A Quantization Property for Static Ginzburg-Landau Vortices

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Abstract : For any critical point of the complex Ginzburg-Landau Functional in dimension 3 we prove that, for large coupling constants $\kappa = \frac{1}{\varepsilon}$*, if the energy of this critical point on a ball of a given radius* r *is relatively small compare to* $r \log \frac{r}{\varepsilon}$ *then the ball of half radius contains no vortex (the modulus of the solution is larger than* $1/2$*). We then show how this property can be applied to describe limiting vortices as* $\varepsilon \to 0$*.*

I Introduction

Since the publication of the book by F. Bethuel, H. Brezis and F. Hélein "Ginzburg-Landau Vortices" [4] the study of solutions of the static complex Ginzburg-Landau Equation in the "large kappa limit" $\kappa = 1/\varepsilon$ (ε small)

$$\Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2) = 0 \tag{I.1}$$

where u goes from a domain Ω of $\mathbf{R^n}$ into $\mathbf{C}$ has been extensively developped. Solutions of (I.1) are critical points of the Ginzburg-Landau Functional

$$E_\varepsilon(u) = \int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 = \int_\Omega e_\varepsilon(u) \tag{I.2}$$

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The major difficulty in studying the asymptotic behavior of solutions of (I.1) as ε tends to zero comes from the existence of zones where u vanishes. Theses zones, often called vortices, are responsible for the lack of compactness of the sequences of solutions of (I.1) in $W^{1,2}$ or any smaller space. This makes the analysis quite involved. In the standard terminology of the above mentioned asymptotic analysis (the so called BBH-asymptotic analysis) the set

$$\mathcal{B} = \{x \in \Omega \quad \text{s.t.} \quad |u|(x) \leq 1/2\}$$

that contains the zero set of u is often called the "bad set". $1/2$ is in fact any number strictly lie in between 0 and 1 (recall that for the suitable boundary conditions considered in [4] - $u_{|\partial\Omega} \in S^1$ - we have $|u| \leq 1$ in Ω). The understanding of the size and the location of $\mathcal{B}$ is the key point for the understanding of the behavior of solutions of (I.1) for small ε and then for the description of Ginzburg-Landau Vortices.

In this paper we give an energy criterium for the existence or non existence of line vortices in 3 dimensions. Precisely our first main result is

Theorem I.1 *[eta-compactness] For any $C > 0$, there exists $\eta > 0$ and ε_0, such that, for any solution of (I.1) verifying $|u| \leq 1$ and $|\nabla|u|| \leq C/\varepsilon$ the following holds : if $\varepsilon \leq \varepsilon_0$, for any $r \leq 1$ and any x in Ω such that $B_r(x) \subset \Omega$*

$$\frac{1}{r}\int_{B_r(x)} e_\varepsilon(u) \leq \eta \ \log\frac{r}{\varepsilon}$$

$$\Longrightarrow \quad |u| \geq 1/2 \quad \text{on } B_{r/2}(x)$$

This result known as the η-compactness property is of course reminiscent to the corresponding $\varepsilon-$regularity properties in the theory of harmonic maps, minimal surfaces, Yang-Mills Monopoles...etc

The proof of the eta-compactness in the case $n = 2$ is quite short and can be found say in [15]. The theorem I.1 already proved in the particular case where u is a minimizer of the Ginzburg-Landau Functional (I.2) in [13] [14] and was extended to the case of minimizers again but for any dimension of the domain in [12].

As we will see bellow the passage from proofs of the eta-compactness property for minimizers, where comparisons arguments are extensively used, to the proof of the same property but for critical points in general requires the introduction of more refined spaces than the natural Sobolev Spaces for variational problems. In particular the use of Hardy and Lorentz Spaces plays a crucial role in our proof bellow of theorem I.1. This is also reminiscent to the situation where one passes from the $\varepsilon-$regularity property for minimizing harmonic maps (see [18]) to the same property for stationary harmonic maps in general (see [8] and [3] and also the regularity of harmonic maps in 2 dimension in [10]). The introduction of interpolation Spaces in the non-linear analysis community goes back to the work of J.L.Lions and E.Magenes. In their work [6] H.Brezis and S.Wainger stressed the importance, in this field, of the Lorentz Spaces . They were first used in the context of PDEs by L.Tartar in [21]. F.Hélein's proof of the regularity of the harmonic maps from a surface into a manifold required the use of the duality $L^{2,\infty} - L^{2,1}$ in two dimension and was perhaps one of the first use of Lorentz Spaces in Calculus of Variations. Not much has been done by the mean of these interpolation spaces in this context since Hélein's work. The originality of the technics introduced in the present paper is perhaps the use of this duality $L^{2,\infty} - L^{2,1}$ in dimension larger than two (larger than the conformal one for the Dirichlet part of the energy). This was made possible through a kind of dimension reduction argument (extraction of a good slice...etc) which permits to finally work in two dimension where these Spaces become more efficient for this type of non linearities.

As in [12], the $\eta-$compactness, theorem I.1, can be used to obtain compactness in $W^{1,p}(\Omega)$, $1 \leq p < 3/2$ for a subsequence of any sequence of solutions u of (I.1) in a bounded smooth domain of $\mathbf{R}^3$ in the following range of energy

$$E_\varepsilon(u) = O(\log \frac{1}{\varepsilon}) \tag{I.3}$$

and with the Dirichlet boundary condition $u_\varepsilon = g_\varepsilon : \partial\Omega \to \mathbf{C}$, where g_ε is a smoothing of any map g into S^1 smooth away from a finite number of points verifying $|\nabla g| \leq \frac{C}{d}$ where d is the distance to the singular points and g_ε verifies assumption $(A2)$ of [12]. Precisely we prove

Theorem I.2 *Let u_ε be a solution of (I.1) on a smooth bounded domain Ω of $\mathbf{R}^3$ with the energy bound given by (I.3) , assume $u_\varepsilon = g_\varepsilon$ on $\partial\Omega$, where g_ε verifies assumptions $(A2)$ of [12], then for any p strictly less than $3/2$,*

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)}$$

is uniformly bounded independently of ε.

Such $W^{1,p}$-compactness is also established in a work in preparation [5] under the assumption that $g \in H^{\frac{1}{2}}(\partial\Omega, S^1)$ only - Observe that our boundary conditions above, smooth maps away from a finite number of points, are dense in $H^{\frac{1}{2}}(\partial\Omega, S^1)$ (see[16]).

We think the $\eta-$compactness, theorem I.1 is probably the key point of the connection between the energy and the vortices of the solutions. In section III bellow, we show (as in [12] for minimizers) that, from the $\eta-$compactness in particular, we can deduce that the energy of solutions are concentrated on a stationary rectifiable one dimensional varifold as ε tends to zero. Precisely we have

Theorem I.3 *Let u_ε, g_ε, Ω be as in theorem I.2 and Ω is in addition convex. Then, by taking sequences if necessary,*

$$\mu_\varepsilon = \frac{e_\varepsilon(u)}{\pi \log \frac{1}{\varepsilon}} dx \rightharpoonup \mu$$

as Radon measures when $\varepsilon \rightarrow 0$. Moreover,

$$\mu = \Theta^1(\mu, x)\ \mathcal{H}^1 \lfloor \Sigma$$

and $V = (\Sigma, \Theta^1(\mu, x))$ is a stationary rectifiable one-dimensional varifold in Ω.

For the pair $(\Sigma, \Theta^1(\mu, x))$ to be a rectifiable varifold we refer to [17] page 77. The structure of one-dimensional stationary rectifiable varifold was completely understood, see [1]. They consist of locally finite many line segment connected at finitely many junctions ; and at each such a junction certain balancing conditions has to be satisfied due to the vanishing of the first variation of the length. .

II Proof of theorem I.1

The argument of the proof of theorem I.1 is decomposed into six main steps

II.1 Monotonicity property of the energy

The Pohozaev formula plays a major role in the proof of the $\eta-$compactness in 2-D (see [15]). Similarly in 3-D the corresponding identity, which is the monotonicity formula bellow, will be used intensively all along the proof.

Lemma II.1 *Let u be a solution of (I.1) then for any $x \in \Omega$ and $B_r(x) \in \Omega$ the following holds*

$$\begin{aligned} &\frac{d}{dr}\left[\frac{1}{r}\int_{B_r(x)} |\nabla u|^2 + \frac{3}{2\varepsilon^2}(1-|u|^2)^2\right] \\ &= \frac{1}{r}\int_{\partial B_r(x)} 2\left|\frac{\partial u}{\partial \nu}\right|^2 + \frac{1}{\varepsilon^2}(1-|u|^2)^2 \end{aligned} \tag{II.1}$$

This lemma is proved in [12] (lemma II.4).

We will use the following notations

$$E_r = \int_{B_r} \hat{e}_\varepsilon(u) = \int_{B_r} |\nabla u|^2 + \frac{3}{2\varepsilon^2}(1-|u|^2)^2$$

$$I_r = \int_{\partial B_r} \hat{e}_\varepsilon(u)$$

$$F_r = \int_{B_r} f_\varepsilon(u) = \int_{B_r} 2\left|\frac{\partial u}{\partial \nu}\right|^2 + \frac{1}{\varepsilon^2}(1-|u|^2)^2$$

$$J_r = \int_{\partial B_r} f_\varepsilon(u)$$

using these notations the monotonicity formula reads

$$\frac{d}{dr}\left[\frac{E_r}{r}\right] = \frac{1}{r}\frac{dF_r}{dr} = \frac{1}{r}J_r \tag{II.2}$$

As a simple consequence of the previous identity the following lemma holds

Lemma II.2 *For any $B_r(x) \subset \Omega$, and u solution of (I.1), we have*

$$\int_{\partial B_r} \hat{e}_\varepsilon(u) \geq \frac{1}{r}\int_{B_r} \hat{e}_\varepsilon(u)$$

II.2 The choice of a "good annulus"

The following lemma holds

Lemma II.3 *Let u be a solution of (I.1) and assume that there exists $4\varepsilon < \rho < 1$ such that*

$$\frac{1}{\rho}\int_{B_\rho} e_\varepsilon(u) \le \eta\ \log\frac{\rho}{\varepsilon}$$

then there exists $r_1 \in [2\varepsilon, \rho]$ such that

$$\frac{1}{r_1}\int_{B_{r_1}} f_\varepsilon(u) \le 8\eta$$

$$\int_{\partial B_{r_1}\cup\partial B_{r_1/2}} f_\varepsilon(u) \le 8\eta$$

Proof of lemma II.3 The proof of lemma II.3 is a simple consequence of the combination of the monotonicity formula (II.1) and the use of the mean value formula.

Integrating (II.2) between ε and ρ we get

$$\int_\varepsilon^\rho \frac{J_r}{r} = \frac{E_\rho}{\rho} - \frac{E_\varepsilon}{\varepsilon} \tag{II.3}$$

This implies in particular

$$\int_\varepsilon^\rho \frac{1}{r}\frac{dF_r}{dr} = \frac{E_\rho}{\rho} - \frac{E_\varepsilon}{\varepsilon}$$

Thus integrating by parts this yields

$$\frac{F_\rho}{\rho} - \frac{F_\varepsilon}{\varepsilon} + \int_\varepsilon^\rho \frac{F_r}{r^2} = \frac{E_\rho}{\rho} - \frac{E_\varepsilon}{\varepsilon}$$

which implies

$$\int_\varepsilon^\rho \frac{F_r}{r^2} \le \frac{E_\rho}{\rho} + \frac{F_\varepsilon}{\varepsilon} - \frac{E_\varepsilon}{\varepsilon} \tag{II.4}$$

From (II.3) we also deduce

$$\int_{2\varepsilon}^\rho \frac{J_{r/2}}{r} \le \frac{E_\rho}{\rho} - \frac{E_\varepsilon}{\varepsilon} \tag{II.5}$$

Combining (II.3), (II.4) and (II.5) we obtain

$$\int_{2\varepsilon}^\rho \frac{J_r}{r} + \frac{J_{r/2}}{r} + \frac{F_r}{r^2} \le 4\frac{E_\rho}{\rho} + \frac{F_\varepsilon}{\varepsilon} - 3\frac{E_\varepsilon}{\varepsilon}$$

Clearly $\frac{F_\varepsilon}{\varepsilon} \leq 3\frac{E_\varepsilon}{\varepsilon}$, so, using the hypothesis of the lemma, we have

$$\int_{2\varepsilon}^{\rho} \frac{1}{r}\left[J_r + J_{r/2} + \frac{F_r}{r}\right]\ dr \leq 4\eta\ \log\frac{\rho}{\varepsilon}$$

Using now the mean value formula this yields the existence of r_1 in $[2\varepsilon, \rho]$ such that

$$J_{r_1} + J_{r_1/2} + \frac{F_{r_1}}{r_1} \leq 8\eta$$

where we have used the fact that $\rho > 4\varepsilon$ that implies $\log 2 \leq \log\frac{\rho}{2\varepsilon}$. which proves the lemma.

We will also need the following soft quantization result :

Lemma II.4 *Let u be a map into $\mathbf{C}$ such that $|\nabla u| \leq C_0/\varepsilon$ and $r \geq 2\varepsilon$ then*

$$\int_{\partial B_r} \frac{(1-|u|^2)^2}{\varepsilon^2} \leq \frac{7^2\pi}{2^{12}C_0^2}$$

$$\Longrightarrow |u| \geq 1/2 \quad \textit{on } \partial B_r$$

Proof of lemma II.4 Assume there exists $y \in \partial B_r$ such that $|u|(y) < 1/2$ since $|\nabla|u|| \leq |\nabla u| \leq C_0/\varepsilon$ we have

$$|u| \geq \frac{3}{4} \qquad \text{in} \qquad B_{\varepsilon/4C_0}$$

Thus $\frac{(1-|u|^2)^2}{\varepsilon^2} \geq 7^2/2^8\varepsilon^2$ on $B_{\varepsilon/4C_0}$. Integrating on $\partial B_r \cap B_{\varepsilon/4C_0}$ we get

$$\int_{\partial B_r} \frac{(1-|u|^2)^2}{\varepsilon^2} \geq \frac{7^2\pi}{2^{12}C_0^2}$$

which contradicts our hypothesis.

We choose r_1 given by lemma II.4 and we choose η small enough ($8\eta \leq 7^2/2^8C_0^2$) so that the conditions of lemma II.4 are fulfilled and we can conclude that

$$|u| \geq \frac{1}{2} \qquad \text{on } \partial B_{r_1} \cup \partial B_{r_1/2} \tag{II.6}$$

We call $B_{r_1}/B_{r_1/2}$ the "good annulus" in which we will work in almost up to the end of the proof.

The aim is to prove that

$$E = \frac{1}{r_1} \int_{B_{r_1}} e_\varepsilon(u) \leq C\eta^{1/2} \tag{II.7}$$

where C is independant on ε (small enough) and u satisfying the hypothesis of the theorem.

In order to get rid of the size r_1 that would appear in all the forthcoming estimates we rescale u in the following way. Let

$$\tilde{u}(x) = u(r_1 x)$$

Then $\tilde{u}$ is a solution of

$$\Delta \tilde{u} + \left(\frac{r_1}{\varepsilon}\right)^2 \tilde{u}(1 - |\tilde{u}|^2) = 0 \tag{II.8}$$

denote also

$$\tilde{\varepsilon} = \frac{\varepsilon}{r_1}$$

we then have

$$\begin{aligned} &\int_{B_1} e_{\tilde{\varepsilon}}(\tilde{u}) = E \\ &\int_{B_1} f_{\tilde{\varepsilon}}(\tilde{u}) \leq 8\eta \\ &\int_{\partial B_1 \cup \partial B_{1/2}} f_{\tilde{\varepsilon}}(\tilde{u}) \leq 8\eta \end{aligned} \tag{II.9}$$

and the hypothesis of the theorem, $|\nabla u| \leq C_0/\varepsilon$ and $|u| \leq 1$ become

$$|\nabla \tilde{u}| \leq \frac{C_0}{\tilde{\varepsilon}} \quad \text{and} \quad |\tilde{u}| \leq 1 \tag{II.10}$$

Observe also that the monotonicity formula implies

$$\int_{B_1} e_{\tilde{\varepsilon}}(\tilde{u}) + \int_{B_1} f_{\tilde{\varepsilon}}(\tilde{u}) = \int_{\partial B_1} e_{\tilde{\varepsilon}}(\tilde{u})$$

$$4 \int_{B_{1/2}} e_{\tilde{\varepsilon}}(\tilde{u}) + 2 \int_{B_{1/2}} f_{\tilde{\varepsilon}}(\tilde{u}) = 2 \int_{\partial B_{1/2}} e_{\tilde{\varepsilon}}(\tilde{u})$$

So that

$$\int_{\partial B_1 \cup \partial B_{1/2}} e_{\tilde{\varepsilon}}(\tilde{u}) \leq 4E \tag{II.11}$$

where we assume that $E \geq 16\eta$ otherwise we go directly to subsection II.6

We will use the notation

$$a \wedge b = (ia, b)$$

for any a and b in $\mathbf{C}$ and where (,) denotes the usual scalar product in $\mathbf{C}$.

II.3 $L^{2,1}$ bound of $u \wedge du$ on spheres in the good annulus.

The aim of this part is to prove the following lemma

Lemma II.5 *There exists C_1 independant of $\tilde{\varepsilon}$, $\tilde{u}$, η or E such that the following holds*

$$\left|\left\{r \in [1/2, 1] \quad \textit{such that} \quad \|\tilde{u} \wedge d\tilde{u}\|_{L^{2,1}(\partial B_r)} \leq C_1 \ (E + E^{1/2})\right\}\right| \geq \frac{1}{3}$$

$L^{2,1}$ is the Lorentz space defined in [20].

Proof of Lemma 2.4. Since $\tilde{u}$ satisfies (II.8) it implies in particular that

$$\tilde{u} \wedge \Delta \tilde{u} = 0$$

in other words, if d^* denotes the adjoint of the exterior differentiation for the L^2 scalar product, the equation above becomes

$$d^*(\tilde{u} \wedge d\tilde{u}) = 0 \tag{II.12}$$

Computing now the laplacian of $\tilde{u} \wedge d\tilde{u}$ we then obtain

$$\begin{aligned} -\Delta(\tilde{u} \wedge d\tilde{u}) \quad &= dd^*(\tilde{u} \wedge d\tilde{u}) + d^*d(\tilde{u} \wedge d\tilde{u}) \\ &= d^*d(\tilde{u} \wedge d\tilde{u}) \end{aligned} \tag{II.13}$$

Our notations imply that

$$d(\tilde{u} \wedge d\tilde{u}) = 2d\tilde{u}_1 \wedge d\tilde{u}_2 = 2\sum_{i=1}^{3} \left[\frac{\partial \tilde{u}_1}{\partial x_{i+1}}\frac{\partial \tilde{u}_2}{\partial x_{i-1}} - \frac{\partial \tilde{u}_1}{\partial x_{i-1}}\frac{\partial \tilde{u}_2}{\partial x_{i+1}}\right] dx_{i+1} \wedge dx_{i-1}$$

(indices i are in $\mathbf{Z}_3$). Using standard extension theorems we obtain the existence of $\hat{u}_1$ and $\hat{u}_2$ in $W^{1,2}(\mathbf{R}^3)$ so that

$$\|\nabla\hat{u}\|_{L^2(\mathbf{R}^3)} \leq C\|\nabla\tilde{u}\|_{L^2(B_1\setminus B_{1/2})} \leq C\ E^{1/2} \tag{II.14}$$

Let ξ be the 2-form in $\mathbf{R}^3$ verifying

$$-\Delta\xi = 2d\hat{u}_1 \wedge d\hat{u}_2$$

Using the result of R.Coifman, P.L.Lions, Y.Meyer and S.Semmes [7], we know that $d\hat{u}_1 \wedge d\hat{u}_2$ is in the Hardy space $\mathcal{H}^1(\mathbf{R}^3)$ with the following bound

$$\|d\hat{u}_1 \wedge d\hat{u}_2\|_{\mathcal{H}^1(\mathbf{R}^3)} \leq C\|\nabla\hat{u}_1\|_2\ \|\nabla\hat{u}_2\|_2 \leq C\ E \tag{II.15}$$

Indeed for instance

$$\frac{\partial\tilde{u}_1}{\partial x_1}\frac{\partial\tilde{u}_2}{\partial x_2} - \frac{\partial\tilde{u}_1}{\partial x_2}\frac{\partial\tilde{u}_2}{\partial x_1} = \begin{pmatrix} \dfrac{\partial\tilde{u}_1}{\partial x_1} \\[2mm] \dfrac{\partial\tilde{u}_1}{\partial x_2} \\[2mm] \dfrac{\partial\tilde{u}_1}{\partial x_3} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial\tilde{u}_2}{\partial x_2} \\[2mm] -\dfrac{\partial\tilde{u}_2}{\partial x_1} \\[2mm] 0 \end{pmatrix}$$

$B = \nabla u_1$ is curl-free and in L^2 and $E = (\partial_2 u_2, -\partial_1 u_2, 0)$ is divergence-free and in L^2. Using now the classical result of C.Fefferman and E.Stein [9] (which can also be a definition of Hardy) we deduce that ξ is in $W^{2,1}(\mathbf{R}^3)$ with the following bound

$$\|\xi\|_{W^{2,1}(\mathbf{R}^3)} \leq C\|\Delta\xi\|_{\mathcal{H}^1(\mathbf{R}^3)} \leq C\ E \tag{II.16}$$

It is clear that $\tilde{u} \wedge d\tilde{u} - d^*\xi$ is an harmonic form on $B_1 \setminus B_{1/2}$. By the mean of (II.11) and (II.16) we control some norm of it on $\partial(B_1 \setminus B_{1/2})$ by $C\,E^{1/2}$, so that on $B_{19/20} \setminus B_{11/20}$ we have

$$\|\tilde{u} \wedge d\tilde{u} - d^*\xi\|_{W^{1,1}(B_{19/20}\setminus B_{11/20})} \leq C\ E^{1/2} \tag{II.17}$$

Combining this identity and (II.16) we finally obtain

$$\|\tilde{u} \wedge d\tilde{u}\|_{W^{1,1}(B_{19/20}\setminus B_{11/20})} \leq C\ (E + E^{1/2}) \tag{II.18}$$

Lemma II.5 is now a simple consequence of the combination of the previous estimate and the embedding of $W^{1,1}(\partial B_r)$ into $L^{2,1}(\partial B_r)$ for any r in $[1/2, 1]$ with a uniformly bounded modulus of continuity (see a presentation of this embedding in 2-D in [22]).

II.4 $L^{2,\infty}$ bound of $u \wedge du$ on spheres in the "good annulus".

Lemma II.6 *There exists C_2 independant of $\tilde{\varepsilon}$, $\tilde{u}$, η and E such that*

$$\left|\left\{r \in [1/2,1] \qquad \text{such that } \|\tilde{u}\wedge d\tilde{u}\|_{L^{2,\infty}(\partial B_r)} \leq C_2\,\eta\right\}\right| \geq \frac{1}{3}$$

Proof of lemma II.6 We will denote the "good annulus" $B_1 \setminus B_{1/2}$ by T. In $T = B_1 \setminus B_{1/2}$ we introduce the following function

$$\begin{cases} a(x) = \dfrac{1}{|\tilde{u}|^2} & \text{for } x \text{ such that } |\tilde{u}|(x) \geq 1/2 \\ a(x) = 4 \end{cases}$$

we have

$$\tilde{u}\wedge d\tilde{u} = a(x)\tilde{u}\wedge d\tilde{u} + (1-a(x))\tilde{u}\wedge d\tilde{u}$$

Observe that first we have

$$\forall x \in T \qquad |1-a(x)| \leq 4|1-|\tilde{u}|^2|$$

Thus

$$\begin{aligned}\int_T |(1-a(x))\tilde{u}\wedge d\tilde{u}|^2 &\leq 16\int_T (1-|\tilde{u}|^2)^2|d\tilde{u}|^2 \\ &\leq 16C_0^2\int_T \frac{(1-|\tilde{u}|^2)^2}{\tilde{\varepsilon}^2}\end{aligned}$$

and using (II.9) the previous inequality implies

$$\int_T |(1-a(x))\tilde{u}\wedge d\tilde{u}|^2 \leq 128C_0^2\,\eta \tag{II.19}$$

Consider now the function H such that

$$\begin{cases} \Delta H = d^*\left[a(x)\tilde{u}\wedge d\tilde{u}\right] \\ \dfrac{\partial H}{\partial \nu} = a(x)\tilde{u}\wedge\dfrac{\partial \tilde{u}}{\partial r} & \text{on } \partial B_1 \\ \dfrac{\partial H}{\partial \nu} = -a(x)\tilde{u}\wedge\dfrac{\partial \tilde{u}}{\partial r} & \text{on } \partial B_{1/2} \end{cases} \tag{II.20}$$

Using (II.12) we obtain

$$\Delta H = d^*((a(x)-1)\tilde{u}\wedge d\tilde{u})$$

Standard elliptic estimates imply

$$\begin{aligned}\left[\int_T |\nabla H|^2\right]^{1/2} &\le C\left[\int_T |(a(x)-1)\tilde{u}\wedge d\tilde{u}|^2\right]^{1/2}\\ &+C\left[\int_{\partial T}\left|\frac{\partial\tilde{u}}{\partial r}\right|^2\right]^{1/2}\end{aligned}$$

Using (II.9) and (II.19) we finally get

$$\|\nabla H\|_{L^2(T)} \le C\ \eta^{1/2} \tag{II.21}$$

Since we have

$$d^*\left[dH - a(x)\tilde{u}\wedge d\tilde{u}\right] = 0 \qquad \text{on } T$$

there exists a 2-form ψ such that

$$a(x)\tilde{u}\wedge d\tilde{u} = dH + d^*\psi \qquad \text{on } T \tag{II.22}$$

Moreover, since $|\tilde{u}|\ge 1/2$ on ∂T, $a(x)=\frac{1}{|\tilde{u}|^2}$ on ∂T and (II.20) implies that the restriction to ∂T of $d*\psi$ is 0. Thus $d*\psi$ is the unique solution (see lemma A.1) of

$$\left\{\begin{array}{ll} dd^*\psi = d\left[a(x)\tilde{u}\wedge d\tilde{u}\right] & \text{on } \partial T\\ & \\ \iota^*_{\partial T} d*\psi = 0 & \end{array}\right. \tag{II.23}$$

if $a(x)=\frac{1}{|\tilde{u}|^2}$, $d\left[a(x)\tilde{u}\wedge d\tilde{u}\right]=0$ so that $d\left[a(x)\tilde{u}\wedge d\tilde{u}\right]$ is supported in the set where $|\tilde{u}|\le 1/2$ and we have, using the fact that $4d[\tilde{u}\wedge d\tilde{u}]$ is made of 2-jacobiens of $\tilde{u}$,

$$\begin{aligned}\int_T |d\left[a(x)\tilde{u}\wedge d\tilde{u}\right]| &\le 12\int_{T\cap\{x;\ |\tilde{u}|<\frac{1}{2}\}}|\nabla\tilde{u}|^2 \le \frac{12C_0^2}{\tilde{\varepsilon}^2}\int_{T\cap\{x;\ |\tilde{u}|<\frac{1}{2}\}} 1\\ &\le \frac{2^6C_0^2}{3^2}\int_T\frac{(1-|\tilde{u}|^2)^2}{\tilde{\varepsilon}^2}\end{aligned} \tag{II.24}$$

Combining this estimate and lemma IV.1 we obtain

$$\|d^*\psi\|_{L^p(T)} \leq C\ \eta \tag{II.25}$$

for any $p < 3/2$.

Since $a(x) = \frac{1}{|\tilde{u}|^2}$ in a neighborhood of ∂T, $d\,[a(x)\tilde{u} \wedge d\tilde{u}] = 0$ in a neighborhood of ∂T in T and if we extend $d\,[a(x)\tilde{u} \wedge d\tilde{u}]$ by 0 outside T on whole $\mathbf{R}^3$, we still have a closed form that we denote by dv. Let $\hat{\psi}$ be the closed 2-form equal to

$$\hat{\psi} = -\frac{1}{4\pi}\ \frac{1}{|x|} \star dv \tag{II.26}$$

($\star$ means the convolution). From (II.24) we deduce that

$$\|dv\|_{L^1(\mathbf{R}^3)} \leq C\ \eta \tag{II.27}$$

Since

$$|d^*\hat{\psi}|(x) \leq C \int_T \frac{|dv|(y)}{|x-y|^2}$$

Using now lemma IV.2, we get the existence of a constant C independant on $\tilde{\varepsilon}$ and η such that

$$\left|\left\{r \in [1/2, 1]\ \text{ such that } \|d^*\hat{\psi}\|_{L^{2,\infty}(\partial B_r)} \leq C\eta\right\}\right| \geq \frac{11}{20} \tag{II.28}$$

Using the closedness of $\hat{\psi}$, (II.23) and (II.26) we have that $d^*(\psi - \hat{\psi})$ is harmonic on T. Combining (II.25) (II.26) and (II.27) we have

$$\|d^*(\psi - \hat{\psi})\|_{L^p(T)} \leq C\eta \tag{II.29}$$

for any $p < 3/2$. Thus by standard result on harmonic function we deduce that

$$\|d^*(\psi - \hat{\psi})\|_{W^{1,2}(B_{19/20} \setminus B_{11/20})} \leq C\ \eta \tag{II.30}$$

So using trace theorems we have for all $11/20 < r < 19/20$

$$\|d^*(\psi - \hat{\psi})\|_{L^{2,\infty}(\partial B_r)} \leq \|d^*(\psi - \hat{\psi})\|_{L^2(\partial B_r)} \leq C\eta \tag{II.31}$$

Combining (II.28) and (II.31) we obtain that

$$\left|\left\{r \in [1/2, 1]\ \text{ such that }\ \|d^*\psi\|_{L^{2,\infty}(\partial B_r)} \leq C\eta\right\}\right| \geq \frac{11}{20} \tag{II.32}$$

Combining (II.19), (II.21) and (II.32) we establish lemma II.5.

Combining now lemma II.4 and lemma II.5 we obtain the following result

Lemma II.7 *There exists C_1 independant of $\tilde{\varepsilon}$, $\tilde{u}$, η or E such that,*

$$\left|\left\{r \in [1/2,1] \quad \text{such that } \|\tilde{u}\wedge d\tilde{u}\|_{L^2(\partial B_r)} \leq C\eta^{1/4}(E^{1/2}+E^{1/4})\right\}\right| \geq \frac{1}{6}$$

Proof of lemma 2.6. The lemma is a simple consequence of the following interpolation inequality which is also an application of the duality between $L^{2,1}$ and $L^{2,\infty}$ (see [20])

$$\|\tilde{u}\wedge d\tilde{u}\|_{L^2(\partial B_r)} \leq \|\tilde{u}\wedge d\tilde{u}\|^{\frac{1}{2}}_{L^{2,1}(\partial B_r)}\|\tilde{u}\wedge d\tilde{u}\|^{\frac{1}{2}}_{L^{2,\infty}(\partial B_r)}$$

II.5 L^2 bound of $|\nabla|\tilde{u}||$ in the good annulus.

Let $1/4 > \delta > 0$ be any constant. We have the following lemma

Lemma II.8 *There exists a constant C depending only on C_0 (the upper-bound of $\tilde{\varepsilon}\|\nabla\tilde{u}\|_\infty$) such that*

$$\int_T |\nabla|\tilde{u}||^2 \leq C\left[\delta E + \frac{\eta}{\delta^2}\right]$$

Proof of lemma II.8. Let χ be the following cut-off function

$$\chi(s) = 1 \text{ for } s \in [0, 1-2\delta] \qquad \chi(s) = 1 \text{ for } s \geq 1-\delta$$

and such that $|\chi'| \leq 2/\delta$. On the set where $|\tilde{u}| \geq 1/2$, $|\tilde{u}|$ satisfies

$$\Delta|\tilde{u}| + |\tilde{u}|\left(\frac{1-|\tilde{u}|^2}{\tilde{\varepsilon}^2} - \frac{|\tilde{u}\wedge d\tilde{u}|^2}{|\tilde{u}|^4}\right) = 0 \tag{II.33}$$

Let multiply (II.33) by $\chi(|\tilde{u}|)(1-|\tilde{u}|)$ and integrate on T, we get

$$\begin{aligned}
&\int_T \chi(|\tilde{u}|)|\nabla|\tilde{u}||^2 + \int_{\partial T}\chi(|\tilde{u}|)(1-|\tilde{u}|)\frac{\partial|\tilde{u}|}{\partial\nu}\\
&+\int_T |\tilde{u}|(1+|\tilde{u}|)\ \chi(|\tilde{u}|)\frac{(1-|\tilde{u}|)^2}{\tilde{\varepsilon}^2} = \int_T \frac{1-|\tilde{u}|}{|\tilde{u}|^3}|\tilde{u}\wedge d\tilde{u}|^2\\
&+\int_T \chi'(|\tilde{u}|)\ (1-|\tilde{u}|)\ |\nabla|\tilde{u}||^2
\end{aligned}$$

If we denote by $\Omega_\delta = \{x \in T \quad ; \quad |\tilde{u}|(x) \geq 1-\delta\}$, we have

$$\begin{aligned}\int_{\Omega_\delta} |\nabla|\tilde{u}||^2 \quad &\leq 4\delta \int_T |\nabla \tilde{u}|^2 + \frac{4C_0^2}{\delta^2} \int_T \frac{(1-|\tilde{u}|^2)^2}{\tilde{\varepsilon}^2} \\ &+ \int_{\partial T} |1-|\tilde{u}||^2 + \int_{\partial T} \left|\frac{\partial |\tilde{u}|}{\partial r}\right|^2\end{aligned}$$

Using (II.9) we deduce

$$\int_{\Omega_\delta} |\nabla|\tilde{u}||^2 \leq 4\delta E + \frac{32C_0^2}{\delta^2}\eta + 8\eta \tag{II.34}$$

Moreover using (II.9) again we have

$$|T \setminus \Omega_\delta| \, \frac{\delta^2}{\tilde{\varepsilon}^2} \leq \int_T \frac{(1-|\tilde{u}|^2)^2}{\tilde{\varepsilon}^2} \leq 8\eta$$

Using also $|\nabla|\tilde{u}||^2 \leq \frac{C_0^2}{\tilde{\varepsilon}^2}$ we obtain

$$\int_{T\setminus\Omega_\delta} |\nabla|\tilde{u}||^2 \leq \frac{8C_0^2}{\delta^2}\,\eta \tag{II.35}$$

Combining (II.34) and (II.35) we prove lemma II.8.

II.6 Conclusion

It is clear that $\|\nabla\tilde{u}\|^2_{L^2(\partial B_r)} = \|\nabla|\tilde{u}|\|^2_{L^2(\partial B_r)} + \|\nabla\frac{\tilde{u}}{|\tilde{u}|} \wedge d\tilde{u}\|^2_{L^2(\partial B_r)}$. For η small enough, by the mean of Lemma II.3 and Lemma II.4, using lemma II.7 lemma II.8 we deduce the existence of $r \in [1/2, 1]$ such that $|\tilde{u}| \geq 1/2$ on ∂B_r and

$$\|\nabla\tilde{u}\|_{L^2(\partial B_r)} + \frac{1}{\varepsilon}\|1-|\tilde{u}|^2\|_{L^2(\partial B_r)} \leq C\left[\eta^{1/4}(E^{1/2}+E^{1/4}) + \delta^{1/2}E^{1/2} + \frac{\eta^{1/2}}{\delta}\right]$$

Going back to the original scale, this yields the existence of $r \in [r_0/2, r_0]$ such that

$$\int_{\partial B_r} \hat{e}_\varepsilon(u) \leq C\eta^{1/2}(E+E^{1/2}) + \delta E + \frac{\eta}{\delta^2}$$

Using lemma II.2 and the monotonicity formula, we have

$$\frac{2}{r_1}\int_{B_{r_1/2}} \hat{e}_\varepsilon(u) \le C\eta^{1/2}(E + E^{1/2}) + \delta E + \frac{\eta}{\delta^2} \tag{II.36}$$

The monotonicity formula combined with (II.9) yields

$$\begin{aligned}
&\frac{1}{r_1}\int_{B_{r_1}} \hat{e}_\varepsilon(u) - \frac{2}{r_1}\int_{B_{r_1/2}} \hat{e}_\varepsilon(u) = \int_{r_1/2}^{r_1} \frac{1}{r}\int_{\partial B_r} f_\varepsilon(u) \\
&= \frac{1}{r_1}\int_{B_{r_1}} f_\varepsilon(u) - \frac{2}{r_1}\int_{B_{r_1/2}} f_\varepsilon(u) + \int_{r_1/2}^{r_1} \frac{1}{r^2}\int_{B_r} f_\varepsilon(u) \\
&\le 16\eta
\end{aligned} \tag{II.37}$$

So (II.36) and (II.37) imply

$$E \le C\left[\eta^{1/2}(E + E^{1/2}) + \delta E + \frac{\eta}{\delta^2} + \eta\right]$$

Let choose $\delta^2 = \eta^{1/2}$. Then we have

$$E \le C\left[\eta^{1/4}E + \eta^{1/2}E^{1/2} + \eta^{1/2}\right]$$

Choosing now $C\eta^{1/4} < 1/2$ we thus have $E \le 4C\eta^{1/2}$. Using the monotonicity formula we obtain

$$\frac{1}{\varepsilon}\int_{B_\varepsilon(x)} \hat{e}_\varepsilon(u) \le 4C\eta^{1/2}$$

In particular we have

$$\int_{B_\varepsilon(x)} (1 - |u|^2)^2 \le 4C\eta^{1/2}\varepsilon^3$$

Since $|\nabla|u|| \le \frac{C_0}{\varepsilon}$, if η is chosen small enough this implies that $|u|(x) > 1/2$.

We have then proven, that, under the hypothesis of theorem I.1 we have $|u|(x) > 1/2$. Taking η a bit smaller we deduce that the same hypothesis imply $|u| > 1/2$ on $B_{\rho/2}$.

III Consequences of the $\eta-$compactness Theorem

III.1 Proof of theorem I.2.

In two dimension an important step to get the $W^{1,p}$ estimate for sequences of solutions of (I.1) was to establish that the bad set $\mathcal{B}_\varepsilon = \{x\ ;\ |u_\varepsilon|(x) \leq 1/2\}$ could be covered by a uniformly bounded number of balls of radius ε (see [4]). In other words we had

$$|\mathcal{B}_\varepsilon| \leq C\varepsilon^2 \tag{III.1}$$

This bound of the size of the bad set was a direct consequence of the Pohozaev formula and the use of the L^∞ estimate of the gradient of u. In fact this bound still holds in 3 dimension for solutions of (I.1) verifying the hypothesis of theorem I.2. Precisely we have

Lemma III.1 *Let u_ε be a solution of (I.1) verifying the hypothesis of theorem I.2, then (III.1) holds.*

Proof of Lemma III.1. In order to simlify the presentation we give the arguments of the proof of (III.1) away from the boundary. The whole proof would require the eta-compactness up to the boundary stated in [12] (Lemma II.8). So Let $0 < \alpha < 1$ and let $K \subset\subset \Omega$, for any $x \in K$ we assign $\varepsilon^\alpha < \rho_x < \text{dist}(K, \partial\Omega)$ such that

$$\frac{1}{\rho_x}\int_{B_{\rho_x}(x)} \frac{(1-|u|^2)^2}{\varepsilon^2} \leq C$$

where C is independent on x and ε. This is obtained by integrating the monotonicity formula (Lemma II.2) like in the begining of the proof of lemma II.3 (we have $\int_{\varepsilon^\alpha}^{\text{dist}(K,\partial\Omega)} F_r/r^2 \leq C\log\frac{1}{\varepsilon}...$). From the $B_{\rho_x}(x)$ we extract a Besicovitch covering $(B_{\rho_i}(x_i))_{i\in I}$ of $\mathcal{B}_\varepsilon$ such that any point of $\cup_{i\in I}B_{\rho_i}(x_i)$ is covered by a uniformly bounded number of balls. Since $|u| \geq 1/2$ on $\mathcal{B}_\varepsilon$, on one hand we have

$$|\mathcal{B}_\varepsilon| \leq C\sum_{i\in I}\int_{B_{\rho_i}(x_i)} (1-|u|^2)^2 \leq C\varepsilon^2\sum_{i\in I}\rho_i$$

On the other hand we have, because of the $\eta-$compactness,

$$\eta\sum_{i\in I}\rho_i\ \log\frac{\rho_i}{\varepsilon}\le\sum_{i\in I}e_\varepsilon(u_\varepsilon)\le C\log\frac{1}{\varepsilon}$$

Since $\rho_i\ge\varepsilon^\alpha$ and $\alpha<1$, the previous inequality implies $\sum_{i\in I}\rho_i$ is uniformly bounded and Lemma III.1 follows at once.

We explain now how the $W^{1,p}$ estimate of theorem I.2 follows from the bound (III.1). Let $a(x)=1/|u|^2$ on $\Omega\setminus\mathcal{B}_\varepsilon$ and $a(x)=4$ otherwise. We proceed to the following Hodge decomposition $a(x)\,u\wedge du=dH+d^*\psi$ where $H=0$ on $\partial\Omega$ (it is clear that both dH and $d^*\psi$ are unique). On one hand we have

$$\left\{\begin{array}{l}\Delta H=d^*(a(x)\ u\wedge du)=d^*\left((a(x)-1)\ u\wedge du\right)\\ \\ H=0\qquad\text{on }\partial\Omega\end{array}\right.$$

Standard estimates imply, for any $p<2$ and $\delta>0$ to be chosen small later,

$$\begin{aligned}\int_\Omega|dH|^p&\le C\int_\Omega|a(x)-1|^p|u\wedge du|^p\\ &\le C\delta^p\textstyle\int_{|a(x)-1|<\delta}|u\wedge du|^p+C\int_{|a(x)-1|>\delta}|u\wedge du|^p\\ &\le C\delta^p\textstyle\int_\Omega|u\wedge du|^p+|\{x\ ;\ |a(x)-1|>\delta\}|\frac{1}{\varepsilon^p}\end{aligned}\tag{III.2}$$

Using the previous lemma we have $|\{x\ ;\ |a(x)-1|>\delta\}|\le C\varepsilon^2$ so finally for H we get

$$\int_\Omega|dH|^p\le C\delta^p\int_\Omega|u\wedge du|^p+C\varepsilon^{2-p}\tag{III.3}$$

On the other hand $d^*\psi$ verifies

$$\left\{\begin{array}{ll}dd^*\psi=d\left(a(x)u\wedge du\right)=8\chi_{\mathcal{B}_\varepsilon}du_1\wedge du_2&\text{on }\Omega\\ \\ d^*\psi=g_\varepsilon\wedge dg_\varepsilon\qquad\text{on }\partial\Omega\end{array}\right.$$

where $\chi_{\mathcal{B}_\varepsilon}$ is the characteristic function of $\mathcal{B}_\varepsilon$. Using the previous Lemma and the L^∞ bound on ∇u we have

$$\int_\Omega|\chi_{\mathcal{B}_\varepsilon}du_1\wedge du_2|\le C$$

where C is independent of ε. So standard elliptic estimates imply for $p < 3/2$

$$\int_\Omega |d^*\psi|^p \leq C_p \tag{III.4}$$

Combining (III.3) and (III.4) we get

$$\int_\Omega |u \wedge du|^p \leq C\delta \int_\Omega |u \wedge du|^p + C_p + C_\delta \varepsilon^{2-p} \tag{III.5}$$

Choosing δ small enough we get the desired L^p estimate for $u \wedge du$ and the one for $d|u|$ can be deduced using similar arguments as the one presented in [14]. This proves Theorem I.2

III.2 Proof of theorem I.3.

By taking subsequences if needed, we may assume $\mu_\varepsilon \rightharpoonup \mu$ as Radon measures on Ω. This comes from the assumption on the upperbound of the total energy $E_\varepsilon(u) \leq K \log \frac{1}{\varepsilon}$ which implies

$$\mu_\varepsilon(\Omega) \leq K$$

As in [12], we let

$$\Sigma = \left\{ x \in \Omega \quad : \quad \Theta^1(\mu, x) > 0 \right\}$$

Here

$$\Theta^1(\mu, x) = \lim_{r \to 0} \frac{\mu(B_r(x))}{r}$$

Due to the monotonicity formula we see that

Lemma III.2 $\Theta^1(\mu, x)$ *exists at every point* $x \in \Omega$ *and it is upper-semi continuous in* x.

<u>We claim</u> $\mathrm{spt}(\mu \cap \Omega) \subset \Sigma$.

Indeed, if $x_0 \in \Omega \setminus \Sigma$, then there is r_0 such that

$$\frac{\mu(B_{r_0}(x_0))}{r_0} < \eta$$

(the constant η comes from the $\eta-$compactness theorem I.1). Then it follows from theorem I.1, that one has, for ε small, that $|u| \geq 1/2$ on $B_{r_0/2}(x_0)$.

We then write $u_\varepsilon(x) = \rho_\varepsilon(x)\exp(i\psi_\varepsilon(x))$ on $B_{r_0/2}(x_0)$. Ginzburg-Landau Equations imply

$$\mathrm{div}(\rho_\varepsilon^2 \nabla\psi_\varepsilon) = 0 \qquad \text{in} \qquad B_{r_0/2}(x_0) \tag{III.6}$$

Elliptic estimates for (III.6) imply

$$\begin{aligned}\int_{B_{r_0/4}(x_0)} \rho_\varepsilon^2 |\nabla\psi_\varepsilon|^2 \, dx \ &\leq \frac{C}{r_0^2}\int_{B_{r_0/2}} |\psi_\varepsilon - \overline{\psi}_\varepsilon|^2 \, dx \\ &\leq C\|\nabla\psi_\varepsilon\|^2_{L^p(B_{r_0/2})} \leq C_* \qquad (p = 6/5)\end{aligned}$$

On the other hand, arguments in section II.5 imply that

$$\int_{B_{r_0/4}(x_0)} |\nabla\rho_\varepsilon|^2 \, dx \leq C_*$$

Therefore $\mu_\varepsilon(B_{r_0/4}(x_0)) \leq C_*/(\pi\log\frac{1}{\varepsilon}) \to 0^+$ as $\varepsilon \to 0^+$. In other words, $\mathrm{spt}\mu \cap B_{r_0/4}(x_0) = \emptyset$. This proves the claim. In the same way one can show that

$$\Sigma = \left\{ x \in \Omega \quad ; \quad \Theta^1(\mu, x) \geq \eta/2 \right\} \tag{III.7}$$

(III.7) implies in particular that $\mathcal{H}^1(\Sigma) \leq C < \infty$.

The rectifiability of Σ (or equivalently the varifold V) follows from the same arguments in [12], see also [11].

Finally to verify that V is a stationary varifold, we follow [2] section 5. Indeed let

$$V_\varepsilon = \delta_{I-2P^tP}(A)\frac{|\nabla u_\varepsilon|^2}{2\pi\log\frac{1}{\varepsilon}} \qquad \text{where } P = \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$$

Then V_ε is a generalised varifold as in [2]. Moreover, the first variation of V_ε can be calculated by

$$\begin{aligned}\delta V_\varepsilon(X) &= \int_\Omega \left[\frac{|\nabla u|^2}{2} I - \nabla u_\varepsilon \otimes \nabla u_\varepsilon\right](x) \ : \ DX(x)dx \\ &= -\frac{1}{\pi\log\frac{1}{\varepsilon}}\int_\Omega \frac{(1-|u|^2)^2}{4\varepsilon^2} \ \ \mathrm{div}\, X(x) \ \ dx\end{aligned}$$

Here $X \in C_0^1(\Omega, \mathbf{R}^n)$. We claim $\delta V_\varepsilon \to 0$ in the sense of distribution that is,

$$-\frac{1}{\pi\log\frac{1}{\varepsilon}} \ \frac{(1-|u|^2)^2}{4\varepsilon^2} \ dx \to 0$$

as Radon measures.

In fact if we set

$$\nu_\varepsilon = \frac{(1-|u|^2)^2}{4\varepsilon^2 \log\frac{1}{\varepsilon}}\, dx \rightharpoonup \nu$$

as Radon measures on Ω. Then $\nu << \mu$.

Moreover, from the monotonicity formula again, one has

$$\frac{\mu(B_r(x))}{r} - \Theta^1(\mu, x) \geq \int_0^r \frac{\nu(B_r)}{r^2} dr$$

This implies in particular that $\liminf \frac{\nu(B_r)}{r} = 0$. Thus $\nu \equiv 0$ since $\nu << \mu$.

In other words, in the terminology of [2] we have $V_\varepsilon \rightharpoonup V$ and V is a stationary generalized varifold. The final conclusion of Theorem I.3 follows by [2] Theorem 3.8.

IV Appendix

Lemma IV.1 *Let Ω be a smooth domain of $\mathbf{R}^3$ such that $\partial\Omega$ is simply connected. Let v be a smooth 1-form on Ω and ψ be a solution of*

$$\begin{cases} dd^*\psi = dv & \text{in } \Omega \\ \\ \iota^*_{\partial\Omega} d * \psi = 0 \end{cases}$$

where $\iota_{\partial\Omega}$ is the embbeding of $\partial\Omega$ in $\mathbf{R}^3$. Then $d^*\psi$ is the unique minimizer of

$$\min_{d^*\psi\in V} \int_\Omega |d^*\psi - v|^2 \tag{IV.1}$$

where V is given by

$$V = \left\{ d^*\psi \quad \text{such that} \quad \psi \in W^{1,2}(\Lambda^2\Omega) \quad \iota^*_{\partial\Omega} d * \psi = 0 \right\}$$

Moreover for any $p < 3/2$ we have

$$\left[\int_\Omega |d^*\psi|^p\right|^{\frac{1}{p}} \leq C_p(\Omega) \int_\Omega |dv| \tag{IV.2}$$

Proof of lemma IV.9. The existence of a minimizer for (IV.9) can be seen in the following way. Let K be the function solution of

$$\left\{\begin{array}{ll}\Delta K = 0 & \text{in } \Omega\\[2ex] \dfrac{\partial K}{\partial \nu} = < v ; \nu > & \text{on } \partial\Omega\end{array}\right. \tag{IV.3}$$

where $<\, ;\, >$ is the duality between 1-forms and vectors. We have

$$d(*(v - dK)) = 0 \qquad \text{in } \Omega$$

Moreover

$$\iota^*_{\partial\Omega} * (v - dK) = 0$$

Since Ω is a subset of $\mathbf{R}^3$ the inclusion map $H_1(\partial\Omega) \to H_1(\Omega)$ is surjective and thus $*(v - dK)$ is exact and there exists a 2-form ψ such that

$$d * \psi = *(v - dK)$$

Because of (IV.3) ψ verifies

$$\left\{\begin{array}{ll}dd^*\psi = dv & \text{in } \Omega\\[2ex] \iota^*_{\partial\Omega} d * \psi = 0 & \end{array}\right. \tag{IV.4}$$

we claim that for any $d^*\xi$ in V

$$\int_\Omega |d^*\psi - v|^2 \le \int_\Omega |d^*\xi - v|^2 \tag{IV.5}$$

Indeed, we have for any $d^*\xi$ in V

$$\int_\Omega (d^*\psi - v).d^*\xi = \int_\Omega (d^*\psi - v) \wedge d * \xi$$
$$= \int_{\partial\Omega} (d^*\psi - v) \wedge *\xi$$

Since $d^*\xi \in V$ and $\partial\Omega$ is simply connected $*\xi = d\gamma$ on $\partial\Omega$ and

$$\int_{\partial\Omega} (d^*\psi - v) \wedge *\xi = \int_{\partial\Omega} d(d^*\psi - v) \wedge \gamma = 0$$

Thus for any $d^*\xi \in V$

$$\int_\Omega (d^*\psi - v).d^*\xi = 0 \tag{IV.6}$$

and (IV.5) is just a projection theorem. We have in fact

$$\int_\Omega |d^*\xi - d^*\psi|^2 + \int_\Omega |d^*\psi - v|^2 = \int_\Omega |d^*\xi - v|^2$$

and the uniqueness and minimality of $d^*\psi$ follow at once.

Let k be any 1-form in $L^q(\Omega)$ for $q > 3$, from the Hodge-Morrey decomposition we deduce the existence of α, β and h such that

$$k = d\alpha + d^*\beta + h$$

where $\alpha \in W_0^{1,q}(\Omega)$, $\beta \in W^{1,q}(\Lambda^2\Omega)$ and h is an harmonic 1-form on Ω, moreover

$$\left\{ \begin{array}{l} \|\beta\|_{W^{1,q}} \leq C\|k\|_{L^q} \\ \\ \iota^*_{\partial\Omega} * \beta = 0 \end{array} \right. \tag{IV.7}$$

We have

$$\begin{aligned} \int_\Omega d^*\psi.k \ &= \int_\Omega d * \psi \wedge d\alpha + \int_\Omega d^*\psi \wedge d * \beta \\ &+ \int_\Omega d * \psi \wedge h \end{aligned}$$

Clearly

$$\int_\Omega d * \psi \wedge d\alpha = \int_{\partial\Omega} d * \psi \wedge \alpha = 0$$

Moreover

$$\begin{aligned} \int_\Omega d * \psi \wedge h \ &= \int_{\partial\Omega} *\psi \wedge h + \int_\Omega *\psi \wedge dh \\ &= \int_{\partial\Omega} *\psi \wedge h \end{aligned}$$

Since $\partial\Omega$ is simply connected and $\iota^*_{\partial\Omega} d * \psi = 0$ we have $*\psi = d\gamma$ on $\partial\Omega$ and

$$\int_{\partial\Omega} *\psi \wedge h = \int_{\partial\Omega} d\gamma \wedge h = 0$$

since h is harmonic. So

$$\begin{aligned}\int_\Omega d^*\psi.k &= \int_\Omega d^*\psi\wedge d*\beta=\int_\Omega dd^*\psi\wedge *\beta\\ &=\int_\Omega dv\wedge *\beta\end{aligned}\tag{IV.8}$$

Where we have used (IV.4) and (IV.7). Since $q>3$ we have an embedding $W^{1,q}\to L^\infty$ and $\|\beta\|_\infty\leq C\|k\|_{L^q}$. From (IV.8) we deduce then that, for any $k\in L^q$,

$$\left|\int_\Omega d^*\psi.k\right|\leq C\|k\|_{L^q}\ \int_\Omega|dv|$$

which implies the desired result and lemma IV.1 is proved.

Lemma IV.2 *For any $g\in L^1(\mathbf{R}^2\times[0,1])$, if we denote*

$$f=\frac{1}{|x|^2}\star g$$

for any $\delta>0$ there exists a subset $E_\delta\subset[0,1]$, verifying

$$|E_\delta|>1-\delta$$

such that

$$\forall r\in E_\delta\qquad \|f\|_{L^{2,\infty}(\mathbf{R}^2\times\{r\})}\leq C_\delta\|g\|_{L^1}$$

where C_δ only depends on δ.

Proof of lemma IV.10 take $g>0$ and fix any $0<\delta<1$

$$\int_{\mathbf{R}^2\times\{r\}}g(y,r)\ dy\quad\in\quad L^1([0,1])$$

Thus, following [19],

$$M_g(s)=\sup_{t>0}\frac{1}{t}\int_{s-t}^{s+t}dr\int_{\mathbf{R}^2\times\{r\}}g(y,r)\ dy$$

is in $L^{1,\text{weak}}([0,1])$: we have for all $\mu>0$

$$\left|\left\{s\quad;\quad\sup_{t>0}\frac{1}{t}\int_{s-t}^{s+t}dr\int_{\mathbf{R}^2\times\{r\}}g(y,r)\ dy\geq\mu\right\}\right|\leq C_0\frac{\|g\|_{L^1}}{\mu}$$

Choose μ_δ such that $C_0 \frac{\|g\|_{L^1}}{\mu_\delta} = \delta$ and denote

$$E_\delta = \{s \quad \text{such that} \quad M_g(s) \leq \mu_\delta\}$$

To simplify notations we assume that $0 \in E_\delta$. We will prove now that for all $\lambda > 0$

$$\big|\{x \in \mathbf{R}^2 \quad \text{such that} \quad f(x,0) \geq \lambda\}\big| \leq C\frac{\|g\|_{L^1}}{\lambda^2}$$

Fix $\lambda > 0$, we have

$$f(x,0) = \int_0^1 \int_{y \in \mathbf{R}^2} \frac{g(y,r)}{|x-y|^2 + r^2} dy\ dr$$

we have

$$f(x,0) = f_1(x) + f_2(x) + f_3(x)$$

where

$$f_1(x) = \int_0^1 \int_{|y-x| \geq 1/\lambda} \frac{g(y,r)}{|x-y|^2 + r^2} dy\ dr$$

$$f_2(x) = \int_{1/\lambda}^1 \int_{|x-y| \leq 1/\lambda} \frac{g(y,r)}{|x-y|^2 + r^2} dy\ dr$$

$$f_3(x) = \int_0^{1/\lambda} \int_{|x-y| \leq 1/\lambda} \frac{g(y,r)}{|x-y|^2 + r^2} dy\ dr$$

First of all we have

$$\begin{aligned} f_1(x) &\leq \int_0^1 \int_{\mathbf{R}^2} \frac{g(y,r)}{\lambda^{-2} + r^2} dy\ dr \\ &\leq \sum_{i=0}^{\infty} 2^{-i} \frac{\lambda^2}{1 + \lambda^2 2^{-2i-2}} \frac{1}{2^{-i}} \int_{2^{-i-1}}^{2^{-i}} \int_{\mathbf{R}^2} g(y,r)\ dy \\ &\leq \int_0^1 \frac{\lambda^2}{1 + \lambda^2 r^2} dr\ \mu_\delta \\ &\leq c_1 \|g\|_{L^1}\ \lambda \end{aligned}$$

Second we have

$$\begin{aligned}
f_2(x) &= \int_{1/\lambda}^{1}\int_{|x-y|\leq 1/\lambda}\frac{g(y,r)}{|x-y|^2+r^2}dy\ dr\\
&\leq \int_{1/\lambda}^{1}\frac{1}{r^2}\left(\int_{\mathbf{R}^2} g(y,r)\ dy\right)\ dr\\
&\leq \sum_{i=0}^{\log_2\lambda} 2^{2i}\int_{2^{-i-1}}^{2-i}\int_{\mathbf{R}^2} g(y,r)\ dy\\
&\leq \int_{1/\lambda}^{1}\frac{dr}{r^2}\mu_\delta\\
&\leq c_2\lambda\ \|g\|_{L^1}
\end{aligned}$$

For estimating the last function f_3 we decompose $f_3(x)$ into the sum $f_3(x) = f_{31}(x) + f_{32}(x)$ where

$$f_{31}(x) = \int_0^{\frac{1}{\lambda}} dr\int_{y\ ;\ |x-y|\leq r}\frac{g(y,r)}{|x-y|^2+r^2}dy$$

We have in one hand

$$\begin{aligned}
&\int_{\mathbf{R}^2} f_{31}(x)\ dx \leq \int_0^{\frac{1}{\lambda}}\frac{dr}{r^2}\int_{y\in\mathbf{R}^2} g(y,r)\ \int_{x\ ;\ |x-y|\leq r} dx\ dy\\
&\leq \frac{1}{\lambda}\left[\lambda\int_0^{\frac{1}{\lambda}}\int_{\mathbf{R}^2} g(y,r)\ dy\ dr\right]\leq\frac{\mu_\delta}{\lambda}
\end{aligned}\tag{IV.9}$$

in the other hand

$$\begin{aligned}
&\int_{\mathbf{R}^2} f_{32}(x)\ dx \leq \int_0^{\frac{1}{\lambda}} dr\int_{y\in\mathbf{R}^2} g(y,r)\ \int_{x\ ;\ r\leq|x-y|\leq\frac{1}{\lambda}}\frac{dx}{|x-y|^2}\ dy\\
&\leq C\int_0^{\frac{1}{\lambda}}\log\frac{1}{\lambda r}\ dr\int_{y\in\mathbf{R}^2} g(y,r)\\
&\leq C\sum_{i\ ;\ 2^{-i}\leq 1/\lambda} 2^{-i}\ \log\frac{1}{2^{-i}\lambda}\ \frac{1}{2^{-i}}\int_0^{2^{-i}}\int_{y\in\mathbf{R}^2} g(y,r)dy\ dr\\
&\leq C\mu_\delta\int_0^{\frac{1}{\lambda}}\log\frac{1}{\lambda r}dr\leq C\frac{\mu_\delta}{\lambda}
\end{aligned}\tag{IV.10}$$

Thus combining (IV.9) and (IV.10) we have $\int_{\mathbf{R}^2} f_3(x) \leq C\frac{\mu_\delta}{\lambda}$ and this imply that

$$\{x \quad ; \quad f_3(x) \geq \lambda\} \leq \frac{\|f_3\|_{L^1}}{\lambda} \leq C\frac{mu_\delta}{\lambda^2}$$

Since the statement of our theorem is linear in g (i.e. f depends linearly on g) we may allways assume that $(c_1 + c_2)\|g\|_{L^1} < 1$. So we have

$$\left\{x \in \mathbf{R}^2 \quad \text{such that} \quad f(x) \geq 3\lambda\right\} \subset \left\{x \in \mathbf{R}^2 \quad \text{such that} \quad f_3(x) \geq \lambda\right\}$$

Thus

$$\left|\left\{x \in \mathbf{R}^2 \quad \text{such that} \quad f(x) \geq \lambda\right\}\right| \leq \frac{C\|g\|_{L^1}}{\lambda^2}$$

which proves the lemma.

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