifenberg [Re], give a rate of decay of the mass of the rescaled current $(\Sigma_r)\llcorner B_1^n$ as r tends to zero. Recall that the function

$$r\longrightarrow M(\Sigma_r\llcorner B_1^n)=\frac{M(\Sigma\llcorner B_r^n)}{r^{n-2}}$$

is increasing for a minimizing 2-current and converges to π times an integer. Namely, it is proved in [Wh]

Corollary 1.2 *Let Σ be a minimizing 2-rectifiable integral current in $B_2^n(0)$, and let*

$$\pi Q=\lim_{r\rightarrow 0}\ M\left(\Sigma_r\llcorner B_1^n\right)\quad .$$

Then there exist two positive constants $c>0$ and $\gamma>0$ such that, for r small enough

$$\frac{M\left(\Sigma\llcorner B_r^n\right)}{r^2}-\pi Q=M\left(\Sigma_r\llcorner B_1^n\right)-\pi Q\le cr^{\gamma}\quad . \tag{1.3}$$

Note that this epiperimetric inequality, which gives a rate of increase of the **upper excess** $r^{-2}M\left(\Sigma\llcorner B_r^n\right)-\pi Q$ as r tends to 0, is very reminiscent to the Morrey type estimates of the form

$$\frac{1}{r^{n-2}}\int_{B_r(0)}|\nabla u|^2\le C\ r^{\gamma}$$

that one establishes in solving regularity problems for solutions to elliptic PDE (see [Gi] for instance).

One of the consequence of this rate of decay of $M(\Sigma_r\llcorner B_1^n)$ is that the limiting tangent cone at 0 is unique : the function $r\longrightarrow\Sigma_r\llcorner B_1^n$ converges in Flat norm to a sum of flat disks passing through the origin (see [Wh]).

Assuming now that we are dilating at a neighborhood point x_0 of 0 such that $\lim_{r\rightarrow 0}\ r^{-2}M\left(\Sigma\llcorner B_r^n(x_0)\right)<\pi Q$. Denote this time μ_{r,x_0} the following dilation $\mu_{x_0,r}(x)=\frac{x-x_0}{r}$ and let $\Sigma_{r,x_0}:=(\mu_{r,x_0})_*\Sigma$. Let r be a small enough radius such that $\frac{1}{2}M\left(\partial(\Sigma_{r,x_0}\llcorner B_1^n(x_0))\right)<\pi Q$ and such that Σ_r is still close in flat norm to the Q disks (i.e. (1.1) holds). Then, since the monotonicity formula tells us that

$$M(\Sigma_{r,x_0}\llcorner B_1^n(x_0))\le\frac{1}{2}M\left(\partial(\Sigma_{r,x_0})\llcorner B_1^n(x_0)\right)\right)\quad ,$$

the additional information provided by the upper-epiperimetric inequality (1.2) is void. Nevertheless one can still expect, in the present situation, that there is some rate of increase for the following quantity we shall call the **lower excess**

$$\pi Q - M(\Sigma_{r,x_0} \llcorner B_1^n(x_0)) = \pi Q - r^{-2} M\left(\Sigma \llcorner B_r^n(x_0)\right)$$

as r tends to zero at least as long as (1.1) holds. This question is answered positively by the following **lower-epiperimetric inequality** which is the main result of the present paper.

THEOREM 1.3 *Given an integer Q, there exists a positive number $\varepsilon_Q > 0$, such that, for any Σ area minimizing 2-rectifiable integral current in $B_2^n(0)$, satisfying $\partial\Sigma \llcorner B_2^n = 0$, assuming there exists Q flat oriented 2-disks $D_1 \ldots D_Q$ passing through zero such that*

$$\mathcal{F}\left(\Sigma_2 \llcorner B_1^n(0) - \sum_{i=1}^{Q} [D_i]\right) \le \varepsilon_Q \quad . \tag{1.4}$$

Then

$$\pi Q - \frac{1}{2} M\left(\partial(\Sigma \llcorner B_1^n(0))\right) \le (1-\varepsilon_Q)\ \left(\pi Q - M(\Sigma \llcorner B_1^n(0))\right) \quad . \tag{1.5}$$

A consequence of the previous theorem is the following decrease of the lower excess : let $R > r$ assuming (1.4) holds for $\Sigma_{2\rho}$ for $R > \rho > r$, one gets

$$Q\pi - R^{-2} M(\Sigma \llcorner B_R(0)) \le \left(\frac{r}{R}\right)^{\gamma} \left[Q\pi - r^{-2} M(\Sigma \llcorner B_r(0))\right] \quad , \tag{1.6}$$

where γ is a positive number. This decrease is responsible to the *"splitting before tilting phenomenon"* for calibrated 2-rectifiable cycles described in the last part of the paper. The observation of this phenomenon is one of the crucial step in the proof of the regularity of almost complex 2-integral currents given in [RT].

As an illustration of a situation fulfilling the assumptions of theorem 1.3, one can consider, for instance, in $\mathbb{C}^2$ the holomorphic curve $\Sigma\ :\ w^2 = z$ which is calibrated by the standard Kähler form $\omega = 1/2(dz \wedge d\overline{z} + dw \wedge d\overline{w})$ and hence area minimizing. For ρ large enough $\mathcal{F}\left(\Sigma_{2\rho} - 2[D_0]\right) \le \varepsilon_2$ where D_0 is the flat disk $B_1^4(0) \cap \{w = 0\}$ and therefore for R and r large enough (1.6) hold. In other words the area density of Σ, for decreasing radii from infinity, get away from 2π with at least the speed given by (1.6).

2 Proof of the lower-epiperimetric inequality.

Let Σ be an area minimizing integral current in $B_2^{n+2}(0)$ with no boundary in $B_2^{n+2}(0)$. For convenience we have replaced n in the statement of the theorem by $n+2$. Let $Q>0$ be an integer. To simplify the presentation we assume that (1.4) holds for $\sum_{i=1}^{Q}[D_i]$ being Q times the disk $D_0 := B_1^{n+2}(0)\cap\{x_i = 0 \text{ for } i\geq 3\}$ where $x_1,\cdots,x_{n+2}$ are the canonical coordinates in $\mathbb{R}^{n+2}$.

It is a standard result whose proof is based on the monotonicity formula saying that for every $Q\in\mathbb{N}$ and $\varepsilon>0$ there exists some $\alpha>0$ such that for Σ being an energy minimizing cycle in $B_2^{n+2}(0)$ whenever

$$\mathcal{F}\left(\Sigma_2 \llcorner B_1^n - Q[D_0]\right)\leq\alpha \quad , \tag{2.1}$$

then

$$d_{\mathcal{H}}\left(supp\{\Sigma\llcorner B_1^{n+2}(0)\};D_0\right)\leq\delta \quad , \tag{2.2}$$

where $d_{\mathcal{H}}$ denotes the Hausdorff distance.

We adopt the notations of [Wh] : Let $T=\partial(\Sigma\llcorner B_1^{n+2}(0))$ We assume that $M(T)\leq 2\pi Q$ (the case $M(T)\geq 2\pi Q$ being the trivial case for the lower-epiperimetric inequality) . T being a finite mass boundary of a rectifiable current it is also rectifiable [Fe] and belongs to the class $\mathbf{I}_1(\partial B_1^{n+2}(0))$ of 1 dimensional integral current. Denote $C(T)$ the 2-integral current in $\mathbf{I}_2(\mathbb{R}^{n+2})$ obtained by taking the whole cone issued from zero whose slice with $\partial B_r^{n+2}(0)$ for any $+\infty>r>0$ equals $< C(T), dist\{\cdot,0\}, r >= \mu(r)_*T$ where μ_r is the dilation map $x\rightarrow rx$ (Observe that $\partial(C(T))=0$ in $\mathbb{R}^{n+2}$). Following [Wh] we introduce the Excess of T

$$EX(T):=M(C(T)\llcorner B_1^2\times\mathbb{R}^n)-\pi Q \quad . \tag{2.3}$$

It is clear that $EX(T)$ is non negative. Denote by X the space of lipshitz maps g from $[0,2\pi Q]$ into $\mathbb{R}^n$ such that $g(0)=g(2\pi Q)$. For such a g we denote by $(\cos\times\sin\times g)_*[0,2\pi Q]$ the current in $\mathbf{I}_1(\partial B_1^2\times\mathbb{R}^n)$ given by the lipshitz closed curve $\theta\rightarrow(\cos\theta,\sin\theta,g(\theta))$. Let $F(x)=\frac{x}{|x|}$ be the projection map onto $\partial B_1^{n+2}(0)$. For any g in X we introduce the current $\mathcal{T}(g)$ in $\mathcal{I}_1(\partial B^{n+2}(0))$ (still following [Wh]) given by

$$\mathcal{T}(g):=F_*(\cos\times\sin\times g)_*[0,2\pi Q] \tag{2.4}$$

We now grant most of the achievements in [Wh]. There exists $\varepsilon>0$ such that, for any δ small enough there exists α such that, if Σ solves (2.1), modulo eventually a very small rotation of the canonical basis of $\mathbb{R}^{n+2}$, there exists $g\in X$ and $R\in\mathbf{I}_2(\partial B_1^2\times\mathbb{R}^n)$ and $H\in\mathbf{I}_2(B_1^2\times\mathbb{R}^n)$ satisfying

i)
$$\|g\|_{W^{1,\infty}} \leq \delta \quad , \tag{2.5}$$

ii)
$$EX(\mathcal{T}(g)) \leq EX(T) \quad , \tag{2.6}$$

iii)
$$M(R) \leq \delta \; EX(T) \quad , \tag{2.7}$$

iv)
$$\partial R = \partial((C(T))\llcorner B_1^2 \times \mathbb{R}^n) - (\cos \times \sin \times g)_*[0, 2\pi Q] \quad , \tag{2.8}$$

v)
$$M(H) - \pi Q \leq (1-\varepsilon) EX(\mathcal{T}(g)) \quad , \tag{2.9}$$

vi)
$$\partial H = \mathcal{T}(g) \quad . \tag{2.10}$$

Properties i)...iv) come from propositions 2.3, 2.7 and 2.8 of [Wh]. Properties v) and vi) being the cylindrical epiperimetric inequality (proposition 2.4 of [Wh]). The passage from the proof of the **upper-epiperimetric** inequality to the proof of the **lower-epiperimetric** inequality is based on the following lemma.

LEMMA 2.1 *For any $Q \in \mathbb{N}$, $\eta > 0$ there exists $\delta > 0$ such that for any integral current $T \in \mathcal{I}_1(\partial B^{n+2}(0))$ satisfying*

$$M(T) \leq 2\pi Q \quad , \tag{2.11}$$

$$\mathcal{F}((C(T))\llcorner B_1^2(0) \times \mathbb{R}^n - Q[D_0]) \leq \delta \quad , \tag{2.12}$$

and
$$d_{\mathcal{H}}\left(supp\{T\}; \partial B_1^2(0) \times \{0\}\right) \leq \delta \quad , \tag{2.13}$$

then
$$EX(T) > \left(\frac{1}{2} - \eta\right) \; [2\pi Q - M(T)] \quad . \tag{2.14}$$

PROOF: T being an integral 1-dimensional cycle, from 4.2.25 [Fe], we deduce that $T = \sum_{i \in I} \Gamma_i$ where I is at most countable, Γ_i is a push-forward of $[S^1]$ (the current of integration over S^1) by a lipshitz maps, each Γ_i is indecomposable and $M(T) = \sum_{i \in I} M(\Gamma_i)$. Let Q_i be the winding of Γ_i about the axis $\{0\} \times \mathbb{R}^n$, because of (2.11), (2.12) and (2.13), it is clear, for δ small enough, that each Q_i is non negative (recall $Q > 0$). Therefore it suffices to prove the lemma when T is the current of integration on a lipshitz closed curve in $\partial B_1^{n+2}(0)$. Thus we assume that T is the push forward of the segment $[0, 2\pi Q + E]$, by the following map

$$\gamma(s) := \left[\sqrt{1+|g|^2(s)}\right]^{-1} (\cos\phi(s), \sin\phi(s), g(s)) \quad ,$$

where $\phi \in \mathbb{R}$, $g \in \mathbb{R}^n$, $\gamma(0) = \gamma(2\pi Q + E)$ and $|\dot\gamma| \equiv 1$ (arc-length parametrized curve). This implies that $E = M(T) - 2\pi Q < 0$ and that the following identity holds

$$1 + |g|^2 = \dot\phi^2 + |\dot g|^2 - \frac{(g \cdot \dot g)^2}{1+|g|^2} \quad . \tag{2.15}$$

From (2.13) we have that $|g| \le \delta$ and, combining this fact with (2.15), we obtain that $|\dot g|^2 \le 1 + 3\delta^2$. Since, from (2.12), the winding of T about $\{0\} \times \mathbb{R}^n$ is Q, we have

$$\begin{aligned}
2\pi Q \quad &\le \int_0^{2\pi Q+E} |\dot\phi| \le (2\pi Q + E)^{\frac{1}{2}} \left[\int_0^{2\pi Q+E} |\dot\phi|^2\right]^{\frac{1}{2}} \\
&\le (2\pi Q + E) \left[1 + \frac{1}{2\pi Q + E} \int_0^{2\pi Q+E} |g|^2 - (1-\delta^2)|\dot g|^2\right]^{\frac{1}{2}} \\
&\le 2\pi Q + E + \frac{1}{2}\int_0^{2\pi Q+E} |g|^2 - (1-\delta^2)|\dot g|^2
\end{aligned} \tag{2.16}$$

where we have used consecutively Cauchy-Schwartz, $|g| \le \delta$ and $\sqrt{1+x} \le 1 + \frac{x}{2}$. Therefore we deduce from (2.16) that

$$2\pi Q - M(T) = -E = |E| \le \frac{1}{2}\int_0^{2\pi Q+E} |g|^2 - (1-\delta^2)|\dot g|^2 \quad . \tag{2.17}$$

We now establish a lower-bound for $M(C(T) \llcorner B_1^2(0) \times \mathbb{R}^n)$. The current $C(T) \llcorner B_1^2(0) \times \mathbb{R}^n$ is given by the push-forward of the current of integration on $[0,1] \times [0, 2\pi Q + E]$ by the map

$$\Psi(t,s) := (t\cos\phi(s), t\sin\phi(s), tg(s)) \quad .$$

A short computation gives that

$$\left|\frac{\partial\Psi}{\partial t} \wedge \frac{\partial\Psi}{\partial s}\right|^2 = t^2 \left[[1+|g|^2]\dot\phi^2 + |\dot g|^2 + |g \wedge \dot g|^2\right] \quad . \tag{2.18}$$

Therefore, using again the fact that $|g| \leq \delta$, we have

$$M(C(T) \llcorner B_1^2(0) \times \mathbb{R}^n) = \int_0^1 \int_0^{2\pi Q+E} \left| \frac{\partial \Psi}{\partial t} \wedge \frac{\partial \Psi}{\partial s} \right| \, dt \, ds$$

$$\geq \frac{1}{2} \int_0^{2\pi Q+E} \left[1 + 2|g|^2 - \delta^2 |\dot{g}|^2\right]^{\frac{1}{2}} \tag{2.19}$$

$$\geq \pi Q + \frac{E}{2} + \frac{1}{2} \int_0^{2\pi Q+E} [1 - O(\delta^2)]|g|^2 - O(\delta^2)|\dot{g}|^2 \quad .$$

Combining (2.17) and (2.19) we obtain that

$$M(C(T) \llcorner B_1^2(0) \times \mathbb{R}^n) \geq \pi Q - \left(\frac{1}{2} - O(\delta^2) \right) E \tag{2.20}$$

This establishes (2.14) for $\eta = O(\delta^2)$ and lemma 2.1 is proved. ∎

Proof of theorem 1.3.

Proof: Let $\mu(\frac{1}{2})$ be the contraction $x \to \frac{x}{2}$ in $\mathbb{R}^{n+2}$. Following [Wh] page 159, we introduce the following current

$$S := \mu\left(\frac{1}{2}\right)_* (R + H) + U \quad ,$$

where H and R are given by (2.5)...(2.10) and

$$U = C(T) \llcorner [B_1^{n+2}(0) \setminus B_{\frac{1}{2}}^2 \times \mathbb{R}^n] \quad .$$

From (2.8) and (2.10) we have $\partial S = T$. Moreover we have using (2.7) and (2.9)

$$\begin{aligned} M(S) &= \frac{1}{4}[M(H) + M(R)] + M(U) \\ &\leq \frac{1}{4}[\pi Q + (1 - \varepsilon + \delta) EX(T)] + M(U) \quad . \end{aligned} \tag{2.21}$$

Taking $\delta = \frac{\varepsilon}{2}$, we have

$$\begin{aligned} M(S) &\leq \frac{1}{4}[\pi Q + EX(T)] - \frac{\varepsilon}{2} EX(T) + M(U) \\ &\leq M(C(T) \llcorner [B_{\frac{1}{2}}^2 \times \mathbb{R}^n]) - \frac{\varepsilon}{2} EX(T) + M(U) \\ &\leq M(C(T) \llcorner B_1^{n+2}(0)) - \frac{\varepsilon}{2} EX(T) \quad . \end{aligned} \tag{2.22}$$

$M(C(T)\llcorner B_1^{n+2}(0)) = \frac{1}{2}M(T)$ and using lemma 2.1, we have that

$$-\frac{\varepsilon}{2}EX(T) \leq -\frac{\varepsilon}{4}(\pi Q - \frac{M(T)}{2}) \quad .$$

Thus we have proved that

$$\pi Q - \frac{M(T)}{2} + \frac{\varepsilon}{4}(\pi Q - \frac{M(T)}{2}) \leq \pi Q - M(S) \leq \pi Q - M(\Sigma\llcorner B_1(0)) \quad .$$

where we have used the minimality of Σ. This finishes the proof of theorem 1.3. ∎

3 The "splitting before tilting" phenomenon.

In the present section we consider, in the unit ball $B_1^{2p}(0)$ of $\mathbb{R}^{2p}$, a non-degenerate symplectic form ω together with a compatible almost complex structure compatible with J : $\omega(\cdot, J\cdot)$ is a scalar product. We assume both ω and J are C^1 and we consider a 2-dimensional rectifiable current Σ satisfying :

i)

$$\partial\Sigma\llcorner B_1^{2p}(0) = 0 \quad . \tag{3.1}$$

ii)

$$\Sigma(\omega) = M(\Sigma) \quad , \tag{3.2}$$

where M is the mass of the current Σ for the metric $\omega(\cdot, J\cdot)$. The last condition ii) simply means that Σ is calibrated by ω. We shall use the following notations : given a point $x_0 \in B_1^{2p}(0)$, for a sufficiently small radius r (depending on $\|\omega\|_{C^1}$ and $\|J\|_{C^1}$), in $B_r(x_0)$ we consider normal coordinates $(x_1, \cdots x_{2p})$ such that for any $1 \leq l \leq p$

$$J_{x_0}\frac{\partial}{\partial x_{2l-1}}(x_0) = \frac{\partial}{\partial x_{2l}}(x_0) \quad .$$

We shall now identify $\Sigma\llcorner B_r(x_0)$ with it's pull back in this new chart. Let then for any $\rho \leq r$, $\lambda^{x_0,\rho}$ be the dilation $\lambda^{x_0,\rho}(x) := \frac{x-x_0}{\rho}$ we denote Σ_{ρ,x_0} the following dilated current in $B_1^{2p}(0)$

$$\Sigma_{\rho,x_0} := (\lambda^{x_0,\rho})_*(\Sigma\llcorner B_\rho(x_0)) \quad .$$

For that current Σ we also denote, for any $x_0 \in B_1^{2p}(0)$, Q_{x_0} the limiting density of Σ at x_0

$$Q_{x_0} := \pi^{-1} \lim_{r\to 0} r^{-2} M(\Sigma \llcorner B_\rho(x_0)) = \pi^{-1} \lim_{r\to 0} M(\Sigma_{\rho,x_0}) \in \mathbb{N} \quad .$$

Considering a point $x_0 \in B_1(0)$ such that $Q_{x_0} = Q$, then using B.White upper-epiperimetric inequality, it is not difficult to see (see [RT] chapter III) that there exists $\delta > 0$, depending only $\|\omega\|_{C^1}$ and $\|J\|_{C^1}$, such that if there exists a flat J_{x_0} holomorphic disk D_0 passing through x_0 and a radius ρ such that $\forall \rho < \delta$

$$\mathcal{F}(\Sigma_{\rho,x_0} \llcorner B_1(0) - Q\ [D_0]) \leq \alpha \tag{3.3}$$

then, no significative tilting is possible while dilating the current further, and for any smaller radius r one has

$$\mathcal{F}(\Sigma_{r,x_0} \llcorner B_1(0) - Q\ [D_0]) \leq C\sqrt{\alpha} \tag{3.4}$$

for some constant C depending only on $\|\omega\|_{C^1}$ and $\|J\|_{C^1}$ (similar result holds of course when one replaces $Q[D_0]$ by the sum of Q arbitrary $J_{x_0}-$holomorphic flat disks passing through x_0 : $\sum_{i=1}^{Q}[D_i]$).

Consider now a point x_0 such that $0 < Q_{x_0} < Q$ and assume still that $\mathcal{F}(\Sigma_{\rho,x_0} \llcorner B_1(0) - Q\ [D_0])$ is very small. Since $Q > Q_{x_0}$ the tangent cone at x_0 contains strictly less than Q disks and therefore while dilating the current further one has to leave this "almost Q-times-a-disk-shape" and reach a splitted situation where $M(\Sigma_{r,x_0}) \leq \pi(Q - \frac{1}{2})$. Now comes naturally the question about what could happen in between : can the current tilt before to loose this "almost $Q-$times-a-disk shape" ? can we have for $r < \rho$ and for a disk D_1 far from D_0 (i.e. $\mathcal{F}([D_0] - [D_1]) \geq \frac{1}{4}$)

$$\mathcal{F}(\Sigma_{r,x_0} \llcorner B_1(0) - Q\ [D_1]) << 1 \qquad ?$$

The following lemma, whose proof is based on the lower-epiperimetric inequality proved in the previous section, answers negatively to that question : the current has to split before to tilt.

Lemma 3.1 **(splitting before tilting)**. *Let $Q \in \mathbb{N}$, $Q \geq 2$. There exists $\alpha > 0$ depending only on Q, $Q\|\omega\|_{C^1}$ and $\|J\|_{C^1}$ such that, for any x_0 satisfying $0 < Q_{x_0} < Q$ and for any radius $0 < \rho < \alpha$ satisfying*

$$\mathcal{F}(\Sigma_{2\rho,x_0} \llcorner B_1(0) - Q\ [D_0]) \leq \alpha \tag{3.5}$$

where D_0 *is a flat* $J_{x_0}-$*holomorphic disk passing through* x_0. *Then, for any* $r < \rho$ *and any* $J_{x_0}-$*holomorphic flat disk,* D_1, *passing through* x_0 *and satisfying*

$$\mathcal{F}([D_0] - [D_1]) \geq \frac{1}{4} \tag{3.6}$$

we have

$$\mathcal{F}(\Sigma_{r,x_0} \llcorner B_1(0) - Q[D_1]) \geq \alpha \quad . \tag{3.7}$$

Moreover, there exists $r_0 < \rho$ *and* K_0 *a constant depending only on* $\|\omega\|_{C^1}$ *and* $\varepsilon_Q^{\pm}$, *the epiperimetric constants, such that*

$$M(\Sigma_{r_0,x_0} \llcorner B_1(0)) = \pi Q - K_0 \alpha \quad , \tag{3.8}$$

$$\mathcal{F}(\Sigma_{r_0,x_0} \llcorner B_1(0) - Q[D_0]) \leq K \ \sqrt{\alpha} \tag{3.9}$$

for some constant K *depending also only on* $\|\omega\|_{C^1}$ *and* $\varepsilon_Q^{\pm}$. *Finally, for any* $J_{x_0}-$*holomorphic disk* D *passing through* 0

$$\forall r \leq r_0 \qquad \mathcal{F}(\Sigma_{r,x_0} \llcorner B_1(0) - Q\ [D]) \geq \alpha \quad . \tag{3.10}$$

Remark 3.2 *We believe this phenomenon should be extended to the general case of area minimizing integer 2-rectifiable currents and should be one crucial step in a new proof of their regularity aside from isolated points.*

Proof: Let ε_Q^+ and ε_Q^- be the two positive constants arising in the respectively upper and lower-epiperimetric inequalities (1.1) and (1.4). We first choose $0 < \alpha^{\frac{1}{3}} \leq min\{\varepsilon_Q^+, \varepsilon_Q^-\}$. Recall that, as in part II, we mixed the notations for ω and it's pull-back by the inverse of the chart map $(x_1, \cdots, x_{2p})$. Denoting $\omega_r = r^{-2}(rx)^*\omega$, it is clear that, as $r \to 0$

$$\|\omega_r - \omega_0\|_\infty \leq 2r\ \|\omega\|_{C^1} \tag{3.11}$$

Let $\rho < \alpha$ and assume (3.5) holds. Let S and R be $3-$ and $2-$currents such that $\Sigma_{2\rho,x_0} \llcorner B_1(0) - Q\ [D_0] = \partial S + R$ and $M(S) + M(R) \leq 2\alpha$. We have

$$\begin{aligned} |(\Sigma_{2\rho,x_0} \llcorner B_1(0) - Q\ [D_0])(\omega_0)| &= |R(\omega_0)| \\ &\leq M(R)\ \|\omega_0\|_* \leq 2\alpha \quad , \end{aligned} \tag{3.12}$$

where $\|\omega_0\|_*$ is the comass of ω_0 which is 1 in g_0. Since D_0 is $J_{x_0}-$holomorphic, $[D_0](\omega_0) = \pi$. Thus

$$|\Sigma_{2\rho,x_0} \llcorner B_1(0)(\omega_0) - \pi Q| \leq 2\alpha \quad . \tag{3.13}$$

Since $\Sigma_{2\rho,x_0}$ is calibrated by $\omega_{2\rho}$, $M(\Sigma_{2\rho,x_0}\llcorner B_1(0)) = \Sigma_{2\rho,x_0}\llcorner B_1(0)(\omega_{2\rho})$. Thus from (3.13) and (3.11) we deduce that

$$|M(\Sigma_{\rho,x_0}\llcorner B_1(0)) - \pi\ Q| \le (2+2\|\omega\|_{C^1})\ \alpha = K_0\ \alpha \quad . \tag{3.14}$$

Assume first that $M(\Sigma_{\rho,x_0}\llcorner B_1(0)) > \pi Q$. Consider now the function $f(r)$ introduced in the proof of lemma III.1 in [RT] (where the notation C has to be replaced by Σ) : $f(r) = M(\Sigma\llcorner B_r(x_0)) - \pi Q r^2$. Let s be the maximal radius such that $f(s) \ge 0$ and let $\tilde{s}$ be the minimal radius $s \le \tilde{s} \le \rho$ such that, for any $r \in [\tilde{s}, \rho]$

$$\mathcal{F}(\Sigma_{2r,x_0}\llcorner B_1(0) - Q\ [D_0]) \le \varepsilon_Q^+ \quad , \tag{3.15}$$

We claim that $\tilde{s} = s$. Indeed, like in the proof of lemma III.1 in [RT], as long as (3.15) holds we can apply the upper-epiperimetric inequality which says that

$$f(r) \le 2r\ (1-\varepsilon_Q^+) f'(r) \quad . \tag{3.16}$$

Using Cauchy-Schwartz and 5.4.3 (2) [Fe] (the explicit formulation of the monotonicity formula) one has for any $\sigma < s$

$$\begin{aligned} & M(F_*\,(\Sigma\llcorner B_s(x_0)\setminus B_\sigma(x_0))) \\ & \le \left[\frac{M(\Sigma\llcorner B_s(0))}{s^2} - \frac{M(\Sigma\llcorner B_\sigma(x_0))}{\sigma^2}\right]^{\frac{1}{2}} \left[\frac{M(\Sigma\llcorner B_s(x_0))}{\sigma^2}\right]^{\frac{1}{2}} \\ & \le \left[\frac{M(\Sigma\llcorner B_s(x_0))}{s^2} - \pi Q\right]^{\frac{1}{2}} \left[\frac{M(\Sigma\llcorner B_s(x_0))}{\sigma^2}\right]^{\frac{1}{2}} \\ & \le \left[\frac{f(s)}{s^2}\right]^{\frac{1}{2}} \left[\frac{s^2}{\sigma^2}\frac{M(\Sigma\llcorner B_s(x_0))}{s^2}\right]^{\frac{1}{2}} \le K\ s^{\frac{\nu}{2}}\ \frac{s}{\sigma} \quad . \end{aligned} \tag{3.17}$$

Integrating then the differential inequality (3.16) and using the inequality (3.17), we obtain for any $r \ge \tilde{s}$

$$M(F_*(\Sigma\llcorner B_\rho(x_0)\setminus B_r(x_0))) \le K\sqrt{\alpha} \quad , \tag{3.18}$$

where we recall the notation $F(x) = x/|x|$ and K is a constant independent of r. Since

$$\partial(F_*(\Sigma\llcorner B_\rho(x_0)\setminus B_r(x_0))) = \partial\Sigma_{\rho,x_0} - \partial\Sigma_{r,x_0} \quad ,$$

we deduce that for any $\tilde{s} \leq r \leq \rho$

$$\mathcal{F}(\Sigma_{\rho,x_0} \llcorner B_1(0) - \Sigma_{r,x_0} \llcorner B_1(0)) \leq K\sqrt{\alpha} \quad . \tag{3.19}$$

Using (3.17) we have

$$\begin{aligned} & M(F_* (\Sigma \llcorner B_{2\rho}(x_0) \setminus B_\rho(x_0))) \leq \\ & \left[\frac{M(\Sigma \llcorner B_{2\rho}(x_0)}{(2\rho)^2} - \frac{M(\Sigma \llcorner B_\rho(x_0))}{\rho^2}\right]^{\frac{1}{2}} \left[\frac{M(\Sigma \llcorner B_{2\rho}(x_0))}{\rho^2}\right]^{\frac{1}{2}} \quad . \end{aligned} \tag{3.20}$$

Thus, using (3.14)

$$M(F_* (\Sigma \llcorner B_{2\rho}(x_0) \setminus B_\rho(x_0))) \leq K \sqrt{\alpha} \quad , \tag{3.21}$$

and arguing like few lines above we have

$$\mathcal{F}(\Sigma_{2\rho,x_0} \llcorner B_1(0) - \Sigma_{\rho,x_0} \llcorner B_1(0)) \leq K\sqrt{\alpha} \quad . \tag{3.22}$$

Combining (3.5), (3.19) and (3.22) we deduce that for any $\tilde{s} \leq r \leq \rho$

$$\mathcal{F}(\Sigma_{2r,x_0} \llcorner B_1(0) - Q\ [D_0]) \leq K\sqrt{\alpha} \quad , \tag{3.23}$$

where K is independent of α. Thus it is clear that we can keep applying the upper-epiperimetric inequality and the arguments above until s (as long as $f(r) > 0$), which means that $\tilde{s} = s$ and we have moreover

$$\mathcal{F}(\Sigma_{s,x_0} \llcorner B_1(0) - Q\ [D_0]) \leq K\sqrt{\alpha} \quad . \tag{3.24}$$

Let K_0 given in (3.14). Choose r_0 such that

$$\pi Q - r_0^{-2} M(\Sigma \llcorner B_{r_0}(x_0)) = 2K_0\alpha \quad . \tag{3.25}$$

For $s \geq r \geq r_0$, we introduce the function $g(r) = \pi Q r^2 - M(\Sigma \llcorner B_r(x_0))$. Let $\tilde{r}_0$ the smallest radius larger than r_0 such that $\forall r \in [\tilde{r}_0, s]$

$$\mathcal{F}(\Sigma_{2r,x_0} \llcorner B_1(0) - Q\ [D_0]) \leq \varepsilon_Q^- \quad . \tag{3.26}$$

Since $\frac{d}{dr} M(\Sigma \llcorner B_r(x_0)) \geq M(\partial(\Sigma \llcorner B_r(x_0)))$, we have that $g'(r) \leq 2\pi Q r - M(\partial(\Sigma \llcorner B_r(x_0)))$. Using now the lower-epiperimetric inequality, that we can apply since (3.26) holds, we obtain

$$\forall r \in [\tilde{r}_0, s] \qquad g'(r) \leq 2(1 - \varepsilon_Q^-) g(r) \quad . \tag{3.27}$$

Integrating this differential inequality between r and $\frac{r}{2}$, we obtain

$$\frac{g(r)}{r^2} \leq \frac{1}{2^{2\varepsilon_Q^-}} \frac{g\left(\frac{r}{2}\right)}{\left(\frac{r}{2}\right)^2} \quad . \tag{3.28}$$

Moreover, from (3.20), we have

$$\begin{aligned} M\left(F_*\left(\Sigma \llcorner (B_r(x_0) \setminus B_{\frac{r}{2}}(x_0))\right)\right) &\leq K\left[\pi Q - \frac{M(\Sigma \llcorner B_{\frac{r}{2}}(x_0))}{\left(\frac{r}{2}\right)^2}\right]^{\frac{1}{2}} \\ &\leq K\left[\frac{g\left(\frac{r}{2}\right)}{\left(\frac{r}{2}\right)^2}\right]^{\frac{1}{2}} \quad . \end{aligned} \tag{3.29}$$

Summing up this inequality between s and $2^{-k_0}s$ for k_0 being an arbitrary integer less than $\log_2 \frac{s}{\tilde{r}_0}$, using (3.28), we get that

$$\begin{aligned} M\left(F_*\left(\Sigma \llcorner (B_s(x_0) \setminus B_{2^{-k_0}s}(x_0))\right)\right) &\leq K \sum_{k=1}^{k_0} \left[\frac{g(2^{-k}s)}{(2^{-k}s)^2}\right]^{\frac{1}{2}} \\ &\leq K \sum_{k=1}^{k_0} \left[\frac{1}{2^{2\varepsilon_Q^-(k_0-k)}} \frac{g(2^{-k_0}s)}{(2^{-k_0}s)^2}\right]^{\frac{1}{2}} \\ &\leq K\sqrt{\alpha} \quad , \end{aligned} \tag{3.30}$$

where we have used the fact that $\varepsilon_Q^- > 0$ and that (3.25) implies $\frac{g(2^{-k_0}s)}{(2^{-k_0}s)^2} \leq 2K_0\alpha$. As before (3.30) gives

$$\mathcal{F}((\Sigma_{s,x_0} - \Sigma_{2^{-k_0}s,x_0}) \llcorner B_1(0)) \leq K\sqrt{\alpha} \quad . \tag{3.31}$$

Choosing then k_0 being the largest integer less than $\log_2 \frac{s}{\tilde{r}_0}$, using one more time the inequality of the type (3.20) between $2^{-k_0}s$ and $\tilde{r}_0$, since $|\frac{2^{-k_0}s}{\tilde{r}_0}| \leq 2$, we get

$$\mathcal{F}((\Sigma_{2^{-k_0}s,x_0} - \Sigma_{\tilde{r}_0,x_0}) \llcorner B_1(0)) \leq K\sqrt{\alpha} \quad . \tag{3.32}$$

Combining now (3.24), (3.31) and (3.32) we obtain

$$\mathcal{F}(\Sigma_{\tilde{r}_0,x_0} \llcorner B_1(0) - Q[D_0]) \leq K\ \sqrt{\alpha} \quad , \tag{3.33}$$

where K only depends on $\|\omega\|_{C^1}$ and $\varepsilon_Q^{\pm}$. Therefore, as above for $\tilde{s}$, since $K\ \sqrt{\alpha} << \varepsilon_Q^-$, we can use the lower-epiperimetric inequality all the way to r_0, which means that $\tilde{r}_0 = r_0$ and we have for any r between 2ρ and r_0

$$\forall r \in [r_0, 2\rho] \qquad \mathcal{F}(\Sigma_{r,x_0} \llcorner B_1(0) - Q[D_0]) \leq K\ \sqrt{\alpha} \quad . \tag{3.34}$$

From (3.25) and the monotonicity formula we have for any $r < r_0$

$$M(\Sigma_{r,x_0} \llcorner B_1(0)) \leq \pi Q - 2K_0\alpha \quad . \tag{3.35}$$

We claim now that for any $J_{x_0}-$holomorphic disk D

$$\forall r \leq r_0 \qquad \mathcal{F}(\Sigma_{r,x_0} \llcorner B_1(0) - Q\ [D]) \geq \alpha \tag{3.36}$$

Indeed, if that would not have been the case, arguing like for proving (3.14) above, we would have obtained $|M(\Sigma_{r,x_0} \llcorner B_1(0)) - \pi Q| \leq K_0\alpha$ and this would have contradict (3.35). Now combining (3.34) and (3.36) we obtain lemma 3.1. ∎

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